PARTIAL WAVE ANALYSIS AND THE HIGH ENERGY BEHAVIOUR OF MANY-PARTICLES AMPLITUDES

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(presented by V. N. Gribov)

A simple method of relativistic partial wave analysis of general many-particle amplitudes is proposed. The expansions thus obtained are used to analyse the asymptotic behaviour of the amplitudes of inelastic processes on the basis of the Regge [1] and Gribov [2] method.

I. — Jacob and Wick [3] have obtained a very elegant form of the partial wave expansion for the amplitude shown in Fig. 1 of the transition of particles 1, β into 2, α:

$$M_{l'm'\,;\,l''m''} = \sum_{L,\,M}(2L+1)\,D^{(L)*}_{M,\,m'}(\mathbf{n}_\alpha)\,D^{(L)}_{M,\,m''}(\mathbf{n}_\beta)\,f^{(l',\,l'')}_{L\,;\,m'm''}(t\,;\,s',\,s'') \quad (1)$$

l', m' and l'', m'' being the spins and the projections of spin of particles α and β on the direction $\mathbf{n}_\alpha$ and $\mathbf{n}_\beta$ of their momenta $\mathbf{p}_\alpha$, $\mathbf{p}_\beta$ in the c.m. system of the reaction. The spin of particles 1,2 (as well as of all others unless otherwise specified) is assumed to be zero. $f^{(l',\,l'')}_{L\,;\,m',\,m''}$ is the partial amplitude with definite helicity, it depends on $s' = p_\alpha^2$, $s'' = p_\beta^2$ and on the energy $t = (p_\beta + p_1)^2 = (\varepsilon_\beta + \varepsilon_1)^2 - (\mathbf{p}_\beta + \mathbf{p}_1)^2$. The notations are similar to those used by Jacob and Wick [3] and by Rose [4]. In particular,

$$D^{(L)}_{M,\,m'}(\mathbf{n}_\alpha) = e^{i(m'-M)\phi_\alpha}\,d^{(L)}_{M,\,m'}(\cos\vartheta_\alpha),$$

where $d^{(L)}_{M,\,0}(\cos\vartheta_\alpha) = P_{LM}(\cos\vartheta_\alpha)$ is the associated Legendre polynomial.

If the particle α or both the particles α and β are considered as composite ones [5, 6]—each consisting of two particles being in a state with definite values of the angular momenta in their c.m. system (α—as consisting of the particles 3 and 4 in a state with definite l', m'; β—of the particles 5 and 6 with definite l'', m'')—then (1) may be considered as the 5 or 6-line vertex shown in Fig. 2 and 3. The 5 and 6-line vertices (Figs. 2, 3), $M_5 = M_{\mathbf{k};\,0,0}$, $M_6 = M_{\mathbf{k}',\mathbf{k}''}$, corresponding to definite values of momenta $\mathbf{p}_i$ of all the particles, will be obviously (*) a linear combination of quantites $M_{l'm';\,l''m''}$:

$$M_{\mathbf{k}'\,;\,l'',\,m''} = \textstyle\sum_{l'm'}(2l'+1)D^{(l')*}_{m',\,0}(\mathbf{n}')M_{l'm';\,l''m''} \quad (2)$$

$$M_{\mathbf{k}'\,;\,\mathbf{k}''} = \textstyle\sum_{l''m''}(2l''+1)D^{(l'')}_{m'',\,0}(\mathbf{n}'')M_{\mathbf{k}'\,;\,l''m''} \quad (3)$$

where $\mathbf{k}' = \mathbf{p}_3' = -\mathbf{p}_4'$ is the momentum of the relative motion of particles 3 and 4 in their c.m. system (all the quantities related to such a c.m. system of a group of particles will be marked by one or two primes), $\mathbf{n}'$ is its direction; $\mathbf{k}'' = \mathbf{p}_5'' = -\mathbf{p}_6''$ and $\mathbf{n}''$ are just the same for the particles 5 and 6.

From (1), (2) and (3) we obtain the following partial wave expansions for the n-line vertex M_n:

$$M_n(s_{ik}) = \sum_{L,\,m',\,m''}(2L+1)d^{(L)}_{m',\,m''}(z)\chi^{(n)}_{L\,;\,m'm''}\,, \quad (4)$$

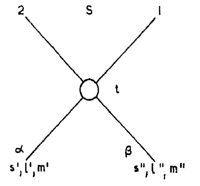

Fig. 1

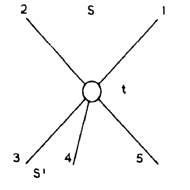

Fig. 2

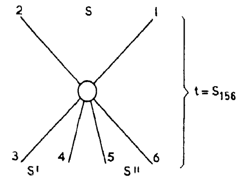

Fig. 3

(*) Just as the wave function $\psi_{\mathbf{k}}$ of particles 3,4 moving with definite $\mathbf{k}$ is the linear combination [3] of the wave functions $\psi_{l'm'}$ of states with definite l', m': $\psi_{\mathbf{k}} = \Sigma_{l'm'}(2l'+1)D^{(l')}_{m',\,0}(\mathbf{n}')\psi_{l',\,m'}$.

where for $n=4$, $n=5$ and $n=6$

$$\chi^{(4)}_{L,m',m''} = \delta_{m',0}\delta_{m'',0}\lambda_L(t)\,,$$

$$\chi^{(5)}_{Lm'm''} = \delta_{m'',0}e^{im'\phi}\textstyle\sum_{l'}(2l'+1)P_{l'm'}(z_3')\varphi^{(l')}_{L,m',m''}(t\,;\,s')\,,$$

$$\chi^{(6)}_{Lm'm''} = e^{i(m'\phi'-m''\phi'')}\textstyle\sum_{l'l''}(2l'+1)(2l''+1)P_{l'm'}(z_3')\times$$

$$\times P_{l''m''}(z_5'')f^{(l',l'')}_{Lm'm''}(t\,;\,s',s'')\,. \qquad (5)$$

Figs. 4, 5 show the configurations of momenta of particles for 5 and 6-lines vertices of Figs. 3 and 4 in the c.m. system of the reaction, the axis OZ being parallel to $\mathbf{n}_\alpha$. z, z_3', z_5'' are the cosines of the angles

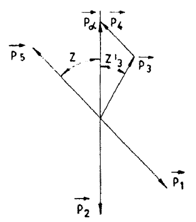

Fig. 4

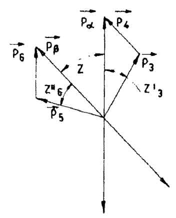

Fig. 5

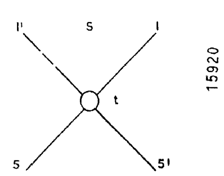

Fig. 6

indicated in Figs. 4, 5; ϕ is the angle between the plane of the vectors $\mathbf{p}_\alpha$, $\mathbf{p}_5$ and that of $\mathbf{p}_\alpha$, $\mathbf{p}_3$ (Fig. 3); ϕ' and ϕ'' in (5) are the angles between the plane $\mathbf{p}_\alpha$, $\mathbf{p}_\beta$ and the planes $\mathbf{p}_\alpha$, $\mathbf{p}_3$ and $\mathbf{p}_\beta$, $\mathbf{p}_5$ correspondingly (Fig. 4). The partial amplitude $f^{(l',0)}_{L;m'0}(t\,;\,s',m_5^2)$ is denoted by $\varphi^{(l')}_{L,m'}(t\,;s')$ for the case of the 5-line vertex of Fig. 2; $\lambda_L(t)$ is the partial amplitude of elastic scattering of the particles 5 and 1 (Fig. 6).

Note that the invariant $s=(p_1-p_2)^2$ is related in all the cases to the cosine z by the equation $s=m_1^2+m_2^2-2\varepsilon_1\varepsilon_2+2p_1p_2z$ where ε_1, ε_2 (and p_1, p_2) are determined by the quantities t, or t, s' or t, s', s''.

The expansions (1)-(5) may be simply generalized to the case when the spins of all the particles differ from zero. Then considering one of the particles as a system of two or more particles (and its spin as the total angular momentum of this system) one can easily obtain the expansions of the type (4)-(5) for the amplitude with any number of lines.

The unitarity conditions for the partial amplitudes with definite helicity have a fairly simple form. If, in particular, the energy t is below the threshold of the production of three particles (i.e. is in the non-physical region) we shall have [7, 8]:

$$\lambda_L(t_+)-\lambda_L(t_-) = 2i\rho(t)\lambda_L(t_+)\lambda_L(t_-)\,,$$

$$\varphi^{(l')}_{L;m'}(t_+\,;\,s')-\varphi^{(l')}_{L;m'}(t_-\,;\,s') = 2i\rho(t)\varphi^{(l')}_{L;m'}(t_+\,;\,s')\lambda_L(t_-)\,,$$

$$f^{(l',l'')}_{L;m'm''}(t_+\,;\,s',s'')-f^{(l',l'')}_{L;m'm''}(t_-\,;\,s',s'') = 2i\rho(t)\varphi^{(l')}_{L,m'}(t_+\,;\,s')\varphi^{(l'')}_{L,m''}(t_-\,;\,s'')\,, \qquad (6)$$

where

$$\rho(t)=\frac{2p_{15}(t)}{\sqrt{t}}=\frac{1}{t}\sqrt{t^2-2t(m_1^2+m_5^2)+(m_1^2-m_5^2)^2}\,,$$

$t_\pm = t\pm i\tau$, $\tau\to 0$, $\tau>0$, t is real, s' and s'' may be any complex numbers.

II. — Expansions (4)-(5) and unitarity conditions (6) provide the basis for analysis of the high energy behaviour of the many particle amplitudes of Figs. 2, 3 in the physical region of the channel in which the incident particles are 1, 2 and the produced ones are 3, 4, 5 (in Fig. 2) or 3, 4, 5, 6 (in Fig. 3). The case will be considered when the energy of this channel is very high ($s\to\infty$), the value of s', s'' and t being fixed. This corresponds to $z\simeq \dfrac{s}{2p_1p_2}\to\infty$. Note that for $z\to\infty$ the functions $d^{(L)}_{m',m''}(z)$ have a simple asymptotic form: $d^{(L)}_{m'm''}(z)\approx C_{L,m'}C_{L,-m''}(z/2)^L$, where

$$C_{L,m'} = i^{m'}\sqrt{\frac{\Gamma(2L+1)}{\Gamma(L+m'+1)\Gamma(L-m'+1)}}$$

and that they possess the property

$$d^{(L)}_{m',m''}(-z_\pm) = e^{\mp i\pi(L-m'')}\cdot d^{(L)}_{m',-m''}(x)\,,$$

where $z_\pm = x\pm i\tau$, x real positive. By taking into account these properties of $d^{(L)}_{m',m''}$ and using Regge [1)] and Gribov [2)] method we find the asymptotic behaviour of the sum (4) for $z_+\to\infty$ to be

$$M_n\approx (z/2)^\alpha\left(i-\mathrm{ctg}\frac{\pi\alpha}{2}\right)\cdot\sum_{m',m''=-\infty}^{\infty} C_{\alpha,m'}C_{\alpha,m''}R^{(n)}_{m',m''} \qquad (7)$$

where $R^{(n)}_{m',-m''}$ is the residue of the pole furthest to the right of the function $\chi^{(n)}_{L;\,m',m''}$ in L:

$$L\to\alpha\,;\qquad \frac{\pi}{2}(2L+1)\chi^{(n)}_{L;\,m',m''}\approx\frac{R^{(n)}_{m',-m''}}{L-\alpha}\,. \tag{8}$$

The function $\chi^{(n)}_{L;\,m',m''}$ was introduced as an analytic continuation into the complex L-plane of the function $\chi^{(n)}_{2\nu;\,m',m''}$ defined by (5) for even integer values of $L=2\nu$ (the part of the sum (4), corresponding to odd values of L is not essential in the high energy limit as in the case [2] of elastic scattering). In obtaining (7)-(8) it was assumed that the part of the function $\chi^{(n)}_{L,m',m''}$ which is essential for the asymptotic form of M_n: (a) is an analytic function of L in the right half plane of L, (b) decreases rapidly when $L\to\infty$. It is fairly probable that the anomalous terms in the dispersion relation for $M_n(s)$ (being not analytic in region $L\to\infty$) do not influence the asymptotic behaviour because they are determined by integrals in s (or in z) over contours which are not of infinite but finite length.

As an analysis of the equations (6) shows (being quite similar to that made by Gribov and Pomeranchuk [9]) the partial amplitudes λ_L, $\varphi^{(l')}_{Lm'}$, $f^{(l',l'')}_{L,m',m''}$ do have the poles of the type (8), the residues being factorized:

$$L\to\alpha(t),\quad\left\{\begin{aligned}&(\pi/2)\cdot(2L+1)\lambda_L(t)\approx\frac{u^2(t)}{L-\alpha(t)}\,,\\&(\pi/2)\cdot(2L+1)\varphi^{(l')}_{L,m'}(t\,;s')\approx\frac{u(t)v_{l',m'}(t,s')}{L-\alpha(t)}\,,\\&(\pi/2)\cdot(2L+1)f^{(l',l'')}_{L,m'm''}(t\,;s',s'')\approx\frac{v_{l'm'}(t,s')v_{l''m''}(t,s'')}{L-\alpha(t)}\,.\end{aligned}\right.$$

Here $u(t)$ is a real quantity and $v_{l'm'}(t,s')\equiv v_{l',-m'}(t,s')$ (as a consequence [3] of space reflection invariance). Using (5) and (7)-(8) we obtain then the following asymptotic values of M_n:

$$s\to\infty\left\{\begin{aligned}&M_4\approx g^2(t)\left(i-\mathrm{ctg}\frac{\pi\alpha}{2}\right)s^{\alpha(t)}\\&M_5\approx g(t)G(t,\mathbf{k}')\left(i-\mathrm{ctg}\frac{\pi\alpha}{2}\right)s^{\alpha(t)}\\&M_6\approx G(t,\mathbf{k}')G(t,\mathbf{k}'')\left(i-\mathrm{ctg}\frac{\pi\alpha}{2}\right)s^{\alpha(t)}\end{aligned}\right. \tag{9}$$

where $g(t)=[2p_1(t)]^{-\alpha(t)}C_{\alpha,0}u(t)$ and

$$G(t,\mathbf{k}')=[2p_2(t,s')]^{-\alpha(t)}\sum_{l'm'}(2l'+1)C_{\alpha,m'}v_{l'm'}(t,s')\,P_{l'm'}(z_3')e^{im'\phi} \tag{10}$$

The amplitudes (9) have the form of contributions from the pole graphs (Fig. 7) the factor $\left(i-\mathrm{ctg}\frac{\pi\alpha}{2}\right)s^{\alpha(t)}$ corresponding to the virtual particle—a " reggeon "—, $g(t)$ to the three-line vertex and $G(t,\mathbf{k}')$ to the four-line vertex. Eq. (9) does not exhaust all information about the asymptotic behaviour of many-particle amplitudes at $s\to\infty$. There is a very interesting case when the energy s is distributed among all the particles, i.e. when s' and s'' increase as $s\to\infty$. In this case the expansions of the type (10) (written down for the case when the fixed invariants are $s_{23}=t'$ and t instead of s' and t) can also be transformed into an integral of Sommerfield-Watson type and the asymptotic value of $G(t,\mathbf{k}')$ for $s'\to\infty$ turns out to be $g(t')\gamma(t',t)s'^{\alpha(t')}$. For the 5-line vertex (Fig. 2) we obtain in this case

$$M_5\approx g(t')\gamma(t',t)g(t)s'^{\alpha(t')}s_{45}^{\alpha(t)}\,,$$

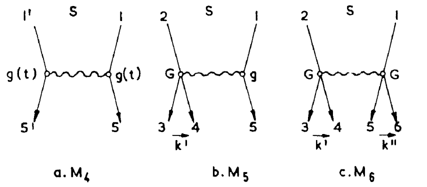

Fig. 7

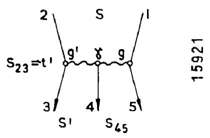

Fig. 8

which corresponds to the pole graph of Fig. 8 with two " reggeons ". Such true non-elastic processes will be considered in detail elsewhere. Here we will deal with " quasi-elastic " processes by which we mean those where final states consist of two well-defined groups of particles having small total c.m. energies s' and s''.

There are a series of results following from (9) which may be checked by the experiment:

1. The dependence of differential cross-sections of " quasi-elastic " processes on s and t is controlled by the same Regge pole as in the case of elastic scattering

$$s\to\infty\begin{cases} d\sigma_4(12\to34) \approx g^4(t)s^{2[\alpha(t)-1]}dt\,, \\ d\sigma_5(12\to345) \approx g^2(t)|G(t,\mathbf{k}')|^2 s^{2[\alpha(t)-1]}dt d\tau'\,, \\ d\sigma_6(12\to3456) \\ \qquad \approx |G(t,\mathbf{k}')|^2|G(t,\mathbf{k}'')|^2 s^{2[\alpha(t)-1]}dt d\tau' d\tau''\,, \end{cases} \tag{11}$$

where $d\tau' = \frac{2k'}{\sqrt{s'}}ds'\cdot\frac{d\mathbf{n}'}{4\pi}$, $d\tau''$ is defined similarly. For fixed s' and s'' at $s\to\infty$ the value of $-t$ is always determined by $-t \approx \frac{s}{2}(1-\cos\vartheta_\alpha) \approx s\cdot\frac{\vartheta_\alpha^2}{4}$, where ϑ_α is the angle of deviation of the total moment $\mathbf{p}_\alpha$ of the produced group of particles from the original direction. Therefore, the factor

$$(s/M^2)^{2[\alpha(t)-1]} \approx \exp\left[-(\vartheta_\alpha^2/2)\alpha_0' s \ln(s/M^2)\right]$$

and the cross-sections (1) decrease rapidly with increase of ϑ_α; only the values ϑ_α of the order $M/\sqrt{s\ln(s/M^2)}$ being probable (M is a quantity of the order of mass of particles; $\alpha_0' = (d\alpha/dt)_{t=0}$ is of the order of $1/M^2$).

2. The differential cross-section of the production of one or two groups of particles with given s' and s'' integrated over ϑ_α (i.e. over $-t$) decrease as $1/\ln(s/M^2)$ when $s\to\infty$.

3. The $\mathbf{k}'$ distribution of the groups of particles produced and, in particular, their mass spectrum is independent of the way the group was produced and in particular does not depend on the nature of the incoming particle.

4. Between the cross-sections of the " quasi-elastic " processes of the type considered here, in the region $s\to\infty$ there exists a number of relations analoguous to those established by Gribov and Pomeranchuk [9] and Gell-Mann [10], (some of them have been mentioned in the paper by Gribov, Ioffe, Rudik and Pomeranchuk [11]). For the case of non-elastic processes all the cross-sections which will be dealt with can be measured directly.

Let us consider for example the process of production of π-mesons on a nucleon under the action of π-mesons or nucleons. The corresponding cross-sections (Fig. 9 a, b) will be denoted by $\sigma_\pi(t, \mathbf{k}')$ and $\sigma_N(t, \mathbf{k}')$. Let $\sigma_{\pi N}(t)$ and $\sigma_{NN}(t)$ be differential cross-sections of πN and NN scattering (Fig. 9 d, e) and $\sigma_N'(t;\ \mathbf{k}',\ \mathbf{k}'')$ the cross-section of the process $N+N\to(N+\pi)'+(N+\pi)''$. Writing down all these cross-sections according to Fig. 9 in the form (11) we obtain the following relations between them:

$$\sigma_\pi(t,\mathbf{k}') = \frac{\sigma_{\pi N}(t)}{\sigma_{NN}(t)}\cdot\sigma_N(t,\mathbf{k}')\,,$$

$$\sigma_{NN}(t)\sigma_N'(t\,;\mathbf{k}',\mathbf{k}'') = \sigma_N(t\,;\mathbf{k}')\sigma_N(t,\mathbf{k}'')$$

(the dependence of σ_i on s is not indicated, the value of s being the same for all the processes). These relations like many similar ones (for other particles) can be directly checked experimentally.

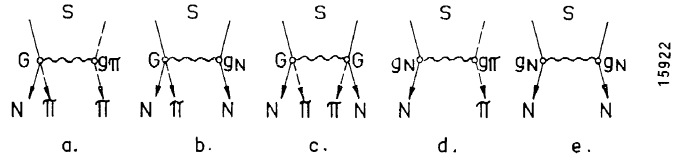

Fig. 9

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THEORY OF HIGH-ENERGY SCATTERING AND MULTIPLE PRODUCTION

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(Invited paper presented by D. Amati)

We wish to report about the different consequences of a model for high-energy interactions [1] in the investigation of which Bertocchi, Ceolin, Duimio and Tonin collaborated with us. This model has been suggested to us by the structure of the strip approximation to the Mandelstam representation [2] and can be simply understood as a generalization to very high energy of the peripheral model. The basic idea is that the main contributions to multiple production are given by a combination of a large number of low-energy processes. The graphs we are considering are shown in Fig. 1.

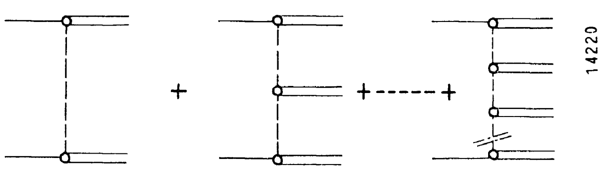

Fig. 1

Each bubble represents a low-energy two-body process. The number of multiperipheral graphs does, of course, increase with increasing energy. We wish to show that the sum of all multiperipheral effects exhibits, in the high-energy limit, particularly simple features, both for elastic scattering and for multiple production.

It is clear that the knowledge of the production amplitude allows to compute not only the production cross-section, but also the imaginary part of the elastic scattering amplitude through the unitarity relation

$$A(p_1 p_1' p_2 p_2') = \langle p_1' p_2' | \operatorname{Im} T | p_1 p_2 \rangle = \sum_n \langle p_1' p_2' | T^+ | n \rangle \langle n | T | p_1 p_2 \rangle \qquad (1)$$

In fig. 2 are shown the multiperipheral diagrams that give the elastic amplitude. They represent the shadow scattering of the multiple production as given by the multiperipheral model.