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A conjecture concerning the q-Onsager algebra

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Abstract

The q-Onsager algebra $\mathcal{O}_q$ is defined by two generators W_0, W_1 and two relations called the q-Dolan/Grady relations. Recently Baseilhac and Kolb obtained a PBW basis for $\mathcal{O}_q$ with elements denoted

$$\{B_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{B_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{B_{n\delta}\}_{n=1}^{\infty}.$$

In their recent study of a current algebra $\mathcal{A}_q$, Baseilhac and Belliard conjecture that there exist elements

$$\{W_{-k}\}_{k=0}^{\infty}, \qquad \{W_{k+1}\}_{k=0}^{\infty}, \qquad \{G_{k+1}\}_{k=0}^{\infty}, \qquad \{\tilde{G}_{k+1}\}_{k=0}^{\infty}$$

in $\mathcal{O}_q$ that satisfy the defining relations for $\mathcal{A}_q$. In order to establish this conjecture, it is desirable to know how the elements on the second displayed line above are related to the elements on the first displayed line above. In the present paper, we conjecture the precise relationship and give some supporting evidence. This evidence consists of some computer checks on SageMath due to Travis Scrimshaw, a proof of the analog conjecture for the Onsager algebra $\mathcal{O}$, and a proof of our conjecture for a homomorphic image of $\mathcal{O}_q$ called the universal Askey-Wilson algebra.

1. Introduction

We will be discussing the q-Onsager algebra $\mathcal{O}_q$ [3,35]. This infinite-dimensional associative algebra is defined by two generators W_0, W_1 and two relations called the q-Dolan/Grady relations; see Definition 3.1 below. One can view $\mathcal{O}_q$ as a q-analog of the universal enveloping algebra of the Onsager Lie algebra $\mathcal{O}$ [19–21,28–31].

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The algebra $\mathcal{O}_q$ originated in algebraic combinatorics [35]. There is a family of algebras called tridiagonal algebras [35, Definition 3.9] that arise in the study of association schemes [34, Lemma 5.4] and tridiagonal pairs [22, Theorem 10.1], [35, Theorem 3.10]. The algebra $\mathcal{O}_q$ is the "most general" example of a tridiagonal algebra [24, Section 1.2]. A finite-dimensional irreducible $\mathcal{O}_q$-module is essentially the same thing as a tridiagonal pair of q-Racah type [35, Theorem 3.10]. These tridiagonal pairs are classified up to isomorphism in [23, Theorem 3.3]. To our knowledge the q-Dolan/Grady relations first appeared in [34, Lemma 5.4].

The algebra $\mathcal{O}_q$ has applications outside combinatorics. For instance, $\mathcal{O}_q$ is used to study boundary integrable systems [2–5,7,10–12,16]. The algebra $\mathcal{O}_q$ can be realized as a left or right coideal subalgebra of the quantized enveloping algebra $U_q(\widehat{\mathfrak{sl}}_2)$; see [4,5,25]. The algebra $\mathcal{O}_q$ is the simplest example of a quantum symmetric pair coideal subalgebra of affine type [25, Example 7.6]. A Drinfeld type presentation of $\mathcal{O}_q$ is obtained in [26], and this is used in [27] to realize $\mathcal{O}_q$ as an ιHall algebra of the projective line. There is an injective algebra homomorphism from $\mathcal{O}_q$ into the algebra $\square_q$ [37, Proposition 5.6], and a noninjective algebra homomorphism from $\mathcal{O}_q$ into the universal Askey-Wilson algebra Δ_q [36, Sections 9,10]. In [5, Section 4] some infinite-dimensional $\mathcal{O}_q$-modules are constructed using q-vertex operators. In [24] the augmented q-Onsager algebra is introduced; this algebra is obtained from $\mathcal{O}_q$ by adding an extra generator. The augmented q-Onsager algebra is used in [17] to derive a Q-operator. In [4] a higher rank generalization of $\mathcal{O}_q$ is introduced, and applied to affine Toda theories with boundaries.

In [15, Theorem 4.5], Baseilhac and Kolb obtain a Poincaré-Birkhoff-Witt (or PBW) basis for $\mathcal{O}_q$. They obtain this PBW basis by using a method of Damiani [18] along with two automorphisms of $\mathcal{O}_q$ that are roughly analogous to the Lusztig automorphisms of $U_q(\widehat{\mathfrak{sl}}_2)$. The PBW basis elements are denoted

$$\{B_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{B_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{B_{n\delta}\}_{n=1}^{\infty}. \tag{1}$$

In mathematical physics, $\mathcal{O}_q$ comes up naturally in the context of a reflection algebra [2,3]. Using a framework of Sklyanin [33], in [10,16] a current algebra $\mathcal{A}_q$ for $\mathcal{O}_q$ is introduced. In [16, Definition 3.1] Baseilhac and Shigechi give a presentation of $\mathcal{A}_q$ by generators and relations. The generators are denoted

$$\{\mathcal{W}_{-k}\}_{k=0}^{\infty}, \qquad \{\mathcal{W}_{k+1}\}_{k=0}^{\infty}, \qquad \{\mathcal{G}_{k+1}\}_{k=0}^{\infty}, \qquad \{\tilde{\mathcal{G}}_{k+1}\}_{k=0}^{\infty}$$

and the relations are given in (20)–(30) below.

We now summarize some recent results about $\mathcal{A}_q$. In [5, Section 3] a reflection algebra is used to obtain a generating function for quantities in a commutative subalgebra of $\mathcal{A}_q$. In [10, 11] some finite-dimensional tensor product representations of $\mathcal{A}_q$ are constructed, and used to create quantum integrable spin chains. The algebra $\mathcal{A}_q$ is used to study the open XXZ spin chain with generic nondiagonal boundary conditions [11,12] and also its thermodynamic limit [5,13,14]. In [13,14] the study of $\mathcal{A}_q$ is combined with the q-vertex operator approach of the Kyoto school, to derive correlation functions and form factors. For the open XXZ spin chain in the thermodynamic limit, the algebra $\mathcal{A}_q$ is used in [6] to classify the non-abelian symmetries for any type of boundary condition. In [8], a limit $q \mapsto 1$ is taken in $\mathcal{O}_q$ to obtain a presentation of the Onsager algebra $\mathcal{O}$ in terms of a non-standard Yang-Baxter algebra. In [9], a similar limiting process is applied to $\mathcal{A}_q$, to obtain a Lie algebra $\mathcal{A}$ that turns out to be isomorphic to $\mathcal{O}$. An explicit isomorphism between $\mathcal{O}$ and $\mathcal{A}$ is established, and explicit relations between the generators of $\mathcal{O}$ and $\mathcal{A}$ are given.

The algebras $\mathcal{A}_q$ and $\mathcal{O}_q$ are both q-analogs of the universal enveloping algebra of $\mathcal{O}$, so it is natural to ask how $\mathcal{A}_q$ is related to $\mathcal{O}_q$. Baseilhac and Belliard investigate this issue in [7]; their results are summarized as follows. In [7, line (3.7)] they show that $\mathcal{W}_0$, $\mathcal{W}_1$ satisfy the q-Dolan/Grady relations. In [7, Section 3] they show that $\mathcal{A}_q$ is generated by $\mathcal{W}_0$, $\mathcal{W}_1$ together with the central elements $\{\Delta_n\}_{n=1}^{\infty}$ defined in [7, Lemma 2.1]. In [7, Section 3] they consider the quotient algebra of $\mathcal{A}_q$ obtained by sending Δ_n to a scalar for all $n \geq 1$. The construction yields an algebra homomorphism Ψ from $\mathcal{O}_q$ onto this quotient. In [7, Conjecture 2] Baseilhac and Belliard conjecture that Ψ is an isomorphism. If the conjecture is true then there exists an algebra homomorphism $\mathcal{A}_q \to \mathcal{O}_q$ that sends $\mathcal{W}_0 \mapsto W_0$ and $\mathcal{W}_1 \mapsto W_1$. In this case there exist elements

$$\{W_{-k}\}_{k=0}^{\infty}, \qquad \{W_{k+1}\}_{k=0}^{\infty}, \qquad \{G_{k+1}\}_{k=0}^{\infty}, \qquad \{\tilde{G}_{k+1}\}_{k=0}^{\infty} \tag{2}$$

in $\mathcal{O}_q$ that satisfy the relations (20)–(30). In order to make progress on the above conjecture, it is desirable to know how the elements (2) are related to the elements in (1). In the present paper, we conjecture the precise relationship and give some supporting evidence. Our conjecture statement is Conjecture 6.2. Our supporting evidence consists of some computer checks on SageMath (see [32]) due to Travis Scrimshaw, a proof of the analog conjecture for the Onsager algebra $\mathcal{O}$, and a proof of the conjecture at the level of the algebra Δ_q mentioned above.

The paper is organized as follows. Section 2 contains some preliminaries. In Section 3 we recall the algebra $\mathcal{O}_q$, and describe the PBW basis due to Baseilhac and Kolb. In Sections 4, 5 we develop some results about generating functions that will be used in Conjecture 6.2. In Section 6 we state Conjecture 6.2 and explain its meaning. In Section 7 we present our evidence supporting Conjecture 6.2. In Section 8 we give some comments. In Appendices A, B we display in detail some equations from the main body of the paper.

2. Preliminaries

Throughout the paper, the following notational conventions are in effect. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Let $\mathbb{F}$ denote a field. Every vector space mentioned is over $\mathbb{F}$. Every algebra mentioned is associative, over $\mathbb{F}$, and has a multiplicative identity.

Definition 2.1. (See [18, p. 299].) Let $\mathcal{A}$ denote an algebra. A *Poincaré-Birkhoff-Witt* (or *PBW*) basis for $\mathcal{A}$ consists of a subset $\Omega \subseteq \mathcal{A}$ and a linear order $<$ on Ω such that the following is a basis for the vector space $\mathcal{A}$:

$$a_1 a_2 \cdots a_n \qquad n \in \mathbb{N}, \qquad a_1, a_2, \ldots, a_n \in \Omega, \qquad a_1 \leq a_2 \leq \cdots \leq a_n.$$

We interpret the empty product as the multiplicative identity in $\mathcal{A}$.

We will be discussing generating functions. Let $\mathcal{A}$ denote an algebra and let t denote an indeterminate. For a sequence $\{a_n\}_{n \in \mathbb{N}}$ of elements in $\mathcal{A}$, the corresponding generating function is

$$a(t) = \sum_{n \in \mathbb{N}} a_n t^n.$$

The above sum is formal; issues of convergence are not considered. We call $a(t)$ the *generating function over $\mathcal{A}$ with coefficients $\{a_n\}_{n \in \mathbb{N}}$*. For generating functions $a(t) = \sum_{n \in \mathbb{N}} a_n t^n$ and

$b(t) = \sum_{n \in \mathbb{N}} b_n t^n$ over $\mathcal{A}$, their product $a(t)b(t)$ is the generating function $\sum_{n \in \mathbb{N}} c_n t^n$ such that $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ for $n \in \mathbb{N}$. The set of generating functions over $\mathcal{A}$ forms an algebra. Let $a(t) = \sum_{n \in \mathbb{N}} a_n t^n$ denote a generating function over $\mathcal{A}$. We say that $a(t)$ is *normalized* whenever $a_0 = 1$. If $0 \neq a_0 \in \mathbb{F}$ then define

$$a(t)^\vee = a_0^{-1} a(t), \tag{3}$$

and note that $a(t)^\vee$ is normalized.

Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity. Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n \in \mathbb{N}.$$

3. The q-Onsager algebra $\mathcal{O}_q$

In this section we recall the q-Onsager algebra $\mathcal{O}_q$. For elements X, Y in any algebra, define their commutator and q-commutator by

$$[X, Y] = XY - YX, \qquad [X, Y]_q = qXY - q^{-1}YX.$$

Note that

$$[X, [X, [X, Y]_q]_{q^{-1}}] = X^3Y - [3]_q X^2YX + [3]_q XYX^2 - YX^3.$$

Definition 3.1. (See [3, Section 2], [35, Definition 3.9].) Define the algebra $\mathcal{O}_q$ by generators W_0, W_1 and relations

$$[W_0, [W_0, [W_0, W_1]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [W_1, W_0], \tag{4}$$

$$[W_1, [W_1, [W_1, W_0]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [W_0, W_1]. \tag{5}$$

We call $\mathcal{O}_q$ the *q-Onsager algebra*. The relations (4), (5) are called the *q-Dolan/Grady relations*.

Remark 3.2. In [15] Baseilhac and Kolb define the q-Onsager algebra in a slightly more general way that involves two scalar parameters c, q. Our $\mathcal{O}_q$ is their q-Onsager algebra with $c = q^{-1}(q - q^{-1})^2$.

Remark 3.3. We clarify how to recover the Onsager algebra $\mathcal{O}$ from $\mathcal{O}_q$ by taking a limit $q \mapsto 1$. To keep things simple, assume that $\mathbb{F} = \mathbb{C}$. In (4), (5) make a change of variables $W_0 = \xi A_0$ and $W_1 = \xi A_1$ with $\xi = \sqrt{-1}(q - q^{-1})/2$. Simplify and set $q = 1$ to obtain

$$[A_0, [A_0, [A_0, A_1]]] = 16[A_0, A_1], \qquad [A_1, [A_1, [A_1, A_0]]] = 16[A_1, A_0].$$

These are the Dolan/Grady relations and the defining relations for $\mathcal{O}$ [9, Section 2.1].

In [15], Baseilhac and Kolb obtain a PBW basis for $\mathcal{O}_q$ that involves some elements

$$\{B_{n\delta+\alpha_0}\}_{n=0}^{\infty}, \qquad \{B_{n\delta+\alpha_1}\}_{n=0}^{\infty}, \qquad \{B_{n\delta}\}_{n=1}^{\infty}. \tag{6}$$

These elements are recursively defined as follows. Writing $B_\delta = q^{-2}W_1W_0 - W_0W_1$ we have

$$B_{\alpha_0} = W_0, \qquad B_{\delta+\alpha_0} = W_1 + \frac{q[B_\delta, W_0]}{(q-q^{-1})(q^2-q^{-2})}, \tag{7}$$

$$B_{n\delta+\alpha_0} = B_{(n-2)\delta+\alpha_0} + \frac{q[B_\delta, B_{(n-1)\delta+\alpha_0}]}{(q-q^{-1})(q^2-q^{-2})} \qquad n \ge 2 \tag{8}$$

and

$$B_{\alpha_1} = W_1, \qquad B_{\delta+\alpha_1} = W_0 - \frac{q[B_\delta, W_1]}{(q-q^{-1})(q^2-q^{-2})}, \tag{9}$$

$$B_{n\delta+\alpha_1} = B_{(n-2)\delta+\alpha_1} - \frac{q[B_\delta, B_{(n-1)\delta+\alpha_1}]}{(q-q^{-1})(q^2-q^{-2})} \qquad n \ge 2. \tag{10}$$

Moreover for $n \ge 2$,

$$B_{n\delta} = q^{-2} B_{(n-1)\delta+\alpha_1} W_0 - W_0 B_{(n-1)\delta+\alpha_1} + (q^{-2}-1) \sum_{\ell=0}^{n-2} B_{\ell\delta+\alpha_1} B_{(n-\ell-2)\delta+\alpha_1}. \tag{11}$$

By [15, Proposition 5.12] the elements $\{B_{n\delta}\}_{n=1}^{\infty}$ mutually commute.

Lemma 3.4. (See [15, Theorem 4.5].) *Assume that q is transcendental over $\mathbb{F}$. Then a PBW basis for $\mathcal{O}_q$ is obtained by the elements (6) in any linear order.*

Remark 3.5. With reference to Remark 3.3, we give the limiting values of the elements (6). In (7)–(11) and the expression for B_δ below (6), make a change of variables

$$B_{n\delta+\alpha_0} = \xi A_{-n}, \qquad B_{n\delta+\alpha_1} = \xi A_{n+1}, \qquad B_{m\delta} = 4\xi^2 B_m$$

for $n \ge 0$ and $m \ge 1$. Simplify and set $q = 1$ to obtain

$$[B_1, A_n] = 2A_{n+1} - 2A_{n-1}, \qquad [A_m, A_0] = 4B_m$$

for $n \in \mathbb{Z}$ and $m \ge 1$. The elements $\{A_n\}_{n\in\mathbb{Z}}$, $\{B_n\}_{n=1}^{\infty}$ form the basis for $\mathcal{O}$ given in [9, Definition 2.1].

Definition 3.6. We define a generating function in the indeterminate t:

$$B(t) = \sum_{n\in\mathbb{N}} B_{n\delta} t^n, \qquad B_{0\delta} = q^{-2} - 1. \tag{12}$$

In Section 6 we will make a conjecture about $B(t)$. In Sections 4, 5 we motivate the conjecture with some comments about generating functions.

4. Generating functions over a commutative algebra

Throughout this section the following notational conventions are in effect. We fix a commutative algebra $\mathcal{A}$. Every generating function mentioned is over $\mathcal{A}$.

The following results are readily checked.

Lemma 4.1. *A generating function $a(t) = \sum_{n\in\mathbb{N}} a_n t^n$ is invertible if and only if a_0 is invertible in $\mathcal{A}$. In this case $(a(t))^{-1} = \sum_{n\in\mathbb{N}} b_n t^n$ where $b_0 = a_0^{-1}$ and for $n \ge 1$,*

$$b_n = -a_0^{-1} \sum_{k=1}^{n} a_k b_{n-k}.$$

Lemma 4.2. *For generating functions $a(t)=\sum_{n\in\mathbb{N}} a_n t^n$ and $b(t)=\sum_{n\in\mathbb{N}} b_n t^n$ the following are equivalent:*

(i) $a(t)=b(qt)b(q^{-1}t)$;
(ii) $a_n=\sum_{i=0}^{n} b_i b_{n-i} q^{2i-n}$ *for* $n\in\mathbb{N}$.

Lemma 4.3. *For a normalized generating function $a(t)=\sum_{n\in\mathbb{N}} a_n t^n$, there exists a unique normalized generating function $b(t)=\sum_{n\in\mathbb{N}} b_n t^n$ such that*

$$a(t)=b(qt)b(q^{-1}t).$$

Moreover for $n\geq 1$,

$$b_n=\frac{a_n-\sum_{i=1}^{n-1} b_i b_{n-i} q^{2i-n}}{q^n+q^{-n}}.$$

Definition 4.4. Referring to Lemma 4.3, we call $b(t)$ the *q-square root* of $a(t)$.

Lemma 4.5. *For generating functions $a(t)=\sum_{n\in\mathbb{N}} a_n t^n$ and $b(t)=\sum_{n\in\mathbb{N}} b_n t^n$ the following are equivalent:*

(i) $a(t)=b\left(\frac{q+q^{-1}}{t+t^{-1}}\right)$;
(ii) $a_0=b_0$ *and for* $n\geq 1$,

$$a_n=\sum_{\ell=0}^{\lfloor (n-1)/2\rfloor}(-1)^{\ell}\binom{n-1-\ell}{\ell}[2]_q^{n-2\ell} b_{n-2\ell}. \tag{13}$$

Proof. Note that for $k\in\mathbb{N}$,

$$(1-t)^{-k-1}=\sum_{\ell\in\mathbb{N}}\binom{k+\ell}{\ell}t^{\ell}. \tag{14}$$

We have

$$b\left(\frac{q+q^{-1}}{t+t^{-1}}\right)=\sum_{n\in\mathbb{N}}\left(\frac{q+q^{-1}}{t+t^{-1}}\right)^n b_n=b_0+\sum_{k\in\mathbb{N}}\left(\frac{q+q^{-1}}{t+t^{-1}}\right)^{k+1} b_{k+1}.$$

We have

$$\frac{q+q^{-1}}{t+t^{-1}}=[2]_q t(1+t^2)^{-1}.$$

By this and (14) we find that for $k\in\mathbb{N}$,

$$\left(\frac{q+q^{-1}}{t+t^{-1}}\right)^{k+1}=[2]_q^{k+1}t^{k+1}\sum_{\ell\in\mathbb{N}}(-1)^{\ell}\binom{k+\ell}{\ell}t^{2\ell}.$$

By these comments

$$
\begin{aligned}
b\left(\frac{q+q^{-1}}{t+t^{-1}}\right) &= b_0 + \sum_{k,\ell\in\mathbb{N}} (-1)^\ell \binom{k+\ell}{\ell} [2]_q^{k+1} b_{k+1} t^{k+1+2\ell} \\
&= b_0 + \sum_{n=1}^{\infty} \sum_{\ell=0}^{\lfloor (n-1)/2\rfloor} (-1)^\ell \binom{n-1-\ell}{\ell} [2]_q^{n-2\ell} b_{n-2\ell} t^n.
\end{aligned}
$$

The result follows. □

Lemma 4.6. *For a generating function $a(t) = \sum_{n\in\mathbb{N}} a_n t^n$, there exists a unique generating function $b(t) = \sum_{n\in\mathbb{N}} b_n t^n$ such that*

$$
a(t) = b\left(\frac{q+q^{-1}}{t+t^{-1}}\right). \tag{15}
$$

Moreover $b_0 = a_0$ and for $n \geq 1$,

$$
b_n = \frac{a_n - \sum_{\ell=1}^{\lfloor (n-1)/2\rfloor} (-1)^\ell \binom{n-1-\ell}{\ell} [2]_q^{n-2\ell} b_{n-2\ell}}{[2]_q^n}.
$$

Proof. This is a routine consequence of Lemma 4.5. □

Definition 4.7. Referring to Lemma 4.6, we call $b(t)$ the *q-symmetrization* of $a(t)$.

We now combine the above constructions.

Proposition 4.8. *Let $a(t) = \sum_{n\in\mathbb{N}} a_n t^n$ denote a normalized generating function. Then for a generating function $b(t) = \sum_{n\in\mathbb{N}} b_n t^n$ the following are equivalent:*

(i) *$b(t)$ is the q-symmetrization of the q-square root of the inverse of $a(t)$;*
(ii) *$b(t)$ is normalized and*

$$
a(t) b\left(\frac{q+q^{-1}}{qt+q^{-1}t^{-1}}\right) b\left(\frac{q+q^{-1}}{q^{-1}t+qt^{-1}}\right) = 1; \tag{16}
$$

(iii) *$b(t)$ is normalized and*

$$
a(qt) b\left(\frac{q+q^{-1}}{q^2t+q^{-2}t^{-1}}\right) = a(q^{-1}t) b\left(\frac{q+q^{-1}}{q^{-2}t+q^{2}t^{-1}}\right); \tag{17}
$$

(iv) *$b_0 = 1$ and for $n \geq 1$,*

$$
0 = [n]_q a_n + \sum_{\substack{j+k+2\ell+1=n,\\ j,k,\ell\geq 0}} (-1)^\ell \binom{k+\ell}{\ell} [2n-j]_q [2]_q^{k+1} a_j b_{k+1}. \tag{18}
$$

Proof. (i) ⇒ (ii) Let $a_1(t)$ denote the inverse of $a(t)$, and let $a_2(t)$ denote the q-square root of $a_1(t)$. By assumption $b(t)$ is the q-symmetrization of $a_2(t)$. The generating function $a(t)$ is normalized, so $a_1(t)$ is normalized by Lemma 4.1. Now $a_2(t)$ is normalized by Lemma 4.3 and Definition 4.4. Now $b(t)$ is normalized by Lemma 4.6 and Definition 4.7. By construction

$$a(t)a_1(t) = 1, \qquad a_1(t) = a_2(qt)a_2(q^{-1}t), \qquad a_2(t) = b\left(\frac{q+q^{-1}}{t+t^{-1}}\right).$$

Combining these equations we obtain (16).
(ii) ⇒ (iii) In the equation (16), replace t by qt and also by $q^{-1}t$. Compare the two resulting equations to obtain (17).
(iii) ⇒ (iv) Write each side of (17) as a power series in t, and compare coefficients.
(iv) ⇒ (i) By assumption, the generating function $b(t)$ is normalized and satisfies (18). Let $b'(t)$ denote the q-symmetrization of the q-square root of the inverse of $a(t)$. From our earlier comments, the generating function $b'(t)$ is normalized and satisfies (18). The equations (18) admit a unique solution, so $b(t) = b'(t)$. □

Definition 4.9. Referring to Proposition 4.8, we call $b(t)$ the *q-expansion of $a(t)$* whenever the equivalent conditions (i)–(iv) are satisfied.

Lemma 4.10. *Let $a(t) = \sum_{n\in\mathbb{N}} a_n t^n$ denote a normalized generating function. Let $b(t) = \sum_{n\in\mathbb{N}} b_n t^n$ denote the q-expansion of $a(t)$. Then for $n \geq 1$ the following hold:*

(i) *b_n is a polynomial in $a_1, a_2, \ldots, a_n$ that has coefficients in $\mathbb{F}$ and total degree n, where we view a_k as having degree k for $1 \leq k \leq n$. In this polynomial the coefficient of a_n is $-[n]_q[2n]_q^{-1}[2]_q^{-n}$.*
(ii) *a_n is a polynomial in $b_1, b_2, \ldots, b_n$ that has coefficients in $\mathbb{F}$ and total degree n, where we view b_k as having degree k for $1 \leq k \leq n$. In this polynomial the coefficient of b_n is $-[n]_q^{-1}[2n]_q[2]_q^{n}$.*

Proof. (i) By (18) and induction on n.
(ii) By (i) above and induction on n. □

5. Generating functions over a noncommutative algebra

Throughout this section the following notational conventions are in effect. We fix an algebra $\mathcal{B}$ that is not necessarily commutative. Every generating function mentioned is over $\mathcal{B}$.

Definition 5.1. A generating function $a(t) = \sum_{n\in\mathbb{N}} a_n t^n$ is said to be *commutative* whenever $\{a_n\}_{n\in\mathbb{N}}$ mutually commute.

Lemma 5.2. *For a commutative generating function $a(t) = \sum_{n\in\mathbb{N}} a_n t^n$ there exists a commutative subalgebra $\mathcal{A}$ of $\mathcal{B}$ that contains a_n for $n \in \mathbb{N}$.*

Proof. Take $\mathcal{A}$ to be the subalgebra of $\mathcal{B}$ generated by $\{a_n\}_{n\in\mathbb{N}}$. □

Referring to Lemma 5.2, we may view $a(t)$ as a generating function over $\mathcal{A}$.

Definition 5.3. Let $a(t) = \sum_{n\in\mathbb{N}} a_n t^n$ denote a generating function that is commutative and normalized. By the *q-expansion of $a(t)$* we mean the q-expansion of the generating function $a(t)$ over $\mathcal{A}$, where $\mathcal{A}$ is from Lemma 5.2. By (18) and Lemma 4.10, the q-expansion of $a(t)$ is independent of the choice of $\mathcal{A}$.

6. Some elements in $\mathcal{O}_q$

In the previous two sections we discussed generating functions. We now return our attention to the q-Onsager algebra $\mathcal{O}_q$. Recall from Section 1 that in [7, Conjecture 2] Baseilhac and Belliard effectively conjecture that there exist elements

$$\{W_{-k}\}_{k\in\mathbb{N}}, \qquad \{W_{k+1}\}_{k\in\mathbb{N}}, \qquad \{G_{k+1}\}_{k\in\mathbb{N}}, \qquad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}} \tag{19}$$

in $\mathcal{O}_q$ that satisfy the following relations. For $k, \ell \in \mathbb{N}$,

$$[W_0, W_{k+1}] = [W_{-k}, W_1] = (\tilde{G}_{k+1} - G_{k+1})/(q+q^{-1}), \tag{20}$$
$$[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = \rho W_{-k-1} - \rho W_{k+1}, \tag{21}$$
$$[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = \rho W_{k+2} - \rho W_{-k}, \tag{22}$$
$$[W_{-k}, W_{-\ell}] = 0, \qquad [W_{k+1}, W_{\ell+1}] = 0, \tag{23}$$
$$[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] = 0, \tag{24}$$
$$[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] = 0, \tag{25}$$
$$[W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{-\ell}] = 0, \tag{26}$$
$$[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] = 0, \tag{27}$$
$$[W_{k+1}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{\ell+1}] = 0, \tag{28}$$
$$[G_{k+1}, G_{\ell+1}] = 0, \qquad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0, \tag{29}$$
$$[\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] = 0. \tag{30}$$

In the above equations $\rho = -(q^2 - q^{-2})^2$. For notational convenience define

$$G_0 = -(q-q^{-1})[2]_q^2, \qquad \tilde{G}_0 = -(q-q^{-1})[2]_q^2. \tag{31}$$

Remark 6.1. Referring to Remark 3.3, we give the limiting values of the elements (19). In (20)–(30), make a change of variables

$$W_{-k} = \xi W'_{-k}, \qquad W_{k+1} = \xi W'_{k+1}, \qquad G_{k+1} = \xi^2 G'_{k+1}, \qquad \tilde{G}_{k+1} = \xi^2 \tilde{G}'_{k+1}$$

for $k \in \mathbb{N}$. Simplify and set $q = 1$. Lines (20)–(22) become

$$[W'_0, W'_{k+1}] = [W'_{-k}, W'_1] = (\tilde{G}'_{k+1} - G'_{k+1})/2, \tag{32}$$
$$[W'_0, G'_{k+1}] = [\tilde{G}'_{k+1}, W'_0] = 16W'_{-k-1} - 16W'_{k+1}, \tag{33}$$
$$[G'_{k+1}, W'_1] = [W'_1, \tilde{G}'_{k+1}] = 16W'_{k+2} - 16W'_{-k} \tag{34}$$

and (23)–(30) remain essentially unchanged. In [9, Definition 4.1] and [9, Theorem 2], Baseilhac and Crampé display a basis $\{W'_{-k}\}_{k\in\mathbb{N}}$, $\{W'_{k+1}\}_{k\in\mathbb{N}}$, $\{\tilde{G}'_{k+1}\}_{k\in\mathbb{N}}$ for $\mathcal{O}$ that satisfies (23)–(34), where $G'_{k+1} = -\tilde{G}'_{k+1}$ for $k \in \mathbb{N}$.

Returning to $\mathcal{O}_q$, it is desirable to know how the elements (19) are related to the elements (6). In this paper we conjecture the precise relationship. We will state the conjecture shortly. Before stating the conjecture, we discuss what is involved. Let us simplify things by writing the elements (19) in terms of W_0, W_1, $\{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$. To do this, we use (21), (22) to recursively obtain W_{-k}, W_{k+1} for $k \geq 1$:

$$
\begin{aligned}
W_{-1} &= W_1 - \frac{[\tilde{G}_1, W_0]_q}{(q^2 - q^{-2})^2}, \\
W_3 &= W_1 - \frac{[\tilde{G}_1, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_2]_q}{(q^2 - q^{-2})^2}, \\
W_{-3} &= W_1 - \frac{[\tilde{G}_1, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_2]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_3, W_0]_q}{(q^2 - q^{-2})^2}, \\
W_5 &= W_1 - \frac{[\tilde{G}_1, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_2]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_3, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_4]_q}{(q^2 - q^{-2})^2}, \\
W_{-5} &= W_1 - \frac{[\tilde{G}_1, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_2]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_3, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_4]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_5, W_0]_q}{(q^2 - q^{-2})^2}, \\
&\cdots
\end{aligned}
$$

$$
\begin{aligned}
W_2 &= W_0 - \frac{[W_1, \tilde{G}_1]_q}{(q^2 - q^{-2})^2}, \\
W_{-2} &= W_0 - \frac{[W_1, \tilde{G}_1]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_2, W_0]_q}{(q^2 - q^{-2})^2}, \\
W_4 &= W_0 - \frac{[W_1, \tilde{G}_1]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_2, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_3]_q}{(q^2 - q^{-2})^2}, \\
W_{-4} &= W_0 - \frac{[W_1, \tilde{G}_1]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_2, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_3]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_4, W_0]_q}{(q^2 - q^{-2})^2}, \\
W_6 &= W_0 - \frac{[W_1, \tilde{G}_1]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_2, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_3]_q}{(q^2 - q^{-2})^2} - \frac{[\tilde{G}_4, W_0]_q}{(q^2 - q^{-2})^2} - \frac{[W_1, \tilde{G}_5]_q}{(q^2 - q^{-2})^2}, \\
&\cdots
\end{aligned}
$$

The recursion shows that for any integer $k \geq 1$, the generators W_{-k}, W_{k+1} are given as follows. For odd $k = 2r + 1$,

$$W_{-k} = W_1 - \sum_{\ell=0}^{r} \frac{[\tilde{G}_{2\ell+1}, W_0]_q}{(q^2 - q^{-2})^2} - \sum_{\ell=1}^{r} \frac{[W_1, \tilde{G}_{2\ell}]_q}{(q^2 - q^{-2})^2}, \tag{35}$$

$$W_{k+1} = W_0 - \sum_{\ell=0}^{r} \frac{[W_1, \tilde{G}_{2\ell+1}]_q}{(q^2 - q^{-2})^2} - \sum_{\ell=1}^{r} \frac{[\tilde{G}_{2\ell}, W_0]_q}{(q^2 - q^{-2})^2}. \tag{36}$$

For even $k = 2r$,

$$W_{-k} = W_0 - \sum_{\ell=0}^{r-1} \frac{[W_1, \tilde{G}_{2\ell+1}]_q}{(q^2 - q^{-2})^2} - \sum_{\ell=1}^{r} \frac{[\tilde{G}_{2\ell}, W_0]_q}{(q^2 - q^{-2})^2}, \tag{37}$$

$$W_{k+1} = W_1 - \sum_{\ell=0}^{r-1} \frac{[\tilde{G}_{2\ell+1}, W_0]_q}{(q^2 - q^{-2})^2} - \sum_{\ell=1}^{r} \frac{[W_1, \tilde{G}_{2\ell}]_q}{(q^2 - q^{-2})^2}. \tag{38}$$

Next we use (20) to obtain the generators $\{G_{k+1}\}_{k \in \mathbb{N}}$:

$$G_{k+1} = \tilde{G}_{k+1} + (q + q^{-1})[W_1, W_{-k}] \qquad (k \in \mathbb{N}). \tag{39}$$

We have expressed the elements (19) in terms of W_0, W_1, $\{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$. Next, we would like to know how the elements $\{\tilde{G}_{k+1}\}_{k\in\mathbb{N}}$ are related to the elements (6). We will discuss this relationship using generating functions.

Recall the generating function $B(t)$ from Definition 3.6. The generating function $B(t)$ is commutative by Definition 5.1 and the comment above Lemma 3.4. By (12) the generating function $B(t)$ has constant term $q^{-2}-1=-q^{-1}(q-q^{-1})$, so by (3) we have

$$B(t)^\vee = -q(q-q^{-1})^{-1}B(t).$$

The generating function $B(t)^\vee$ is commutative and normalized, so we may speak of its q-expansion as is Definition 5.3.

Conjecture 6.2. *Define the elements*

$$\{W_{-k}\}_{k\in\mathbb{N}}, \qquad \{W_{k+1}\}_{k\in\mathbb{N}}, \qquad \{G_{k+1}\}_{k\in\mathbb{N}}, \qquad \{\tilde{G}_{k+1}\}_{k\in\mathbb{N}} \tag{40}$$

in $\mathcal{O}_q$ as follows:

(i) *the generating function $\tilde{G}(t)^\vee$ is the q-expansion of $B(t)^\vee$, where $\tilde{G}(t)=\sum_{n\in\mathbb{N}}\tilde{G}_n t^n$ and $\tilde{G}_0$ is from (31);*
(ii) *the elements $\{W_{-k}\}_{k\in\mathbb{N}}$, $\{W_{k+1}\}_{k\in\mathbb{N}}$ satisfy (35)–(38);*
(iii) *the elements $\{G_{k+1}\}_{k\in\mathbb{N}}$ satisfy (39).*

Then the elements (40) satisfy (20)–(30).

We have some comments about the q-expansion of $B(t)^\vee$. We mentioned above that $B(t)$ is commutative, so by Lemma 5.2 there exists a commutative subalgebra $\mathcal{A}$ of $\mathcal{O}_q$ that contains $B_{n\delta}$ for $n\in\mathbb{N}$. So $B(t)$ is over $\mathcal{A}$. The q-expansion of $B(t)^\vee$ is over $\mathcal{A}$, and described as follows. For the moment let $\tilde{G}(t)=\sum_{n\in\mathbb{N}}\tilde{G}_n t^n$ denote any generating function over $\mathcal{A}$ such that $\tilde{G}_0$ satisfies (31). By Proposition 4.8 and Definitions 4.9, 5.3 we find that

$$\tilde{G}(t)^\vee \text{ is the } q\text{-expansion of } B(t)^\vee$$

if and only if

$$B(t)\tilde{G}\left(\frac{q+q^{-1}}{qt+q^{-1}t^{-1}}\right)\tilde{G}\left(\frac{q+q^{-1}}{q^{-1}t+qt^{-1}}\right) = -q^{-1}(q-q^{-1})^3[2]_q^4 \tag{41}$$

if and only if

$$B(qt)\tilde{G}\left(\frac{q+q^{-1}}{q^2t+q^{-2}t^{-1}}\right) = B(q^{-1}t)\tilde{G}\left(\frac{q+q^{-1}}{q^{-2}t+qt^{-2}}\right) \tag{42}$$

if and only if for $n\geq 1$,

$$0=[n]_q B_{n\delta}\tilde{G}_0 + \sum_{\substack{j+k+2\ell+1=n,\\ j,k,\ell\geq 0}} (-1)^\ell \binom{k+\ell}{\ell}[2n-j]_q[2]_q^{k+1}B_{j\delta}\tilde{G}_{k+1}. \tag{43}$$

In Appendix A we display (43) in detail for $1\leq n\leq 8$.

7. Supporting evidence for Conjecture 6.2

In this section we give some supporting evidence for Conjecture 6.2.

Our first type of evidence is from checking via computer. The algebra $\mathcal{O}_q$ has been implemented in the computer package SageMath (see [32]) by Travis Scimshaw. Using this package Scrimshaw defined the elements (40) for $0 \le k \le 5$ using (43) along with (35)–(38) and (39). He then had SageMath verify the relations among (20)–(30) that involved these defined elements.

Our next type of evidence concerns the analog of Conjecture 6.2 for the Onsager algebra $\mathcal{O}$. Consider the equation (41). In that equation we compute the limit $q \mapsto 1$ in two steps: (i) make a change of variables as before; (ii) simplify the result and set $q = 1$.

Step (i): We express our generating functions as

$$B(t) = q^{-2} - 1 + 4\xi^2 \mathcal{B}(t), \qquad \mathcal{B}(t) = \sum_{n=1}^{\infty} B_n t^n, \tag{44}$$

$$\tilde{G}(t) = -(q - q^{-1})[2]_q^2 + \xi^2 \tilde{G}'(t), \qquad \tilde{G}'(t) = \sum_{n=1}^{\infty} \tilde{G}'_n t^n. \tag{45}$$

Evaluating (41) using (44), (45) and $\xi^2 = -(q - q^{-1})^2/4$ we obtain

$$\begin{aligned}&\Big(q^{-2} - 1 - (q - q^{-1})^2 \mathcal{B}(t)\Big)\left(-(q - q^{-1})[2]_q^2 - \frac{(q - q^{-1})^2}{4}\tilde{G}'\left(\frac{q + q^{-1}}{qt + q^{-1}t^{-1}}\right)\right)\\ &\times \left(-(q - q^{-1})[2]_q^2 - \frac{(q - q^{-1})^2}{4}\tilde{G}'\left(\frac{q + q^{-1}}{q^{-1}t + qt^{-1}}\right)\right) = -q^{-1}(q - q^{-1})^3[2]_q^4.\end{aligned}$$

Step (ii): For the above equation, let D denote the left-hand side minus the right-hand side. After expanding D and doing some cancellation, we find that D is equal to $-(q - q^{-1})^4[2]_q^2/2$ times

$$2[2]_q^2 \mathcal{B}(t) + \frac{1}{2q}\tilde{G}'\left(\frac{q + q^{-1}}{qt + q^{-1}t^{-1}}\right) + \frac{1}{2q}\tilde{G}'\left(\frac{q + q^{-1}}{q^{-1}t + qt^{-1}}\right) \tag{46}$$

plus $(q - q^{-1})^5$ times some additional terms. Dividing D by $(q - q^{-1})^4$ and then setting $q = 1$, we find that (41) becomes

$$8\mathcal{B}(t) + \tilde{G}'\left(\frac{2}{t + t^{-1}}\right) = 0. \tag{47}$$

Equation (47) matches the equation on the right in [9, Line (4.8)]. By that citation the equation (47) is satisfied by the basis for $\mathcal{O}$ described in Remark 6.1. We have verified the analog of Conjecture 6.2 that applies to $\mathcal{O}$.

Our next type of evidence has to do with the universal Askey-Wilson algebra Δ_q [36, Definition 1.2]. This algebra is defined by generators and relations. The generators are A, B, C. The relations assert that each of the following is central in Δ_q:

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \qquad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \qquad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}}.$$

For the above three central elements, multiply each by $q + q^{-1}$ to get α, β, γ. Thus

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}} = \frac{\alpha}{q + q^{-1}}, \tag{48}$$

$$B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}} = \frac{\beta}{q + q^{-1}}, \tag{49}$$

$$C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} = \frac{\gamma}{q + q^{-1}}. \tag{50}$$

Each of α, β, γ is central in Δ_q. By [36, Corollary 8.3] the center of Δ_q is generated by $\alpha, \beta, \gamma, \Omega$ where

$$\Omega = qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma. \tag{51}$$

The element Ω is called the Casimir element. By [36, Theorem 8.2] the elements $\alpha, \beta, \gamma, \Omega$ are algebraically independent. We write $\mathbb{F}[\alpha, \beta, \gamma, \Omega]$ for the center of Δ_q.

Next we summarize from [36, Section 3] how the modular group $\mathrm{PSL}_2(\mathbb{Z})$ acts on Δ_q as a group of automorphisms. By [1] the group $\mathrm{PSL}_2(\mathbb{Z})$ has a presentation by generators ϱ, σ and relations $\varrho^3 = 1$, $\sigma^2 = 1$. By [36, Theorems 3.1, 6.4] the group $\mathrm{PSL}_2(\mathbb{Z})$ acts on Δ_q as a group of automorphisms in the following way:

u	A	B	C	α	β	γ	Ω
$\varrho(u)$	B	C	A	β	γ	α	Ω
$\sigma(u)$	B	A	$C + \frac{[A,B]}{q-q^{-1}}$	β	α	γ	Ω

For notational convenience define

$$C' = C + \frac{[A, B]}{q - q^{-1}}. \tag{52}$$

Applying σ to (48)–(50) and using the above table, we obtain

$$B + \frac{qAC' - q^{-1}C'A}{q^2 - q^{-2}} = \frac{\beta}{q + q^{-1}}, \tag{53}$$

$$A + \frac{qC'B - q^{-1}BC'}{q^2 - q^{-2}} = \frac{\alpha}{q + q^{-1}}, \tag{54}$$

$$C' + \frac{qBA - q^{-1}AB}{q^2 - q^{-2}} = \frac{\gamma}{q + q^{-1}}. \tag{55}$$

Next we explain how Δ_q is related to $\mathcal{O}_q$. By [36, Theorem 2.2] the algebra Δ_q has a presentation by generators A, B, γ and relations

$$A^3B - [3]_q A^2BA + [3]_q ABA^2 - BA^3 = (q^2 - q^{-2})^2(BA - AB), \tag{56}$$

$$B^3A - [3]_q B^2AB + [3]_q BAB^2 - AB^3 = (q^2 - q^{-2})^2(AB - BA), \tag{57}$$

$$A^2B^2 - B^2A^2 + (q^2 + q^{-2})(BABA - ABAB) = (q - q^{-1})^2(BA - AB)\gamma, \tag{58}$$

$$\gamma A = A\gamma, \qquad \gamma B = B\gamma. \tag{59}$$

The relations (56), (57) are the q-Dolan/Grady relations. Consequently there exists an algebra homomorphism $\natural : \mathcal{O}_q \to \Delta_q$ that sends $W_0 \mapsto A$ and $W_1 \mapsto B$. This homomorphism is not injective by [36, Theorem 10.9].

For the elements (6) and (40) we retain the same notation for their images under $\natural$. We will show that for Δ_q the elements (40) satisfy the relations (20)–(30).

For the algebra Δ_q define

$$\Psi(t) = B(t) + 1 - q^{-2}, \tag{60}$$

where $B(t)$ is from Definition 3.6. By (12) we have $\Psi(t) = \sum_{n=1}^{\infty} B_{n\delta} t^n$. By [38, Corollary 5.7] the elements $\{B_{n\delta}\}_{n=1}^{\infty}$ are contained in the subalgebra of Δ_q generated by $\mathbb{F}[\alpha, \beta, \gamma, \Omega]$ and C. Consequently the elements $\{B_{n\delta}\}_{n=1}^{\infty}$ commute with C, so $\Psi(t)$ commutes with C. By this and [38, Line (5.19)] we find that

$$\Psi(t)\big(qt + q^{-1}t^{-1} + C\big)\big(q^{-1}t + qt^{-1} + C\big) \tag{61}$$

is equal to $1 - q^{-2}$ times

$$\Omega - \frac{(t+t^{-1})\alpha\beta}{(t-t^{-1})^2} - \frac{\alpha^2+\beta^2}{(t-t^{-1})^2} - (t+t^{-1})\gamma + (q+q^{-1})(t+t^{-1})C + C^2.$$

Upon eliminating $\Psi(t)$ from (61) using (60), we find that

$$B(t)\big(qt + q^{-1}t^{-1} + C\big)\big(q^{-1}t + qt^{-1} + C\big) \tag{62}$$

is equal to $1 - q^{-2}$ times

$$\Omega - \frac{(t+t^{-1})\alpha\beta}{(t-t^{-1})^2} - \frac{\alpha^2+\beta^2}{(t-t^{-1})^2} - (t+t^{-1})\gamma - (qt+q^{-1}t^{-1})(q^{-1}t+qt^{-1}).$$

Define

$$N(t) = \frac{B(t)}{q^{-2}-1}\,\frac{qt+q^{-1}t^{-1}+C}{qt+q^{-1}t^{-1}}\,\frac{q^{-1}t+qt^{-1}+C}{q^{-1}t+qt^{-1}}. \tag{63}$$

By the above comments

$$N(t) = 1 + N_1(t)\Omega + N_2(t)\alpha\beta + N_3(t)(\alpha^2+\beta^2) + N_4(t)\gamma, \tag{64}$$

where

$$N_1(t) = \frac{-1}{(qt+q^{-1}t^{-1})(q^{-1}t+qt^{-1})}, \tag{65}$$

$$N_2(t) = \frac{t+t^{-1}}{(t-t^{-1})^2(qt+q^{-1}t^{-1})(q^{-1}t+qt^{-1})}, \tag{66}$$

$$N_3(t) = \frac{1}{(t-t^{-1})^2(qt+q^{-1}t^{-1})(q^{-1}t+qt^{-1})}, \tag{67}$$

$$N_4(t) = \frac{t+t^{-1}}{(qt+q^{-1}t^{-1})(q^{-1}t+qt^{-1})}. \tag{68}$$

Evaluating (65)–(68) using

$$\frac{1}{qt+q^{-1}t^{-1}} = \sum_{n\in\mathbb{N}} (-1)^n q^{2n+1} t^{2n+1},$$

$$\frac{1}{q^{-1}t+qt^{-1}} = \sum_{n\in\mathbb{N}} (-1)^n q^{-2n-1} t^{2n+1},$$

$$\frac{1}{(t-t^{-1})^2} = \sum_{n\in\mathbb{N}} n t^{2n}$$

we find that the functions $N_1(t)$, $N_2(t)$, $N_3(t)$, $N_4(t)$ are power series in t with zero constant term. By this and (64), we may view $N(t)$ as a normalized generating function over $\mathbb{F}[\alpha, \beta, \gamma, \Omega]$.

Definition 7.1. Define a generating function $Z(t) = \sum_{n\in\mathbb{N}} Z_n t^n$ over $\mathbb{F}[\alpha, \beta, \gamma, \Omega]$ such that $Z_0 = q^{-2} - q^2$ and $Z(t)^\vee$ is the q-expansion of $N(t)$.

The notation $Z(t)^\vee$ is explained in (3). The q-expansion concept is explained in Proposition 4.8 and Definition 4.9. By these explanations and Definition 7.1,

$$N(t)Z\left(\frac{q+q^{-1}}{qt+q^{-t}t^{-1}}\right)Z\left(\frac{q+q^{-1}}{q^{-1}t+qt^{-1}}\right) = (q^2 - q^{-2})^2. \tag{69}$$

Proposition 7.2. *For the algebra Δ_q,*

$$\tilde{G}(t) = Z(t)(q + q^{-1} + tC). \tag{70}$$

Proof. Define the generating function $\tilde{\mathcal{G}}(t) = Z(t)(q + q^{-1} + tC)$. We show that $\tilde{G}(t) = \tilde{\mathcal{G}}(t)$. Let $\mathcal{A}$ denote the subalgebra of Δ_q generated by $\mathbb{F}[\alpha, \beta, \gamma, \Omega]$ and C. Note that $\mathcal{A}$ is commutative. By construction $\tilde{\mathcal{G}}(t)$ is over $\mathcal{A}$. By our comments below (60), the generating function $B(t)$ is over $\mathcal{A}$. By the discussion around (41), it suffices to show that

$$B(t)\tilde{\mathcal{G}}\left(\frac{q+q^{-1}}{qt+q^{-1}t^{-1}}\right)\tilde{\mathcal{G}}\left(\frac{q+q^{-1}}{q^{-1}t+qt^{-1}}\right) = -q^{-1}(q-q^{-1})^3[2]_q^4. \tag{71}$$

Using (63) and (69),

$$\begin{aligned}
&B(t)\tilde{\mathcal{G}}\left(\frac{q+q^{-1}}{qt+q^{-1}t^{-1}}\right)\tilde{\mathcal{G}}\left(\frac{q+q^{-1}}{q^{-1}t+qt^{-1}}\right) \\
&\quad = [2]_q^2 B(t) Z\left(\frac{q+q^{-1}}{qt+q^{-1}t^{-1}}\right)\frac{qt+q^{-1}t^{-1}+C}{qt+q^{-1}t^{-1}} Z\left(\frac{q+q^{-1}}{q^{-1}t+qt^{-1}}\right)\frac{q^{-1}t+qt^{-1}+C}{q^{-1}t+qt^{-1}} \\
&\quad = [2]_q^2 (q^{-2}-1) N(t) Z\left(\frac{q+q^{-1}}{qt+q^{-1}t^{-1}}\right) Z\left(\frac{q+q^{-1}}{q^{-1}t+qt^{-1}}\right) \\
&\quad = [2]_q^2 (q^{-2}-1)(q^2-q^{-2})^2 \\
&\quad = -q^{-1}(q-q^{-1})^3[2]_q^4.
\end{aligned}$$

We have shown (71), and the result follows. □

Define the generating functions

$$W^-(t) = \sum_{n\in\mathbb{N}} W_{-n} t^n, \qquad W^+(t) = \sum_{n\in\mathbb{N}} W_{n+1} t^n.$$

By (35)–(38) we obtain

$$W^+(t) = \frac{t[\tilde{G}(t), A]_q + [B, \tilde{G}(t)]_q}{(t^2-1)(q^2-q^{-2})^2}, \tag{72}$$

$$W^-(t) = \frac{[\tilde{G}(t), A]_q + t[B, \tilde{G}(t)]_q}{(t^2-1)(q^2-q^{-2})^2}. \tag{73}$$

Lemma 7.3. *For the algebra* Δ_q*,*

$$W^+(t) = Z(t)\,\frac{(q-q^{-1})(\alpha+\beta t) - (q^2-q^{-2})(t-t^{-1})B}{(q^2-q^{-2})^2(t-t^{-1})}, \tag{74}$$

$$W^-(t) = Z(t)\,\frac{(q-q^{-1})(\alpha t+\beta) - (q^2-q^{-2})(t-t^{-1})A}{(q^2-q^{-2})^2(t-t^{-1})}. \tag{75}$$

Proof. To obtain (74), eliminate $\tilde{G}(t)$ from (72) using (70), and evaluate the result using (48), (49). Equation (75) is similarly obtained. □

Define the generating function

$$G(t) = \sum_{n\in\mathbb{N}} G_n t^n.$$

Using (39) we obtain

$$G(t) = \tilde{G}(t) + t(q+q^{-1})[B, W^-(t)]. \tag{76}$$

Lemma 7.4. *For the algebra* Δ_q *we have*

$$G(t) = Z(t)(q+q^{-1}+tC'), \tag{77}$$

where C' *is from* (52)*.*

Proof. Eliminate $\tilde{G}(t)$ from (76) using (70). Eliminate $W^-(t)$ from (76) using (75), and evaluate the result using (52). □

Let s denote an indeterminate that commutes with t.

Lemma 7.5. *For the algebra* Δ_q *we have*

$$[A, W^+(t)] = [W^-(t), B] = t^{-1}(\tilde{G}(t) - G(t))/(q+q^{-1}),$$
$$[A, G(t)]_q = [\tilde{G}(t), A]_q = \rho W^-(t) - \rho t W^+(t),$$
$$[G(t), B]_q = [B, \tilde{G}(t)]_q = \rho W^+(t) - \rho t W^-(t),$$
$$[W^-(s), W^-(t)] = 0, \qquad [W^+(s), W^+(t)] = 0,$$
$$[W^-(s), W^+(t)] + [W^+(s), W^-(t)] = 0,$$
$$s[W^-(s), G(t)] + t[G(s), W^-(t)] = 0,$$
$$s[W^-(s), \tilde{G}(t)] + t[\tilde{G}(s), W^-(t)] = 0,$$
$$s[W^+(s), G(t)] + t[G(s), W^+(t)] = 0,$$
$$s[W^+(s), \tilde{G}(t)] + t[\tilde{G}(s), W^+(t)] = 0,$$
$$[G(s), G(t)] = 0, \qquad [\tilde{G}(s), \tilde{G}(t)] = 0,$$
$$[\tilde{G}(s), G(t)] + [G(s), \tilde{G}(t)] = 0,$$

where $\rho = -(q^2-q^{-2})^2$*.*

Proof. These relations are routinely verified using Proposition 7.2 and Lemmas 7.3, 7.4 along with (48), (49), (53), (54). □

Theorem 7.6. *In the algebra Δ_q the elements* (40) *satisfy the relations* (20)–(30).

Proof. This is a routine consequence of Lemma 7.5. □

8. Comments

In the previous section we gave some supporting evidence for Conjecture 6.2. In this section we assume that Conjecture 6.2 is correct, and provide more information about how the elements (40) are related to the elements (6). We will give a variation on (35)–(38).

Using Appendix A and $B_\delta = q^{-2}W_1W_0 - W_0W_1$ we obtain

$$\tilde{G}_1 = -qB_\delta = [W_0, W_1]_q. \tag{78}$$

Lemma 8.1. *For $k \in \mathbb{N}$,*

(i) $[\tilde{G}_{k+1}, W_0]_q = (q - q^{-1})W_0\tilde{G}_{k+1} - q^2[B_\delta, W_{-k}]$,
(ii) $[W_1, \tilde{G}_{k+1}]_q = (q - q^{-1})W_1\tilde{G}_{k+1} + [B_\delta, W_{k+1}]$.

Proof. (i) Observe that

$$[\tilde{G}_{k+1}, W_0]_q = (q - q^{-1})W_0\tilde{G}_{k+1} + q[\tilde{G}_{k+1}, W_0].$$

By (26) and (78),

$$[\tilde{G}_{k+1}, W_0] = [\tilde{G}_1, W_{-k}] = -q[B_\delta, W_{-k}].$$

The result follows.
(ii) Observe that

$$[W_1, \tilde{G}_{k+1}]_q = (q - q^{-1})W_1\tilde{G}_{k+1} - q^{-1}[\tilde{G}_{k+1}, W_1].$$

By (28) and (78),

$$[\tilde{G}_{k+1}, W_1] = [\tilde{G}_1, W_{k+1}] = -q[B_\delta, W_{k+1}].$$

The result follows. □

Lemma 8.2. *For $n \geq 1$,*

$$W_{-n} = W_n - \frac{(q - q^{-1})W_0\tilde{G}_n}{(q^2 - q^{-2})^2} + \frac{q^2[B_\delta, W_{1-n}]}{(q^2 - q^{-2})^2}, \tag{79}$$

$$W_{n+1} = W_{1-n} - \frac{(q - q^{-1})W_1\tilde{G}_n}{(q^2 - q^{-2})^2} - \frac{[B_\delta, W_n]}{(q^2 - q^{-2})^2}. \tag{80}$$

Proof. Use the equations on the right in (21), (22) along with Lemma 8.1. □

We recall some notation from [15]. For a negative integer k define

$$B_{k\delta+\alpha_0} = B_{(-k-1)\delta+\alpha_1}, \qquad B_{k\delta+\alpha_1} = B_{(-k-1)\delta+\alpha_0}.$$

We have

$$B_{r\delta+\alpha_0} = B_{s\delta+\alpha_1} \qquad (r, s \in \mathbb{Z}, \quad r+s=-1). \tag{81}$$

Lemma 8.3. *For $n \in \mathbb{Z}$,*

$$\frac{q[B_\delta, B_{n\delta+\alpha_0}]}{(q-q^{-1})(q^2-q^{-2})} = B_{(n+1)\delta+\alpha_0} - B_{(n-1)\delta+\alpha_0}, \tag{82}$$

$$\frac{q[B_\delta, B_{n\delta+\alpha_1}]}{(q-q^{-1})(q^2-q^{-2})} = B_{(n-1)\delta+\alpha_1} - B_{(n+1)\delta+\alpha_1}. \tag{83}$$

Proof. Use (7)–(10) and (81). □

Proposition 8.4. *For $n \in \mathbb{N}$ the following hold in $\mathcal{O}_q$:*

$$W_{-n} = -(q-q^{-1})^{-1} \sum_{k=0}^{n} \sum_{\ell=0}^{k} \binom{k}{\ell} q^{k-2\ell} [2]_q^{-k-2} B_{(k-2\ell)\delta+\alpha_0} \tilde{G}_{n-k}, \tag{84}$$

$$W_{n+1} = -(q-q^{-1})^{-1} \sum_{k=0}^{n} \sum_{\ell=0}^{k} \binom{k}{\ell} q^{2\ell-k} [2]_q^{-k-2} B_{(k-2\ell)\delta+\alpha_1} \tilde{G}_{n-k}. \tag{85}$$

Proof. We use induction on n. First assume that $n=0$. Then (84), (85) hold. Next assume that $n \geq 1$. To obtain (84), evaluate the right-hand side of (79) using induction along with (81), (82). To obtain (85), evaluate the right-hand side of (80) using induction along with (81), (83). □

In Appendix B we display (84), (85) in detail for $0 \leq n \leq 7$.

Referring to (84) and (85), if we express each term $\tilde{G}_{n-k}$ as a polynomial in $B_\delta, B_{2\delta}, \ldots, B_{(n-k)\delta}$ using (43), then we effectively write W_{-n} and W_{n+1} in the PBW basis for $\mathcal{O}_q$ given in Lemma 3.4. Unfortunately the resulting formula are not pleasant.

CRediT authorship contribution statement

Paul Terwilliger is the sole author and responsible for all aspects of the paper.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

For the q-Onsager algebra $\mathcal{O}_q$ we use (43) to obtain $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_8$ in terms of $B_\delta, B_{2\delta}, \ldots, B_{8\delta}$.

Recall that

$$B_{0\delta} = q^{-2} - 1, \qquad \tilde{G}_0 = -(q - q^{-1})[2]_q^2.$$

$\tilde{G}_1$ satisfies

$0 =$

	$[2]_q B_{0\delta}$	$[1]_q B_{1\delta}$
$\tilde{G}_0$	0	1
$[2]_q \tilde{G}_1$	1	0

$\tilde{G}_2$ satisfies

$0 =$

	$[4]_q B_{0\delta}$	$[3]_q B_{1\delta}$	$[2]_q B_{2\delta}$
$\tilde{G}_0$	0	0	1
$[2]_q \tilde{G}_1$	0	1	0
$[2]_q^2 \tilde{G}_2$	1	0	0

$\tilde{G}_3$ satisfies

$0 =$

	$[6]_q B_{0\delta}$	$[5]_q B_{1\delta}$	$[4]_q B_{2\delta}$	$[3]_q B_{3\delta}$
$\tilde{G}_0$	0	0	0	1
$[2]_q \tilde{G}_1$	-1	0	1	0
$[2]_q^2 \tilde{G}_2$	0	1	0	0
$[2]_q^3 \tilde{G}_3$	1	0	0	0

$\tilde{G}_4$ satisfies

$0 =$

	$[8]_q B_{0\delta}$	$[7]_q B_{1\delta}$	$[6]_q B_{2\delta}$	$[5]_q B_{3\delta}$	$[4]_q B_{4\delta}$
$\tilde{G}_0$	0	0	0	0	1
$[2]_q \tilde{G}_1$	0	-1	0	1	0
$[2]_q^2 \tilde{G}_2$	-2	0	1	0	0
$[2]_q^3 \tilde{G}_3$	0	1	0	0	0
$[2]_q^4 \tilde{G}_4$	1	0	0	0	0

$\tilde{G}_5$ satisfies

$0 =$

	$[10]_q B_{0\delta}$	$[9]_q B_{1\delta}$	$[8]_q B_{2\delta}$	$[7]_q B_{3\delta}$	$[6]_q B_{4\delta}$	$[5]_q B_{5\delta}$
$\tilde{G}_0$	0	0	0	0	0	1
$[2]_q \tilde{G}_1$	1	0	-1	0	1	0
$[2]_q^2 \tilde{G}_2$	0	-2	0	1	0	0
$[2]_q^3 \tilde{G}_3$	-3	0	1	0	0	0
$[2]_q^4 \tilde{G}_4$	0	1	0	0	0	0
$[2]_q^5 \tilde{G}_5$	1	0	0	0	0	0

$\tilde{G}_6$ satisfies

$0 =$

	$[12]_q B_{0\delta}$	$[11]_q B_{1\delta}$	$[10]_q B_{2\delta}$	$[9]_q B_{3\delta}$	$[8]_q B_{4\delta}$	$[7]_q B_{5\delta}$	$[6]_q B_{6\delta}$
$\tilde{G}_0$	0	0	0	0	0	0	1
$[2]_q \tilde{G}_1$	0	1	0	−1	0	1	0
$[2]_q^2 \tilde{G}_2$	3	0	−2	0	1	0	0
$[2]_q^3 \tilde{G}_3$	0	−3	0	1	0	0	0
$[2]_q^4 \tilde{G}_4$	−4	0	1	0	0	0	0
$[2]_q^5 \tilde{G}_5$	0	1	0	0	0	0	0
$[2]_q^6 \tilde{G}_6$	1	0	0	0	0	0	0

$\tilde{G}_7$ satisfies

$0 =$

	$[14]_q B_{0\delta}$	$[13]_q B_{1\delta}$	$[12]_q B_{2\delta}$	$[11]_q B_{3\delta}$	$[10]_q B_{4\delta}$	$[9]_q B_{5\delta}$	$[8]_q B_{6\delta}$	$[7]_q B_{7\delta}$
$\tilde{G}_0$	0	0	0	0	0	0	0	1
$[2]_q \tilde{G}_1$	−1	0	1	0	−1	0	1	0
$[2]_q^2 \tilde{G}_2$	0	3	0	−2	0	1	0	0
$[2]_q^3 \tilde{G}_3$	6	0	−3	0	1	0	0	0
$[2]_q^4 \tilde{G}_4$	0	−4	0	1	0	0	0	0
$[2]_q^5 \tilde{G}_5$	−5	0	1	0	0	0	0	0
$[2]_q^6 \tilde{G}_6$	0	1	0	0	0	0	0	0
$[2]_q^7 \tilde{G}_7$	1	0	0	0	0	0	0	0

$\tilde{G}_8$ satisfies $0 =$

	$[16]_q B_{0\delta}$	$[15]_q B_{1\delta}$	$[14]_q B_{2\delta}$	$[13]_q B_{3\delta}$	$[12]_q B_{4\delta}$	$[11]_q B_{5\delta}$	$[10]_q B_{6\delta}$	$[9]_q B_{7\delta}$	$[8]_q B_{8\delta}$
$\tilde{G}_0$	0	0	0	0	0	0	0	0	1
$[2]_q \tilde{G}_1$	0	−1	0	1	0	−1	0	1	0
$[2]_q^2 \tilde{G}_2$	−4	0	3	0	−2	0	1	0	0
$[2]_q^3 \tilde{G}_3$	0	6	0	−3	0	1	0	0	0
$[2]_q^4 \tilde{G}_4$	10	0	−4	0	1	0	0	0	0
$[2]_q^5 \tilde{G}_5$	0	−5	0	1	0	0	0	0	0
$[2]_q^6 \tilde{G}_6$	−6	0	1	0	0	0	0	0	0
$[2]_q^7 \tilde{G}_7$	0	1	0	0	0	0	0	0	0
$[2]_q^8 \tilde{G}_8$	1	0	0	0	0	0	0	0	0

Appendix B

For the q-Onsager algebra $\mathcal{O}_q$ we use (84), (85) to obtain $\{W_{-n}\}_{n=0}^{7}$ and $\{W_{n+1}\}_{n=0}^{7}$ in terms of $\{B_{n\delta+\alpha_0}\}_{n=0}^{7}$, $\{B_{n\delta+\alpha_1}\}_{n=0}^{7}$, $\{\tilde{G}_n\}_{n=0}^{7}$. Recall that $\tilde{G}_0 = -(q-q^{-1})[2]_q^2$.

We have

$$W_0 = B_{\alpha_0} = -(q-q^{-1})^{-1}[2]_q^{-2} B_{\alpha_0}\tilde{G}_0.$$

W_{-1} is equal to $-(q-q^{-1})^{-1}[2]_q^{-3}$ times

	$\tilde{G}_0$	$[2]_q \tilde{G}_1$
$q^{-1} B_{\alpha_1}$	1	0
B_{α_0}	0	1
$q B_{\delta+\alpha_0}$	1	0

W_{-2} is equal to $-(q-q^{-1})^{-1}[2]_q^{-4}$ times

	$\tilde{G}_0$	$[2]_q \tilde{G}_1$	$[2]_q^2 \tilde{G}_2$
$q^{-2} B_{\delta+\alpha_1}$	1	0	0
$q^{-1} B_{\alpha_1}$	0	1	0
B_{α_0}	2	0	1
$q B_{\delta+\alpha_0}$	0	1	0
$q^2 B_{2\delta+\alpha_0}$	1	0	0

W_{-3} is equal to $-(q-q^{-1})^{-1}[2]_q^{-5}$ times

	$\tilde{G}_0$	$[2]_q \tilde{G}_1$	$[2]_q^2 \tilde{G}_2$	$[2]_q^3 \tilde{G}_3$
$q^{-3} B_{2\delta+\alpha_1}$	1	0	0	0
$q^{-2} B_{\delta+\alpha_1}$	0	1	0	0
$q^{-1} B_{\alpha_1}$	3	0	1	0
B_{α_0}	0	2	0	1
$q B_{\delta+\alpha_0}$	3	0	1	0
$q^2 B_{2\delta+\alpha_0}$	0	1	0	0
$q^3 B_{3\delta+\alpha_0}$	1	0	0	0

W_{-4} is equal to $-(q-q^{-1})^{-1}[2]_q^{-6}$ times

	$\tilde{G}_0$	$[2]_q \tilde{G}_1$	$[2]_q^2 \tilde{G}_2$	$[2]_q^3 \tilde{G}_3$	$[2]_q^4 \tilde{G}_4$
$q^{-4} B_{3\delta+\alpha_1}$	1	0	0	0	0
$q^{-3} B_{2\delta+\alpha_1}$	0	1	0	0	0
$q^{-2} B_{\delta+\alpha_1}$	4	0	1	0	0
$q^{-1} B_{\alpha_1}$	0	3	0	1	0
B_{α_0}	6	0	2	0	1
$q B_{\delta+\alpha_0}$	0	3	0	1	0
$q^2 B_{2\delta+\alpha_0}$	4	0	1	0	0
$q^3 B_{3\delta+\alpha_0}$	0	1	0	0	0
$q^4 B_{4\delta+\alpha_0}$	1	0	0	0	0

W_{-5} is equal to $-(q-q^{-1})^{-1}[2]_q^{-7}$ times

	$\tilde{G}_0$	$[2]_q\tilde{G}_1$	$[2]_q^2\tilde{G}_2$	$[2]_q^3\tilde{G}_3$	$[2]_q^4\tilde{G}_4$	$[2]_q^5\tilde{G}_5$
$q^{-5}B_{4\delta+\alpha_1}$	1	0	0	0	0	0
$q^{-4}B_{3\delta+\alpha_1}$	0	1	0	0	0	0
$q^{-3}B_{2\delta+\alpha_1}$	5	0	1	0	0	0
$q^{-2}B_{\delta+\alpha_1}$	0	4	0	1	0	0
$q^{-1}B_{\alpha_1}$	10	0	3	0	1	0
B_{α_0}	0	6	0	2	0	1
$qB_{\delta+\alpha_0}$	10	0	3	0	1	0
$q^2B_{2\delta+\alpha_0}$	0	4	0	1	0	0
$q^3B_{3\delta+\alpha_0}$	5	0	1	0	0	0
$q^4B_{4\delta+\alpha_0}$	0	1	0	0	0	0
$q^5B_{5\delta+\alpha_0}$	1	0	0	0	0	0

W_{-6} is equal to $-(q-q^{-1})^{-1}[2]_q^{-8}$ times

	$\tilde{G}_0$	$[2]_q\tilde{G}_1$	$[2]_q^2\tilde{G}_2$	$[2]_q^3\tilde{G}_3$	$[2]_q^4\tilde{G}_4$	$[2]_q^5\tilde{G}_5$	$[2]_q^6\tilde{G}_6$
$q^{-6}B_{5\delta+\alpha_1}$	1	0	0	0	0	0	0
$q^{-5}B_{4\delta+\alpha_1}$	0	1	0	0	0	0	0
$q^{-4}B_{3\delta+\alpha_1}$	6	0	1	0	0	0	0
$q^{-3}B_{2\delta+\alpha_1}$	0	5	0	1	0	0	0
$q^{-2}B_{\delta+\alpha_1}$	15	0	4	0	1	0	0
$q^{-1}B_{\alpha_1}$	0	10	0	3	0	1	0
B_{α_0}	20	0	6	0	2	0	1
$qB_{\delta+\alpha_0}$	0	10	0	3	0	1	0
$q^2B_{2\delta+\alpha_0}$	15	0	4	0	1	0	0
$q^3B_{3\delta+\alpha_0}$	0	5	0	1	0	0	0
$q^4B_{4\delta+\alpha_0}$	6	0	1	0	0	0	0
$q^5B_{5\delta+\alpha_0}$	0	1	0	0	0	0	0
$q^6B_{6\delta+\alpha_0}$	1	0	0	0	0	0	0

W_{-7} is equal to $-(q-q^{-1})^{-1}[2]_q^{-9}$ times

	$\tilde{G}_0$	$[2]_q\tilde{G}_1$	$[2]_q^2\tilde{G}_2$	$[2]_q^3\tilde{G}_3$	$[2]_q^4\tilde{G}_4$	$[2]_q^5\tilde{G}_5$	$[2]_q^6\tilde{G}_6$	$[2]_q^7\tilde{G}_7$
$q^{-7}B_{6\delta+\alpha_1}$	1	0	0	0	0	0	0	0
$q^{-6}B_{5\delta+\alpha_1}$	0	1	0	0	0	0	0	0
$q^{-5}B_{4\delta+\alpha_1}$	7	0	1	0	0	0	0	0
$q^{-4}B_{3\delta+\alpha_1}$	0	6	0	1	0	0	0	0
$q^{-3}B_{2\delta+\alpha_1}$	21	0	5	0	1	0	0	0
$q^{-2}B_{\delta+\alpha_1}$	0	15	0	4	0	1	0	0
$q^{-1}B_{\alpha_1}$	35	0	10	0	3	0	1	0
B_{α_0}	0	20	0	6	0	2	0	1
$qB_{\delta+\alpha_0}$	35	0	10	0	3	0	1	0
$q^2B_{2\delta+\alpha_0}$	0	15	0	4	0	1	0	0
$q^3B_{3\delta+\alpha_0}$	21	0	5	0	1	0	0	0
$q^4B_{4\delta+\alpha_0}$	0	6	0	1	0	0	0	0
$q^5B_{5\delta+\alpha_0}$	7	0	1	0	0	0	0	0
$q^6B_{6\delta+\alpha_0}$	0	1	0	0	0	0	0	0
$q^7B_{7\delta+\alpha_0}$	1	0	0	0	0	0	0	0

$$W_1 = B_{\alpha_1} = -(q - q^{-1})^{-1}[2]_q^{-2} B_{\alpha_1} \tilde{G}_0.$$

W_2 is equal to $-(q - q^{-1})^{-1}[2]_q^{-3}$ times

	$\tilde{G}_0$	$[2]_q \tilde{G}_1$
$q^{-1} B_{\delta+\alpha_1}$	1	0
B_{α_1}	0	1
$q B_{\alpha_0}$	1	0

W_3 is equal to $-(q - q^{-1})^{-1}[2]_q^{-4}$ times

	$\tilde{G}_0$	$[2]_q \tilde{G}_1$	$[2]_q^2 \tilde{G}_2$
$q^{-2} B_{2\delta+\alpha_1}$	1	0	0
$q^{-1} B_{\delta+\alpha_1}$	0	1	0
B_{α_1}	2	0	1
$q B_{\alpha_0}$	0	1	0
$q^2 B_{\delta+\alpha_0}$	1	0	0

W_4 is equal to $-(q - q^{-1})^{-1}[2]_q^{-5}$ times

	$\tilde{G}_0$	$[2]_q \tilde{G}_1$	$[2]_q^2 \tilde{G}_2$	$[2]_q^3 \tilde{G}_3$
$q^{-3} B_{3\delta+\alpha_1}$	1	0	0	0
$q^{-2} B_{2\delta+\alpha_1}$	0	1	0	0
$q^{-1} B_{\delta+\alpha_1}$	3	0	1	0
B_{α_1}	0	2	0	1
$q B_{\alpha_0}$	3	0	1	0
$q^2 B_{\delta+\alpha_0}$	0	1	0	0
$q^3 B_{2\delta+\alpha_0}$	1	0	0	0

W_5 is equal to $-(q - q^{-1})^{-1}[2]_q^{-6}$ times

	$\tilde{G}_0$	$[2]_q \tilde{G}_1$	$[2]_q^2 \tilde{G}_2$	$[2]_q^3 \tilde{G}_3$	$[2]_q^4 \tilde{G}_4$
$q^{-4} B_{4\delta+\alpha_1}$	1	0	0	0	0
$q^{-3} B_{3\delta+\alpha_1}$	0	1	0	0	0
$q^{-2} B_{2\delta+\alpha_1}$	4	0	1	0	0
$q^{-1} B_{\delta+\alpha_1}$	0	3	0	1	0
B_{α_1}	6	0	2	0	1
$q B_{\alpha_0}$	0	3	0	1	0
$q^2 B_{\delta+\alpha_0}$	4	0	1	0	0
$q^3 B_{2\delta+\alpha_0}$	0	1	0	0	0
$q^4 B_{3\delta+\alpha_0}$	1	0	0	0	0

W_6 is equal to $-(q - q^{-1})^{-1}[2]_q^{-7}$ times

	$\tilde{G}_0$	$[2]_q\tilde{G}_1$	$[2]_q^2\tilde{G}_2$	$[2]_q^3\tilde{G}_3$	$[2]_q^4\tilde{G}_4$	$[2]_q^5\tilde{G}_5$
$q^{-5}B_{5\delta+\alpha_1}$	1	0	0	0	0	0
$q^{-4}B_{4\delta+\alpha_1}$	0	1	0	0	0	0
$q^{-3}B_{3\delta+\alpha_1}$	5	0	1	0	0	0
$q^{-2}B_{2\delta+\alpha_1}$	0	4	0	1	0	0
$q^{-1}B_{\delta+\alpha_1}$	10	0	3	0	1	0
B_{α_1}	0	6	0	2	0	1
qB_{α_0}	10	0	3	0	1	0
$q^2B_{\delta+\alpha_0}$	0	4	0	1	0	0
$q^3B_{2\delta+\alpha_0}$	5	0	1	0	0	0
$q^4B_{3\delta+\alpha_0}$	0	1	0	0	0	0
$q^5B_{4\delta+\alpha_0}$	1	0	0	0	0	0

W_7 is equal to $-(q-q^{-1})^{-1}[2]_q^{-8}$ times

	$\tilde{G}_0$	$[2]_q\tilde{G}_1$	$[2]_q^2\tilde{G}_2$	$[2]_q^3\tilde{G}_3$	$[2]_q^4\tilde{G}_4$	$[2]_q^5\tilde{G}_5$	$[2]_q^6\tilde{G}_6$
$q^{-6}B_{6\delta+\alpha_1}$	1	0	0	0	0	0	0
$q^{-5}B_{5\delta+\alpha_1}$	0	1	0	0	0	0	0
$q^{-4}B_{4\delta+\alpha_1}$	6	0	1	0	0	0	0
$q^{-3}B_{3\delta+\alpha_1}$	0	5	0	1	0	0	0
$q^{-2}B_{2\delta+\alpha_1}$	15	0	4	0	1	0	0
$q^{-1}B_{\delta+\alpha_1}$	0	10	0	3	0	1	0
B_{α_1}	20	0	6	0	2	0	1
qB_{α_0}	0	10	0	3	0	1	0
$q^2B_{\delta+\alpha_0}$	15	0	4	0	1	0	0
$q^3B_{2\delta+\alpha_0}$	0	5	0	1	0	0	0
$q^4B_{3\delta+\alpha_0}$	6	0	1	0	0	0	0
$q^5B_{4\delta+\alpha_0}$	0	1	0	0	0	0	0
$q^6B_{5\delta+\alpha_0}$	1	0	0	0	0	0	0

W_8 is equal to $-(q-q^{-1})^{-1}[2]_q^{-9}$ times

	$\tilde{G}_0$	$[2]_q\tilde{G}_1$	$[2]_q^2\tilde{G}_2$	$[2]_q^3\tilde{G}_3$	$[2]_q^4\tilde{G}_4$	$[2]_q^5\tilde{G}_5$	$[2]_q^6\tilde{G}_6$	$[2]_q^7\tilde{G}_7$
$q^{-7}B_{7\delta+\alpha_1}$	1	0	0	0	0	0	0	0
$q^{-6}B_{6\delta+\alpha_1}$	0	1	0	0	0	0	0	0
$q^{-5}B_{5\delta+\alpha_1}$	7	0	1	0	0	0	0	0
$q^{-4}B_{4\delta+\alpha_1}$	0	6	0	1	0	0	0	0
$q^{-3}B_{3\delta+\alpha_1}$	21	0	5	0	1	0	0	0
$q^{-2}B_{2\delta+\alpha_1}$	0	15	0	4	0	1	0	0
$q^{-1}B_{\delta+\alpha_1}$	35	0	10	0	3	0	1	0
B_{α_1}	0	20	0	6	0	2	0	1
qB_{α_0}	35	0	10	0	3	0	1	0
$q^2B_{\delta+\alpha_0}$	0	15	0	4	0	1	0	0
$q^3B_{2\delta+\alpha_0}$	21	0	5	0	1	0	0	0
$q^4B_{3\delta+\alpha_0}$	0	6	0	1	0	0	0	0
$q^5B_{4\delta+\alpha_0}$	7	0	1	0	0	0	0	0
$q^6B_{5\delta+\alpha_0}$	0	1	0	0	0	0	0	0
$q^7B_{6\delta+\alpha_0}$	1	0	0	0	0	0	0	0

References

[1] R.C. Alperin, $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}_2 \star \mathbb{Z}_3$, Am. Math. Mon. 100 (1993) 385–386.

[2] P. Baseilhac, An integrable structure related with tridiagonal algebras, Nucl. Phys. B 705 (2005) 605–619, arXiv:math-ph/0408025.

[3] P. Baseilhac, Deformed Dolan-Grady relations in quantum integrable models, Nucl. Phys. B 709 (2005) 491–521, arXiv:hep-th/0404149.

[4] P. Baseilhac, S. Belliard, Generalized q-Onsager algebras and boundary affine Toda field theories, Lett. Math. Phys. 93 (2010) 213–228, arXiv:0906.1215.

[5] P. Baseilhac, S. Belliard, The half-infinite XXZ chain in Onsager's approach, Nucl. Phys. B 873 (2013) 550–584, arXiv:1211.6304.

[6] P. Baseilhac, S. Belliard, Non-Abelian symmetries of the half-infinite XXZ spin chain, Nucl. Phys. B 916 (2017) 373–385, arXiv:1611.05390.

[7] P. Baseilhac, S. Belliard, An attractive basis for the q-Onsager algebra, Preprint, arXiv:1704.02950.

[8] P. Baseilhac, S. Belliard, N. Crampé, FRT presentation of the Onsager algebras, Lett. Math. Phys. 108 (2018) 2189–2212, arXiv:1709.08555.

[9] P. Baseilhac, N. Crampé, FRT presentation of classical Askey-Wilson algebras, Lett. Math. Phys. 109 (2019) 2187–2207, arXiv:1806.07232.

[10] P. Baseilhac, K. Koizumi, A new (in)finite dimensional algebra for quantum integrable models, Nucl. Phys. B 720 (2005) 325–347, arXiv:math-ph/0503036.

[11] P. Baseilhac, K. Koizumi, A deformed analogue of Onsager's symmetry in the XXZ open spin chain, J. Stat. Mech. Theory Exp. (10) (2005) P10005 (electronic), arXiv:hep-th/0507053.

[12] P. Baseilhac, K. Koizumi, Exact spectrum of the XXZ open spin chain from the q-Onsager algebra representation theory, J. Stat. Mech. Theory Exp. (9) (2007) P09006 (electronic), arXiv:hep-th/0703106.

[13] P. Baseilhac, T. Kojima, Correlation functions of the half-infinite XXZ spin chain with a triangular boundary, J. Stat. Mech. (2014) P09004, arXiv:1309.7785.

[14] P. Baseilhac, T. Kojima, Form factors of the half-infinite XXZ spin chain with a triangular boundary, Nucl. Phys. B 880 (2014) 378–413, arXiv:1404.0491.

[15] P. Baseilhac, S. Kolb, Braid group action and root vectors for the q-Onsager algebra, Transform. Groups 25 (2020) 363–389, arXiv:1706.08747.

[16] P. Baseilhac, K. Shigechi, A new current algebra and the reflection equation, Lett. Math. Phys. 92 (2010) 47–65, arXiv:0906.1482v2.

[17] P. Baseilhac, Z. Tsuboi, Asymptotic representations of augmented q-Onsager algebra and boundary K-operators related to Baxter Q-operators, Nucl. Phys. B 929 (2018) 397–437, arXiv:1707.04574.

[18] I. Damiani, A basis of type Poincare-Birkhoff-Witt for the quantum algebra of $\widehat{\mathfrak{sl}}_2$, J. Algebra 161 (1993) 291–310.

[19] B. Davies, Onsager's algebra and superintegrability, J. Phys. A, Math. Gen. 23 (1990) 2245–2261.

[20] B. Davies, Onsager's algebra and the Dolan-Grady condition in the non-self-dual case, J. Math. Phys. 32 (1991) 2945–2950.

[21] L. Dolan, M. Grady, Conserved charges from self-duality, Phys. Rev. D 25 (1982) 1587–1604.

[22] T. Ito, K. Tanabe, P. Terwilliger, Some algebra related to P- and Q-polynomial association schemes, in: Codes and Association Schemes, Piscataway NJ, 1999, Amer. Math. Soc., Providence RI, 2001, pp. 167–192, arXiv:math.CO/0406556.

[23] T. Ito, P. Terwilliger, Tridiagonal pairs of q-Racah type, J. Algebra 322 (2009) 68–93, arXiv:0807.0271.

[24] T. Ito, P. Terwilliger, The augmented tridiagonal algebra, Kyushu J. Math. 64 (2010) 81–144, arXiv:0904.2889.

[25] S. Kolb, Quantum symmetric Kac-Moody pairs, Adv. Math. 267 (2014) 395–469, arXiv:1207.6036.

[26] M. Lu, W. Wang, A Drinfeld type presentation of affine $\imath$quantum groups I: split ADE type, Preprint, arXiv:2009.04542.

[27] M. Lu, S. Ruan, W. Wang, $\imath$Hall algebra of the projective line and q-Onsager algebra, Preprint, arXiv:2010.00646.

[28] L. Onsager, Crystal statistics. I. A two-dimensional model with an order-disorder transition, Phys. Rev. (2) 65 (1944) 117–149.

[29] J.H.H. Perk, Star-Triangle Relations, Quantum Lax Pairs, and Higher Genus Curves, Proceedings of Symposia in Pure Mathematics, vol. 49, Amer. Math. Soc., Providence, RI, 1989, pp. 341–354.

[30] J.H.H. Perk, The early history of the integrable chiral Potts model and the odd-even problem, J. Phys. A 49 (15) (2016) 153001, arXiv:1511.08526.

[31] S.S. Roan, Onsager's algebra, loop algebra and chiral Potts model, Max Plank Institute for Mathematics, Bonn, 1991, Preprint MPI 91–70.

[32] The Sage Developers, Sage Mathematics Software (Version 9.2). The Sage Development Team, http://www.sagemath.org, 2020.
[33] E.K. Sklyanin, Boundary conditions for integrable quantum systems, J. Phys. A 21 (1988) 2375–2389.
[34] P. Terwilliger, The subconstituent algebra of an association scheme III, J. Algebraic Comb. 2 (1993) 177–210.
[35] P. Terwilliger, Two relations that generalize the q-Serre relations and the Dolan-Grady relations, in: Physics and Combinatorics 1999, Nagoya, World Scientific Publishing, River Edge, NJ, 2001, pp. 377–398, arXiv:math.QA/0307016.
[36] P. Terwilliger, The universal Askey-Wilson algebra, SIGMA 7 (2011) 069, arXiv:1104.2813.
[37] P. Terwilliger, The q-Onsager algebra and the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$, Linear Algebra Appl. 521 (2017) 19–56, arXiv:1506.08666.
[38] P. Terwilliger, The q-Onsager algebra and the universal Askey-Wilson algebra, SIGMA 14 (2018) 044, arXiv:1801.06083.