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Modified $F(R,T)$-Gravity Model Coupled with Magnetized Strange Quark Matter Fluid

Mohd Danish Siddiqi [1,†], Meraj Ali Khan [2,*,†] and Ibrahim Al-Dayel [2,†]

[1] Department of Mathematics College of Science, Jazan University, P.O. Box 277, Jazan 45142, Saudi Arabia; msiddiqi@jazanu.edu.sa
[2] Department of Mathematics and Statistics, Imam Muhammad Ibn Saud Islamic University, Riyadh 11566, Saudi Arabia; iaaldayel@imamu.edu.sa
* Correspondence: mskhan@imamu.edu.sa
† These authors contributed equally to this work.

Abstract: This research note presents the properties of the $F(R,T)$-gravity model in combination with magnetized strange quark matter. We obtain the equation of state for the magnetized strange quark matter in the $F(R,T)$-gravity model endowed with the Lagrangian through of Ricci curvature. We also examine the Ricci solitons supported by a time-like conformal vector field in $F(R,T)$-gravity, attached with magnetized strange quark matter fluid. Within this ongoing research, we give an estimate of the total quark pressure and total density in the phantom barrier and the radiation epochs of the Universe. Finally, using Ricci solitons, we study the various energy conditions, some black holes criteria, and Penrose's singularity theorem for magnetized strange quark matter fluid spacetime coupled with the $F(R,T)$-gravity model.

Keywords: $F(R,T)$-gravity; Ricci soliton; magnetized strange quark matter; conformal vector field; black holes

MSC: 53C44; 53B30; 53C50; 53C80

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1. Introduction

Einstein's gravitational field equations present a standard method for examining known cosmic dynamics [1,2]. The most reliable description of observed data is given by Einstein's field equation. The $F(R,T)$-gravity theory was formulated to tackle issues like the late-time acceleration of the Universe and the problems of dark matter and inflation in cosmological models.

In response to this scenario, several mathematicians and physicists have crafted advanced gravitational theories like $F(R)$-gravity [3–5], $F(G)$-gravity [6], and $F(R,T)$-gravity theory [7], among others. These theories originated from the Einstein–Hilbert action. Diverging from Einstein's conventional gravity theory, these models hold promise as potential accurate approximations to quantum gravity [8]. Utilizing the Einstein–Hilbert Lagrangian density, General Relativity (GR) can be expanded into $F(R)$ gravity, where $F(R)$ is a function of the Ricci scalar R. Neutron stars of considerable size within $F(R)$ gravity are characterized by higher-order curvature effects; refer to works such as [9–11] for more details. However, concerns have been raised about the applicability of $F(R)$ gravity due to its limitations in achieving equilibrium within the solar system and its inability to support various astronomical models, including stable star structures (refer to [12,13] for further insights).

Harko et al. [14] introduced a more inclusive gravity model known as $F(R, T)$-gravity theory, where they consider the Lagrangian to be an arbitrary function of T and R, with T denoting the trace of the energy–momentum tensor. This framework proved effective in describing the rapid expansion of the Universe during late cosmic times.
The notion of a quark star or a compact star, supported by the degenerate pressure of quark matter, has been proposed for stars smaller than neutron stars. Many researchers have delved into the characteristics of such quark stars. Alcock et al. [15] and Haensel et al. [16] suggest that a few neutron stars could potentially be exotic stars composed entirely of strange materials. Cheng et al. [17] investigated the properties of strange quark stars.

In this model, quarks are perceived as a degenerate Fermi gas restricted to the spatial area occupied by vacuum energy density. The quark content within this model comprises electrons, massless quarks u, massive quarks s, and quarks d.

According to the bag model, in scenarios where quarks are massless and non-interacting, quark pressure is anticipated.

$$\rho_q = 3p_q, \tag{1}$$

where the energy density of quarks is denoted by ρ_q. Furthermore, the total pressure p_M and total energy density ρ_M are expressed as

$$\rho_M = \rho_q + \mathcal{B}_c, \quad p_M = p_q - \mathcal{B}_c. \tag{2}$$

Ultimately, the equation of the state is employed to generate quark matter ($\mathcal{E}o\mathcal{S}$):

$$p_M = \frac{\rho_M - 4\mathcal{B}_c}{3}, \tag{3}$$

Here, $\mathcal{B}_c$ is referred to as the bag constant. This concept is in line with the phenomenological bag model of quark matter, where the quark confinement is described by an energy term that is proportional to the volume.

Substantial alterations in the properties of strange matter occur when the electromagnetic scale coincides with nuclear scales. In a study by [18], a phenomenological bag model has been employed to examine quark matter in a strong magnetic field, revealing that the stability of strange quark matter is enhanced when the magnetic field surpasses a specific critical strength. Literature references [18,19] highlight that the presence of a magnetic field induces pressure anisotropy, with the bag model being considered the most suitable approach for investigating magnetized strange quark matter (MSQM) as discussed in [20–22].

Moreover, the extension of the quasiparticle model has been applied to analyze MSQM in [23], unveiling a bag function dependent on density and magnetic field that achieves the maximum saturation density at the quantum chromodynamic (QCD) scale parameter [24]. General relativity (GR) and cosmology are both employed to represent a time-constrained, 4-dimensional connected Lorentzian manifold [25,26].

When the Ricci tensor exhibits a specific form, these spacetimes are referred to as perfect fluid spacetimes within quasi-Einstein Lorentzian manifolds [26,27].

$$Ric = A_1 d + A_2 \eta \otimes \eta \tag{4}$$

In this scenario, scalars A_1 and A_2 are included, d is the Lorentzian metric, while the 1-form η is metrically equivalent to a unit time-like vector field. Furthermore, a Lorentzian spacetime manifold is one that accommodates a vector field resembling time [28].

The concept of a generalized quasi-Einstein manifold (GQE) is elaborated in the manner outlined in [29,30].

Definition 1. *A non-flat Riemannian manifold (M^n, d) $(n > 2)$ is termed a generalized quasi-Einstein Lorentzian manifold (GQE) when its Ricci tensor S of type $(0,2)$ is non-zero and meets the condition:*

$$Ric = A_1 d + A_2 \eta \otimes \eta + A_3 \theta \otimes \theta \tag{5}$$

wherein A_1, A_2, and A_3 are smooth scalar functions of which $A_2 \neq 0$, $A_3 \neq 0$ and η, θ are 1-from such that

$$d(p,\gamma) = \eta(p), \qquad d(p,\zeta) = \theta(p)$$

for any vector filed $p \in \chi(M^n, d)$.

The unit vectors γ and ζ corresponding to 1-form η and θ are orthogonal to each other. In addition, γ and ζ are the generators of the manifold. If $A_3 = 0$, then (M^n, d) reduces to a perfect fluid spacetime.

Spacetime geometry, especially in the context of the General Theory of Relativity (GR), is intricately linked to the symmetry of physical matter. The solutions to field equations are often streamlined by the presence of symmetries in the metric. One notable form of symmetry is represented by solitons, which are connected to the geometric evolution of spacetime.

In 1988, Hamilton [31] introduced the concept of Ricci flow. The Ricci soliton emerges as the ultimate limit of solutions to the Ricci flow. The behavior of self-similar solutions or Ricci solitons often arises as limits of dilations of singularities in Ricci flow. They can be viewed as fixed points of Ricci flow, as a dynamical system, and the spaces of Riemannian metrics.

The Ricci flow equation is detailed in [31].

$$\frac{\partial}{\partial t} d = -2Ric, \tag{6}$$

A Riemannian manifold (M, d) that allows for a smooth vector field $\mathcal{F}$ is termed a Ricci soliton as per [32].

$$\frac{1}{2}\mathfrak{L}_{\mathcal{F}} d + \lambda d + Ric = 0, \tag{7}$$

where the Lie derivative the Ricci tensor, and a real number are denoted by $\mathfrak{L}_{\mathcal{F}}$, Ric, and λ, respectively. A Ricci soliton (RS) is said to be expanding, stable, or declining, with reference to (7), depending on whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

Researchers in [33] delved into spacetime through the lens of the Ricci soliton. Additionally, Venkatesha and Aruna discussed Ricci solitons within the context of ideal fluid spacetime in [34]. Siddiqi et al. (refer to [35–37]) scrutinized spacetime utilizing solitons across a range of methodologies.

Currently, the investigation of quark matter fluid represents a captivating research area. Within General Relativity (GR), the exploration of quark matter fluid is conducted under various assumptions. Mak and Harko [38] have employed conformal motion to analyze quark matter within spherically symmetric spacetime. Agarwal and Pawar [39] delved into a cosmological model incorporating quark matter within the context of $F(R,T)$-gravity theory. In a recent study in 2022, Siddiqi et al. [40,41] delved into the characteristics of $F(R,T)$-gravity featuring perfect fluid matter and accommodating Ricci solitons, Yamabe solitons, Einstein solitons, gradient Ricci solitons, and gradient Yamabe solitons. Building upon prior investigations, we explore an $F(R,T)$-gravity model within a magnetized strange quark matter fluid spacetime that admits a Ricci soliton in this paper.

2. Gravitational Field Equation of $F(R,T) = F_1(R) + F_2(T)$-Gravity Model Attached with Magnetized Strange Quark Matter Fluid

In this section, we delve into the $F(R,T) = F_1(R) + F_2(T)$-gravity model coupled with quark matter fluid. Given that this model hinges on the physical characteristics of the magnetized strange quark matter fluid, we can formulate a range of theoretical models across different values of R and T, as outlined in [14]. For illustrative purposes, we will consider the following model.

$$F(R,T) = F_1(R) + F_2(T), \tag{8}$$

where the functions of R and T are represented by $F_1(R)$ and $F_2(T)$, respectively.

The adjusted Einstein–Hilbert action term is considered as:

$$\Pi_E = \frac{1}{16\pi} \int [\mathbf{L}_m + F(R,T))] \sqrt{(-d)}\, d^4x, \tag{9}$$

where the Lagrangian is symbolized as $\mathbf{L}_m$. The energy tensor of the matter is given by

$$T_{ab} = \frac{-2\delta(\sqrt{-d})\mathbf{L}_m}{\sqrt{-d}\,\delta^{ab}}. \tag{10}$$

Assume that $\mathbf{L}_m$ is not contingent on its derivatives but is solely reliant on d_{ab}. The variation in action (9) concerning d_{ab} suggests

$$F'_1(R)Ric_{ab} - \frac{1}{2}F_1(R)d_{ab} + (d_{ab}\nabla_c\nabla^c - \nabla_a\nabla_b)F'_1(R) \tag{11}$$

$$= 8\pi T_{ab} - F'_2(T)T_{ab} - F'_2(T)\Xi_{ab} + \frac{1}{2}F_2(T)d_{ab}.$$

where $F_1'(R) = \frac{\partial F(R,T)}{\partial R}$ and $F_2'(T) = \frac{\partial F(T,T)}{\partial T}$.

Standard notation is employed here; ∇_a and $\Box \equiv \nabla_c\nabla^c$ represent the d'Alembert operator and covariant derivative, respectively. Additionally, we have

$$\Xi_{ab} = -2T_{ab} + d_{ab}\mathbf{L}_m - 2d^{lk}\frac{\partial^2 \mathbf{L}_m}{\partial d^{ab}\partial d^{lk}}. \tag{12}$$

The field equation of the standard $F(R)$-gravity model can be obtained anew by setting $F_2(T)$ to zero.

Consider a quark matter fluid characterized by the total pressure p_M, total energy density ρ_M, and velocity vector η^α. Given our flexibility in selecting $\mathbf{L}_m$, we set $\mathbf{L}_m = -p_M$.

The energy–momentum tensor for magnetized strange quark (MSQ) matter fluid is defined as per reference [23,42].

$$T_{ab}^{(MSQ)} = (p_M + \rho_M)\eta_a\eta_b + \left(\frac{h^2}{2} - p_M\right)d_{ab} - \theta_a\theta_b, \tag{13}$$

where

$$\eta^a\nabla_b\eta_a = 0, \quad \eta_a \cdot \eta^a = 1. \tag{14}$$

Moreover, where $\eta^a = (0,0,0,1)$ serves as the four-velocity vector, the magnetic flux h^2 aligns in the x-direction due to the condition $\eta_a\theta_a = 0$. In this context, p_M stands for the proper pressure, and ρ_M denotes the energy density.

By utilizing Equations (12) and (13), we derive the variation of the stress energy–momentum tensor for the magnetized strange quark fluid as follows.

$$\Xi_{ab} = -2T_{ab} - p_M d_{ab}. \tag{15}$$

After adopting (8) and (11), we obtain

$$F_1'(R)Ric_{ab} = \frac{1}{2}F_1(R)d_{ab} + 8\pi T_{ab} - F_2'(T)T_{ab} - F_2'(T)\Xi_{ab} + \frac{1}{2}F_2(T)d_{ab}. \tag{16}$$

In view of (13)–(15), the gravitational field equation for magnetized strange quark matter fluid in $F(R,T) = F_1(R) + F_2(T)$-gravity model (16) becomes

$$Ric_{ab} = \frac{1}{F_1'(R)}\left\{\frac{1}{2}(F_1(R) + F_2(T)) + (8\pi + F_2'(T)\left(\frac{h^2}{2} - p_M\right)\right\}d_{ab} \tag{17}$$

$$+\frac{1}{F_1'(R)}\left\{(\rho_M + p_M + h^2)(8\pi + F_2'(T))\right\}\eta_a\eta_b + \frac{1}{F_1'(R)}\left\{(8\pi + F_2'(T))\right\}\theta_a\theta_b,$$

Upon contraction, Equation (17) simplifies to

$$R = \frac{2[F_1(R) + F_2(T)]}{F_1'(R)} + \frac{(32\pi + 4F_2'(T)}{F_1'(R)}[(\rho_M - 3p_M) + h^2]. \tag{18}$$

Therefore, for a spacetime (M^4, d) containing magnetized strange quark matter fluid within $F(R,T) = F_1(R) + F_2(T)$-gravity, the Ricci tensor takes on the following structure:

$$Ric_{ab} = \alpha d_{ab} + \beta\eta_a\eta_b + \gamma\theta_a\theta_b, \tag{19}$$

where

$$\alpha = \frac{1}{F_1'(R)}\left\{\frac{1}{2}(F_1(R) + F_2(T)) + (8\pi + F_2'(T)\left(\frac{h^2}{2} - p_M\right)\right\}, \tag{20}$$

$$\beta = \frac{1}{F_1'(R)}\left\{(\rho_M + p_M + h^2)(8\pi + F_2'(T))\right\}, \quad \gamma = \frac{1}{F_1'(R)}\left\{(8\pi + F_2'(T))\right\}. \tag{21}$$

We assume that a and b are non-simultaneously zero throughout the document. A similar approach was employed in [40] to establish the formulation of the Ricci tensor. For the sake of coherence, we present the proof as well. As a result, we obtain

Theorem 1. *The Ricci tensor for the spacetime of magnetized strange quark matter fluid in the $F(R,T) = F_1(R) + F_2(T)$-gravity model is given by*

$$Ric_{ab} = \frac{1}{F_1'(R)}\left\{\frac{1}{2}(F_1(R) + F_2(T)) + (8\pi + F_2'(T)\left(\frac{h^2}{2} - p_M\right)\right\}d_{ab}$$

$$+\frac{1}{F_1'(R)}\left\{(\rho_M + p_M + h^2)(8\pi + F_2'(T))\right\}\eta_a\eta_b + \frac{1}{F_1'(R)}\left\{(8\pi + F_2'(T))\right\}\theta_a\theta_b.$$

Corollary 1. *The scalar curvature for the $F(R,T) = F_1(R) + F_2(T)$-gravity model featuring magnetized strange quark matter fluid is defined as:*

$$R = \frac{2[F_1(R) + F_2(T)]}{F_1'(R)} + \frac{(32\pi + 4F_2'(T))}{F_1'(R)}[(\rho_M - 3p_M) + h^2].$$

In a notation devoid of indices, Equation (17) can now be represented as

$$Ric = \alpha d + \beta\eta \otimes \eta + \gamma\theta \otimes \theta. \tag{22}$$

By referencing Equations (1) and (17), we arrive at the subsequent outcome.

Theorem 2. *A spacetime (M^4, d) in $F(R,T) = F_1(R) + F_2(T)$-gravity featuring magnetized strange quark matter fluid represents a generalized quasi-Einstein spacetime.*

Now, employing Equation (18), we derive

$$p_M = -\frac{\rho_M}{3} - \frac{1}{3}\left\{h^2 - \frac{R}{F_1'}(R) - 2(F_1(R) + F_2(T))(32\pi + 4F_2'(T))\right\}. \tag{23}$$

Referring to [43], for the fact that $p_M = \rho_M + F(r)$ is the equation of state for dark energy, with t being the cosmic time and $F(r)$ being a function of the scale factor "r". The author also demonstrated that whereas $\omega < -1$ and $\omega > -1$ indicate a shift from phantom to non-phantom, $\omega = \frac{p_M}{\rho_M} = -1$ yields a phantom barrier.

Theorem 3. *If the matter of $F(R,T) = F_1(R) + F_2(T)$-gravity model is magnetized strange quark matter fluid, then $\mathcal{E}o\mathcal{S}$ is given by (23).*

Assume that the strange quark matter's $\mathcal{E}o\mathcal{S}$ is $p_M = \frac{(\rho_M - 4\mathcal{B}_c)}{3}$. This finding, when used with Equation (23), yields

$$p_M = \left\{\frac{RF_1'(R) - 2(F_1(R) + F_2(T))}{6(32\pi + 4F_2'(T))} - h^2\right\} - \frac{2}{3}\mathcal{B}_c. \tag{24}$$

$$\rho_M = \left\{\frac{RF_1'(R) - 2(F_1(R) + F_2(T))}{2(32\pi + 4F_2'(T))} - h^2\right\} + 2\mathcal{B}_c. \tag{25}$$

Thus, we turn up the following outcomes.

Corollary 2. *If the magnetized strange quark matter fluid in the $F(R,T) = F_1(R) + F_2(T)$-gravity model obeys the $\mathcal{E}o\mathcal{S}$ (23) for strange quark matter. Then (24) and (25) determines the total pressure p_M and the total energy density ρ_M, respectively.*

In the case of phantom barrier, $\rho_M = -p_M = \left\{h^2 - \frac{RF_1'(R) - 2(F_1(R)+F_2(T))}{2(32\pi + 4F_2'(T))}\right\} + \frac{2}{3}\mathcal{B}_c$. As a result, we may infer

Corollary 3. *If a phantom barrier type source of matter is used in a $F(R,T) = F_1(R) + F_2(T)$-gravity model filled with magnetized strange quark matter fluid, the total pressure and total energy density are calculated as*

$$\rho_M = -p_M = \left\{h^2 - \frac{RF_1'(R) - 2(F_1(R) + F_2(T))}{2(32\pi + 4F_2'(T))}\right\} + \frac{2}{3}\mathcal{B}_c. \tag{26}$$

3. Ricci Soliton on $F(R,T) = F_1(R) + F_2(T)$-Gravity Model Attached with Magnetized Strange Quark Matter

In this section, we analyze the Ricci soliton (RS) within the $F(R,T)$-gravity model, coupled with magnetized strange quark matter. The conformal vector field ζ in this scenario acts as the timelike velocity vector field.

As defined by Kuhnel and Rademacher [44], a conformal vector field $\mathcal{F}$ (CVF) on a Lorentzian spacetime manifold (M^4, d) is given by:

$$\mathfrak{L}_{\mathcal{F}} d = 2\Omega d \tag{27}$$

wherein Ω is a smooth function on M^4. $\mathcal{F}$ is homothetic when Ω is constant, and is Killing if $\Omega = 0$. Next, adopting $\mathcal{F} = \zeta$, Equation (7) becomes

$$(\mathfrak{L}_{\zeta} d)(a,b) + 2Ric(a,b) + 2\lambda d(a,b) = 0. \tag{28}$$

In light of (27), we gain

$$Ric(a,b) + (\lambda + \Omega) d(a,b) = 0. \tag{29}$$

Substituting (22) into the preceding equation, we obtain

$$(\alpha + \lambda + \Omega) d(a,b) + \beta \eta(a) \eta(b) = 0. \tag{30}$$

By substituting $a = b = \zeta$ into (30), we deduce

$$\lambda = \frac{1}{F_1'(R)} \left\{ \frac{1}{2}(F_1(R) + F_2(T)) + (8\pi + F_2'(T) \left(\frac{h^2}{2} - p_M \right) \right\} \tag{31}$$
$$- \left(\frac{1}{F_1'(R)} \left\{ (\rho_M + p_M + h^2)(8\pi + F_2'(T)) \right\} + \Omega \right).$$

As such, we obtain the subsequent outcome.

Theorem 4. *If a spacetime (M^4, g) in the $F(R,T) = F_1(R) + F_2(T)$-gravity model attached with magnetized strange quark matter fluid admits a RS (d, ζ, λ) with a CVF ζ, then RS is growing, stable, or decreasing, referring as*

1. $\frac{1}{F_1'(R)} \left\{ \frac{1}{2}(F_1(R) + F_2(T)) + (8\pi + F_2'(T) \left(\frac{h^2}{2} - p_M \right) \right\}$
   $> \frac{1}{F_1'(R)} \left\{ (\rho_M + p_M + h^2)(8\pi + F_2'(T)) \right\} + \Omega,$
2. $\frac{1}{F_1'(R)} \left\{ \frac{1}{2}(F_1(R) + F_2(T)) + (8\pi + F_2'(T) \left(\frac{h^2}{2} - p_M \right) \right\}$
   $= \frac{1}{F_1'(R)} \left\{ (\rho_M + p_M + h^2)(8\pi + F_2'(T)) \right\} + \Omega,$ *and*
3. $\frac{1}{F_1'(R)} \left\{ \frac{1}{2}(F_1(R) + F_2(T)) + (8\pi + F_2'(T) \left(\frac{h^2}{2} - p_M \right) \right\}$
   $< \frac{1}{F_1'(R)} \left\{ (\rho_M + p_M + h^2)(8\pi + F_2'(T)) \right\} + \Omega,$ *respectively, provided* $\frac{1}{F_1'(R)} \neq 0$.

4. Energy Conditions in $F(R,T) = F_1(R) + F_2(T)$-Gravity Model Attached with Magnetized Strange Quark Matter Fluid Admits Ricci Soliton

Referring to [45], we determine whether the criterion is met by the Ricci tensor Ric in the spacetime described by Equation (32), where

$$Ric(\zeta, \zeta) > 0, \tag{32}$$

holds for all timelike vector fields $\zeta \in \chi(M^4)$. In such cases, Equation (32) is termed the time-like convergence condition (TCC).

From Equation (17), we have

$$Ric(\zeta,\zeta) = \alpha + \beta.$$

The spacetime in question is valid if it satisfies the time-like convergence condition (TCC), meaning $Ric(\zeta,\zeta) > 0$.

$$\frac{1}{F_1'(R)}\left\{\frac{1}{2}(F_1(R)+F_2(T)) + (8\pi + F_2'(T)\left(\frac{h^2}{2} - p_M\right)\right\} \tag{33}$$

$$> \frac{1}{F_1'(R)}\left\{(\rho_M + p_M + h^2)(8\pi + F_2'(T))\right\} + \Omega.$$

The spacetime adheres to the cosmological strong energy condition (SEC) [46]. With the provided details and considering Equation (33), we can conclude that

Theorem 5. *If a spacetime (M^4, d) in the $F(R,T) = F_1(R) + F_2(T)$-gravity model attached with magnetized strange quark matter fluid admits a RS (d,ζ,λ) with with a CVF ζ and satisfies TCC, then RS is growing.*

Remark 1. *In 1973, Hawking and Ellis demonstrated that [47]:*

(i) The time-like convergence condition (TCC) implies the cosmological strong energy condition (SEC) and the null convergence condition (NCC).
(ii) The strong energy condition (SEC) implies the null energy condition (NEC).

Consequently, it follows that TCC implies NCC as well.

Using Theorem 5 and Remark 1 together, we turn up the following outcomes:

Theorem 6. *If a spacetime (M^4, d) in the $F(R,T) = F_1(R) + F_2(T)$-gravity model attached with magnetized strange quark matter fluid admits a growing RS (d,ζ,λ) with a CVF ζ, if (33) holds, then the quark matter fluid spacetime (M^4, d) in the $f(R,T)$ satisfies SEC.*

Corollary 4. *In the framework of the $F(R,T) = F_1(R) + F_2(T)$-gravity model coupled with magnetized strange quark matter fluid, if a spacetime (M^4, d) allows for an expanding RS geometry (d,ζ,Λ) characterized by a conformal vector field ζ, and if condition (33) is met, then the spacetime of the quark matter fluid in the $f(R,T)$ model satisfies the null convergence condition (NCC).*

Corollary 5. *In the context of the $F(R,T) = F_1(R) + F_2(T)$-gravity model coupled with magnetized strange quark matter fluid, if a spacetime (M^4, d) allows for an RS geometry (d,ζ,λ) characterized by a conformal vector field (CVF) ζ and satisfies the strong energy condition (SEC), then the Ricci tensor Ric in the expanding RS geometry belongs to the second Segre type [47].*

The Universe's evolution and the formation of galaxies were significantly influenced by dark matter, which is implied by gravitational effects that general relativity cannot account for unless there is more matter than can be seen [48].

Various changes to the standard rules of general relativity are advocated by dark matter. These consist of entropic gravity, tensor–vector–scalar gravity, and modified Newtonian dynamics. None of the modified gravity theories that have been put forth thus far are able to account for all of the observational data simultaneously, indicating that dark matter of some kind will still be necessary even if gravity must be changed [48].

Remark 2. *Strong energy conditions are mathematically imposed boundary restrictions that aim to reflect the idea that energy should be positive rather than physical constraints. However, negative pressure is a characteristic of the dark energy component. Dark energy slows the motion of large-scale structures while speeding up the Universe's expansions because it defies gravity [49]. The SEC violation is correlated with this negative.*

Moreover, a necessary condition for the late accelerated expansion of the Universe is that the parameter of the dark energy equation (EoS) $\frac{p_M}{\rho_M} = \omega$ must be negative, i.e., $\omega < 0$. Therefore, if $Ric(\zeta, \zeta) \leq 0$, then the violation of SEC is necessary. Therefore, (33) holds only for non-expanding solitons such that

$$\frac{1}{F_1'(R)}\left\{\frac{1}{2}(F_1(R) + F_2(T)) + (8\pi + F_2'(T)\left(\frac{h^2}{2} - p_M\right)\right\} \tag{34}$$

$$\leq \frac{1}{F_1'(R)}\left\{(\omega + \frac{h^2}{\rho_M})(8\pi + F_2'(T))\right\} + \Omega.$$

The right-hand side of (34) disappears in the case of dark energy, $\omega = -\frac{h^2}{\rho_M} = -1$. Thus, we can articulate the following outcome.

Theorem 7. *Let the source of matter be dark energy in the $F(R, T) = F_1(R) + F_2(T)$-gravity model attached with magnetized strange quark matter fluid, which admits a non-growing RS $(d, \zeta, \lambda \leq 0)$ with a CVF ζ, then SEC violates.*

Moreover, we gain an interesting result with the violation of strong energy conditions.

Theorem 8. *If the source of matter is dark energy in the $F(R, T) = F_1(R) + F_2(T)$-gravity model attached with magnetized strange quark matter fluid admits a non-growing RS $(d, \zeta, \lambda \leq 0)$ with a CVF ζ, then the magnetic flux h in equal to the total density ρ_M.*

5. Application of Singularity Theorem in $F(R, T) = F_1(R) + F_2(T)$-Gravity Model Attached with Magnetized Strange Quark Matter Fluid Admits a Ricci Soliton

Remark 3. *Based on Penrose's singularity theorem, Vilenkin and Wall ([50]) showed that the spacetime M satisfies the null convergence condition (NCC), indicating the presence of black holes and a trapped surface outside these black holes within M.*

In view of Theorem 5, Remarks 1 and 2, and Corollary 4, we can articulate the upcoming result.

Theorem 9. *If a spacetime (M^4, d) in the $F(R, T) = F_1(R) + F_2(T)$-gravity model coupled with magnetized strange quark matter fluid accommodates a growing RS geometry (d, ζ, Λ) characterized by a conformal vector field (CVF) ζ, and if the spacetime (M^4, g) satisfies the null convergence condition (NCC), then the spacetime of the magnetized strange quark matter fluid in (M^4, d) includes black holes with a trapped surface outside these black holes within the $F(R, T)$-gravity model.*

Corollary 6. *If a spacetime (M^4, d) in the $F(R, T) = F_1(R) + F_2(T)$-gravity model coupled with magnetized strange quark matter fluid features a growing RS geometry (d, ζ, Λ) with a Killing vector field ζ, and if the spacetime (M^4, d) satisfies the null convergence condition (NCC), then the spacetime of the magnetized strange quark matter fluid in (M^4, d) includes black holes with a trapped surface outside these black holes within the $F(R, T)$-gravity model.*

6. Solitonic Solutions for Magnetic Dilaton

The inclusion of a hypothetical particle known as the magnetic dilaton in the standard model. The magnetic dilaton is a scalar field that appears in gravity theories with additional dimensions [51] and in the field theory when conformal symmetry is spontaneously violated [52].

Certain cosmological issues are clarified by the magnetic dilaton field. Magnetic dilaton field theory provides an explanation for inflation and the existence of the cosmological constant [53]. Additionally, using a magnetic dilaton to study black holes could lead to new insights on black hole thermodynamics [54]. Additionally, composite Higgs models contain magnetic dilaton. In this case, a magnetic dilaton might act as a link between dark matter particles and the standard model.

Multidimensional theories of gravity, such as the superstring theory [55] and the 5-dimensional Kaluza–Klein theory [51], which combines gravitational and electromagnetic interactions, also predict magnetic dilaton. The diagonal elements of a multidimensional metric tensor d_{ab}, which correspond to the coordinates of additional spacetime dimensions, are where a magnetic dilaton is regarded in these theories as a scalar field. While elements d_{00}, d_{11}, d_{22}, and d_{33} reflect ordinary 4-dimensional spacetime, the magnetic dilaton field in a 5-dimensional scenario might be found in metric tensor element d_{44} [51].

Maxwell-dilaton theory [56], which specifically derives from the multidimensional theories of gravity [55], takes the magnetic dilaton into account. Electromagnetic fields can produce magnetic dilation, which can be found in investigations looking for particles that resemble axion [57].

In some astrophysical objects, magnetic dilatons [58], produced by different arrangements of electromagnetic fields may be investigated. Investigating revolving neutron stars (pulsars and magnetars) as dilaton sources is particularly noteworthy. These objects are known to generate coherent electromagnetic radiation and have strong magnetic fields between 10^8 and 10^{15} Gauss [59]. A detailed discussion of how revolving neutron stars produce dilatons may be found in [60].

The creation of magnetic dilaton during magnetic dipole radiation transmission in a galactic magnetic field. For relatively long distances $L_c oh$, where $L_c oh$ is the magnetic field's coherence length, the galactic magnetic field can be regarded as uniform and constant [61]. L_{coh} = approx. 100 pc for the galactic magnetic field [62]. This is why galactic magnetic fields can contribute significantly to dilaton formation even though they are only about 10^{-5} Gauss [61].

The following is an expression for the density of the Lagrange function for the magnetic dilaton interacting with an electromagnetic field [60]:

$$\mathcal{L} = C_0(\partial\Psi) + C_1 e^{-2K\psi}\mathcal{F}_{ab}\mathcal{F}^{ab}, \tag{35}$$

where the Maxwell tensor is $\mathcal{F}_{ab}$ and the gauge constants are C_0, C_1, and K. A massless scalar field Ψ is said to be a magnetic dilaton.

Superstring theory predicts the value of the constant $K = 1$. $K = \sqrt{3}$ is the value obtained from the 5-dimensional Kaluza–Klein theory result [51].

The density of the Lagrange function (35) yields the field equations, which take the following form:

$$\partial_k\partial^k\Psi = \frac{-C_1 K}{C_0} e^{-2K\Psi}\mathcal{F}_{ab}\mathcal{F}^{ab}, \tag{36}$$

$$\partial_a[e^{-2K\Psi}\mathcal{F}^{ab}] = 0. \tag{37}$$

The self-interacting magnetic dilaton field Ψ is represented by Equation (36). An electromagnetic field equation is shown in Equation (37). It also explains how the electromagnetic is affected by the magnetic dilaton field.

In the spacetime, the field Equations (36) and (37), will look like this with this approximation:

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi = \frac{C_1 K}{C_0}\mathcal{F}_{ab}\mathcal{F}^{ab}, \tag{38}$$

$$\partial_a \mathcal{F}^{ab} = 0. \tag{39}$$

Maxwellian electrodynamics without charges or currents is described by Equation (39).

The following is a representation of the invariant $\mathcal{F}_{ab}\mathcal{F}^{ab}$ in terms of the electric field E and magnetic field B:

$$\mathcal{F}_{ab}\mathcal{F}^{ab} = 2[B^2 - E^2]. \tag{40}$$

The Equation (38) will be written as follows in this case:

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi = \frac{C_1 K}{C_0}2[B^2 - E^2]. \tag{41}$$

The source of the magnetic dilaton field can only be electromagnetic fields for which the invariant (39) is not equal to zero, per Equations (38) and (41) in the approximation under consideration. As a result, the magnetic dilaton field can be produced during the propagation of electromagnetic waves in an external electromagnetic field or in the vicinity of an electromagnetic source.

However, Kong and Liu studied the hyperbolic Ricci flow in 2010 [63]. This flow is composed of a system of second-order nonlinear evolution of partial differential equations. The wave characteristics of metrics and manifold curvatures are described by hyperbolic Ricci flow. Furthermore, gravity has the ability to create waves. Gravity waves generate spacetime rippling that spreads over the cosmos. A gravitational wave is essentially an oscillation of spacetime curvature that is traveling away from Earth.

For example, the hyperbolic Ricci flow [63], which is thus inspired by the Ricci flow, is characterized by the resulting evolution equation

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}d(t) = -Ric(t)d(t), \quad d_0 = d(0), \quad \frac{\partial}{\partial t}d_{ij} = \Delta\Psi, \tag{42}$$

Therefore, a hyperbolic Ricci soliton is a self-similar solution of hyperbolic Ricci flow that is characterized as:

Definition 2 ([64])**.** *A hyperbolic Ricci soliton is a semi-Riemannian manifold (M^n, g) if, and only if, there is a vector field ζ on M and real scalars μ and λ such that*

$$\frac{1}{2}\mathfrak{L}_\zeta\mathfrak{L}_\zeta g + \lambda\mathfrak{L}_\zeta g + Ric = \mu d, \tag{43}$$

where Ric is the Ricci curvature of M.

Now, in light of Equations (41)–(43), we can express the geometric flow for magnetic dilaton such that

$$\frac{\partial^2}{\partial t^2}\Psi = -c^2\Delta\Psi + 2c^2\frac{C_1 K}{C_0}[B^2 - E^2]. \tag{44}$$

Thus, the soliton is a self-similar solution of geometric flow (44) that is expressed as:

$$\frac{1}{2}\mathfrak{L}_\zeta\mathfrak{L}_\zeta\Psi + c^2\Delta\Psi - 2c^2\frac{C_1 K}{C_0}[B^2 - E^2] + \lambda\mathfrak{L}_\zeta\Psi = \mu\Psi. \tag{45}$$

where metric d_Ψ induced by Ψ and d_Ψ depends nonlinearly on Ψ.

In (45), λ and μ show the types of solitons for magnetic dilaton and the rate of the underlying type, respectively. Moreover, μ represents the rate of change in the solutions and has geometric meaning as well. Depending on the constant μ, the soliton rate of change can be either shrinking, expanding, or approximately stable, regardless of whether $\mu < 0$, $\mu > 0$, or $\mu = 0$. Still, it is an open problem to discuss the rate of change of μ of hyperbolic Ricci solitons for Thorems 4–7 (see the chart in Figure 1).

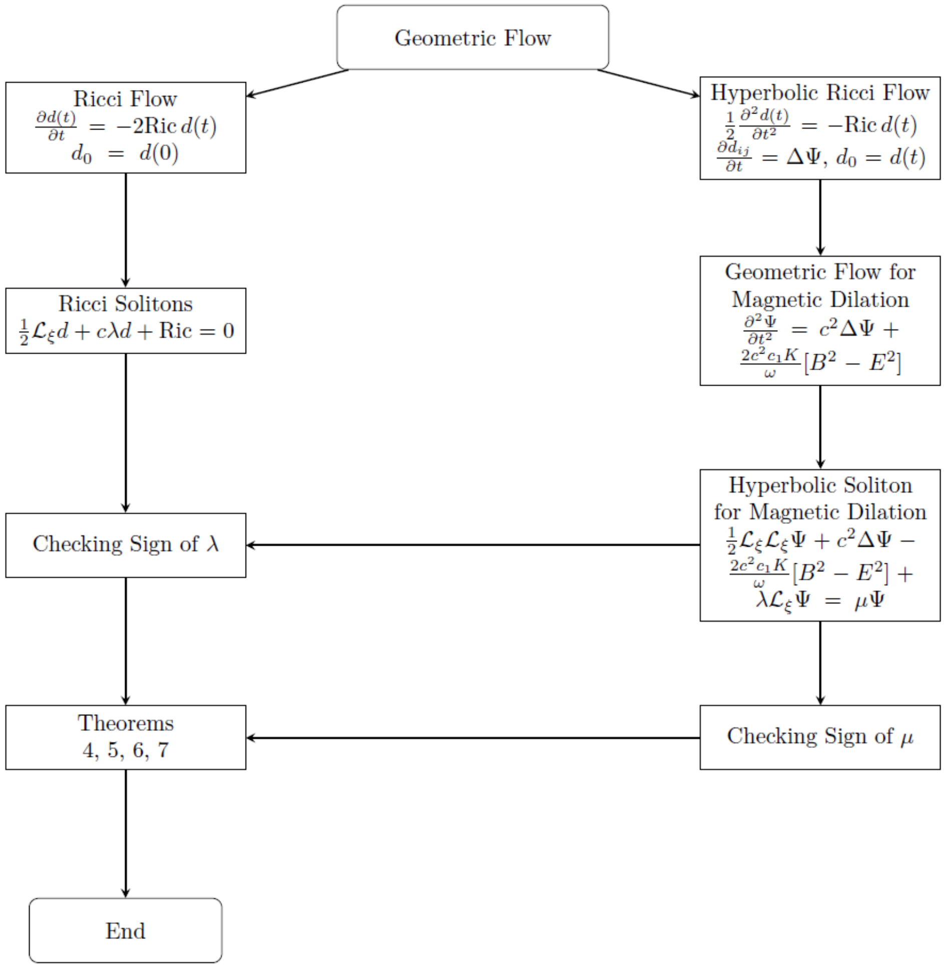

Figure 1. Analysis chart for the Ricci solitons and the solitons for magnetic dilaton.

7. Conclusions

$F(R,T)$-gravity theory is effective in representing the fast expansion of the Universe throughout late cosmic eras.

We have revealed an extensive variety of phenomena from our study of the properties of magnetized strange quark matter in $F(R,T)$-gravity, especially its interesting coupling to Ricci solitons with a conformal vector field. We derive the gravitational field equation of the $F(R,T) = F_1(R) + F_2(T)$-gravity model attached with magnetized strange quark matter fluid and derive the expression for the Ricci scalar for the spacetime of magnetized strange quark matter fluid in the specific $F(R,T)$-gravity model.

Moreover, we gain the equation of state for magnetized strange quark matter fluid attached with the $F(R,T)$-gravity framework and determine the total quark density and total pressure during the radiation and phantom barrier epochs of the Universe. Furthermore, we delve into the characteristics of magnetized strange quark matter in $F(R,T)$-gravity, highlighting its association with Ricci solitons endowed with a time-like conformal vector field. After that, we provide certain criteria for the magnetized strange quark matter combined with the given $F(R,T)$-gravity model, taking into account the black holes and different energy circumstances in terms of Ricci solitons.

Finally, we obtain a singularity theorem, which is based on the application of Penrose's singularity theorem for a spacetime that satisfies the null convergence condition indicating the presence of black holes, and explore a solitonic method on magnetic dilaton.

Some aspects of $F(R,T)$-gravity, and important other field theory aspects:

Researchers have investigated the singularity in the research of black holes using contour integrals and assessed it in the energy component using Fourier transforms; some theories, such as transistors and perturbation theory, are even highly helpful in this regard. In addition to being commonly utilized to give dark matter abundances a geometric origin, the $F(R,T)$-gravity model has garnered more interest recently in the context of astrophysical applications. Furthermore, it has been suggested as an explanation for cosmic inflation.

The $F(R,T)$-gravity theory, which generalizes the EH Lagrangian to be an arbitrary function of the Ricci scalar R and energy–momentum tensor T, is one of the most extensively researched variations of Einstein's classical explanation of gravitational waves. The technical nature of these theories restricts analytical knowledge to systems with high symmetry or perturbations of such systems, despite the fact that significant progress has been made in grasping the consequences of such theories. Numerical relativity has made significant contributions to our understanding of space–time dynamics in classical general relativity in recent years, and it offers some optimism that concerns in $F(R,T)$ gravity will be similarly applicable.

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