Black holes and wormholes in light of Weyl transformations*

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A lot can be learned about black holes and wormholes by re-scaling spacetime itself without changing the coordinates used to describe it. Such a conformal transformation is called a Weyl transformation. It takes spacetime from a given frame — called Einstein frame — to a conformal frame — called Jordan frame. Such a transformation reveals that horizons and wormholes might appear/disappear in the conformal frame even if they were absent/present in the original frame. It arises both from the simple prescription for defining black holes and wormholes, as well as from the more sophisticated definitions. In addition, some definitions might be transformed into one another under Weyl transformations.

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1. Introduction

A Weyl transformation consists in re-scaling spacetime without changing its coordinates. It is described by a simple transformation that takes an original spacetime metric $g_{\mu\nu}$ into a new metric $\tilde{g}_{\mu\nu}$ such that,

$$\tilde{g}_{\mu\nu} = e^{2\Omega} g_{\mu\nu}, \tag{1}$$

where $\Omega(x)$ is spacetime-dependent function, everywhere regular and non-vanishing.

On the other hand, various definitions of black holes and wormholes have been given in the literature. They range from what could be considered as a simple "prescription" to what could be called "sophisticated" definitions.

2. The simple "prescription"

This prescription works only for spherically symmetric metrics that depend only on the time coordinate t and a radial coordinate. The latter could be chosen to be the physical areal radius R that multiplies $\mathrm{d}\Theta^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2$. The radial coordinate could also be chosen to be any other real parameter r on which all the components of the metric would depend.

Using R as the radial coordinate, a spherically symmetric metric is written as,

$$\mathrm{d}s^2 = -A(t,R)\mathrm{d}t^2 + B(t,R)\mathrm{d}R^2 + R^2\mathrm{d}\Theta^2, \tag{2}$$

where, $A(t,R)$ and $B(t,R)$ are functions of the time coordinate t and the areal radius R. The simple prescription for defining black holes and wormholes consists then

*This talk is based on the published paper 1 (see Ref. 2 and the references therein) in checking whether the following algebraic equation,

$$g^{\mu\nu}\nabla_\mu R \nabla_\mu R = 0, \tag{3}$$

has a single- or a double-root. The single-root solution is identified with the location of the closed 2-surface of a black hole horizon, whereas the double-root is identified with the location of the 2-surface of a wormhole throat. Although this procedure is incomplete and imprecise, it is useful in practice since it gives a quick way of localizing the radial coordinate of an apparent horizon and/or a wormhole throat.

2.1. *Weyl transformation and the simple prescription*

The Weyl conformal transformation (1) changes the prescription (3) and the metric (2) into the following ones, respectively, to be used in the conformal frame,[2]

$$\mathrm{d}\tilde{s}^2 = -e^{2\Omega}A(t,R)\mathrm{d}t^2 + e^{2\Omega}B(t,R)\mathrm{d}R^2 + \tilde{R}(t)^2\mathrm{d}\Theta^2, \tag{4}$$

$$\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu \tilde{R}\tilde{\nabla}_\mu \tilde{R} = 0. \tag{5}$$

Here, $\tilde{R}(t) = e^{2\Omega}R(t)$ is the areal radius in te conformal frame. Using the metric (4), after having rewritten it entirely in terms of the pair $(\tilde{t}, \tilde{R})$ — which is in itself a very tedious task 2 — and then applying the conformal prescription (5), one finds,

$$\frac{1}{B}(\Omega_{,R}R + 1)^2 - \frac{1}{A}\Omega_{,t}^2 R^2 = 0. \tag{6}$$

While this condition does give the locations of the various possible black hole horizons and wormhole throats in the static case, it does not describe in detail their real origin in the dynamical case. As shown in Ref. 1, if one uses instead an arbitrary parameter r as the radial coordinate and writes the metric (2) as,

$$\mathrm{d}s^2 = -A(t,r)\mathrm{d}t^2 + B(t,r)\mathrm{d}r^2 + R(t,r)^2\mathrm{d}\Theta^2, \tag{7}$$

Weyl's conformal transformation (1) turns the latter into the following form,

$$\mathrm{d}\tilde{s}^2 = -e^{2\Omega}A(t,r)\mathrm{d}t^2 + e^{2\Omega}B(t,r)\mathrm{d}r^2 + \tilde{R}(t,r)^2\mathrm{d}\Theta^2. \tag{8}$$

Applying the conformal prescription (5) on this last form of the metric gives after a pretty short and a simple calculation the following condition for detecting black holes and/or wormholes,[1]

$$\frac{1}{B}(\Omega_{,r}R + R_{,r})^2 - \frac{1}{A}(\Omega_{,t}R + R_{,t})^2 = 0. \tag{9}$$

We clearly see form this condition how and when a black hole and/or a wormhole might arise in the conformal frame. In fact, contrary to Eq. (6), the condition (9) contains all the necessary information on the time and space dependences of both the areal radius R and the conformal exponent Ω.

3. "Sophisticated" definitions for general spacetimes

Besides the simple prescription (3) for detecting black hole horizons and wormholes, various other definitions are also given in the literature. Since these other definitions are rather valid for any spacetime and involve the concept of null vectors, trapped surfaces, and geodesic expansions, we call them the sophisticated definitions. An important fundamental feature of the sophisticated definitions is that, in contrast to the simple prescription (3), these more rigorous definitions do not allow a black hole horizon to coincide with a wormhole throat. These definitions make a clear distinction between the two concepts.

3.1. *Black hole horizon*

A generic black hole horizon is defined as being the future outer trapping horizon.[3] Such a statement is translated into the following three conditions to be satisfied on the 2-surface H of the horizon,[3]

$$\theta_+|_H = 0, \qquad \theta_-|_H < 0, \qquad \partial_-\theta_+|_H < 0. \tag{10}$$

The quantities $\theta_\pm$ represent the expansion of the outgoing (ingoing) null geodesics, the tangent vectors of which are denoted $l^\mu_\pm$, respectively. The partial derivatives, $\partial_\pm$, stand for derivative with respect to an affine parameter $u^\pm$ along the geodesic $l^\mu_\pm$.

3.2. *Hochberg-Visser wormhole*

A simple, covariant, and quasilocal definition of a wormhole throat, as defined in Ref. 4, and which we shall call a Hochberg-Visser wormhole, is that of a marginally anti-trapped surface. This definition does not involve any information about the faraway region outside the throat. It simply consists of the hypersurface foliated by compact spatial 2-surfaces S on which the following conditions are satisfied,[4]

$$\theta_\pm|_S = 0 \qquad \text{and} \qquad \partial_\pm\theta_\pm|_S > 0. \tag{11}$$

3.3. *Hayward wormhole*

Another simple, covariant, and quasilocal definition for wormholes is the one given in Ref. 5, and which we shall call here a Hayward wormhole. According to this definition, a wormhole throat is a *timelike* hypersurface foliated by a non-vanishing minimal spatial 2-surface S on a null hypersurface, *i.e.*, a timelike trapping horizon. This statement is formally expressed by the following three conditions,[5]

$$\theta_\pm|_S = 0, \qquad \partial_\pm\theta_\pm|_S > 0, \qquad \partial_\mp\theta_\pm|_S < 0. \tag{12}$$

In other words, Hochberg-Visser wormholes include spacelike hypersurfaces and therefore are not necessarily Hayward wormholes.

3.4. *Maeda-Harada-Carr wormhole*

This wormhole is a 2-surface required to be extremal on a spacelike hypersurface. Using a null coordinate foliation, a spherically symmetric metric takes the form $\mathrm{d}s^2 = -2e^{2f}\mathrm{d}u\mathrm{d}v + R^2\mathrm{d}\Theta^2$, where u and v are the null coordinates and $f = f(u, v)$ is a function of these. The Maeda-Harada-Carr wormhole then consists of the 2-sphere S, of radius $R = R(u, v)$, which is extremal and minimal in a spacelike radial direction ζ^μ. Formally, this translate into the following two conditions,[6]

$$R_{|A}\zeta^A\big|_S = 0 \qquad \text{and} \qquad R_{|AB}\zeta^A\zeta^B\big|_S > 0. \tag{13}$$

A vertical bar with the subscript, $|_A$, stands for a covariant derivative with respect to the two-metric g_{AB} of the two-dimensional spacetime spanned by the null vectors ∂_u and ∂_v.

3.5. *Tomikawa-Izumi-Shiromizu wormhole*

As in the case of the Maeda-Harada-Carr wormhole, the minimality of the 2-surface S representing a Tomikawa-Izumi-Shiromizu wormhole is imposed on a spacelike hypersurface. In contrast to all the above definitions, however, what is required for this wormhole is the vanishing, not of the expansions themselves, but of the difference $\theta_+ - \theta_-$ between the outgoing and ingoing expansions. This translates into the following two conditions,[7]

$$\theta_+ - \theta_-|_S = 0 \qquad \text{and} \qquad (\partial_+ - \partial_-)(\theta_+ - \theta_-)|_S > 0. \tag{14}$$

4. Weyl transformation and the sophisticated definitions

In order to find the affect of a Weyl transformation on each of the above more rigorous definitions, one only needs to figure out how the null tangents $l^\mu_\pm$, the transverse metric $h_{\mu\nu}$, and the null expansions $\theta_\pm$ transform under (1). It is straightforward to show that $\tilde{h}_{\mu\nu} = e^{2\Omega}h_{\mu\nu}$, $\tilde{l}^\mu_\pm = e^{-\Omega}l^\mu_\pm$ and that $\tilde{\theta}_\pm = e^{-\Omega}\left(\theta_\pm + 2\partial_\pm\Omega\right)$.[1] With these simple transformations at hand, it is easy to find the behavior under Weyl's conformal transformation of all the above sophisticated black hole and wormhole definitions.

4.1. *Conformal black holes*

The definition (10) conformally transforms into the following three conditions,[1]

$$\begin{gathered} \theta_+ + 2\partial_+\Omega\big|_H = 0, \qquad \theta_- + 2\partial_-\Omega\big|_H < 0, \\ \partial_-\theta_+ + 2\partial_-\partial_+\Omega\big|_H < 0. \end{gathered} \tag{15}$$

From the first equality, we learn that in the absence of a black hole in the original frame, it is possible for a black hole horizon to arise in the new frame provided only that the conformal exponent Ω does vary with u^+. On the other hand, to have a black hole in the conformal frame if one already exists in the original frame, Ω must be independent of u^+.

4.2. *Conformal Hochberg-Visser wormhole*

The definition (11) of such a wormhole conformally transforms into the following two conditions,[1]

$$\theta_\pm + 2\partial_\pm\Omega\big|_S = 0 \quad \text{and} \quad \partial_\pm\theta_\pm + 2\partial_\pm\partial_\pm\Omega\big|_S > 0. \tag{16}$$

From the first equality we see that to have a Hochberg-Visser wormhole in the new frame if one already exists in the old frame, the conformal exponent must again be independent of the parameter $u^\pm$.

4.3. *Conformal Hayward wormhole*

The definition (12) of such a wormhole conformally transforms into the following three conditions,[1]

$$\begin{gathered}\theta_\pm + 2\partial_\pm\Omega\big|_S = 0, \qquad \partial_\pm\theta_\pm + 2\partial_\pm\partial_\pm\Omega\big|_S > 0,\\ \partial_\mp\theta_\pm + 2\partial_\mp\partial_\pm\Omega\big|_S < 0.\end{gathered} \tag{17}$$

Like for the Hochberg-Visser wormhole, if a Hayward wormhole already exists in the old frame another one might arise in the new frame if Ω is indepednet of u^+. Recall that a Hayward wormhole is necessarily a Hochberg-Visser wormhole but the converse is not true. However, if the conformal factor is chosen such tha the last inequality in (17) is not satisfied but the second inequality is, then a Hayward wormhole transforms into a pure Hochberg-Visser wormhole.

4.4. *Conformal Maeda-Harada-Carr wormhole*

The definition (13) of such a wormhole conformally transforms into the following three conditions,[1]

$$\begin{aligned}&\zeta^A\left(R_{|A} + R\Omega_{|A}\right)\Big|_S = 0\\ &\zeta^A\zeta^B\left(R_{|AB} + R\Omega_{|AB} - R\Omega_{|A}\Omega_{|B}\right)\\ &\qquad + \zeta^A\zeta^B g_{AB}\left(R_{|C}\Omega^{|C} + R\Omega_{|C}\Omega^{|C}\right)\Big|_S > 0.\end{aligned} \tag{18}$$

Although these conditions are more complicated, it is clear that a Maeda-Harada-Carr wormhole can arise or disappear in the conformal frame depending on one's choice of Ω. For a worked out example, though, see Ref. 1.

4.5. *Conformal Tomikawa-Izumi-Shiromizu wormhole*

The definition (14) of such a wormhole conformally transforms into the following two conditions,[1]

$$\begin{gathered}\theta_+ - \theta_- + 2\left(\partial_+\Omega - \partial_-\Omega\right)\Big|_S = 0,\\ \left(\partial_+ - \partial_-\right)\left[\theta_+ - \theta_- + 2\left(\partial_+\Omega - \partial_-\Omega\right)\right]\Big|_S > 0.\end{gathered} \tag{19}$$

Again, we see that a Tomikawa-Izumi-Shiromizu wormhole can arise in the conformal frame, even if in the original frame there was none, provided that the conformal factor is chosen to satisfy both conditions (19). Worked out examples are found in Ref. 1.

5. Conclusion

A conformal transformation can make black holes and wormholes appear or disappear just by judiciously choosing the conformal factor Ω. A conformal transformation thus does have a non-trivial effect on a given physical concept just as it was already shown in Refs. 8, 9 regarding the fate of the concept of a quasilocal mass.

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