Mass, tension and thermodynamics of Kaluza-Klein black holes

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Abstract

We study the thermodynamics of charged black holes with squashed S^3 horizons in five-dimensional Einstein-Maxwell theory. The free energy of the black hole is obtained on two different backgrounds; one is four-dimensional Minkowski times S^1 and the other is the Kaluza-Klein monopole spacetime. The black hole has different values for the Hamiltonian, the Abbott-Deser mass and the Komar mass, and each mass satisfies thermodynamic first law. It is shown that the Hamiltonian, these masses and the free energy can be related each other in terms of Legendre transformations with respect to some thermodynamic variables. We find a new couple of thermodynamic variables that represents the squashing of the outer horizon. It gives a work term to each first law. In stead of them, the gravitational tension and its conjugate are also a couple of thermodynamic variables and give a work term to each first law except for that of the Komar mass, though the Komar mass satisfies Smarr's formula and would be thought of as a thermodynamic mass. In this case, the Komar mass can not be related to the free energy and other masses using Legendre transformations. We also investigate thermodynamic stability of the black hole in many thermodynamic environment. In environment with fixed gravitational tension, the most globally stable state is the Kaluza-Klein monopole. We find a phase transition between the black hole and the four-dimensional Minkowski times S^1 in an enviromnent with fixed temperature, electric charge and gravitational tension. In isolated system, smaller extra dimension is entropically favored, and the five-dimensional Reissner-Nordstörm black hole has the smallest entropy in the sequence of the black holes with the same mass and the same electric charge. Therefore, the Reissner-Nordstörm black hole might be the most unstable state in the sequence.

1 Introduction and Summary

It has a long history to formulate black hole thermodynamics in terms of quantum theory of gravity. Especially, the Euclidean path integral for gravitational field in saddle-point approximation is successful and gives the Bekenstein-Hawking formula for black hole entropy [1]. The Euclidean path integral method is consistent formulation for black hole thermodynamics and has a feature that it gives a thermodynamic free energy. The free energy is approximately given as the Euclidean classical action of the solution times its temperature.

Recently, spacetime structure of charged static black holes with squashed S^3 horizons in five-dimensional Einstein-Maxwell theory was studied by one of the authors and his collaborator [2]. The metric has three parameters characterizing mass, electric charge and the deformation of the S^3 horizon from round S^3 or the size of the extra dimension. The squashed S^3 can be thought of as a twisted S^1 fiber over S^2 base space with different circumference radius; the circumference radius of S^1 does not equal that of S^2. The spacetime asymptotes to a twisted S^1 bundle over four-dimensional Minkowski spacetime, which is not flat spacetime and not solution of the Einstein equation. The size of S^1 is asymptotically constant and, if its size is adequately small, the spacetime is effectively asymptotically four-dimensional spacetime with one extra dimension. In this sense, the black hole has the properties of Kaluza-Klein black holes. Furthermore, the solution includes the Kaluza-Klein monopole spacetime in a no horizon limit, which is originally found by Gross and Perry [3], and Sorkin [4].

The thermodynamics of this black hole was firstly investigated by Cai et al.[5]. They showed that the mass defined by the counter-term method equals the Abbott-Deser (AD) mass [6] on the locally

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flat spacetime to which the black hole spacetime asymptotes. It was shown that these masses satisfy a thermodynamic first law if one parameter is treated as a constant. In the investigation, they assumed that entropy, temperature, electric charge and electric potential difference are thermodynamic variables for the black hole. Furthermore, they discussed that there might exist other thermodynamic variables representing the deformation of the horizon and its conjugate generalized pressure.

In this paper, we study the thermodynamics of the black hole in more detail. The free energy of the black hole can be evaluated using traditional background subtraction method on two different backgrounds. The free energy $(F^{BH} - F^{flat})$ is evaluated on four-dimensional Minkowski times S^1 spacetime which is a flat spacetime obtained by neglecting the twisting of the S^1 fiber in the locally flat spacetime. The other free energy $(F^{BH} - F^{GPS})$ is evaluated on the Kaluza-Klein monopole spacetime. We consider the Hamiltonian which is formulated by Hawking and Horowitz [7] and the Komar mass in addition to the AD mass. We find that the Hamiltonian is a Legendre transform of the free energy with respect to temperature,

$$H^{B-f} = (F^{BH} - F^{flat}) + TS, \quad \text{and} \quad H^{B-G} = (F^{BH} - F^{GPS}) + TS \tag{1}$$

in both cases of the flat background and the Kaluza-Klein monopole background. In the above equation, H^{B-f} and H^{B-G} are the Hamiltonian of the black hole on the flat and the Kaluza-Klein monopole background, respectively. T is temperature defined by the Hawking formula and S is entropy given by the Bekenstein-Hawking formula. The AD mass on the flat background M_{AD} is a Legendre transform of the Hamiltonian with respect to the electric potential difference

$$M_{AD} = H^{B-f} + Q\Phi, \tag{2}$$

where Q is the electric charge and Φ is the electric potential difference between that at the outer horizon and at the spatial infinity. The Hamiltonian and the AD mass satisfy the following thermodynamic first laws:

$$\begin{aligned} dH^{B-f} &= TdS - Qd\Phi + \mathcal{T}_f dL \quad &(3)\\ dH^{B-G} &= TdS - Qd\Phi + \mathcal{T}_G dL \quad &(4)\\ dM_{AD} &= TdS + \Phi dQ + \mathcal{T}_f dL, \quad &(5)\end{aligned}$$

where $\mathcal{T}_f$ and $\mathcal{T}_G$ are graviational tension [8] evaluated on the flat spacetime background and the Kaluza-Klein monopole background, respectively. L is its conjugate and given by the period of the compact fifth dimension, or equivalently, the size of S^1 fiber at the spatial infinity. The Legendre transformations with respect to these variables are summarized as follows:

$$(F^{BH} - F^{flat}) \xrightarrow{T\to S} H^{B-f} \xrightarrow{\Phi\to Q} M_{AD}^{B-f}$$

for the flat background and

$$(F^{BH} - F^{GPS}) \xrightarrow{T\to S} H^{B-G} \xrightarrow{\Phi\to Q} W_1$$

for the Kaluza-Klein monopole background. W_1 is defined as $W_1 := H^{B-G} + Q\Phi$.

On the other hand, the Komar mass can not be related to other masses or the free energy in terms of Legendre transformation with respect to those already known thermodynamic variables. However, since the Komar mass satisfies Smarr's formula [9]

$$M_K - Q\Phi = \frac{3}{2}TS. \tag{6}$$

then, it would be thought of as a thermodynamic mass and would be related to the free energy by a Legendre transformation with respect to some thermodynamic variable. In order to do so, we introduce a couple of thermodynamic variables (X, Y_f) or (X, Y_G), depending on choice of the background. X is proportional to square of L and Y_f or Y_G is its conjugate on the flat background and on the Kaluza-Klein monopole background, respectively. Especially, Y_G can be written as

$$Y_G = -\frac{\pi}{16G}\frac{2}{R_+ r_+}\left(\frac{r_+}{2} - R_+\right)^2, \tag{7}$$

where

$$R_+ = \frac{r_+}{2}\sqrt{\frac{r_\infty^2 - r_-^2}{r_\infty^2 - r_+^2}} \tag{8}$$

is the circumference radius of S^2 base space at the outer horizon and $r_+/2$ is the circumference radius of S^1 fiber there. Therefore, Y_G vanishes when the size of S^2 at the outer horizon equals that of S^1 and the horizon is a full-orbed three sphere as in the case of the extreme black hole. In this way, Y_G represents the squashing of the horizon. On the other hand, Y_f does not have this property because the flat background does not have any spherically symmetric limit unlike the case of the Kaluza-Klein monopole background. With these new variables, the Komar mass satisfy the following first laws:

$$dM_K = TdS + \Phi dQ + XdY_f, \quad \text{or} \quad dM_K = TdS + \Phi dQ + XdY_G. \tag{9}$$

The last terms are new thermodynamic work terms. Furthermore, the Komar mass can be related to the free energy by Legendre transformation with these new variables as

$$\begin{aligned} F^{BH} - F^{flat} &= M_K - TS - Q\Phi - XY_f, && (10)\\ F^{BH} - F^{GPS} &= M_K - TS - Q\Phi - XY_G. && (11) \end{aligned}$$

Then, the Legendre transformations are summarized as follows:

$$\begin{tikzcd}
(F^{BH} - F^{fl}) \arrow[r, "T\to S"] & H^{B-f} \arrow[r, "X_f\to Y_f"] \arrow[d, "\Phi\to Q"'] & W_2 \arrow[d, "\Phi\to Q"'] \\
 & M_{AD}^{B-f} \arrow[r, "X_f\to Y_f"] & M_K
\end{tikzcd}$$

for the flat background and

$$\begin{tikzcd}
(F^{BH} - F^{GPS}) \arrow[r, "T\to S"] & H^{B-G} \arrow[r, "X\to Y_G"] \arrow[d, "\Phi\to Q"'] & W_2 \arrow[d, "\Phi\to Q"'] \\
 & W_3 \arrow[r, "X\to Y_G"] & M_K
\end{tikzcd}$$

for the Kaluza-Klein monopole background. From equation (6), it is found that $W_2 = \frac{3}{2}TS$.

The new quantities X and its conjugate Y_f or Y_G also give a new work term to the first laws of the Hamiltonian and the AD mass.

$$\begin{aligned} dH^{B-f} &= TdS - Qd\Phi + Y_f dX && (12)\\ dH^{B-G} &= TdS - Qd\Phi + Y_G dX && (13)\\ dM_{AD} &= TdS + \Phi dQ + Y_f dX. && (14) \end{aligned}$$

Thus, thermodynamic first law for each Hamiltonian or AD mass is not unique. For example, H^{B-f} is a function of (S, Φ, X) as well as of (S, Φ, L). On the other hand, the gravitational tension does not give any work term to the first law of the Komar mass. And, it is found that any Legendre transformation with respect to the gravitational tension or its conjugate quantity can not relate the Komar mass to the free energy and the other masses.

We also investigate thermodynamic stability of the black hole. There are many thermodynamic environment depending choice of a set of independent thermodynamic variables. The specific heat of the black hole is apt to be negative and then it is unstable under thermal fluctuation. For near extremal black hole in environment with fixed electric charge, it tends to be positive and then, the black hole is locally thermodynamically stable. On the other hand, in environment with fixed electric potential difference, the specific heat is always negative and the black hole is always locally thermodynamically unstable. In environment with fixed X or L, the most globally stable state that has the minimum free energy is the flat spacetime whereas, in environment with fixed Y, the most globally stable state is degenerate both for the flat spacetime and the Kaluza-Klein monopole. With fixed gravitational tension $\mathcal{T}$, the Kaluza-Klein monopole is the most globally stable state. It is found that, there is a phase transition between the black hole with positive specific heat and the flat spacetime in environment with fixed T, Q and $\mathcal{T}$. In this situation, the most stable state is the Kaluza-Klein monopole spacetime and the black hole is thought of as a meta-stable state. These result are summarized in the following table:

Environment	The most stable state	local stability of the BH	phase transition
(T, Q, L) (T, Q, X)	flat	only near extremal	none
(T, Q, Y)	flat = KK monopole	only nearextremal	none
$(T, Q, \mathcal{T})$	KK monopole	not only near extremal	BH ↔ flat
(T, Φ, L) (T, Φ, X)	flat	alway unstable	none
(T, Φ, Y)	flat= KK monopole	alway unstable	none
$(T, \Phi, \mathcal{T})$	KK monopole	alway unstable	none

In isolated system, we have an interest in thermodynamic stability with respect to the size of the extra dimension, that is the size of the S^1 fiber at the spatial infinity L. We consider the sequence of the solutions with the same mass and the same electric charge. Then, the entropy of the black hole with the same mass and the same electric charge is a monotonically decreasing function with respect to the size of the extra dimension

$$\left(\frac{\partial S}{\partial L}\right)_{M,Q} < 0, \quad \text{or} \quad \left(\frac{\partial S}{\partial X}\right)_{M,Q} < 0. \tag{15}$$

Therefore, smaller extra dimension is entropically favored. In the limit that L becomes infinity, the black hole becomes the five-dimensional Reissner-Nordstörm (RN) black hole. Therefore, the five-dimensional RN black hole has the minimum entropy. In this sense, it might be the most unstable state in the sequence of the black hole solutions with the same mass and the same electric charge.

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