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JT gravity and the asymptotic Weil–Petersson volume

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ABSTRACT

A path integral in Jackiw–Teitelboim (JT) gravity is given by integrating over the volume of the moduli of Riemann surfaces with boundaries, known as the “Weil–Petersson volume,” together with integrals over wiggles along the boundaries. The exact computation of the Weil–Petersson volume $V_{g,n}(b_1, \dots, b_n)$ is difficult when the genus g becomes large. We utilize two partial differential equations known to hold on the Weil–Petersson volumes to estimate asymptotic behaviors of the volume with two boundaries $V_{g,2}(b_1, b_2)$ and the volume with three boundaries $V_{g,3}(b_1, b_2, b_3)$ when the genus g is large. Furthermore, we present a conjecture on the asymptotic expression for general $V_{g,n}(b_1, \dots, b_n)$ with n boundaries when g is large.

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1. Introduction

Jackiw–Teitelboim (JT) gravity [1,2] is a two-dimensional (2d) quantum gravitational theory. In the case of bosonic JT gravity, the path integral is essentially determined [3] from the volume of the moduli of hyperbolic Riemann surfaces, otherwise known as the “Weil–Petersson volume,” and the path integrals over “wiggles” along boundaries of Riemann surfaces [4–6] when these surfaces have a boundary. Schwarzian theory [7–9] controls the wiggles. Therefore, Sachdev–Ye–Kitaev (SYK) models [10,7,9] are related to JT gravity, as the SYK models are described by the one-dimensional Schwarzian theory at low energies. Recent progress of JT gravity can be found, e.g., in [11,3,12–24].

The genus g partition function with n boundaries, $Z_{g,n}(\beta_1, \dots, \beta_n)$, is given by an integral over a function involving the Weil–Petersson volume of the moduli of Riemann surfaces of genus g with n boundaries [3].

A Hermitian-matrix integral computes the JT gravity path integral [3]. This is owing to the fact that Mirzakhani’s recursion relation [25], which holds for the Weil–Petersson volumes of the moduli of the Riemann surface on the JT gravity side, corresponds [26] to the topological recursion of Eynard and Orantin [27], which yields the genus expansion of a Hermitian-matrix integral as discussed in [28].

In JT gravity, connected correlators are given as the sums over genus g JT path integrals [3]:

$$\langle Z(\beta_1) \dots Z(\beta_n) \rangle_c \simeq \sum_{g=0}^{\infty} \frac{Z_g(\beta_1, \dots, \beta_n)}{(e^{S_0})^{2g-2+n}}. \quad (1)$$

The genus g partition function with n boundaries, $Z_g(\beta_1, \dots, \beta_n)$ is an integral over a function involving the Weil–Petersson volume $V_{g,n}$ [3]

$$Z_g(\beta_1, \dots, \beta_n) = \alpha^n \prod_{i=1}^n \int_0^{\infty} b_i db_i V_{g,n}(b_1, \dots, b_n) \prod_{j=1}^n Z_{\text{Sch}}^{\text{trumpet}}(\beta_j, b_j). \quad (2)$$

Therefore, connected correlators and path integrals in JT gravity can be computed when the Weil–Petersson volume $V_{g,n}(b_1, \dots, b_n)$ is known (together with Schwarzian theory along the boundaries). For this reason, the evaluation of the Weil–Petersson volume $V_{g,n}(b_1, \dots, b_n)$ has a physical importance.

In this work, we aim to estimate the asymptotic behavior of the Weil–Petersson volume $V_{g,2}(b_1, b_2)$ with genus g and two boundaries of geodesic lengths b_1 and b_2 when g is large. We also estimate the asymptotic behavior of the Weil–Petersson volume with three boundaries, $V_{g,3}(b_1, b_2, b_3)$, when g is large. Furthermore, we extend these results to propose a conjecture concerning an asymptotic expression for $V_{g,n}(b_1, \dots, b_n)$. This study focuses on bosonic JT gravity.

In mathematics, Mirzakhani’s recursion relation [25] is known for the Weil–Petersson volume $V_{g,n}(b_1, \dots, b_n)$, which provides a method for computing the Weil–Petersson volume recursively. Although computational results are known up to some finite genus g and up to some finite number of boundaries n , the computa-

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tion becomes difficult with increasing g . In this note, we study the asymptotic behavior of the Weil–Petersson volume $V_{g,n}(b_1, \dots, b_n)$ when the genus g of Riemann surfaces becomes large.¹

Mirzakhani's recursion relation expresses the derivative of $b_1 V_{g,n}$ in terms of $V_{g',n'}$, where $2g' + n'$ is less than $2g + n$. However, this expression necessarily involves $V_{g-1,n+1}$. For this reason, when g is large, it is considerably difficult to deduce the asymptotic expression for $V_{g,n+1}$ from that of $V_{g,n}$ recursively. To resolve this difficulty, we utilize partial differential equations [30,31] that hold for the Weil–Petersson volume $V_{g,n}$. This approach yields $V_{g,n+1}$ from $V_{g,n}$ up to the leading order in g , when the genus g is sufficiently large. This approach might be useful in evaluating the JT path integral and connected correlators in bosonic JT gravity.

This report is structured as follows. Section 2 summarizes our strategy to estimate the asymptotic behavior of the Weil–Petersson volume $V_{g,n}$ ($n \geq 2$) when the genus g is large. In Section 3.1, we provide an explicit calculation for $V_{g,2}(b_1, b_2)$ when the genus g is large. A leading term in g is obtained, and a subleading correction $\sim 1/g$ is also discussed. In Section 3.2, we estimate the asymptotic expression for $V_{g,3}(b_1, b_2, b_3)$, namely the Weil–Petersson volume with genus g and three geodesic boundaries of lengths b_1, b_2, b_3 , when g is large. We also provide a conjecture on the asymptotic expression for $V_{g,n}(b_1, \dots, b_n)$ when g is large. In Section 4 we comment on the asymptotic Weil–Petersson volume when the geodesic lengths b_1, \dots, b_n are large, and we mention a related question. When $n = 1$, the Weil–Petersson volume in this region was predicted in [3]. Section 5 closes with concluding remarks and an outlook on some open problems.

2. Summary of our strategy to estimate asymptotic

$V_{g,n}(b_1, \dots, b_n)$

As noted in the Introduction, we analyze the asymptotic behavior of the Weil–Petersson volume $V_{g,n}(b_1, \dots, b_n)$ when the genus g is large. Here, $V_{g,n}(b_1, \dots, b_n)$ denotes the Weil–Petersson volume of the moduli of genus g Riemann surfaces with n boundaries of geodesic lengths b_1, \dots, b_n . The evaluation of the Weil–Petersson volume enables the computation of the JT path integral and of the connected correlators [3].

In performing this analysis, if the asymptotic expression for the volume $V_{g,n}$ is known for a large enough g , can the asymptotic expression be deduced for the volume $V_{g,n+1}$?

In principle, Mirzakhani's recursion formula [25] provides a method for computing $V_{g,n}$ for every g and n , starting from $V_{0,3}$ and $V_{1,1}$, recursively. However, computing $V_{g,n}$ precisely and directly from the recursion formula is considerably difficult when g is large. When the asymptotic expression for $V_{g,n}$ is concerned, there is the additional difficulty of estimating $V_{g,n+1}$ from $V_{g,n}$ from the recursion formula. Mirzakhani's recursion formula expresses $\partial_{b_1} b_1 V_{g,n+1}(b_1, \dots, b_{n+1})$ as the sum of the integrals of $V_{g,n}$ times a function, the products $V_{g_1, n_1} V_{g_2, n_2}$ times a function (where g_1 and g_2 add up to g , and n_1 and n_2 add to $n + 2$), and $V_{g-1, n+2}$ times a function. Therefore, to estimate the asymptotic behavior of $V_{g,n+1}$ from $V_{g,n}$ via the recursion formula, one needs to know the asymptotic behavior of $V_{g-1, n+2}$ where g is large.

To avoid this difficulty, our approach utilizes partial differential equations that hold for $V_{g,n}(b_1, \dots, b_n)$, as deduced in [30,31]. The Weil–Petersson volumes satisfy the following partial differential equations [30,31]:

$$\begin{aligned} \partial_{n+1} V_{g,n+1}(b_1, \dots, b_n, 2\pi i) &= 2\pi i(2g - 2 + n) V_{g,n}(b_1, \dots, b_n) \\ \partial_{n+1}^2 V_{g,n+1}(b_1, \dots, b_n, 2\pi i) &= \sum_{j=1}^n b_j \partial_j V_{g,n}(b_1, \dots, b_n) \\ &\quad - (4g - 4 + n) V_{g,n}(b_1, \dots, b_n). \end{aligned} \quad (3)$$

Here, ∂_j represents the derivative with respect to b_j , where $j = 1, \dots, n + 1$.

Because we focus on the asymptotic expressions for $V_{g,n}$, we impose the following asymptotic conditions on the genus g and the geodesic lengths of the boundaries, b_1, \dots, b_n :

$$g \gg 1 \quad g \gg b_i \quad (i = 1, \dots, n). \quad (4)$$

Then, we may replace the second differential equation in (3) with the following reduced equation under the asymptotic conditions (4):

$$\partial_{n+1}^2 V_{g,n+1}(b_1, \dots, b_n, 2\pi i) = -(4g - 4 + n) V_{g,n}(b_1, \dots, b_n). \quad (5)$$

The partial differential equations (3) impose highly nontrivial constraints on the asymptotic expressions for the Weil–Petersson volumes $V_{g,n}$. For example, when $n = 0$, the first differential equation in (3) becomes [30,31]

$$V'_{g,1}(2\pi i) = 2\pi i(2g - 2) V_{g,0}. \quad (6)$$

Asymptotic expressions for $V_{g,0}$ and $V_{g,1}(b)$ were predicted in [3] from the matrix-integral analysis using a contour integral for large g . One can verify that, when $g \gg 1$ and $g \gg b$, the formulas for $V_{g,0}$ and $V_{g,1}(b)$ in [3] satisfy the equation (6). This provides a consistency check of the asymptotic expressions for $V_{g,0}$ and $V_{g,1}(b)$ predicted in [3].

Here, we estimate $V_{g,2}(b_1, b_2)$ when g is large by applying (3), (5) to the asymptotic expression for $V_{g,1}(b)$ obtained in [3]. The differential equations in (3) and (5) are highly effective for estimating $V_{g,2}$ when this is asymptotic in g . Furthermore, applying (3), (5) to the deduced asymptotic expression for $V_{g,2}(b_1, b_2)$, we also estimate $V_{g,3}(b_1, b_2, b_3)$ when g is large. The iteration of this process leads us to a conjecture on the asymptotic expression for $V_{g,n}(b_1, \dots, b_n)$ when g is large.

One can compute large genus contributions to the JT path integral and the connected correlators from the deduced expressions.

3. Asymptotic behavior of the Weil–Petersson volume

$V_{g,n}(b_1, \dots, b_n)$ with large genus g

3.1. Asymptotic $V_{g,2}(b_1, b_2)$

We estimate the asymptotic expression for $V_{g,2}(b_1, b_2)$ when the genus g is large, using the partial differential equations (3) deduced in [30,31]. When the Riemann surface has two boundaries, the differential equations (3) become

$$\begin{aligned} \partial_2 V_{g,2}(b_1, 2\pi i) &= 2\pi i(2g - 1) V_{g,1}(b_1) \\ \partial_2^2 V_{g,2}(b_1, 2\pi i) &= b_1 \partial_1 V_{g,1}(b_1) - (4g - 3) V_{g,1}(b_1). \end{aligned} \quad (7)$$

As stated in Section 2, we impose asymptotic conditions on the genus g and geodesic lengths b_1, b_2 as follows:

$$g \gg 1 \quad g \gg b_1, b_2. \quad (8)$$

The second equation in (7) becomes reduced to the following equation under these conditions:

¹ There are conjectures by Zograf [29] for $V_{g,n}$.

$$\partial_2^2 V_{g,2}(b_1, 2\pi i) = -(4g - 3)V_{g,1}(b_1). \tag{9}$$

From a physical argument on 2d topological gravity under the condition $g \gg 1$ and $g \gg b_1$, the asymptotic form of $V_{g,1}(b_1)$ is predicted to become [3]

$$V_{g,1}(b_1) \sim \frac{4(4\pi^2)^{2g-\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}} \Gamma(2g - \frac{3}{2}) \frac{\sinh(\frac{b_1}{2})}{b_1}. \tag{10}$$

From this $V_{g,1}(b_1)$, we estimate the asymptotic expression for $V_{g,2}(b_1, b_2)$ when the conditions (8) are satisfied. It is well known in mathematics that any $V_{g,n}(b_1, \dots, b_n)$ is symmetric about b_1, \dots, b_n . This implies in particular that $V_{g,2}$ must be symmetric about b_1 and b_2 . Utilizing this symmetry property of the Weil–Petersson volume and the first equation in (7), one is naturally led to considering the following asymptotic expression² for $V_{g,2}(b_1, b_2)$:

$$V_{g,2}(b_1, b_2) \sim 4\sqrt{\frac{2}{\pi}} (4\pi^2)^{2g-1} \Gamma(2g - \frac{3}{2})(2g - 1) \frac{\sinh(\frac{b_1}{2})}{b_1} \frac{\sinh(\frac{b_2}{2})}{b_2}. \tag{11}$$

When g is large ($g \gg 1$), one can replace $\Gamma(2g - \frac{3}{2})(2g - 1)$ with $\Gamma(2g - \frac{1}{2})$ because

$$\frac{\Gamma(2g - \frac{3}{2})(2g - 1)}{\Gamma(2g - \frac{1}{2})} = 1 + \frac{1}{4g - 3} \rightarrow 1 \tag{12}$$

as g tends toward infinity. Thus, a natural asymptotic expression for $V_{g,2}(b_1, b_2)$ under the conditions (8) is

$$V_{g,2}(b_1, b_2) \sim 4\sqrt{\frac{2}{\pi}} (4\pi^2)^{2g-1} \Gamma(2g - \frac{1}{2}) \frac{\sinh(\frac{b_1}{2})}{b_1} \frac{\sinh(\frac{b_2}{2})}{b_2}. \tag{13}$$

The coefficient is chosen to satisfy the first equation in (7).

Confirming whether the asymptotic expression (13) actually satisfies the reduced differential equation (9) yields a nontrivial check. One can confirm that (13) indeed satisfies equation (9) as follows: when our expression (13) is substituted into the left-hand side of (9), we obtain

$$\begin{aligned} &\partial_2^2 V_{g,2}(b_1, 2\pi i) \\ &\sim 4\sqrt{\frac{2}{\pi}} (4\pi^2)^{2g-1} \Gamma(2g - \frac{1}{2}) \frac{\sinh(\frac{b_1}{2})}{b_1} \partial_2^2 \frac{\sinh(\frac{b_2}{2})}{b_2} \Big|_{b_2=2\pi i} \tag{14} \\ &= -4\sqrt{\frac{2}{\pi}} (4\pi^2)^{2g-2} \Gamma(2g - \frac{1}{2}) \frac{\sinh(\frac{b_1}{2})}{b_1}. \end{aligned}$$

Substituting (10) into the right-hand side of (9) yields

² Because the Weil–Petersson volume $V_{g,n}(b_1, \dots, b_n)$ is given as an integral of the exterior power of the Weil–Petersson symplectic form over the moduli space of complete hyperbolic surfaces of the genus g with n boundaries, one naturally expects that the asymptotic expression for the Weil–Petersson volume is given in terms of hyperbolic functions in the large g limit with $g \gg b_1, b_2$. Other seemingly straightforward candidate functions (in terms of hyperbolic functions symmetric under exchange of b_1 and b_2) do not satisfy equations (7) to the leading order in g .

$$\begin{aligned} &-(4g - 3)V_{g,1}(b_1) \\ &= -\frac{4(4\pi^2)^{2g-\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}} \Gamma(2g - \frac{3}{2})(4g - 3) \frac{\sinh(\frac{b_1}{2})}{b_1} \\ &= -4\sqrt{\frac{2}{\pi}} (4\pi^2)^{2g-2} \frac{4g - 3}{2} \Gamma(2g - \frac{3}{2}) \frac{\sinh(\frac{b_1}{2})}{b_1} \tag{15} \\ &= -4\sqrt{\frac{2}{\pi}} (4\pi^2)^{2g-2} \Gamma(2g - \frac{1}{2}) \frac{\sinh(\frac{b_1}{2})}{b_1}, \end{aligned}$$

because $2\frac{4(4\pi^2)^{2g-\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}} = 4\sqrt{\frac{2}{\pi}} (4\pi^2)^{2g-2}$, and $\frac{4g-3}{2} \Gamma(2g - \frac{3}{2}) = \Gamma(2g - \frac{1}{2})$. This confirms that the left- and right-hand sides of (9) are equal when $V_{g,2}(b_1, b_2)$ is expressed as (13). We thus confirmed that (13) satisfies the reduced equation (9).

The expression for $V_{g,2}(b_1, b_2)$ (13) is consistent with Conjecture 1 given by Zograf in [29]. This can be confirmed by comparing the Weil–Petersson volume with intersection numbers using [32]

$$\begin{aligned} V_{g,2}(b_1, b_2) &= \int_{\overline{\mathcal{M}}_{g,2}} \exp(2\pi^2 \kappa_1 + \frac{1}{2}(b_1^2 \psi_1 + b_2^2 \psi_2)) \\ &= \sum_{3g-1 \geq i+j \geq 0} \frac{(2\pi^2)^{3g-1-i-j}}{i!j!(3g-1-i-j)!} (\frac{b_1^2}{2})^i \\ &\quad \times (\frac{b_2^2}{2})^j < \psi_1^i \psi_2^j \kappa_1^{3g-1-i-j} >, \tag{16} \end{aligned}$$

³ then setting b_1 and b_2 to zero. (i, j on the right-hand side of (16) are non-negative integers, and their sum ranges from 0 to $3g - 1$.) Zograf’s conjectures in [29] for the Weil–Petersson volume when the genus g is large are partially proved rigorously in [35–37]. The fact that (13) is consistent with Conjecture 1 in [29] also supports our expression.

We compared the precise computations of Peter Zograf of $V_{g,2}(b_1, b_2)$ up to $g = 18$ [38] with our expression (13). When $V_{g,2}(b_1, b_2)$ is expanded in $b_1^{2i} b_2^{2j}$ as

$$V_{g,2}(b_1, b_2) = \sum_{3g-1 \geq i+j \geq 0} c_{gij} b_1^{2i} b_2^{2j}, \tag{17}$$

(where $c_{gij} = \frac{(2\pi^2)^{3g-1-i-j}}{i!j!(3g-1-i-j)!} (\frac{1}{2})^{i+j} < \psi_1^i \psi_2^j \kappa_1^{3g-1-i-j} > [32]$),⁴ we compared the coefficients of $b_1^{2i} b_2^{2j}$ obtained by expanding the expression (13) in terms of $b_1^{2i} b_2^{2j}$, which we denote by c_{gij}^{asymp} , with the precise coefficient computed by Zograf, which we denote by $c_{gij}^{precise}$.

For example, when one sets $b_1 = b_2 = 0$, one obtains the coefficient c_{g00} . This corresponds to $\frac{(2\pi^2)^{3g-1}}{(3g-1)!} < \kappa_1^{3g-1} >$. We compared the constant term in (13) with the results by Zograf [38]. The error percentages obtained from

$$1 - \frac{c_{g00}^{precise}}{c_{g00}^{asymp}} \tag{18}$$

are within 6% for $2 \leq g \leq 18$. The precision of the expression (13) improves as genus g increases: for $6 \leq g \leq 18$, the error percentages are less than 2%, and for $12 \leq g \leq 18$, the error percentages are less than 1%.

³ The first Miller–Morita–Mumford class κ_1 is cohomologous to $\frac{1}{2\pi^2}$ times the Weil–Petersson symplectic form ω , $\kappa_1 = \frac{\omega}{2\pi^2}$, owing to results in [33,34].

⁴ The right-hand side of equation (17) is not summed over g . We placed g in the subscript of the coefficient c_{gij} to indicate that it depends on the genus g .

We also compared the coefficients of $b_1^{2i}b_2^{2j}$ for $0 \leq i, j \leq 2$ (where i and j are not simultaneously zero) obtained from the asymptotic expression (13) with the results computed by Zograf [38] when $g = 18$. Because coefficients are symmetric under the exchange of i and j , we only consider the case $i \geq j$ here. For all the cases $(i, j) = (1, 0), (1, 1), (2, 0), (2, 1), (2, 2)$, the error percentages obtained from

$$1 - \frac{C_{gij}^{\text{precise}}}{C_{gij}^{\text{asympt.}}} \quad (19)$$

are less than 10%. Except for the case $(i, j) = (2, 2)$, the error percentages are less than 6%. The comparison seems to suggest that genus g required for a precision of agreement, say error percentage of less than 2%, increases as i and j rise.

It is worth noting that the asymptotic expression (13) for $V_{g,2}(b_1, b_2)$ yields a leading term in g . There is a correction term of order $\sim 1/g$. One can see this as follows: The Weil–Petersson volume $V_{g,n}(b_1, \dots, b_n)$ also satisfies an integral equation [30,31]:

$$V_{g,n+1}(b_1, \dots, b_n, 2\pi i) = \sum_{j=1}^n \int_0^{b_j} b_j V_{g,n}(b_1, \dots, b_n) db_j. \quad (20)$$

When there are two boundaries, this equation takes the following particular form:

$$V_{g,2}(b_1, 2\pi i) = \int_0^{b_1} b_1 V_{g,1}(b_1) db_1. \quad (21)$$

Integrating the result for $V_{g,1}(b_1)$ (10) in [3], we expect

$$V_{g,2}(b_1, 2\pi i) = \frac{4^2 (4\pi^2)^{2g-\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}} \Gamma(2g - \frac{3}{2}) \sinh^2(\frac{b_1}{4}). \quad (22)$$

However, expression (13) for $V_{g,2}(b_1, b_2)$ vanishes when $b_2 = 2\pi i$.⁵ This suggests that there is a subleading correction term $\sim 1/g$ to the leading term (13), $\frac{4^2 (4\pi^2)^{2g-\frac{3}{2}}}{2\pi^{\frac{3}{2}}} \Gamma(2g - \frac{3}{2}) f(b_1, b_2)$, where $f(b_1, 2\pi i) = \sinh^2(\frac{b_1}{4})$ and $f(b_1, b_2)$ does not depend on g . The function $f(b_1, b_2)$ must be symmetric under exchange of b_1 and b_2 .

Therefore, from the above argument, we deduce the following more precise asymptotic expression for $V_{g,2}$ under (8):

$$V_{g,2}(b_1, b_2) \sim 4\sqrt{\frac{2}{\pi}} (4\pi^2)^{2g-1} \Gamma(2g - \frac{1}{2}) \frac{\sinh(\frac{b_1}{2}) \sinh(\frac{b_2}{2})}{b_1 b_2} + \frac{4^2 (4\pi^2)^{2g-\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}} \Gamma(2g - \frac{3}{2}) f(b_1, b_2). \quad (23)$$

The form of the function $f(b_1, b_2)$ is left undetermined.

⁵ With our asymptotic expression (13) for $V_{g,2}(b_1, b_2)$, we have $\frac{V_{g,2}}{V_{g,2}} \sim \frac{\Gamma(2g-\frac{3}{2})}{\Gamma(2g-\frac{1}{2})} = \frac{2}{4g-3}$. Additionally, $g \gg b_1$ under (8). Therefore, we may set $\int_0^{b_1} b_1 V_{g,1}(b_1) db_1$ to zero to the leading order in g . Based on this reasoning, for leading order in g , the vanishing of (13) when b_2 assumes the value $2\pi i$ does not suggest an inconsistency here.

3.2. Asymptotic $V_{g,n}(b_1, \dots, b_n)$

Now, we would like to estimate the asymptotic expression for general $V_{g,n}(b_1, \dots, b_n)$ when condition (4) is satisfied. First, we estimate the Weil–Petersson volume with three boundaries $V_{g,3}$ for $g \gg 1, g \gg b_1, b_2, b_3$. For this situation, $V_{g,3}$ must satisfy the following two differential equations [30,31]:

$$\begin{aligned} \partial_3 V_{g,3}(b_1, b_2, 2\pi i) &= 2\pi i \cdot 2g V_{g,2}(b_1, b_2) \\ \partial_3^2 V_{g,3}(b_1, b_2, 2\pi i) &= -(4g - 2)V_{g,2}(b_1, b_2). \end{aligned} \quad (24)$$

We used the reduced equation (5) for the second equation in (24) owing to the conditions $g \gg 1, g \gg b_1, b_2, b_3$.

By a similar argument to that presented in Section 3.1, we exploit symmetry to estimate the asymptotic expression for $V_{g,3}(b_1, b_2, b_3)$:

$$V_{g,3}(b_1, b_2, b_3) \sim 8\sqrt{\frac{2}{\pi}} (4\pi^2)^{2g} \Gamma(2g + \frac{1}{2}) \frac{\sinh(\frac{b_1}{2}) \sinh(\frac{b_2}{2}) \sinh(\frac{b_3}{2})}{b_1 b_2 b_3}. \quad (25)$$

Assuming that $V_{g,2}(b_1, b_2)$ is given by (13) under (8), expression (25) satisfies the two equations in (24) (when g tends toward infinity). Furthermore, expression (25) is consistent with Conjecture 1 in [29], as can be confirmed by setting b_1, b_2, b_3 in (25) to zero. These results support our expression (25) to some degree.

Similar to the Weil–Petersson volume with two boundaries $V_{g,2}(b_1, b_2)$, $V_{g,3}(b_1, b_2, b_3)$ must satisfy the integral equation [30,31]:

$$V_{g,3}(b_1, b_2, 2\pi i) = \int_0^{b_1} b_1 V_{g,2}(b_1, b_2) db_1 + \int_0^{b_2} b_2 V_{g,2}(b_1, b_2) db_2. \quad (26)$$

With our expression (25) for $V_{g,3}(b_1, b_2, b_3)$, the left-hand side in (26) vanishes, while the right-hand side does not vanish when (13) is substituted into $V_{g,2}(b_1, b_2)$. This suggests that there is a subleading correction $\sim 1/g$, similar to the situation discussed in Section 3.1. Integrating (13) times b_1 and b_2 , we deduce that the subleading correction is of the form

$$(4\pi^2)^{2g-1} \Gamma(2g - \frac{1}{2}) f(b_1, b_2, b_3), \quad (27)$$

where $f(b_1, b_2, b_3)$ is symmetric in b_1, b_2, b_3 and $f(b_1, b_2, b_3)$ does not depend on g .

Iteration of this computation leads to the following conjecture on the asymptotic expression for $V_{g,n}(b_1, \dots, b_n)$ under the conditions (4):

$$V_{g,n}(b_1, \dots, b_n) \sim 2^n \sqrt{\frac{2}{\pi}} (4\pi^2)^{2g+n-3} \Gamma(2g + n - \frac{5}{2}) \prod_{i=1}^n \frac{\sinh(\frac{b_i}{2})}{b_i}. \quad (28)$$

As g tends toward infinity, this expression satisfies the two differential equations mentioned in Section 2:

$$\begin{aligned} \partial_{n+1} V_{g,n+1}(b_1, \dots, b_n, 2\pi i) &= 2\pi i(2g - 2 + n) V_{g,n}(b_1, \dots, b_n) \\ \partial_{n+1}^2 V_{g,n+1}(b_1, \dots, b_n, 2\pi i) &= -(4g - 4 + n)V_{g,n}(b_1, \dots, b_n). \end{aligned} \quad (29)$$

Here, we used the reduced equation for the second equation in (29) owing to the conditions (4), as previously explained in Section 2. Furthermore, setting b_1, b_2, \dots, b_n to zero, the consistency of expression (28) with Conjecture 1 in [29] is confirmed.

$V_{g,n}(b_1, \dots, b_n)$ must also satisfy the integral equation (20) [30, 31]. This suggests that there is a subleading correction $\sim 1/g$ to expression (28), similar to $V_{g,2}(b_1, b_2)$ and $V_{g,3}(b_1, b_2, b_3)$, which we have already discussed.

We make a remark here that the asymptotic Weil–Petersson volumes can also be evaluated from the matrix integral perspective.

Correlation function of resolvents $\langle R(E_1)R(E_2)\dots R(E_n) \rangle_c$ is expanded as follows [3]:

$$\langle R(E_1)R(E_2)\dots R(E_n) \rangle_c \simeq \sum_{g=0}^{\infty} e^{(-2g+2-n)S_0} R_{g,n}(E_1, \dots, E_n). \tag{30}$$

Functions $W_{g,n}$ ⁶ are defined as [39,3]:

$$W_{g,n}(z_1, \dots, z_n) = 2^n (-1)^n \cdot z_1 \dots z_n \cdot R_{g,n}(-z_1^2, -z_2^2, \dots, -z_n^2). \tag{31}$$

Then, the following equality holds [26]:

$$W_{g,n}(z_1, \dots, z_n) = \prod_{i=1}^n \int_0^{\infty} b_i db_i e^{-b_i z_i} V_{g,n}(b_1, \dots, b_n) \tag{32}$$

Using the definition of $W_{g,n}$ (31), the equality (32) can be rewritten as follows:

$$R_{g,n}(-z_1^2, -z_2^2, \dots, -z_n^2) = (-1)^n \frac{1}{2^n z_1 \dots z_n} \prod_{i=1}^n \int_0^{\infty} b_i db_i e^{-b_i z_i} V_{g,n}(b_1, \dots, b_n). \tag{33}$$

Multiplying the two sides in the equation (33) by $e^{(-2g+2-n)S_0}$ and summing up over genus g , the following equality is obtained:

$$\langle R(-z_1^2)R(-z_2^2)\dots R(-z_n^2) \rangle_c = (-1)^n \frac{1}{2^n z_1 \dots z_n} \times \prod_{i=1}^n \int_0^{\infty} b_i db_i e^{-b_i z_i} \sum_g e^{(-2g+2-n)S_0} V_{g,n}(b_1, \dots, b_n). \tag{34}$$

The case with one boundary, $n = 1$, was discussed, and the volume $V_{g,1}(b)$ was evaluated from the expectation value of the resolvent $\langle R(-z^2) \rangle$ in [3]. The equation (34) yields straightforward generalization to an arbitrary number of boundaries. We expect that by using an argument similar to that given in [3], computing correlation function of the resolvents and applying inverse Laplace transform, the Weil–Petersson volume $V_{g,n}(b_1, \dots, b_n)$ can also be estimated from the matrix integral perspective.

4. Asymptotic behavior of the Weil–Petersson volume $V_{g,n}(b_1, \dots, b_n)$ with large b

Mirzakhani proved [25] that the Weil–Petersson volume $V_{g,n}(b_1, \dots, b_n)$ is a polynomial in b given by

⁶ The functions $W_{g,n}$ satisfy a recursion relation [28] that corresponds to Mirzakhani’s recursion relation via Laplace transform [26].

$$V_{g,n}(b_1, \dots, b_n) = \sum_{|\alpha| \leq 3g-3+n} c_g(\alpha) b^{2\alpha}, \tag{35}$$

where b on the right-hand side represents $b = (b_1, b_2, \dots, b_n)$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, and $\alpha_i, i = 1, \dots, n$ are non-negative integers. $b^{2\alpha}$ is defined as $b^{2\alpha} = b_1^{2\alpha_1} b_2^{2\alpha_2} \dots b_n^{2\alpha_n}$. The coefficient $c_g(\alpha)$ is positive: $c_g(\alpha) > 0$, and $c_g(\alpha)$ is in $\pi^{6g-6+2n-2|\alpha|} \cdot \mathbb{Q}$. $|\alpha|$ is defined as $|\alpha| = \sum_{i=1}^n \alpha_i$ in (35).⁷

With regard to the theorem of Mirzakhani, it is worth making a remark about the Weil–Petersson volume in the limit where the boundary lengths b become large, i.e., $b \gg g \gg 1$. It follows from Mirzakhani’s theorem that in the large b regime ($b_1, \dots, b_n \gg g \gg 1$), the highest-order terms in the polynomial (35) are dominant. Therefore, in the regime $b_1, \dots, b_n \gg g \gg 1$, the Weil–Petersson volume should be approximated by the sum of these highest-order terms in b :

$$V_{g,n}(b_1, \dots, b_n) \simeq \sum_{|\alpha|=3g-3+n} c_g(\alpha) b^{2\alpha}, \tag{36}$$

where the coefficients $c_g(\alpha)$ with $|\alpha| = 3g - 3 + n$ are positive and are in \mathbb{Q} .

However, the method discussed in this report, which employs partial differential equations (3) to estimate the asymptotic Weil–Petersson volume, does not apply in the large b regime to determine the coefficients in (36). This is owing to the fact that the conditions $b_1, \dots, b_n \gg g \gg 1$ are imposed on b .

It might be interesting to consider if there is a method for estimating the asymptotic expression for the Weil–Petersson volume $V_{g,n}(b_1, \dots, b_n)$ in the regime $b_1, \dots, b_n \gg g \gg 1$, from the volume $V_{g,n-1}(b_1, \dots, b_{n-1})$. This is equivalent to estimating the coefficients $c_g(\alpha)$ in (36) for asymptotic $V_{g,n}$, when the coefficients $c_g(\alpha)$ are known for asymptotic $V_{g,m}$, $m < n$, in this regime.

The asymptotic Weil–Petersson volume for any genus for the case of one boundary length becoming large is computed in Appendix A of [40]. This corresponds to the situation $b_1 \gg g, b_1 \gg 1$. For this situation, the term with $\alpha = (3g - 3 + n, 0, \dots, 0)$ yields a unique leading term, and the coefficient $c_g(3g - 3 + n, 0, \dots, 0)$ of the leading term is given by $\frac{1}{(24)^g \cdot 2^{3g-3+n} \cdot g! (3g-3+n)!}$ [40].

5. Concluding remarks and open problems

Herein, we utilized partial differential equations satisfied by the Weil–Petersson volume [30,31] to estimate the asymptotic Weil–Petersson volumes $V_{g,2}(b_1, b_2)$ and $V_{g,3}(b_1, b_2, b_3)$ for large genus g ($g \gg 1$ and $g \gg b_1, b_2, b_3$). We also conjectured the asymptotic expression for the volume $V_{g,n}(b_1, \dots, b_n)$ for general n , when the genus g is large ($g \gg 1$ and $g \gg b_1, \dots, b_n$). The obtained asymptotic expressions satisfy the partial differential equations deduced in [30,31] to leading order in g . We also confirmed that, when the b_i all vanish, our asymptotic expressions for the volumes ($V_{g,2}(b_1, b_2)$ with two boundaries, $V_{g,3}(b_1, b_2, b_3)$ with three boundaries, and $V_{g,n}(b_1, \dots, b_n)$ in general) are all consistent with Conjecture 1 for the asymptotic Weil–Petersson volumes specified by Zograf in [29]. We also compared our asymptotic expression (13) for $V_{g,2}(b_1, b_2)$ with the precise computational results by Zograf [38]. The comparison showed good agreement.

The Weil–Petersson volumes yield the intersection numbers of cohomology classes on the moduli of Riemann surfaces [32]. For

⁷ The coefficients $c_g(\alpha)$ in (35) are expressed as intersection numbers of cohomology classes on the moduli of Riemann surfaces in [32]. When $n = 2$, the equation (35) corresponds to the equations (16) and (17).

example, Mirzakhani proved [32] Witten's conjecture [41]⁸ by relating the intersection numbers of cohomology classes on the moduli of Riemann surfaces to the coefficients of the Weil–Petersson volume in (35). The asymptotic expressions for the Weil–Petersson volumes obtained in this note provide information on the intersection numbers of certain cohomology classes on the moduli of Riemann surfaces. It might be interesting to compare the known intersection numbers of cohomology classes with our results. Physically, they include the correlation functions of 2d topological gravity [45].

The asymptotic Weil–Petersson volumes $V_{g,0}$ and $V_{g,1}(b)$ were predicted in [3] in the context of bosonic JT gravity, by using the density of eigenvalues via topological recursion. As discussed in [3], their method of using topological recursion⁹ to predict asymptotic Weil–Petersson volumes may extend to general $V_{g,n}(b_1, \dots, b_n)$. If so, it might also be interesting to study whether our expressions agree with the results obtained from the approach of topological recursion.

If the method of topological recursion succeeds in evaluating the asymptotic Weil–Petersson volume $V_{g,n}(b_1, \dots, b_n)$ in the regime $b_1, \dots, b_n \gg g \gg 1$, the saddle-point approximation should yield the approximation of the Weil–Petersson volume $V_{g,n}(b_1, \dots, b_n)$ in the form (36) discussed in Section 4. This provides a non-trivial check of the result given by the topological recursion, if it is applied to evaluate the Weil–Petersson volume $V_{g,n}(b_1, \dots, b_n)$ for general n in the regime $b_1, \dots, b_n \gg g \gg 1$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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⁸ Witten's conjecture was proved in [42]. Proofs of Witten's conjecture can also be found in [43,44].

⁹ Discussions of a related approach can be found, e.g., in [46–48].