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MSC DISSERTATION

# A Review of Soft Black Hole Hair and Its Potential as a Solution to the Information Paradox

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*in*

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Department of Physics

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# Declaration of Authorship

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# *Abstract*

Faculty of Natural Sciences
Department of Physics

Master of Science

**A Review of Soft Black Hole Hair and Its Potential as a Solution to the Information Paradox**

by Matthew STAFFORD

The Information Paradox is the quintessential example of the fundamental disconnect between General Relativity and Quantum Field Theory. A recent proposal by Strominger, Hawking and Perry claims to have made the first steps towards a new resolution of the Paradox through the use of 'soft black hole hair', stored on the event horizon in the form of zero energy photons and gravitons. This project provides a comprehensive review of the proposal and assesses its viability as a solution to the information paradox. The two key results of the proposition, the existence of black hole hair and the degeneracy of the vacuum, are explicitly re-derived through the use of asymptotic symmetry analysis in massless QED and gravitational scattering.

Although the proposal does invalidate two key assumptions underlying the Information Paradox, it does not provide a current solution. The two sets of soft hair are not deemed physical. The proposal does however construct a solid foundation for future research in the area, including the search for new types of soft hair.

# *Acknowledgements*

I would like to thank all of the staff in the Theoretical Physics Department for such an engaging year. I would particularly like to thank Professor Jerome Gauntlett, as his excellent lecture course sparked my interest in black hole physics and Professor Toby Wiseman for his continued support as an advisor throughout the duration of the project,

# Contents

# 1 Introduction

Our current understanding of the universe contains four fundamental forces: Electromagnetism, Weak Nuclear, Strong Nuclear and Gravity. The first three are described by Quantum Field Theories (QFTs) while the latter is understood through General Relativity (GR). These are two of the most successful theories ever created in physics, passing every experimental test thrown at them so far. However, there is an underlying problem. At the fundamental level these two theories are incompatible with one another.

Quantum Field Theory, the result of unifying Special Relativity with Quantum Mechanics, has become the most effective theory describing physics on the subatomic scale, culminating in the Standard Model of Particle Physics. Although QFT is extremely successful there are still many things which are yet to be understood. Calculations are plagued by divergences, much of the underlying mathematics is not rigorous and a full, non-perturbative theory is yet to be discovered. Finally, experimental results tell us that neutrinos, predicted to be massless, actually carry a very tiny mass, suggesting the Standard Model in its current state is incomplete.

General Relativity, first proposed by Einstein in 1915 is our best current theory describing the universe at macroscopic scales. Unlike QFT, the gravitational field is spacetime itself and the 'force' of gravity is due to the warping of this spacetime. GR, like QFT, also comes with issues. The theory is classical, contrasting both QFT and the general consensus that the universe is fundamentally quantum. GR also predicts the existence of singularities, points where spacetime itself is ill defined, at these points all of known physics breaks down[1].

The incompatibilities between and issues within both GR and QFT point to the existence of a more fundamental underlying framework from which these two arise as approximations. The search for this unification is collectively known as the study of "Quantum Gravity" and has occupied theoretical physicists for the past 100 years. One of the main struggles with a search for Quantum Gravity is the lack of experimental guidance. Quantum Gravity is expected to become significant on the Planck energy scale ($\approx 10^{19}$ GeV), much higher than any particle accelerator could hope to achieve. Because of this, physicists have to rely on thought experiments to help guide the theory's development.

[1] This is due to all of current physics relying upon an underlying smooth differentiable manifold, there are some interesting avenues of research which treat spacetime as fundamentally discrete

## 1.1 Black Holes

One place where Quantum Gravity is thought to be significant is at the singularities predicted by GR, particularly within black hole formation and evaporation. As seen in Figure 1.1 the structure of a black hole is fundamentally different in the classical and quantum cases. This disconnect is nowhere more pronounced than in the case of the **Information Paradox**, making black holes the ideal "laboratory" for thought experiments revolving around Quantum Gravity.

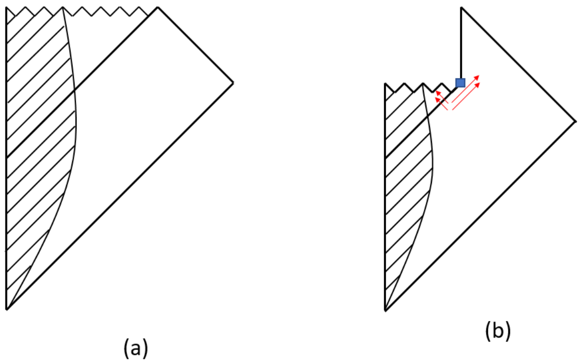


FIGURE 1.1: Penrose diagrams illustrating the fundamental causal difference between classical (a) and quantum (b) black hole formation. In the classical case once the black hole has formed it will remain in existence for all time. The quantum picture is significantly different, the black hole emits Hawking radiation leading to its eventual evaporation [32]. What happens once the black hole reaches the Planck scale is still open for debate and will require a theory of Quantum Gravity. Penrose diagrams follow conventions found in *Wald* [72].

### 1.1.1 Classical Black Holes

Black holes are the product of runaway gravitational collapse of a system. The simplest case of classical black hole formation can be described by the collapse of spherically symmetric, pressure free dust to form a Schwarzschild black hole. For background reading on classical black hole formation see Chapter **(xx)** of *Wald* [72].

Birkhoff's theorem states the only spherically symmetric solution to the Vacuum Eisenstein Equations is the Schwarzschild solution. In $(t, r, \theta, \phi)$ coordinates this is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (1.1)$$

in units where $G = c = 1$. $M$ is the only free parameter and can be interpreted as the total mass of the black hole, the only distinguishing characteristic of a Schwarzschild black hole is its mass. This metric has two singularities at $r = 0$ and $r = 2M$. The

$r = 2M$ singularity is a coordinate singularity and can be removed by an appropriate coordinate transformation. The singularity at $r = 0$ however is a physical singularity, this can be shown by looking at the curvature of the manifold as one approaches $r = 0$ given by the Kretschmann scalar [72]

$$R^{abcd}R_{abcd} \propto \frac{M^2}{r^6}$$

which diverges as $r \to 0$.

Originally thought to be a product of a very specific situation, it was later shown that physical singularities formed under generic conditions. These are the famous *Singularity Theorems* [34]. In this sense, General Relativity both predicts its own downfall and provides a hint on where to look for an improved theory of gravity.

One could ask what black holes are formed under more generic collapse situations. It was shown through a series of theorems, collectively known as **"No-Hair" Theorems** [15, 72], that the unique, stationary, asymptotically flat solution to the Einstein-Maxwell equations is a member of the Kerr-Newman family describing charged, rotating black holes. Therefore, regardless of the collapse process, all stationary physical black holes are characterised by only the total mass $M$, angular momentum $J$ and total electric charge $Q$. From the perspective of General Relativity, black holes are the some of the simplest macroscopic objects in the known universe. This story significantly changes when looking at quantum effects.

### 1.1.2 Semi-Classical Black Holes: Hawking Radiation

In a series of papers by Hawking, a semi-classical treatment of black holes was undertaken. Here a quantum field[2] was introduced upon the curved spacetime of a classical black hole which lead to the discovery of Hawking radiation [32].

The key idea behind Hawking radiation is the effect quantising a field in a non-stationary spacetime has upon the notion of a "particle". Consider a pair of observers, one in free fall and the other accelerating. Both can define a local notion of energy and the vacuum state however, if the inertial observer is in their local vacuum state the accelerating observer in general experiences a bath of thermal radiation. This is known as the Unruh effect [70][3].

In the case of black hole formation, early and late time observers are separated by a complicated time dependent spacetime which gives rise to a relative acceleration between them, see Figure 1.2. For a given vacuum state defined on A, B would experience thermal radiation being produced outside the horizon. Once the black hole has settled down the spacetime outside the horizon is approximately stationary (Fig 1.2 (iii)). Hawking showed that even at these late times in the black hole evolution, radiation was still observed due to infinite time dilation as one approached

[2]The usual treatments in textbooks and lecture notes deal with scalar fields, however the calculation has been generalised to spinors, vectors etc..

[3]See [21] for a review of the Unruh effect and its applications.

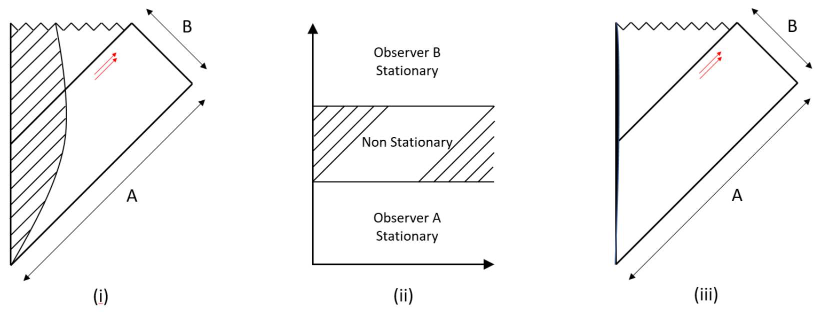


FIGURE 1.2: The production of Hawking radiation from a black hole. The collapsing matter in Fig (i) is described by a non-stationary spacetime separating early and late time observers A and B (Fig (ii)). Even after formation is long complete, B still sees radiation due to the time dilation effects of the horizon (Fig (iii)).

the horizon, particles would take an arbitrarily long time to escape to infinity [69]. This is known as Hawking radiation. The black hole is not black at all, but radiates a thermal particle spectrum with associated temperature

$$T_H = \frac{\hbar}{8\pi GM} \tag{1.2}$$

This is the Hawking temperature for a black hole. The semiclassical nature of the original Hawking calculation does not take into account the back reaction effects of the radiation onto the spacetime, the mass of the black hole remains constant.

### 1.1.3 Back Reaction and Evaporation

To fully understand the back reaction effects of radiation upon the metric, a full theory of Quantum Gravity is needed. However, up until the black hole becomes Planckian in size - where it is assumed Quantum Gravity effects become important - one can proceed with some approximate calculations.

It is widely thought that, although not in the leading order Hawking calculation, that radiation does in fact have correlations with the black hole state. For example, for every quanta of Hawking radiation carrying energy $E$ away from the black hole it is expected that that mass of the black hole decreases by the corresponding amount. Therefore the black hole will lose mass and evaporate. Since the temperature is inversely proportional to mass, the evaporation rate increases towards the end of the black hole's life until it reaches the Planck scale [30].

The analysis can be extended to Kerr-Newman black holes parametrised by the total mass M, angular momentum J and electric charge Q. These quantities are exactly conserved between the radiation and black hole as it preferentially radiates off particles with corresponding charge and angular momentum. The effect of these correlations is to carry some information away from the black hole during the evaporation process [63]. If one were able to collect all of the Hawking radiation corresponding to black hole evaporation they should in principle be able to reconstruct these three properties. However, due to the No-Hair theorem black holes do not possess any further distinguishing properties and no more information can be recovered detailing the formation process.

## 1.2 Entropy and the Information Puzzle

By considering the second law of thermodynamics in the presence of black holes, Bekenstein proposed that a black hole must have an entropy and suggested the generalised second law $\delta S_{BH} + \delta S_{out} \geq 0$ [8]. This led to the Bekenstein-Hawking entropy of a black hole [7]

$$S_{BH} = \frac{c^3 k_B A}{4G\hbar} \tag{1.3}$$

where $G$ is the gravitational constant, $k_B$ Boltzmann's constant and $A$ the area of the black hole horizon, measured in Planck units. This promoted the laws of black hole mechanics to fully thermodynamical laws. Equation (1.3) contains all of the fundamental physical constants, suggesting that a true understanding of this equation would lead to deep revelations about a unified theory of gravity with quantum theory, another hint that black holes are the ideal "laboratories" to test new ideas.

From a thermodynamical perspective, entropy can be thought of as the measure of disorder of a system. For a black hole of one solar mass the entropy is of the order $10^{77}$, around 20 orders of magnitude greater than that of a similar star. This suggests that we currently live in a very low entropy, highly ordered state. Adopting the interpretation of statistical physics, the entropy of a system in thermal equilibrium is given by

$$S = k_B \log(N) \tag{1.4}$$

where N is the number of microstates corresponding to a given marcrostate, this is known as the Boltzmann Entropy [47]. Equating this to the BH entropy we can then determine the number of microstates corresponding to a black hole macrostate parametrised by $M$, $J$ and $Q$. For the solar mass black hole $N$ is of the order **(xxx)**! The huge entropy can be attributed to the number of possible combinations of collapse that lead to the same black hole.

Finally, through ties with information theory, entropy can also be interpreted as the amount of information stored within a system. Bekenstein associated a limit to the amount of entropy/information available in a finite area, known as the Bekenstein Bound [6]. In the case of black holes, the Hawking Bekenstein entropy matches the upper limit of the bound suggesting black holes have the most efficient information storage capabilities of any objects in the known universe. This suggests that black holes are some of the most complex objects in the known universe, in sharp contrast to the classical picture, see Figure 1.3. Since the entropy is proportional to the horizon area a natural suggestion is that the information is transferred to the horizon of the Black Hole during formation, however due to the no hair theorem this is not possible [47].

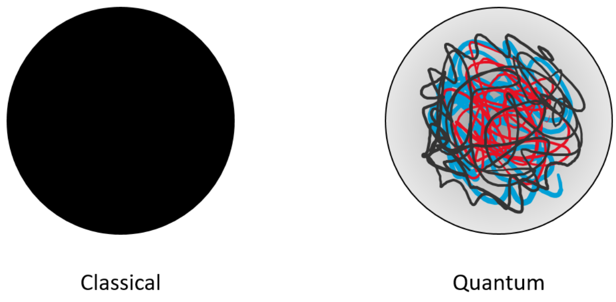


FIGURE 1.3: The Entropy Puzzle: The BH Entropy suggests that black holes carry a huge number of quantum microstates yet the classical picture prevents them from existing on the horizon. Where does all the information get stored?

If the information passes beyond the horizon, then the causal structure of the black hole demands that it end up at the singularity along with all collapsing matter. Once the black hole evaporates what happens to this information? This is one way of phrasing the Information Paradox proposed by Hawking [31], the central topic of this project.

## 1.3 The Information Paradox

In this section we follow the original argument presented by Hawking in [31] and reviewed in [46, 47, 45] which led to the conclusion that information is lost during the process of black hole formation and evaporation.

### 1.3.1 Information and Classical Black Holes

Consider the case of a Schwarzschild black hole. A brick and encyclopaedia of exactly the same mass are then thrown into the black hole. Once the item passes the horizon the total mass of the black increases by a corresponding amount. To an observer outside of the horizon, how can one tell if it was the brick or encyclopaedia thrown into the black hole?

Due to the event horizon[4] we cannot see the singularity and from the No-Hair theorem the only parameter needed to describe the black hole is the overall mass. Therefore there is no possible way for an external observer to know what went into the formation process. The next logical question to ask is: where is all the

[4]The Cosmic Censorship conjecture by Penrose suggests no physical naked singularities exist, they are always shrouded behind a horizon [56].

information distinguishing the brick from the encyclopaedia? Clearly we started out with a lot more characterising information than just the total mass.

In the case of a classical black hole the answer is simple, it is hidden behind the event horizon for all time. The information is there, just cannot be retrieved by an external observer. For classical black holes there is no information paradox.

### 1.3.2 Information and Quantum Black Holes

Now consider quantum effects. Assuming that classical formation and quantum emission can be temporally separated, quantum effects do not become significant until sometime after the formation process has completed. In the case of the brick and encyclopaedia, the black hole settles down to Schwarzschild long before Hawking emission becomes relevant.

Hawking radiation can be interpreted as production of an entangled pair of particles, one with negative and the other with positive energy with respect to an observer at infinity [31]. The negative energy particle always falls beyond the horizon, the positive particle is released out to infinity with a probability matching thermal emission at temperature $T_H$. A pair of Hawking quanta can be described by a pure entangled quantum state of the form [5] [47]

$$|\psi\rangle_1 = \frac{1}{\sqrt{2}}\left( |0\rangle_{b_1} \otimes |0\rangle_{c_1} + |1\rangle_{b_1} \otimes |1\rangle_{c_1} \right) \tag{1.5}$$

where the $b_1$ quanta is the Hawking radiation which escapes to infinity and the $c_1$ quanta fall into the black hole. This can be expressed as a density matrix $\rho = |\psi\rangle_1 \langle\psi|_1$. An external observer cannot see the part of the state corresponding to the infalling particle, therefore the internal states have to be traced out to produce a mixed density matrix $\rho$ of the form

$$\sum_n p_n |\psi_n\rangle \langle\psi_n| \tag{1.6}$$

The entanglement entropy, given by $S_{ent} = -\mathrm{Tr}\rho \ln \rho$ is a measure of the degree of entanglement between two subsystems of a quantum state [47]. Calculating the entanglement entropy between the infalling and outgoing quanta defined (2.1) gives a value of $S_{ent} = \ln 2$. As more quanta are emitted the entanglement entropy continues to increase. However, as long as the black hole exists, the underlying state is still pure, we have just coarse grained the state according to the information that is unknown about the system.

Eventually, the black hole evaporates. The semi-classical approximations remain valid until the black hole reaches a Planckian size. At this point, assuming no information has been released beforehand[6] there remain two main possibilities:

[5]Note that this is just a simplified example to illustrate the effect of Hawking radiation upon entanglement entropy. More accurate calculations can be found within the references.

[6]This strictly isn't true, the black hole does carry information about M, J and Q. These charges are conserved as part of the evaporation process and hence a tiny amount of information does get

1. **Remnants**
   Once the black hole becomes Planckian in size, some Quantum Gravity effect turns off Hawking radiation. All of the information which fell into the black hole remains in the remnant. However, this would require a Planck sized object capable of storing an arbitrary amount of information detailing the collapsing matter, clearly violating the Bekenstein bound. A review on black hole remnants can be found in [16].

2. **Information Loss**
   The black hole evaporates and reverts to the unique vacuum. The internal part of the entangled radiation state no longer exists and has to be forcibly removed by tracing over it. The Hawking radiation is still entangled but it has nothing to be entangled with, the resultant state is mixed and described by a density matrix. Pure to Mixed evolution is not unitary and violates one of the postulates of quantum theory. Furthermore, all information about the collapsing state is lost and one has no way of knowing what went into the formation process. Physics loses its predictive power.

If one assumes remnants do not exist, as Hawking did, then information is lost and quantum theory must be wrong. This is the Information Paradox. It is widely accepted that there must be some flaw in the original argument and that information is returned from the black hole during the evaporation process. Since the original paper was published there have been numerous attempts to find fault with its reasoning, most notably in looking at corrections to the leading order radiation calculation that could restore unitary evolution. These attempts have so far proved to be unsuccessful and the information paradox remains unsolved to this day, for further reading on the Information Paradox see [46, 47, 45].

### 1.3.3 Information and Page Time

There is another associated problem during the black hole evaporation process, first argued by Page [52] and reviewed within [45]. The Bekenstein-Hawking entropy (1.3) saturates the Bekenstein bound and therefore black holes are the at the upper limit of information storage. The entanglement entropy of a subsystem tells you how much information is unknown about the overall state. Therefore the BH entropy must be strictly greater than the entanglement entropy to satisfy the Bekenstein bound.

If we initially start from a pure state then $S_{ent} = 0$. As Hawking radiation is produced, the overall state becomes more entangled and increases the entanglement entropy. This suggests more information is hidden behind the black hole as it evolves. At the same time the evaporation is causing the area and hence Bekenstein-Hawking entropy of the black hole to decrease. At some point the entanglement

released. Due to the No-Hair theorem the black hole and radiation products cannot carry any further distinguishing features.

entropy is going to become greater than the Bekenstein-Hawking entropy and a contradiction is reached. This happens when the Bekenstein-Hawking entropy has been halved and is known as the Page time, long before quantum effects turn on.

## 1.4 A Potential Solution?

In 2015, Hawking posted a note proposing a potential resolution to the information paradox [33]. In it he suggested that complete information could be stored on the horizon of a black hole in the form of soft[7] particle excitations, particularly soft photons and gravitons. Since then a number of papers released by Strominger, Hawking, Perry et. al. have provided more structure to the proposal [36, 35]. The proposal, from now on referred to as the SHP proposal, has received widespread attention and some controversy within the academic community, particularly as to whether it lives up to the original claim.

### 1.4.1 The Infrared Triangle

Recently, it has been found that what were once considered three separate subjects: asymptotic symmetries, soft theorems and the memory effect are all in fact different faces of the same underlying physical system [63]. Here we provide a brief introduction to each:

1. **Asymptotic Symmetries**
   This is the study of non trivial symmetry groups at the boundaries of spacetime. These symmetry groups preserve a set of pre-defined boundary conditions on any dynamical fields in theory, for more detail see Section 2.3. The most famous case of asymptotic symmetry analysis is probably that of Bondi, van de Burg, Metzner and Sachs who, in studying the asymptotic symmetry group preserving asymptotic flatness conditions discovered an infinite dimensional group now known as the BMS group [9]. This group will play an important role in Sections 4 and 5.

2. **Soft Theorems**
   For any theory containing massless force carrying particles, typically photons and, in any proposed quantum gravity theory, gravitons there are infrared divergences. This divergences occur because there is no lower bound on the energy needed to excite the massless particle, for any scattering process an infinite number of soft quanta can be produced. A classic example is soft bremsstrahlung from an accelerating electron, see *Peskin and Schroeder* [58] for further details. First discovered by Weinberg and Low [75] soft theorems are relations between quantum scattering amplitudes of processes with and without these soft quanta insertions; they are key to taming infrared divergences. For a more detailed discussion on soft theorems see *Weinberg* [76].

[7]Soft means zero frequency / energy excitations, they will be formally discussed in later sections.

3. **Memory Effects**
   Memory effects are most well known in terms of gravity and were originally discovered by Zeldovich and Polnarev in 1974 [77]. When gravitational waves pass through a pair of inertial detectors a permanent relative shift in position is produced between them, this shift is in principle experimentally measurable and therefore provide potential results to test the ideas of asymptotic symmetries and soft theorems [63]. Although not discussed within this project, links to works related to memory effects will be presented where relevant.

This relation was coined "The Infrared Triangle" and can be seen to repeat throughout all of physics. For example, in Section 2 it is shown that the asymptotic symmetry group of QED is equivalent to the Weinberg soft photon theorem and, although not discussed here, an associated electromagnetic memory effect. The real power in this association is predicting the existence of new areas of physics. For example, if one discovered a new asymptotic symmetry group, then we now know that there must be an associated soft theorem and memory effect, potentially revealing new interpretations and insights. An example of this is the equivalence between a soft pion theorem and a new associated memory effect [29].

### 1.4.2 The SHP Proposal

The key ingredient of the Strominger, Hawking and Perry proposal is the discovery of overall, reduced asymptotic symmetry groups which act on both the future and past boundaries of spacetime simultaneously. Via Noether's theorem, any symmetry group comes equipped with associated conserved charges which act to constrain any scattering process. If one considers black hole formation and evaporation as a scattering process then the asymptotic constraints must be respected throughout, see Figure 1.4.

The first claim is these conservation laws constrain the resultant Hawking radiation and therefore, by studying the radiation, one can distinguish black holes from one and other using more than just the mass, angular momentum and charge predicted by the No-Hair theorem. This is the claim that black holes carry hair. Moreover, if the Hawking radiation is further constrained then it in fact carries more information away from the black hole than previously predicted, potentially with significant impact upon the information paradox argument.

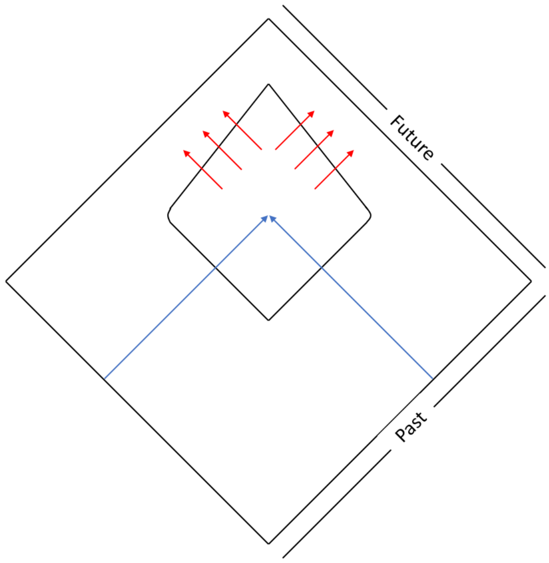


FIGURE 1.4: Causal Structure of Black Hole Formation and Evaporation. Starting from the vacuum a null shockwave goes in to form a black hole, eventually the black hole completely evaporates and reverts back to the vacuum. Viewed as a scattering process this can be completely characterised by data evolving from the Past to the Future, the black hole evolution must respect any constraints placed upon these boundaries [63].

The second claim is the black hole hair, and hence information, is stored on the event horizon. Any matter falling through the horizon to the singularity has the action of inducing the asymptotic symmetry transformation upon the horizon itself. Thanks to the Infrared Triangle, this action can be interpreted as zero energy soft quanta being stored on the horizon in a form of holography. This provides a mechanism for which hair can be transferred to and stored upon the black hole with zero energy cost. From this proposal three main questions arise:

1. Is the black hole hair physical?
2. Does the horizon information storage capacity match that predicted by the Bekenstein Hawking entropy formula?
3. Is there a process in which information is carried away from the black hole, restoring unitarity?

Any proposed resolution to the Information Paradox must be able to answer these questions in a satisfactory manner. In judging the successes of the SHP proposal we will return to and attempt to answer these questions in Section 6.

## 1.5 Project Aims and Structure

The aim of this project is twofold:

1. To provide a comprehensive review of the SHP proposal, detailing calculations and expanding upon the underlying physics to bring any newcomers to the field completely up to date.

2. To assess the proposals validity towards resolving the information paradox.

The structure of the rest of the project is as follows:

In section two we re-derive the infinite dimensional asymptotic symmetry group of massless QED. It is shown that the symmetry group corresponds to "large gauge transformations" acting non-trivially at the null boundaries of spacetime. Associated conserved charges are constructed and shown, through careful treatment of the symplectic form, to correctly generate the infinitesimal symmetry action and constrain any QED scattering process. The charges can be split into "soft" and "hard" terms. Upon quantisation, the soft charges have the action of exciting zero energy "soft" photons on the boundaries. The vacuum state is shown to be infinitely degenerate, in contrary to the assumption in Hawking's original argument.

In section three the symmetries are applied to black hole spacetimes, particularly the collapse of null, uncharged matter to form a Schwarzschild black hole. The black hole horizon can generically carry large gauge charge, although in the classical it does not. Under a quantum treatment it is found that large gauge transformed black holes are distinguishable from one another, constituting to an infinite head of soft electric hair; a second violation to the assumptions in Hawking's argument.

Sections four and five repeat the analysis for the gravitational case. The asymptotic symmetry group of asymptotically flat spacetimes, the BMS group, is introduced and reviewed. The techniques developed in section three are then applied to the supertranslation subgroup and are seen to correspond to soft graviton excitations at null infinity. The recently discovered superrotation symmetry is also discussed. The action on of supertranslations on the vacuum is shown to produce a degenerate vacuum state with non-zero angular momentum. The symmetries are then applied to black hole spacetimes and it is shown that black holes also carry an infinite head of classical supertranslation hair in addition to the quantum soft electric hair.

In section six the implications of these symmetries are discussed in the case of the information paradox. We return to the questions proposed above and see if the proposal can sufficiently answer them. Current research, outstanding issues and proposed future avenues of research are all discussed. We ultimately find that the proposal does not provide a resolution to the information paradox but opens up the door and provides a framework for future research, such as potential superrotation hair.

Finally, many of the calculations in the original works and this project relies upon the symplectic structure of Hamiltonian Field Theories and General Relativity. A brief introduction and review of this structure is provided in Appendix A with links to further resources for those who are unfamiliar with the topic.

# 2 Large Gauge Symmetry

We begin by reviewing the recent discovery of the asymptotic symmetry group of Massless QED. Although the calculations are much less technical than the gravitational case in Section 4, many of the key subtle conceptual points carry directly over. Therefore in this section we will perform all calculations explicitly to build up the formalism and understanding required to tackle the gravitational case. The prescription for the analysis is as follows:

1. Derive the asymptotic symmetry group of Massless QED acting on the future null boundary $\mathcal{I}^+$.

2. Construct the conserved charges associated to this symmetry group.

3. Develop the symplectic form of QED on the null boundaries of spacetime[1]. Use this form to verify that the conserved charges correctly generate the infinitesimal asymptotic symmetry action.

4. Complete exactly the same analysis on the past null boundary $\mathcal{I}^-$ to find a second, independent asymptotic symmetry group.

5. Find an overall asymptotic symmetry group acting on the past and future null boundaries simultaneously through specifying an appropriate matching conditions between the past and future data.

The vast majority of the analysis can be completed classically, Sections 2.1 to 2.6 perform this analysis, re-deriving and elaborating upon results published in [38, 39, 63]. The results are then discussed in Section 2.7. Finally, to understand some key conceptual points we must include quantum effects, these are discussed in Section 2.8.

## 2.1 Preliminaries

Here we provide a brief review of electromagnetism, specify any required notation for future calculations and complete some basic metric computations for use later.

[1]For those unfamiliar with symplectic forms, recall the necessary background can be found in Appendix A

### 2.1.1 Recap of Electromagnetism

The action for Electrodynamics is [60]

$$= -\frac{1}{4e^2}\int \sqrt{-g}F_{\mu\nu}F^{\mu\nu} + S_M \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the antisymmetric electromagnetic field strength tensor, $S_M$ is a generic matter action and the constant $e^2$ is introduced to align with notation in the literature and express the electric charge $Q_E$ is an integer number. Performing a variation of the action yields the Euler Lagrange equations of motion

$$\begin{aligned} \nabla^\mu F_{\mu\nu} = e^2 j_\nu \qquad j^\nu = -\frac{\delta S_M}{\delta A_\nu} \\ d\star F = e^2\star j \qquad F = dA \end{aligned} \tag{2.2}$$

where in the first line the equations are expressed in coordinate notation and the second in form notation. $dB$ denotes the exterior derivative of differential form $B$ and $\star B$ denotes the Hodge dual. This theory is invariant under the local $U(1)$ gauge transformations where the gauge and matter fields transform as

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\lambda(x) \qquad \phi(x) \to e^{iq\lambda(x)}\phi(x) \tag{2.3}$$

and hence has a local U(1) symmetry [60]. Following Noether's procedure one can construct an associated conserved electric charge inside a two-sphere at infinity [63]

$$Q_E = \frac{1}{e^2}\int_{S^2_\infty}\star F \tag{2.4}$$

### 2.1.2 Metric Conventions

In coordinates $(t, r, \theta, \phi)$ the line element for Minkowski space is expressed as

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + r^2\sin^2\theta d\phi^2) \tag{2.5}$$

We want to study the behaviour of incoming and outgoing massless fields, propagating from $(t = -\infty, r = \infty)$ to $(t = \infty, r = \infty)$. To do this one can 'bring in' infinity to a finite boundary by an appropriate coordinate transformation and conformal rescaling, originally completed by Penrose [72]. The compactification process leaves the causal structure of spacetime intact. This structure can be represented by a Penrose diagram, for Minkowski space this is displayed in Figure 2.1.

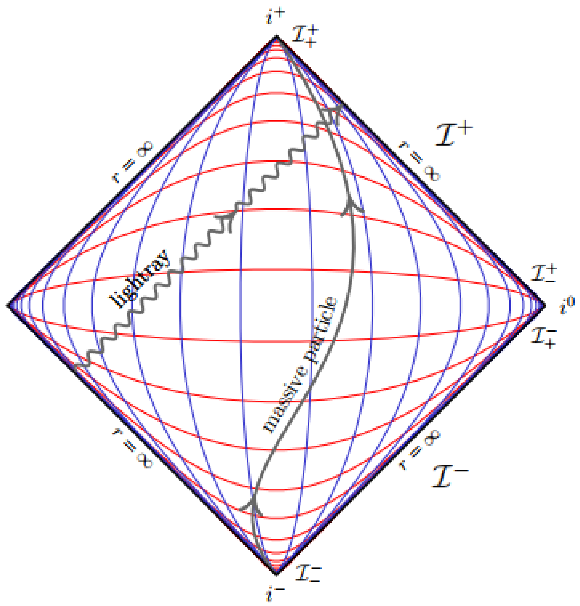


FIGURE 2.1: Penrose Diagram of Minkowski Spacetime in Four Dimensions. For further reading on Penrose diagrams and conformal compactification see *Wald* [72]. Image sourced from [63].

- $i^{\pm}$ is future/past timelike infinity. All trajectories of massive particles start at $i^-$ and end at $i^+$, denoted by smooth worldline.
- $i^0$ is spacelike infinity.
- $\mathcal{I}^{\pm}$ is future/past null infinity. Topologically they are $\mathbb{R} \times \mathcal{S}^2$. The boundaries $\mathcal{I}^+_+, \mathcal{I}^+_-, \mathcal{I}^-_+, \mathcal{I}^-_-$ are the future of the future, past of the future, future of the past and past of the past respectively. All massless radiation is characterised by 45 degree trajectories starting at $\mathcal{I}^-$ and ending at $\mathcal{I}^+$, shown here by the wavy worldline.
- Slices of constant $r$ are given by the blue lines, slices of constant $t$ by the red lines.

As we will primarily be studying the behaviour of radiation at the null boundaries of spacetime we will adopt the following coordinate systems. In a neighbourhood of $\mathcal{I}^+$ we use retarded coordinates $(u, r, z, \bar{z})$

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \tag{2.6}$$

where $u = t - r$ and $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$ is the metric on the conformal sphere in stereographic coordinates where $z = e^{i\phi}\tan\frac{\theta}{2}$. $z$ runs over the complex plane; $z = 0$ is the north pole, $z\bar{z} = 1$ is the equator and $z = \infty$ is the south pole [63]. The inverse metric components are

$$g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{\gamma^{z\bar{z}}}{r^2} \\ 0 & 0 & \frac{\gamma^{z\bar{z}}}{r^2} & 0 \end{pmatrix} \tag{2.7}$$

and the non-zero Christoffel symbols are

$$\Gamma^z_{rz} = \frac{1}{r} \qquad \Gamma^u_{z\bar{z}} = r\gamma_{z\bar{z}} \qquad \Gamma^r_{z\bar{z}} = -r\gamma_{z\bar{z}} \qquad \Gamma^z_{zz} = \partial_z ln(\gamma_{z\bar{z}}) \qquad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \partial_{\bar{z}} ln(\gamma_{z\bar{z}}) \tag{2.8}$$

where the last two are also the Christoffel symbols for the unit 2-sphere. Null trajectories are given by curves of constant $(u, z, \bar{z})$. In this coordinate system $\mathcal{I}^+$ is naturally parametrised by $(u, z, \bar{z})$ with $r \to \infty$ and the boundary spheres $\mathcal{I}^+_\pm$ by taking $u \to \pm\infty$ and keeping $(z, \bar{z})$ constant. In a neighbourhood of $\mathcal{I}^-$ we use advanced coordinates $(v, r, z, \bar{z})$

$$ds^2 = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \tag{2.9}$$

where $v = t + r$ and $z_{adv} = -\frac{1}{\bar{z}_{ret}}$, that is the $(z, \bar{z})$ advanced coordinates are **antipodal** to those in retarded coordinates. A free plane wave travelling from $\mathcal{I}^-$ with $(z, \bar{z})$ in advanced coordinates will propagate to $\mathcal{I}^+$ with the same value $(z, \bar{z})$ in retarded coordinates [38], see Figure 2.2.
The inverse metric components are

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{\gamma^{z\bar{z}}}{r^2} \\ 0 & 0 & \frac{\gamma^{z\bar{z}}}{r^2} & 0 \end{pmatrix}$$

and the non-zero Christoffel symbols are

$$\Gamma^z_{rz} = \frac{1}{r} \qquad \Gamma^v_{z\bar{z}} = -r\gamma_{z\bar{z}} \qquad \Gamma^r_{z\bar{z}} = -r\gamma_{z\bar{z}} \qquad \Gamma^z_{zz} = \partial_z ln(\gamma_{z\bar{z}}) \qquad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \partial_{\bar{z}} ln(\gamma_{z\bar{z}})$$

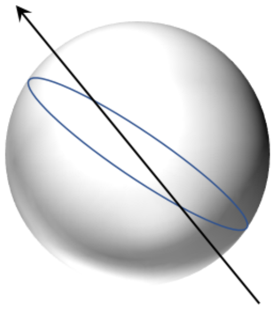

FIGURE 2.2: A Diagram to Illustrate the Antipodal Relation of Advanced and Retarded Coordinates. Any null ray travelling though Minkowski space starts at $(z, \bar{z})$ in Advanced coordinates (2.9), passes through the origin $r = 0$ and emerges at antipodal point given by the same value of $(z, \bar{z})$ in Retarded coordinates (2.6).

## 2.2 Antipodal Matching Conditions: A Simple Example

To aid conceptually with the following analysis we first discuss the behaviour of the Liénard-Wiechert solution at null infinity for $n$ particles with charge $Q_k$ moving at constant 4-velocity $U_k^\mu = \gamma_k(1, \overrightarrow{\beta}_k)$ with $U_k^2 = -1$ and $\gamma_k^2 = \frac{1}{1-\beta_k^2}$ [63]. The radial electric field is

$$F_{rt}(\overrightarrow{x}, t) = \frac{e^2}{4\pi} \sum_{k=1}^{n} \frac{Q_k \gamma_k (r - t\hat{x} \cdot \overrightarrow{\beta}_k)}{|\gamma_k^2 (t - r\hat{x} \cdot \overrightarrow{\beta}_k)^2 - t^2 + r^2|^{3/2}} \tag{2.10}$$

where $r^2 = \overrightarrow{x} \cdot \overrightarrow{x}$ and $\overrightarrow{x} = r\hat{x}$. $\hat{x} = \hat{x}(z, \bar{z})$ determines a point on the unit 2-sphere.

We are now going to take this solution to spatial infinity $i^0$ via two different paths and compare the results, see Figure 2.3. First we will go to $\mathcal{I}^+$ along a constant $(u, \hat{x})$ path before approaching $\mathcal{I}^+_-$ by taking the limit $u \to -\infty$, denoted by the red path. Secondly we will go to $\mathcal{I}^-$ along a constant $(v, \hat{x})$ path before approaching $\mathcal{I}^-_+$ by taking the limit $v \to \infty$, denoted by the blue path.

Expressing the solution in terms of retarded time by making the substitution $t = u + r$ one finds

$$\begin{aligned} F_{ru} &= \frac{e^2}{4\pi} \sum_{k=1}^{n} \frac{Q_k \gamma_k (r - (u+r)\hat{x} \cdot \overrightarrow{\beta}_k)}{|\gamma_k^2 ((u+r) - r\hat{x} \cdot \overrightarrow{\beta}_k)^2 - (u+r)^2 + r^2|^{3/2}} \\ &= \frac{e^2}{4\pi} \sum_{k=1}^{n} \frac{Q_k \gamma_k (r(1 - \hat{x} \cdot \overrightarrow{\beta}_k) + \mathcal{O}(1))}{|\gamma_k^2 ((u+r)^2 - 2r(u+r)\hat{x} \cdot \overrightarrow{\beta}_k + r^2(\hat{x} \cdot \overrightarrow{\beta}_k)^2) - u^2 + 2ur|^{3/2}} \\ &= \frac{e^2}{4\pi} \sum_{k=1}^{n} \frac{Q_k \gamma_k (r(1 - \hat{x} \cdot \overrightarrow{\beta}_k) + \mathcal{O}(1))}{|\gamma_k^2 r^2 (1 - \hat{x} \cdot \overrightarrow{\beta}_k)^2 + \mathcal{O}(r)|^{3/2}} \end{aligned} \tag{2.11}$$

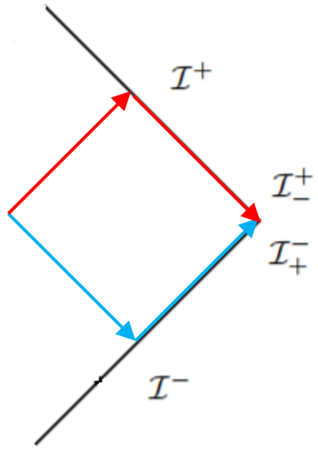


FIGURE 2.3: The two paths to $i^0$ considered in the Lienard-Wiechert solution.

where we have discarded all but the leading order terms in $r$. Now, taking the limits and following a similar calculation for the second path we find to leading order

$$\begin{aligned}\lim_{r\to\infty} F_{ru} = F_{ru}\Big|_{\mathcal{I}^+_-} &= \frac{e^2}{4\pi r^2}\sum_{k=1}^{n}\frac{Q_k}{\gamma_k^2(1-\hat{x}\cdot\overrightarrow{\beta}_k)^2} \\ \lim_{r\to\infty} F_{rv} = F_{rv}\Big|_{\mathcal{I}^-_+} &= \frac{e^2}{4\pi r^2}\sum_{k=1}^{n}\frac{Q_k}{\gamma_k^2(1+\hat{x}\cdot\overrightarrow{\beta}_k)^2}\end{aligned} \tag{2.12}$$

where the first equality arises because the expressions are independent of $u$ and $v$. Note that the solution is discontinuous as $i^0$ is approached via different routes. However the limits are equal under the transformation $\hat{x} \to -\hat{x}$, that is the solutions are **antipodally equal**.

Writing this using the convention for advanced and retarded coordinates detailed in section 1

$$F_{ru}(z,\bar{z})\big|_{\mathcal{I}^+_-} = F_{rv}(z,\bar{z})\big|_{\mathcal{I}^-_+} \tag{2.13}$$

Using the general definition for a conserved charge (2.4) we can then define

$$Q^+_\epsilon = \int_{\mathcal{I}^+_-} \epsilon \star F = Q^-_\epsilon = \int_{\mathcal{I}^-_+} \epsilon \star F \tag{2.14}$$

for any function $\epsilon(z,\bar{z})$ satisfying the condition $\epsilon(z,\bar{z})|_{\mathcal{I}^+_-} = \epsilon(z,\bar{z})|_{\mathcal{I}^-_+}$. Therefore there are now an infinite number of conserved charges living at the boundary of spacetime, one for every function $\epsilon(z,\bar{z})$.We could continue on down this path, discovering the symmetry transformation that these charges generate, indeed this method is discussed heavily in [63]. However, to more closely align with the analysis of gravitational systems later in the text we will now diverge and discuss an alternative method of identifying the conserved charges using asymptotic symmetries.

## 2.3 Asymptotic Falloff Conditions and Large Gauge Symmetries

*From this point onwards all work will be carried out in a region near $\mathcal{I}^+$ in retarded coordinates only. Analogous results for $\mathcal{I}^-$ can be found in Section 2.6.* As mentioned in the previous section, one could start with a symmetry group, derive the conserved charges and verify these charges generate the appropriate field transformations in a canonical formulation. To achieve this one needs a way to determine the initial symmetry transformation. Asymptotic symmetry analysis provides a rough prescription to achieve this with relatively little work via the following method [63]:

1. Define boundary falloff conditions on any dynamical fields in the theory. This is more of an intuitive process than an exact one. In the case of electromagnetism we want to produce physically reasonable finite energy situations. This leads to the conditions

2. Derive the allowed gauge symmetries. This is any transformation which preserves the previously defined boundary conditions

3. Remove any trivial transformations. These are the transformations that do not act on the physical phase space of a given theory.

In the case of massless electrodynamics one needs to define appropriate boundary conditions on the gauge field $A_\mu$. The first conditions come from a gauge choice, in this case the retarded radial gauge

$$A_r = 0 \qquad A_u|_{\mathcal{I}^+} = 0 \tag{2.15}$$

The field strength component $F_{ur}$ defines the long range electric field, for finite energy configurations $F_{ur} \sim \mathcal{O}(r^{-2})$. Hence in this gauge $A_u \sim \mathcal{O}(r^{-1})$. The stress-energy tensor for electrodynamics is [72]

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_{bd}g^{dc} - \frac{1}{4}g_{ab}F_{de}F^{de}\right) \tag{2.16}$$

The energy density is given by the $T_{00}$ component and in $(u, r, z, \bar{z})$ coordinates is

$$T_{uu} = \frac{1}{4\pi}\left(F_{uc}F_{ud}g^{dc} + ...\right) \sim F_{uz}F_{u\bar{z}}\frac{\gamma^{z\bar{z}}}{r^2} + ... \tag{2.17}$$

where "..." indicates further subleading terms in $r$. For the energy flux to be finite $T_{uu} \sim \mathcal{O}(\frac{1}{r^2})$ [38]. $F_{uz}$ must therefore be $\mathcal{O}(1)$ in $r$ and this in turn forces $A_z$ and $A_{\bar{z}}$ to be $\mathcal{O}(1)$. In summary, the minimum falloff conditions for the gauge field are

$$\begin{aligned} A_u \sim & \quad \mathcal{O}(r^{-1}) \\ A_r = & \quad 0 \\ A_z \sim & \quad \mathcal{O}(1) \end{aligned} \tag{2.18}$$

All of the field components are either constant or die off when approaching $\mathcal{I}^+$ by taking $r \to \infty$ and keeping $(u, z, \bar{z})$ constant. Therefore in a region close to $\mathcal{I}^+$ the gauge field components can be expanded in powers of $\frac{1}{r}$, this is known as an *Asymptotic Expansion*.

$$A_u(u,r,z,\bar{z}) = \sum_{n=1}^{\infty} \frac{A_u^{(n)}(u,z,\bar{z})}{r^n} \qquad A_z(u,r,z,\bar{z}) = \sum_{n=0}^{\infty} \frac{A_z^{(n)}(u,z,\bar{z})}{r^n} \tag{2.19}$$

In this notation $A_\mu^{(n)}$ denotes the $r^{-n}$'th order term in the asymptotic expansion. Using the gauge field falloff conditions (2.18), the leading order field strength components become

$$\begin{aligned} F_{ur} \sim & \quad \mathcal{O}(r^{-2}) = -\partial_r A_u = \frac{A_u^{(1)}}{r^2} \\ F_{uz} \sim & \quad \mathcal{O}(1) = \partial_u A_z = \partial_u A_z^{(0)} \\ F_{rz} \sim & \quad \mathcal{O}(r^{-2}) = \partial_r A_z = -\frac{A_z^{(1)}}{r^2} \\ F_{z\bar{z}} \sim & \quad \mathcal{O}(1) = \partial_z A_{\bar{z}}^{(0)} - \partial_{\bar{z}} A_z^{(0)} \end{aligned} \tag{2.20}$$

Now that we have the required falloff conditions (2.18) the allowed gauge transformations (2.3) can be determined. Any gauge transformation must respect these conditions, in this case the allowed transformations are

$$\partial_r \epsilon = 0 \qquad \partial_u \epsilon \sim \mathcal{O}(r^{-1}) \qquad \partial_z \epsilon \neq 0 \tag{2.21}$$

Therefore at $\mathcal{I}^+$ the residual allowed gauge transformations are of the form $\epsilon = \epsilon(z, \bar{z})$. These are known as *large gauge transformations* as, unlike standard gauge transformations, they do not tend to the identity at infinity. In later sections we will see that these transformations are also non-trivial. To summarise, the asymptotic symmetry transformation for U(1) gauge theory is

$$\delta_\epsilon A_u = \delta_\epsilon A_r = 0 \qquad \delta_\epsilon A_z = \partial_z \epsilon(z, \bar{z}) \tag{2.22}$$

## 2.4 Conserved Charges

As with any symmetry, we would like to calculate the associated conserved charges. By applying Noether's procedure we find an infinite number of conserved charges of the form [38]

$$Q_\epsilon^+ = \frac{1}{e^2} \int_{\mathcal{I}_-^+} \epsilon \star F \tag{2.23}$$

These charges are all defined at the "past of the future" $\mathcal{I}^+_-$, a boundary of $\mathcal{I}^+$ with the topology of a two sphere. There are an infinite number of them as the function $\epsilon(z,\bar{z})$ is arbitrary. We can apply Stokes's theorem to convert the form of (2.23) to a bulk integral over a Cauchy surface which has $\mathcal{I}^+_-$ as one of its boundaries [72]. A natural choice is $\mathcal{I}^+$, applying Stokes's theorem one finds

$$\begin{aligned} \cancel{\frac{1}{e^2}\int_{\mathcal{I}^+_+}\epsilon\star F} + \frac{1}{e^2}\int_{\mathcal{I}^+_-}\epsilon\star F &= \frac{1}{e^2}\int_{\mathcal{I}^+} d(\epsilon\star F) \\ &= \frac{1}{e^2}\int_{\mathcal{I}^+} d\epsilon\wedge\star F + \int_{\mathcal{I}^+}\epsilon\star j \end{aligned} \tag{2.24}$$

where the boundary term at $\mathcal{I}^+_+$ cancels under the assumption that we are dealing with massless fields only. Under this assumption no charged matter can ever reach $i^+$ and the charge dies off as $i^+$ is approached along $\mathcal{I}^+$. In the second line we made use of the Leibniz property of the exterior derivative and substituted in the equation of motion (2.2).

We can also express the charges (2.23) in terms of retarded coordinates. $\mathcal{I}^+_-$ is the $u\to-\infty$ limit of of $\mathcal{I}^+$ and therefore has coordinates $(-\infty,\infty,z,\bar{z})$. When evaluating the integral only the $z,\bar{z}$ component of $\star F$ will contribute, (2.23) becomes

$$Q^+_\epsilon = \frac{1}{e^2}\int_{\mathcal{I}^+_-} d^2z\ \epsilon(\star F)_{z\bar{z}} \tag{2.25}$$

Calculating the $z,\bar{z}$ component of $\star F$ we find

$$\begin{aligned} \star F_{z\bar{z}} &= \frac{1}{2}\varepsilon_{z\bar{z}cd}F^{cd} \\ &= \varepsilon_{z\bar{z}ur}g^{u\alpha}g^{r\beta}F_{\alpha\beta} \\ &= \sqrt{g}F_{ru} \\ &\simeq \gamma_{z\bar{z}}F^{(2)}_{ru}\Big|_{\mathcal{I}^+_-} \end{aligned} \tag{2.26}$$

where $\varepsilon_{abcd} = \sqrt{g}\varepsilon(a,b,c,d)$ is the volume 4-form and $\epsilon(a,b,c,d)$ is the Levi-Civita symbol. Recall from (2.20) that $F_{ru}$ is $\mathcal{O}(r^{-2})$ to leading order. In the last line this cancels with the $r^2$ coming from $\sqrt{g} = r^2\gamma_{z\bar{z}}$. Substituting this into $Q^+_\epsilon$ gives

$$Q^+_\epsilon = \frac{1}{e^2}\int_{\mathcal{I}^+_-} d^2z\ \gamma_{z\bar{z}}\epsilon F^{(2)}_{ru} \tag{2.27}$$

We can also express (2.23) in retarded coordinates over all $\mathcal{I}^+$. To do so we need the leading order $u$ component Maxwell equation

$$\nabla^\mu F_{\mu u} = \cancel{\nabla^u F_{uu}} + \nabla^r F_{ru} + \nabla^z F_{zu} + \nabla^{\bar{z}} F_{\bar{z}u} = e^2 j_u \tag{2.28}$$

where the first term is zero due to the antisymmetry properties of $F$. Working through each term individually

$$\begin{aligned}\nabla^r F_{ru} &= \nabla_r F_{ru} - \nabla_u F_{ru} \\ &= \partial_r F_{ru} - \cancel{\Gamma^{\alpha}_{rr} F_{\alpha u}} - \cancel{\Gamma^{\alpha}_{ru} F_{r\alpha}} - \left(\partial_u F_{ru} - \cancel{\Gamma^{\alpha}_{ur} F_{\alpha u}} - \cancel{\Gamma^{\alpha}_{uu} F_{r\alpha}}\right) \\ &= \mathcal{O}(r^{-3}) - \partial_u \frac{F^{(2)}_{ru}}{r^2}\end{aligned} \tag{2.29}$$

where in the second line the terms cancel because those Christoffel components are zero. The leading order of $F_{ru}$ is $r^{-2}$, its $r$ derivative is therefore $\mathcal{O}(r^{-3})$ and when working to leading order we can throw it away. The next component is

$$\begin{aligned}\nabla^z F_{uz} &= g^{z\bar{z}} \nabla_{\bar{z}} F_{uz} \\ &= g^{z\bar{z}} \left(\partial_{\bar{z}} F_{uz} - \cancel{\Gamma^{\alpha}_{\bar{z}u} F_{\alpha z}} - \Gamma^{\alpha}_{\bar{z}z} F_{u\alpha}\right) \\ &= \frac{\gamma^{z\bar{z}}}{r^2} \left(\partial_{\bar{z}} F^{(0)}_{uz} + \mathcal{O}(r^{-1})\right) \\ &= \frac{1}{r^2} D^z F^{(0)}_{uz}\end{aligned} \tag{2.30}$$

where $D_z$ is the covariant derivative on the 2-sphere and is raised/lowered using $\gamma_{z\bar{z}}$. We can switch from partial to covariant derivative here as none of the 2-sphere Christoffel symbols contribute. Substituting these into (2.28) and working to leading order we find

$$\partial_u F^{(2)}_{ru} + D^z F^{(0)}_{uz} + D^{\bar{z}} F^{(0)}_{u\bar{z}} + e^2 j^{(2)}_u = 0 \tag{2.31}$$

We are now ready to express (2.23) as an integral over all of $\mathcal{I}^+$ by first integrating by parts over $u$ and substituting in the leading order equation of motion.

$$\begin{aligned}\cancel{Q^+_\epsilon\big|_{\mathcal{I}^+_+}} - Q^+_\epsilon &= \cancel{\frac{1}{e^2}\int_{\mathcal{I}^+} d^2z du \gamma_{z\bar{z}} (\nabla_u \epsilon) F^2_{ru}} + \frac{1}{e^2}\int_{\mathcal{I}^+} d^2z du \gamma_{z\bar{z}} \epsilon \nabla_u F^{(2)}_{ru} \\ Q^+_\epsilon &= \frac{1}{e^2}\int_{\mathcal{I}^+} d^2z du\, \epsilon (D_{\bar{z}} F^{(0)}_{uz} + D_z F^{(0)}_{u\bar{z}}) + \int_{\mathcal{I}^+} d^2z du\, \gamma_{z\bar{z}} \epsilon j^{(2)}_u\end{aligned} \tag{2.32}$$

where we recall the charge relaxes to zero at $\mathcal{I}^+_+$ and $\epsilon(z,\bar{z})$ is independent of $u$. Integrating the first term by parts again over the two-sphere yields

$$\begin{aligned}Q^+_\epsilon &= -\frac{1}{e^2}\int_{\mathcal{I}^+} d^2z du\, (\partial_{\bar{z}} \epsilon F^{(0)}_{uz} + \partial_z \epsilon F^{(0)}_{u\bar{z}}) + \int_{\mathcal{I}^+} d^2z du\, \gamma_{z\bar{z}} \epsilon j^{(2)}_u \\ &= Q^{S+}_\epsilon + Q^{H+}_\epsilon\end{aligned} \tag{2.33}$$

There are no boundary terms as the manifold we are integrating over (the two-sphere) is compact. We see that by expressing the charge as an integral over $\mathcal{I}^+$ it has naturally split into two parts, a 'soft charge' $Q^{S+}_\epsilon$ and a 'hard charge' $Q^{H+}_\epsilon$.

The hard charge contains the leading order component of the boundary current

$j_u^{(0)}$, integrated over $\mathcal{I}^+$ this is just the total charge flux due to massless matter fields passing through null infinity. However, in this case there is now an arbitrary function $\epsilon(z, \bar{z})$ which acts as a weight when integrating over the two-sphere. The soft charge term compensates for this and produces an overall conserved charge $Q_\epsilon^+$. Note neither the soft nor hard charges are individually conserved, only the sum. Further interpretation of the soft charge term is given later in Section 2.8.

## 2.5 Symplectic Structure at $\mathcal{I}^+$

From their construction, the conserved charges (2.23) should generate the large gauge transformations determined in Section 2.3, we would like to verify that this is the case. To do so we will need the Poisson Brackets, or equivalently the symplectic form for EM on $\mathcal{I}^+$[2]. We follow the procedure discussed in Section A.3 for the Maxwell action (2.1)

First, we need to perform a variation of the Maxwell Lagrangian and read off the pre symplectic current density according to (A.12)

$$\begin{aligned} \delta\mathcal{L} &= \left(\frac{\partial\mathcal{L}}{\partial A_\mu} - \partial_\alpha \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\mu}) \right)\delta A_\mu + \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\mu)}\delta A_\mu\right) \\ \implies \Theta^\alpha &= \frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\mu)}\delta A_\mu = F^{\alpha\mu}\delta A_\mu \end{aligned} \tag{2.34}$$

We can now calculate the pre-symplectic current using (A.13) and write it as a wedge product over the symplectic manifold. Note, the wedge product is NOT a wedge product between forms on spacetime, it is a wedge product between the variation differential 1-forms on the symplectic manifold.

$$\begin{aligned} \omega^\alpha &= \delta_1 F^{\alpha\mu}\delta_2 A_\mu - \delta_2 F^{\alpha\mu}\delta_1 A_\mu \\ &= \delta F^{\alpha\mu} \wedge \delta A_\mu \end{aligned} \tag{2.35}$$

Integrating this expression over a chosen Cauchy surface $\Sigma$ yields the pre-symplectic form (A.15)

$$\Omega_\Sigma = \int_\Sigma d\Sigma^\alpha \, \delta F_{\alpha\mu} \wedge \delta A^\mu \tag{2.36}$$

where $d\Sigma^\alpha$ is the induced volume element multiplied by the unit normal to the surface $n^\mu$. In form notation (2.36) becomes [63]

$$\Omega_\Sigma = -\frac{1}{e^2}\int_\Sigma \delta(\star F) \wedge \delta A \tag{2.37}$$

As we are working under the assumption of no massive matter fields, $\mathcal{I}^+$ forms a Cauchy surface for the spacetime. We can therefore integrate over $\mathcal{I}^+$ to obtain an expression for the symplectic form in terms of the gauge field components. The integrand is a differential 3-form, when we choose to integrate over $\mathcal{I}^+$ any component

[2] In introduction to the symplectic structure of classical field theories can be found in Appendix A

containing $r$ will not contribute. Writing out the $u, z, \bar{z}$ component in full, calculating the Hodge dual components and working to leading order yields[3] [63]

$$\Omega_{\mathcal{I}^+} = \frac{1}{e^2}\int dud^2z \left(\delta F^{(0)}_{uz}\wedge\delta A^{(0)}_{\bar{z}} + \delta F^{(0)}_{u\bar{z}}\wedge\delta A^{(0)}_{z}\right) \tag{2.38}$$

where we recall from (2.18) and (2.20) that $F_{uz}$, $A_z$ etc. are all $\mathcal{O}(1)$. If we integrate the second term of (2.38) by parts over $u$ and require the gauge field $A_z$ relax to zero as $u \to \pm\infty$) then we find

$$\Omega_{\mathcal{I}^+} = \frac{2}{e^2}\int dud^2z \left(\delta F^{(0)}_{uz}\wedge\delta A^{(0)}_{\bar{z}}\right) + \cancel{\left[\int d^2z\, \delta A^{(0)}_{\bar{z}}\wedge\delta A^{(0)}_{z}\right]^{u=\infty}_{-\infty}} \tag{2.39}$$

where integrating by parts shifted the $u$ partial derivative in the second term to the field $A^{(0)}_z$ and we have made use of the antisymmetry properties of the wedge product. This is, up to a constant factor, the canonical form of the symplectic structure detailed in Section A.3 for the configuration field $A_{\bar{z}}$ and associated conjugate momentum $F^{(0)}_{uz}$. Therefore, using the canonical definition of a Poisson bracket (A.9) we find

$$\frac{2}{e^2}\left[\partial_u A^{(0)}_z(u,z,\bar{z}), A^{(0)}_{\bar{w}}(u',w,\bar{w})\right] = i\int du''d^2x \left(\frac{\delta\partial_u A^{(0)}_z}{\delta A^{(0)}_{\bar{x}}}\frac{\delta A^{(0)}_{\bar{w}}}{\delta\partial_{u''}A^{(0)}_x} - \frac{\delta\partial_u A^{(0)}_z}{\delta\partial_{u''}A^{(0)}_x}\frac{\delta A^{(0)}_{\bar{w}}}{\delta A_{\bar{x}}}\right) \tag{2.40}$$

Evaluating the functional derivatives produces a string of delta functions which can be reduced by completing the integrals

$$\begin{aligned} RHS(2.40) &= -i\int du''d^2x\, \delta(u-u'')\delta^2(z-x)\delta(u'-u'')\delta^2(w-x)\delta^{u''}_u\delta^x_z\delta^{\bar{x}}_{\bar{w}} \\ &= -i\delta(u-u')\delta^2(z-w) \\ \implies &\left[\partial_u A^{(0)}_z(u,z,\bar{z}), A^{(0)}_{\bar{w}}(u',w,\bar{w})\right] = -\frac{ie^2}{2}\delta(u-u')\delta^2(z-w) \end{aligned} \tag{2.41}$$

where the factor of $i$ is included when going from classical bracket to quantum commutator. We find that we have recovered the commutators on the radiative (non-zero frequency) phase space in [3]. Integrating over $u$ and setting any integration constant to 0 to satisfy antisymmetry properties of the bracket we find

$$\left[A^{(0)}_z(u,z,\bar{z}), A^{(0)}_{\bar{w}}(u',w,\bar{w})\right] = -\frac{ie^2}{4}sign(u-u')\delta^2(z-w) \tag{2.42}$$

$$\left[A^{(0)}_z(u,z,\bar{z}), \partial_{u'}A^{(0)}_{\bar{w}}(u',w,\bar{w})\right] = \frac{ie^2}{2}\delta(u-u')\delta^2(z-w) \tag{2.43}$$

where $sign(x)$ is the well sign function and $\frac{\partial sign(x)}{\partial x} = \frac{1}{2}\delta(x)$. We will now proceed naively to see what transformation $Q^+_\epsilon$ (2.33) generates on the gauge field using

[3] See page 103 of [63] for an explicit calculation

these brackets.

$$\begin{aligned}
[Q_\epsilon^+, A_z^{(0)}(u,z,\bar{z})] &= -\frac{1}{e^2}\int d^2w du' [\cancel{\partial_{\bar{w}}\epsilon(w,\bar{w})F_{u'w}^{(0)}} + \partial_w\epsilon(w,\bar{w})F_{u'\bar{w}}^{(0)}, A_z^{(0)}] \\
&= \frac{1}{e^2}\int d^2w du' \partial_w \epsilon(w,\bar{w}) [A_z^{(0)}, \partial_{u'} A_{\bar{w}}^{(0)}] \\
&= \frac{1}{\cancel{e^2}}\int d^2w du' \partial_w \epsilon(w,\bar{w}) \frac{i\cancel{e^2}}{2}\delta(u-u')\delta^{(2)}(z-w) \\
&= \frac{i}{2}\partial_z\epsilon(z,\bar{z})
\end{aligned} \tag{2.44}$$

The hard charge commutes with the gauge field and does not contribute, therefore we only need to evaluate the soft term. In the first line we notice the $F_{u'w}$ term commutes with the gauge field $A_z$ and does not contribute. The transformation generated on the gauge field is off by a factor of 2, therefore something must be wrong with either the charge or the brackets. This was first properly dealt with in [38] which we now review.

Under the assumption that no magnetic monopoles exist we have the following constraint [38]

$$F_{z\bar{z}}\Big|_{\mathcal{I}^+_\pm} = \left[\partial_z A_{\bar{z}}^{(0)} - \partial_{\bar{z}} A_z^{(0)}\right]\Big|_{\mathcal{I}^+_\pm} = 0 \tag{2.45}$$

Substituting this constraint into the original bracket (2.42) reveals that the bracket is non-zero and therefore does not respect the above constraint. As this constraint only applies to the boundaries $\mathcal{I}^+_\pm$ the commutator (2.42) is still valid for all values of $u, u' \neq \pm\infty$. The Poisson Brackets must be modified according to Dirac's procedure [24]. Direct modification of the brackets was undertaken in [38] however, we choose to adopt an alternate approach in which the symplectic form is modified detailed in [63]. We have the freedom to do this as the symplectic form and Dirac brackets are equivalent to one another, see Appendix A for further details.

We now return to the soft charge term

$$Q_\epsilon^{S+} = -\frac{1}{e^2}\int_{\mathcal{I}^+} d^2z du \,(\partial_{\bar{z}}\epsilon F_{uz}^{(0)} + \partial_z \epsilon F_{u\bar{z}}^{(0)}) \tag{2.46}$$

The field strength component can be written as an $\omega \to 0$ limit of

$$F_{uz}^{(0)}(\omega) = \int_{-\infty}^{\infty} du\, F_{uz}^{(0)} e^{i\omega u} \tag{2.47}$$

which is a Fourier transform of the field strength with frequency $\omega$ [63]. Therefore the modes present in the soft term are of zero frequency / energy. Defining these zero modes

$$N_z \equiv \int_{-\infty}^{\infty} du\, F_{uz}^{(0)} = A_z^+ - A_z^- \tag{2.48}$$

where $A_z^\pm = A_z^{(0)}|_{\mathcal{I}^+_\pm}$ we see that the zero modes are completely characterised by fields living at the boundaries of $\mathcal{I}^+$, precisely where the commutator (2.42) is not valid. We can express the field $N_z$ as a real scalar defined on the boundaries by considering its curl [63].

$$\begin{aligned}\partial_{\bar{z}} N_z - \partial_z N_{\bar{z}} &= \int_{-\infty}^{\infty} du \left[ \partial_{\bar{z}} F_{uz}^{(0)} - \partial_z F_{u\bar{z}}^{(0)} \right] \\ &= -\int_{-\infty}^{\infty} du \partial_u F_{z\bar{z}}^{(0)} \\ &= -F_{z\bar{z}}^{(0)} \Big|_{\mathcal{I}_-^+}^{\mathcal{I}_+^+} \\ &= 0\end{aligned} \tag{2.49}$$

where in the second line we applied the Bianchi Identity $\partial_{[\alpha} F_{\beta\gamma]}$ and then recalled the magnetic monopole assumption (2.45). $N_z$ can therefore be expressed as

$$N_z = e^2 \partial_z N(z, \bar{z}) \tag{2.50}$$

One can easily check that (2.50) satisfies (2.49). Using these definitions we can rewrite the soft charge purely in terms of the boundary field N.

$$\begin{aligned}Q_\epsilon^{S+} &= -\frac{1}{e^2} \int_{\mathcal{I}^+} d^2z du \left( \partial_{\bar{z}} \epsilon F_{uz}^{(0)} + \partial_z \epsilon F_{u\bar{z}}^{(0)} \right) \\ &= -\frac{1}{e^2} \int d^2z \left( \partial_{\bar{z}} \epsilon N_z + \partial_z \epsilon N_{\bar{z}} \right) \\ &= -\int d^2z \left( \partial_{\bar{z}} \epsilon \partial_z N + \partial_z \epsilon \partial_{\bar{z}} N \right) \\ &= 2 \int d^2z \, N \partial_z \partial_{\bar{z}} \epsilon\end{aligned} \tag{2.51}$$

where in the last line we have integrated each term by parts again over the two sphere. Studying the original symplectic form (2.39) we note that there is currently no symplectic partner for the zero modes described by N.

When deriving the radiative symplectic form we made the assumption $A_z^{(0)} = 0$ at the boundaries of $\mathcal{I}^+$. In general this is not true and therefore suggests a starting point in modifying the symplectic form. We begin by separating out the boundary and the bulk parts of $A_z^{(0)}$ by [63]

$$A_z^{(0)} = \hat{A}_z(u, z, \bar{z}) + \partial_z \phi(z, \bar{z}) \tag{2.52}$$

where $\hat{A}_z = 0$ at $\mathcal{I}_\pm^+$ and

$$\partial_z \phi = \frac{1}{2} \left[ A_z^+ + A_z^- \right] \tag{2.53}$$

The choice of $\partial_z \phi$ will soon be justified as it becomes the conjugate momentum to the zero mode $N$. We now substitute equation (2.52) into the expression for the symplectic form (2.38)

$$\begin{aligned}\Omega_{\mathcal{I}^+} &= \frac{1}{e^2}\int dud^2z\,\left(\delta F_{uz}^{(0)}\wedge\delta A_{\bar{z}}^{(0)}+\delta F_{u\bar{z}}^{(0)}\wedge\delta A_{z}^{(0)}\right)\\ &= \frac{1}{e^2}\int dud^2z\,\left(\delta F_{uz}^{(0)}\wedge\delta\left(\hat{A}_{\bar{z}}+\partial_{\bar{z}}\phi\right)+\delta F_{u\bar{z}}^{(0)}\wedge\delta\left(\hat{A}_{z}+\partial_{z}\phi\right)\right)\end{aligned} \tag{2.54}$$

Analysing the terms containing $\hat{A}_z$ first.

$$\begin{aligned}\Omega_{rad} &= \frac{1}{e^2}\int dud^2z\,\left(\delta F_{uz}^{(0)}\wedge\delta\hat{A}_{\bar{z}}+\delta F_{u\bar{z}}^{(0)}\wedge\delta\hat{A}_{z}\right)\\ &= \frac{1}{e^2}\int dud^2z\,\left(\delta\partial_u\hat{A}_{z}\wedge\delta\hat{A}_{\bar{z}}+\delta\partial_u\hat{A}_{\bar{z}}\wedge\delta\hat{A}_{z}\right)\\ &= \frac{1}{e^2}\int dud^2z\,\left(\delta\partial_u\hat{A}_{z}\wedge\delta\hat{A}_{\bar{z}}-\delta\hat{A}_{\bar{z}}\wedge\delta\partial_u\hat{A}_{z}\right)+\cancel{\left[\int d^2z\,\delta\hat{A}_{\bar{z}}\wedge\delta\hat{A}_{z}\right]_{-\infty}^{u=\infty}}\\ &= \frac{2}{e^2}\int dud^2z\,\left(\delta\partial_u\hat{A}_{z}\wedge\delta\hat{A}_{\bar{z}}\right)\end{aligned} \tag{2.55}$$

In the second line we made use of $\partial_z\phi(z,\bar{z})$ being independent of $u$. We then integrated the second term by parts over $u$, where the fields $\hat{A}_z$ are defined to be zero at the boundaries of $\mathcal{I}^+$. Finally, we made use of the antisymmetry properties of the wedge product of variations on the symplectic manifold. This has recovered the radiative phase space symplectic form (2.39) except now it is only defined on the bulk of $\mathcal{I}^+$. We can use the previously calculated commutators (2.42-2.43) and just replace $A_z^{(0)}$ with $\hat{A}_z$ to find

$$\begin{aligned}\left[\partial_u\hat{A}_z(u,z,\bar{z}),\hat{A}_{\bar{w}}(u',w,\bar{w})\right] &= -\frac{ie^2}{2}\delta(u-u')\delta^2(z-w)\\ \left[\hat{A}_z(u,z,\bar{z}),\hat{A}_{\bar{w}}(u',w,\bar{w})\right] &= -\frac{ie^2}{4}sign(u-u')\delta^2(z-w)\end{aligned} \tag{2.56}$$

Next we analyse the terms containing $\partial_z\phi$

$$\begin{aligned}\Omega_{zero} &= \frac{1}{e^2}\int dud^2z\,\left(\delta F_{uz}^{(0)}\wedge\delta\partial_{\bar{z}}\phi+\delta F_{u\bar{z}}^{(0)}\wedge\delta\partial_{z}\phi\right)\\ &= \frac{1}{e^2}\int d^2z\,\left(\delta N_z\wedge\delta\partial_{\bar{z}}\phi+\delta N_{\bar{z}}\wedge\delta\partial_{z}\phi\right)\\ &= \int d^2z\,\left(\delta\partial_z N\wedge\delta\partial_{\bar{z}}\phi+\delta\partial_{\bar{z}}N\wedge\delta\partial_{z}\phi\right)\\ &= 2\int d^2z\,\left(\delta\partial_{\bar{z}}N\wedge\delta\partial_{z}\phi\right)\\ &= -2\int d^2z\,\left(\delta\partial_{z}\phi\wedge\delta\partial_{\bar{z}}N\right)\end{aligned} \tag{2.57}$$

In the second line we made use of $\partial_z\phi(z,\bar{z})$ being independent of $u$, completed the integral over $u$ and used the definition of $N_z$ (2.48). We then applied (2.50) to introduce the boundary field $N$, integrated the first term by parts twice over the two-sphere and made use of the antisymmetry of the wedge product. Up to a constant factor, this is the canonical form of the symplectic structure. Therefore $\partial_{\bar{z}}N$ contained in the soft electric charge is the canonical momentum conjugate to the zero modes described by $\partial_z\phi$. We can apply the canonical form of the Poisson bracket (A.9) to find

$$
\begin{aligned}
-2\big[\partial_z\phi(z,\bar{z})\partial_{\bar{w}}N(w,\bar{w})\big] &= i\int d^2x\left(\frac{\delta\partial_z\phi(z)}{\cancel{\delta\partial_{\bar{x}}N(x)}}\frac{\cancel{\delta\partial_{\bar{w}}N(w)}}{\delta\partial_x\phi(x)} - \frac{\delta\partial_z\phi(z)}{\delta\partial_x\phi(x)}\frac{\delta\partial_{\bar{w}}N(w)}{\delta\partial_{\bar{x}}N(x)}\right)\\
&= -i\int d^2x\,\delta^2(z-x)\delta^2(w-x)\\
&\implies \big[\partial_z\phi(z,\bar{z}),\partial_{\bar{w}}N(w,\bar{w})\big] = \frac{i}{2}\delta^2(z-w)
\end{aligned}
\tag{2.58}
$$

Therefore we find a new commutator which applies only to fields living on $\mathcal{I}^+_\pm$. We have extended the phase space described by the symplectic form to include the zero frequency modes and their conjugates found in the soft charge. Using these new commutators we can generate the action of $Q^{S+}_\epsilon$ (**??**) on the leading order gauge field $A^{(0)}_z$.

$$
\big[Q^{S+}_\epsilon, A^{(0)}_z(u,z,\bar{z})\big] = \big[Q^{S+}_\epsilon, \hat{A}_z(u,z,\bar{z})\big] + \big[Q^{S+}_\epsilon, \partial_z\phi(z,\bar{z})\big] \tag{2.59}
$$

From (2.51) we see that $Q^{S+}_\epsilon$ can be expressed purely in terms of boundary fields which all commute with $\hat{A}_z$, therefore the first term does not contribute. Evaluating the second term we find

$$
\begin{aligned}
\big[Q^{S+}_\epsilon, \partial_z\phi(z,\bar{z})\big] &= \big[2\int d^2w\,N(w,\bar{w})\partial_w\partial_{\bar{w}}\epsilon(w,\bar{w}), \partial_z\phi(z,\bar{z})\big]\\
&= -2\int d^2w\,\partial_w\epsilon(w,\bar{w})\big[\partial_{\bar{w}}N(w,\bar{w}),\partial_z\phi(z,\bar{z})\big]\\
&= i\partial_z\epsilon(z,\bar{z})
\end{aligned}
\tag{2.60}
$$

where in the second line we have integrated by parts over the variable $\bar{w}$ and pulled out the terms which do not contribute to the commutator. The result of (2.58) is then substituted in and the integral over the Dirac distribution is completed. Equation (2.59) becomes

$$
\big[Q^{S+}_\epsilon, A^{(0)}_z(u,z,\bar{z})\big] = i\partial_z\epsilon(z,\bar{z}) \tag{2.61}
$$

We have therefore seen that the soft charge generates the large gauge transformations (2.22) discovered through Asymptotic Symmetry analysis in Section 2.3.

Finally, for any theory containing massless charged matter fields we need to check that $Q^+_\epsilon$ generates the correct transformation (2.3) upon the fields. This is the role of the hard charge term which contains the conserved current $j^\mu$ with action upon the a field $\Phi_k$ with charge $Q_k$ [63]

$$
\big[Q^+_\epsilon, \Phi_k\big] = iQ_k\epsilon(z,\bar{z})\Phi \tag{2.62}
$$

This transformation is just the usual addition of a phase upon the matter field for a $U(1)$ symmetry. Therefore we have seen that $Q^+_\epsilon$ generates the appropriate asymptotic large gauge symmetry acting on $\mathcal{I}^+$ and also imparts a phase on any massless charged matter fields.

Finally, we need to categorise the overall phase space of fields upon $\mathcal{I}^+$. We currently have the data $\left(F_{uz}^{(0)}, A_z^{(0)}, \partial_z\phi(z,\bar{z}), N(z,\bar{z})\right)$. First, note that both $\partial_z\phi$ and $N$ are constructed from the boundary terms of $A_z^{(0)}$ and are therefore fully determined. From the definition of the field strength tensor and the gauge condition $A_u\big|_{\mathcal{I}^+} = 0$ we also have

$$F_{uz}^{(0)} = \partial_u A_z^{(0)}$$

$A_z^{(0)}$ is determined by the field strength up to integration constants. Finally, recalling the magnetic constraint (2.45)

$$F_{z\bar{z}}\big|_{\mathcal{I}^+_\pm} = \partial_z A_{\bar{z}}^\pm - \partial_{\bar{z}} A_z^\pm = 0$$

This can be solved by $A_z^\pm = e^2\partial_z\psi^\pm(z,\bar{z})$ for some unconstrained field $\psi$ on the two-sphere [38]. The physical phase space of the theory can therefore be described using the fields

$$\left(F_{uz}, \psi^+(z,\bar{z}), \psi^-(z,\bar{z})\right) \tag{2.63}$$

Let us now summarise results obtained so far on $\mathcal{I}^+$: We started by imposing falloff conditions on the gauge field $A_\mu$ to ensure physically realistic (i.e. finite energy) situations, through asymptotic symmetry analysis an associated non-trivial asymptotic gauge symmetry was found which preserved these conditions. This symmetry then lead to an infinite number of conserved charges, one for every function $\epsilon(z,\bar{z})$ defined on the two-sphere. When expressing the charge as an integral over all of $\mathcal{I}^+$ it separated out into soft and hard terms. The hard term was related to the flux of charged massless matter passing through $\mathcal{I}^+$ while the soft term was associated with zero energy modes of the gauge field defined at the boundaries $\mathcal{I}^+_\pm$. Through careful analysis of the symplectic form it was shown that the infinite number of charges correctly generate the asymptotic gauge symmetry.

## 2.6 Results from $\mathcal{I}^-$, Antipodal Matching and the Overall Symmetry Group

The analysis contained within previous sections can equally be applied to $\mathcal{I}^-$ to find an independent gauge symmetry group and associated conserved charges. Here we summarise the key results, obtained from [38, 63].

In a region near $\mathcal{I}^-$ one uses advanced coordinates $(v, r, z, \bar{z})$ (2.9) and works in the advanced radial gauge

$$B_v\big|_{\mathcal{I}^-} = 0 \qquad B_r = 0 \tag{2.64}$$

This leads to the following asymptotic falloff conditions of the fields

$$B_v = \mathcal{O}(r^{-1}) \quad B_r = 0 \quad B_z = \mathcal{O}(1) \tag{2.65}$$

and an infinite dimensional residual gauge symmetry parametrised by $\epsilon^-$

$$\delta_{\epsilon^-} B_z = \partial_z \epsilon^-(z,\bar{z}) \tag{2.66}$$

Using this gauge symmetry we can construct the associated conserved charges as an integral over $\mathcal{I}^-_+$ and convert this to an integral over all $\mathcal{I}^-$ using Stokes's Theorem

$$Q^-_{\epsilon^-} = \frac{1}{e^2}\int_{\mathcal{I}^-_+} \epsilon^- \star G = \frac{1}{e^2}\int_{\mathcal{I}^-} d\epsilon^- \wedge \star G + \int_{\mathcal{I}^-} \epsilon^- \star j = Q^{S-}_{\epsilon^-} + Q^{H-}_{\epsilon^-} \tag{2.67}$$

where $G_{\mu\nu}$ is the field strength tensor of the gauge field $B_\mu$. Again, the charge can be split out into a soft term relating to zero frequency modes of the gauge field and hard terms relating to charged massless matter originating from $\mathcal{I}^-$. The above soft charges can be expressed in advanced coordinates and we find they are characterised completely by fields defined at the boundaries $\mathcal{I}^+_\pm$, see equations (2.51) for the analogous calculation at $\mathcal{I}^+$.

Performing the symplectic analysis one eventually finds the following commutators between the conserved charges and leading order fields.

$$\begin{aligned}
\left[\partial_v \hat{B}_z(v,z,\bar{z}), \hat{B}_{\bar{w}}(v',w,\bar{w})\right] &= -\frac{ie^2}{2}\delta(v-v')\delta^2(z-w) \\
\left[\partial_z \phi(z,\bar{z}), \partial_{\bar{w}} N(w,\bar{w})\right] &= \frac{i}{2}\delta^2(z-w) \\
\left[Q^{S-}_{\epsilon}, B^{(0)}_z(v,z,\bar{z})\right] &= \left[Q^{S-}_{\epsilon}, \hat{B}_z(v,z,\bar{z})\right] + \left[Q^{S-}_{\epsilon}, \partial_z \phi(z,\bar{z})\right]
\end{aligned} \tag{2.68}$$

Working through the algebra one eventually finds that the large gauge charges on initial data at $\mathcal{I}^-$ correctly generates the asymptotic symmetry

$$\left[Q^-_{\epsilon^-}, B^{(0)}_z(v,z,\bar{z})\right] = i\partial_z \epsilon^-(z,\bar{z}) \tag{2.69}$$

this takes on exactly the same form as (2.59) with the arbitrary function $\epsilon^-(z,\bar{z})$. The physical phase space is defined analogously to (2.63) and contains the fields

$$\left(G_{vz}, \chi^+(z,\bar{z}, \chi^-(z,\bar{z})\right) \tag{2.70}$$

where $\chi^\pm$ solves the magnetic monopole constraint for the field $B_\mu$.

Finally, to have a well defined classical scattering problem one needs a map from the initial Cauchy data (2.70) on $\mathcal{I}^-$ to the final data (2.63) on $\mathcal{I}^+$. Currently, any map initial data is only known up to the two large gauge transformations (LGTs) defined independently on $\mathcal{I}^+$ and $\mathcal{I}^-$. Therefore one must introduce a restriction on the functions $\epsilon$ and $\epsilon^-$. There needs to be an understanding of how large gauge transformed initial data is mapped to large gauge transformed final data. This is achieved in [38] by enforcing Lorentz invariance upon any relation between $\epsilon$ and $\epsilon^-$. In particular, we want to look at how the boundary data at $\mathcal{I}^-_+$ and $\mathcal{I}^+_-$ transforms under a Lorentz $SL(2,\mathbb{C})$ transform. The resultant condition is

$$\epsilon(z,\bar{z}) = \epsilon^{-}(z,\bar{z}) \tag{2.71}$$

remembering that the coordinates $(z,\bar{z})$ are antipodally identified between advanced and retarded coordinates. The original pair of symmetry groups $(LGT)^{+} \times (LGT)^{-}$ has been broken down into a residual group $LGT^{0}$ defined by the antipodal matching condition and acts upon both $\mathcal{I}^{+}$ and $\mathcal{I}^{-}$ simultaneously. This recovers the matching condition seen in the simple case described in Section 2.2 and introduces the overall charge conservation constraint

$$Q_{\epsilon}^{+} = Q_{\epsilon}^{-} \tag{2.72}$$

Any scattering process between $\mathcal{I}^{-}$ and $\mathcal{I}^{+}$ must respect these infinite number of constraints. We have shown that classical electromagnetism in flat space admits an infinite dimensional large gauge symmetry group, parametrised by a function $\epsilon(z,\bar{z})$ on the two-sphere, acting non-trivially on the boundary of spacetime.

## 2.7 Classical Discussion

### 2.7.1 Classical Properties of Large Gauge Transformations

Firstly, calculating the Dirac Bracket between different charges $Q_{\epsilon}^{+}$ and $Q_{\epsilon'}^{+}$ we find

$$\begin{aligned}
[Q_{\epsilon}^{+}, Q_{\epsilon'}^{+}] &= [Q_{\epsilon}^{S+} + Q_{\epsilon}^{H+}, Q_{\epsilon'}^{S+} + Q_{\epsilon'}^{H+}] \\
&= [Q_{\epsilon}^{S+}, Q_{\epsilon'}^{S+}] + [Q_{\epsilon}^{S+}, Q_{\epsilon'}^{H+}] + [Q_{\epsilon}^{H+}, Q_{\epsilon'}^{s+}] + [Q_{\epsilon}^{H+}, Q_{\epsilon'}^{H+}]
\end{aligned} \tag{2.73}$$

The soft terms only contain the zero modes $N$ and the hard terms contain the conserved current $j$. Recalling the commutators (2.56,2.58), there are no non zero contributions containing only terms in $N$ and $j$. Therefore $\left[Q_{\epsilon}^{+}, Q_{\epsilon'}^{+}\right] = 0$ and the charges satisfy an abelian algebra.

All of the analysis so far has been purely classical. Recall the action of $Q$ on the gauge and charged matter fields

$$\delta_{\epsilon} A_{z} = \partial_{z}\epsilon(z,\bar{z}) \qquad \delta_{\epsilon}\Phi = iQ\epsilon(z,\bar{z})\Phi \tag{2.74}$$

The field strength tensor $F_{\mu\nu}$ and matter action $S_{M}$ are both invariant under these transformations, therefore any effects of the large gauge transformation are classically unobservable [38]. Because the charge algebra is abelian, acting on a system with a large gauge transformation does not introduce large gauge charges. We can see this directly by considering the vacuum $A_{\mu} = 0 \implies F_{\mu\nu} = 0$. From the definition of large gauge charge (2.63)

$$Q_\epsilon^+ = \frac{1}{e^2}\int_{\mathcal{I}_-^+} \epsilon \star F = 0 \tag{2.75}$$

for all $\epsilon(z,\bar{z})$. Now act on the spacetime with a large gauge transformation $A_z = 0 \to \partial_z \epsilon(z,\bar{z})$. Because the field strength is invariant under the transform $F'_{uz} = F'_{rz} = 0$ all the large gauge charges remain unchanged.

### 2.7.2 Vacuum Degeneracy

Although the overall Lagrangian is invariant under these gauge transformations we have just shown the vacuum, defined by $A_\mu = 0$, is not, except for the $\epsilon = const$ charge which corresponds to overall charge conservation. This is an example of spontaneous symmetry breaking. For an introduction to spontaneous symmetry breaking in field theories see Chapter 4 of [60]. Proceeding classically we see that the action of Q on the vacuum is to introduce the massless field $\phi$, corresponding to zero energy photon modes. $\phi$ transforms non-trivially under the large gauge symmetries according to (2.60) and therefore, as per Goldstone's theorem, $\phi$ is the massless Goldstone boson introduced by the symmetry breaking [63].

One could, in principle, keep acting on the vacuum with Q and generating soft photons living on the boundaries $\mathcal{I}$ without changing the energy of the vacuum. Therefore the vacuum in fact has an infinite degeneracy and one can transition between vacua by acting with large gauge transformations. This is the first key result of the proposal: **the vacuum in electromagnetism is not unique. [38, 63]** However, the field strength is invariant under large gauge transforms, hence there is no way to classically distinguish the vacua from each other; all of the large gauge charges must be zero. To understand the true significance of large gauge transforms one needs to move to the quantum case.

## 2.8 Quantum Discussion

### 2.8.1 Quantisation and Soft Photons

One of the steps in moving to a quantum theory is to first promote the charges $Q_\epsilon$ to quantum operators $\hat{Q}_\epsilon$ and enforce the Dirac Brackets as canonical commutation relations, for notational simplicity we have omitted the hat notation. To help interpret the action of $Q_\epsilon$ upon states is useful to connect with the familiar notion of plane wave expansions and creation/annihilation operators.

We assume that near $\mathcal{I}^\pm$ interactions are sufficiently weak so the gauge field becomes free. The outgoing field $A_\mu(x)$, near $\mathcal{I}^+$, in a Minkowski background with signature $(-1,1,1,1)$ then has an approximate plane wave expansion of the form [38, 58]

$$A_\mu(x) = e \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} \left[ \varepsilon_\mu^{*\alpha}(\vec{q}) a_\alpha^{out}(\vec{q}) e^{iq\cdot x} + \varepsilon_\mu^{\alpha}(\vec{q}) a_\alpha^{out}(\vec{q})^\dagger e^{-iq\cdot x} \right] \tag{2.76}$$

As this is a massless field $q^\mu q_\mu = 0$. $\alpha$ denotes the helicity of the photon, $\varepsilon$ is a polarisation vector and $a/a^\dagger$ are the standard annihilation/creation operators for the gauge field. We want to express the field $A_z^{(0)}(u,z,\bar{z})$ at $\mathcal{I}^+$ in this form. This requires changing coordinates from $(t,x,y,z)$ to $(u,r,z,\bar{z})$ and, recalling the asymptotic expansion in Section 2.3, that $A_\mu^{(0)}$ is the leading order term in the expansion of $A_\mu$ as $r \to \infty$. Using this it is shown that [38, 63]

$$A_z^{(0)}(u,z,\bar{z}) = \frac{-i}{8\pi^2} \frac{\sqrt{2}e}{1+z\bar{z}} \int_0^\infty d\omega_q \left[ a_+^{out}(\omega_q \hat{x}) e^{-i\omega_q u} - a_-^{out}(\omega_q \hat{x})^\dagger e^{i\omega_q u} \right] \tag{2.77}$$

where $\hat{x}(z,\bar{z})$ is a unit vector pointing to a point on the two sphere. We see that $A_z^{(0)}$ creates a negative helicity photon exiting $\mathcal{I}^+$ through the angle denoted by $\hat{x}$ and annihilates a positive frequency photon exiting through the same angle. Taking the hermitian conjugate of $A_z^{(0)}$ we see that $A_{\bar{z}}^{(0)}$ has the opposite effect, it creates a positive helicity and annihilates a negative helicity photon. The same process can also be carried out at $\mathcal{I}^-$ to express $A_z^{(0)}(v,z,\bar{z})$ in terms of incoming photon creation/annihilation operators.

We can now use equation (2.77) to understand the action of $Q_\epsilon^{S+}$ in terms of creation/annihilation operators. Recall from (2.51) $Q_\epsilon^{S+}$ can be expressed as

$$Q_\epsilon^{S+} = -2 \int_{S^2} d^2z \, \partial_{\bar{z}} \epsilon \partial_z N \tag{2.78}$$

where

$$e^2 \partial_z N = N_z = \int_{-\infty}^{\infty} du \, F_{uz}^{(0)} = \lim_{\omega \to 0} \int_{-\infty}^{\infty} du \, F_{uz}^{(0)} e^{i\omega u} \equiv \lim_{\omega \to 0} N_z^\omega \tag{2.79}$$

Therefore we need to expand $N_z^\omega$ out into plane wave modes and then take the zero frequency limit. Substituting (2.77) into the expression for $N_z^\omega$ we find

$$\begin{aligned} N_z^\omega &= \int_{-\infty}^{\infty} du \, \partial_u A_z^{(0)} e^{i\omega u} \\ &= \frac{-1}{8\pi^2} \frac{\sqrt{2}e}{1+z\bar{z}} \int_{-\infty}^{\infty} du \int_0^\infty d\omega_q \, \omega_q \left[ a_+^{out}(\omega_q \hat{x}) e^{i(\omega-\omega_q)u} - a_-^{out}(\omega_q \hat{x})^\dagger e^{i(\omega+\omega_q)u} \right] \\ &= \frac{-1}{4\pi} \frac{\sqrt{2}e}{1+z\bar{z}} \int_0^\infty d\omega_q \, \omega_q \left[ a_+^{out}(\omega_q \hat{x}) \delta(\omega-\omega_q) - a_-^{out}(\omega_q \hat{x})^\dagger \delta(\omega+\omega_q) \right] \end{aligned} \tag{2.80}$$

where we have used

$$\int_{-\infty}^{\infty} du \, e^{-i(\omega-\omega_q)u} = (2\pi)\delta(\omega-\omega_q) \tag{2.81}$$

Note that due to the integration limits, for $\omega > 0$ only the first term can contribute and for $\omega < 0$ only the second. The $\omega \to 0$ limit of $N_z^\omega$ is then defined by [38]

$$N_z = \lim_{\omega \to 0^+} \frac{1}{2}(N_z^\omega + N_z^{-\omega}) \tag{2.82}$$

Noting that $\omega$ arrives to zero from a positive value we can then substitute (2.80) into the above expression and evaluate the integral over $\omega_q$ to find

$$N_z = \frac{-1}{8\pi}\frac{\sqrt{2}e}{1+z\bar{z}} \lim_{\omega \to 0^+} \omega \left[ a_+^{out}(\omega\hat{x}) + a_-^{out}(\omega\hat{x})^\dagger \right] \tag{2.83}$$

This can now be substituted into the operator expression for $Q_\epsilon^{S+}$ to find

$$\begin{aligned} Q_\epsilon^{S+} &= \frac{-2}{e^2}\int_{S^2} d^2z\, \partial_{\bar{z}}\epsilon N_z \\ &= \frac{\sqrt{2}}{4\pi e}\int_{S^2} d^2z\, \frac{1}{1+z\bar{z}}\partial_{\bar{z}}\epsilon \lim_{\omega \to 0^+} \omega \left[ a_+^{out}(\omega\hat{x}) + a_-^{out}(\omega\hat{x})^\dagger \right] \end{aligned} \tag{2.84}$$

Studying this now provides a qualitative idea on the action of the soft charge $Q_\epsilon^{S+}$. We see that it creates and annihilates zero frequency photons, of opposite helicity, with polarisation given by $\partial_{\bar{z}}\epsilon$ at points on the conformal two-sphere at $\mathcal{I}^+$. These are the soft photons.

As a specific example we can act with $Q_\epsilon^{S+}$ on the outgoing vacuum state ${}_{out}\langle 0|$

$${}_{out}\langle 0| Q_\epsilon^{S+} = \frac{\sqrt{2}}{4\pi e}\int_{S^2} d^2z\, \frac{1}{1+z\bar{z}}\partial_{\bar{z}}\epsilon \lim_{\omega \to 0^+} \omega\; {}_{out}\langle 0| a_+^{out}(\omega\hat{x}) \tag{2.85}$$

We qualitatively see that the action on the vacuum is to insert an outgoing positive helicity soft photon at the boundary $\mathcal{I}^+$, this is the general quantum action of the large gauge charges. The vacuum state is not annihilated unless $\epsilon = const$. This is the quantum statement of spontaneous symmetry breaking we saw earlier. Because the photon carries zero energy, the new state is energetically degenerate to the original. We can obtain an infinite vacuum degeneracy by simply modifying the soft photon count. To recover the classical statement that all large gauge charges are zero in the vacuum case consider the expectation value of $Q_\epsilon^{S+}$ [38]

$${}_{out}\langle 0| Q_\epsilon^{S+} |0\rangle_{out} = 0 \tag{2.86}$$

Equation (2.86) is zero because $Q_\epsilon^{S+}$ acts on the bra state to produce an orthogonal state containing the soft photon.

Finally, a couple of points that we have skipped over in this qualitative analysis.

1. All of the above can be repeated at $\mathcal{I}^-$ to give corresponding results for incoming states.

2. Recalling the expression for $N_z$ (2.83) we see that the term is taking a limit of $\omega$ to zero while simultaneously multiplying the expression by $\omega$, why isn't this

identically zero? It is shown in [63] that these factors of $\omega$ cancel with poles contained within elements of the $\mathcal{S}$ matrix.

3. It looks as though the soft photon has two degrees of freedom, given by it's direction on the two sphere and polarisation. However, we have seen in the classical case on spontaneous symmetry breaking that only one Goldstone mode is produced. It is shown in [38] that the helicites of soft photons are not actually independent, therefore reducing the degrees of freedom back down to one.

### 2.8.2 Symmetries, Ward Identities and Soft Theorems

We now turn to the discussion of symmetries in quantum field theories so see how the classical large U(1) gauge symmetry carries over to the quantum case, we follow the treatment in [38]. This is done via a Ward-Takahashi identity. Ward-Takahashi identities relate correlation functions to each other due to a global or gauge symmetry, they are essentially the quantum equivalent of Noether's theorem [23]. For a general discussion of Ward-Takahashi identities see Chapter 7 of *Peskin and Schroeder* [58]. We are interested in the Ward identity that arises due to the large gauge symmetry and conserved charges from Section (2.3)

Let us begin by defining the relevant notation [38]:

1. An in state $|in\rangle \equiv |z_1^{in}, ..., z_k^{in}\rangle$ consists of n charged particles with in individual charges $q_k^{in}$ localised at a point $z_k^{in}$ on the conformal sphere at $\mathcal{I}^-$.

2. An out state $\langle out| \equiv \langle z_1^{out}, ..., z_k^{out}|$ consists of m charged particles with in individual charges $q_k^{out}$ localised at a point $z_k^{out}$ on the conformal sphere at $\mathcal{I}^+$.

In the classical case a well defined scattering involved finding a map from a complete set of initial data on the Cauchy surface $\mathcal{I}^-$ to a complete set of final data on the Cauchy surface $\mathcal{I}^+$. The quantum equivalent is to find the $\mathcal{S}$ matrix $\sim \lim_{T\to\infty} e^{iHT}$ which takes you from a complete set of in states to the corresponding out states. A generic scattering is then denoted

$$\langle out| \mathcal{S} |in\rangle \tag{2.87}$$

and large gauge charge conservation is stated as

$$\langle out| \left(Q_\epsilon^+ \mathcal{S} - \mathcal{S} Q_\epsilon^-\right) |in\rangle = 0 \tag{2.88}$$

where $Q_\epsilon^\pm$ are now quantum operators which act on the in/out states. Because $\epsilon(z, \bar{z})$ can take any form on the two sphere we will actually acquire an infinite number of Ward Identities. Also recall that the charges $Q_\epsilon^\pm$ are antipodally matched from (2.72) however, it is useful to treat them separately as $Q_\epsilon^\pm$ use appropriate notation for acting on the out/in states respectively.

Recalling the expression for $Q_\epsilon^-$ in Section (2.6) we find [63, 38]

$$\begin{aligned} Q_\epsilon^- |in\rangle &= Q_\epsilon^{S-} |in\rangle + Q_\epsilon^{H-} |in\rangle \\ &= 2\int d^2z\, \partial_z \partial_{\bar{z}} \epsilon N^- |in\rangle + Q_\epsilon^{H-} |in\rangle \\ &= 2\int d^2z\, \partial_z \partial_{\bar{z}} \epsilon N^- |in\rangle + \sum_{k=1}^{n} Q_k^{in} \epsilon(z_k^{in}, \bar{z}_k^{in}) |in\rangle \end{aligned} \tag{2.89}$$

Likewise the large gauge charge acting on the $\langle out|$ state is

$$\langle out| Q_\epsilon^+ = 2\int d^2z\, \partial_z \partial_{\bar{z}} \epsilon \langle out| N + \sum_{k=1}^{m} Q_k^{out} \epsilon(z_k^{out}, \bar{z}_k^{out}) \langle out| \tag{2.90}$$

Substituting these expressions into the expression for large gauge charge conservation one then finds

$$\begin{gathered} 2\int d^2z\, \partial_z \partial_{\bar{z}} \epsilon \langle out| N(z,\bar{z})\mathcal{S} - \mathcal{S}N^-(z,\bar{z}) |in\rangle = \\ \left[ \sum_{k=1}^{n} Q_k^{in} \epsilon(z_k^{in}, \bar{z}_k^{in}) - \sum_{k=1}^{m} Q_k^{out} \epsilon(z_k^{out}, \bar{z}_k^{out}) \right] \langle out| \mathcal{S} |in\rangle \end{gathered} \tag{2.91}$$

Recall from the previous section that the action of the soft charge was to introduce a polarised soft photon at null infinity. Therefore these infinite number of Ward identities simply relate the scattering with a soft photon insertion to the scattering to a scattering without the soft photon multiplied by some factor that is the difference of weighted charges. It has been shown that these infinite number of Ward identities are actually equivalent to the Weinberg soft photon theorem [63, 38, 75]. Therefore, the large gauge transformations do not correspond to a new undiscovered phenomenon but link directly to existing symmetries discovered by completely different techniques.

## 2.9 Omitted Topics

Finally, there are a number of other interesting avenues of research which have not been discussed here as they do not directly apply to information paradox calculations we wish to study. A brief description of each is provided below with links to further reading if desired:

1. **Massive QED**

   In all of the above calculations we made the simplifying assumption that no massive matter fields were present in the theory. This allowed $\mathcal{I}^\pm$ to be defined as Cauchy surfaces to define initial/final data upon. Physically this is not the case, fields carry mass in the Standard Model. Massive particles follow timelike trajectories from $i^-$ to $i^+$, therefore one needs to include these in defining Cauchy surfaces. This is achieved by considering a hyperbolic slicing of Minkowski space near past/future timelike infinity [39, 63]. It is shown that the asymptotic symmetry for massive QED generates Ward Identities which are in turn equivalent to the soft photon theorem for massive QED. The simplifying assumption will continue in the analysis of black hole spacetimes and hence we do not need to take these considerations into account.

2. **Non-Abelian Gauge Theories**

   So far we have only considered the case of U(1) abelian gauge theory. One could naturally ask if the symmetries extend to more general non-abelian theories such as SU(2) and SU(3) found in the Standard Model? The answer is yes, but only when the gauge symmetry is both non-higgsed and unconfined [63]. For the case of physical black holes formed from standard model matter there are no soft 'gluons' nor asymptotic gauge symmetries. Therefore they are not discussed within this project.

3. **Electromagnetic Memory Effect**

   The third point of the 'Infrared Triangle' introduced in Section (1.3). The presence of an asymptotic symmetry group and soft theorem for QED suggest that there also exists a corresponding "Electromagnetic Memory Effect". This is indeed the case and there are a number of papers proposing experiments on how this could in principle be measured [67, 55]. Note that unlike the gravitational memory effect, this is a purely quantum phenomenon as the memory effect manifests itself as the change in relative phases between two test particles.

# 3 Electric Hair on Black Holes

The asymptotic analysis can now be applied to black holes in asymptotically flat spacetimes. An intuitive definition of asymptotic flatness is that the boundary of the spacetime 'looks like' the boundary of Minkowski space, therefore the work completed at $\mathcal{I}^{\pm}$ in Chapter 2 applies here. We begin by performing a classical analysis, applying the same techniques developed in the previous section. As with the flat space case, to fully understand the implications of large gauge symmetries one must include quantum effects, this is discussed in Section 3.2. Calculations follow discussions first presented in [35, 63].

## 3.1 Classical Large Gauge Symmetry in Black Hole Spacetimes

To keep calculations simple, the authors chose to study the collapse of neutrally charged null matter to form a Schwarzschild black hole. The spacetime is described by the Vaidya metric in advanced coordinates[71, 35].

$$ds^2 = -\left(1 - \frac{2M\Theta(v)}{r}\right)dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \qquad \Theta(v) = \begin{cases} 0 & v < 0 \\ 1 & v \geq 0 \end{cases} \tag{3.1}$$

The choice of advanced coordinates can be extended into the interior of spacetime and therefore be used to describe the event horizon $\mathcal{H}^+$. This metric describes a null shockwave at advanced time $v = 0$ collapsing to form a Schwarzschild black hole. One can see for $v < 0$ the metric describes flat space, afterwards it becomes Schwarzschild in advanced coordinates. The Penrose diagram for this spacetime is shown in Figure 3.1.

The event Horizon starts at $r = 0$ and expands as a function of advanced time until it reaches the Schwarzschild radius $r = 2M$ according to [63]

$$r_H(v) = \begin{cases} \frac{v}{2} + 2M & v \leq 0 \\ 2M & v > 0 \end{cases} \tag{3.2}$$

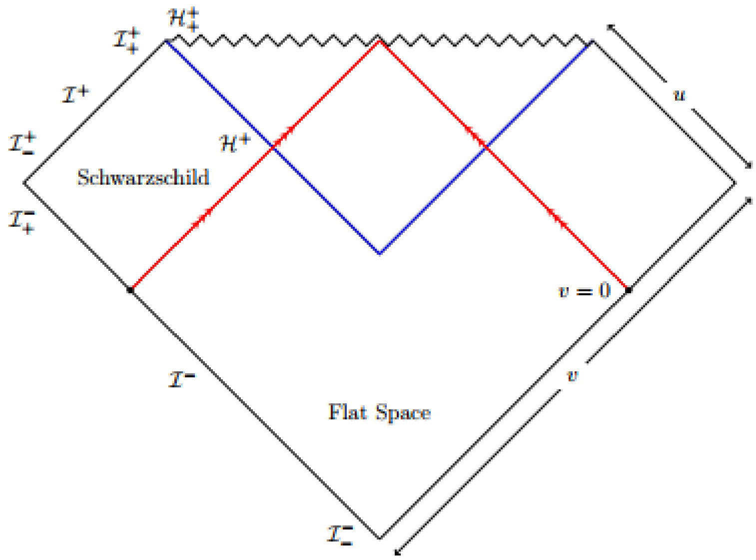


FIGURE 3.1: A Penrose Diagram of the Vaidya Spacetime: A null shockwave of uncharged, non-rotating matter (in red) collapses at advanced time $v = 0$ to form a classical Schwarzschild black hole. The event horizon, $\mathcal{H}^+$ is given by the blue line. For $v < 0$ the spacetime is Minkowski, for $v \geq 0$ is is Schwarzschild. Diagram obtained from [35]

We can now study the implications of the large gauge transformations upon this spacetime. Recall the infinite number of conserved charges for any antipodally identified function $\epsilon(z, \bar{z})$ defined upon the two-sphere.

$$Q^+_\epsilon = \int_{\mathcal{I}^+_-} \epsilon \star F = Q^-_\epsilon = \int_{\mathcal{I}^-_+} \epsilon \star F \tag{3.3}$$

Using stokes's theorem these integrals can be rewritten over any Cauchy surface with matching boundaries. In Section (2.4) this was applied to flat spacetime containing only massless fields, therefore both $\mathcal{I}^+$ and $\mathcal{I}^-$ could be chosen as appropriate Cauchy surfaces to integrate over. In the case of the Vaidya metric this is no longer the case.

For the past charges $Q^-_\epsilon$ we can once again use $\mathcal{I}^-$ as a Cauchy surface and immediately recover the previous results. However, $\mathcal{I}^+$ no longer acts as a Cauchy surface as matter can now pass through the horizon and end at the spacelike singularity. Assuming no massive matter fields are present a new Cauchy surface $(\mathcal{I}^+ \cup \mathcal{H}^+)$ can be taken and we now find

$$Q^+_\epsilon = \int_{\mathcal{I}^+_-} \epsilon \star F = \int_{\mathcal{I}^+} d(\epsilon \star F) + \int_{\mathcal{H}^+} d(\epsilon \star F) = Q^{\mathcal{I}^+}_\epsilon + Q^{\mathcal{H}^+}_\epsilon \tag{3.4}$$

The future conserved charges now contain contributions from both $\mathcal{I}^+$ and $\mathcal{H}^+$, therefore the black hole horizon must contribute to the overall large gauge charge. It was shown in Section (2.3) that the action of $Q_\epsilon^{\mathcal{I}^+}$ was to generate large gauge transformations which act non-trivially on the phase space at the asymptotic boundary of spacetime. We will now show that the same is the case for $Q_\epsilon^{\mathcal{H}^+}$. A full explicit calculation can be found on page 132 of [63]. Here we review the key steps and elaborate on certain conceptual points.

The horizon charge term can be expanded out into a soft and hard charge contribution by applying the Leibnitz rule and Maxwell Equations of Motion to give

$$Q_\epsilon^{\mathcal{H}^+} = \frac{1}{e^2}\int_{\mathcal{H}^+} d\epsilon \wedge \star F + \int_{\mathcal{H}^+} \epsilon \star j \tag{3.5}$$

To calculate the action on the gauge fields we only need the soft term. This requires extending the quantities $\epsilon$ and $F$ in from the boundary to the horizon. $\epsilon$ is brought in to be constant on null trajectories defined by slices of constant $v$,$\partial_v \epsilon = 0$ throughout the bulk and $d\epsilon$ in component notation is

$$d\epsilon = \partial_z \epsilon dz + c.c. \tag{3.6}$$

The components of $\star F$ also need to be computed on Horizon for this metric. The volume 4-form $\varepsilon$ for the Vaidya metric is

$$\varepsilon_{vrz\bar{z}} = \sqrt{-g} = ir^2\gamma_{z\bar{z}} = ig_{z\bar{z}} \tag{3.7}$$

Working through the component $(\star F)_{vz}$ we find

$$\begin{aligned}(\star F)_{vz} =& \varepsilon_{vzr\bar{z}} g^{r\alpha} g^{\bar{z}\beta} F_{\alpha\beta} \\ =& -ig_{z\bar{z}} g^{z\bar{z}} \left(g^{rr}F_{rz} + g^{rv}F_{vz}\right) \\ =& -i\left(\left(1 - \frac{2M\Theta(v)}{r}F_{rz}\right) + F_{uz}\right)\end{aligned} \tag{3.8}$$

After calculating all components in the same manner we find

$$\begin{aligned}\star F =& iF_z^z dv \wedge dr - ir^2 \gamma_{z\bar{z}} F_{vr} dz \wedge d\bar{z} \\ &+ \left[ iF_{rz} dr \wedge dz - i\left[F_{vz} + \left[1 - \frac{2M\Theta(v)}{r}\right] F_{rz}\right] dv \wedge dz + c.c \right]\end{aligned} \tag{3.9}$$

The form now needs to be evaluated on $\mathcal{H}^+$. To do this note that the event horizon is parametrised by advanced time $v$, therefore

$$dr = \frac{\partial r_H}{\partial v} dv = \frac{1}{2}\left(\Theta(-v) - v\delta(-v)\right) \tag{3.10}$$

Setting $r = r_H(v)$ and substituting the above expression into equation 3.9 yields

$$\star F\Big|_{\mathcal{H}^+} = -ir_H^2\gamma_{z\bar{z}}\widetilde{F}_{vr}dz\wedge d\bar{z} + \left[\frac{i}{2}\Theta(-v)\widetilde{F}_{rz}dv\wedge dz - i\left[\widetilde{F}_{vz} + \left[1-\frac{2M\Theta(v)}{r_H}\right]\widetilde{F}_{rz}\right]dv\wedge dz + c.c\right] \tag{3.11}$$

where the first term in equation (3.9) is lost because $dr \propto dv$ and $\widetilde{F}_{ab}$ denotes the field strength component evaluated on the horizon. This expression reduces to [63]

$$\star F\Big|_{\mathcal{H}^+} = -ir_H^2\gamma_{z\bar{z}}\widetilde{F}_{vr}dz\wedge d\bar{z} + \left\{-i\left[\widetilde{F}_{vz} + \frac{1}{2}\Theta(-v)\widetilde{F}_{rz}\right]dv\wedge dz + c.c\right\} \tag{3.12}$$

The soft charge (3.5) can now be expressed in component notation using equations (3.6) and (3.12)

$$\begin{aligned} Q_\epsilon^{\mathcal{H}^{S+}} &= \frac{1}{e^2}\int_{\mathcal{H}^+} d\epsilon\wedge\star F \\ &= \frac{1}{e^2}\int_{\mathcal{H}^+} -i\partial_{\bar{z}}\epsilon\left[\widetilde{F}_{vz} + \frac{1}{2}\Theta(-v)\widetilde{F}_{rz}\right]d\bar{z}\wedge dv\wedge dz + c.c \\ &= \frac{1}{e^2}\int_{\mathcal{H}^+} d^2zdv\,\partial_{\bar{z}}\epsilon\left[\widetilde{F}_{vz} + \frac{1}{2}\Theta(-v)\widetilde{F}_{rz}\right] + \partial_z\epsilon\left[\widetilde{F}_{v\bar{z}} + \frac{1}{2}\Theta(-v)\widetilde{F}_{r\bar{z}}\right] \\ &= \frac{1}{e^2}\int_{\mathcal{H}^+} d^2z\,\partial_{\bar{z}}\epsilon\,N_z + \partial_z\epsilon\,N_{\bar{z}} \end{aligned} \tag{3.13}$$

where in going from the second to third line we have taken account of the antisymmetry of the wedge product and then defined the "zero modes" $N_z$ by

$$N_z = \int dv\left[\widetilde{F}_{vz} + \frac{1}{2}\Theta(-v)\widetilde{F}_{rz}\right] \tag{3.14}$$

In the advanced radial gauge (Section 2.6) these modes can once again be expressed completely in terms of boundary contributions. To calculate the action of $Q_\epsilon^{\mathcal{H}^{S+}}$ we once again need the symplectic form for Electromagnetism and its associated Dirac brackets acting at $\mathcal{H}^+$. Separating out the boundary and bulk contributions on $\mathcal{H}^+$ as in Section 2.5 we find

$$\widetilde{A}_z = \hat{A}_z(v,z,\bar{z}) + \frac{1}{2}(A_z^+ + A_z^-) = \hat{A}_z(v,z,\bar{z}) + C_z \tag{3.15}$$

where $A_z^\pm$ are the boundary terms of the gauge field on $\mathcal{H}^+$. Directly following the calculation in Section 2.5 and inserting the above into the symplectic form we find

$$\Omega_{\mathcal{H}} = \frac{2}{e^2}\int dvd^2z\partial_v\delta\hat{A}_z\wedge\delta\hat{A}_{\bar{z}} - \frac{1}{e^2}\int d^2z\left[\delta C_z\wedge\delta N_{\bar{z}} + \delta C_{\bar{z}}\wedge\delta N_z\right] \tag{3.16}$$

and therefore the bulk-bulk and boundary-boundary commutation relations are

$$\begin{aligned}\left[\hat{A}_z(v,z,\bar{z}),\hat{A}_{\bar{z}}(v',w,\bar{w})\right] =& -\frac{ie^2}{4}\Theta(v-v')\delta^2(z-w)\\ \left[C_z(z,\bar{z}),N_{\bar{w}}(w,\bar{w})\right] =& ie^2\delta^2(z-w)\end{aligned} \tag{3.17}$$

Finally, using these commutators we can evaluate the action $Q_\epsilon^{\mathcal{H}^{S+}}$ has upon the bulk and boundary fields

$$\begin{aligned}\left[Q_\epsilon^{\mathcal{H}^{S+}},\hat{A}_z\right] = 0, \qquad \left[Q_\epsilon^{\mathcal{H}^{S+}},N_z\right] = 0, \qquad &\left[Q_\epsilon^{\mathcal{H}^{S+}},C_z\right] = i\partial_z\epsilon(z,\bar{z})\\ \implies &\left[Q_\epsilon^{\mathcal{H}^{S+}},\widetilde{A}_z\right] = i\partial_z\epsilon(z,\bar{z})\end{aligned} \tag{3.18}$$

Therefore we see the soft horizon charges generate a large gauge transformation acting upon the Horizon of the Black Hole. Recalling from Section 2.7 we note that because the LGT group is abelian, acting upon the horizon will not introduce any large gauge charges. Once again, the classical action of large gauge transformations upon the Black Hole horizon is unobservable.

The No-Hair theorem also states that any Black Hole is determined by only is mass, angular momentum and total electric charge. Therefore classically $Q_\epsilon^{\mathcal{H}} = 0 \quad \forall\, \epsilon \neq const$. Although the black hole horizon can in principle carry large gauge charge, classically it does not [35]. To understand the significance of the large gauge symmetry one needs to move to the quantum case.

Finally, in this section we defined the large gauge charge (3.4) on the horizon. Applying Stoke's theorem in reverse will produce a large gauge charge defined upon the boundaries of the horizon. However, in the quantum case the horizon will eventually evaporate, meaning that we cannot really define the charge there. Quantum mechanically the horizon charges are not well defined [63, 36]

## 3.2 Quantum Analysis and Soft Electric Hair

Recall in Section 2.8, we showed for the vacuum that although all classical large gauge charges were zero, the operator action of the charges was non-trivial. Soft photons were annihilated / created upon the null boundaries of spacetime. The vacuum states were therefore energetically degenerate but in fact physically inequivalent.

The quantum action of a large gauge transform is to introduce soft photons upon the surface it is defined on. We have shown above that in a black hole spacetime, large gauge transformations act on both the horizon and boundary of spacetime. Therefore the quantum action of the horizon transform is to introduce soft photons on the horizon, transforming it into an energetically degenerate but physically inequivalent state. If one could find a way to differentiate between these two states then the black hole would need to be characterised by more than just M, J and Q and the No-Hair theorem would be quantum mechanically violated.

### 3.2.1 Distinguishing Between Large Gauge Transformed Black Holes

One way to see if we can distinguish between transformed black holes is to look at the resultant Hawking radiation, the argument follows that presented in [35]. Recall from Section 1.3 that there is thought to be a correlation between any charges a black hole carries and its resultant Hawking radiation. For example, if the black hole carried large gauge charge we would expect the radiation to carry this away to $\mathcal{I}^+$ due to overall charge conservation enforced by the $LGT^0$ group. We have just shown that classically, black holes cannot carry large gauge charge, however the quantum action of the charge on the horizon is still non-trivial. We can still attempt to distinguish between the evaporation product of a black hole state $|M\rangle$ and its large gauge transformed state $Q_\epsilon^{\mathcal{H}} |M\rangle$.

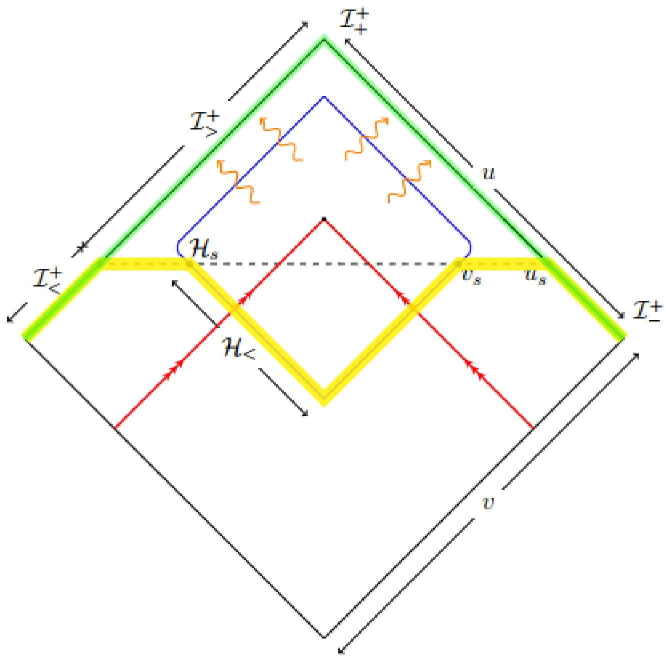


FIGURE 3.2: A Penrose Diagram Displaying the Formation and Evaporation of a Black Hole. The red null shockwave collapses to form the black hole. Future Cauchy surfaces may be taken to include a horizon contribution (the yellow path) or not (the green path). Diagram source: [35]

Consider the Penrose diagram depicted in Figure 3.1 of a black hole which has formed and completely evaporated due to Hawking radiation. Here we detail a number of simplifying assumptions to aid with calculations

1. The collapse forming the black hole is due to a spherically symmetric null shockwave of uncharged matter, depicted by the red lines in Figure 3.1. This is the same as the shockwave which went into forming the classical Schwarzschild black hole in Section 3.1.

2. During the black hole formation phase, there are no quantum effects taking place. It is treated as purely classical. The apparent horizon associated to the classical formation is given by $\mathcal{H}_<$ with future boundary $\mathcal{H}_s$.

3. The points $u_s$,$v_s$ are chosen to be at a time long after classical formation has completed but long before the black hole begins to radiate. This temporal separation of quantum and classical effects is key in the original argument for information loss [31].

4. A long time after formation is complete Hawking radiation begins to emit from the black hole. This allows us to temporally separate $\mathcal{I}^+$ into $\mathcal{I}^+_<$ which receives no Hawking radiation and $\mathcal{I}^+_>$ which receives the total energy of the black hole.

5. Outside of the black hole, the spacelike slice between $v_s$ and $u_s$ is completely empty . That is, after formation is complete but before radiation begins the space outside the black hole is a vacuum. This assumption allows $\mathcal{I}^+_< \cup \mathcal{H}_<$ to be treated as an approximate Cauchy surface, denoted by the yellow path in Figure 3.1.

6. Given an incoming pure black hole state $|M\rangle$ there is some unitary evolution process throughout the evaporation of the black hole to another pure state $|X\rangle$ living on $\mathcal{I}^+$. There is currently no known unitary evolution process which satisfies this[1].

Now, consider the pure incoming quantum black hole state $|M\rangle$. This is defined completely in terms of a Hilbert space on $\mathcal{H}_<$ and is uncorrelated with the vacuum defined on $\mathcal{I}^+_<$ due to assumptions 2-4. This state is then allowed to evolve (assumption 6) to another pure state $|X\rangle$, defined completely on $\mathcal{I}^+_>$ with total energy equal to the original black hole mass.

Alternatively, one can act on the incoming state with a large gauge transformation on the horizon, denoted by $Q_\epsilon^{\mathcal{H}<}$. The new state

$$|M'\rangle = Q_\epsilon^{\mathcal{H}<} |M\rangle \tag{3.19}$$

is then just an energetically degenerate state with an additional soft photon living on the horizon[2]. At this point, if one invoked the No-Hair theorem it would state, in this case, that the black hole only depends upon its total mass $M$. Therefore the evolution of $|M'\rangle$ should be exactly the same as that of $|M\rangle$, all other defining characteristics of the state, such as the soft photon, are lost. Using charge conservation of the $\mathrm{LGT}^0$ group, the authors of [35] showed this is not the case.

The future conserved large gauge charges $Q_\epsilon^+$ can be evaluated over any closed Cauchy surface on the spacetime. Both the surface $(\mathcal{I}^+_< \cup \mathcal{I}^+_>)$ and $(\mathcal{I}^+_< \cup \mathcal{H}_<)$ are

[1]Although there is currently no known process for this, the assumption is that any theory of Quantum Gravity will obey the unitary evolution postulate.

[2]As the original state had no charged matter only the soft term of the charge contributes

equally valid choices of surface (assumption 5). Therefore

$$Q_\epsilon^+ = Q_\epsilon^{\mathcal{I}^+_<} + Q_\epsilon^{\mathcal{I}^+_>} = Q_\epsilon^{\mathcal{I}^+_<} + Q_\epsilon^{\mathcal{H}<} \tag{3.20}$$

where the charges $Q_\epsilon^{\mathcal{I}^+_{</>}}$ are volume integrals consisting of the usual hard and soft term defined using the same techniques as previous examples. Immediately we see that $Q_\epsilon^{\mathcal{I}^+_>} = Q_\epsilon^{\mathcal{H}<}$.

Now assume that $|M'\rangle$ evolves to some other pure state $|X'\rangle$ on $\mathcal{I}^+_>$. Therefore

$$\begin{aligned} |M\rangle &\to |X\rangle \\ \implies |M'\rangle = Q_\epsilon^{\mathcal{H}<}|M\rangle &\to Q_\epsilon^{\mathcal{I}^+_>}|X\rangle = |X'\rangle \end{aligned} \tag{3.21}$$

That is, we can distinguish between a black hole with a soft photon insertion on the horizon and one without just by looking at its evaporation products. The No-Hair theorem does not hold in the quantum mechanical case and we reach the second main result of the proposal: **Black holes can carry soft quantum electric hair** [35].

### 3.2.2 Physical Soft Hair Excitations

Finally, even though we have found that black holes can in principle carry soft electric hair it needs to be shown that this can be achieved by a physical process. This is carried out in [63, 35] where an asymmetric null charged shockwave is sent into the black hole after formation but before evaporation sets in. The shockwave is constructed in such a way as to ensure the net charge inserted into the black hole is zero. The divergence free charge current is given by

$$j_v = \frac{Y_{lm}(z,\bar{z})}{r^2}\delta(v-v_0) \tag{3.22}$$

where $Y_{lm}$ are spherical harmonics satisfying

$$D^2 Y_{lm}(z,\bar{z}) = -l(l+1)Y_{lm}(z,\bar{z}) \tag{3.23}$$

and we recall $D$ is the covariant derivative on the two-sphere raised/lowered by $\gamma_{z\bar{z}}$. Assuming that there is no initial radial electric field $F^{(2)}_{vr}|_{\mathcal{I}^-} = 0$, $A_z\big|_{\mathcal{I}^-} = 0$ and working in the $A_v = 0$ gauge one can solve the $v$ component Maxwell equation on $\mathcal{I}^-$ [38]

$$\partial_v F^{(2)}_{rv} + \gamma^{z\bar{z}}\left(\partial_z F_{\bar{z}v^{(0)}} + \partial_{\bar{z}} F^{(0)}_{zv}\right) = e^2 Y_{lm}(z,\bar{z})\delta(v-v_0) \tag{3.24}$$

to find [63]

$$A_z = \partial_z\left[\theta(v-v_0)\frac{e^2}{l(l+1)}Y_{lm}(z,\bar{z})\right] \tag{3.25}$$

Note that for $v < v_0$ the gauge field is just zero, after it is just a gauge transformation with parameter

$$\epsilon = \frac{e^2}{l(l+1)}Y_{lm}(z,\bar{z}) \tag{3.26}$$

As $A_z$ is either zero or pure gauge, the field strength remains invariant and the action of the shock wave does not introduce any electric charge. All the shockwave does is produce a large gauge transformation acting on both the horizon and null infinity. From our previous quantum interpretation this means that the shockwave can excite a soft photon on the horizon of the black hole in a physically reasonable manner. Finally, note that although in principle soft photons with an arbitrarily small localisation could exist, there is no physical way to do this. The minimum localisation possible of a soft photon is the Planck length [35]. This suggest that the maximum number of soft photons, or hairs, one can excite on a black hole is related to the area of the black hole in Planck units. The same as the BH-Entropy.

## 3.3 Discussion

### 3.3.1 Summary of Results

By studying the electromagnetic asymptotic symmetry group upon flat and black hole spacetimes, SHP have uncovered two key results of significance to the information paradox. The first is that, at the quantum mechanical level, black holes can carry hair. This hair manifests itself as excitations of soft photons living on the event horizon of the black hole, hence the name 'soft hair'. The hair is excited whenever charged matter crosses the event horizon, storing some information of the formation process beyond M, J and Q. The presence of this hair, coupled with the $LGT^0$ scattering constraints, must in turn constrain any outgoing Hawking radiation from the black hole as it evaporates, imparting more information than previously thought upon the evaporation products.

Although the symmetry group is infinite dimensional, it is thought that only a finite amount of physical hair can be excited on the horizon, proportional to the area in Planck units. This is important as it provides a hint towards some UV cut-off of the soft modes. An infinite amount of quantum hair, implying an infinite number of constraints on the Hawking radiation. This would suggest the entropy and the information storage capability of the black hole is infinite, which is thought to not be the case.

The second key result is that the vacuum is not unique at all, but carries an infinite degeneracy. Vacuum states can be transitioned between using large gauge transformations acting at the future and past null boundaries of spacetime. Accounting for this vacuum degeneracy in QED has shown that the theory is in fact IR finite [41]. The ability to distinguish between vacuum states also means that it can carry non-trivial information. Its ability to store information may prove useful in recovering a pure quantum state once the black hole has completely evaporated.

### 3.3.2 Implications towards the Information Paradox

It has been shown that, although not infinite, the black hole can carry a vast quantity of soft electric hair. We can now return to our questions posed in Section 1.4. First, does this hair have the capacity to store all of the information in black hole formation? By considering a couple of simple thought experiments we can deduce that the answer is likely to be no. The soft photons are excited by electrically charged matter crossing the horizon. What would be the case if we constructed the hole purely from uncharged matter, such as the brick/encyclopaedia example? No large gauge transformation would take place implanting no soft electric hair. There is still a lot of non trivial information about the matter which has fallen through the event horizon and is once again irretrievable.

Another flaw is to consider multiple shells of charged matter crossing the horizon. Surely it would be possible to construct two asymmetrically charged shells of matter which have the same net effect as a single shell with the same overall angular profile. How does one distinguish between these two black holes if the net large gauge transform is the same? Clearly something more is needed.

Finally, soft black hole hair is currently a gauge theory dependent phenomenon. It was briefly mentioned in Section 2.9 that the non-abelian gauge theories in the Standard Model do not have an associated asymptotic symmetry group. If we chose a theory with other large gauge symmetry content (such as Yang-Mills) then this would lead to, via the same arguments, associated large gauge hair and soft particles on the horizon. If the entropy is related to the amount of hair which can be excited upon the horizon then theories with different matter content would provide vastly different ideas of entropy, this introduces a species problem and contradicts with the notion that the BH entropy is independent of the particle spectrum of a given theory [57]. Therefore a logical place to look for new hair would be gravity itself. This is what is discussed in the next Section.

# 4 Gravitational Asymptotic Symmetries

In this section we begin by reviewing work which was originally completed by Bondi, van de Burg, Metzner and Sachs [9] who studied the asymptotic symmetry group of null infinity for asymptotically flat spacetimes. Comprehensive reviews of the BMS group may be found in [49, 1]. We then move onto the more recent discovery where, considering a certain class of spacetime called Christodoulou-Klainerman spaces, an overall version of the BMS group emerges which enforces an infinite number of constraints upon any vacuum-vacuum gravitational scattering process [65, 37]. Furthermore, when quantised the symmetries correspond to a set of soft theorems known as "Soft Graviton Theorems" [75]. Many of the logical arguments presented in this section closely follow those presented in the large gauge case. Due to the lengthy nature of calculations in General Relativity we proceed mostly by analogy, discussing the structure of the calculations but leaving much of the algebra to existing literature.

## 4.1 Asymptotic Symmetries at $\mathcal{I}^+$

Recalling Section 2.3, we are looking for the allowed, non-trivial symmetry transformations acting upon our theory. Unlike the gauge case, where the transformations were acting on dynamical fields evolving over a fixed spacetime, the metric components themselves are now the dynamical objects. Therefore we are looking for the allowed, non-trivial diffeomorphisms acting on a spacetime. We mean non-trivial in the sense that coordinate transformations - just like traditional gauge transformations - are excluded. This can be most easily done by completely fixing the coordinate system, just as one would fix a gauge. As we are interested in asymptotic symmetries at future/past null infinity a logical choice of coordinates is those suited to outgoing/incoming null rays. Bondi developed a coordinate system $(u, r, \Theta^A)$ in this exact fashion, based upon null hypersurfaces $S(x^\mu) = u$ of constant retarded time $u = t - r$.

### 4.1.1 Metric Conventions and Coordinate Fixing

A general 4D metric comes with 10 free parameters. Out of these ten, four are related to transformations of the four coordinates. Coordinates in general relativity are not physical, therefore if we are interested in diffeomorphisms which change the physical spacetime we must remove this coordinate transformations by fixing four constraints on the metric, completed below [49].

The normal co-vector to a hypersurface $S$ is

$$n_\mu = -\partial_\mu S \tag{4.1}$$

As we are dealing with a null hypersurface, the normal vector to the surface is also tangent, hence $n^2 = 0$. The minus sign ensures the tangent $n^\mu$ is future pointing. From the null condition we find

$$n^2 = g^{ab} n_a n_b = g^{uu} = 0 \tag{4.2}$$

As one travels along the null rays out to $\mathcal{I}^+$ we fix the angular coordinate $x^A$

$$n^a \partial_a x^A = -g^{ab} \partial_b u \partial_a x^A = -g^{uA} = 0 \tag{4.3}$$

We can use these conditions to find constraints on the metric components, using $g_{ab} g^{bc} = \delta_a^c$

$$\begin{aligned} \delta_r^u &= 0 = g^{uc} g_{cr} = g^{ur} g_{rr} \implies g_{rr} = 0 \\ \delta_A^u &= 0 = g^{uc} g_{cA} = g^{ur} g_{rA} \implies g_{rA} = 0 \end{aligned} \tag{4.4}$$

This provides three of the four required constraints on the metric. The final constraint comes from restricting the angular components of the metric $g_{AB} = r^2 h_{AB}$ where $h_{AB}$ is a metric of the unit 2-sphere. These coordinates can be applied to any four dimensional Lorentzian metric to produce a line element of the form [63, 49]

$$ds^2 = -U du^2 - 2e^{2\beta} du dr + g_{AB} \left( dx^A + \frac{1}{2} U^A du \right) \left( dx^B + \frac{1}{2} U^B du \right) \tag{4.5}$$

where $U, \beta, U^A$ and $g_{AB}$ are functions of $(u, r, x^a)$. By enforcing that any diffeomorphism preserve the structure of (4.5) we eliminate all trivial transformations in the form of coordinate transforms. We now need define some conditions upon the metric to find the allowed symmetry transformations.

### 4.1.2 Asymptotic Flatness

Recall in Section (3.1) we provided an intuitive definition of *Asymptotic Flatness* in the sense that the space-time *'looks like Minkowski'* at infinity. Thanks to the Penrose conformal compactification process one can define infinity as a boundary of some associated compactified spacetime. Therefore the definition can be presented as conditions on the boundary of the spacetime compared to the boundary of Minkowski.

A balancing act has to be struck on how restrictive these boundary conditions are, just as in the gauge case in Section 2.3. In the gauge theory case fields were evolving through a spacetime geometry, the boundary conditions were then applied directly to the fields themselves. In general relativity things are much more complex, the field components we wish to restrict determine the evolution of the spacetime the restrictions are defined against. Fortunately, this is now well understood, reviews of asymptotic flatness conditions can be found in *Ashketar* [3] and *Wald* [72]. The definition for asymptotic flatness is

**Definition 4.1.1.** "A spacetime manifold equipped with metric $(M, g)$ is said to be *Asymptotically Flat* if it satisfies the following conditions [59]:

1. There exists a Penrose completion $(\bar{M}, \bar{g})$ such that $\bar{g} = \Omega^2 g$ for some positive function $\Omega$ on $M$. At the boundary of $\bar{M}$ $\Omega = 0$ and $d\Omega \neq 0$.

2. The boundary of $\bar{M}$. $\partial\bar{M}$ is the disjoint union of $\mathcal{I}^+$ and $\mathcal{I}^-$, each have the structure of $\mathbb{R} \times S^2$

3. No past / future directed causal curve beginning in $M$ crosses $\mathcal{I}^+$ / $\mathcal{I}^-$

4. $\mathcal{I}^\pm$ are complete."

Condition one ensures that there exists a boundary to compare with the boundary of compactified flat space, two places the topological structure requirements upon the boundaries, three ensures that the boundaries are in fact null and finally four ensures that all generators of $\mathcal{I}^\pm$ are complete[1]. These conditions impose boundary restrictions on the form the unphysical metric can take.These boundary restrictions on the unphysical metric can be translated back to falloff conditions on the physical metric components. In $(u, r, z, \bar{z})$ coordinates the asymptotic expansion of an asymptotically flat metric in a region near $\mathcal{I}^+$ is given by [63, 65]

$$\begin{aligned} ds^2 = &- du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &+ \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + D^zC_{zz}dudz + D^{\bar{z}}C_{\bar{z}\bar{z}}dud\bar{z} + ... \end{aligned} \tag{4.6}$$

where '...' indicates further terms subleading in $r$, we will return to these terms when discussing *superrotations* later. Note that the first three terms of the metric are just Minkowski followed by subleading corrections. By inspection of the line element (4.6) we now have our desired falloff conditions for preservation of asymptotic flatness:

[1] In retarded Bondi coordinates this means that the coordinate $u$ has a range $(-\infty, \infty)$.

$$\begin{aligned} g_{uu} &= -1 + \mathcal{O}(r^{-1}) \\ g_{ur} &= -1 + \mathcal{O}(r^{-2}) \\ g_{uz} &= \mathcal{O}(1) \\ g_{rr} &= 0 \\ g_{rz} &= 0 \\ g_{zz} &= \mathcal{O}(r) \\ g_{z\bar{z}} &= r^2\gamma_{z\bar{z}} + \mathcal{O}(1) \end{aligned} \tag{4.7}$$

Finally, we define the metric coefficients $m_B$ and $C_{zz}$. The function $m_B(u, z, \bar{z})$ is known as the Bondi mass aspect. It can be interpreted as the energy carried away through the angle $(z, \bar{z})$, integrating over the two sphere yields the Bondi mass. Finally, integrating over $\mathcal{I}^+$ gives quantifies the total energy radiated away from the spacetime through null infinity due to gravitational radiation. As one approaches $\mathcal{I}^+_-$ the Bondi Mass will match the ADM mass defined at $i^0$. $C_{zz}$ behaves analogously to $A_\mu$ in the gauge case, it can be thought of as a tensor potential for gravitational radiation. The Bondi News Tensor, defined as $N_{zz} = \partial_u C_{zz}$ then behaves analogously to the field strength tensor for gravitational radiation.

### 4.1.3 The BMS Group

We now need to find the transformations which preserve both the conditions in (4.7) (allowed) and (4.5) (non-trivial). To do this we need to look at "large diffeomorphisms" acting at the boundaries of spacetime. Under a diffeomorphism generated by a vector $\zeta$, the metric components infinitesimally transform as

$$g_{ab} \to g_{ab} + \mathcal{L}_\zeta g_{ab}$$

where $\mathcal{L}_\zeta g_{ab}$ is the Lie derivative of the metric with respect to the vector $\zeta$ and is given by

$$\mathcal{L}_\zeta g_{ab} = \zeta^\rho \partial_\rho g_{ab} + g_{\rho b}\partial_a \zeta^\rho + g_{a\rho}\partial_b \zeta^\rho \tag{4.8}$$

We need to find the structure of the generator which preserves all of the asymptotic boundary conditions. To do this one needs to calculate the Lie derivative of each metric component and, from the boundary conditions, restrict the form of generating vector $\zeta$ such that the conditions are preserved at $\mathcal{I}^+$.

This analysis was originally performed by BMS [9] where they discovered the result that the asymptotic symmetry group of asymptotically flat spacetime is not the Poincare group, but an infinite dimensional group called the BMS$^+$ group acting on future null infinity. The original BMS$^+$ consists of an $SL(2, \mathbb{C})$ Lorentz subgroup and an infinite dimensional extension of the usual 4 spacetime translations, called supertranslations. It has been shown since that the Lorentz subgroup also has an infinite dimensional extension, called superrotations [5], discussion of these is deferred to Section 4.6. For now we will follow the original BMS analysis.

We are primarily interested in the infinite dimensional supertranslation subgroup. Boosts and rotations can be eliminated from the BMS generators by imposing the falloff conditions $\zeta^u, \zeta^r \sim \mathcal{O}(1)$ and $\zeta^z, \zeta^{\bar{z}} \sim \mathcal{O}(r^{-1})$ [63]. A full derivation of the form of the supertranslation generating vectors can be found on page 120 of [63]. Here we present a couple of calculations to give a flavour of the methodology. Consider the infinitesimal transformation of $g_{rr}$

$$\begin{aligned} \mathcal{L}_\zeta g_{rr} &= \cancel{\zeta^\rho \partial_\rho g_{rr}} + 2 g_{r\rho} \partial_r \zeta^\rho = 0 \qquad (g_{rr} = g_{rz} = 0) \\ &\implies 2 g_{r\rho} \partial_r \zeta^\rho = 0 \\ &\implies 2 g_{ru} \partial_r \zeta^u = 0 \\ &\implies \partial_r \zeta^u = 0 \\ &\implies \zeta^u = \zeta^u(u, z, \bar{z}) \end{aligned} \tag{4.9}$$

We see this already yields a restriction on the coordinate dependence of $\zeta^u$. A further condition upon this component can be discovered by considering the leading order transformation of the $g_{ru}$ component

$$\mathcal{L}_\zeta g_{ru} = \zeta^\rho \partial_\rho g_{ru} + g_{r\rho} \partial_u \zeta^\rho + g_{u\rho} \partial_r \zeta^\rho \tag{4.10}$$

this has to respect the condition presented in (4.7), specifically the transformed metric has to be $\mathcal{O}(r^{-2})$. Working to leading order

$$\begin{aligned} \mathcal{L}_\zeta g_{ru} &= \cancel{\zeta^\rho \partial_\rho g_{ru}} + g_{ru} \partial_u \zeta^u + \cancel{g_{uu} \partial_r \zeta^u} + \cancel{g_{ur} \partial_r \zeta^r} + \cancel{g_{uA} \partial_r \zeta^A} \\ &= g_{ru} \partial_u \zeta^u + \mathcal{O}(r^{-2}) \\ &\implies \partial_u \zeta^u = 0 \\ &\implies \zeta^u = \zeta^u(z, \bar{z}) \end{aligned} \tag{4.11}$$

where the first and last terms in line one are $\mathcal{O}(r^{-2})$, they satisfy the falloff conditions and can therefore be ignored. This has restricted the $u$ component of $\zeta$ to be a function $f(z, \bar{z})$ defined on the two-sphere only. Completing this process for all components and enforcing the Bondi coordinate conditions eventually reveals the generators of supertranslations to take the form [65]

$$\zeta = f \partial_u - \frac{1}{r}\left(D^{\bar{z}} f \partial_{\bar{z}} + D^z f \partial_z\right) + D^z D_z f \partial_r + \ldots \qquad f = f(z, \bar{z}) \tag{4.12}$$

where '...' indicates further subleading terms.

We can now use this to determine how the mass aspect, $C_{zz}$ and the News tensor all transform under supertranslations. To calculate the infinitesimal variation of the Bondi mass aspect under a supertranslation we need to consider the variation of the

$\mathcal{O}(r^{-1})$ term in the $g_{uu}$ component.

$$\mathcal{L}_\zeta g_{uu} = \zeta^\rho \partial_\rho g_{uu} + 2 g_{\rho u} \partial_u \zeta^\rho \tag{4.13}$$

working with the first term and retaining only the $\mathcal{O}(r^{-1})$ contribution

$$\begin{aligned} \zeta^\rho \partial_\rho g_{uu} &= \zeta^\rho \partial_\rho \left( -1 + \frac{2m_B}{r} + ... \right) \\ &= \zeta^u \partial_u \left( \frac{2m_B}{r} \right) + \mathcal{O}(r^{-2}) \\ &= \frac{2f}{r} \partial_u m_B + \mathcal{O}(r^{-2}) \end{aligned} \tag{4.14}$$

Expanding and analysing the second term gives

$$2 g_{\rho u} \partial_u \zeta^\rho = \cancel{2 g_{uu} \partial_u \zeta^u} + 2 g_{ru} \partial_u \zeta^r + \cancel{(g_{zu} \partial_u \zeta^z + c.c.)} \tag{4.15}$$

where the three terms cancel because $\zeta^u, \zeta^z$ are both independent of $u$ at orders up to and including $r^{-1}$. To calculate the $\mathcal{O}(r^{-1})$ variation we need the next subleading contribution of $\zeta^r$. This is [63]

$$-\frac{1}{2} D_z f D_z C^{zz} - \frac{1}{4} C^{zz} D_z^2 f + c.c. \tag{4.16}$$

Substituting (4.16) into (4.15) then yields

$$2 g_{ru} \partial_u \zeta^r = \frac{1}{2} (2 D_z f D_z N^{zz} + N^{zz} D_z^2 f + c.c.) \tag{4.17}$$

Putting everything together and canceling off the factor of two yields

$$\mathcal{L}_\zeta m_B = f \partial_u m_B + \frac{1}{4} (2 D_z f D_z N^{zz} + N^{zz} D_z^2 f + c.c.) \tag{4.18}$$

The same technique can be applied to $C_{zz}$ to find

$$\begin{aligned} \mathcal{L}_\zeta C_{zz} &= f \partial_u C_{zz} - 2 D_z^2 f \\ \mathcal{L}_\zeta N_{zz} &= f \partial_u N_{zz} \end{aligned} \tag{4.19}$$

(4.18) and (4.19) describe how final data describing the spacetime transform under infinitesimal supertranslations at $\mathcal{I}^+$. Analogous results hold for initial data on $\mathcal{I}^-$ discussed in Section 4.4. We will return to these variations later in Section 4.3 when discussing supertranslation symmetry generators.

### 4.1.4 What are Supertranslations?

As previously mentioned, supertranslations are an infinite dimensional extension to the usual four spacetime translations contained within the Poincaré group. Studying the form of the generating vector (4.12) we see that in the case of $f = const$ the action reduces to shifts in retarded time $u$ at $\mathcal{I}^+$. For a general $f$ at $\mathcal{I}^+$ we see that the $u$ translation is now angle dependent, therefore each null generator on $\mathcal{I}^+$ is shifted by a different amount depending upon it's $(z, \bar{z})$ coordinates[65].

As $f$ is a function defined upon the two sphere, it can be expanded out in terms of spherical harmonics $Y(l.m)$. Furthermore, one can also recover the standard spatial translations when taking $f$ as the $l = 1$ spherical harmonics. Just like the standard spacetime translations, the supertranslations form an Abelian Lie algebra to leading order. One can see this by taking the Lie bracket between two general supertranslation vectors [63].

Finally, although the supertranslations correspond to a diffeomorphism on the spacetime, they actually transform between physically different spacetimes [63]. Unlike the large gauge symmetries, supertranslations have clear classical effects. This key difference will arise again in Chapter 6 when discussing supertranslation hair on black holes.

## 4.2 Supertranslation Conserved Charges

As with any symmetry group, associated conserved charges can be constructed. Normally is done by applying Noether's theorem. However in the case of BMS transformations at null infinity this does not work, the charges are not actually conserved due to fluxes in gravitational radiation passing through $\mathcal{I}^+$. A general treatment for defining conserved quantities in these situations was developed in [73], see *Flanagan* [25] for a review. In retarded Bondi coordinates the supertranslation charges can be expressed as [65, 63]

$$T^+(f) = \frac{1}{4\pi G} \int_{\mathcal{I}^+_-} d^2z \, \gamma_{z\bar{z}} f(z, \bar{z}) m_B \tag{4.20}$$

This looks remarkably similar to the Gauge case expressed in equation (2.27). Stokes's theorem can once again be applied to express (4.20) as an integral over some Cauchy surface with boundary $\mathcal{I}^+_-$. As with the gauge case, we will assume there are no massive fields present so $\mathcal{I}^+$ constitutes a Cauchy surface. Also, since no massive fields can carry energy out to $i^+$ one would expect all the energy to be radiated away through $\mathcal{I}^+$. The mass aspect $m_B = 0$ as $\mathcal{I}^+_+$ is approached along $\mathcal{I}^+$ and the boundary term at $\mathcal{I}^+_+$ has no contribution. Integrating by parts over $u$ yields

$$\frac{1}{4\pi G} \int_{\mathcal{I}^+} du d^2z \, \gamma_{z\bar{z}} f(z, \bar{z}) \partial_u m_B = \cancel{T^+(f)|_{\mathcal{I}^+_+}} - T^+(f) - \cancel{\frac{1}{4\pi G} \int_{\mathcal{I}^+} du d^2z \, \gamma_{z\bar{z}} \partial_u f(z, \bar{z}) m_B} \tag{4.21}$$

where the second term cancels as $f = f(z, \bar{z})$ deduced in the previous section.

So far, no equations of motion have been specified to determine evolution of the data. Dynamical evolution in General Relativity is governed by the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T^M_{\mu\nu} \tag{4.22}$$

where $R_{\mu\nu}$ is the Ricci Tensor, $R$ the Ricci scalar and $T^M_{\mu\nu}$ is the matter stress energy tensor. As in the Gauge case, we will work to leading order in large $r$. Working through the algebra, the leading order $uu$ component of (4.22) reads [63]

$$\partial_u m_B = \frac{1}{4}\left[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right] - T_{uu} \tag{4.23}$$

where

$$T_{uu} = \frac{1}{4}N_{zz}N^{zz} + 4\pi G \lim_{r\to\infty}\left[r^2 T^M_{uu}\right]$$

is the total energy flux due to matter and gravitational radiation. Equation (4.23) can then be substituted into (4.21) to find

$$\begin{aligned} T^+(f) &= \frac{1}{4\pi G}\int_{\mathcal{I}^+} dud^2z\, \gamma_{z\bar{z}} f(z,\bar{z})\left(T_{uu} - \frac{1}{4}\left[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right]\right) \\ &= \frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z f(z,\bar{z})\gamma_{z\bar{z}}\left(N_{zz}N^{zz} - \left[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right]\right) \end{aligned} \tag{4.24}$$

where we have assumed that no matter fields are present i.e. $T^M_{\mu\nu} = 0$ for simplicity.

Equation (4.24) has a term both quadratic in the Bondi News and a linear contribution. Comparing with the form of the large gauge charge in Section 2.5 we recall that the linear (soft) term in $F_{\mu\nu}$ could be expressed completely in terms of fields living at the boundaries of $\mathcal{I}^+$. Considering the resemblance so far between the gauge and gravity cases it would be interesting to see if the same applied here. To do so we need to motivate boundary conditions on $m_B, C_{zz}$ and $N_{zz}$ at $\mathcal{I}^+_\pm$.

So far we have been dealing with asymptotically flat spacetimes which are vacuum in both the far past and future. These spaces have been defined and categorised by Christodoulou and Klainerman, denoted CK spaces [19]. Here we quote the key properties on the metric components, reviewed in [65].

1. The Bondi News tensor must fall off at a rate $N_{zz}(u) \sim |u|^{-3/2}$ or greater as $u \to \pm\infty$. This ensures both $N_{zz}$ and $C_{zz}$ decay sufficiently for finite energy configurations.

2. The Bondi mass aspect has boundary values $m_B|_{\mathcal{I}^+_+} = 0$ and $m_B|_{\mathcal{I}^+_-} = M$ where $M$ is the ADM mass.

3. $\left[D_z^2 C_{\bar{z}\bar{z}} - D_{\bar{z}}^2 C_{zz}\right]_{\mathcal{I}^+_\pm} = 0$.

Condition three can be satisfied by

$$C_{zz}|_{\mathcal{I}^+_\pm} = C_{zz}(\pm\infty, z, \bar{z}) = D_z^2 C^\pm(z,\bar{z}) \tag{4.25}$$

for some real function $C^{\pm}$ defined on the boundaries of $\mathcal{I}^+$. One can also consider

$$\begin{aligned}\int_{-\infty}^{\infty} du N_{zz} &= \int_{-\infty}^{\infty} du \partial_u C_{zz} \\ &= C_{zz}|_{\mathcal{I}^+_+} - C_{zz}|_{\mathcal{I}^+_-} \\ &= D_z^2 (C^+(z,\bar{z}) - C^-(z,\bar{z})) \\ &\equiv D_z^2 N(z,\bar{z})\end{aligned} \tag{4.26}$$

for another real function $N(z,\bar{z})$. Again this structure is very similar to that found in the gauge case in Section 2.5 where zero frequency modes were defined completely in terms of gauge fields at the boundary.

We can now return to the expression for the supertranslation charge (4.24) to see if the linear term can be expressed in terms of boundary fields. The linear term is

$$\begin{aligned}&-\frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z f(z,\bar{z})\gamma_{z\bar{z}}\left[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right] \\ &= -\frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z f(z,\bar{z})\gamma^{z\bar{z}}\left[D_z^2 N_{\bar{z}\bar{z}} + D_{\bar{z}}^2 N_{zz}\right] \\ &= -\frac{1}{16\pi G}\int d^2z f(z,\bar{z})\gamma^{z\bar{z}}\int_{-\infty}^{\infty} du\left[D_z^2 N_{\bar{z}\bar{z}} + D_{\bar{z}}^2 N_{zz}\right] \\ &= -\frac{1}{8\pi G}\int d^2z f(z,\bar{z})\gamma^{z\bar{z}} D_{\bar{z}}^2 D_z^2 N(z,\bar{z})\end{aligned} \tag{4.27}$$

where in line two we lowered the indices on the Bondi News using $\gamma_{z\bar{z}}$ twice, in line three the $u$ integral was brought in as all other terms only depended upon the angular coordinates and finally, equation (4.26) was used to express the charge purely in terms of the boundary field $N(z,\bar{z})$. Insight can again be drawn from the previous gauge case. Once promoted to an operator the soft gauge charge, linear in the field strength, was seen to excite zero energy soft photons. It turns out that the term in (4.27) excites soft zero energy gravitons when quantised [37]

$$T^+(f) = \frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z f(z,\bar{z})\gamma_{z\bar{z}} N_{zz} N^{zz} - \frac{1}{8\pi G}\int d^2z f(z,\bar{z})\gamma^{z\bar{z}} D_{\bar{z}}^2 D_z^2 N(z,\bar{z}) \tag{4.28}$$

## 4.3 Symplectic Structure at $\mathcal{I}^+$

We now need to verify that the supertranslation charge (4.28) correctly generates the symmetry transforms from section 4.1. To do this one needs the Poisson Brackets or Symplectic Form for General Relativity. As with the gauge field case, one needs to follow the process presented in Appendix A except this time the field variations are variations upon the metric components themselves. Derivations of the Symplectic Form of radiative modes for General Relativity can be found in [44, 22, 3]. In retarded Bondi coordinates it is [65]

$$\Omega = \frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z \gamma^{z\bar{z}} \delta C_{zz} \wedge \overset{\leftrightarrow}{\partial}_u \delta C_{\bar{z}\bar{z}} \tag{4.29}$$

where the wedge product is once again over the symplectic manifold and $\alpha\overleftrightarrow{\partial_u}\beta$ denotes $\alpha(\partial_u\beta) - (\partial_u\alpha)\beta$. Examining the structure of this symplectic form we see that $C_{zz}$ and $N_{\bar{z}\bar{z}}$ are canonically paired with one an other, as are the complex conjugates. Working with the $N_{zz}, C_{\bar{z}\bar{z}}$ pair we can calculate the Poisson Bracket associated to the form

$$-\frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z\gamma^{z\bar{z}}\delta N_{zz}\wedge\delta C_{\bar{z}\bar{z}} \tag{4.30}$$

$$\begin{aligned}\{C_{\bar{w}\bar{w}}(u',w), N_{xx}(u'',x)\} &= -16\pi G\int dud^2z\gamma_{z\bar{z}}\left(\frac{\delta C_{\bar{w}\bar{w}}}{\delta C_{\bar{z}\bar{z}}}\frac{\delta N_{xx}}{\delta N_{zz}} - \frac{\delta N_{xx}}{\cancel{\delta C_{\bar{z}\bar{z}}}}\frac{\cancel{\delta C_{\bar{w}\bar{w}}}}{\delta N_{zz}}\right)\\ &= -16\pi G\int dud^2z\gamma_{z\bar{z}}\delta(u'-u)\delta^2(w-z)\delta(u''-u)\delta^2(x-z)\\ &= -16\pi G\gamma_{w\bar{w}}\delta(u'-u'')\delta^2(w-z)\end{aligned} \tag{4.31}$$

where we have identified $C_{\bar{z}\bar{z}}$ as the configuration variable and $N_{zz}$ its conjugate momentum, allowing us to apply the standard form of the Poisson Bracket. Differentiating with respect to $u'$ and relabelling variables then gives

$$\{N_{\bar{z}\bar{z}}(u,z), N_{ww}(u',w)\} = -16\pi G\gamma_{z\bar{z}}\partial_u\delta(u-u')\delta^2(w-z) \tag{4.32}$$

This is the Poisson Bracket on the radiative (non-zero frequency) phase space for General Relativity. This bracket can now be used to calculate the action of $T^+(f)$ upon the Bondi News, as the second term in (4.28) is expressed purely in terms of boundary fields it does not contribute[2] and we find

$$\begin{aligned}\{T^+(f), N_{ww}(u',w)\} &= \frac{1}{16\pi G}\int dud^2zf\gamma_{z\bar{z}}\{N_{zz}N^{zz}, N_{ww}\}\\ &= \frac{1}{16\pi G}\int dud^2zfN_{zz}\gamma^{z\bar{z}}\{N_{\bar{z}\bar{z}}, N_{ww}\}\end{aligned} \tag{4.33}$$

where we have lowered $N^{zz}$ twice with $\gamma_{z\bar{z}}$ and pulled out $N_{zz}$ from the bracket[3]. The news Dirac bracket (4.32) can then be substituted into (4.33) to give

$$\begin{aligned}\{T^+(f), N_{ww}\} &= -\int dud^2zfN_{zz}\partial_u\delta(u-u')\delta^2(w-z)\\ &= \int dud^2zf\partial_uN_{zz}\delta(u-u')\delta^2(w-z)\\ &= f(w,\bar{w})\partial_{u'}N_{ww}\end{aligned} \tag{4.34}$$

where in the second line we integrated by parts over the variable $u$. The boundary terms do not contribute due to the falloff conditions on $N_{zz}$ as $u\to\pm\infty$. This precisely matches the variation of the news tensor under a supertranslation calculated

[2]From studying the symplectic form (4.29) we see that the boundary fields are not paired with the Bondi news and therefore will fully commute. That we currently have no symplectic partner for $N$ is a hint that the initial calculations are going to fail (c.f. Gauge case)

[3]As $N_{zz}$ is not paired with $N_{ww}$ in the symplectic form (4.29) it commutes and we can therefore pull it out of the bracket.

in (4.19).

To calculate the action upon the expanded phase space containing non radiative modes $C_{zz}$ we first need to integrate the bracket (4.32) over $u$ and $u'$ to find

$$\{C_{\bar{z}\bar{z}}(u,z), C_{ww}(u',w)\} = 8\pi G \gamma_{z\bar{z}} sign(u-u')\delta^2(w-z) \tag{4.35}$$

this can now be used to find $\{T^+(f), C_{ww}\}$. Now both terms in (4.28) will contribute to the bracket. Working through the quadratic term:

$$\begin{aligned}
&\{\frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z f(z,\bar{z})\gamma_{z\bar{z}} N_{zz}N^{zz}, C_{ww}\} \\
&= \frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z f(z,\bar{z})\gamma^{z\bar{z}} N_{zz}\{N_{\bar{z}\bar{z}}, C_{ww}\} \\
&= \frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z f(z,\bar{z})\gamma^{z\bar{z}} \partial_u C_{zz}\partial_u\{C_{\bar{z}\bar{z}}, C_{ww}\} \\
&= \int_{\mathcal{I}^+} dud^2z f(z,\bar{z})\partial_u C_{zz}\delta(u-u')\delta^2(z-w) \\
&= f(w,\bar{w})\partial_{u'}C_{ww}
\end{aligned} \tag{4.36}$$

where we have followed the same method as presented for the Bondi News bracket. Calculating the linear term[4]:

$$\begin{aligned}
&\{-\frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z f(z,\bar{z})\gamma_{z\bar{z}}[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}]\} \\
&= -\frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z f(z,\bar{z})\gamma^{z\bar{z}}\{D_z^2\partial_u C_{\bar{z}\bar{z}} + \cancel{D_{\bar{z}}^2\partial_u C_{zz}}, C_{ww}\} \\
&= -\frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z f(z,\bar{z})\gamma^{z\bar{z}} D_z^2\partial_u\{C_{\bar{z}\bar{z}}, C_{ww}\} \\
&= -\int_{\mathcal{I}^+} dud^2z f(z,\bar{z})D_z^2\delta(u-u')\delta^2(z-w)
\end{aligned} \tag{4.37}$$

Integrating by parts over the two-sphere and evaluating the integrals over the Dirac distributions then yields

$$\begin{aligned}
&= -\int_{\mathcal{I}^+} dud^2z D_z^2 f(z,\bar{z})\delta(u-u')\delta^2(z-w) \\
&= -\int_{\mathcal{I}^+} dud^2z D_z^2 f(z,\bar{z})\delta(u-u')\delta^2(z-w) \\
&= -D_w^2 f(w,\bar{w})
\end{aligned} \tag{4.38}$$

Combining these two terms gives the overall action of $T^+(f)$ on $C_{ww}$

$$\{T^+(f), C_{ww}\} = f(w,\bar{w})\partial_{u'}C_{ww} - D_w^2 f(w,\bar{w}) \tag{4.39}$$

which does **not** match the transformation found in (4.19), the second term is off by a factor of two. The bracket (4.35) does not respect the boundary conditions

[4] We have returned to the expression of the charge as fields over the bulk as this is where the bracket (**??**) is defined.

defined for CK spaces in Section 4.2 and is not valid at the boundaries $u = \pm\infty$. Recall from equation (4.28) the linear term be expressed purely in terms of boundary fields, explaining why the calculation above fails. The bracket must therefore be modified at the boundaries according to Dirac's procedure to respect the constraints [24]. This is completed in [37] and the modified Dirac brackets are shown to generate the correct transformations

$$\begin{aligned}\{T^+(f), N_{zz}\} &= f\partial_u N_{zz}\\ \{T^+(f), C_{zz}\} &= f\partial_u C_{zz} - 2D_z^2 f\\ \{T^+(f), N(z,\bar{z})\} &= 0\\ \{T^+(f), C^-(z,\bar{z})\} &= -2f\end{aligned} \tag{4.40}$$

Just as in the Gauge case, we have found an infinite number of conserved supertranslation charges, one for each arbitrary function $f(z,\bar{z})$ defined upon the two-sphere. These charges can once again be split into a 'soft' linear term and a 'hard' quadratic term when defined over the Cauchy surface $\mathcal{I}^+$. Through careful treatment of the Dirac brackets these charges were shown to correctly generate the action of a supertranslation on the metric components.

Finally, we need to categorise the overall phase space of fields upon $\mathcal{I}^+$. We currently have the data $(C_{zz}, N_{zz}, C(z,\bar{z}), N(z,\bar{z}), m_B)$. First, note that N is just the difference between the boundary fields $C^+$ and $C^-$ and is therefore fully determined. $C^+$ and $C^-$ are boundary terms of the bulk field $C_{zz}$ which is in turn constrained by

$$N_{zz} = \partial_u C_{zz}$$

Integrating this over $u$ determines $C_{zz}$ in terms of the Bondi News up to some integration constant, which can be determined from the initial condition $C_{zz}\big|_{\mathcal{I}^+_-} = D_z^2 C^-$ derived from conditions upon CK spaces. Finally, the Bondi mass aspect is constrained by the equation of motion (4.23), which is fully described in terms of the News Tensor once again up to an initial condition. The boundary conditions upon the mass aspect are another result of working in a CK space, $m_B\big|_{\mathcal{I}^+_-}$ may be taken as the initial condition. Therefore a complete set of data on the Cauchy surface $\mathcal{I}^+$ can be given by [63]:

$$\left(N_{zz}(u,z,\bar{z}), C_{zz}(z,\bar{z})\big|_{\mathcal{I}^+_-}, m_B\big|_{\mathcal{I}^+_-}\right) \tag{4.41}$$

## 4.4 Antipodal Matching and Overall Symmetry Group

All of the analysis presented above was completed at future null infinity ($\mathcal{I}^+$). The same calculations can be completed on initial data located at $\mathcal{I}^-$ to find a seperate, infinite dimensional asymptotic symmetry group $\text{BMS}^-$. These results can be found in [37, 65]. In a region near past null infinity, advanced coordinates are used with the asymptotic expansion

$$\begin{aligned} ds^2 = &-dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &+ \frac{2m_B^-}{r}dv^2 + rD_{zz}dz^2 + rD_{\bar{z}\bar{z}}d\bar{z}^2 - D^z D_{zz}dudz - D^{\bar{z}} D_{\bar{z}\bar{z}}dud\bar{z} + \ldots \end{aligned} \tag{4.42}$$

where the Bondi news tensor $M_{zz}$ is related to the metric component $D_{zz}$ by

$$M_{zz} = \partial_v D_{zz} \tag{4.43}$$

Supertranslations acting at $\mathcal{I}^-$ are then generated by

$$\zeta = f^-\partial_v + \frac{1}{r}\left(D^{\bar{z}} f^- \partial_{\bar{z}} + D^z f^- \partial_z\right) - D^z D_z f^- \partial_r + \ldots \qquad f^- = f^-(z,\bar{z}) \tag{4.44}$$

This leads to a supertranslation symmetry group with generators/charges given by

$$T^-(f^-) = \frac{1}{4\pi G}\int_{\mathcal{I}^-_+} d^2z\ \gamma_{z\bar{z}} f^-(z,\bar{z}) m_B^- \tag{4.45}$$

The mass aspect $m_B^-$ is constrained by the $vv$ component of the Einstein Equations in advanced coordinates

$$\begin{aligned} \partial_v m_B^- &= \frac{1}{4}(D_z^2 M^{zz} + D_{\bar{z}}^2 M^{\bar{z}\bar{z}}) + T_{vv} \\ T_{vv} &= \frac{1}{4} M_{zz} M^{zz} + 4\pi G \lim_{r\to\infty}\left[r^2 T^M_{vv}\right] \end{aligned} \tag{4.46}$$

Integrating (4.45) by parts over $v$ and substituting in the constraint equation (4.46) expresses $T^-(f)$ as an integral over all $\mathcal{I}^-$.

$$\begin{aligned} T^-(f) &= \frac{1}{4\pi G}\int_{\mathcal{I}^-} d^2z dv\ \gamma_{z\bar{z}} f^-(z,\bar{z})(T_{vv} + \frac{1}{4}[D_z^2 M^{zz} + D_{\bar{z}}^2 M^{\bar{z}\bar{z}}]) \\ &= \frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z f^-(z,\bar{z})\gamma_{z\bar{z}}\left(M_{zz}M^{zz} + \left[D_z^2 M^{zz} + D_{\bar{z}}^2 M^{\bar{z}\bar{z}}\right]\right) \end{aligned} \tag{4.47}$$

Finally, a complete set of initial data on $\mathcal{I}^-$ is given by

$$\left(M_{zz}(u,z,\bar{z}), D_{zz}(z,\bar{z})\big|_{\mathcal{I}^-_+}, m_B^-\big|_{\mathcal{I}^-_+}\right) \tag{4.48}$$

As in the gauge case, we would like to find an overall symmetry group for gravitational scattering, which acts on $\mathcal{I}^-$ and $\mathcal{I}^+$ simultaneously. Currently, any scattering process is only known up to independent BMS$^+$ and BMS$^-$ transformations. To do this one must match the initial data (4.48) to the final data (4.41) in both a Lorentz and CPT invariant manner. This was completed in [65] where the below conditions were determined

$$\begin{aligned} m_B^-\big|_{\mathcal{I}^-_+} &= m_B\big|_{\mathcal{I}^+_-} \\ D_{zz}(z,\bar{z})\big|_{\mathcal{I}^-_+} &= C_{zz}(z,\bar{z})\big|_{\mathcal{I}^+_-} \\ \implies f^-(z,\bar{z}) &= f(z,\bar{z}) \end{aligned} \tag{4.49}$$

these conditions break down the separate BMS symmetry groups into a single overall group acting over the whole spacetime simultaneously. The supertranslation generators / conserved charges are then given by

$$T^{-}(f^{-}) = T^{+}(f) = T(f) \tag{4.50}$$

This overall, infinite dimensional symmetry group BMS$^0$ then acts to apply an infinite number of constraints upon any asympotically flat, vacuum to vacuum gravitational scattering.

## 4.5 Discussion

### 4.5.1 Summary of Results

Let us summarise the results obtained in this section so far. BMS discovered, by defining asymptotic flatness as a set of falloff conditions upon the metric components, that there existed two infinite dimensional asymptotic symmetry groups which preserve asymptotic flatness. The BMS$^{\pm}$ groups consist of a Lorentz subgroup and an infinite dimensional, abelian subgroup called supertranslations. The action of a general supertranslation at $\mathcal{I}^{\pm}$ is to shift the advanced/retarded time at every point on the two sphere, determined by an arbitrary function $f(z, \bar{z})$. Conserved charges for the supertranslations can be constructed and were shown to correctly generate the supertranslation action upon metric components. Finally, by limiting to CK spaces and using their properties, matching conditions on the phase space data and functions $f, f^{-}$ were deduced and an overall asymptotic symmetry group BMS$^0$ was found, which constrains any vacuum to vacuum gravitational scattering process.

### 4.5.2 Interpreting $T(f)$

The previous section identified an infinite number of conserved charges constraining any scattering process. The first point of discussion is to identify what these charges actually conserve. Recalling the expressions for $T^{+}(f)$ (4.24) and $T^{-}(f)$ (4.47) and applying the matching conditions we have

$$\begin{aligned} &\int_{\mathcal{I}^+} dud^2z\, \gamma_{z\bar{z}} f(z,\bar{z}) (T_{uu} - \frac{1}{4}[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}]) \\ = &\int_{\mathcal{I}^-} dvd^2z\, \gamma_{z\bar{z}} f(z,\bar{z}) (T_{vv} + \frac{1}{4}[D_z^2 M^{zz} + D_{\bar{z}}^2 M^{\bar{z}\bar{z}}]) \end{aligned} \tag{4.51}$$

For the simplifying case $f = \delta^2(z - w)$ then the integrals over the two sphere can be completed and we are left with [65]

$$\int_{\mathcal{I}^+} du\, \gamma_{z\bar{z}} (T_{uu} - \frac{1}{4}[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}]) = \int_{\mathcal{I}^-} dv\, \gamma_{z\bar{z}} (T_{vv} + \frac{1}{4}[D_z^2 M^{zz} + D_{\bar{z}}^2 M^{\bar{z}\bar{z}}]) \tag{4.52}$$

This equality expresses energy conservation at every angle. The total inbound energy through $\mathcal{I}^-$ at an angle $z$ (in advanced coordinates) is equal to the total energy out through $\mathcal{I}^+$ through the antipodal point $z$ (in retarded coordinates). Note that - just like the large gauge case - the soft and hard charges are not individually conserved [65].

### 4.5.3 Symmetry Breaking and Vacuum Degeneracy

Recall the action of a supertranslation upon the metric components (**??** - **??**)

$$\begin{aligned}
\mathcal{L}_\zeta m_B &= f\partial_u m_B + \frac{1}{4}\left(2D_z f D_z N^{zz} + N^{zz} D_z^2 f + c.c.\right) \\
\mathcal{L}_\zeta C_{zz} &= f\partial_u C_{zz} - 2D_z^2 f \\
\mathcal{L}_\zeta N_{zz} &= f\partial_u N_{zz}
\end{aligned} \tag{4.53}$$

What happens if we choose to supertranslate Minkowski space, given by $C_{zz} = N_{zz} = m_B = 0$?[5] Both $m_B$ and $N_{zz}$ are invariant however, $C_{zz}$ shifts to a non-zero value determined by the generic function $f(z,\bar{z})$. Therefore, the action of a supertranslation changes Minkowski space to another, physically inequivalent, asymptotically flat spacetime. Treating Minkowski as the vacuum, this is another example of spontaneous symmetry breaking, analogous to the large gauge case in Section 2.7. For any broken symmetry there must be a corresponding Goldstone mode. Recalling equations (4.40) we see that the boundary field $C^-(z,\bar{z})$ is introduced and transforms non-trivially under the action of a supertranslation. $-\frac{1}{2}C^-$ is the Goldstone mode corresponding to the broken symmetry, conjugate to the soft graviton mode $N$ [37]. This is analogous to the EM case, where to close the symplectic form we had to introduce the field $\phi(z,\bar{z})$ conjugate to the soft photon mode.

Unlike the gauge case, where the symmetry group was abelian, supertranslated vacua can be classically distinguished from one and other. The supertranslations do not commute with any of the Lorentz generators, therefore charges associated with the Lorentz group such as angular momentum may change under a supertranslation [63]. Although the action of a supertranslation doesn't change the energy in the spacetime it can change the other classical charges, such as angular momentum, therefore the degenerate vacua in General Relativity are classically distinguishable from one and other. Discussions of vacua with non-zero angular momentum can be found in [17].

### 4.5.4 Quantisation

Recalling the Infrared Triangle equivalence, discussed in Section 1.4, one would expect the Asymptotic $\text{BMS}^0$ symmetry is equivalent to a soft theorem upon quantisation. We can attempt to quantise the theory by promoting the charge to an

[5]One can see these conditions specify flat space by returning to the asymptotic expansion of the metric (4.6) in substituting in the conditions

operator, this was completed in [37, 65] where, after expressing the soft term in a plane wave basis, it was shown to annihilate/create zero energy 'soft' gravitons. Gravitons are the hypothetical force carrying particles associated with quantisation of the gravitational field. Although we do not have a consistent theory of quantum gravity, some properties have been deduced. It is thought to be a spin 2, massless particle; see Chapter two of *Kiefer* [43] for further background reading on attempts to quantise the gravitational field.

The massless property of the graviton both explains the long range of gravity and subjects it to the same infrared divergences found in QED. The lack of a lower energy bound required to excite a graviton means zero energy 'soft' gravitons can be excited. Any physical 'hard' gravitational scattering process comes with an infinite dressing of soft gravitons. Just as in the electromagnetic case, Weinberg developed an associated soft graviton theorem which relates scattering amplitudes with and without soft graviton insertions [75].

Finally, the infinite number of conserved charges $T(f)$ carry over to the quantum case of in/out states and $\mathcal{S}$ matrix evolution. In [65] it was shown the classical supertranslation scattering invariance carries over to a series of Ward identities relating scattering processes with soft graviton insertions to those without. Furthermore, in [37] their equivalence to the Weinberg soft graviton theorem was proven.

## 4.6 Recent Developments: Superrotations

The final section in this chapter concerns superrotations, a proposed infinite dimensional extension of the Lorentz group contained within BMS. As we will not be dealing with superrotation symmetry on the black holes[6] only a brief overview and key results are provided. Much of the logical structure follows that of the supertranslation and large gauge cases.

We begin by discussing the Lorentz subgroup of standard BMS. It is shown in [51] that the connected Lorentz group is isomorphic to $SL(2,\mathbb{C})$, the group of conformal transformations on the celestial sphere. The general generating vector of the Lorentz group as an $SL(2,\mathbb{C})$ transformation is [42]

$$\zeta_Y = (1+\frac{u}{2r})Y^z\partial_z - \frac{u}{2r}D^{\bar{z}}D_zY^z\partial_{\bar{z}} - \frac{1}{2}(u+r)D_zY^z\partial_r + \frac{u}{2}D_zY^z\partial_u + c.c \tag{4.54}$$

where $Y^z$ is a two-dimensional vector field defined upon the conformal celestial sphere. As we are interested in transformations at $\mathcal{I}^+$ this simplifies to

$$\zeta_Y\big|_{\mathcal{I}^+} = Y^z\partial_z + \frac{u}{2}D_zY^z\partial_u + c.c \tag{4.55}$$

BMS showed [9], following the same argument as was completed for supertranslations in Section 4.1, that these transformations preserved the asymptotic flatness

[6] it will become evident why soon

(4.7) and Bondi Gauge conditions if $\partial_{\bar{z}} Y^z = 0$. In general this has a solution $Y^z = z^n$ but BMS restricted $Y^z$ to the globally defined solutions $Y^Z = 1, z, z^2, i, iz, iz^2$ only [63]. This reduced the infinite number of generating vectors (4.55) to the 6 usual Lorentz generators associated with boosts and rotations.

Barnich and Troessaert argued to relax these conditions on the basis that on a local scale the symmetries are perfectly well defined regardless of its global properties [5, 4]. This resulted in the infinite dimensional superrotation subgroup. The infinitesimal action of superrotations upon the metric components can be calculated using the same method described for the supertranslations in Section 4.1. Performing the required calculations one finds [42, 63]

$$\begin{aligned}\mathcal{L}_{\zeta_Y} C_{zz} &= \frac{u}{2} D \cdot Y N_{zz} + Y \cdot D C_{zz} - \frac{1}{2} D \cdot Y C_{zz} + 2 D_z Y^z C_{zz} - u D_z^3 Y^z \\ \mathcal{L}_{\zeta_Y} N_{zz} &= \frac{u}{2} D \cdot Y N_{zz} + Y \cdot D N_{zz} + 2 D_z Y^z N_{zz} - D_z^3 Y^z\end{aligned} \tag{4.56}$$

where the News tensor variation is found by simply taking the $u$ derivative of the $C_{zz}$ variation.

As with the supertranslation and large gauge case, there should be an infinite number of generators / conserved charges associated to the superrotation symmetry. To discuss the superrotation charges, we briefly need to diverge back to the asympotic expansion of the metric (4.6). If one goes to the next order in $\frac{1}{r}$ then we find [63]

$$ds^2 = (4.6) + \frac{1}{r}\left(\frac{4}{3}(N_z + u\partial_z m_B) - \frac{1}{4}\partial_z(c_{zz}C^{zz})\right) dudz + c.c. + ... \tag{4.57}$$

where $N_z(u, z, \bar{z})$ is known as the Angular Momentum Aspect, contracting it with a rotational vector field and integrating over the two sphere will yield the total angular momentum. In retarded Bondi coordinates, the superrotation charges acting on $\mathcal{I}^+$ are then [25, 63]

$$Q_Y = \frac{1}{8\pi G}\int_{\mathcal{I}^+_-} d^2z (Y_{\bar{z}} N_z + Y_z N_{\bar{z}}) \tag{4.58}$$

We can once again apply the usual method of using Stokes's theorem and a constraint equation to express these charges as integrals over all of $\mathcal{I}^+$. The constraint equation for $N_z$ is given by the $uz$ component of Einstein's Equation [63]

$$\begin{aligned}\partial_u N_z &= \frac{1}{4}\partial_z(D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}}) - u\partial_u\partial_z m_B - T_{uz} \\ T_{uz} &\equiv 8\pi G \lim_{r\to\infty}\left[r^2 T^M_{uz}\right] - \frac{1}{4}\partial_z(C_{zz}N^{zz}) - \frac{1}{2}C_{zz}D_z N^{zz}\end{aligned} \tag{4.59}$$

Integrating equation (4.58) by parts yields

$$\frac{1}{8\pi G}\int_{\mathcal{I}^+} dud^2z (Y_{\bar{z}}\partial_u N_z + Y_z \partial_u N_{\bar{z}}) = -Q_Y - \frac{1}{8\pi G}\int_{\mathcal{I}^+} dud^2z(\partial_u Y_{\bar{z}} N_z + \partial_u Y_z N_{\bar{z}}) \tag{4.60}$$

where we have assumed that the contribution at $\mathcal{I}^+_+$ relaxes to zero and the second term cancels as the components $Y^z(z,\bar{z})$ are functions of the two sphere only. One then substitutes in the constraint equations for $N_z$ (4.59) and $m_B$ (4.23). After some considerable algebra the charge can eventually be expressed as [63]

$$\begin{aligned} Q_Y &= Q_H + Q_S \\ Q_S &= -\frac{1}{16\pi G}\int_{\mathcal{I}^+} dud^2z\left(D_z^3 Y^z u N_{\bar{z}}^z + D_{\bar{z}}^3 Y^{\bar{z}} u N_z^{\bar{z}}\right) \\ Q_H &= \frac{1}{8\pi G}\int_{\mathcal{I}^+} dud^2z\left(Y_{\bar{z}} T_{uz} + Y_z T_{u\bar{z}} + u\partial_z Y_{\bar{z}} T_{uu} + u\partial_{\bar{z}} Y_z T_{uu}\right) \end{aligned} \tag{4.61}$$

Again we see that the charge has split out into a linear soft term and a quadratic hard term. Using the Dirac brackets defined in (4.40) one can then verify that that charges correctly generate the infinitesimal action upon the metric components (4.56).

Equivalent analysis can also be carried out on $\mathcal{I}^-$ to find the extended BMS$^-$ group. In advanced coordinates the next order metric expansion is

$$ds^2 = (4.42) - \frac{1}{r}\left(\frac{4}{3}(M_z + u\partial_z m_B^-) - \frac{1}{4}\partial_z(D_{zz}D^{zz})\right)dvdz + c.c. + ... \tag{4.62}$$

with conserved charges

$$Q_Y^- = \frac{1}{8\pi G}\int_{\mathcal{I}^+_-} d^2z(Y_{\bar{z}}M_z + Y_z M_{\bar{z}}) \tag{4.63}$$

where $M_z$ is the incoming angular momentum aspect on past null infinity. In [42] it was shown, using the properties of CK spaces that there is an extra antipodal matching condition relating the AM aspects to one another

$$N_z(z,\bar{z})|_{\mathcal{I}^+_-} = M_z(z,\bar{z})|_{\mathcal{I}^-_+} \tag{4.64}$$

and therefore the infinite number of superrotation charges are antipodally conserved.

$$Q_Y = Q_Y^- \tag{4.65}$$

Therefore, by allowing superrotations we have identified another infinity of constraints upon any gravitational scattering process. When $Y^z$ is one of the original 6 conformal killing vectors this matching condition corresponds to conservation of the BORT center of mass [63]. If $Y^z$ is chosen to be a delta function $\delta^2(z-w)$ then the matching condition expresses conservation of angular momentum through every angle, just as supertranslations equate energy through every angle.

Upon quantization the symmetries were shown to be equivalent to an infinite number of Ward identities [42]. Therefore given the infrared triangle there should be an associated soft theorem for this symmetry, providing evidence for a second soft theorem associated to gravity. This is an example of the power of the Infrared Triangle results, the superrotation symmetry pointed to a previously unknown soft theorem. The sub-leading soft graviton theorem was discovered and shown to be equivalent to the superrotation Ward identities in [13, 12]. To complete the triangle, there should also be an associated memory effect. This is known as Gravitational Spin memory and is discussed in [55].

Finally a brief note on the action of the symmetry upon the metric components. If one looks at the transformations (4.56) as $u \to \pm\infty$ we see that the news tensor is transformed to a vector on the two sphere and that the component $C_{zz}$ diverges. This clearly doesn't respect the boundary conditions for CK spaces defined in Section 4.2 [63]. Therefore, the phase space must be extended to close off the action of the symmetry. This has proved to be a very difficult piece of work and studies are ongoing, see Section 6.2 for discussion of recent developments.

# 5 BMS Hair on Black Holes

We are now in a position where results obtained from the previous section can be applied to black hole spacetimes. We are interested to see if, like the gauge case, the existence of the overall extended $BMS^0$ group predicts the existence of previously unknown black hole hair. This work was independently completed in [36, 20]. In this section we review key points of the treatment presented in [36, 63]. Much of the argument follows the same logical structure as the large gauge case presented in Section 3. However, due to the structure of the BMS group there are some key physical differences which we will highlight. Finally, many of the calculations in the BMS case can be extremely long winded. We limit this section to presenting logical understanding behind the calculations and refer the reader to to the literature for further detail.

## 5.1 Classical Supertranslation Hair

We begin by discussing the Extended BMS transformations upon classical black hole spacetimes, the simplest case of which is eternal Schwarzschild illustrated in Figure 5.1. However, because the superrotation subgroup is currently not fully understood it cannot yet be applied to black holes. Therefore the analysis in [36] is restricted to supertranslations. In Chapter 4 all of the asymptotic symmetry analysis was completed within regions close to $\mathcal{I}^{\pm}$, this needs to be extended into the bulk spacetime.

An appropriate coordinate choice is to use Advanced Bondi coordinates $(v, r, \Theta^A)$ as these cover both $\mathcal{I}^-$ and the event horizon $\mathcal{H}^+$. They are not valid in a region close to $\mathcal{I}^+$, one would need to work in a different gauge to analyse this area of the spacetime. The Schwarzschild metric in advanced coordinates is

$$ds^2 = -Vdv^2 + 2dvdr + r^2\gamma_{AB}d\Theta^A d\Theta^B \qquad V \equiv 1 - \frac{2M}{r} \tag{5.1}$$

where $\gamma_{AB}$ is any metric on the unit two-sphere. To analyse Supertranslation symmetry in the presence of black holes we need to understand how the metric components transform. Recall the infinitesimal action of a diffeomorphism generated by $\zeta$ upon a metric component is given by its Lie derivative

$$g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_\zeta g_{\mu\nu} \tag{5.2}$$

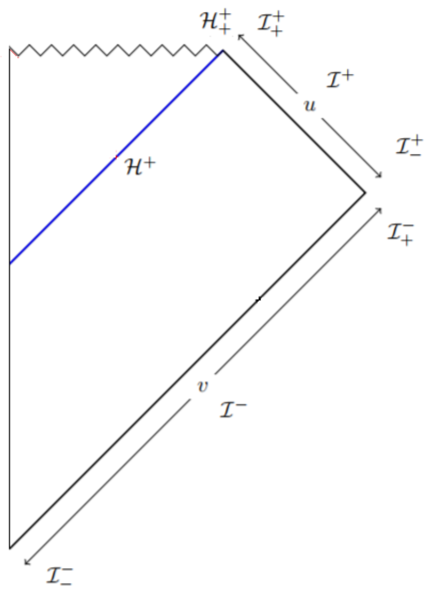


FIGURE 5.1: Penrose diagram of an Eternal Schwarzschild Black Hole

The supertranslation generator (4.44) can then be extended into the bulk to find [36]

$$\zeta = f\partial_v + \frac{1}{r}D^A f \partial_A - \frac{1}{2}D^2 f \partial_r \qquad f = f(\Theta^A) \tag{5.3}$$

To find the infinitesimally transformed Schwarzschild metric we have to compute the Lie derivative of each component

$$\begin{aligned}
\mathcal{L}_\zeta g_{vv} &= \zeta^\rho \partial_\rho g_{vv} + 2 g_{v\rho}\partial_v \zeta^\rho \\
&= \zeta^r \partial_r g_{vv} + 2\cancel{g_{vv}\partial_v \zeta^v} + 2\cancel{g_{vr}\partial_v \zeta^r} \\
&= \frac{MD^2 f}{r^2} \\
\\
\mathcal{L}_\zeta g_{vr} &= \cancel{\zeta^\rho \partial_\rho g_{vr}} + g_{v\rho}\partial_r \zeta^\rho + g_{\rho r}\partial_v \zeta^\rho \\
&= 0 \\
\\
\mathcal{L}_\zeta g_{Av} &= \cancel{\zeta^\rho \partial_\rho g_{Av}} + g_{A\rho}\partial_v \zeta^\rho + g_{v\rho}\partial_A \zeta^\rho \\
&= \cancel{g_{AC}\partial_v \zeta^C} + g_{vv}\partial_A \zeta^v + g_{rv}\partial_A \zeta^r \\
&= -D_A\left(Vf + \frac{1}{2}D^2 f\right) \\
\\
\mathcal{L}_\zeta g_{AB} &= \zeta^\rho \partial_\rho g_{AB} + g_{A\rho}\partial_B \zeta^\rho + g_{\rho B}\partial_A \zeta^C \\
&= 2r D_A D_B f - r\gamma_{AB} D^2 f \\
\\
\mathcal{L}_\zeta g_{rr} &= \mathcal{L}_\zeta g_{rA} = 0
\end{aligned} \tag{5.4}$$

Adding these to the line element 5.1 gives us the transformed Schwarzschild metric

$$\begin{aligned} ds^2 = &- \left(V - \frac{MD^2 f}{r^2}\right) dv^2 + 2dvdr + -D_A(2Vf + D^2 f) dvd\Theta^A \\ &+ (r^2\gamma_{AB} + 2rD_A D_B f - r\gamma_{AB} D^2 f) d\Theta^A d\Theta^B \end{aligned} \tag{5.5}$$

where the event horizon is now located at $r = 2M + \frac{1}{2}D^2 f$ [36]. For a black hole to have supertranslation hair we must be able to distinguish between the original metric (5.1) and the supertranslated metric (5.5). The analysis deals strictly with infinitesimal transformations, therefore any hair discovered will be linearised. We attempt to do this by seeing how the classical supertranslation charges change under the supertranslation. Recall the expression for a supertranslation charge (4.45)

$$T^-(f^-) = \frac{1}{4\pi G} \int_{\mathcal{I}_+^-} d^2\Theta \, \sqrt{\gamma} f(\Theta^A) m_B^- \tag{5.6}$$

where we have restored covariance in the angular coordinates. As the supertranslation charge is dependent upon the Bondi mass aspect we need to see how this transforms under a supertranslation. Applying the method developed in Section 4.1 we need the $\mathcal{O}(r^{-1})$ variation of the $vv$ component of the Schwarzschild metric. Reading off from (5.4) we see that there is no order $\mathcal{O}(r^{-1})$ variation and therefore

$$\mathcal{L}_\zeta m_B^- = 0 \tag{5.7}$$

Supertranslating a black hole does not add any supertranslation charge. This is because the supertranslation algebra is abelian, mirroring the large gauge case discussed in section 2.7.

Unlike the gauge case, the extended BMS group also contains the superrotation subgroup which does not commute with the supertranslations. Therefore we can attempt to distinguish between black holes using the superrotation charges (4.63) they carry. Recalling the superroation charge

$$Q_Y^- = \frac{1}{8\pi G} \int_{\mathcal{I}_+^-} d^2\Theta \, \sqrt{\gamma} Y^A M_A \tag{5.8}$$

where once again we have not specified explicit two-sphere coordinates. To see if a supertranslation changes the superrotation charge we need to calculate the variation of the angular momentum aspect $M_A$. Reading off from (4.62) we see that the variation of $M_A$ corresponds to the $\mathcal{O}(r^-1)$ variation of the $vA$ metric components. The $\mathcal{O}(r^-1)$ variation of $g_{vA}$ can be read off from (5.4) and matching coefficients with (4.62) gives

$$\begin{aligned} \mathcal{L}_\zeta - \frac{2}{3} M_A &= 2MD_A f \\ \implies \mathcal{L}_\zeta M_A &= -3D_A f \end{aligned} \tag{5.9}$$

Plugging (5.9) into (5.8) gives us the change in superrotation charge under an infinitesimal supertranslation

$$\Delta Q_Y^- = -\frac{3}{8\pi G}\int_{\mathcal{I}_+^-} d^2\Theta\ \sqrt{\gamma} Y^A D_A f \tag{5.10}$$

This result states that, using the superrotation charges, we can classically distinguish between black holes which differ by a supertranslation. Here we arrive at another key result of the SHP proposal: **the black hole carries an infinite amount of classical supertranslation hair.** This differs from the gauge case in Section 2.7, where to understand the implications of the asymptotic symmetry we had to include quantum effects.

The results here paint a drastically different picture to the one presented by the No-Hair theorem, where the usual interpretation is that an uncharged black hole can only carry the 10 classical Poincaré hairs. While on the surface it looks like the No-Hair theorem is violated this discrepancy is actually down a common misinterpretation of the theorems. The mathematical statement of the no hair theorem is that "all stationary black hole solutions are diffeomorphic to Kerr spacetimes" [63]. Any BMS transformation is diffeomorphic to the original Kerr family, however the action of the transformation maps between physically inequivalent spacetimes. Therefore the presence of an infinite amount of soft hair does not violate the No-Hair theorem, it was just that these new overall transformations were unknown until recently.

## 5.2 Physical Hair

The existence of an infinite amount of soft hair isn't useful unless it can be excited by a physical process. In [36, 63] the authors show that classical linearised supertranslation hair can be implanted upon a black hole by sending in an asymmetric linearised shockwave of matter at advanced time $v_0$ with energy momentum density

$$T_{vv} = \frac{\mu + T(\Theta)}{4\pi r^2}\delta(v - v_0) \tag{5.11}$$

They then solve this for the linearised metric and show that it is diffeomorphic to Schwarzschild both before and after the shockwave passes. Furthermore, the action of the shockwave is to increase the mass of the black hole by $\mu$ and induce a supertranslation upon its horizon. Therefore, any matter falling across the horizon of a black hole has the effect of implanting supertranslation hair in the form of non-trivial superrotation charges.

## 5.3 Classical Horizon Charges

Recall Section 3.1 where, in the presence of a classical black hole, the future conserved large gauge charge had to have a contribution on the event horizon. The same argument can be applied here. The overall BMS group provides two infinite sets of constraints relating the past and future supertranslation/superrotation charges. In the case of vacuum to vacuum scattering with no massive fields present, it was shown in Section 4.3 that $\mathcal{I}^{\pm}$ could be chosen as Cauchy surfaces and the charges would split into a soft term, associated to soft graviton annihilation/creation, and a hard term which contained the usual stress energy tensor contribution weighted by an arbitrary function.

In the eternal Schwarzschild spacetime illustrated by Figure 5.1, $\mathcal{I}^+$ no longer forms a Cauchy surface. Assuming no massive fields are present $\mathcal{I}^+ \cup \mathcal{H}^+$ can be chosen to be a Cauchy surface and the conservation of supertranslation charge (4.50) becomes

$$T^-(f) = T^+(f) = T^{\mathcal{I}^+}(f) + T^{\mathcal{H}^+}(f) \tag{5.12}$$

There is now a contribution to the overall charge from both future null infinity and the event horizon. In Section 4.3 we demonstrated that $T^{\mathcal{I}^+}(f)$ successfully generated supertranslations upon $\mathcal{I}^+$, one would expect the same to the the case for $T^{\mathcal{H}^+}(f)$. It was shown in [36] that this is indeed the case using roughly the same method applied to the gauge example in Section 3.1. First, the horizon charge is expressed as a boundary charge integral with contributions from $H^+_{\pm}$ [1]. It is then converted to an integral over $\mathcal{H}^+$ using the usual Stoke's theorem and constraint equation method. Once the charge has been computed, the symplectic form and Dirac brackets then have to be constructed at the horizon. Due to the presence of both zero-modes and coordinate freedom this is a technically complicated process, the same fundamental ideas we have developed do apply. Finally, once this has been completed the action of the charge upon the metric components can be verified and shown to correctly generate supertranslations on the horizon.

[1] Only eternal Schwarzschild has both future and past boundaries. In the EM case we discussed a vaiyada black hole which didn't have a past boundary. In the quantum case, the horizon eventually evaporates and therefore does not have a future boundary either. Therefore when taking into account quantum effects there is no sensible definition of 'horizon charge' [63]

# 6 Discussion and Conclusions

## 6.1 Summary of Results

The work completed by Strominger, Hawking, Perry and collaborators, reviewed over the course of this project, has produced two very significant results. These are:

1. **The Vacuum is not unique.**
   In both the Electromagnetism and BMS cases it was shown that the vacuum is not unique but infinitely degenerate. Furthermore the different vacua were found to be able to store information in the form of non trivial angular momentum charge. One can transition between the different vacua by acting with a large gauge or Supertranslation.

2. **Black Holes can Carry Soft Hair.**
   The work has revealed that, even at the classical level, a black hole can carry an infinite amount of soft Supertranslation hair. Supertranslations act on the spacetime to insert graviton zero modes upon the horizon. Supertranslated black holes can then be distinguished from one and other by considering the change in classical superrotation charges, constituting the definition of hair. At the quantum level we find, in addition to quantum supertranslation hair, that black holes can also carry soft large gauge hair. This suggests that the no hair theorems do not hold when considering quantum black holes. Both types of hair can be implanted upon the horizon by a physical process. Uncharged matter crossing the horizon has the effect of introducing supertranslation hair. When the matter is charged it will also have the effect of implanting large gauge hair upon the horizon.

Vacuum uniqueness and the no-hair theorems were two of the key underlying assumptions in the original information loss argument [31]. Both of these assumptions have been questioned and now appear to be invalidated. Clearly this is going to have some bearing upon the information paradox, whether it is enough to actually provide a solution is discussed next.

## 6.2 Applicability to the Information Paradox

We begin by revisiting the information flow of black hole formation and evaporation in light of the new results. Before formation and after evaporation the spacetime is a vacuum, therefore the overall BMS$^0$ and large gauge conservation laws are in effect. For simplicity we restrict ourselves to uncharged, non-rotating matter to form a Schwarzschild black hole. During the formation process, any matter crossing the horizon acts to induce a supertranslation on the spacetime. The effect of this supertranslation is to implant hair upon the horizon in terms of soft gravitons. The soft gravitons carry zero energy and therefore the excitation process is completed without an energy cost [63]. As the black hole evaporates, any quantum of Hawking radiation is constrained by not only by conservation of energy but also conservation of BMS charges. By constraining the radiation, more information is imparted upon it and carried away from the black hole. Eventually, once the black hole has evaporated, the spacetime becomes one of the degenerate vacuum states carrying non-zero BMS charges.

There is potential here for the information paradox to be solved, however to constitute a full solution we must return to the questions posed in Section 1.4 and see if they are adequately answered. First of all, the soft hair must be physical in the sense that when excited it carries non-trivial information about the formation process. Secondly, the hair must be able to carry enough information to account for the Bekenstein-Hawking entropy. If there isn't enough then even with the hair there is still an information loss problem[1]. Finally, the overall evaporation process must result in a pure quantum state to restore unitary evolution.

### 6.2.1 Is the Hair Physical?

Since the soft hair papers [35, 36] were first published a number of response papers have argued that neither the large gauge nor Supertranslation hair is physical [48, 11, 10, 28, 26]. The authors argue that any physical incoming/outgoing hard particle should come with an associated cloud of soft quanta and therefore the overall state is the superposition of the hard state with every possible soft dressing arrangement [26]. When considering these physical dressed states, factorisation theorems may be applied to decouple the soft and hard parts of the $\mathcal{S}$ matrix from each other. The soft scattering essentially becomes trivial to the evaporation process. The appearance of seemingly non-trivial scattering constraints is then down to considering undressed, non physical states in the scattering process. A useful analogy from [11] states *"It is clear that soft (long-wavelength) particles cannot affect hard scattering, for otherwise experiments at the LHC could not be analysed without detailed knowledge of the cosmic microwave background*.

The work presented in [11] shows that the factorisation method is applicable in both the classical and quantum cases. In the classical case they show that the initial data on $\mathcal{I}^-$ and the final data on $\mathcal{I}^+$ can be 'dressed' by performing canonical BMS transformations upon them. After they have transformed to the dressed fields, the symplectic transformation corresponding to time evolution is calculated and they

[1] Essentially we would be in the same position as before where information about the mass, charge and angular momentum was carried away but nothing else.

show that, even after applying the soft constraints, the hard evolution is still completely unconstrained. That is, the existence of any conditions upon the soft modes tell us nothing about the evolution of the hard modes. The same methodology is shown to apply to the quantum case of in/out states and $\mathcal{S}$ matrix evolution.

With both Supertranslation and large gauge hair being ruled out as non-physical, they cannot be significant to the information paradox. Therefore, the remaining hope for the proposition is to find another form of black hole hair that is physical. The first place to look for new hair is still within the BMS group, specifically the superrotation subgroup discussed in Section 4.6. It is expected that the superrotation charges generate an overall local antipodal Virasoro symmetry, restricting to globally defined solutions would then return the $SL(2,\mathbb{C})$ Lorentz subgroup in BMS [36]. Unfortunately, the superrotation symmetry is currently not fully understood at a global level. Until this work has been completed superrotations cannot be applied to black hole spacetimes. Here we provide a brief overview of some of the work attempting to resolve this:

1. In [66] it is argued that a finite superrotation does not act between two globally asymptotically flat spacetimes, rather it maps an asymptotically flat spacetime to a spacetime which is only locally asymptotically flat. Globally this spacetime has isolated defects corresponding to the singularities in the superrotation conformal killing vector. The defects are associated to cosmic strings at null infinity which is broken by black hole pair nucleation. This provides a potential physical interpretation of superrotation action.

2. In [14] the Infrared Triangle equivalence is used to motivate the existence of sub-subleading soft theorems and associated asymptotic symmetries. New large transformations are found which are divergent at the boundaries, suggesting that they do not preserve asymptotic flatness at all. Although in very early stages, these new symmetries may eventually be able to be understood and contribute additional hair.

### 6.2.2 Does the Hair Carry Enough Information?

From the research detailed in the previous section, currently it looks like there is no known physical hair on the black hole horizon. However, it is useful to proceed regardless as lessons learned from the Supertranslation and large gauge cases may be applicable to any physical hair discovered in the future, such as the potential superrotation hair. The overall storage capacity can naturally be broken down into two parts:

1. How much soft hair is present on the black hole horizon?

2. What is the information storage capacity of each soft hair?

From Section 3.2 and references therein we suspect there must be some sort of UV cut-off on soft hair excitations upon the horizon, quantum hair localised to scales smaller than the Planck area cannot be physically excited. This is reassuring as without a UV cut-off the soft hair would be subject to UV divergences where an infinite amount of hair could exist on the horizon, resulting in an infinite storage capacity and violating the Bekenstein Bound. Furthermore, the cut-off suggests the amount of hair is linked to the horizon area in Planck units tying in with the definition of Hawking-Bekenstein entropy. However, currently there is no systematic way to define a UV cut-off and remove disallowed modes. As far as the author is aware, at the time of writing no work has been published attempting to tackle this issue.

To answer the second question, one needs to look at the complexity of the hair being excited on the black hole horizon. In Section 2.8 it was mentioned that a soft photon carries a single degree of freedom. This initially suggests that an excited soft hair carries a single degree of freedom, therefore the overall number of degrees of freedom and hence entropy of the hair is directly tied to the horizon area [35].

Work completed in [68] suggests that simple soft excitations are not the whole story. The authors have managed to show that more structured, twisted soft hair implants can be excited upon the horizon. Furthermore, they suggest that the superposition of these coherent, excited states on the horizon can correspond to a string of QBITs storing information as a hologram on the horizon. The increased complexity of the horizon hair naturally leads to increased information storage. The work was completed for large gauge hair but it was suggested that the analysis could also extend to Supertranslation hair.

Interesting studies would be look at the information storage capabilities of soft photons and gravitons to determine the maximum amount of information which could be encoded into a single quanta of hair. This, combined with a systematic cut off for the maximum number of hairs on the horizon could then provide an upper bound on the information storage capacity of the black hole. The final goal would be to then completely reconstruct the BH entropy formula using this soft hair.

### 6.2.3 Returning Information and Restoring Unitarity

Recall the entropy problem discussed in Section 1.4. During a black hole evaporation process the entanglement entropy strictly increases with Hawking radiation given off, while the Hawking-Bekenstein entropy decreases due to the reduction in horizon area. This leads to a contradiction at the Page time where entanglement entropy becomes larger than the Hawking-Bekenstein entropy and violates the Bekenstein bound. The proposal attempts to deal with this contradiction by suggesting gradual information return through correlations between the Hawking quanta and black hole hair.

If we assume the unitary evolution postulate of quantum theory holds, then any initial pure state going to form a black hole must eventually evaporate into a final state which is also pure. In terms of entanglement entropy $S_{ent}(u = \pm\infty) = 0$, see Figure 6.1. However, we know that black hole evaporation increases entanglement entropy with each pair of Hawking quanta released. Therefore a process must exist

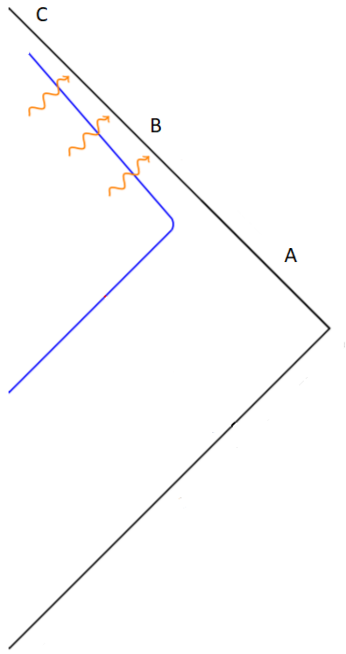


FIGURE 6.1: Hawking Radiation and Entanglement Entropy. For an initial pure state going in to form the black hole the entanglement entropy at (A) is zero. If the overall evolution process is unitary then at late times the state should also be pure and the entanglement entropy at (C) is zero. To fully understand black hole evolution we should also be able to calculate the entanglement entropy at the arbitrary point (B).

which restores unitarity during the evaporation; unless $S_{ent} = 0$ at the point of complete evaporation the information paradox remains unresolved. Furthermore, if we have really understood the black hole evaporation process, we should be able to calculate the entanglement entropy $S_{ent}(u)$ of the released radiation at an arbitrary retarded time $u$ [62]. Again, this problem can be split into two separate parts:

1. A unitary version of the Hawking radiation process needs to be discovered.

2. The function $S_{ent}(u)$ should be a well defined, calculable function.

The potential solution to the first problem suggested by *Page* in [54] and reviewed in [53] involves correlations between early and late time hard Hawking radiation. Consider the example discussed in [48]. An initially stationary black hole emits a quanta of Hawking radiation at early times with 3-momentum **P**. By linear momentum conservation the ingoing Hawking quanta must now carry 3-momentum **-P**, hence the black hole now carries momentum **-P**. Further Hawking quanta will now preferentially emit radiation with the associated momentum, enforcing correlations between the early and late time quanta. If there is enough physical hair present, then in principle the radiation could be constrained enough to release all the information back to the vacuum and restore unitarity. For the case of an initial pure state, the entanglement entropy would initially be zero, increase due to the early time Hawking radiation to some maximum and then decrease due to the late time correlations [40]. Although this sounds feasible in principle, the evolution operator nor the final state are yet to be calculated.

Strominger proposed an alternative method in [61]. The paper suggests that quantum purity may be restored by the emission of soft quanta associated with any hard scattering process, here we briefly review the argument. The original Hawking argument suggests, for a given initial pure state $|\Psi_{in}\rangle$ after evaporation we are left with a density matrix of the form [31]

$$|\Psi_{in}\rangle \rightarrow \rho_{Hawking} = \sum_{\alpha} \rho_{\alpha} |H_{\alpha}\rangle \langle H_{\alpha}| \tag{6.1}$$

where the occupation of numbers of the states $|H_{\alpha}\rangle$ are given by an approximately thermal spectrum. Strominger proposes that the outgoing state be modified to one of the form [61]

$$\mathcal{S} |\Psi_{in}\rangle = \sum_{\alpha} c_{\alpha} |H_{\alpha}\rangle |S_{\alpha}\rangle = |\Psi_{out}\rangle \tag{6.2}$$

where the Hilbert space has been decomposed into hard and soft parts for some arbitrary cut off. Note that the structure of the soft spectrum is currently not understood, this would have to be dealt with for the proposition to hold. It is then assumed that the soft dressing states are orthonormal and by tracing over the soft quanta - which a hypothetical detector couldn't receive - the hard Hawking result is obtained [61]. In the paper an example of a test model was given: a 4D theory containing only the graviton as a stable particle. Although no such theory exists the techniques developed in the paper may also be applicable to theories with particle content more closely resembling the Standard Model, this is suggested for future research.

An attempt at dealing with the second issue has been made in [40]. Returning to Figure 6.1 we notice that the state defined on the $u < b$ section of $\mathcal{I}^+$ describes the Hawking radiation of a partially evaporated black hole. The author attempts to apply the idea behind the Bousso bound[2], which relates the change in area of a null surface to the entropy flux through the surface, to future null infinity $\mathcal{I}^+$. This could provide a method of defining the entanglement entropy of the outgoing state at an arbitrary time, in practice is there are many technical difficulties to overcome. One of which is that the area, and change in area, of a cross sectional cut at $\mathcal{I}^+$ are both infinite. The author works through these problems and proposes an "area-entropy bound" upon $\mathcal{I}^+$.

## 6.3 Outstanding Issues and Future Research

It is clear to see that great progress has been made in tackling the information paradox over recent years. However there are still a number of outstanding issues which

[2]See the references in [40] for sources on the Bousso Bound.

need to be dealt with in future research to have any hope of a full resolution to the paradox, some are discussed in [57] and noted here:

1. **There is no physical soft hair.**
   Currently the most significant issue, all of the newly discovered soft hair is not thought to be of significance to the information paradox. For this proposal to get off the ground, a new type of non-trivial hair must be discovered. The obvious hope lies within the extended BMS group in the form of Superrotations. To apply superrotations to black hole spacetimes the symmetry group and it's phase space must first be fully understood. This is a clear avenue for research and is currently very active.

2. **Information Flow.**
   This is more of a conceptual issue around the proposed information flow. The action of matter falling through the horizon excites soft hair and hence stores information. The question is whether this information is copied or transferred from the original matter. If it were transferred then there must be some process that strips the infalling matter of its information in some sort of 'Quantum Bleaching' effect [64]. This would violate the principle of equivalence, any observer falling through the horizon shouldn't notice anything special. If the information is copied from the infalling matter to the horizon, this would double the amount of information present and is not allowed by a 'No Cloning' theorem [64]. So, how does the information get from the infalling matter to the horizon in an agreeable way?

3. **Global Symmetries.**
   All of the symmetries discussed in the proposal are local only. There is no similar structure for global symmetries such as baryon conservation number therefore information describing global charges is still lost. Surely a resolution to the information paradox should encode and return this information too?

4. **What Happens at the Singularity?**
   While the process does give a potential solution to how information is returned from the black hole, it still doesn't shed any light on what happens once the BH reaches Planckian scales just before evaporation. Understanding this will be important to restoring unitarity of the overall evolution and quantum gravity.

## 6.4 Conclusions

While ultimately the study of soft black hole hair hasn't resolved the information paradox, by invalidating two of the key assumptions in Hawking's original information loss argument [31] it has opened up a number of new avenues for research. Furthermore, a comprehensive framework has been developed to analyse and understand any new potential types of soft hair arising in the future. Whether the information paradox can still be formulated without these assumptions remains to be seen. The author suspects that this will not be the case and, whether through soft black hole hair or other means, a method will be found that gradually returns information back to the vacuum as it evaporates; preserving the unitary evolution postulate of quantum theory.

# A Hamiltonian Formalism and Symplectic Structures

Much of the literature around the SHP proposal discusses field theories through a symplectic formalism. This appendix is designed to give a lightning overview of the symplectic structure of field theories in the Hamiltonian formalism, links to further reading are cited throughout. We begin with a brief recap of discrete Hamiltonian Mechanics where the phase space and Poisson bracket structure are introduced. The symplectic structure of Hamiltonian Mechanics is then discussed before generalising to field theories.

## A.1 A Brief Review of Hamiltonian Mechanics

The Hamiltonian form of mechanics is an alternative formalism to that of Lagrangian Mechanics. Most undergraduate classical mechanics textbooks will provide a comprehensive overview, here we just note down the formulae needed for later sections, using *Nakahara* as a reference [50]. Starting from an action

$$S = \int dt L(q, \dot{q}) \tag{A.1}$$

the Hamiltonian is defined by a Legendre transform on the Lagrangian of the form

$$(q, p) = \sum_k p_k \dot{q}_k - L(q, \dot{q}) \tag{A.2}$$

where $p_k = \frac{\partial L(q,\dot{q})}{\partial \dot{q}_k}$ is the associate conjugate momentum to $q_k$. The space covered by the set of coordinates $\{q_k\}$ is called the **Configuration Space** and the space covered by both $(q_k, p_k)$ is the **Phase Space**. It is easy to see that given a $k$ dimensional configuration space, the phase space will have dimension $2k$. Applying the variational principle to the Hamiltonian then yields Hamilton's equations of motion.

$$\frac{\partial H}{\partial p_k} = \dot{q}_k, \quad \frac{\partial H}{\partial q_k} = -\dot{p}_k \tag{A.3}$$

The equations of motion can also be expressed using a Lie bracket structure called the **Poisson Bracket**. The Poisson bracket between two functions on phase space is given by

$$[A,B] = \sum_k \left( \frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial q_k} \right) \tag{A.4}$$

Using the Poisson bracket the equations of motion then become

$$\frac{\mathrm{d}p_k}{\mathrm{d}t} = [p_k, H] \quad \frac{\mathrm{d}q_k}{\mathrm{d}t} = [q_k, H] \tag{A.5}$$

Finally, by **Noether's Theorem** any transformation that leaves the Hamiltonian invariant leads to an associated conserved charge Q. This conserved charge then acts as the generator of the infinitesimal symmetry transformation.

$$q_i \to q_i' = q_i + \epsilon f_k(q) \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = [Q, H] = 0 \qquad \epsilon[q_i, Q] = \epsilon f_i(q) = \delta q_k \tag{A.6}$$

Throughout this work there are a number of occasions where we want to verify that a derived conserved charge generates the correct asymptotic symmetry transformation. We do this by first calculating the bracket structure and then applying these brackets to our charges and fields, see Sections (xx) for specific examples. When quantising a given theory, two key steps are to promote the symmetry generators to quantum operators and derive the canonical commutation relations. For canonical coordinates the Poisson brackets can be promoted to canonical commutation relations simply by including a factor of $i\hbar$ in the result. In natural units $\hbar = 1$ and is therefore suppressed.

## A.2 Symplectic Structures

We begin this section by defining the symplectic manifold, symplectic structure and canonical coordinates. It is then shown that the Phase Space of Hamiltonian mechanics naturally admits a symplectic structure, this structure allows us to construct Poisson brackets in a coordinate free manner. In the case of canonical coordinates we recover the standard definition of a Poisson bracket between two functions on the phase space. A comprehensive review on the symplectic formalism of classical mechanics can be found within *Arnold* [2]. To proceed we will need the following definitions:

**Definition A.2.1.** [2] Let V be an n-dimensional manifold:

1. A **tangent vector** to V at the point **x** is the velocity vector of a curve in V

   $$\dot{\mathbf{x}} = \lim_{x \to 0} \frac{\phi(t) - \phi(0)}{t}$$

   where $\phi(0) = \mathbf{x}$, $\phi(t) \in V$.

2. The set of all tangent vectors to V at **x** forms the **tangent space to V at** $x$**, denoted** $T_x V$.

3. A 1-form on the tangent space to V at a point **x** is called a **cotangent vector to V at x**.

4. The set of all cotangent vectors to V at **x** forms an n-dimensional vector space dual to the tangent space $T_x V$ called the **cotangent space to V at x**, denoted $T^* V_x$.

5. The union of the cotangent spaces to the manifold at all of its points is called the **cotangent bundle of V** and is denoted $T^* V$. $T^* V$ has a natural structure of a differential manifold of dimension 2n.

**Definition A.2.2.** [2] Let $M^{2n}$ be an even-dimensional differentiable manifold. A **symplectic Structure or Form** on $M^{2n}$ is a closed differential 2-form $\Omega$ on $M^{2n}$ such that:

1. $\Omega$ is closed, $d\Omega = 0$

2. $\Omega$ is non-degenerate: $\forall \zeta \neq 0 \, \exists \eta : \, \Omega(\zeta, \eta) \neq 0 \quad (\zeta, \eta \in T_x M$, the tangent space to the Manifold at $x$).

$(M^{2n}, \Omega)$ is called a *Symplectic Manifold.* If the symplectic form has the structure $\Omega = \sum_1^n dp^i \wedge dq^i$ then the set of coordinates $(q^1, ..q^n, p^1.., p^n)$ are said to be **canonical**. [18]

In the case of classical mechanics, a configuration manifold is described by the set $(q^k)$. The set of velocities $\dot{q}^k$ forms a tangent space and the conjugate momenta $p_k$ forms a cotangent space at $q$. The phase space $(p_k, q^k)$ is therefore a cotangent bundle of the configuration space.

**Theorem A.2.1.** *[2] The cotangent bundle $T^* V$ has a natural symplectic structure. In the local coordinates $(p_k, q^k)$, the symplectic form is given by $\Omega = \sum_k dp_k \wedge dq^k$*

Using Theorem A.2.1 it can immediately be seen that all classical mechanical systems naturally admit an underlying structure. Furthermore, from the structure of the symplectic form and Definition A.2.2 we see that the coordinates $(q^k, p_k)$ are canonical. For canonical coordinates, the symplectic form pairs configuration variables with their conjugate momenta.

For canonical symplectic forms we can also define the *canonical one-form* $\theta = q^\mu dp_\mu$ such that $\Omega = d\theta$. The Poisson bracket on phase space can then constructed in a coordinate independent manner via the definition [50]

$$i_{X_f}(i_{X_g}\Omega) = \{f, g\} \tag{A.7}$$

where

$$X_f = \frac{\partial f}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial f}{\partial q^\mu}\frac{\partial}{\partial p_\mu} \tag{A.8}$$

is the **Hamiltonian vector field** and $i_X$ is the **interior product** of a differential form. When the coordinates are canonical we recover the standard definition of the bracket (A.4) [50]. This is a key result we will use throughout the work, if we find that the symplectic form has a canonical structure we will simply apply the standard Poisson bracket definition.

## A.3 Hamiltonian Field Theories

To extend the above formalism to field theories on a general manifold $M$ we note the following changes, see [27] for further details.

1. The discrete configuration variables $q_k(t)$ become continuous fields $\phi_A(t, \mathbf{x})$ where A is a discrete index labelling the field. The manifold of field configurations is denoted $\mathcal{F}$.

2. Sums over discrete coordinates are replaced with an integral over some Cauchy surface $\Sigma$ [27]

3. The action is now given in terms of a *Lagrangian Density* $\mathcal{L}$

$$S = \int d^4x \sqrt{-g}\mathcal{L}(g_{\mu\nu}; R_{\mu\nu\rho\sigma}, \nabla_\mu R_{\nu\rho\sigma\tau}, ...; \phi_A(x), \nabla_\mu \phi_A(x), ...)$$

   where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor and "..." indicates a finite number of higher order derivatives of the configuration fields [74].

4. All partial derivatives are replaced by functional derivatives of the fields. Therefore the conjugate momentum $p_k(t)$ becomes a conjugate field

$$\pi^A(x) = \frac{\delta \mathcal{L}}{\delta \dot{\phi}_A(x)}$$

5. Before obtaining a non-degenerate symplectic form on the physical phase space one must first construct a pre-symplectic form on the configuration manifold $\mathcal{F}$. This form is degenerate whenever a gauge symmetry exists in the theory [22]. The physical phase space and non-degenerate symplectic form can then be obtained by a reduction procedure detailed in [44].

Applying the above to our definitions of the canonical pre-symplectic two-form and Poisson bracket we find

$$\begin{aligned} \Omega &= \int_\Sigma \delta\pi^A \wedge \delta\phi_A \\ \{A(\pi, \phi), B(\pi, \phi)\} &= \int_\Sigma \frac{\delta A}{\delta\phi}\frac{\delta B}{\delta\pi} - \frac{\delta A}{\delta\pi}\frac{\delta B}{\delta\phi} \end{aligned} \tag{A.9}$$

To best illustrate how one constructs a pre-symplectic form from a given field theory we will work through an example following the methods presented in [44, 22].

Consider a single scalar field in flat space with metric $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$. The Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi) \tag{A.10}$$

Performing a variation of the action yields

$$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial\phi} - \nabla_\mu\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\phi)}\right)\delta\phi + \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\phi)}\delta\phi\right) \tag{A.11}$$

Where the first term vanishes when $\phi$ satisfies the E-L equations of motion. If we compare variation to the definition provided by *Wald* [44] we can read off the **pre-symplectic potential current density** $\Theta^\mu$

$$\delta\mathcal{L} = E_a\delta\phi^a + \nabla_\mu\Theta^\mu \quad \Longrightarrow \quad \Theta^\mu = \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\phi)}\delta\phi = -\eta^{\mu\nu}\nabla_\nu\phi\delta\phi = -\nabla^\mu\phi\delta\phi \tag{A.12}$$

**The pre-symplectic current** $\omega^\mu$ is then calculated by an anti-symmetrised variation of $\Theta^\mu$ [44]

$$\omega^\mu = \delta_1\Theta^\mu(\phi, \delta_2\phi) - \delta_2\omega^\mu(\phi, \delta_1\phi) \tag{A.13}$$

Applying this to our scalar field example we find

$$\omega^\mu == -(\delta_1\nabla^\mu\phi\delta_2\phi - 1 \leftrightarrow 2) = -\delta\nabla^\mu\phi \wedge \delta\phi \tag{A.14}$$

where in the last expression we have written the current as a wedge product between variations on the symplectic manifold. this is NOT a wedge product between the two fields. The pre-symplectic form is then defined by integrating $\omega$ over a Cauchy Surface

$$\begin{aligned}\Omega_\Sigma(\phi, \delta_1\phi, \delta_2\phi) &= \int d\Sigma_\mu\omega^\mu \\ &= \int d^3x\, \delta\dot{\phi} \wedge \delta\phi\end{aligned} \tag{A.15}$$

In the last equality a spacelike Cauchy surface $t = 0$ was chosen. Recalling the conjugate momentum to a scalar field is given by $\dot{\phi}$ we see that we have recovered the canonical symplectic form for a scalar field theory. As the symplectic form is canonical we can apply the standard Poisson bracket definition (A.9) to find

$$\begin{aligned}\{\phi(\mathbf{u}), \dot{\phi}(\mathbf{v})\} &= \int d^3x\, \frac{\delta\phi(\mathbf{u})}{\delta\phi(\mathbf{x})}\frac{\delta\dot{\phi}(\mathbf{v})}{\delta\dot{\phi}(\mathbf{x})} - \frac{\delta\phi(\mathbf{u})}{\delta\dot{\phi}(\mathbf{x})}\frac{\delta\dot{\phi}(\mathbf{v})}{\delta\phi(\mathbf{x})} \\ &= \int d^3x\, \delta^3(\mathbf{u}-\mathbf{x})\delta^3(\mathbf{v}-\mathbf{x}) \\ &= \delta^3(\mathbf{u}-\mathbf{v}) \\ \{\phi(\mathbf{u}), \phi(\mathbf{v})\} &= \{\dot{\phi}(\mathbf{u}), \dot{\phi}(\mathbf{v})\} = 0\end{aligned} \tag{A.16}$$

as expected.

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