Physical and Mathematical Behavior of Black Diring Solutions

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Abstract

We consider some characteristics of the solutions of five-dimensional black dirings. First we confirm the equivalence of the two different solution-sets of the black dirings(one was generated by the authors with the solitonic method similar to the Backlund transformation and the other was by Evslin and Krishnan with the inverse scattering method). Then we show some physical properties of the systems of black diring: especialy the existence of thermo-dynamical systems of black diring and their properties.

1 Introduction

Previously we discovered that the five-dimensional S^1-rotating black rings appeared first in Ref. [1] can be superposed in concentric way and succeeded to construct regular black diring systems as the simplest case(called diring I) [2]. For the construction above we used the solitonic method similar to the Backlund transformation that was developed by us for the first time to generate non-trivial five-dimensional axisymmetric spacetimes with asymptotic flatness [3–5]. Following the above work, Evslin and Krishnan constructed another diring solution-set(called diring II) [6]. They used the inverse scattering method that was modified by Pomeransky to treat the higher dimensional case(hereafter abbreviated to PISM) [7]. However, because of the complexity of their expressions, the study to confirm the equivalence of these two diring solution-sets and further investigation of the physics of the diring systems still remain to be done. Here we give some answeres to these problems.

For the system of the dirings, we consider five-dimensinal spacetimes with three commuting Killing vector fields: a time-like Killing field and two axial Killing fields. We assume further that one of the axial Killing fields is orthogonal to the other Killing fields. So the line-elements adopted here is reduced to

$$ds^2 = G_{tt}(dx^0)^2 + 2G_{t\psi}dtd\psi + G_{\psi\psi}(d\psi)^2 + G_{\phi\phi}(d\phi)^2 + e^{2\nu}\left(d\rho^2 + dz^2\right) , \tag{1}$$

where the metric coefficients are the function of (ρ, z) and $\det G = -\rho^2$ is imposed. Owing to the assumptions we can say that any type of angular momentums corresponding to ϕ-rotaion is zero.

In the first half of the paper we show the complete equivalence of these two different representations with the aid of the facts established by Hollands and Yazadjiev, which concern the uniqueness of higher-dimensional black holes. In the latter half we give some physical quantities of the dirings. Using these quatities, we clarify some physical properties of the regular black diring systems. Especially we show the existence of thermo-dynamical diring systems and some peculiar properties of the thermal systems.

2 Equivalence of diring I and diring II

The strategy adopted here is the following. First, reconstructing the solution-set of diring I with the PISM we show that the difference between the diring I and diring II comes from the difference of the corresponding seeds. Then we give the moduli-parameters and physical quantities to identify the solution-sets respectively. Using these quantities we confirm the equivalence of these two solution-sets with the aid of the facts established by Hollands and Yazadjiev [8]. Once the equivalence is established, we can use the more convenient representation among the diring I and diring II according to problems we face.

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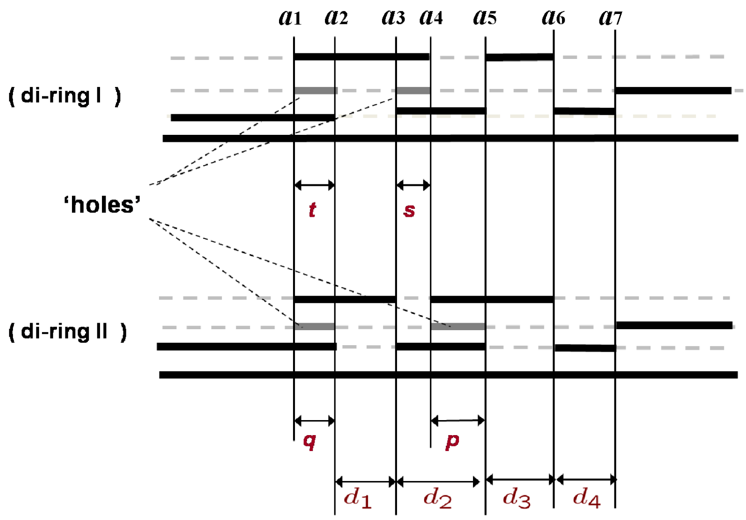

Figure 1: Rod-structures describing the seeds of the dirings. The upper and lower rod-diagrams correspond to the diring I and diring II respectively. Black rods interpreted to have 1/2 line mass density, while gray rods(holes) correspond to −1/2 line mass density. Two solitons are removed and recovered at the positions a_1 and a_4.

A key mathematical fact to establish the equivalence of diring I and diring II is in the work by Hollands and Yazadjiev which have discussed the uniqueness of five dimensional stationary black holes with axial $U(1)^2$-symmetry. Originally they have considered the sytems of single black hole, but their discussions can be applied to the systems of multiple black holes so that the statement is still valid with some modification. It is described that two different systems of multiple black holes are isometirc when all the rod-lengths and the Komar angular momentums coincide with each other. It should be noticed that validity of the proof seem to hold, whether conical singularities on the axes exist or not. So the above statement remains useful for the spacetimes with conical singularities. For the systems of black diring, two Komar angular momentums corresponding to the ϕ-rotation are zero so that the other two Komar angular momentums corresponding to the ψ-rotation are essential to determine the solution, once the rod-lengths are fixed. We can say further that two independent physical quantities can be used to determine the solution in place of two Komar angular momentums at least locally.

The diring I and diring II are generated by PISM from the corresponding seeds respectively. The rod structures described in Fig.1 show the seeds that are used to generate diring I and diring II. The parameters (s,t) and (p,q) inscribed on the Fig.1 mean lengths of the gray rods(holes) and have the following range:

$$0 \le s \le d_2\,,\ 0 \le t\,, \tag{2}$$

$$0 \le p \le d_2\,,\ 0 \le q\,. \tag{3}$$

Following the procedure of PISM, first two solitons with trivial BZ-parameters are removed at the positions a_1 and a_4 and then the solitons are recovered with non-trivial BZ-parameters at the same positions. Here $(b_{\rm I},\, c_{\rm I})$ and $(b_{\rm II},\, c_{\rm II})$ are assigned to BZ-parameters of diring I and II respectively. After adjusting the BZ-parameters in the following way

$$\left\{ b_{\rm I} = \pm\left(\frac{2a_{21}a_{61}a_{71}}{a_{31}a_{51}}\right)^{1/2},\ c_{\rm I} = \pm\left(\frac{2a_{42}a_{64}a_{74}}{a_{43}a_{54}}\right)^{1/2} \right\} \tag{4}$$

$$\left\{ b_{\rm II} = \pm\left(\frac{2a_{31}a_{61}a_{71}}{a_{21}a_{51}}\right)^{1/2},\ c_{\rm II} = \pm\left(\frac{2a_{43}a_{64}a_{74}}{a_{42}a_{54}}\right)^{1/2} \right\}, \tag{5}$$

we obtain the regular diring systems respectively up to conical singularities. The symbols a_{ij} is defined as $a_i - a_j$. The moduli-parameters to provide the solution-sets of diring I and II are $\{s,t,d_1,d_2,d_3,d_4\}_{\rm I}$ and $\{p,q,d_1,d_2,d_3,d_4\}_{\rm II}$ respectively. Other physical quantities of the dirings can be represented with these

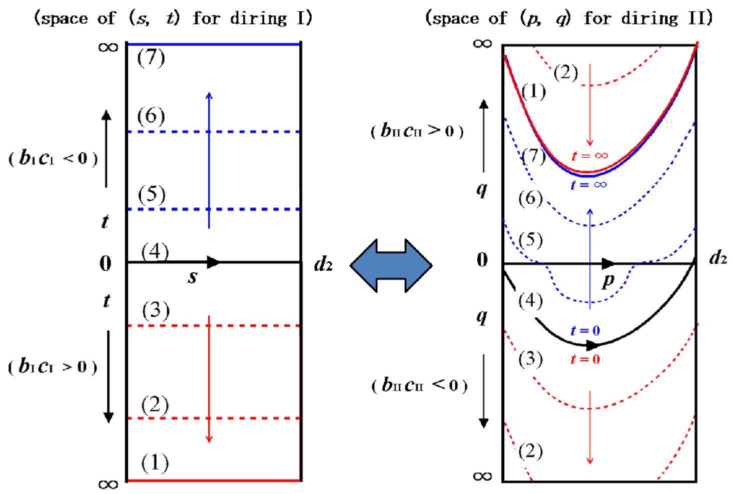

Figure 2: Correspondence between the space of (s,t) for diring I and the space of (p,q) for diring II. Each horizontal line with a number from (1) to (7) is mapped into the curve with the same number.

parameters. For example, ADM masses: $m_{\rm I}$ and $m_{\rm II}$, angles assuring regular axis for $[a_6, a_7]$: $(\Delta\phi_{\rm R})_{\rm I}$ and $(\Delta\phi_{\rm R})_{\rm II}$ and angles assuring regular axis for $[a_3, a_5]$: $(\Delta\phi_{\rm L})_{\rm I}$ and $(\Delta\phi_{\rm L})_{\rm II}$ are given as follows

$$m_{\rm I} = \frac{3\pi}{4}(a_{41}+a_{65}) + \frac{3\pi}{2}a_{41}\left[\frac{a_{41}}{(b_{\rm I}-c_{\rm I})^2} - \frac{b_{\rm I}}{b_{\rm I}-c_{\rm I}}\right], \qquad m_{\rm II} = \frac{3\pi}{4}(a_{31}+a_{64}), \tag{6}$$

$$\left(\frac{\Delta\phi_{\rm R}}{2\pi}\right)^2_{\rm I} = \frac{a_{73}^2 a_{76}(a_{74}b_{\rm I} - a_{71}c_{\rm I})^2}{a_{71}^2 a_{72}^2 a_{74} a_{75}(b_{\rm I}-c_{\rm I})^2}, \qquad \left(\frac{\Delta\phi_{\rm R}}{2\pi}\right)^2_{\rm III} = \frac{a_{71}a_{74}a_{73}a_{76}}{a_{72}^2 a_{75}^2}, \tag{7}$$

$$\left(\frac{\Delta\phi_{\rm L}}{2\pi}\right)^2_{\rm I} = \frac{a_{53}^2 a_{73}^2 a_{62}(a_{51}a_{64}a_{74}b_{\rm I} - a_{54}a_{61}a_{71}c_{\rm I})^2}{a_{72}^2 a_{51}a_{52}a_{54}a_{61}a_{63}a_{64}a_{71}a_{74}(b_{\rm I}-c_{\rm I})^2}, \tag{8}$$

$$\left(\frac{\Delta\phi_{\rm L}}{2\pi}\right)^2_{\rm II} = \left(\frac{a_{53}a_{62}a_{73}}{a_{41}^2 a_{52}^2 a_{63}^2 a_{72}^2}\right)\left[a_{21}a_{31}a_{54}a_{64}a_{74} + a_{42}a_{51}(a_{43}a_{61}a_{71} + a_{21}a_{54}b_{\rm II}c_{\rm II})\right]. \tag{9}$$

Owing to the mathematical facts mentioned above, we can conclude that two dirings are isometrically equivalent when the coditions $(d_1, d_2, d_3, d_4)_{\rm I} = (d_1, d_2, d_3, d_4)_{\rm II}$ and $(m_{\rm I}, (\Delta\phi_{\rm R})_{\rm I}) = (m_{\rm II}, (\Delta\phi_{\rm R})_{\rm II})$ can be imposed. Once the conditions are satisfied, remarkably simple relations between (s,t) and (p,q) are extracted from the mass equality: $m_{\rm I}(s,t) = m_{\rm II}(p,q)$ and regular angle equality: $(\Delta\phi_{\rm R})_{\rm I} = (\Delta\phi_{\rm R})_{\rm II}$ as $(p,q) = (p(s,t), q(s,t))$. Using these equations we can confirm that the correspondence between (s,t)-space and (p,q)-space is onto and one-to-one If infinities on the spaces are included. Figure 2 shows the correspondence between (s,t)-space and (p,q)-space. Finally we can say that the systems of diring I and diring II are completely equivalent.

3 Thermo-dynamical black diring

The equivalence have been established in the previous section. So we can use the more convenient representation among the diring I and diring II according to problems we face. The existence of regular dirings have been already confirmed in Ref. [2] and Ref. [6]. In the rest we show some results of physical properpties of the systems of diring, especially the existence of thermo-dynamical dirings and their peculier properties. To assure that the multi-system becomes a thermal system, the surface gravities (κ_L, κ_R) and rotational angular velocities (ω_L, ω_R) of the inner and outer black rings of the system must be equal:

$$\kappa_L = \kappa_R, \qquad \omega_L = \omega_R. \tag{10}$$

We investigate whether thermo-dynamical diring systems exist or not by solving the above conditions numerically. The results are shown in Figure 3. The red line in the left picture of Fig.3 shows the

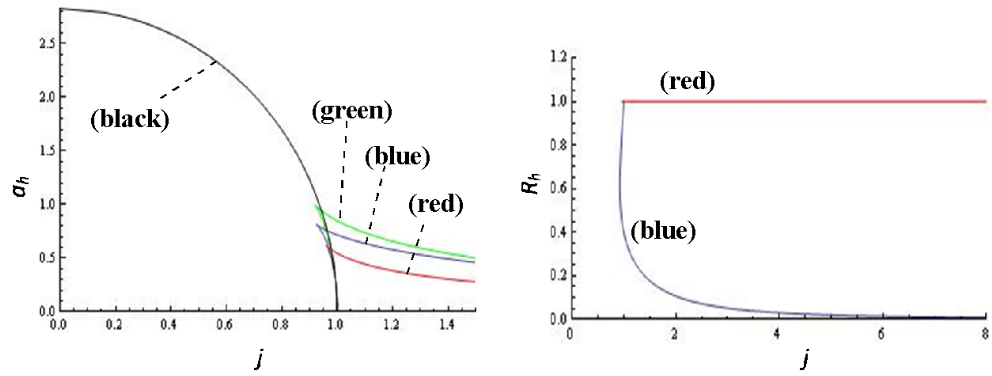

Figure 3: Left: Total area (a_h) as a function of angular momentum (j). All the quantities are normalized by ADM mass. Black, green, blue and red lines correspond to MP black hole (MP-BH), black ring (BR), black saturn (BS) and black diring (BD), respectively. Right: Ratio of the area of inner BR or BH to the area of outer BR (R_h) as a function of j. Green and blue lines correspond to BS and BR.

existence of thermo-dynamical black dirings. Other thermal systems: Meyres-Perry black holes, black rings, black saturns are also shown for comparing with the dirings. The right picture of Fig.3 shows the behavior of the ratio of the horizon-area of inner black ring or black hole to the horizon-area of outer black ring. The ratio of the system of diring holds a constant value one, while the ratio of the system of black saturn quickly vanishes as the angular momentum j increases. This means that in the system of black saturn the influence of the inner black hole quickly vanishes and in the diring the interaction of the inner and outer rings remains.

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