Lorentz-Finsler geometry and Einstein equations

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Thesis supervised by

Dr. Miguel Sánchez Caja and Dr. Miguel Ángel Javaloyes Victoria

for the

PhD Program in Mathematics

of the Universities of Granada, Almería, Cádiz, Jaén and Málaga

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What we know is a drop, what we don't know is an ocean.
(Attributed to Isaac Newton)

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Resumen en español

La geometría de Lorentz-Finsler generaliza los productos escalares lorentzianos a una *norma de Lorentz* en cada punto de una variedad. Estas normas lorentzianas aún permiten definir vectores temporales, la longitud de curvas de este género y (bajo ciertas condiciones de regularidad) vectores luminosos y causalidad. Todo ello da lugar a un espaciotiempo de Finsler. Además de por su interés matemático, en las dos últimas décadas tal generalización ha experimentado un fuerte auge de interés por sus posibles papeles en física. Citemos, solo a modo de muestra, la *relatividad muy especial* de Cohen y Glashow (2006) o las *relaciones de dispersión modificadas* (2007 en adelante), las cuales son centrales en fenomenología de la gravedad cuántica. Para las aplicaciones a gravedad clásica o cosmología, una cuestión acuciante permanece sin consenso: ¿Cuál es la generalización más apropiada de la ecuación de campo de Einstein al ambiente finsleriano?

La presente Memoria contribuye a la resolución de este problema en diferentes aspectos. Su punto de vista es que, matemáticamente, deben tenerse en cuenta dos principios. El primero es la preponderancia de las ecuaciones variacionales en el espíritu del formalismo de Hilbert de 1915. El segundo es que la conexión de una variedad desempeña un papel independiente al de su métrica, conduciendo a formalismos tipo Hilbert-Palatini. Con mayor motivo, además, en el caso de extensiones relativistias finslerianas, pues ahí conviven al menos tres conceptos naturales que generalizan al de conexión afín. Mencionemos también la necesidad de entender sistemáticamente las consecuencias de estos formalismos, en forma de leyes de conservación, y las posibles fuentes de materia/energía para sus ecuaciones. Teniendo todo esto en cuenta como motivación, tras un capítulo introductorio (que pretendemos sirva como guía de lectura), desarrollamos cuatro capítulos. Cada uno de ellos reproduce en su totalidad una de las publicaciones de investigación que compendian la presente Memoria y que describimos con detalle a continuación.

El Capítulo 1 corresponde al artículo [JSV1]. Se dedica a un estudio sistemático de las conexiones anisotrópicas ∇, las cuales formalizan la idea intuitiva de tener una conexión afín clásica con un valor distinto para cada observador en cada punto. Estas ∇ habían sido implícitamente introducidas por autores como Matthias (1980), Shen (2001) o Rademacher (2004) y desarrolladas por Javaloyes (2014 en adelante), pero su rol en geometría pseudo-Finsler (entiéndase, la obtenida reemplazando arriba la signatura lorentziana por una cualquiera) no estaba completamente establecido. Aquí, determinamos tal rol al demostrar que el conjunto de todas las ∇ está en biyección canónica con el de las conexiones finslerianas ∇^* (o sea, lineales en el fibrado vertical) que verifican la propiedad de ser verticalmente triviales. Más aún, obtenemos el panorama completo de relaciones entre las conexiones anisotrópicas y otras más estándar:

1) Cada ∇ induce naturalmente una conexión nolineal (o sea, una distribución horizontal sobre el fibrado tangente), y viceversa.
2) Prefijada una conexión nolineal, cada ∇^* (verticalmente trivial o no) se proyecta naturalmente sobre una ∇, que aparece como la "parte horizontal" de ∇^*.
3) En presencia de una métrica de pseudo-Finsler L, la conexión nolineal vendrá determinada por el spray de las L-geodésicas, obteniéndose que las ∇^* de Hashiguchi y Berwald se proyectan en la ∇ de este último y las ∇^* de Cartan y Chern-Rund se proyectan sobre la ∇ de Chern. (Notemos que las ∇^* de Berwald y Chern son verticalmente triviales pero las de Hashiguchi y Cartan no.)

Existe un transporte paralelo asociado a ∇ que tampoco había sido objeto de un estudio específico previo, lo que nos lleva a dilucidar este aspecto de la geometría de conexiones anisotrópicas. Consistentemente con la intuición de estas, encontramos la siguiente interpretación. Para transportar un vector o tensor a lo largo de una curva general α, primero debemos elegir un cierto campo vectorial V a lo largo de ella (nótese que el tangente a α puede no ser admisible). La elección natural es transportar paralelamente un vector v ("observador instantáneo"), de modo que el campo resultante V sobre α resulte paralelo con respecto a la dirección determinada por el propio V. De esta manera, el observador v elegido sirve como referencia para propagar el resto de objetos. Concluimos que este "transporte paralelo adaptado al observador" es apropiado, pues demostramos que ∇ se recupera a partir del transporte de tensores anisotrópicos. También que la preservación del tensor fundamental g mediante paralelismos caracteriza a la conexión de Chern, lo que lleva a interpretarla como la *conexión de Levi-Civita* de la métrica L.

El Capítulo 2 corresponde al artículo [JSV2]. En él, generalizamos el funcional de Einstein-Hilbert-Palatini a uno que depende de una métrica de pseudo-Finsler y una conexión nolineal arbitrarias. Este funcional extiende de manera natural el formalismo puramente métrico desarrollado por Pfeifer et al. (2012/19), considerando la clase de conexiones más adecuada para definir la curvatura de Ricci (que, en el caso finsleriano, es una función escalar sobre la proyectivización del fibrado tangente). Obtenemos sus ecuaciones de Euler-Lagrange para variaciones de la métrica y de la conexión, y demostramos que el estudio de sus soluciones puede reducirse al de aquellas con torsión 0. Tras esta reducción, vemos que diferentes conexiones, si resuelven la correspondiente ecuación, necesariamente tienen diferentes pregeodésicas, en fuerte contraste con el formalismo de Palatini relativista clásico. Esto, y la interpretación física, nos lleva a preguntarnos:

- Dada la métrica L, ¿existe una única conexión?
- En tal caso, ¿se recupera la misma ecuación que en el formalismo métrico de Pfeifer?

Dedicamos los resultados principales a responder a ambas cuestiones. La gran dificultad reside en que, en el lugar de lo que en el formalismo de Palatini clásico eran meras ecuaciones algebraicas, aparece un sistema de EDPs en cada espacio tangente. Con el fin de extraer toda la información posible de dicho sistema, hemos usado distintas herramientas analíticas nada triviales:

1) Nuestro resultado principal de unicidad muestra que la conexión es única si se asume que su restricción a cada espacio tangente es analítica (como ocurre trivialmente en el caso de las métricas de Lorentz y conexiones afines clásicas). Además, las geodésicas luminosas de la conexión siempre coinciden con las de L. Todo esto lo obtenemos mediante un argumento original de divisibilidad por potencias de L que difiere fuertemente de enfoques habituales tipo Cauchy-Kovalevskaya. De hecho, es aplicable en signatura indefinida pero no en el caso definido positivo, donde se usará un argumento totalmente distinto (véase el ítem 3)).
2) En el caso Lorentz-Finsler, el principio del máximo de Hopf permite sustituir la analiticidad del ítem anterior por diferenciabilidad en todo el cono causal futuro, siempre que el tensor de Cartan medio de L se anule idénticamente.
3) En caso de que L sea una métrica riemanniana, el conocimiento de los valores propios del laplaciano de una esfera implica que la conexión debe ser la de Levi-Civita.
4) Más aún, cada una de estas tres técnicas también permite demostrar que nuestra ecuación para variaciones de L se reduce a que se anule idénticamente la curvatura de Ricci de la conexión (aunque esta no tiene por qué ser ninguna de las canónicamente asociadas a L). Compárese con el caso clásico, en el que este resultado se obtenía simplemente tomando la traza de la ecuación de Einstein en el vacío y la conexión necesariamente era (esencialmente) Levi-Civita.

Como conclusiones, si el tensor de Landsberg medio de L se anula idénticamente, recuperamos la situación análoga a la del formalismo de Palatini clásico, así como la ecuación de Pfeifer et al. Pero si no lo hace, el panorama es remarcablemente distinto, pues la conexión solución necesariamente tendrá pregeodésicas diferentes a las de L.

El Capítulo 3 corresponde a [Vill] y trata sobre la extensibilidad del clásico teorema de Schur a geometría (pseudo-)Finsler. Para la curvatura bandera, distintas demostraciones habían sido dadas por Berwald (1947), del Riego (1973), Matsumoto (1986) y Z. Szilasi (2009). Pero para la curvatura de Ricci, continúa siendo un problema abierto. El principal avance en esta dirección fue obtenido por Robles en 2003, cuando demostró la extensión del teorema a la familia de métricas de Finsler-Randers. Aquí, de manera complementaria, lo demostramos para cualquier métrica de Finsler cuyo tensor de Landsberg medio sea 0. Nuestra técnica está inspirada por los desarrollos del Capítulo 2, pues puede verse como aplicación del teorema de Noether a la invariancia por difeomorfismos de nuestros funcionales.

De hecho, suponiendo que la curvatura de Ricci es una función en la variedad base, obtenemos una identidad que expresa su diferencial en cada punto como una integral en la correspondiente indicatriz de términos construidos a partir del tensor de Landsberg medio. Así, en principio estamos caracterizando exactamente cuándo el teorema de Schur se extiende. Terminamos demostrando un resultado que unifica todos los teoremas tipo Schur que se conocían para métricas de pseudo-Finsler, y analizando las implicaciones sobre el problema general.

El Capítulo 4 corresponde a [JSV3]. Aquí, por primera vez, se realiza un estudio sistemático de las posibles fuentes físicas de las ecuaciones de Einstein finslerianas en la forma de un tensor de impulso-energía T. Razonamos que, genéricamente, T será anisotrópico. En efecto, hay diversos motivos que nos llevan a ello en un espaciotiempo de Finsler, como por ejemplo, la violación de la simetría Lorentz predicha en ámbitos cuánticos, la medición de un fluido por distintos observadores o el formalismo lagrangiano. Para ilustrar la idea, presentamos distintos ejemplos de tensores de este tipo, obtenidos a partir de la función de distribución de un gas cinético o de lagrangianos tipo Hilbert-Finsler. Tras esto, nuestro principal objetivo es estudiar la ley de conservación infinitesimal $\mathrm{div}(T) = 0$.

Ello nos requiere introducir todo un cálculo de corchetes y derivadas de Lie anisotrópicos, usando la conexión nolineal. Hecho este desarrollo, definimos geométricamente la divergencia de un campo vectorial anisotrópico y (enfatizando la naturalidad de nuestra propuesta) posteriormente la de T. Demostramos un teorema de la divergencia anisotrópico que generaliza otros de Rund (1975) y Minguzzi (2017), y que permite dar una interpretación física de div(T) y, por tanto, de div(T) $= 0$ como ley de conservación. En efecto, en regiones de espaciotiempo de volumen (finsleriano) pequeño, dado un campo vectorial anisotrópico X, el momento total (calculado integrando $T_V(X_V)$, donde V es un campo de observadores) se conserva entre dos instantes salvo por su flujo a través de la frontera espacial. Finalmente, obtenemos los siguientes ejemplos de leyes globales a partir de la infinitesimal div(T) $= 0$:

1) En un espacio afín con una norma de Lorentz. En él, cada campo vectorial paralelo produce conservación del momento en esa dirección entre dos instantes de tiempo.
2) En un espaciotiempo de Finsler globalmente hiperbólico. En él, el momento total en una dirección se conserva entre dos hipersuperficies de Cauchy cuando se dan condiciones de decaimiento apropiadas en el infinito espacial.

[JSV1] M. Á. Javaloyes, M. Sánchez y Fidel F. Villaseñor. Anisotropic connections and parallel transport in Finsler spacetimes. En *Developments in Lorentzian geometry* (actas de *GeLoCor2021*), Springer Proceedings in Mathematics & Statistics vol. 389 (2022).

[JSV2] M. Á. Javaloyes, M. Sánchez y Fidel F. Villaseñor. The Einstein-Hilbert-Palatini formalism in pseudo-Finsler geometry. *Adv. Theor. Math. Phys.* vol. 26, no. 10, pp. 3563–3631 (2022).

[Vill] Fidel F. Villaseñor. Schur theorem for the Ricci curvature of any weakly Landsberg Finsler metric. En prensa en *Israel J. Math.*; preprint: https://arxiv.org/abs/2304.08933 (2023).

[JSV3] M. Á. Javaloyes, M. Sánchez y Fidel F. Villaseñor. On the significance of the stress-energy tensor in Finsler spacetimes. *Universe* vol. 8, no. 2, artículo 63 (2022).

Summary

Lorentz-Finsler geometry generalizes the Lorentzian scalar products to a Lorentz norm at each point of a manifold. These Lorentzian norms still allow for the definition of timelike vectors, the length of curves of this type, and (under certain regularity conditions) lightlike vectors and causality. All of this gives rise to a Finsler spacetime. In addition to its mathematical interest, over the past two decades, such a generalization has experienced a strong surge of interest for its potential roles in physics. To mention just a few examples, there is Cohen and Glashow's *very special relativity* (2006) or *modified dispersion relations* (2007 onwards), which are central to quantum gravity phenomenology. For applications to classical gravity or cosmology, an urgent question remains without consensus: What is the most appropriate generalization of the Einstein field equation to the Finslerian ambient?

The present Report contributes to the resolution of this problem in different aspects. Its point of view is that, mathematically, two principles must be taken into account. The first is the preponderance of variational equations in the spirit of the Hilbert formalism of 1915. The second is that the connection of a manifold plays a role independent of that of its metric, leading to Hilbert-Palatini type formalisms. This is even more so in the case of Finslerian relativistic extensions, since there coexist at least three natural concepts that generalize that of affine connection. Let us also mention the need to understand systematically the consequences of these formalisms, in the form of conservation laws, and the possible matter/energy sources for their equations. With all this in mind as a motivation, after an introductory chapter (which we intend to serve as a reading guide), we develop four chapters. Each of them reproduces in its entirety one of the research publications that make up this Report and which we describe in detail below.

Chapter 1 corresponds to the article [JSV1]. It is devoted to a systematic study of anisotropic connections ∇, which formalize the intuitive idea of having a classical affine connection with a different value for each observer at each point. These ∇'s had been implicitly introduced by authors such as Matthias (1980), Shen (2001) or Rademacher (2004) and developed by Javaloyes (2014 onwards), but their role in pseudo-Finsler geometry (i.e., the one obtained by replacing above the Lorentzian signature by an arbitrary one) was not fully established. Here, we determine such a role by proving that the set of all ∇'s is in canonical bijection with that of Finslerian connections ∇^* (i.e., linear in the vertical fibration) which satisfy the property of being vertically trivial. Moreover, we obtain the full picture of relations between anisotropic

connections and other more standard ones:

1) Each ∇ naturally induces a nonlinear connection (i.e., a horizontal distribution over the tangent bundle), and vice versa.
2) Once a nonlinear connection is prefixed, every ∇^* (vertically trivial or not) is naturally projected onto a ∇, which appears as the “horizontal part” of ∇^*.
3) In the presence of a pseudo-Finsler metric L, the nonlinear connection will be determined by the spray of the L-geodesics, obtaining that the ∇^*'s of Hashiguchi and Berwald project onto the ∇ of the latter and the ∇^*'s of Cartan and Chern-Rund project onto the ∇ of Chern. (Note that the Berwald and Chern ∇^*'s are vertically trivial, but the Hashiguchi and Cartan ones are not.)

There exists a parallel transport associated with ∇ that had not been the subject of a specific study before, leading us to clarify this aspect of the geometry of anisotropic connections. Consistently with the intuition of these, we find the following interpretation. To transport a vector or tensor along a general curve α, we must first choose a certain vector field V along it (note that the tangent to α may not be admissible). The natural choice is to parallel transport a vector v ("instantaneous observer"), so that the resulting field V along α is parallel with respect to the direction determined by V itself. In this way, the chosen observer v serves as a reference to propagate the rest of the objects. We conclude that this "parallel transport adapted to the observer" is appropriate, as we prove that ∇ is recovered from the transport of anisotropic tensors. Also, the preservation of the fundamental tensor g by parallelism characterizes the Chern connection, which leads to interpreting it as the *Levi-Civita connection* of the metric L.

Chapter 2 corresponds to the article [JSV2]. In it, we generalize the Einstein-Hilbert-Palatini functional to one that depends on an arbitrary pseudo-Finsler metric and nonlinear connection. This functional naturally extends the purely metric formalism developed by Pfeifer et al. (2012/19), considering the class of connections that are most suitable for defining the Ricci curvature (which, in the Finslerian case, is a scalar function on the projectivization of the tangent bundle). We obtain its Euler-Lagrange equations for variations of the metric and of the connection, and prove that the study of their solutions can be reduced to that of those with 0 torsion. After this reduction, we observe that different connections, if they solve the corresponding equation, necessarily have different pregeodesics, in stark contrast to the classical relativistic Palatini formalism. This, along with the physical interpretation, leads us to ask:

- Given the metric L, does there exist a unique connection?
- If so, does one recover the same equation as in Pfeifer's metric formalism?

We dedicate the main results to answering both questions. The great difficulty lies in the fact that, in place of mere algebraic equations as in the classical Palatini formalism, a system of PDEs appears on each tangent space. In order to extract all possible information from this system, we have used various nontrivial analytical tools:

1) Our main uniqueness result shows that the connection is unique if we assume that its restriction to each tangent space is analytic (as trivially occurs in the case of classical Lorentz metrics and affine connections). Furthermore, the lightlike geodesics of the connection always coincide with those of L. We obtain all of this through an original argument of divisibility by powers of L that differs significantly from typical approaches of Cauchy-Kovalevskaya type. In fact, it is applicable in indefinite signature but not in the positive definite case, where a completely different argument will be used (see item 3)).
2) In the Lorentz-Finsler case, Hopf's maximum principle allows one to replace the analyticity from the previous item with smoothness on all of the future causal cone, provided that the mean Cartan tensor of L vanishes identically.
3) In the case in which L is a Riemannian metric, the knowledge of the eigenvalues of the Laplacian on a sphere implies that the connection must be the Levi-Civita one.
4) Furthermore, each of these three techniques also allows one to prove that our equation for variations of L reduces to the Ricci curvature of the connection vanishing identically (although this connection does not have to be any of the ones canonically associated with L). Compare this with the classical case, where the same result was obtained simply by taking the trace of the Einstein equation in vacuum and the connection necessarily was (essentially) the Levi-Civita one.

As conclusions, if the mean Landsberg tensor of L vanishes identically, we recover a situation analogous to that of the classical Palatini formalism, as well as the equation by Pfeifer et al. But if it does not, the picture is remarkably different, as the connection in the solution will necessarily have different pregeodesics than those of L.

Chapter 3 corresponds to [Vill] and deals with the extensibility of the classical Schur theorem to (pseudo-)Finsler geometry. For the flag curvature, various proofs had been given by Berwald (1947), del Riego (1973), Matsumoto (1986), and Z. Szilasi (2009). However, for the Ricci curvature, it remains an open problem. The main advance in this direction was achieved by Robles in 2003, when she proved the extension of the theorem to the family of Finsler-Randers metrics. Here, complementarily, we prove it for any Finsler metric whose mean Landsberg tensor is 0. Our technique is inspired by the developments of Chapter 2, as it can be seen as an application of Noether's theorem to the diffeomorphism invariance of our functionals.

In fact, assuming that the Ricci curvature is a function on the base manifold, we obtain an identity that expresses its differential at each point as an integral over the corresponding indicatrix of terms constructed from the mean Landsberg tensor. Thus, in principle we are characterizing exactly when the Schur theorem extends. We end by proving a result that unifies all known Schur-type theorems for pseudo-Finsler metrics and analyzing the implications for the general problem.

Chapter 4 corresponds to [JSV3]. Here, for the first time, a systematic study is conducted of the possible physical sources for Finslerian Einstein equations in the form of an stress-energy tensor T. We argue that, generically, T will be anisotropic. Indeed, there are various reasons leading to this in a Finsler spacetime, such as the violation of Lorentz symmetry predicted in quantum realms, the measurement of a fluid by different observers, or the Lagrangian formalism. To illustrate the idea, we present different examples of such tensors, obtained from the distribution function of a kinetic gas or Hilbert-Finsler type Lagrangians. After this, our main objective is to study the infinitesimal conservation law $\mathrm{div}(T) = 0$.

This requires introducing an entire calculus of anisotropic Lie brackets and derivatives, using the nonlinear connection. Once this development is done, we geometrically define the divergence of an anisotropic vector field and (emphasizing the naturalness of our proposal) subsequently that of T. We prove an anisotropic divergence theorem that generalizes others by Rund (1975) and Minguzzi (2017), allowing for a physical interpretation of $\mathrm{div}(T)$ and, therefore, of $\mathrm{div}(T) = 0$ as a conservation law. Indeed, in regions of small (Finslerian) spacetime volume, given an anisotropic vector field X, the total momentum (computed by integrating $T_V(X_V)$, where V is an observer field) is conserved between two instants except for its flow across the spatial boundary. Finally, we obtain the following examples of global laws from the infinitesimal $\mathrm{div}(T) = 0$:

1) In an affine space with a Lorentz norm. Here, each parallel vector field provides conservation of momentum in that direction between two instants of time.
2) In a globally hyperbolic Finsler spacetime. Here, the total momentum in a direction is conserved between two Cauchy hypersurfaces when appropriate decay conditions are satisfied at spatial infinity.

[JSV1] M. Á. Javaloyes, M. Sánchez and Fidel F. Villaseñor. Anisotropic connections and parallel transport in Finsler spacetimes. In *Developments in Lorentzian geometry* (proceedings of *GeLoCor2021*), Springer Proceedings in Mathematics & Statistics vol. 389 (2022).

[JSV2] M. Á. Javaloyes, M. Sánchez and Fidel F. Villaseñor. The Einstein-Hilbert-Palatini formalism in pseudo-Finsler geometry. *Adv. Theor. Math. Phys.* vol. 26, no. 10, pp. 3563–3631 (2022).

[Vill] Fidel F. Villaseñor. Schur theorem for the Ricci curvature of any weakly Landsberg Finsler metric. To appear in *Israel J. Math.*; preprint: https://arxiv.org/abs/2304.08933 (2023).

[JSV3] M. Á. Javaloyes, M. Sánchez and Fidel F. Villaseñor. On the significance of the stress-energy tensor in Finsler spacetimes. *Universe* vol. 8, no. 2, article 63 (2022).

About the content

This is a thesis on pseudo-Finsler geometry, the theory of connections and the interplay of these topics with mathematical physics. Its main focus is on the Lorentz-Finsler case, the one that makes it possible to generalize and extend Einstein's general theory of relativity. But it also contains applications to the positive definite case which are of a more purely geometric interest. Concretely, the thesis is composed by the following research publications:

- *Anisotropic connections and parallel transport in Finsler spacetimes* (reference [1] below and Chapter 1).
- *The Einstein-Hilbert-Palatini formalism in pseudo-Finsler geometry* (reference [2] below and Chapter 2).
- *Schur theorem for the Ricci curvature of any weakly Landsberg Finsler metric* (reference [3] below and Chapter 3).
- *On the significance of the stress-energy tensor in Finsler spacetimes* (reference [4] below and Chapter 4).

Observe that the last two papers have been displayed in the opposite order to that of their first appearances due to thematic reasons that will be made clear in the Introduction and conclusions chapter. There, we endeavor to provide thoughtful commentary on the articles in order to facilitate their reading. But we also have the aim of giving updated views, with a broader perspective than that at their original time of writing. So, we provide: (A) A historical and conceptual introduction to the subject, (B) A discussion of the problems that we tackle, (C) A summary of the results of each of the above, (D) Their interrelations and subsequent conclusions, and (E) A list of open questions.

Remark 0.1. Each Chapter reproduces its corresponding article in the version published or accepted by the respective journal, up to some occasional update of reference, correction of grammatical typo or very minor stylistic change. The only substantial changes are the following:

- pp. 107 and 132 Text added in Rem. 2.7 (C) and the proof of Th. 2.3 resp., to correct a previous mathematical typo in the latter. See also footnote 13.
- pp. 146 onwards Sign of the Ricci scalar changed (mathematical typo). See also footnote 10.

- pp. 197 and 201 Footnotes added to match the latest version https://doi.org/10.48550/arXiv.2202.10801.

Before starting, we include a table collecting the main concepts used in the introductory chapter and in the articles. To prevent possible confusions, besides their definition, it includes the majority of the notations associated with each of them in a comparative manner between the Introduction and conclusions and Chapters 1 to 4. Let us use this chance to remark that throughout the thesis, Einstein's summation convention holds on the indices $i, j, k, l, a, b, c, d, e \in \{1, \dots, n\}$, where $n \geq 2$. (Still, this can always be understood e.g. in the spirit of Penrose's abstract index notation [5].) For these indices, the "subscript with a dot" notation $_{\cdot i}$, $_{\cdot j}$, $_{\cdot k}$, ... is ubiquitous in the thesis and means a *vertical derivative* ∂_{y^i}, ∂_{y^j}, ∂_{y^k}, ... of the corresponding object. All objects are *smooth*, meaning $\mathscr{C}^\infty$, unless stated otherwise.

Term	Def.	Intr.	Ch. 1	Ch. 2	Ch. 3	Ch. 4
Connected smooth manifold	–	M	=	=	=	=
Dimension of M	–	n	=	=	=	=
Point of M	–	x	p	=	=	x
Vector in T_xM	–	y	v	=	=	y
Coordinates on M	–	x^i	=	=	=	=
Natural coordinates on T_xM	§4.2	y^i	=	=	=	=
Set of admissible directions	Def. 1.1	A	=	=	=	=
(Pos.) projectivization of A	§2.2.1	–	–	$\mathbb{P}^+A$	$\mathbb{P}^+\mathrm{T}M, (\mathrm{T}M)^+$	–
Admissible vector field	§1.3.1	$V(x), V^i(x)$	V	–	–	V
Vertical bundle of A	§1.3.1	$\mathrm{V}A$	=	=	–	$\mathrm{V}A$
Set of A-anisotropic tensors	§1.3.1	–	$\mathcal{T}^r_s(M_A)$	=	–	$\mathcal{T}^r_s(M_A)$
Canonical (Liouville) field	(1.3)	$\mathbb{C}, y^i$	$=, \mathbb{C}^{\mathrm{V}}$	=	$\mathbb{C}, y^i$	=
Vertical derivative	Def. 1.3	${}_{\cdot i}, \partial_{y^i}$	$=, \dot{\partial}_i$	=	=	=
Pseudo-Finsler metric	Def. 2.7	L, F	=	=	=	L
(Standard) Finsler metric	Def. 3.1	F	–	L, F	F	–
Fundamental tensor	Def. 2.7	g_{ij}	=	=	=	=
Indicatrix at x	§1.2.1	Σ_x	Σ_p	Σ^F	–	Σ_p
Null cone at x	§1.2.1	$\mathcal{C}_x, \partial A_x$	$\mathcal{C}_p$	∂A_p	–	$\mathcal{C}_p$
Cartan tensor	§4.2.3	–	C_{ijk}	C_{ijk}	C_{ijk}	C_{ijk}
Mean Cartan tensor	(4.3)	–	–	C_i	C_i	$\mathrm{C}^{\mathfrak{m}}, \mathrm{C}_i$
Hilbert form	(3.3)	–	–	–	–	ω_i
Projectivized L-volume form	§3.2.2	$\underline{d\mu}, d\Sigma$	–	$\underline{d\mu}$	$d\Sigma^+$	–
Anisotropic L-volume form	(4.18)	$d\mathrm{Vol}$	–	–	–	$d\mathrm{Vol}$
Anisotropic connection	Def. 1.4	∇, Γ^i_{jk}	=	=	–	∇, Γ^i_{jk}
Vertical bundle connection	(1.22)	$\nabla^*, \left((\Gamma^{\mathrm{H}})^i_{jk}, (\Gamma^{\mathrm{V}})^i_{jk}\right)$	=	–	–	–
Spray	Def. 1.8	G^i	=	–	–	–
Nonlinear connection	§1.4.1	N^i_j	$=, D, \nu$	N^i_j	–	D, N^i_j
Torsion of Γ^i_{jk}	Def. 1.5	–	Tor^i_{jk}	–	–	Tor^i_{jk}
Torsion of N^i_j	(2.9)	Tor^i_{jk}	–	Tor^i_{jk}	–	–
Curvature of N^i_j	(2.9)	–	–	$\mathcal{R}^i_{jk}$	–	$\mathcal{R}^i_{jk}$
Ricci curvature (or scalar) of N^i_j	(2.9)	Ric	–	Ric	$=, \mathrm{Ric}/L$	–
Spray of L	(2.14)	$\left(G^L\right)^i$	G^i	$\left(G^L\right)^i$	G^i	–
Nonlinear connection of L	(4.4)	$\left(G^L\right)^i_{\cdot j}$	$\mathring{N}^i_j, \mathring{\nu}$	$\left(\mathrm{N}^L\right)^i_j, \left(G^L\right)^i_{\cdot j}$	$G^i_{\cdot j}$	N^i_j
Chern covariant derivative	(4.5)	${}_{\vert i}$	–	–	${}_{\vert i}$	–
Landsberg tensor	§2.2.4	P_{ijk}	$\mathcal{L}$	Lan_{ijk}	P_{ijk}	Lan_{ijk}
Mean Landsberg tensor	(4.7)	P_i	–	Lan_i	P_i	$\mathrm{Lan}^{\mathfrak{m}}, \mathrm{Lan}_i$
Anisotropic Lie bracket	Def. 4.2	$\mathfrak{l}^{\mathrm{H}}_Z$	–	–	–	$\mathfrak{l}^{\mathrm{H}}_Z$
Anisotropic Lie derivative	Def. 4.2	–	–	–	–	$\mathfrak{L}^{\mathrm{H}}_Z$

Introduction and conclusions

(A) Finsler, pseudo-Finsler and Lorentz-Finsler geometries

It is well known[1] that Finsler geometry was actually originated by Riemann. In fact, in his famous 1854 dissertation [7], he defined manifolds with a length measure for curves: in our modern notation,

$$\ell[\gamma] = \int F(\gamma, \frac{\mathrm{d}\gamma}{\mathrm{d}t})\,\mathrm{d}t, \tag{1}$$

where $F(x, y)$ is a homogeneous function of degree 1 in the y variable. Riemann even suggested $F(x, y) = \sqrt[4]{H_{abcd}(x)y^a y^b y^c y^d}$, and only after that he focused on the *quadratic* case $F(x, y)^2 = g_{ab}(x)y^a y^b$. This has led Chern to appropriately describe Finsler geometry as "Riemannian geometry without the quadratic restriction" [8]. Its true development began with Paul Finsler's 1918 PhD thesis [9], and with later foundational works by, among others, Berwald [10], Cartan [11], Chern [12] or Rund [13]. Nowadays, it is a classical field in differential geometry after many celebrated results. Just to mention a few, let us cite those by Berwald [14], Szabó [15], Akbar-Zadeh [16], Bryant [17], Bao-Robles-Shen [18, 19, 20] or Asanov [21].

Accordingly, pseudo-Finsler geometry is "pseudo-Riemannian[2] geometry without the quadratic restriction". To be precise, the signature distinction of *Riemannian* vs. *pseudo-Riemannian* is now embodied in the convexity properties in the y variable that the function $L(x, y) = F(x, y)^2$ has. This is measured by the *fundamental tensor*, of

[1] See the preface of [6], which is a standard reference for the field.

[2] We opt for this term instead of the equivalent *semi-Riemannian* for consistency with the commonly used *pseudo-Finsler*.

components

$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}(x,y), \tag{2}$$

and in turn impacts the behavior of the length functional (1) (optimization, triangle inequalities... See [22, §3].) One is thus led to the notion of a *pseudo-Finsler metric* ((2) is nondegenerate). Provided that appropriate y-global conditions hold (see e.g. Def. 2.8), the most distinguished cases[3] are those of a *Finsler* one ((2) is positive definite) and a *Lorentz-Finsler* one ((2) is of signature $(+,-,\dots,-)$). The latter is our main interest.

Due to the homogeneity of the function L, a Lorentz-Finsler metric will be determined by its *(future timelike) indicatrices*[4] $\Sigma_x := \{y\colon\ L(x,y)=1\} \subset \mathrm{T}_x M$, which admit a clear geometric interpretation. Indeed, Finsler geometry generalized the Riemannian tangent ellipsoids to any strongly convex hypersurface enclosing 0_x. Analogously, the Lorentzian hyperboloids are now generalized to any embedded Σ_x that is radial and strongly concave with respect to minus the position vector field $-\mathbb{C}|_{\mathrm{T}_x M}$. The domain of L will be taken to be the set $A \subset \mathrm{T}M \setminus \mathbf{0}$ whose *fiber* $A_x := A \cap \mathrm{T}_x M$ is the union of all the rays joining 0_x with a point of Σ_x. These objects were introduced by Beem [25], but his definition was challenged in the subsequent literature:

1) The *spacelike* directions (which with our convention would have $L < 0$) were deemed physically meaningless, apparently starting with Asanov [26], and we do not include them in our definition.
2) Some examples lack a *null cone* $\mathcal{C}_x := \{y\colon\ L(x,y)=0\}$ to which Σ_x is asymptotic, and in many others in which $\mathcal{C}_x$ exists, it has nonsmoothness or degeneracy issues. However, consistently with our foundational standpoint, we will follow [27] (where, implicitly, pathololological cones are regarded as limits of non pathological ones). Therefore, it will be natural for us to assume that L extends to $\overline{A}$ in a regular manner with $\partial A = \bigcup_{x\in M} \partial(A_x) = \bigcup_{x\in M} \mathcal{C}_x \subset \mathrm{T}M \setminus \mathbf{0}$. (Incidentally, this turns out to be the convenient hypothesis for the main results of Ch. 2.)

All in all, one obtains a picture of the following type at each point x of a *Finsler spacetime* (M, L):

[3] The book [23], and implicitly [24], covers the general case, but, as far as we know, no signature-specific development has been carried out outside the definite and Lorentzian realms.

[4] Had we taken $(-,+,\dots,+)$ to be our Lorentzian signature, we would have had to put "$L(x,y) = -1$" in the definition of the indicatrix. As in the thesis we always have to work essentially on the indicatrix bundle $\Sigma := \bigcup_{x\in M} \Sigma_x$, our choice makes computations simpler.

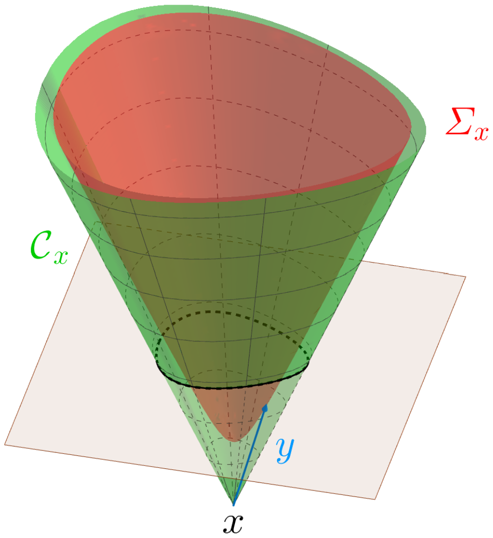

Figure by David Moya

From the physical viewpoint, the cone structure ∂A is enough to develop all of causality theory [28], while (1) would be the proper time measured by the observer with worldline γ. In view of all this, the possibility of extending the rest of interpretations of special and general relativity to Finsler spacetimes becomes apparent, and it is important to make clear that this is no mere academic exercise of generalization. Rather, the interest in this topic has grown quickly in the last years [29, 30, 31, 32, 33, 34, 27, 35, 36, 37, 38, 39, 40, 41, 42] from specific motivations, such as the following.

- The Ehlers-Pirani-Schild axiomatics for spacetime [43] actually allow for certain Lorentz-Finsler metrics when carefully analyzed [44], e.g. of the form

$$L(x, y) = e^{2u(x,y)} g_{ab}(x) y^a y^b.$$

- Beyond this, usual postulates on observers' measurements can be regarded as linear idealizations, which when non-linearized, can give rise to Finsler spacetimes. See [45] for a thorough discussion.
- As shown in [46], Cohen and Glashow's *very special relativity* proposal [47] leads to metrics of Bogoslovsky [48] type:

$$L(x, y) = \left(g_{ab}(x) y^a y^b\right)^{1-\beta} \left(\mathcal{A}_c(x) y^c\right)^{2\beta}.$$

- In quantum gravity phenomenology, to each *modified dispersion relation* (a function on $\mathrm{T}^* M$), one can attach a dual pseudo-Finsler metric [49].

- Tied to the previous point, Kostelecky's *standard model extension* [50] yields examples of the form

$$L(x,y) = \left\{ \sqrt{g_{ij}(x)y^i y^j} + \mathcal{A}_k(x)y^k \pm \sqrt{\left(\mathcal{B}_a(x)y^a\right)^2 - g^{bc}(x)\mathcal{B}_b(x)\mathcal{B}_c(x)g_{de}(x)y^d y^e} \right\}^2 .$$

We refer the reader to [51] for a general review of the emergence of Lorentz-Finsler geometry in physics.

(B) Einstein equations and generalized connections

In order for the physical interpretation of Finsler spacetimes to be fully consistent, an extension of the Einstein equation [52] is required. By this, we understand a PDE system that relates the geometry of the Lorentz-Finsler metric with the matter and energy content of the spacetime. For a Lorentzian metric $g_{ij}(x)$ [53] with Ricci tensor $\mathrm{ric}^g_{ij}(x)$, the classical Einstein eq. (see e.g. [54, Ch. 17] for details) is

$$\mathrm{ric}^g_{ij}(x) - \frac{1}{2}\left(g^{ab}\mathrm{ric}^g_{ab}\right)(x)\, g_{ij}(x) + \Lambda g_{ij}(x) = \kappa T_{ij}(x). \tag{3}$$

The extension of this equation will determine the dynamics of the Finslerian gravitational field. To be precise, the worldlines of free-falling particles must be given by the geodesics associated with L, or a connection related with L, and these would act as gravitational potentials.

There is disagreement among authors about what the most appropriate extension is, with plenty of independent proposals having been made. As $L(x,y)$ and $g_{ij}(x,y)$ are defined on the open subset $A \subseteq \mathrm{T}M \setminus \mathbf{0}$ of the previous section, so are the proposals generically. Roughly, they can be classified into two[5] types [51, §V]:

1) **Tensorial equations formally analogous to** (3), such as the following. Asanov [55] obtained (3) but for an *osculating metric tensor* $\bar{g}_{ij}(x) := g_{ij}(x, V(x))$. Miron [56] considered the standard Einstein eq. for a Sasaki-type metric constructed from (2). Li and Chang [57] argued from the Finslerian Bianchi identities in a particular case.

[5]Besides other approaches, such as that of the reference [9] in [34], we may be omitting some examples in our account of types 1) and 2).

2) **Homogeneous scalar equations.** The idea being that as the fundamental variable is the function L, a single PDE of the same type should be given for it. This thesis' study falls mostly into this class, so let us comment on the examples more detailedly.

In 1993, Rutz [58] proposed the vacuum equation to be the vanishing of the *Ricci curvature*, $\mathrm{Ric}^L(x, y) = 0$. (Note that Ric^L is a scalar function on A that in the classical case $L(x, y) = g_{ab}(x)y^a y^b$ adopts the form $\mathrm{Ric}^L(x, y) = \mathrm{ric}^g_{ab}(x)y^a y^b$.) In 2012, Pfeifer and Wohlfart [59] considered what seems to be the simplest possible variational principle and derived the more complicated equation

$$(n+2)\,\mathrm{Ric}^L - g^{ab}\mathrm{Ric}^L_{\cdot a \cdot b}\, L - 2g^{ab}\left\{P_{a|b} - P_a P_b + \left(y^c P_{a|c}\right)_{\cdot b}\right\} = 0. \tag{4}$$

(It must be mentioned that (4) had appeared before in the positive definite context and with a different interpretation [60].) Quite interestingly, in 2019, Hohmann, Pfeifer and Voicu [61] proved that Rutz's eq. is not variational and that (4) is its so-called *canonical variational completion*. Moreover, in [62], the same authors constructed a well-defined coupling of (4) with the distribution function (1PDF) of a kinetic gas within a framework for Finslerian field theories.

With the aim of contributing to the problem of the selection of some Finslerian Einstein equations, the thesis' approach has two starting points:

- From the viewpoints of modern physics [63] and geometric variational problems, it seems reasonable to restrict the search to Euler-Lagrange eqs. of geometric functionals.
- Then, it should be taken into account that (3) can actually be derived in different ways, particularly from the *Palatini formalism*.[6] Indeed, in it, one considers the *Einstein-Hilbert-Palatini functional*

$$\mathscr{F}[g_{ij}, \Gamma^i_{\ jk}] := \int g^{ab}(x)\,\mathrm{ric}_{ab}(x)\, d\mathrm{Vol}(x) \tag{5}$$

(where ric_{ij} is the Ricci tensor of the arbitrary affine connection $\Gamma^i_{\ jk}$), which has the (gauge) symmetry

$$\Gamma^i_{\ jk}(x) \mapsto \Gamma^i_{\ jk}(x) + \mathcal{A}_j(x)\,\delta^i_k. \tag{6}$$

[6]This name (common in textbooks [64], see also [65, App. E.1]) is unfortunate, since the method was discovered by Einstein [66]. Still, we use it for distinction from general *metric-affine formalisms* [67], in which eventually matter would couple to the independent connection. In this case, the results in § (C2) concerning (9) could suffer modifications.

The critical points of (5) are given by (3) (for $\Lambda = 0$ and $T_{ij} = 0$) together with the condition

$$\Gamma^i_{jk}(x) = (\Gamma^g)^i_{jk}(x) + \mathcal{A}_j(x)\,\delta^i_k, \tag{7}$$

where $\mathcal{A}_i$ is arbitrary. Then, the torsion-freeness requirement, or the metric compatibility one, fixes the gauge to be $\mathcal{A}_i = 0$, yielding only the Levi-Civita connection. We refer to [68] for details.

This second point becomes even more relevant in the Finslerian setting, for there are several nonequivalent notions generalizing that of an affine connection, see [69] for an account. A first question thus arises: What is the geometric or physical meaning of these different generalized connections?

Then, even within each of these nonequivalent classes, there is typically more than one connection canonically associated with the metric. (For instance, within the class of anisotropic connections, one has *Chern's* and *Berwald's* [70, §2.6 and 2.7], and within that of vertical bundle connections, one has *Chern's*, *Cartan's*, *Berwald's* and *Hashiguchi's* [71, §5.1], [5, p. 39], possibly among others.) Hence, for Finslerian Palatini theories, analogous to (5): Do they select a unique connection? Is it one of the canonical ones? Do they lead to the same equation (4) as Pfeifer's *metric formalism*?

Having established (4) or its Palatini extensions: What are the conservation laws that follow from symmetries of their functionals? Do they lead to interesting geometric results, as it happens in the classical case? (The conservation law following from the Einstein-Hilbert action is the contracted second Bianchi identity, which in turn implies the Schur theorems [72, Th. 1.97 and 1.89].)

Lastly, what are the matter/energy sources to which (4) or its Palatini counterparts couple? What is their physical interpretation and relation with conservation laws?

Remark 0.2. Before examining our results and what they tell us about these problems, let us say a few words on our terminology. We are referring as a pseudo-Finsler *metric* to the function L, even though, of course, it is not a metric tensor. Moreover, below, we will talk about *anisotropic* and *nonlinear* connections, despite that, technically, one could have "isotropic anisotropic connections" and "linear nonlinear connections". While some readers may find these choices awkward, we made them, first, to have simple and comfortable names available, and, second, to match what we find to be the most common usages in the literature on (pseudo-)Finsler geometry.

(C) Summary of our results

Next, we will display the main developments of each of the four articles [1, 2, 3, 4]. In order to assist the reader, we will do so with a unified notation, consistent with the one used in this introductory chapter and that may differ from some of the notation used along the next Chapters. Moreover, we shall give refinements or alternative statements of some of the results, reflecting an up-to-date understanding of them.

(C1) In Chapter 1 (reference [1])

We review the notion of[7] *(homogeneous) A-anisotropic connections* ∇ (Def. 1.4), objects determined by Christoffel symbols that depend on the direction (i.e., $\Gamma^i_{\ jk}$ with $\Gamma^i_{\ jk}(x,\lambda y) = \Gamma^i_{\ jk}(x,y)$ for $(x,y) \in A$ and $\lambda > 0$). Their role in relation to pseudo-Finsler geometry was not totally clear before, so we prove that they are the "horizontal parts" of the Finsler connections. Besides this one, we find the correspondences that exist between the ∇'s and the other standard objects: nonlinear connections and sprays on A. On top of that, we determine exactly what notion of parallel transport ∇ provides.

(Homogeneous) nonlinear connections are objects of components $N^i_j(x,y)$ with $N^i_j(x,\lambda y) = \lambda N^i_j(x,y)$ for $\lambda > 0$. As discussed in §1.4.1, they can be defined in different ways, such as distributions of *horizontal subspaces*

$$\mathrm{H}_{(x,y)}A = \mathrm{Span}\left\{ \partial_{x^i}\big|_{(x,y)} - N^a_i(x,y)\, \partial_{y^a}\Big|_{(x,y)} \ :\ i = 1,\dots,n \right\} \subset \mathrm{T}_{(x,y)}A$$

or as nonlinear covariant derivative operators (1.18). The result on their relation with anisotropic connections, cf. Th. 1.2, is the following.

Theorem 0.1. *There is a bijection between the set of anisotropic connections $\nabla (\equiv \Gamma^i_{\ jk})$ and that of pairs formed by a nonlinear connection N^i_j and a 0-homogeneous A-anisotropic tensor $Q^i_{\ jk}$ with $Q^i_{\ ja}y^a = 0$. Indeed,*

$$\nabla \leftrightarrow (N^i_j, Q^i_{\ jk}); \qquad N^i_j := \Gamma^i_{\ ja}y^a, \quad Q^i_{\ jk} := \Gamma^i_{\ jk} - N^i_{\ j\cdot k} = -\Gamma^i_{\ ja\cdot k}y^a.$$

In particular:

[7] Here, A is as in § (A). Keep in mind the notational difference between A and $\mathcal{A}_i$; the choice for the latter was in order to follow [68].

(i) *One has natural maps given by* $(N^i_j, Q^i_{jk}) \mapsto N^i_j$ *and* $N^i_j \mapsto \nabla^N \equiv (N^i_j, 0)$.
(ii) *The classical affine connections can be described either as anisotropic connections of the form* $\Gamma^i_{jk}(x)$ *or as nonlinear ones of the form* $\Gamma^i_{ja}(x)y^a$.

(Homothety invariant) linear connections on the vertical bundle $\mathrm{V}A \to A$ are also known as *Finsler connections* [7]. They are studied in §1.5 and can be defined as covariant derivatives ∇^* as in (1.22), among other ways. For the result that relates them with anisotropic connections, namely Prop. 1.3 together with Th. 1.3, one needs to fix an auxiliary nonlinear connection $\mathring{N}^i_j$:

Theorem 0.2. *There is a bijection between the set of vertical bundle connections* ∇^* *and that of pairs formed by an anisotropic connection and a* (-1)*-homogeneous* A*-anisotropic* $(1,2)$ *tensor. Indeed,*

$$\nabla^* \leftrightarrow \left(\left(\Gamma^{\mathrm{H}}\right)^i_{jk}, \left(\Gamma^{\mathrm{V}}\right)^i_{jk}\right);$$

$$\left(\Gamma^{\mathrm{H}}\right)^i_{jk} \partial_{y^i} := \nabla^*_{\partial_{x^j} - \mathring{N}^a_j \partial_{y^a}} \partial_{y^k}, \quad \left(\Gamma^{\mathrm{V}}\right)^i_{jk} \partial_{y^i} := \nabla^*_{\partial_{y^j}} \partial_{y^k}.$$

In particular:

(i) *One has natural maps given by* $\left(\left(\Gamma^{\mathrm{H}}\right)^i_{jk}, \left(\Gamma^{\mathrm{V}}\right)^i_{jk}\right) \mapsto \nabla \equiv \left(\Gamma^{\mathrm{H}}\right)^i_{jk}$ *and* $\nabla \equiv \Gamma^i_{jk} \mapsto (\Gamma^i_{jk}, 0)$.
(ii) *The tensor* $\left(\Gamma^{\mathrm{V}}\right)^i_{jk}$ *is independent of the chosen* $\mathring{N}^i_j$*, and so is the set of vertical bundle connections with* $\left(\Gamma^{\mathrm{V}}\right)^i_{jk} = 0$*, called* vertically trivial. *Moreover, for this class of* ∇^**'s, the anisotropic connection* $\left(\Gamma^{\mathrm{H}}\right)^i_{jk}$ *is also independent of the chosen* $\mathring{N}^i_j$.

Sprays, discussed in §1.6.1, are vector fields of the form $y^i\partial_{x^i} - 2G^i\partial_{y^i}$ with $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$ for $(x, y) \in A$ and $\lambda > 0$. They are characterized by the facts that their integral curves are the derivatives of curves on M, called the *geodesics* of the spray, and any affine reparametrization of a geodesic is again a geodesic. In Prop. 1.4 and §1.6.2, we show the following.

Theorem 0.3. *There is a bijection between the set of nonlinear connections N^i_j and that of pairs formed by a spray and a 1-homogeneous A-anisotropic $(1,1)$ tensor. Indeed,*

$$N^i_j \leftrightarrow (G^i, J^i_j); \qquad G^i := \frac{1}{2} N^i_a y^a, \quad J^i_j := N^i_j - G^i_{\cdot j}.$$

In particular, one has natural maps given by $(G^i, J^i_j) \mapsto G^i$ and $G^i \mapsto (G^i, 0)$. Moreover:

(i) *J^i_j determines and is determined by the* torsion *of N^i_j, namely* $\mathrm{Tor}^i_{jk} := N^i_{j\cdot k} - N^i_{k\cdot j}$.

(ii) *When in Th. 0.2 one chooses $\mathring{N}^i_j \equiv ((G^L)^i, 0)$ with $(G^L)^i$ the* canonical spray *of a pseudo-Finsler metric L, one obtains the following identifications (in which P^i_{jk} is the* Landsberg tensor*):*

Vertical bundle ∇^*	$\equiv$	$\left((\Gamma^{\mathrm{H}})^i_{jk}, (\Gamma^{\mathrm{V}})^i_{jk}\right)$
Berwald	$\equiv$	$((G^L)^i_{\cdot j\cdot k}, 0)$
Hashiguchi	$\equiv$	$((G^L)^i_{\cdot j\cdot k}, C^i_{jk})$
Chern(-Rund)	$\equiv$	$((G^L)^i_{\cdot j\cdot k} - P^i_{jk}, 0)$
Cartan	$\equiv$	$((G^L)^i_{\cdot j\cdot k} - P^i_{jk}, C^i_{jk})$,

which then project onto

Anisotropic ∇	$\equiv$	Γ^i_{jk}	$\equiv$	(N^i_j, Q^i_{jk})
Berwald	$\equiv$	$(G^L)^i_{\cdot j\cdot k}$	$\equiv$	$((G^L)^i_{\cdot j}, 0)$
Chern	$\equiv$	$(G^L)^i_{\cdot j\cdot k} - P^i_{jk}$	$\equiv$	$((G^L)^i_{\cdot j}, -P^i_{jk})$.

(Note that these results recover part of those of [29, 69, 71], as well as some of the constructions of [73], from a unifying viewpoint.)

In the second part of the article, we turn our attention to the parallel transport associated with an anisotropic connection ∇, which had not been studied in the previous literature. As noted in §1.7.1, the anisotropic covariant derivative along a curve[8] $\alpha(t) \in$

[8]The notational switch from γ to α is intentional. Indeed, in § (A), by γ we denoted the worldline of an observer, so an L-timelike curve, whereas here we are developing the theory for an arbitrary one, α. Hence the need of supplying the additional observer field V. The case $(\alpha, V) = (\gamma, \frac{d\gamma}{dt})$ had already been considered in [24, Def. 7.3.1]. The general case may be important from the physical standpoint, e.g. when needing to compare energy-momentum vectors at different points of the spacetime.

M (Def. 1.9, eventually given as in Prop. 1.5) needs of an *observer field* $V(t) \in A$. We propose to choose it such that $\frac{D^V V}{dt} = 0$, so that the *parallel transport of the instantaneous observer* $y = V(t_0)$ is defined to be $\mathcal{P}_{t_0,t_1}(y) := V(t_1)$ (Def. 1.11). Then, the *parallel transport* of $z \in T_{\alpha(t_0)}M$ *with respect to* y is defined as $\mathcal{P}^y_{t_0,t_1}(z) := Z(t_1) \in T_{\alpha(t_1)}M$ provided that $\frac{D^V Z}{dt} = 0$ and $Z(t_0) = z$ (Def. 1.12).

The main point of this construction is that it allows one to transport a whole A-anisotropic (r,s) tensor T from $\alpha(t_0)$ to $\alpha(t_1)$ (as in §1.7.2) and compute its covariant derivative, thus retrieving the anisotropic connection ∇ from this parallel transport (Prop. 1.6). In case that $T = g$ for a pseudo-Finsler metric L, it characterizes when ∇ is the Chern connection of L (Prop. 1.7). As this one is the only *torsion-free* anisotropic connection that parallelizes g, it is sometimes referred to as the *Levi-Civita–Chern connection* of L (see also Th. 1.4). All in all:

Theorem 0.4. *Let T be an A-anisotropic (r,s) tensor, $X = \frac{d\alpha}{dt}(0) \in T_{\alpha(0)}M$ for some curve $\alpha(t)$, and $y \in A_{\alpha(0)}$. If $\mathcal{P}_t(T)$ denotes the parallel transport of $T|_{A_{\alpha(t)}}$ to $\alpha(0)$, then*

$$\left(\nabla_X T\right)_y = \frac{d}{dt}\mathcal{P}_t(T)_y\Big|_{t=0}.$$

Assume now that ∇ is torsion-free and let L be a pseudo-Finsler metric. Then its fundamental tensor is preserved by parallel transport *(that is, for all y, w, z and t,*

$$g_{\mathcal{P}_t(y)}(\mathcal{P}^y_t(w), \mathcal{P}^y_t(z)) = g_y(w,z))$$

if and only if ∇ is the (Levi-Civita–)Chern connection of L.

(C2) In Chapter 2 (reference [2])

When looking for a Finslerian Einstein equation, there is a point of the utmost importance to take into account. Namely, the huge variety of possibilities that we have just seen in attaching a generalized connection to a Finsler spacetime. One is thus naturally led to the Palatini formalism, of which we develop a Finslerian version, see §2.3. As this is the first theory of this kind, we will stay at the level of nonlinear connections, but our study may serve as a basis for the study of functionals dependent on, e.g., anisotropic connections.

We begin by introducing the definition of *proper* pseudo-Finsler metrics (Def. 2.8), which will turn out to be the suitable ones for our results. (Sometimes, this condition can be weakened, see Rem. 2.22.) Then, we consider the functional that led to Pfeifer's equation (4),

$$\mathcal{S}[L, N^i_j] := \int g^{ab}\mathrm{Ric}_{\cdot a \cdot b}\, \underline{d\mu}, \tag{8}$$

in which now N^i_j is independent of L.[9] In Th. 2.1, we find that its Euler-Lagrange equation for variations of N^i_j is[10]

$$\left(\frac{n+2}{2}\frac{\nabla^N_c L}{L} - g^{ab}\nabla^N_c g_{ab} + \mathrm{Tor}^a_{ac}\right)\left(y^c\delta^j_i - y^j\delta^c_i\right) - \mathrm{Tor}^j_{ia}y^a = 0, \tag{9}$$

while the equation for variations of L is

$$(n+2)\,\mathrm{Ric} - Lg^{ab}\mathrm{Ric}_{\cdot a\cdot b} = 0. \tag{10}$$

The first outstanding feature of (8) is a gauge symmetry that extends (6). Indeed, in Lem. 2.2, we see that (8) is invariant under

$$N^i_j(x,y) \mapsto N^i_j(x,y) + \mathcal{A}_j(x,y)\, y^i$$

whenever $\mathcal{A}_i$ is 0-hom. ($\mathcal{A}_i(x,\lambda y) = \mathcal{A}_i(x,y)$ for $\lambda > 0$). Consequently (for a fixed L), the set $\mathrm{Sol}_L(A)$ of solutions of (9) can be thought of as formed by gauge classes $\left[N^i_j\right]$, but in Th. 2.2, we find that $\mathcal{A}_i$ can be uniquely prescribed yielding a unique torsion-free (or *symmetric*) representative in each class. Working with such a representative is the same as working with a 2-homogeneous A-anisotropic vector field $\mathcal{Z}^i$:

$$N^i_j \leftrightarrow \mathcal{Z}^i; \qquad \mathcal{Z}^i := \frac{1}{2}\left\{N^i_a y^a - \left(G^L\right)^i\right\}, \quad N^i_j := \left(G^L\right)^i_{\cdot j} + \mathcal{Z}^i_{\cdot j}.$$

Indeed, this follows from Th. 0.3, and then $\mathcal{Z}^i$ can be thought of as an element of the set $\mathrm{Sol}^{\mathcal{S}\mathrm{ym}}_L(A)$. Moreover, upon defining $\sigma^{\mathcal{Z}}$ and $K^{\mathcal{Z}}_i$ as in (2.38) and (2.39) resp., we find (Lem. 2.3) that the equation (9) reads

$$\mathcal{Z}^i - 2\sigma^{\mathcal{Z}}y^i + Lg^{ia}\left(\sigma^{\mathcal{Z}}_{\cdot a} + K^{\mathcal{Z}}_a\right) = -\frac{2}{n+2}Lg^{ia}P_a, \tag{11}$$

[9] In the notation of Def. 2.9, what we are displaying here would be $\mathcal{S}_\star$. We choose to write (8) because of its formal similarity with the Einstein-Hilbert(-Palatini) functional. Anyway, it is variationally equivalent to $\int L^{-1}\mathrm{Ric}\,\underline{d\mu}$ (see Prop. 2.6), which is the one actually used for the computations.

[10] There is a typo along [2, App. B]. There, we started by writing "$\mathrm{Ric}(t) = \delta_b(t)N^c_a(t)\left(\delta^a_c y^b - y^a\delta^b_c\right)$" instead of $\mathrm{Ric}(t) = \delta_b(t)N^c_a(t)\left(y^a\delta^b_c - \delta^a_c y^b\right)$, which resulted in an overall minus sign with respect to the correct variation formula. At any rate, this does not affect the Euler-Lagrange eq., of which here we display the form obtained in that App. B for geometric intuition. The form in Th. 2.1 is just (9) written in terms of $\mathcal{J}^i_j := N^i_j - \left(G^L\right)^i_{\cdot j}$.

$$(n+2)\,\sigma^{\mathcal{Z}} - 2C_a\mathcal{Z}^a - \mathcal{Z}^a_{\cdot a} = 0, \tag{12}$$

the latter interchangeable with

$$(n-2)\,\sigma^{\mathcal{Z}} - Lg^{ab}\left(\sigma^{\mathcal{Z}}_{\cdot a\cdot b} + K^{\mathcal{Z}}_{a\cdot b}\right) = \frac{2}{n+2}Lg^{ab}P_{a\cdot b}. \tag{13}$$

Cor. 2.1 contains some simple facts about the solutions. Two of them, $\mathcal{Z}^i$ and $\widehat{\mathcal{Z}}^i$, have the same *pregeodesics* if and only if $\widehat{\mathcal{Z}}^i = \mathcal{Z}^i$. The canonical connection $\left(G^L\right)^i_{\cdot j}$ (given by $\mathcal{Z}^i = 0$) is a solution if and only if L is *weakly Landsberg*, that is, $P_i = 0$ (see also Rem. 2.15).

In view of this and also of physical interpretations, it is important to ask if in fact there is only one solution of (11)+(12)/(13) (analogously as in the classical Palatini formalism, recall (7)). In such a case, the follow-up question is what does the equation (10) look like in terms of it. These are approached by realizing that (11)+(12)/(13) is actually a PDE system on each fiber $A_x = A \cap \mathrm{T}_xM$. (This is because the only derivatives of the unknown $\mathcal{Z}^i$ that appear on it are vertical ones, i.e., with respect to the coordinates y^i of $y \in A_x$.) There, several analytic tools can be employed to study the system, and thus, in §2.5, we prove the main theorems of this article. We do so assuming that[11] dim M is at least 3 and by means of three techniques.

The first of these is what we call *(smooth) division by powers of L on ∂A*, developed in Lem. 2.5 and Rem. 2.19. It provides the main uniqueness results of the thesis: Th. 2.3 and Cor. 2.2. But also, a way of simplifying (10) that did not appear in the article (cf. Th. 2.7). We take this chance to state these results together and to use hypotheses that take into account the refinements in[12] Rem. 2.22. (We shall also see that, technically, one typically does not need all the strength of the *proper solutions* of Def. 2.14.)

Theorem 0.5. *Let $B \subset \mathrm{T}M \setminus \mathbf{0}$ be any set such that $B_x (:= B \cap \mathrm{T}_xM)$ is a nonempty subset of $\partial(A_x) \subset \mathrm{T}_xM \setminus 0_x$ for all $x \in M$.*

(i) *Let $\mathcal{Z}^i$ and $\widehat{\mathcal{Z}}^i$ be solutions of* (11)+(12)/(13) *(defined on A) and assume that for each x, the restrictions $\mathcal{Z}^i\big|_{A_x}$ and $\widehat{\mathcal{Z}}^i\big|_{A_x}$ admit analytic extensions to $A_x \cup B_x$. Then*

$$\widehat{\mathcal{Z}}^i = \mathcal{Z}^i.$$

[11] In dimension 2, the theory appears to be substantially different and remains to be investigated.
[12] Notice the notational difference: the B of Rem. 2.22 would correspond to $A \cup B$ here.

(ii) *Let* N^i_j *be a solution of* (10) *and assume that each* $\mathrm{Ric}|_{A_x}$ *extends analytically to* $A_x \cup B_x$. *Then*

$$\mathrm{Ric} = 0.$$

Some points about this result must be stressed:

(a) The analyticity that it employs has nothing to do with any analytic structure whatsoever on M or $A \subset \mathrm{T}M \setminus \mathbf{0}$, much less with the solutions being fully analytic there. It is just the plain notion of analyticity of functions on subsets of the vector space $\mathrm{T}_x M$. (In the terminology of Def. 2.15, we would say that the objects are *fiberwise analytic on* $A \cup B$.)

(b) Eventually, it could be applied in situations in which $\mathcal{Z}^i|_{A_x}$ and $\widehat{\mathcal{Z}}^i\Big|_{A_x}$ are not analytic at any direction in $\partial(A_x)$, nor even smooth there. The only condition required to prove that $\widehat{\mathcal{Z}}^i\Big|_{A_x} = \mathcal{Z}^i|_{A_x}$ is the analyticity of $\widehat{\mathcal{Z}}^i\Big|_{A_x} - \mathcal{Z}^i|_{A_x}$ on A_x and on B_x, which could even consist of a single null ray.[13]

(c) The proof of (ii) is analogous to that of Lem. 2.5 together with Th. 2.3, just using the equation (10) in place of (11)+(12)/(13) (regarding Ric as the unknown).

Therefore, it should be clear that the division technique is radically different from other analyticity approaches to PDE systems, mainly Cauchy–Kovalevskaya's. In fact, no analogues in the literature are known to us.

The division technique also has as consequences the nonexistence result in Rem. 2.21 and the consistency of the null geodesics of any solution of (9) with those of $\left(G^L\right)^i$, namely Th. 2.4.

Theorem 0.6. *Let* $N^i_j = \left(G^L\right)^i_{\cdot j} + \mathcal{Z}^i_{\cdot j}$ *be a solution of* (11)+(12)/(13) *and assume that it extends smoothly to* $\overline{A}$. *Then, for each* $(x_0, y_0) \in \partial A$, *the unique (parametrized) geodesic of* N^i_j *with initial condition* (x_0, y_0) *coincides with the corresponding geodesic of* $\left(G^L\right)^i$. *In particular, it remains always null.*

[13] In the proof of Th. 2.3, we correct a subtle typo in that of [2, Th. 5.8]. Indeed, there, the existence of a null direction is asserted just because L satisfies the definition of proper metric of indefinite signature. But that actually requires of some nontrivial argument (Rem. 2.7 (C)). Alternatively, in Th. 0.5, such existence is assumed (in the form of B_x being nonempty).

Our second technique is the use of Hopf's maximum principle for elliptic PDEs [74], developed in Lem. 2.6. It is due to the metric induced by g_{ij} on Σ_x and requires L to be (properly) Lorentz-Finsler. (Interestingly, it cannot be of other indefinite signatures because then one would not have a Riemannian metric on Σ_x, and it cannot be definite because then the sign of the zeroth order term of (2.56) would be the wrong one.) The results are Th. 2.5, Cor. 2.3 and Th. 2.7, which we can state as follows.

Theorem 0.7. *Assume that L is Lorentz-Finsler and that the set of Th. 0.5 is taken to be $B = \partial A$.*

(i) *If $C_i = 0$, then the analyticity requirement of Th. 0.5* (i) *can be replaced with mere smoothness on $A_x \cup B_x = \overline{A_x} \subset \mathrm{T}_x M \setminus 0_x$, and the conclusion becomes*

$$\hat{\mathcal{Z}}^i = \mathcal{Z}^i = 0.$$

(ii) *The analyticity requirement of Th. 0.5* (ii) *can always be replaced by smoothness of* Ric *on $\overline{A_x}$, with the same conclusion.*[14]

Remark 0.3. Let us mention a very general version of Th. 0.7 (i). In the system (11)+(12)/(13), the definitions (2.38) and (2.39) can be interpreted as a change of variables, so that (σ, K_i) becomes the unknown. The result says that if L is properly Lorentz-Finsler and smoothness is assumed on $\overline{A_x}$, then σ is determined by K_i. In the particular cases of Th. 2.5 and Cor. 2.3 (the $C_i = 0$ one), it is determined to be $\sigma = 0$, and then $\mathcal{Z}^i_{(\sigma,K)} = 0$.

The third technique stems from the knowledge of the eigenvalues of the Laplacian of a round sphere, allowing one to prove a positive definite analogue of Th. 0.7. Indeed, by Deicke's theorem [75] the only Finsler metrics with $C_i = 0$ are the Riemannian ones, so Th. 2.6 is natural.

Theorem 0.8. *Assume that L is quadratic, namely $L(x, y) = g_{ab}(x) y^a y^b$, with g_{ij} a Riemannian metric. Then, the only solution of* (11)+(12)/(13) *that is smooth on all of* $\mathrm{T}M \setminus \mathbf{0}$ *is*

$$\mathcal{Z}^i = 0.$$

[14] This maximum principle technique allows one to get $\mathrm{Ric} = 0$ from more general equations than (10), see Th. 2.7 and Rem. 2.24.

Moreover, the equation (10) *reduces to*

$$\mathrm{Ric}^L = 0.$$

Finally, in §2.5.3, we prove that when we restrict ourselves to classical objects (i.e., pseudo-Riemannian metrics and affine connections), the solutions of our system (9)+(10) are exactly the same as those of the classical Palatini formalism (Th. 2.8).

(C3) In Chapter 3 (reference [3])

This article is about the extension of the Schur theorem[15] for the Ricci curvature to pseudo-Finsler geometry. Remarkably, this remains an open problem while the corresponding pseudo-Riemannian and *flag curvature* theorems are relatively easy to prove [14]. The only fairly general class (say, containing the Riemannian one) of Finsler metrics for which the problem is solved is Randers', due to Robles [20, Lem. 3.4]. Here, complementarily, we manage to extend the Ricci-Schur theorem to any Finsler metric that is weakly Landsberg. We do so using tools inspired by our previous article.

Whenever a variational principle is used to derive some equations, it is important to understand what "conservation laws"[16] follow from Noether's theorems [79]. In the case of general relativity (5), the relevant symmetry group is $\mathrm{Diff}(M)$, yielding the twice contracted second Bianchi identity [80, §3.3.3]. In the Finslerian setting, the group is the same and a version of the corresponding conservation law had already been found in [62, Th. 31]. In the classical case, the identity implies the Schur theorem, which makes the conservation law even more interesting in the Finslerian one. Indeed, the use of the Finslerian Bianchi identities for the present problem has been reported to fail [20], leading us to follow an alternative strategy.

We begin by clarifying our notions of *Einstein* pseudo-Finsler metrics (Def. 3.2) and of integration on the fibers of the indicatrix bundle Σ of a Finsler metric, see §3.2.2. (Equiv., those of the *projectivized tangent bundle* $(\mathrm{T}M)^+$, Rem. 3.5.) The latter is based upon a local splitting of $d\Sigma$ into a volume form on each fiber times a volume form on the base, namely $d\Sigma = d\Sigma_x \wedge \mathrm{d}^{(n)}x$. Still, its main feature is that averages with respect to $d\Sigma_x$ are independent of the choice of chart.

[15] We opted for this terminology instead of the perhaps more widespread *Schur lemma* [76, Prop. 7.19] because the result is strictly stronger than Schur's theorem for the sectional curvature [77].

[16] There are issues why interpreting the output of Noether's second theorem (the one at play in our case) as a true conservation law becomes tricky, see e.g. [78, §5.3], but we will abuse this terminology for simplicity.

In §3.3, after reviewing the corresponding pseudo-Riemannian case (3.13) and following [61, Prop. 7], we obtain a conservation law from $\bar{\mathcal{S}}[L] := \mathcal{S}[L, \left(G^L\right)^i_{\cdot j}]$ (where $\mathcal{S}$ is given by (8)) in Lem. 3.1. Due to the natural lifting of each $\varphi \in \mathrm{Diff}(M)$ to a $\Phi^+ \in \mathrm{Diff}((\mathrm{T}M)^+)$, the derivation of (3.21) crucially requires the metric to be *(standard)* Finsler, see Rem. 3.8 (cf. [62, Th. 31]). Still, (3.21) turns out not to be enough to prove the Schur theorem that we aim for, which is the reason behind introducing a second functional, namely $\mathcal{I}[f, L] := \int f \, d\Sigma$. We find the conservation law corresponding to the $\mathrm{Diff}(M)$-invariance of this one in Lem. 3.2.

In §3.4, we combine the obtained (3.21) and (3.23) to produce the main results of this article. These are an identity that in principle characterizes when the Schur theorem holds (Th. 3.2) and its consequence for weakly Landsberg metrics (Cor. 3.1). Interestingly enough, the identity also provides some nontrivial information in the case of dimension $n = 2$, something that does not happen for its pseudo-Riemannian counterpart. All in all:

Theorem 0.9. *Assume that the Finsler metric L is Einstein, namely $\mathrm{Ric}^L(x, y) = \rho(x) L(x, y)$ for some function ρ on M^n. Then*

$$(n-2)\, \partial_{x^i} \rho(x) = -2n \frac{\displaystyle\int_{\Sigma_x} \mathfrak{P}_{|0}(x, y)\, g_{ia}(x, y) y^a \, d\Sigma_x(y)}{\displaystyle\int_{\Sigma_x} 1 \, d\Sigma_x(y)} \tag{14}$$

for all $x \in M$, where

$$\mathfrak{P}_{|0} = g^{ab} \left(P_{a|b|0} - 2 P_a P_{b|0} - P_{a|0 \cdot b|0} \right)$$

with P_i being the mean Landsberg tensor and $_|$ denoting Chern's covariant derivative ($_{|0} := {}_{|c} y^c$). Therefore, if $\mathfrak{P}_{|0} = 0$ (in particular, if L is weakly Landsberg) and $n \geq 3$, then ρ is constant.

This result is not extensible to pseudo-Finsler metrics, nor to conic Finsler ones, as it requires that $L \neq 0$ on all of $\mathrm{T}M \setminus \mathbf{0}$. In §3.4.2, we also review the main Schur-type theorems known in the pseudo-Finsler case and we realize that they can be refined and unified into a single general statement (Th. 3.3), namely:

Theorem 0.10. *Assume that the pseudo-Finsler metric L is Einstein, $n \geq 3$ and the Ricci scalar* $\mathrm{Ric}^L(x,y)$ *is quadratic*[17] *on y. Then, L is either globally* Ricci-flat *or globally pseudo-Riemannian with constant* $\frac{\mathrm{Ric}^L}{L}$.

We end the article in §3.5, with a comparison of our results with all the other ones in the literature on the topic. Here, we also comment on the possible outcomes of the problem of a general Schur theorem for the Finslerian Ricci curvature.

(C4) In Chapter 4 (reference [4])

Motivated by the problem of how equations such as (4) or (10) couple to the matter/energy content of spacetime, here we study the Lorentz-Finsler generalization of the stress-energy tensor T_{ij} [81, §4.4]. The first part of the article elucidates the different approaches that lead to such a tensor. The second part gives a precise mathematical and physical meaning to T_{ij} having 0 divergence. An appendix analyzes the difficulties for physical interpretations of measurements if the richness of the Finslerian ambient is fully taken into account.

In §4.3.1, we straightforwardly extend the relativistic interpretations of the energy-momentum of particles, dusts and other fluids, resulting in a well-defined L-timelike vector at each point $x \in M$. Perhaps the main postulate here is that the rest space of an instantaneous observer $y \in \Sigma_x$ is defined by $\mathrm{T}_y\Sigma_x$, and their Euclidean measurements are carried out with the restriction of $g_{ij}(x,y)$ to that subspace. This leads to a violation of (infinitesimal) Lorentz invariance.[18] In §4.3.2, we argue that then there is no reason why the different energy/momentum density components, pressures and stresses should form a well-defined tensor $T_{ij}(x)$ on T_xM. Rather, in general, they can be assembled into an *anisotropic stress-energy tensor* $T_{ij}(x,y)$. We give an example illustrating that the information provided by a single fixed observer need not be

[17] This happens, for instance, in the important particular case that L is *Berwald*, meaning that $\left(G^L\right)^i(x,y)$ is of the form $\frac{1}{2}\Gamma^i_{ab}(x)y^a y^b$.

[18] The observer y will choose a $g_{ij}(x,y)$-orthonormal basis $\left\{e_0 = y, e_1, \ldots, e_{n-1}\right\}$ for physical measurements, whereas $y' \in \Sigma_x$ will choose a $g_{ij}(x,y')$-orthonormal one, $\left\{e_0 = y', e'_1, \ldots, e'_{n-1}\right\}$. Therefore, generically, the change of basis will not be in the Lorentz group $\mathrm{O}(1,n-1)$, but only in $\mathrm{GL}(n)$ (which still ensures the well-definedness of intrinsic objects in T_xM). Closely tied to this is the fact that there seems to be a handful of nonequivalent geometric procedures to measure the relative speed between observers (and, eventually, light), discussed in App. 4.6. This–at least apparent–freedom may be the source of certain ambiguities.

enough to determine T_{ij} (Ex. 4.3.1). Yet for any such observer, the various components retain their usual physical meaning.

In §4.3.3, we analyze the different variational prescriptions for a classical stress-energy tensor $T_{ij}(x)$, to see to what extent they carry over to the Lorentz-Finsler case. Noether's prescription still makes sense on an affine space, yielding objects analogous to (4.8), but Hilbert's one now yields homogeneous scalar functions (4.10) on A. From these functions, there would be various procedures to derive a 2-tensor comparable to $T_{ij}(x)$. Indeed, among possibly others, these would be (4.11) (cf. Ex. 4.3.1), (4.12) or the one obtained from the 1PDF of a kinetic gas in Ex. (4.3.2), namely (4.13). In any case, the generic situation ends up being that of an anisotropic stress-energy tensor $T_{ij}(x, y)$, as above.

In the second part of the article, with the motivation of studying the conservation of such an anisotropic tensor, we introduce a whole calculus of anisotropic Lie brackets, Lie derivatives and divergences. Given the horizontal distribution $\mathrm{H}A$ of a nonlinear connection (eventually, the canonical one, $\left(G^L\right)^i_{\cdot j}$), we start in §4.4 with the *anisotropic Lie bracket* $\mathfrak{L}^{\mathrm{H}}_Z$ associated with an anisotropic vector field $Z^i(x, y)$. This is a tensor derivation (Th. 4.1) with the key properties Th. 4.1 (D) and Prop. 4.1 when it acts on anisotropic differential forms. Upon considering the form $d\mathrm{Vol} := \sqrt{\left|\det g_{ij}(x, y)\right|}\,\mathrm{d}^{(n)}x$, they make it appropriate to define the *divergence* of Z^i by

$$\mathrm{div}(Z)\, d\mathrm{Vol} := \mathfrak{L}^{\mathrm{H}}_Z(d\mathrm{Vol})$$

(Def. 4.3). This is done without any particular choice of anisotropic or vertical bundle connection whatsoever (recall Th. 0.3). Still, Prop. 4.2 shows why it is most naturally expressed in terms of the Chern connection, thereby obtaining (4.35) in an *unbiased* way. The main result in §4.4.3 is a general anisotropic divergence theorem (Th. 4.2), which, after a study of the different choices of unit normal (Rem. 4.5), recovers the theorems by Rund [82] and Minguzzi [83].

Theorem 0.11. *If $V^i(x)$ is an A-valued vector field defined on an open subset $U \subseteq M$ and $\overline{D} \subseteq U$ is a smooth enough domain such that* $\mathrm{Supp}(Z_V) \cap \overline{D}$ *is compact, then*

$$\int_D \mathrm{div}(Z)_V\, d\mathrm{Vol}_V + \int_D \left(C_a Z^b + Z^b_{\cdot a}\right)_V \mathrm{D}_b V^a\, d\mathrm{Vol}_V = \int_{\partial D} \iota_{Z_V}(d\mathrm{Vol}_V),$$

where C_i is the mean Cartan tensor *and* $\mathrm{D}_j V^i := \partial_{x^j} V^i + \left(G^L\right)^i_{\cdot j}(V(x))$. *Moreover, expressing*

$$\int_{\partial D} \iota_{Z_V}(d\mathrm{Vol}_V) = \int_{\partial D} g_V(\widehat{N}_V, \widehat{N}_V)\, g_V(\widehat{N}_V, Z_V)\, d\sigma_V$$

with $\widehat{N}_V$ the unit normal induced by the pseudo-Riemannian metric g_V, one obtains Rund's divergence theorem [82, (3.17)], while expressing

$$\int_{\partial D} \iota_{Z_V}(d\mathrm{Vol}_V) = \int_{\partial D} silon_\xi\, L(\xi)\, g_\xi(\xi, Z_V)\, d\Sigma_V^\xi$$

with ξ the Finslerian normal, one recovers Minguzzi's one [83, Th. 1]

§4.5.1 finally introduces the *divergence* of an anisotropic tensor[19] $T^i_j(x,y)$ (Def. 4.4). This time, the Chern connection is directly used, since it appears as the only Finslerian connection that can guarantee (4.14), cf. §4.5.2. Some conditions are explored so that $\mathrm{div}(T(X)) = \mathrm{div}(T)(X)$ for an anisotropic vector field $X^i(x,y)$ (Prop. 4.3 and Rem. 4.6), which would allow one to simplify the form of the divergence theorem in Cor. 4.1. But, more importantly, from this one follows an infinitesimal interpretation of $\mathrm{div}(T)$, namely Rem. 4.10:

Theorem 0.12. *Let $x \in M$ and $y \in A_x$ be fixed, and let also $w \in \mathrm{T}_x M$. Consider local extensions V of y and X of w such that $\nabla_j V^i(x,y) = \nabla_j X^i(x,y) = 0$ (where ∇ is the Chern anisotropic connection), which always exist. If $\left\{D_m\right\}$ is a sequence of open subsets of M containing x such that $\mathrm{Vol}_V(D_m) \overset{m\to\infty}{\longrightarrow} 0$, then*

$$\mathrm{div}(T)_y(w) = \lim_{m\to\infty} \frac{\int_{\partial D_m} \iota_{T(X)_V}(d\mathrm{Vol}_V)}{\mathrm{Vol}_V(D_m)}.$$

Thus, the meaning of the covariant conservation $\mathrm{div}(T) = 0$ of our anisotropic stress-energy tensor is the following. If an observer y measures the net creation or destruction of energy-momentum $T(X)$ in a small enough spacetime region D_m, then it will

[19] A (1, 1) tensor was considered in order to enhance the interpretation (4.14), but $\mathrm{div}(T)_i$ can naturally be thought of as the divergence of the $T_{ij} = g_{ia} T^a_j$ above, see Rem. 4.7. (The symmetry of T_{ij} is never required for our analysis.)

be practically 0. (Even though, as a Finslerian effect, different observers y and y' will measure different conserved quantities $T(X)_V$ and $T(X)_{V'}$.)

As an illustration, in §4.5.3.1 and §4.5.3.2, we give two examples of global conservation laws obtained from $\mathrm{div}(T) = 0$. The first one is due to the use of parallel vector fields on an affine space equipped with a Lorentz norm, Cor. 4.3:

Theorem 0.13. *Assume that M is an affine space with a Lorentz-Finsler metric independent of the position, and let $V \in \mathfrak{X}(M)$ be parallely identifiable with a vector $y \in A_x$. If T^i_j is an anisotropic tensor with* $\mathrm{div}(T) = 0$ *and X^i is an anisotropic vector field with one of the properties of Prop. 4.3 (C), then*

$$\int_{\{t_1\}\times\Omega} T_V^\flat(V, X_V)\, d\sigma_V - \int_{\{t_0\}\times\Omega} T_V^\flat(V, X_V)\, d\sigma_V$$
$$= \int_{]t_0,t_1[\times\partial\Omega} T_V^\flat(\hat{n}_V, X_V)\, d\sigma_V.$$

Here, $\overline{\Omega}$ is a regular enough domain in the spacelike hyperplane $\mathcal{R} := x + \left\{ z \in \vec{M} : g_y(y,z) = 0 \right\}$, so that $(M, g_y) \equiv (\mathbb{R} \times \mathcal{R}, \mathrm{d}t^2 + g_y\big|_{\mathcal{R}})$ and the geometric objects $(-)^\flat$, *$\hat{n}$ and $d\sigma$ are derived from g_y.*

The second example is the conservation of energy-momentum between two *Cauchy hypersurfaces* of a *globally hyperbolic* Finsler spacetime, provided that appropriate decay rates hold at spatial infinity, Cor. 4.4:

Theorem 0.14. *In the setup of §4.5.3.2 (in which $\mathcal{S}_0, \mathcal{S}_1 \subset U$ are two nonintersecting Cauchy hypersurfaces), let $V^i(x)$ be an A-valued vector field defined on U (mapping each compact exhaustion $\{\Omega_{0,m}\}$ of $\mathcal{S}_0$ to another one, $\{\Omega_{1,m}\}$, of $\mathcal{S}_1$). Let T^i_j be an anisotropic tensor and X^i be an anisotropic vector field such that the hypotheses of Cor. 4.2 hold (on the region enclosed by $\mathcal{S}_0$ and $\mathcal{S}_1$) and so does the decay condition* (4.52). *Then*

$$\int_{\Omega_{1,m}} \iota_{T(X)_V}(d\mathrm{Vol}_V) + \int_{\Omega_{0,m}} \iota_{T(X)_V}(d\mathrm{Vol}_V) \longrightarrow 0 \quad (m \longrightarrow \infty).$$

(D) **Conclusions from the results**

The results of § (C1) settle the interrelations between the nonequivalent generalizations of affine connections. They also determine what each of them provides. Above the base notion of sprays, which provide geodesics,[20] there appear three levels of structure. Indeed, first, projecting over the sprays appear the nonlinear connections (Th. 0.3), which yield the parallel transport of instantaneous observers y. Second, projecting over the nonlinear connections appear the anisotropic ones (Th. 0.1), which yield the parallel transport of general anisotropic tensors. And third, projecting over the anisotropic connections appear the vertical bundle (linear) ones (Th. 0.2). Moreover, in each of the three levels, accompanying the projection, there is an injection in the reverse direction, which may be interpreted as a canonical choice of the additional structure, adding zero further information.

From the physics standpoint, employing one of the levels versus the others would be done according to which of the given elements, parallel transports among them, are deemed physically relevant. For example, assume that the scalar products $g_y(w, z)$ with $y \in A$ and $w, z \in \mathrm{T}_{\pi(y)}M$ are regarded as physical. Then Th. 0.4 tells one that the Chern connection is the convenient one (despite not coming from the canonical injections $\left(G^L\right)^i \mapsto \left(G^L\right)^i_{\cdot j} \mapsto \left(G^L\right)^i_{\cdot j \cdot k}$, which result in Berwald's). This is the case, e.g., in our interpretation in § (C4). There, consistently with the parallel transport, the Chern covariant derivative is found as the natural one for computing divergences.

Following this viewpoint, in § (C2), one may regard (8) as the appropriate functional, as (besides geodesics) nonlinear connections provide a good notion of Ricci curvature.[21] Ths. 0.5 to 0.8 suggest that in this theory, there is a unique torsion-free connection $\left(G^L\right)^i_{\cdot j} + \mathcal{Z}^i_{\cdot j}$ (equiv., a unique class of pregeodesics) as a solution of the equations. In the weakly Landsberg case $P_i = 0$, this is $\mathcal{Z}^i = 0$, and when one substitutes it into (10), this becomes $\mathrm{Ric}^L = 0$. Moreover, just the same happens[22] to the

[20]Physically, these would be perhaps the most elementary ingredient of dynamics on a spacetime: the trajectories of point particles in free fall. This suggests that a *(projective class of)* spray(s) [24, §4.1 I] would yield the most general mathematical formalization of the weak equivalence principle [81, §7.1].

[21]Still, our analysis may serve as a basis for the study of more complicated functionals, eventually depending on anisotropic or even vertical bundle connections, cf. [84]. Besides, despite our discussion, it must be pointed out that any independent connection admits an alternative interpretation. Indeed, in view of the minimal coupling principle of the Palatini formalism [68, (8)], it is totally possible to interpret the dynamical connection $N^i_j \neq \left(G^L\right)^i_{\cdot j}$ as just an auxiliar field. Physical particles would then follow geodesics of $\left(G^L\right)^i$.

[22]Recall that the techniques that prove Th. 0.5 (ii) and Th. 0.7 (ii) also work for (4) provided that

Pfeifer-Wohlfart-Hohmann-Voicu equation (4). When $P_i \neq 0$, necessarily $\mathcal{Z}^i \neq 0$, but not much more is known about the solution. Only that it still shares its null geodesics with $\left(G^L\right)^i$ (Th. 0.6) and that (10) still reduces to Ric $= 0$ (for the connection $\left(G^L\right)^i_{\cdot j} + \mathcal{Z}^i_{\cdot j} \neq \left(G^L\right)^i_{\cdot j}$, see Th. 0.5 (ii) or Th. 0.7 (ii)). In summary,

- $\boxed{P_i = 0}$ The same conclusions as in the classical Palatini formalism [68] hold true, and our proposed Finslerian Einstein equation coincides with (4), i.e., $\mathrm{Ric}^L = 0$.
- $\boxed{P_i \neq 0}$ Conclusions analogous to those of the classical case necessarily fail. The situation also appears substantially different from that of (4): for starters, the actual solution $\left(G^L\right)^i_{\cdot j} + \mathcal{Z}^i_{\cdot j}$ must have different pregeodesics from those of the metric spray $\left(G^L\right)^i_{\cdot j}$.

In § (C3), we find a geometric application of the conservation law obtained from the symmetry of (8) with respect to $\mathrm{Diff}(M)$-variations of L (through $\left(G^L\right)^i_{\cdot j}$). Partially extending the classical pseudo-Riemannian case, this leads us to prove the Schur theorem for the Ricci curvature of any weakly Landsberg Finsler metric in Th. 0.9. Even though in fact we get the general identity (14), the extensibility of the Schur theorem outside the $P_i = 0$ case remains unsettled. Indeed, it might be the case that the right hand side of (14) vanishes identically for *every* Einstein Finsler metric. The pseudo-Finsler result Th. 0.10 suggests that many possibilities are open:

1) There may exist an argument local on A that eventually proves such a Schur theorem for any pseudo-Finsler metric.
2) There may not be such an argument, but the aforementioned right hand side may vanish identically in the Einstein Finsler case, rendering the theorem's scope dependent on the signature (and on the Finsler metric being y-global).
3) Both 1) and 2) may be false, so there may exist Finsler metrics with $\frac{\mathrm{Ric}^L}{L}(x,y)$ independent of y but nonconstant.

Asking about the description of the matter/energy content of a Finsler spacetime, in § (C4), as a first goal, we conclude that an anisotropic stress-energy tensor T^i_j is the object that one should study generically. (Indeed, approaches such as the description of fluids, Lorentz violation or different variational procedures lead to it.) As a main mathematical goal, we get a new divergence theorem for anisotropic vector fields

$g^{ab}\left\{P_{a|b} - P_a P_b + \left(y^c P_{a|c}\right)_{\cdot b}\right\} = 0$. Note that the left hand side of this is the $\mathfrak{P}$ of Th. 0.9.

without having to choose any particular Finslerian connection (Th. 0.11). Then, one can see that defining $\mathrm{div}(T)$ with the Chern one is the only way of guaranteeing that (4.14) holds in general. This divergence has a precise physical meaning (Th. 0.12), and so does the (covariant) conservation law[23] $\mathrm{div}(T) = 0$. Moreover, this infinitesimal conservation allows one to obtain global laws in the circumstances known from classical relativity (Ths. 0.13 and 0.14). This solidifies its physical relevance.

(E) **Problems for the future**

We began in § (C1) with an account, which we believe is essentially complete, of the different levels of generalizations of affine connections on $A \subseteq \mathrm{T}M \setminus \mathbf{0}$. While its mathematical situation is clear, the parallel transport associated with an A-anisotropic connection deserves further studying, the problem now being the following.

> 1) **Model some concrete and relevant physical situations with the anisotropic parallel transport on a spacetime. What differences emerge in the predictions from doing so with the Chern connection, the Berwald one or any other?**

On our Finslerian Palatini formalism of § (C2), our main results were of uniqueness of torsion-free solution $N^i_j = \left(G^L\right)^i_{\cdot j} + \mathcal{Z}^i_{\cdot j}$ of the equation (9). However, we proved this assuming that $\mathcal{Z}^i$ is fiberwise analytic (Th. 0.5), that $C_i = 0$ (Th. 0.7), or that L is Riemannian (Th. 0.8). Our understanding leads us to conjecture that the result is true in general, so a first problem is:

> 2) **Provided that L is (properly) Lorentz-Finsler and $n \geq 3$, prove that if $\mathcal{Z}^i$ and $\widehat{\mathcal{Z}}^i$ are solutions of** (11)+(12)/(13) **such that $\mathcal{Z}^i\big|_{A_x}$ and $\widehat{\mathcal{Z}}^i\big|_{A_x}$ are smooth and they extend smoothly to $\overline{A_x} \subset \mathrm{T}_x M \setminus 0_x$, then**
>
> $$\widehat{\mathcal{Z}}^i\Big|_{A_x} = \mathcal{Z}^i\big|_{A_x}.$$

[23]Not to be mistaken with the one discussed in (C3), obtained from (8). The difference between the two conservation laws might suggest that T^i_j is more closely related to theories of a different kind from (8), such as e.g. [84].

By contrast, we did not really obtain existence results besides knowing that $\mathcal{Z}^i = 0$ is a solution if (and only if) $P_i = 0$. So[24]:

3) **In the above circumstances, prove (or disprove) that indeed there is a $\mathcal{Z}^i$ that solves (11)+(12)/(13) at x with $\mathcal{Z}^i\big|_{A_x}$ smooth and extending smoothly to at least some nonempty subset of $\partial(A_x) \subset \mathrm{T}_x M \setminus 0_x$.**

This appears as a difficult problem in general, since it requires analyzing, on $\Sigma_x \subset A_x$, a PDE system of a form that we could not find in the literature, and then producing a global solution. Motivated by this, finding at least the solutions with cosmological symmetry is currently work in progress, joint with C. Pfeifer. Anyway, for completing the comparison of (10) with Pfeifer's equation (4) beyond the $P_i = 0$ case:

4) **For general solutions $N^i_j = \left(G^L\right)^i_{\cdot j} + \mathcal{Z}^i_{\cdot j}$ of (9), express Ric in terms of Ric^L. Do then the Palatini formalism (10) and the metric one (4) have the same solutions?**

(We do not really expect (10) and (4) to be equivalent. Indeed, before expressing $\mathrm{Ric} = \mathrm{Ric}^L + \dots$, our eq. can be reduced to $\mathrm{Ric} = 0$ as in Ths. 0.5 and 0.7, whereas Pfeifer's in principle cannot be so.) Let us end this part with some (quite general) possible research directions:

5) **Study the initial value problem for (4), or for $\mathrm{Ric} = 0$ (i.e., (10)) after solving (11)+(12)/(13) and writing $\mathrm{Ric} = \mathrm{Ric}^L + \dots$**

6) **Analyze the above problems when the dimension of M is $n = 2$ (for which Ths. (0.5) to (0.8) all fail for different reasons and the functional (8) has the symmetry $L(x, y) \mapsto e^{2u(x)} L(x, y)$).**

7) **Develop the Finslerian Palatini formalism for the next levels of generalized connections, starting e.g. with the functional of anisotropic**

[24] As commented in §2.5, from general theory, one may already expect multiplicity of solutions defined on open subsets of each A_x. But if the existence and uniqueness proposed here are true, one might glue the solution on each A_x to get a unique global solution on A. If this looks unreasonable at first glance, one may think that the requirement of smooth extensibility to each $\partial(A_x)$ automatically yields a boundary condition. Indeed, by Rem. 2.19, one knows that $\mathcal{Z}^i$ must be 0 on $\partial A \subset \mathrm{T}M \setminus \mathbf{0}$ (cf. the division technique in § (C2)).

ones $\widetilde{\mathcal{S}}[L,\Gamma^i_{jk}] := \int g^{ab}\mathrm{ric}^{\Gamma}_{ab}\,\underline{d\mu}$. Does it select the Chern connection, the Berwald one, both or none of them?

From the conclusions corresponding to § (C3), the obvious open problem was the general validity of the Finslerian Schur theorem. But let us be more specific:

8) **Are there additional identities that simplify the right hand side of** (14)**, allowing for better results? Could it be always** 0**?**

9) **If y-globality is not assumed, do counterexamples appear to our Schur theorem for weakly Landsberg metrics?**

On the other hand, taking into account that we only obtained the Noether identity corresponding to the purely metric version of our functional:

10) **Compute the conservation law from** (8) **for** $\mathrm{Diff}(M)$ **variations of a Finsler metric and an arbitrary nonlinear connection N^i_j. Does it yield new geometric results?**

To finish, while from § (C4) we concluded that matter is properly described by the anisotropic T^i_j, there were difficulties in relating its measurements by different observers. They make nontrivial the following:

11) **What is the correct expression for $T^i_j(x,y)$ in the case of e.g. a perfect fluid? What about that of a scalar field, or an electromagnetic one? (However this is modeled on a Finsler spacetime, see [85, §4.2] for a proposal.)**

Then, while we thoroughly analyzed the condition $\mathrm{div}(T) = 0$, we did not *derive* it. Therefore, as a last problem[25]:

[25] As discussed in §4.3.3:

(a) In the case of an affine space with a Lorentz norm, a homogeneous but anisotropic Lagrangian and Noether's procedure would allow to derive this conservation law.
(b) By contrast, Hilbert's procedure from general relativity does not carry over in its usual form to the Finslerian setting. Instead, it produces (average) conservation laws such as (3.21).

It is not clear either that one can invoke some kind of equivalence principle in order to reduce the problem to the affine space case so that $\mathrm{div}(T)_i(x,y) = \partial_{x^a} T^a_i(x,y) = 0$.

12) **Are there principles implying that** $\mathrm{div}(T) = 0$ **on any Finsler spacetime, as in the classical case? Can this be obtained from the coupling of the anisotropic stress-energy tensor to some Finslerian gravity theory?**

The difficulty in coupling T_{ij} (or some of its components) to functionals such as (8), or its metric version, motivates exploring further ones. New Finslerian gravity theories can be defined, stemming from alternative variational principles to the ones treated here. This is also work in progress, joint with this thesis' supervisors, M. Á. Javaloyes and M. Sánchez.

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1 Anisotropic connections and parallel transport in Finsler spacetimes

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Abstract

The general notion of anisotropic connections ∇ is revisited, including its precise relations with the standard setting of pseudo-Finsler metrics, i.e., the metric nonlinear connection and the (linear) Finslerian connections. In particular, the vertically trivial Finsler connections are canonically identified with anisotropic connections. So, these connections provide a simple intrinsic interpretation of a part of any Finsler connection closer to the Koszul formulation in M. Moreover, a new covariant derivative and parallel transport along curves is introduced, taking first a self-propagated vector (*instantaneous observer*) so that it serves as a reference for the propagation of the others. The covariant derivative of any anisotropic tensor is given by the natural derivative of a curve of tensors obtained by parallel transport along a curve and, in the case of pseudo-Finsler metrics, this is used to characterize the Levi-Civita–Chern anisotropic connection as the one that preserves the length of parallely propagated vectors.

Keywords — *Finsler spaces and spacetimes, anisotropic connections, sprays, nonlinear connections, Finsler connections, parallel transport, Levi-Civita–Chern connection.*

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1.1 Introduction

The standard geometric picture for the description of a (pseudo-)Finsler metric $L = F^2 : A(\subset TM) \to \mathbb{R}$ comprises two elements: a (nonlinear) connection ν on the fibration $A \to M$ and a linear connection $\nabla^* \equiv ((\Gamma^{\mathrm{H}})^k_{ij}, (\Gamma^{\mathrm{V}})^k_{ij})$ on the vertical vector bundle $\mathrm{V}A \to A$. The former is canonically associated with the spray determined by L, whose integral curves are the critical points of the energy functional. However, there are quite a few non-equivalent choices for the latter (Berwald, Cartan, Chern, Hashiguchi...). Motivated by the complexity of this and other settings, some researchers have introduced the concept of *anisotropic connection*, a generalization of the (pseudo-)Riemannian setting which incorporates in a natural way the direction dependent geometric structures of Finsler geometry [13, 20, 24, 25].

Recently, one of the authors has developed systematically the *anisotropic calculus* [11, 12], namely, how to make computations with an anisotropic connection, which can be seen as a natural and intuitive generalization of the usual Koszul connections. Some applications have been obtained in [10, 17]. In the present article, we revisit this notion, showing precisely its relations with the other elements of the standard setting and providing a further insight on its associated parallel transport.

More precisely, in §1.2 we introduce heuristically the notions of pseudo-Finsler metric, by looking for general ways of measuring the lengths of curves, and Finsler spacetime, by stressing geometric elements related with measurements. In §2.3, anisotropic tensor fields on M are introduced and the concept of anisotropic connection ∇ is defined. First, ∇ is regarded as a type of covariant derivative which applies to usual vector fields X, Y on M so that it provides an anisotropic vector field $\nabla_X Y$. Then, the usual rules of derivations are discussed so that ∇_X can be applied to any anisotropic tensor. We emphasize some issues which will become relevant later such as homogeneity (natural invariance by homotheties), the notion of torsion or the affine structure of the space of all the anisotropic connections.

In §1.4 and §1.5 we make a detailed study of the relation between anisotropic connections ∇ and, resp., (nonlinear) connections ν on $A \to M$ and linear connections ∇^* on $\mathrm{V}A \to A$ (ν and ∇^* are not assumed to come from any Finsler function L a priori). A detailed correspondence is given for the former, in particular:

> Any anisotropic connection ∇ with Christoffel symbols Γ^a_{ij} is characterized by a pair composed by a nonlinear connection $N^a_i = \Gamma^a_{ij} y^j$ and a

> tensor Q satisfiying $Q^a_{ij} y^j = 0$. In the homogeneous case, all nonlinear connections can be obtained from anisotropic ones (Th. 1.2).

The relation between an anisotropic ∇ and a linear ∇^* becomes subtler. Indeed, if we are given an auxiliary nonlinear connection $\overset{o}{\nu}$, then ∇^* can be determined by specifying the covariant derivatives (of the sections of $\mathrm{V}A \to A$) with respect to the $\overset{o}{\nu}$-horizontal and vertical directions. This is standard in Finsler geometry, and the connections with vanishing vertical derivatives are called *vertically trivial* here; clearly, they are independent of $\overset{o}{\nu}$. Such trivial connections can be put in one to one correspondence with the anisotropic connections by using $\overset{o}{\nu}$. However, this correspondence also becomes independent of $\overset{o}{\nu}$. Summing up:

> There is a natural bijection between vertically trivial connections ∇^* and anisotropic connections ∇. It identifies the homothety invariant ∇^*'s with the homogeneous ∇'s (Prop. 1.3, Th. 1.3).

In §1.6 we focus on the pseudo-Finsler case. As explained therein, the last result above becomes essential for the identification of anisotropic connections in the pseudo-Finsler setting. Indeed, the 2-homogeneity of L leads to the homogeneity of the involved linear connection ∇^* (and the canonical nonlinear one $\overset{o}{\nu}$). Some Finsler connections such as Berwald or Chern are vertically trivial and, thus, directly identifiable with anisotropic connections. Moreover, the non vertically trivial ones, as Cartan or Hasiguchi, will project on vertically trivial ones (by using $\overset{o}{\nu}$). So, anisotropic connections provide the non-vertical part of any Finslerian connection, expressed tidily as Koszul-type derivations on M. As already pointed out in [11], the metric L allows one to select a unique Levi-Civita anisotropic connection, which is then identifiable to Chern's.

In §1.7, we introduce the covariant derivative D_γ and parallel transport along curves γ for any anisotropic connection ∇. Taking into account the dependence of ∇ on the direction, one can choose a reference W (a vector field on γ which takes values on $A \subset TM$) as in [2, page 121], to define its associated covariant derivative D^W_γ and W-parallel transport, which will behave as the usual (isotropic) one. This parallel transport is of crucial importance, as it can be used to define in a very natural way the covariant derivative of tensors, but the dependence on the direction inherent to Finslerian geometry introduces additional subtleties. As a first step, one can parallel transport the observer, which in Finsler spacetimes is interpreted as the (timelike) direction on the tangent bundle where we are doing the computations. This is defined using a parallel observer determined by $D_\gamma^V V = 0$, a nonlinear equation whose solutions may not be extended on the whole γ. However, they do extend in the most interesting cases, such as the standard Finsler and the Lorentz-Finsler ones. Once we have a parallel observer along a curve, we can make the parallel transport of any other vector using as a reference this parallel observer. The parallel transport of the observer coincides with the one provided by a nonlinear connection (see for example [19, Ch. VII], [1, §2.1.6], [25, page 103], [4, §2.1], [8, Def. 1.4] and [27, §7.6]). However, as far as we know, the second parallel transport with respect to an observer has not been considered in literature.

It turns out that the most economical way to codify all the information of the covariant derivatives along curves in a smooth setting with natural assumptions is with an *anisotropic connection*, which allows for covariant derivatives of any kind of tensor (see Th. 1.1). These covariant derivatives were introduced in [12] from a rather abstract viewpoint as tensor derivations which satisfy the Leibniz rule of the tensor product and commute with contractions. To enhance the geometric meaning of these covariant derivatives, we will show in Th. 1.6 that they coincide with the (usual) derivative in a vector space of the curve of tensors obtained with parallel transport (and therefore using only covariant derivatives along curves). Finally, one can wonder if given a pseudo-Finsler metric, there is an anisotropic connection with a parallel transport which preserves the length of vectors. We show in §1.7.3 that this connection exists and it is the Levi-Civita–Chern anisotropic connection, which can be identified (as a vertically trivial linear connection) with the classical Chern connection.

1.2 General background

1.2.1 Pseudo-Finsler metrics

Let us pose the following problem. Given a manifold M, we want to define a general smooth structure that allows us to measure the length of curves. It seems quite natural that this length should be defined as

$$\ell(\gamma) := \int_a^b F(\dot\gamma(s))\, ds$$

for some function $F : TM \to \mathbb{R}$ in the tangent bundle TM. From a geometrical viewpoint, this definition should not depend on the parametrization of γ, which can be achieved by requiring that F is a homogeneous function of degree 1 when restricted to any tangent space. Moreover, if one wants to include relativistic measures and to remain in the smooth realm, it is better to consider the square $L = F^2$, because otherwise one would find many examples where F is not smooth on lightlike vectors, namely, vectors $v \in TM$ where $F(v) = 0$. Indeed, this happens when one considers the one-homogeneous function $F : \mathbb{R}^4 \to \mathbb{R}$ given by

$$F(\tau, v^1, v^2, v^3) = \sqrt{\tau^2 - (v^1)^2 - (v^2)^2 - (v^3)^2},$$

which is non-smooth on the lightlike vectors.

We will make two additional assumptions on L.

(i) The first one is that L is not necessarily defined in the whole tangent bundle TM, but only in some directions. Sometimes, there are some forbidden directions because of some constraints of the problem, or as in General Relativity, because only trajectories with directions on a cone (say, the future-directed timelike one) will become relevant. Therefore, we will choose as domain of L a subset A in TM which is conic, to permit arbitrary positive reparametrizations of the curves, and open for the sake of simplicity, even though the boundary of A can be considered in different ways (as in wind Riemannian metrics [5] or Finsler spacetimes [15]).

(ii) The second one is related to the (vertical) Hessian of L,

$$g_v(u, w) := \frac{1}{2}\frac{\partial^2}{\partial t \partial s} L(v + tu + sw)|_{t=s=0},$$

which, as we will see later, can be thought as the best scalar product approximation of L. We will assume that this scalar product is nondegenerate for every $v \in A$ but not necessarily Euclidean (positive definite). Nondegeneracy will be essential to obtain the existence and uniqueness of the covariant derivative.

Summing up, the following notion of a *pseudo-Finsler metric* collects all the conditions above for a very general definition of length of curves.

Definition 1.1. *Let M be an n-manifold, $\pi : TM \to M$ the natural projection of TM onto M and $A \subset TM \setminus \mathbf{0}$ an open subset of TM which is conic (namely, for every $v \in A$ and $\lambda > 0$, $\lambda v \in A$) and satisfies $\pi(A) = M$. A smooth function $L : A \to \mathbb{R}$ is a* pseudo-Finsler metric *if*

1. *L is positive homogeneous of degree 2, that is, $L(\lambda v) = \lambda^2 L(v)$ for every $v \in A$ and $\lambda > 0$.*
2. *The fundamental tensor of L, namely g_v defined by*

$$g_v(u, w) := \frac{1}{2}\frac{\partial^2}{\partial t \partial s} L(v + tu + sw)|_{t=s=0}$$

for any $v \in A$, and any $u, w \in T_{\pi(v)}M$, is nondegenerate.

In this definition we have excluded the zero section from A. As A is open and conic, the only case in which the zero section could be contained in A is when it is the whole tangent bundle. But even in this case, there are problems with the zero section, because L can be C^2 on the zero section only if it comes pointwise from a scalar product.[1]

Given a pseudo-Finsler metric $L : A \to \mathbb{R}$ on a manifold M, for every $p \in M$, we define the *indicatrix* at p as

$$\Sigma_p = \{v \in T_pM \cap A : L(v) = 1\},$$

(sometimes the indicatrix of $-L$ may be of interest too) and the *lightcone* as

$$C_p = \{v \in T_pM \cap A : L(v) = 0\}.$$

Given $p \in M$ and $v \in T_pM$, let us discuss why g_v is the best scalar product approximation of L at $v \in A$. Assume for example that $L(v) = 1$, which can be assumed by homogeneity if $L(v) > 0$. Recall that the restriction

$$g_v|_\Sigma : T_v\Sigma_p \times T_v\Sigma_p \to \mathbb{R}$$

coincides with second fundamental form of Σ_p with respect to the opposite of the position vector v computed with the affine connection of T_pM (see for example [14, Eq. (2.3)]). Moreover, one has that v is g_v-orthogonal to $T_v\Sigma_p$ and, by homogeneity (applying Euler's theorem), that $g_v(v, v) = L(v)$. This implies that

$$\Sigma^{g_v} = \{w \in T_pM : g_v(w, w) = 1\}$$

satisfies $T_v\Sigma^{g_v} = T_v\Sigma_p$, and the second fundamental form of Σ^{g_v} at v with respect to the opposite of the position vector v coincides with that of Σ_p.

[1] Indeed, if g is one half the Hessian of L at the 0 vector of each tangent space, then $L(v) = g(v, v)$ for every $v \in TM$, see [28, Prop. 4.1].

1.2.2 Finsler spacetimes and its restspace

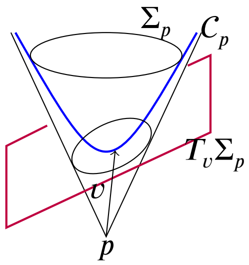
To generalize the definition of spacetime in a certain manifold M, the following observations are in order:

1. We need to measure the length of curves to obtain the elapsed time along the trajectory. By the discussion in the previous section, this leads us to consider a pseudo-Finsler metric $L : A \subset TM \setminus 0 \to \mathbb{R}$.
2. Locally, it must approximate the Lorentz-Minkowski spacetime. This implies that for every $v \in A$, the scalar product g_v must be of Lorentz type since, as argued above, g_v is the best approximation of L around v.
3. There have to be some vectors with zero length, which are the directions of light rays.
4. Moreover, these lightlike directions must be the limit of the timelike directions, therefore, their boundary.

Definition 1.2. *A* Finsler spacetime *is an n-manifold M, $n \geq 2$, endowed with a pseudo-Finsler metric $L : A \to (0, +\infty)$ such that*

(i) L is a Lorentz-Finsler metric, i. e., its indicatrix is strongly concave or equivalently the index of g_v is $n-1$.
(ii) L extends as zero to the closure $\bar{A}$ of A in $TM \setminus 0$ and this extension is smooth with nondegenerate g_v.
(iii) For every $p \in M$, $A_p := A \cap T_pM$ is connected, convex and salient, i. e., if $v \in A_p$ then $-v \notin A_p$. (In fact, the last two conditions follow from the other hypotheses, see [15, Rem. 3.6].)

Moreover, the future-directed timelike unit vectors of the indicatrix $\Sigma_p = \{v \in T_pM : L(v) = 1\}$ are used to model the instantaneous observers, while the vectors in the null cone $C_p = \{v \in T_pM : L(v) = 0\}$ are the lightlike future-directed vectors. The tangent space $T_v\Sigma_p = \{w \in T_pM : g_v(v, w) = 0\}$ is interpreted as the instantaneous restspace of v. Even if we assume that L is defined only in $\bar{A}$, it is possible to extend L (in a non-unique way) to the whole tangent bundle (see [21]). For more details about the interpretation of the restspace see [3] (part (4) after Rem. 9).



An *observer* in a Finsler spacetime is a (future-directed) unit timelike curve, namely $\gamma : I = (a, b) \to (M, L)$ such that $\dot\gamma(s) \in A$ and $L(\dot\gamma(s)) = 1$ for all $s \in I$. The *(instantaneous) restspace of the observer* at $s \in (a, b)$ is $T_{\dot\gamma(s)}\Sigma_{\gamma(s)}$. There are two natural metrics in this restspace. The first one is given by

$$g_{\dot\gamma(s)}|_\Sigma : T_{\dot\gamma(s)}\Sigma_{\gamma(s)} \times T_{\dot\gamma(s)}\Sigma_{\gamma(s)} \to \mathbb{R}.$$

It is the fundamental tensor restricted to Σ, which is a definite metric. The direction $\dot\gamma(s)$ is $g_{\dot\gamma(s)}$-orthogonal to $T_{\dot\gamma(s)}\Sigma$. As we have said above, this metric is the best approximation of L with a scalar product in the direction of $\dot\gamma(s)$ as the restriction $g|_\Sigma$ is the second fundamental form of Σ with respect to the opposite to the position vector (using the natural affine connection in T_pM).

The other metric is a Finsler metric with indicatrix $S_{\dot\gamma(s)}$, where $S_v = T_v\Sigma \cap \mathcal{C}_p$ (the set of velocities of light in the restspace of v). It is unclear which one is more suitable to measure spacelike distances, and indeed, the choice of metric could depend on the type of measure.

1.3 Anisotropic connections

1.3.1 Anisotropic tensor fields and their vertical derivatives

We will denote by $x = (x^1, \dots, x^n)$ local coordinates on some open subset U of M and (with a slight abuse of notation) by $(x, y) = (x^1, \dots, x^n, y^1, \dots, y^n)$ the natural ones induced on $TU \subset TM$. Let us denote by T^*M the cotangent bundle of the manifold M and by $\overset{r)}{\bigotimes} TM \otimes \overset{s)}{\bigotimes} T^*M$ the classical vector bundle of tensors of type (r, s) over M. Recall that an (r, s)-tensor field on M is a smooth section of this bundle, and let $\mathcal{T}^r_s(M)$ be the space of all such tensor fields. We use the simplified notation for smooth functions and vector fields on M: $\mathcal{T}^0_0(M) = \mathcal{F}(M)$ and $\mathcal{T}^1_0(M) = \mathfrak{X}(M)$

resp. Now, let $\pi_A^*(\overset{r)}{\bigotimes} TM \otimes \overset{s)}{\bigotimes} T^*M)$ be the bundle over A pullbacked by the natural projection $\pi_A : A \to M$. A smooth section $T : A \to \pi_A^*(\overset{r)}{\bigotimes} TM \otimes \overset{s)}{\bigotimes} T^*M)$ of this bundle is called an A-*anisotropic tensor field* on M, and let $\mathcal{T}_s^r(M_A)$ the space of such fields (also by convention, $\mathcal{T}_0^0(M_A) = \mathcal{F}(A)$). In natural coordinates, with summation in repeated indices and $\partial_a \equiv \partial_{x^a}$,

$$T_v = T^{a_1,\ldots,a_r}_{b_1,\ldots,b_s}(x,y)\, \partial_{a_1}\Big|_x \otimes \ldots \otimes \partial_{a_r}\Big|_x \otimes \mathrm{d}x^{b_1}|_x \otimes \ldots \otimes \mathrm{d}x^{b_s}|_x, \qquad v \in A \cap \mathrm{T}U; \quad (1.1)$$

here, (x, y) and x are, resp., the coordinates of v and $\pi_A(v)$ (the functions $T^{a_1,\ldots,a_r}_{b_1,\ldots,b_s}$ transform tensorially under changes of coordinates). Recall that, naturally, $\mathcal{T}_s^r(M_A)$ becomes a module over the ring $\mathcal{F}(A)$ and the tensor products and contractions induce further operations on sections, as in the case of usual tensor fields on M. In particular, $\mathcal{T}_0^1(M_A)$ and $\mathcal{T}_1^0(M_A)$ will be called resp. the sets of *anisotropic vector fields* and *1-forms* on M.

We emphasize a particularity of anisotropic vector fields. The elements $X \in \mathcal{T}_0^1(M_A)$ are (smooth) sections of the pullback bundle $\pi_A^*(TM) \to A$. This bundle is naturally isomorphic to the *vertical bundle* $\mathrm{V}A \to A$, where

$$\mathrm{V}_v A := \mathrm{Ker}(T_v \pi_A) = \mathrm{Span}\left\{ \partial_{y^i}|_v : \ i \in \{1, \ldots, n\} \right\} \subset T_v A.$$

Thus, $X \in \mathcal{T}_0^1(M_A)$ can be identified with a vertical vector field X^{V} on A, called the *vertical lift* of X, such that

$$X_v = X^i(x,y)\, \partial_{x^i}|_x \in T_{\pi(v)}M \leftrightarrow X_v^{\mathrm{V}} = X^i(x,y)\, \partial_{y^i}|_{(x,y)} \in \mathrm{V}_v A. \qquad (1.2)$$

Moreover, there is a *canonical anisotropic vector field*:

$$\mathbb{C} = y^i\, \partial_{x^i} \in \mathcal{T}_0^1(M_A), \qquad \mathbb{C}_v := v \in T_{\pi(v)}M. \qquad (1.3)$$

Its vertical lift $\mathbb{C}^{\mathrm{V}}$ is usually called the *Liouville vector field*, and both $\mathbb{C}$ and $\mathbb{C}^{\mathrm{V}}$ are actually smooth on the whole TM. It is also worth pointing out that there is a natural inclusion

$$\mathcal{T}_s^r(M) \hookrightarrow \mathcal{T}_s^r(M_A), \qquad T \mapsto \tilde{T}, \qquad (1.4)$$

just putting the components of $\tilde{T}$ in (1.1) as independent of directions and equal to those of T. The tensor $\tilde{T}$ will be called *isotropic* and we will not distinguish between T and $\tilde{T}$ when there is no possibility of confusion. Finally, we will say that a local vector field $V \in \mathfrak{X}(U)$ is *A-admissible* (where $U \subset M$ is an open subset) if $V_p \in A$ for all $p \in U$, i.e., V is a local section of the fibered manifold $A \to M$.

Notice that, at each $v \in A$, the fiber of the bundle $\pi_A^*(\overset{r)}{\bigotimes} TM \otimes \overset{s)}{\bigotimes} T^*M)$ becomes the space of all the (r,s)-tensors at $p = \pi_A(v)$. As this is a single vector space, the derivative of any curve in it is well defined. Thus, given an anisotropic tensor $T \in \mathcal{T}^r_s(M_A)$, we can define its *vertical derivative* at $v \in A$ in any direction $w \in T_pM$, $p = \pi(v)$, as follows:

$$(\dot{\partial}_w T)_v := \frac{d}{dt} T_{v+tw}|_{t=0},$$

which is again a tensor on T_pM. As the map $w \mapsto (\dot{\partial}_w T)_v$ is linear, we can naturally introduce an $(r, s+1)$ tensor field as follows (we write directly the obvious expression in coordinates).

Definition 1.3. *Given an A-anisotropic tensor $T \in \mathcal{T}^r_s(M_A)$, its* vertical derivative *$\dot{\partial}T \in \mathcal{T}^r_{s+1}(M_A)$ is given (locally) by*

$$(\dot{\partial}T)_v = \partial_{y^{b_{s+1}}} T^{a_1,\dots,a_r}_{b_1,\dots,b_s}(x,y)\, \partial_{a_1}\Big|_x \otimes \dots \otimes \partial_{a_r}\Big|_x \otimes \mathrm{d}x^{b_1}|_x \otimes \dots \otimes \mathrm{d}x^{b_s}_x \otimes \mathrm{d}x^{b_{s+1}}_x$$

in any natural coordinates as in (1.1).

1.3.2 Basic notion of anisotropic connection

As with other kinds of connections, anisotropic connections can be defined in different ways. We introduce them in the spirit of the Koszul formulation of connections, namely as covariant derivatives (on a restricted domain of vector fields first, which is extended later). Thus, we refer to any of their characterizations also as *anisotropic* (or, more properly, *A-anisotropic*) *covariant derivatives.* Anyway, we will prove in §1.5 that such derivatives can be identified with certain linear connections on a suitable bundle (which appears naturally in the Finslerian setting).

Definition 1.4. *An A-*anisotropic connection *(or* covariant derivative*) is a map*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{T}^1_0(M_A), \qquad (X,Y) \mapsto \nabla_X Y,$$

such that

1. $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$ *for all* $X, Y, Z \in \mathfrak{X}(M)$,

2. *$\nabla_X(fY) = (X(f)Y)\circ\pi_A + (f\circ\pi_A)\nabla_X Y$ for all $f \in \mathcal{F}(M)$; $X, Y \in \mathfrak{X}(M)$,*
3. *$\nabla_{fX+hY} Z = (f\circ\pi_A)\nabla_X Z + (h\circ\pi_A)\nabla_Y Z$ for all $f, h \in \mathcal{F}(M)$ and $X, Y, Z \in \mathfrak{X}(M)$.*

We say that an anisotropic connection ∇ is homogeneous (of degree zero)*, or* invariant by homotheties, *if for every $v \in A$ and $\lambda > 0$, $(\nabla_X Y)_{\lambda v} = (\nabla_X Y)_v$ (that is, $\nabla_X Y = \nabla_X Y \circ h_\lambda$ where $h_\lambda : A \to A$ is the homothety given by $h_\lambda(v) = \lambda v$).*

As in the case of affine connections, ∇ has a local nature. We can eventually use the notation $\nabla^v_X Y := (\nabla_X Y)_v$ and, consistently,

$$\nabla^V_X Y := (\nabla_X Y)\circ V \in \mathfrak{X}(U)$$

for any A-admissible local vector field $V \in \mathfrak{X}(U)$. By using coordinates, we can express ∇ in terms of its *Christoffel symbols* $\Gamma^a{}_{ij} : TU \cap A \to \mathbb{R}$, which are defined by

$$(\nabla_{\partial_i}\partial_j)_v \; (= \nabla^v_{\partial_i}\partial_j) = \Gamma^a{}_{ij}(v)\, \partial_a|_{\pi(v)}, \qquad \text{that is,} \qquad \nabla^V_{\partial_i}\partial_j = \left(\Gamma^a{}_{ij}\circ V\right)\partial_a. \tag{1.5}$$

Clearly, the homogeneity of ∇ is then equivalent to the 0-homogeneity of its Christoffel symbols, $\Gamma^a{}_{ij}(\lambda v) = \Gamma^a{}_{ij}(v)$, $\lambda > 0$. The following properties of these symbols are proven as in the standard case of affine connections.

Proposition 1.1. (1) Under a change of coordinates $(U, x) \rightsquigarrow (\bar{U}, \bar{x})$, the Christoffel symbols $\Gamma^m{}_{kl}$ and $\bar{\Gamma}^a{}_{ij}$ are related by

$$\bar{\Gamma}^a{}_{ij}(\bar{x}, \bar{y}) = \frac{\partial \bar{x}^a}{\partial x^m}(x)\left(\frac{\partial^2 x^m}{\partial \bar{x}^i \partial \bar{x}^j}(x) + \frac{\partial x^k}{\partial \bar{x}^i}(x)\,\frac{\partial x^l}{\partial \bar{x}^j}(x)\,\Gamma^m{}_{kl}(x, y)\right) \tag{1.6}$$

(2) Conversely, given any local choice of functions Γ^k_{ij} for a coordinate atlas satisfying the cocycle transformation (1.6), there exists a unique anisotropic connection ∇ whose Christoffel symbols are these functions. Moreover, if the functions are 0-homogeneous in y, then the produced ∇ is homogeneous too.

(3) Any (classical, affine) Koszul connection on M induces naturally an anisotropic one with Christoffel symbols independent of y, for any open conic domain $A \subset TM$ which naturally projects onto the whole M.

(4) Given an anisotropic connection ∇ with Christoffel symbols Γ^a_{ij} for each coordinates (U, x), the choice of functions Γ^a_{ji} for each (U, x) yields a new connection $\hat{\nabla}$, and ∇ is called *symmetric* if $\nabla = \hat{\nabla}$.

1.3.3 Extension to a covariant derivative of anisotropic tensors

Note first that the (anisotropic) covariant derivatives of vector fields can be extended to (anisotropic) covariant derivatives of tensor fields on M. That is, using the product and contraction rules of tensor derivations, there is a unique extension of $\nabla : \mathfrak{X}(M)\times \mathfrak{X}(M) \to \mathcal{T}^1_0(M_A)$ to a covariant derivative operator

$$\nabla : \mathfrak{X}(M)\times \mathcal{T}^r_s(M) \to \mathcal{T}^r_s(M_A), \qquad (X,T)\mapsto \nabla_X T, \tag{1.7}$$

such that

$$\nabla_X f = X(f)\circ\pi_A \qquad (\text{i.e. } \nabla_{\partial_i} f = \partial_i f\circ\pi_A) \tag{1.8}$$

for all $f \in \mathcal{F}(M) = \mathcal{T}^0_0(M)$. For example, if $\omega \in \mathcal{T}^0_1(M)$, then $\nabla_X\omega$ is defined by

$$(\nabla_X\omega)(\tilde{Y}) = X(\omega(Y)) - \tilde{\omega}(\nabla_X Y)$$

(recall (1.4), in particular $\tilde{\omega}_v = \omega_{\pi_A(v)} \in T^*_{\pi_A(v)}M$). In coordinates,

$$\nabla^V_{\partial_i} dx^k = -(\Gamma^k_{\ ij}\circ V)\, dx^j.$$

Next, we will go beyond extending the operators ∇_X to an *(anisotropic) covariant derivative of A-anisotropic tensor fields*

$$\nabla : \mathfrak{X}(M)\times \mathcal{T}^r_s(M_A) \to \mathcal{T}^r_s(M_A), \qquad (X,T)\mapsto \nabla_X T, \tag{1.9}$$

in a natural way (again, as usual, the same symbol ∇ will be used). The key to get this new extension of ∇ is to find a definition of $\nabla_X h$ when $h \in \mathcal{F}(A)$, that is, to find a natural extension of (1.8). The appropriate choice will be

$$(\nabla_X h)(v) = X_p(h\circ V) - \dot{\partial}_{(\nabla_{X_p} V)} h, \tag{1.10}$$

where $p = \pi_A(v)$ and $V \in \mathfrak{X}(M)$ is such that $V_p = v$.

Lemma 1.1. The definition of $\nabla_X h$ in (1.10) is independent of the choice of V. Moreover, if $h = f\circ\pi_A$ for some $f \in \mathcal{F}(M)$, then $\nabla_X h$ is equal to $\nabla_X f$ in (1.8).

Proof. It is enough to check that the expression (1.10) written in coordinates is independent of V. Let $V = V^j\partial_j$, $X = X^i\partial_i \in \mathfrak{X}(U)$. Then

$$\begin{aligned} X(h\circ V) =&\ X^i\left(\frac{\partial h}{\partial x^i}\circ V\right) + X^i\left(\frac{\partial h}{\partial y^j}\circ V\right)\frac{\partial V^j}{\partial x^i}, \\ \dot{\partial}_{(\nabla_X V)} h =&\ \left([X(V^k) + X^i V^j\left(\Gamma^k_{ij}\circ V\right)]\circ\pi_A\right)\dot{\partial}_k h, \end{aligned}$$

in the latter using that $\nabla_X V = X(V^i)\partial_i + X^i V^j \nabla_{\partial_i}\partial_j$. So, (1.10) reads at each v of coordinates (x, y):

$$\nabla_X h = \left(X^i \circ \pi_A\right)\left(\frac{\partial h}{\partial x^i} - y^j \Gamma^k_{ij}\frac{\partial h}{\partial y^k}\right), \tag{1.11}$$

which is independent of the chosen V, as required. ∎

Even though this lemma ensures the consistency of the definition of $\nabla_X h$, its meaning is not so evident. Algebraically, it ensures a sort of chain rule for $X(h \circ V)$. Anyway, we will give a further interpretation (see Rem. 1.4). As a summary of this subsection, we obtain

Theorem 1.1. *Let ∇ be an A-anisotropic connection and $X \in \mathfrak{X}(M)$. The operator $\nabla_X : Y \in \mathcal{T}^1_0(M_A) \mapsto \nabla_X Y \in \mathcal{T}^1_0(M_A)$ determines a unique tensor derivation of the tensor algebra $\mathcal{T}(M_A) = \bigoplus_{r,s\geq 0} \mathcal{T}^r_s(M_A)$ such that $\nabla_X h$ is given by* (1.10) *for $h \in \mathcal{F}(A)$. If $T \in \mathcal{T}^r_s(M_A)$ has the coordinate expression* (1.1), *then the components of $\nabla_k T := \nabla_{\partial_k} T$ are*

$$\begin{aligned} T^{a_1,\dots,a_r}_{b_1,\dots,b_s|k} &:= (\nabla_k T)^{a_1,\dots,a_r}_{b_1,\dots,b_s} \\ &= \partial_k T^{a_1,\dots,a_r}_{b_1,\dots,b_s} - \Gamma^i_{kj}\, y^j\, \dot\partial_i T^{a_1,\dots,a_r}_{b_1,\dots,b_s} + \sum_{l=1}^{r} \Gamma^{a_l}_{kj_l} T^{a_1,\dots,j_l,\dots,a_r}_{b_1,\dots,b_s} - \sum_{l=1}^{s} \Gamma^{i_l}_{kb_l} T^{a_1,\dots a_r}_{b_1,\dots,i_l,\dots b_s}, \end{aligned}$$

where $\partial_k = \partial/\partial x^k$, $\dot\partial_k = \partial/\partial y^k$.

The proof can be carried out by following the indications above. Anyway, full computations can be found in [11], where the following intrinsic version of the last displayed formula (regarding T as a $\mathcal{F}(A)$-multilinear map) can also be found in [11, Th. 11]: for any $v \in A$ and (local) extension $V \in \mathfrak{X}(U)$ of v,

$$\begin{aligned} (\nabla_X T)_v(\theta^1,\dots,\theta^r, X_1,\dots,X_s) = {} & X_{\pi(v)}(T_V(\theta^1,\dots,\theta^r,X_1,\dots,X_s)) \\ & - (\dot\partial T)_v(\theta^1,\dots,\theta^r,X_1,\dots,X_s,\nabla^V_X V), \\ & - \sum_{i=1}^{r} T_v(\theta^1,\dots,\nabla_X\theta^i,\dots,\theta^r,X_1,\dots,X_s) \\ & - \sum_{j=1}^{s} T_v(\theta^1,\dots,\theta^r,X_1,\dots,\nabla_X X_j,\dots,X_s), \end{aligned} \tag{1.12}$$

where $X, X_1, \dots, X_s \in \mathfrak{X}(M)$ and $\theta^1,\dots,\theta^r \in \mathfrak{X}^*(M)$.

Remark 1.1. One can even extend ∇ to a map

$$\nabla : \mathcal{T}_0^1(M_A) \times \mathcal{T}_s^r(M_A) \to \mathcal{T}_s^r(M_A), \qquad (X,T) \mapsto \nabla_X T,$$

just making it $\mathcal{F}(A)$-linear with respect to the first variable.

Remark 1.2. We have seen that the domain of an anisotropic connection can be extended from vector fields $X, Y \in \mathfrak{X}(M)$ to anisotropic vector fields in $\mathcal{T}_0^1(M_A)$. Additionally, multilinear maps over anisotropic tensor fields valued on anisotropic vector fields can be regarded as anisotropic tensor fields.

A relevant example appears when two anisotropic connections $\bar{\nabla}, \nabla$ are considered. Their difference $Q = \bar{\nabla} - \nabla$ is naturally an $\mathcal{F}(M)$-multilinear map

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathcal{T}_0^1(M_A),$$

which can be uniquely extended by $\mathcal{F}(A)$-multilinearity to an anisotropic tensor field $Q \in \mathcal{T}_2^1(M_A)$ (recall the embedding $\mathfrak{X}(M) = \mathcal{T}_0^1(M) \hookrightarrow T_0^1(M_A)$ in (1.4)). Moreover, Q can also be regarded as an $\mathcal{F}(A)$-multilinear map

$$\mathcal{T}_0^1(M_A) \times \mathcal{T}_0^1(M_A) \to \mathcal{T}_0^1(M_A), \text{ or } \mathcal{T}_0^1(M_A) \times \mathcal{T}_0^1(M_A) \times \mathcal{T}_1^0(M_A) \to \mathcal{F}(A).$$

Applying the previous discussion to the case of the connection $\hat{\nabla}$ obtained from ∇ with Christoffel symbols $\hat{\Gamma}^k_{ji} = \Gamma^k_{ij}$ (see Prop. 1.1), the following definition becomes consistent.

Definition 1.5. *The* torsion *of an anisotropic connection* ∇ *is the tensor* Tor $\in \mathcal{T}_2^1(M_A)$ *whose components are*

$$\mathrm{Tor}^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}.$$

The following consequence is also straightforward now.

Corollary 1.1. The space of all anisotropic connections on M has the structure of an affine space with associated vector space $\mathcal{T}_2^1(M_A)$.

Moreover, the homogeneous anisotropic connections form an affine subspace with associated vector space the subspace of $\mathcal{T}_2^1(M_A)$ composed by its 0-homogeneous tensors (i.e., those which satisfy $Q(\lambda v) = Q(v)$, $\lambda > 0$).

1.4 Anisotropic vs nonlinear connections

Any anisotropic connection induces a nonlinear connection on $\pi_A : A \to M$. Let us start recalling the framework of the latter adapted to our case.

The coordinates (x, y) on TU induce naturally coordinates $(x, y, \dot{x}, \dot{y})$ on $TA \subset T(TM)$. Then, the vertical bundle $\mathrm{V}A \subset TA$ is the subbundle of $TA \to A$ composed by the elements with $\dot{x} = 0$, and so $(x, y, \dot{y})$ is a local coordinate system for $\mathrm{V}A$. Clearly, $\mathrm{V}A$ is naturally identifiable with the pullback bundle $\pi_A^*(TM)$, and, consequently, the (smooth) sections of the bundle $\mathrm{V}A \to A$ can be regarded as A-anisotropic vector fields on M (recall (1.2)).

1.4.1 Setting for nonlinear connections

There are several ways to define a connection on the fibered manifold $A \to M$, commonly called a *nonlinear connection* in the Finsler geometry literature. One way is to provide a vector bundle homomorphism $\nu : TA \to \mathrm{V}A$ such that $\nu|_{\mathrm{V}A}$ is the identity. Then, the *horizontal distribution* $\mathrm{H}A := \mathrm{Ker}\, \nu$ characterizes ν and gives a decomposition $TA = \mathrm{H}A \oplus \mathrm{V}A$, which can also be used as an alternative definition of the nonlinear connection. One has the following representations in coordinates:

$$\nu\left(\dot{x}^i \left.\frac{\partial}{\partial x^i}\right|_{(x,y)} + \dot{y}^a \left.\frac{\partial}{\partial y^a}\right|_{(x,y)}\right) = \left(\dot{y}^a + N^a_i(x, y)\dot{x}^i\right) \left.\frac{\partial}{\partial y^a}\right|_{(x,y)},$$

$$\mathrm{H}_v A = \mathrm{Span}\left\{ \left.\frac{\delta}{\delta x^i}\right|_v := \left.\frac{\partial}{\partial x^i}\right|_v - N^a_i(v) \left.\frac{\partial}{\partial y^a}\right|_v \ : \ i \in \{1, \dots, n\} \right\}, \tag{1.13}$$

where the smooth functions N^a_i are defined for $v \in A \cap TU$. When taking a new induced chart $(T\bar{U}, (\bar{x}, \bar{y}))$, the new connection coefficients $\bar{N}^a_i$ are related to the old ones by

$$\bar{N}^a_i(\bar{x}, \bar{y}) = \frac{\partial \bar{x}^a}{\partial x^b}(x) \left(\frac{\partial^2 x^b}{\partial \bar{x}^i \partial \bar{x}^j}(x)\, \bar{y}^j + \frac{\partial x^k}{\partial \bar{x}^i}(x)\, N^b_k(x, y) \right) \tag{1.14}$$

over $T(U \cap \bar{U})$. One has that $\mathrm{H}_v A$ can be identified with $T_v A / \mathrm{V}_v A = T_{\pi(v)} M$, which allows one to identify any $X \in \mathcal{T}^1_0(M_A)$ with a horizontal vector field X^{H} on A, called the *horizontal lift* of X, determined by

$$X_v = X^i(x, y)\, \partial_{x^i}|_x \in T_{\pi(v)} M \leftrightarrow X^{\mathrm{H}}_v = X^i(x, y)\, \delta_{x^i}|_{(x,y)} \in \mathrm{H}_v A, \tag{1.15}$$

where we have simplified the notation $\delta_{x^i} := \delta / \delta x^i$ in (1.13).

Remark 1.3. (a) Conversely, any covering of charts of M endowed with a set of functions satisfying the cocycle transformation (1.14) determines unequivocally a nonlinear connection of $A \to M$. Incidentally, we recover a standard fact from the theory of fibered manifolds: a nonlinear connection is the same thing as a section of the 1-jet bundle $\mathbf{J}^1A \to A$; see [18, §17] for instance. If, by means of such a section, for the selected 1-jet at $v \in A$ one puts

$$N_i^a(v) = -\frac{\partial V^a}{\partial x^i}(\pi(v))$$

(where V is a local extension of v that determines the jet), then it is straightforward to see that these N_i^a's satisfy (1.14).

(b) It makes sense to assume that the nonlinear connection on $A \to M$ is *positive homogeneous* in the sense that the distribution $\mathrm{H}A$ is invariant under homotheties $h_\lambda : A \to A$, that is, if $Th_\lambda : TA \to TA$ is the tangent map (differential) of h_λ, then for each $\lambda > 0$ and $v \in A$,

$$(Th_\lambda)_v(\mathrm{H}_v) = \mathrm{H}_{\lambda v}. \tag{1.16}$$

Equivalently, the connection coefficients satisfy

$$N_i^a \circ h_\lambda = \lambda N_i^a \qquad \left(N_i^a(x, \lambda y) = \lambda N_i^a(x, y)\right).$$

As an integral curve $(x(t), y(t))$ of the horizontal vector field δ_{x^i} satisfies $dy^a/dx^i = -N_i^a(x, y)$, the 1-homogeneity of the functions N_i^a characterizes when $(x(t), \lambda y(t))$ $(= h_\lambda(x(t), y(t)))$ is also an integral curve. In this case, the relations

$$\delta_{x^i}|_{\lambda v} = \left(Th_\lambda\right)_v(\delta_{x^i}|_v), \qquad \partial_{y^i}|_{\lambda v} = \lambda^{-1}\left(Th_\lambda\right)_v(\partial_{y^i}|_v)$$

(0-homogeneity of δ_{x^i} and (-1)-homogeneity of ∂_{y^i}, see [4, §1.5]) are also satisfied,

$$N_i^a(x, y) = \partial_{y^j} N_i^a(x, y) y^j \tag{1.17}$$

(by Euler's theorem) and $\partial_{y^j} N_i^a(x, y)$ is 0-homogeneous in y (i.e., invariant under $h_\lambda : (x, y) \mapsto (x, \lambda y)$).

It is worth pointing out that a nonlinear connection ν induces a (nonlinear) covariant derivative of sections of $A \to M$ defined on open subsets $U \subset M$. Namely, let $\mathfrak{X}^A(U)$ be the set of all A-admissible vector fields on U (which behaves in a similar way as a module on the positive functions on U when A is convex at each point) and let $W : U \to A$ be an element of it; in coordinates, $W(x) = (x^i, W^a(x))$. The *ν-covariant derivative* of W is[2] $\nu \circ TW$, so that for any $X \in \mathfrak{X}(U)$ and in natural coordinates $(x, y, \dot{x}, \dot{y})$,

$$\nu \circ TW(X) \equiv (x^i, W^a(x), 0, \left(D_X W\right)^a(x)),$$

where

$$\left(D_X W\right)^a(x) = X^i \left(\frac{\partial W^a}{\partial x^i} + N_i^a(W(x))\right).$$

We denote the section $x \mapsto (x^i, \left(D_X W\right)^a(x))$ (which is a vector field on U) by $D_X W$. This induces a map

$$D : \mathfrak{X}(U) \times \mathfrak{X}^A(U) \to \mathfrak{X}(U), \qquad (X, W) \to D_X W, \tag{1.18}$$

which is linear in X but, in general,[3] is not linear in W. Notice that DW characterizes $\nu \circ TW$ (so any of them can be called *ν-covariant derivative*), while D detemines the functions N_i^a and, so, the connection ν.

The case of Koszul connections.

The nonlinearity of a connection refers to the nonlinearity of its covariant derivative (1.18). When this derivative is actually linear, the connection is called *linear* too (as usual, the name "nonlinear" must be understood in the sense of "non-necessarily linear"). Anyway, to be more specific, we will use the name *Koszul connection* for the linear connections on $TM \to M$ as in Prop. 1 (3). These (also named *affine connections* in the literature) are locally determined by their Christoffel symbols, which depend only on $x \equiv p \in M$. Finally, to deal with the Koszul case, recall first the following elementary technical result.

Lemma 1.2. Assume that the functions $N_i^a : A \cap TU \to \mathbb{R}$ are 1-homogeneous and can be smoothly extended to 0. Then there exist smooth functions $\Gamma_{ij}^k : U \to \mathbb{R}$ such that

$$N_i^a(x, y) = \Gamma_{ij}^a(x)\, y^j. \tag{1.19}$$

In particular, $N_{i \cdot j}^a(x) := \partial_{y^j} N_i^a(x, y) = \Gamma_{ij}^a(x)$ and each N_i^a can be naturally extended to TU.

[2]It is worth pointing out that some authors such as Shen, Dahl or Miron-Bucataru, consider an alternative covariant derivative by using the *flip* automorphism of TM, namely $(x, y, \dot{x}, \dot{y}) \mapsto (x, \dot{x}, y, \dot{y})$, see for example [22, §3.2]. This is avoided here, due to the different role of A and TM in our approach.

[3]Technically, it is *never* linear, as $A \subset TM \setminus \mathbf{0}$ is not a vector bundle, but it could be the restriction to A (i.e., to A-valued vector fields) of a map $\mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$ linear in the second component, as we will see in Prop. 1.2.

Proof. Just recall that for each point $p \in U$ and for any indices a, i, the function f defined on T_pM by $f(v) = N_i^a(v)$ is linear, as $df|_0(v) = \lim_{\lambda \searrow 0} f(\lambda v)/\lambda = f(v)$ for each v by homogeneity. ∎

Proposition 1.2. A nonlinear connection on $A \to M$ induces a Koszul connection (that is, the map (1.18) is extended to $\mathfrak{X}(U) \times \mathfrak{X}(U)$ and, then, linear in the second component) when its horizontal distribution can be smoothly extended to the zero section (i.e., when its coefficients N_i^a's satisfy (1.19) in every coordinate chart).

Most of the framework of linear connections explained here for $A \to M$ can be extended to more general bundles in a standard way and will be used without further comments in §1.6.

1.4.2 Interplay between anisotropic connections and nonlinear ones

Consider an anisotropic connection ∇ as in Def. 1.4. It can be proved that ∇ induces a horizontal distribution i. e., a nonlinear connection by using an intrinsic approach [11, §3.1]. However, we would like to emphasize the following direct relation with the cocycle trasformation associated with the Christoffel symbols Γ^a_{ij} in (1.5).

Theorem 1.2. *(1) An anisotropic connection ∇ defines canonically a nonlinear connection ν^∇ whose coefficients N_i^a with respect to a chart are*[4]

$$N_i^a(x,y) = \Gamma^a_{ij}(x,y)\, y^j. \tag{1.20}$$

If ∇ is homogeneous then ν^∇ is also homogeneous.

(2) A nonlinear connection ν defines canonically an anisotropic connection ∇^ν whose Christoffel symbols Γ^a_{ij} with respect to a chart are

$$\Gamma^a_{ij}(x,y) = \dot\partial_j N_i^a(x,y)\,(= N^a_{i\cdot j}(x,y)).$$

If ν is homogeneous then ∇^ν is also homogeneous and $\nu^{(\nabla^\nu)} = \nu$. In particular, the map $\nabla \mapsto \nu^\nabla$ is onto when it is restricted to the sets of homogeneous anisotropic and nonlinear connections.

(3) For any homogeneous nonlinear connection ν, the set $\{\nabla$ anisotropic connection $:$ $\nu^\nabla = \nu\}$ is

$$\{\nabla^\nu + Q \;:\; Q \in \mathcal{T}^1_2(M_A) \text{ and } Q^k_{ij} y^j = 0\}.$$

[4] In fact, there would also be a second nonlinear connection $\Gamma^a_{ji}(x,y)\, y^j$. In our terms, this one would be $\nu^{\tilde\nabla}$ for $\tilde\nabla = \nabla - \mathrm{Tor}$ (recall Rem. 1.2).

(4) On the set of Koszul covariant derivatives (restricted to A), the map $\nabla \mapsto \nu^\nabla$ is injective and its image consists precisely of the (restrictions to A of the) linear connections on $TM \to M$.

Proof. (1) Notice that the functions N_i^a satisfy the cocycle transformation (1.14). Then, the 0-homogeneity of the Christoffel symbols $\Gamma^a_{ij}(x, y)$ gives the 1-homogeneity of the connection coefficients $N_i^a(x, y)$.

(2) The cocycle (1.14) for $N_i^a(x, y)$ implies the cocycle (1.6) for Γ^a_{ij}. Then, the homogeneity of ν implies (1.17) and thus (1.20).

(3) Straightforward from part (2) and (1.20).

(4) For such a ∇, the Christoffel symbols are direction independent and ν^∇ is linear. Given a second $\bar\nabla$ with $\nu^{\bar\nabla} = \nu^\nabla$, the difference $Q = \bar\nabla - \nabla$ is also direction independent and $Q^k_{ij}(x) y^j = 0$, which implies $Q = 0$ by taking derivatives with respect to each y^l. ∎

In Th. 1.2 (2), one sees in coordinates that the torsion of ∇^ν (recall Def. 1.5) coincides with the *torsion of* ν, which can be defined intrisically [27, (7.8.10)].

Remark 1.4. Finally, we can give the promised interpretation of the definition of $\nabla_X h$ in (1.10), (1.11). Indeed, any $X_p = X^i(x)\,\partial_i|_x \in T_pM$, $p \equiv x$, gives a horizontal lift $X^{\mathrm{H}}_v = X^i(x)\,\delta/\delta x^i|_{(x,y)} \in \mathrm{H}_vA$ $(\subset T_vA)$ for any $v \equiv (x, y)$ with $\pi_A(v) = p$. So, substituting the expressions in the formulas of $\mathrm{H}_{(x,y)}A$ and (1.20), we find

$$X^H_v(h) = X^i(\pi(v)) \left(\partial_i h - \Gamma^k_{ij}\, y^j \dot\partial_k h\right)(v) = (\nabla_X h)_v.$$

The last equality is in agreement with (1.11).

1.5 Anisotropic vs linear connections

Next, our goal will be to identify anisotropic connections with a class of linear connections on the vector bundle $\mathrm{V}A \to A$.

As shown at the beginning of §1.4, $\mathrm{V}A$ $(\subset TA)$ admits as natural coordinates $(x, y, \dot y)$, which will be relabeled here as (x, y, z). Thus, we can write $z^a\,\partial_{y^a}|_{(x,y)} \in \mathrm{V}_{(x,y)}A$. The tangent bundle $T(\mathrm{V}A)$ includes the vertical subbundle $\mathrm{V}(\mathrm{V}A)$, whose fiber $\mathrm{V}_w(\mathrm{V}A)$ is generated by the $\partial_{z^a}|_{(x,y,z)}$, where (x, y, z) are the coordinates of $w \in \mathrm{V}A$.

1.5.1 Linear connections on $\mathrm{V}A \to A$

In order to define an *(Ehresmann) connection* on $\mathrm{V}A \to A$, we have to provide a smooth horizontal decomposition $T(\mathrm{V}A) = \mathrm{H}(\mathrm{V}A) \oplus \mathrm{V}(\mathrm{V}A)$. Notice first that any positive homothety h_λ on A induces a natural morphism

$$h_{\lambda_*} = \left(Th_\lambda\right)\Big|_{\mathrm{V}A} : \mathrm{V}A \to \mathrm{V}A, \qquad (x, y, z) \mapsto (x, \lambda y, \lambda z).$$

The new horizontal distribution (and then the Ehresmann connection itself) is called *invariant by positive homotheties* if it is preserved by the tangent map of h_{λ_*}, i.e., if for $w \in \mathrm{V}A$,

$$(Th_{\lambda_*})_w \left(\mathrm{H}_w(\mathrm{V}A)\right) = \mathrm{H}_{h_{\lambda_*}(w)}(\mathrm{V}A). \tag{1.21}$$

In what follows, we will focus on the particular case when a *linear connection* ν^* is given on $\mathrm{V}A \to A$. The (linear) covariant derivative operator associated with ν^* will be denoted by ∇^*. As the sections of $\mathrm{V}A \to A$ are identified with the anisotropic vector fields in $\mathcal{T}_0^1(M_A)$ (recall the vertical isomorphism (1.2)), ∇^* becomes a map

$$\nabla^* : \mathfrak{X}(A) \times \mathcal{T}_0^1(M_A) \to \mathcal{T}_0^1(M_A), \qquad (W, Z) \mapsto \nabla^*_W Z. \tag{1.22}$$

(The reader may appreciate the similarities and differences between (1.22) and the anisotropic covariant derivative (1.9), the latter with $(r, s) = (1, 0)$.) Moreover, it is straightforward but tedious to prove (from the definition of ∇^* in terms of the corresponding horizontal decomposition) that the homothety invariance of ∇^* is characterized as follows. If a section $Z^\mathrm{V} : A \to \mathrm{V}A$ is 0-homogeneous (meaning $Z^\mathrm{V} \circ h_\lambda = \lambda^{-1} h_{\lambda_*} \circ Z^\mathrm{V}$ as in [4, §1.5]), then

$$\nabla^*_{Th_\lambda(W)} Z^\mathrm{V} = \lambda^{-1} Th_\lambda(\nabla^*_W Z^\mathrm{V}). \tag{1.23}$$

To specify ∇^* by means of its Christoffel symbols, one has to choose a basis for $\mathfrak{X}(A)$ and another one for $\mathcal{T}_0^1(M_A)$. A possible choice would be the one associated with coordinates, namely $\{\partial_i|_{(x,y)}, \dot\partial_i|_{(x,y)}\} = \{\partial_{x^i}|_{(x,y)}, \partial_{y^i}|_{(x,y)}\}$ for the former and[5] $\{\partial_j|_x\} \equiv \{\dot\partial_j|_{(x,y)}\}$ for the latter. However, in case we have a prescribed nonlinear connection $\overset{o}{\nu}$ on $A \to M$, a more convenient choice than $\{\partial_i, \dot\partial_i\}$ may be $\{\delta_i, \dot\partial_i\}$:

$$\delta_i|_{(x,y)} = \frac{\delta}{\delta x^i}\bigg|_{(x,y)} = \partial_{x^i}|_{(x,y)} - \overset{o}{N}{}^a_i\,(x, y)\, \partial_{y^a}|_{(x,y)}, \qquad \dot\partial_j|_{(x,y)} = \partial_{y^j}|_{(x,y)}.$$

[5] Keep in mind that this identification also corresponds exactly to the vertical isomorphism (1.2).

This happens in the pseudo-Finsler case, where $\overset{o}{\nu}$ is provided by the geodesic spray and, thus, is homogeneous. This last property of $\overset{o}{\nu}$ will be assumed for the sake of simplicity, even though actually it will be important only when the homogeneity of the Christoffel symbols is involved. From the homogeneity of $\overset{o}{\nu}$, δ_i is 1-homogeneous, namely $\delta_i|_{(x,\lambda y)} = \left(Th_\lambda\right)_{(x,y)}(\delta_i|_{(x,y)})$, while $\dot{\partial}_i$ is 0-homogeneous, namely $\dot{\partial}_i|_{(x,\lambda y)} = \lambda^{-1}\left(Th_\lambda\right)_{(x,y)}(\dot{\partial}_i|_{(x,y)})$.

Definition 1.6. *The* horizontal and vertical Christoffel symbols *of* ∇^* *with respect to a prescribed homogeneous nonlinear connection* $\overset{o}{\nu}$ *in the coordinates* (U,x) *are the functions* $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$, $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$ *determined on* $A\cap TU$ *by*

$$\left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x,y)\,\partial_a|_x = \nabla^*_{\delta_i|_{(x,y)}}\partial_j (\equiv \nabla^*_{\delta_i|_{(x,y)}}\dot{\partial}_j = \left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x,y)\,\dot{\partial}_a|_{(x,y)}),$$

$$\left(\Gamma^{\mathrm{V}}\right)^a_{ij}(x,y)\,\partial_a|_x = \nabla^*_{\dot{\partial}_i|_{(x,y)}}\partial_j (\equiv \nabla^*_{\dot{\partial}_i|_{(x,y)}}\dot{\partial}_j = \left(\Gamma^{\mathrm{V}}\right)^a_{ij}(x,y)\,\dot{\partial}_a|_{(x,y)}).$$

Proposition 1.3. (1) The Christoffel symbols $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$, $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$ of ∇^* with respect to $\overset{o}{\nu}$ satisfy:

(a) The cocycle for $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$ (resp., $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$) under a change of coordinates coincides with the one for the Christoffel symbols Γ^a_{ij} of an A-anisotropic connection (1.6) (resp., the one of an A-anisotropic $(1,2)$ tensor). In particular, if all the $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$'s vanish for some coordinates on U, then they vanish for any coordinates therein.

(b) If the linear connection ∇^* is invariant by homotheties then, for all $\lambda > 0$

(b1) $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x,\lambda y) = \left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x,y)$ (0-homogeneity), and

(b2) $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}(x,\lambda y) = \lambda^{-1}\left(\Gamma^{\mathrm{V}}\right)^a_{ij}(x,y)$ ((-1)-homogeneity).

(2) Conversely, once a homogeneous nonlinear connection $\overset{o}{\nu}$ is prescribed, any local choice of functions $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$, $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$ satisfying (a) for a coordinate atlas determines a unique linear connection ∇^*, whose Christoffel symbols with respect to $\overset{o}{\nu}$ in that atlas coincide with the original $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$, $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$. Moreover, if (consistently with (b) above) the functions $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$, $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$ are chosen to be, resp., with 0 and (-1)-homogeneity in y, then the produced ∇^* is invariant by homotheties.

Proof. (1) The cocycles of $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$ and $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$ can be checked from their definitions using the transformation laws

$$\bar{\delta}_i = \frac{\partial x^k}{\partial \bar{x}^i}\delta_k, \qquad \bar{\dot{\partial}}_i = \frac{\partial x^k}{\partial \bar{x}^i}\dot{\partial}_k, \qquad \partial_j = \frac{\partial \bar{x}^l}{\partial x^j}\bar{\partial}_l$$

(recall that the last x^j's are the ones on M and not those on TM). Then, using only the definition of $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$ and $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$, the 1-homogeneity of the δ_i's, and the 0-homogeneity of the $\dot\partial_i$'s, the following identities hold true:

$$\left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x,\lambda y)\,\dot\partial_a\big|_{(x,\lambda y)} = \nabla^*_{\delta_i|_{(x,\lambda y)}}\dot\partial_j = \nabla^*_{(Th_\lambda)_{(x,y)}(\delta_i|_{(x,y)})}\dot\partial_j, \tag{1.24}$$

$$\begin{aligned}&\left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x,y)\,\dot\partial_a\big|_{(x,y)}\\ &= \left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x,y)\lambda\left(Th_\lambda\right)^{-1}_{(x,y)}(\dot\partial_a\big|_{(x,\lambda y)}) = \left(Th_\lambda\right)^{-1}_{(x,y)}(\lambda\left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x,y)\,\dot\partial_a\big|_{(x,\lambda y)}).\end{aligned} \tag{1.25}$$

Now, in case that ∇^* is invariant by homotheties, one can use (1.23) with $Z^{\mathrm{V}} = \dot\partial_j$. From (1.23), (1.24) and (1.25) it follows that

$$\begin{aligned}\left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x,\lambda y)\,\dot\partial_a\big|_{(x,\lambda y)} &= \lambda^{-1}\left(Th_\lambda\right)_{(x,y)}(\nabla^*_{\delta_i|_{(x,y)}}\dot\partial_j)\\ &= \lambda^{-1}\left(Th_\lambda\right)_{(x,y)}(\left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x,y)\,\dot\partial_a\big|_{(x,y)})\\ &= \lambda^{-1}\left(Th_\lambda\right)_{(x,y)}\{\left(Th_\lambda\right)^{-1}_{(x,y)}(\lambda\left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x,y)\,\dot\partial_a\big|_{(x,\lambda y)})\}\\ &= \left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x,y)\,\dot\partial_a\big|_{(x,\lambda y)},\end{aligned}$$

thus proving (b1). An analogous calculation proves (b2).

(2) Knowing the cocycles that the Christoffel symbols of such a ∇^* should satisfy, it is possible to define ∇^* by $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$, $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$ on each coordinate chart and, as usual, assert that the local definitions patch together to form a global linear connection on $\mathrm{V}A \to A$. Moreover, (1.24) and (1.25) are still valid for this ∇^*, and so are the analogous identities for the $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$'s. So, if the $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$'s are 0-homogeneous and the $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$'s are (-1)-homogeneous, then one can use those identities to show that

$$\nabla^*_{(Th_\lambda)_{(x,y)}(\delta_i|_{(x,y)})}\dot\partial_j = \lambda^{-1}\left(Th_\lambda\right)_{(x,y)}(\nabla^*_{\delta_i|_{(x,y)}}\dot\partial_j), \tag{1.26}$$

$$\nabla^*_{(Th_\lambda)_{(x,y)}(\dot\partial_i|_{(x,y)})}\dot\partial_j = \lambda^{-1}\left(Th_\lambda\right)_{(x,y)}(\nabla^*_{\dot\partial_i|_{(x,y)}}\dot\partial_j). \tag{1.27}$$

By using that $\{\delta_i,\dot\partial_i\}$ is a basis for $\mathfrak{X}(A)$, $\{\dot\partial_i\}$ is a basis for the vertical vector fields, and the two expressions $\nabla^*_{(Th_\lambda)_{(x,y)}(W_{(x,y)})}Z^{\mathrm{V}}$ and $\lambda^{-1}\left(Th_\lambda\right)_{(x,y)}(\nabla^*_{W_{(x,y)}}Z^{\mathrm{V}})$ are linear in W and Leibnizian in Z^{V}, the identities (1.26) and (1.27) prove that ∇^* satisfies (1.23), hence that ∇^* is invariant by homotheties. ∎

1.5.2 Anisotropic connections as vertically trivial linear connections

For a linear connection ∇^* on $\mathrm{V}A \to A$, the vanishing of all $\left(\Gamma^{\mathrm{V}}\right)^{a}_{ij}$'s involves only vertical derivatives and, so, it is an intrinsic property (independent also of $\overset{o}{\nu}$ in Prop. 1.3 (1)(a)). This makes possible the following definition.

Definition 1.7. *Let ∇^* be a linear connection on $\mathrm{V}A \to A$. We say that ∇^* is* vertically trivial *if its vertical Christoffel symbols vanish ($\left(\Gamma^{\mathrm{V}}\right)^{a}_{ij} = 0$ everywhere).*

Remark 1.5. From Prop. 1.3, it is clear that any homogeneous nonlinear connection $\overset{o}{\nu}$ induces a projection of the set of all ∇^*'s onto the set of vertically trivial connections, such that $(\left(\Gamma^{\mathrm{H}}\right)^{a}_{ij}, \left(\Gamma^{\mathrm{V}}\right)^{a}_{ij}) \mapsto (\left(\Gamma^{\mathrm{H}}\right)^{a}_{ij}, 0)$.

Theorem 1.3. *Let $\overset{o}{\nu}$ be a homogeneous nonlinear connection on $A \to M$. The map between the sets of the vertically trivial and the A-anisotropic connections, defined locally by*

$$\begin{array}{ccc} \{\textit{vertically trivial } \nabla^*\textit{'s on } \mathrm{V}A \to A\} & \longrightarrow & \{A\textit{-anisotropic } \nabla\textit{'s}\}, \\ (\left(\Gamma^{\mathrm{H}}\right)^{a}_{ij}, \left(\Gamma^{\mathrm{V}}\right)^{a}_{ij} = 0) & \mapsto & \Gamma^{a}_{ij} = \left(\Gamma^{\mathrm{H}}\right)^{a}_{ij}, \end{array}$$

is well defined and bijective, and it also identifies the homothety invariant ∇^*'s with the homogeneous ∇'s.

Moreover, this map does not depend on the choice of $\overset{o}{\nu}$. So, there exists a natural identification between vertically trivial and anisotropic connections, and it preserves the homothety invariance of the connections.

Proof. The first part is straightforward from Prop. 1.3. For the independence of $\overset{o}{\nu}$, recall that when chosing a second ν', the differences $\delta'_i - \delta_i$ are vertical and thus $\nabla^*_{\delta'_i - \delta_i}\partial_j = 0$ as ∇^* is vertically trivial. For the last assertion, recall that $A \to M$ always admits a homogeneous nonlinear connection (for example, the one determined by a pseudo-Finsler metric on A or, in particular, by a Riemannian metric on M). ∎

1.6 Anisotropic versus Finsler connections

When a pseudo-Finsler metric L is given on A, the standard approach focuses on two geometric structures.[6] The first one is its geodesic spray on A and, thus, its associated homogeneous nonlinear connection on $A \to M$; these are canonically constructed

[6] We recommend the nice essays by Dahl [7] and Minguzzi [22] for background.

from L. The second one is a covariant derivative on $\mathrm{V}A \to A$ invariant by homotheties. However, a priori there is not a canonical choice for the latter. Let us explain briefly the interplay of anisotropic connections with these two structures.

Recall that, as L is 2-homogeneous, the nonlinear connections and the covariant derivatives to be considered here will be homogeneous (invariant by homotheties). In particular, all the conclusions of Th. 1.2 will be applicable and, for example, the torsion of a nonlinear connection ν can be identified with the torsion of the corresponding anisotropic connection ∇^ν.

1.6.1 The metric spray

Definition 1.8. *A* spray *on A is a (smooth) section $G : A \to TA$ which satisfies: (a) G is a* second order differential equation *(or s. o. vector field), that is, it can be written as*[7]*:*

$$G = y^i \frac{\partial}{\partial x^i} - 2G^a \frac{\partial}{\partial y^a},$$

and (b) The G^a's are 2-homogeneous, i. e., $G^a \circ h_\lambda = \lambda^2 G^a$ for $\lambda > 0$.[8]

We summarize some basic relations between sprays and homogeneous nonlinear connections (analogous to Th. 1.2) following [22, §3.4]. Recall that the integral curves $(x(t), y(t))$ of G, satisfy $dx^i/dt = y^i$ and

$$\frac{dy^a}{dt} + 2G^a(x(t), y(t)) = 0; \tag{1.28}$$

their projections to M are usually called *geodesics* of G. On the other hand, the *geodesics* of a nonlinear connection ν are its autoparallel curves: in terms of (1.18), $D_{\dot\gamma}\dot\gamma = 0$, whereas on coordinates, $dx^i/dt = y^i$ and

$$\frac{dy^a}{dt} + N^a_i(x(t), y(t))y^i(t) = 0. \tag{1.29}$$

Proposition 1.4. (1) A homogeneous nonlinear connection ν defines a natural spray $G = \mathbb{C}^{\mathrm{H}}$ (the ν-horizontal lift of the canonical anisotropic field, recall (1.3) and (1.15)). In coordinates,

$$G = y^i \frac{\partial}{\partial x^i} - y^i N^a_i \frac{\partial}{\partial y^a}.$$

[7] Intrinsically, $T\pi_A \circ G$ is the identity in A.

[8] More intrinsically, $[\mathbb{C}^{\mathrm{V}}, G] = G$, where $\mathbb{C}^{\mathrm{V}}$ is the Liouville vector field on A. In terms of coordinates, $\mathbb{C}^{\mathrm{V}}_v = y^i \partial_{y^i}|_{(x,y)}$ (recall (1.3) and (1.2)).

The integral curves of G are the *geodesics* of ν.

(2) A spray G in A induces a natural homogeneous nonlinear connection $\overset{o}{\nu}$ with coefficients

$$\overset{o}{N}{}^a_i = \frac{\partial G^a}{\partial y^i} = G^a_{\cdot i}\,.$$

Then, $G = \overset{o}{\mathbb{C}}{}^{\mathrm{H}}$ ($\overset{o}{\nu}$-horizontal lift of $\mathbb{C}$) and $\overset{o}{\nu}$ is torsion-free.

(3) The geodesics of a homogeneous nonlinear connection ν are the integral curves of a spray G iff $G = \mathbb{C}^{\mathrm{H}}$.

(4) Given a homogeneous nonlinear connection ν, consider the natural spray $G = \mathbb{C}^{\mathrm{H}}$ and the nonlinear connection $\overset{o}{\nu}$ associated with this G. Then the difference $\nu - \overset{o}{\nu}$ is in $\mathcal{T}_1^{\,1}(M_A)$ with components

$$N^a_i - G^a_{\cdot i} = \frac{1}{2}\mathrm{Tor}^a_{ij}y^j,$$

where Tor is the torsion of ∇^ν (see part (2) of Th. 1.2). Moreover, if this difference vanishes, then actually Tor vanishes.

(5) Any homogeneous nonlinear connection is determined by its geodesics and torsion.

Proof. (1) These G^a's satisfy the cocycle transformation required to form a second order equation and their 2-homogeneity comes from the 1-homogeneity of N^a_i.

(2) From the cocycle of a second order vector field, the $\overset{o}{N}{}^a_i$'s satisfy (1.14). The 1-homogeneity of $\overset{o}{\nu}$ comes from the 2-homogeneity of the G^a's. (Then $G = \overset{o}{\mathbb{C}}{}^{\mathrm{H}}$ is nothing but the Euler relation for the G^a's.) The components of the torsion tensor of $\overset{o}{\nu}$ are $G^a_{\cdot i\cdot j} - G^a_{\cdot j\cdot i} = G^a_{\cdot i\cdot j} - G^a_{\cdot i\cdot j} = 0$.

(3) Recall the geodesic equations (1.28) and (1.29). Their solutions coincide if and only if $y^i N^a_i = 2G^a$.

(4) This is a straightforward computation taking into account that $G^a = N^a_i y^i/2$ and $\mathrm{Tor}^a_{ij} = N^a_{i\cdot j} - N^a_{j\cdot i}$. Thus, if $\mathrm{Tor}^a_{ij}y^j = 0$, then $\nu = \overset{o}{\nu}$ and its torsion vanishes due to (3).

(5) This follows directly from (4). ∎

Remark 1.6. As a consequence of Th. 1.2, the $\overset{o}{\nu}$ associated with G can be always obtained from a canonical homogeneous anisotropic connection $\nabla^{\overset{o}{\nu}}$ (item (2) of the Th.). This $\nabla^{\overset{o}{\nu}}$ is the so-called *Berwald anisotropic connection*. The remaining anisotropic connections that yield G would be controlled by an anisotropic tensor Q satisfying $Q^a_{ij} y^i y^j = 0$.

The spray canonically associated with a pseudo-Finsler *metric* L, called its *metric spray*, can be introduced in the following intrinsic way (see [4, Th. 5.4.2]). The 1-form

$$\hat{d}L := \frac{\partial L}{\partial y^i} \mathrm{d}x^i \in \mathfrak{X}^*(A)$$

is globally well-defined and, due to the condition (2) in Def. 1.1, the 2-form $\mathrm{d}\hat{d}L$ is nondegenerate. Thus, there exists a unique vector field G on A such that

$$\imath_G \mathrm{d}\hat{d}L = -\frac{1}{2}\mathrm{d}L,$$

where $\imath$ is the interior product operator. This G is indeed a spray and its geodesics are those of (M, L) (the critical points of the energy functional). It is well-known [25, (4.30)] that its components are $G^a = \gamma^a_{ij} y^i y^j$, where

$$\gamma^a_{ij} = \frac{1}{2} g^{ak} \left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right)$$

are the so-called *formal Christoffel symbols*. From them, the *metric nonlinear connection* (that is, the $\overset{o}{\nu}$ in Prop. 1.4 (2)) is the connection whose coefficients are

$$\overset{o}{N}{}^a_i = \gamma^a_{ij} y^j - g^{aj} C_{ijk} \gamma^k_{lm} y^l y^m,$$

where $C_{ijk} = \partial_{y^k} g_{ij}/2$ is the Cartan tensor of L (which measures how far g is from being pseudo-Riemannian).

Next our aim is to select an anisotropic Levi-Civita connection for a pseudo-Finsler metric L, rethinking the role of the Chern connection.

Theorem 1.4. *Given a pseudo-Finsler metric L and being its fundamental tensor g, there exists a unique anisotropic conection ∇ that is torsion-free and* metric*, i.e., such that $\nabla g = 0$. It is characterized by the Koszul-type formula*

$$\begin{aligned} 2g_v(\nabla^V_X Y, Z) = &\left(X(g_V(Y,Z)) - Z(g_V(X,Y)) + Y(g_V(X,Z)) \right)(\pi(v)) \\ &+ g_v([X,Y],Z) + g_v([Z,X],Y) - g_v([Y,Z],X) \\ &- 2C_v(Y,Z,\nabla^V_X V) - 2C_v(Z,X,\nabla^V_Y V) + 2C_v(X,Y,\nabla^V_Z V), \end{aligned} \tag{1.30}$$

where $v \in A$, $X, Y, Z \in \mathfrak{X}(M)$, $V \in \mathfrak{X}^A(U)$ is any local extension of v and C is the Cartan tensor defined above. Its Christoffel symbols are

$$\Gamma^a_{ij} = \frac{1}{2} g^{ak} \left(\frac{\delta g_{ki}}{\delta x^j} + \frac{\delta g_{kj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^k} \right) \tag{1.31}$$

(the δ_j are the ones of the associated nonlinear connection ν^∇). This unique ∇ is called the Levi-Civita anisotropic connection of L. *The corresponding vertically trivial linear connection is the Chern connection.*

Proof. Taking into account that $\dot{\partial} g = 2C$, it follows that

$$(\nabla_X g)_v(Y, Z) = X(g_V(Y, Z))(\pi(v)) - g_v(\nabla^V_X Y, Z) - g_v(Y, \nabla^V_X Z) - 2C_v(Y, Z, \nabla^v_X V)$$

and using that $\nabla g = 0$, as well as the above formula permuting X, Y, Z, one gets (1.30). To get (1.31), observe that $\nabla g = 0$ in coordinates means

$$\delta_k g_{ij} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il} = 0,$$

which is equivalent to the structure equations of the Chern connection (see [2]), and therefore its Christoffel symbols coincide with those of Chern's as well as its vertically trivial associated connection.

∎

Remark 1.7. We have seen that there are two distinguished anisotropic connections associated with a pseudo-Finsler metric, the Berwald one (see Rem. 1.6) and the Levi-Civita–Chern one (see Th. 1.4). The difference between them is a tensor $\mathcal{L}^\sharp$ metrically equivalent to the Landsberg tensor of L, which satisfies $\mathcal{L}^\sharp_v(u, v) = 0$ for all $v \in A$ and $u \in T_{\pi(v)}M$ (see [25]). This property is essential as it guarantees that both connections have the same associated nonlinear connection $\overset{o}{\nu}$ (the metric nonlinear connection of L). Indeed, the anisotropic connections differing in a symmetric tensor Q with this property from the Chern or Berwald ones are exactly the torsion-free anisotropic connections having $\overset{o}{\nu}$ as their associated nonlinear connection (recall Th. 1.2 (3)). The properties of this family of connections as tools for the study of pseudo-Finsler metrics were collected in [12], where they are referred to as *distinguished connections*.

1.6.2 The Finslerian linear connections

The linear connections associated with a pseudo-Finsler metric L live in the bundle $\mathrm{V}A \to A$. As we have seen, L determines the metric spray G and, thus, the *metric* *nonlinear connection* $\overset{o}{\nu}$, which plays the role of the prescribed auxiliary connection seen in §1.5. Then, Prop. 1.3 and Th. 1.3 are applicable. As a summary, one has:

1. The linear connections ∇^* used in pseudo-Finsler geometry are defined in the vector bundle $\mathrm{V}A \to A$ and they are homothetically invariant.
2. Such a ∇^* can be specified by means of the Christoffel symbols with respect to the metric nonlinear connection $\overset{o}{\nu}$ (Prop. 1.3), namely: $\left(\Gamma^{\mathrm{H}}\right)^k_{ij}$, which are 0-homogeneous, and $\left(\Gamma^{\mathrm{V}}\right)^k_{ij}$, which are (-1)-homogeneous.
3. The vertically trivial ∇^*'s are in natural correpondence with the homogeneous A-anisotropic connections on M (Th. 1.3). Using $\overset{o}{\nu}$, the non-vertically trivial ∇^*'s project onto the vertically trivial ones (Rem. 1.5).
4. The most frequent choices of ∇^* in Finsler Geometry have the following horizontal and vertical parts:
   - Berwald and Hashiguchi: $\left(\Gamma^{\mathrm{H}}\right)^k_{ij} = N^k_{i\cdot j} := \dot{\partial}_j N^k_i$, where the N^j_i's come from $\overset{o}{\nu}$. The Berwald connection is vertically trivial and the Hashiguchi connection has $\left(\Gamma^{\mathrm{V}}\right)^k_{ij} = C^k_{ij} = g^{kl} C_{ijl}$.
   - Chern-Rund and Cartan: $\left(\Gamma^{\mathrm{H}}\right)^k_{ij} = \Gamma^k_{ij}$ as in (1.31). The Chern-Rund connection is vertically trivial, and in the case of the Cartan connection, $\left(\Gamma^{\mathrm{V}}\right)^k_{ij} = C^k_{ij}$.

1.7 Parallel transport and anisotropic connections

Next, let us go back to our observers' viewpoint in §2.2 to introduce parallel transport and show how an anisotropic connection can be recovered from it.

1.7.1 Observers and parallel transport

The most natural way to compare vectors in different tangent spaces of a manifold is by making use of parallel transport along a curve. Depending on what we want to study, this parallel transport should preserve certain properties of vectors. In general, we cannot ensure the preservation of the indicatrix of a pseudo-Finsler metric by a linear map, because the indicatrices at different points may be not linearly equivalent. Perhaps the best idea is to require the preservation of their best approximations by a scalar product, namely, g_v. As there is a dependence on v, we will need different parallel transports for every $v \in A$. Summing up, the Christoffel symbols will depend also on the direction, so the covariant derivative along a curve $\gamma : I \to M$ needs a reference vector field $W \in \mathfrak{X}(\gamma)$:

$$D_\gamma^W : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma),$$

where $\mathfrak{X}(\gamma)$ denotes the module of smooth vector fields along γ. Moreover, we will assume that the dependence on W is pointwise, in the sense that at the instant t_0, D_γ^W depends only on $W(t_0)$. Thinking about what happens in a Finsler spacetime, where all the computations depend on the observer, we will make first the parallel transport of the observer along γ by searching for a vector field V such that

$$D_\gamma^V V = 0,$$

with the natural requirement that $L \circ V = 1$. Finally, we are ready to parallel transport vectors along γ using a parallel vector field Z, defined by

$$D_\gamma^V Z = 0.$$

As we will see later, if we require this parallel transport to preserve also the metric g_v of the restspace, then the covariant derivative comes from the Levi-Civita–Chern anisotropic connection. Geodesics are recovered as the curves with autoparallel velocity, namely

$$D_\gamma^{\dot\gamma} \dot\gamma = 0.$$

In particular, $L \circ \dot\gamma = const$.

Of course, when we speak about a covariant derivative, we are assuming that it satisfies the natural properties of a derivative. Let us put this on rigorous basis. In the following, given a smooth curve $\gamma : [a,b] \to M$, $\mathcal{F}(I)$ will denote the ring of smooth real functions defined on $I = [a,b]$. Recall that A denotes a conic open subset of TM, $\pi : TM \to M$ is the natural projection and $\gamma^*(TM)$ is the pullbacked of this bundle by means of the curve $\gamma : [a,b] \to M$.

Definition 1.9. *An* anisotropic covariant derivative D_γ *in A along a curve γ : $[a,b] \to M$ is a map*

$$D_\gamma : \gamma^*(A) \times \mathfrak{X}(\gamma) \to TM, \qquad (v, X) \mapsto D_\gamma^v X \in T_{\pi(v)}M$$

with a smooth dependence on v, such that if $\pi(v) = \gamma(t_0)$ with $t_0 \in [a,b]$, then

(i) $D_\gamma^v(X+Y) = D_\gamma^v X + D_\gamma^v Y;\ X, Y \in \mathfrak{X}(\gamma)$,

(ii) $D^v_\gamma(fX) = \frac{df}{dt}(t_0)X(t_0) + f(t_0)D^v_\gamma X; f \in \mathcal{F}(I), X \in \mathfrak{X}(\gamma)$.

Remark 1.8. Let $\pi_\gamma : \gamma^*(A) \to [a,b]$ be the pullback fibered manifold obtained from $A \to M$ by $\gamma : [a,b] \to M$. The formal similarity of our definition of D_γ with Def. 1.4 for anisotropic connections can be stressed by redefining: (a) the domain of D_γ as $\mathfrak{X}(\gamma)$ and (b) its codomain as the sections of the pullback bundle $\pi^*_\gamma(TM) \to \gamma^*(A)$ obtained from $\gamma^*(TM) \to [a,b]$ by $\pi_\gamma : \gamma^*(A) \to [a,b]$.

The notion of anisotropic connection gathers the information of the covariant derivatives along different curves. In fact, there is a unique covariant derivative along curves determined by an anisotropic connection (see [12, Prop. 2.7]).

Proposition 1.5. Given a smooth curve $\gamma : [a,b] \to M$, an A-anisotropic connection ∇ determines the unique A-anisotropic covariant derivative along γ with the following property: if $X \in \mathfrak{X}(M)$, then $D^v_\gamma(X_\gamma) = \nabla^v_{\dot\gamma}X$, where $X_\gamma := X\circ\gamma$.

Proof. Indeed, given a local chart (U,x) on M, we can express this covariant derivative in terms of the Christoffel symbols of the A-anisotropic connection ∇, which are defined as the functions $\Gamma^k{}_{ij} : TU \cap A \to \mathbb{R}$ determined by

$$\nabla^v_{\partial_i}\partial_j = \Gamma^k{}_{ij}(v)\,\partial_k|_{\pi(v)}.$$

It is easy to check that if $X = X^k\partial_k$, then

$$D^W_\gamma X = (\dot X^i + (\Gamma^i{}_{jk}\circ W)\dot\gamma^j X^k)\,\partial_i. \tag{1.32}$$

This provides coordinate expressions for the covariant derivative. ∎

Moreover, given a curve $\gamma : [a,b] \to M$, if one fixes the reference vector field $W \in \mathfrak{X}(\gamma)$ and $t_1, t_2 \in [a,b]$, then it is possible to define a parallel transport

$$P^W_{t_1,t_2} : T_{\gamma(t_1)}M \to T_{\gamma(t_2)}M$$

in such a way that $P^W_{t_1,t_2}(z) = Z(t_2)$, where $Z \in \mathfrak{X}(\gamma)$ is such that $D^W_\gamma Z = 0$ and $Z(t_1) = z$.

Remark 1.9. The parallel transport $P^W_{t_1,t_2}$ shares all the natural properties of the parallel transport with respect to an affine connection (it is a well-defined linear isomorphism, invariant under reparametrizations of γ including the reversal of the sign). Indeed, as the value of the Christoffel symbols is determined by W, which is fixed, (1.32) yields an equation for the transport of the same type as in the affine case.

There is a different type of parallel transport which is not always well-defined, namely, when the goal is to find a vector field V along γ such that $D_\gamma^V V = 0$. The existence of this parallel transport is not guaranteed along the whole curve unless we have some control on the Christoffel symbols.

Definition 1.10. *A smooth curve $\gamma : [a,b] \to M$ is* parallel transport complete *if for every $v \in A \cap T_{\gamma(a)}M$, there exists a (unique) A-admissible vector field $V \in \mathfrak{X}(\gamma)$ such that $D_\gamma^V V = 0$ and $V(a) = v$. Consistently with §1.3.1, here A-admissible means that $V(t) \in A$ for all $t \in [a,b]$.*

Remark 1.10. From standard results of ODE's one has that, for every $v \in A \cap T_{\gamma(a)}M$, there exists some $\epsilon > 0$ such that a parallel V as above is well-defined in $[a, a+\epsilon]$. Moreover, all the curves are parallel transport complete in the following two cases: (1) when $A = TM \setminus \mathbf{0}$ and the anisotropic connection is homogeneous, and (2) in the case of a Finsler spacetime (M, L) with a distinguished connection (for the latter, notice that the anisotropic connection is homogeneous and $L(V)$ is constant for any parallely transported vector V, so that it cannot abandon A). From now on, we will restrict ourselves to work with curves where this parallel transport is defined everywhere.

Let us define the parallel transports which have a geometric meaning to compare what happens in different points of the manifold.

Definition 1.11 (Instantaneous observer's parallel transport). *Let ∇ be an A-anisotropic connection and $\gamma : [a,b] \to M$ a parallel transport complete curve. For each $t_1, t_2 \in [a,b]$, the* instantaneous observer's parallel transport *is the map*

$$P_{t_1,t_2} : A \cap T_{\gamma(t_1)}M \to A \cap T_{\gamma(t_2)}M$$

given by $P_{t_1,t_2}(v) = V(t_2)$, for $V \in \mathfrak{X}(\gamma)$ satisfying $V(t_1) = v$ and $D_\gamma^V V = 0$.

This parallel transport coincides the one obtained from the nonlinear connection which appears in many classical textbooks devoted to Finsler geometry as [19, Ch. VII], [1, §2.1.6], [25, page 103], [4, §2.1], [8, Def. 1.4] and [27, §7.6]. In some other textbooks there is an additional notion of parallel transport taking as a reference vector the velocity of the curve (see [25, Def. 7.3.1],[26, §5.3] and [6, Ch. 4]). Recall that we defined instantaneous observers in the setting of Finsler spacetimes as vectors $v \in A$ of unit length $L(v) = 1$. Of course, the constraint on the length is not relevant for the transport. In the case of general anisotropic connections such a restriction makes no sense but we have maintained the name of observers to stress the relativistic geometric intuitions.

Definition 1.12 (Parallel transport with respect to an instantaneous observer). *Let ∇ be A-anisotropic connection and $\gamma : [a,b] \to M$ a parallel transport complete curve. For each $t_1, t_2 \in [a,b]$ and observer $v \in T_{\gamma(t_1)}M \cap A$, the* parallel transport with respect to v *is the map*

$$P^v_{t_1,t_2} : T_{\gamma(t_1)}M \to T_{\gamma(t_2)}M$$

obtained as $P^v_{t_1,t_2}(w) = W(t_2)$, where $W \in \mathfrak{X}(\gamma)$ satisfies that $W(t_1) = w$ and $D^V_\gamma W = 0$ with V satisfying $D^V_\gamma V = 0$ and $V(t_1) = v$.

Recall that, as V is fixed by v, this parallel transport satisfies the natural properties of the transport explained in Rem. 1.9. See [23] for a general treatment of parallelism.

1.7.2 Recovering the anisotropic connection from the transport

First observe that given a smooth curve $\gamma : [a,b] \to M$, we can define the paralell transport of covectors with respect to the instantaneous observer $v \in T_{\gamma(a)}M \cap A$,

$$P^v_{a,b} : T_{\gamma(a)}M^* \to T_{\gamma(b)}M^*,$$

as

$$P^v_{a,b}(\theta)(w) = \theta(P^v_{b,a}(w))$$

for any $\theta \in T_{\gamma(a)}M^*$ and $w \in T_{\gamma(b)}M$, so that a parallel covector field on a parallel vector field will be constant. Imposing commutativity with the tensor product, this parallel transport can be extended to arbitrary (r,s)-tensors.

Next, we write explicitly such a transport regarding the tensors as multilinear maps. Consider an A-anisotropic tensor $T \in \mathcal{T}^r_s(M_A)$ and a curve $\gamma : [a,b] \to M$. Then we can define the parallel transport $P_{t_1,t_2}(T)_v$ for any $t_1, t_2 \in [a,b]$ as the map

$$P_{t_1,t_2}(T)_v : T^*_{\pi(v)}M \times \overbrace{\cdots}^{r} \times T^*_{\pi(v)}M \times T_{\pi(v)}M \times \overbrace{\cdots}^{s} \times T_{\pi(v)}M \to \mathbb{R}$$

given by

$$\begin{aligned} &P_{t_1,t_2}(T)_v(\theta^1, \dots, \theta^r, v_1, \dots, v_s) \\ &:= T_{P_{t_2,t_1}(v)}(P^v_{t_2,t_1}(\theta^1), \dots, P^v_{t_2,t_1}(\theta^r), P^v_{t_2,t_1}(v_1), \dots, P^v_{t_2,t_1}(v_s)). \end{aligned}$$

In particular, we can define a curve of anisotropic tensors in $T_{\pi(v)}M$:

$$P_t(T) = P_{t,a}(T).$$

Our next goal is to compare this parallel transport with the covariant derivative of any A-anisotropic tensor, as given in Th. 1.1 and formula (1.12). Namely, being $P_t(T)$ a curve in a vector space, let us relate its natural derivative with $\nabla_{\dot\gamma(a)}T$.

Proposition 1.6. Given an A-anisotropic tensor $T \in \mathcal{T}^r_s(M_A)$, an A-anisotropic connection ∇ and a regular curve $\gamma : [a,b] \to M$, it holds that

$$(\nabla_{\dot\gamma(a)}T)_v = \frac{d}{dt}P_t(T)_v|_{t=a},$$

for any $v \in T_{\gamma(a)}M \cap A$.

Proof. Assume first that $r = 0$. Recall that we can compute $(\nabla_{\dot\gamma(a)}T)_v(v_1,\dots,v_s)$ choosing an A-admissible extension V of v and arbitrary extensions $X, X_1,\dots,X_s$ of $\dot\gamma(a), v_1,\dots,v_s$, respectively. In particular, these extensions can be chosen in such a way that $\nabla^V_X V = \nabla^V_X X_j = 0$ along γ for all $j = 1,\dots,s$. With these choices,

$$\begin{aligned}(\nabla_{\dot\gamma(a)}T)_v(v_1,\dots,v_s) &= X_{\gamma(a)}(T_V(X_1,\dots,X_s)) \\ &= \frac{d}{dt}(T_{V_{\gamma(t)}}((X_1)_{\gamma(t)},\dots,(X_s)_{\gamma(t)}))|_{t=a}. \end{aligned} \tag{1.33}$$

Finally, observe that $V_{\gamma(t)} = P_{a,t}(v)$, since $V(t) := V_{\gamma(t)}$ is a parallel observer along γ (recall that $\nabla^V_X V = 0$) and $V(a) = V_{\gamma(a)} = v$. Moreover, $(X_i)_{\gamma(t)} = P^v_{a,t}(v_i)$ since $\nabla^V_X X_i = 0$ and $(X_i)_{\gamma(a)} = v_i$. Replacing these quantities in (1.33), we get

$$(\nabla_{\dot\gamma(a)}T)_v(v_1,\dots,v_s)|_{t=a} = \frac{d}{dt}T_{P_{a,t}(v)}(P^v_{a,t}(v_1),\dots,P^v_{a,t}(v_s))|_{t=a} = \frac{d}{dt}P_t(T)_v(v_1,\dots,v_s)|_{t=a},$$

as required.

For the general case $r > 0$, observe that given the covector fields $\theta^1,\dots,\theta^r$, it is possible to choose one-forms ω^i such that $\nabla^V_X\omega^i = 0$ and $(\omega^i)_{\gamma(a)} = \theta^i$. Then, applying the proposition for $r = 0$, it can be shown that $(\omega^i)_{\gamma(t)} = P_{a,t}(\theta^i)$. The result follows analogously to the case $r = 0$ by computing the covariant derivative with V as a reference vector and $\omega^1,\dots,\omega^r, X_1,\dots,X_s$ as above. ∎

It is worth pointing out that, as only the parallel transport close to $t = a$ is needed for each chosen v, the result can be applied even if the curve is not parallel transport complete (see Rem. 1.10).

1.7.3 Levi-Civita–Chern connection of a Finsler spacetime

Let (M, L) be a pseudo-Finsler manifold with $L : A \to \mathbb{R}$. We aim to find an A-anisotropic connection that defines a parallel transport which preserves some metric properties. As we explained in §1.7, the parallel transport of an instantaneous observer should preserve the L-length and the parallel transport with respect to an instantaneous observer should preserve the fundamental tensor g_v, namely, for any curve $\gamma : [a,b] \to M$ and $v \in A \cap T_{\gamma(a)}M$ such that the parallel transport of v is well-defined along γ,

$$g_{P_{a,b}(v)}(P^v_{a,b}(u), P^v_{a,b}(w)) = g_v(u,w) \tag{1.34}$$

for all $u, w \in T_{\pi(v)}M$. Observe that this implies in particular that $L(v) = L(P_{a,b}(v))$, as

$$L(v) = g_v(v,v) = g_{P_{a,b}(v)}(P^v_{a,b}(v), P^v_{a,b}(v)) = L(P^v_{a,b}(v)) = L(P_{a,b}(v)).$$

Proposition 1.7. Let (M, L) be a pseudo-Finsler manifold. Then its Levi-Civita–Chern A-anisotropic connection is the only torsion-free connection with a parallel transport that preserves the fundamental tensor of L.

Proof. Observe that by Prop. 1.6, the fact that the parallel transport of ∇ preserves the fundamental tensor g as in (1.34) is equivalent to $\nabla g = 0$, and therefore ∇ is the Levi-Civita–Chern connection of (M, L). ∎

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2 The Einstein-Hilbert-Palatini formalism in pseudo-Finsler geometry

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Abstract

A systematic development of the so-called Palatini formalism is carried out for pseudo-Finsler metrics L of any signature. Substituting in the classical Einstein-Hilbert-Palatini functional the scalar curvature by the Finslerian Ricci scalar constructed with an independent nonlinear connection N, the affine and metric equations for (N, L) are obtained. In Lorentzian signature with vanishing mean Landsberg tensor Lan_i, both the Finslerian Hilbert metric equation and the classical Palatini conclusions are recovered by means of a combination of techniques involving the (Riemannian) maximum principle and an original argument about divisibility and fiberwise analyticity. Some of these findings are also extended to classical Riemannian solutions by using the eigenvalues of a Laplacian. When $\mathrm{Lan}_i \neq 0$, the Palatini conclusions fail necessarily, however, a good number of properties of the solutions remain. The framework and proofs are built up in detail.

Keywords — *Finsler spacetimes, Palatini formalism, Hilbert action, uniqueness of partially analytic solutions, Finsler-Einstein equations, nonlinear connections, geodesics.*

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2.1 Introduction

Recently, the interest in Finslerian modifications of General Relativity has grown [6, 8, 9, 10, 14, 16, 19, 22, 32, 34, 37, 41, 47] motivated in part by the role of Finsler Geometry in the Standard-Model Extension [13, 30, 31] and Lorentz violation. The search for an extension of the Einstein equations to this setting emerges as a fundamental issue. A first way to find them is to consider Finslerian generalizations of the Einstein tensor $\mathbf{G}$, having several alternatives [35, 42, 48, 51, 54]. A second way is provided by Hilbert's variational approach, developed by Hohmann, Pfeifer, Voicu and Wohlfarth [21, 22, 46], these authors take the natural generalization $\mathcal{S}$ of the Hilbert functional. This $\mathcal{S}$ is given by the integral of the 0-homogeneized (Finslerian) Ricci scalar of any Lorentz-Finsler metric L for a given manifold M (see [23] for a general framework dealing with action functionals of arbitrary homogeneous fields). The corresponding Euler-Lagrange equation leads to a scalar which, when restricted to Lorentzian metrics, yields naturally a tensor field; this tensor is not exactly equal to $\mathbf{G}$, but it still leads to the same vacuum equations for such metrics. The aim of the present article is to deepen in the variational approach to the Einstein equations by considering the so-called Palatini formalism[1] for pseudo-Finsler metrics of arbitrary signature, paying special attention to the Lorentzian and positive definite cases. Let us notice that there are also some works that study Finslerian Einstein manifolds with a variational approach, such as [11] (which overcomes certain issues encountered in[2] [1]). In particular, in [11] the authors use a similar functional to that of [21, 46] but dividing by the total volume in a positive definite setting. Another different approach is the one in [3], where, indeed, the author explores several possibilities, using in particular the concept of osculation. Finally, beyond pseudo-Finsler geometry, in [53] variational equations for any Sasaki-type metric on the tangent bundle of M are derived by taking the Palatini formalism into account.

Recall that the classical Palatini approach considered the affine connection ∇ and the pseudo-Riemanian metric g as independent variables for the Hilbert functional and, given g, it recovered its Levi-Civita connection ∇^g as the unique symmetric solution of the Euler-Lagrange affine equation for ∇ (the properties of the non-symmetric ones are also known [7]). This was a milestone for the mathematical foundations of Relativity because it ensured that the connection ∇ which describes gravity is the

[1]This is the usual name in textbooks, even though the approach was actually invented in 1925 by Einstein [15]. Anyway, the name is maintained here so that it is distinguished from more general metric-affine formalisms.

[2]See D. Bao's report in Mathematical Reviews, MR1365208 (99m:53130).

same one as the connection ∇^g which provides the critical points of the length or energy functionals for curves. Thus, light rays and free falling particles are unequivocally described by this unique connection. In the Finslerian setting, to ensure such a consistency is a much more prioritary task, because there is a huge freedom when looking for associated (linear or nonlinear) connections.

Consistently, here we will maintain the functional $\mathcal{S}$ but its variables will be the nonlinear connection N and the pseudo-Finsler metric L. Notice that no other kind of (linear) Finsler connection is required for the construction of the Ricci scalar. That is, (N, L) is enough for our functional and we remain formally close to the classical Palatini setting, thus obtaining coupled *affine* (2.19) and *metric* (2.20) Palatini equations. However, further functionals should be tractable with the basic ingredients that we will develop.

The central question is, given L, to what extent its associated nonlinear N^L is the unique affine solution N. In the pseudo-Riemannian case, a simple argument shows that all of these can be written as $\nabla^g + \mathcal{A} \otimes \mathrm{Id}$, where the arbitrary 1-form $\mathcal{A} \equiv \mathcal{A}_i(x)$ ($\mathrm{Id} \equiv \delta^i_j$ is the identity tensor) determines the torsion [7]. In the Finslerian case, the torsion part of N becomes $\mathcal{A} \otimes \mathbb{C}$ with $\mathcal{A} \equiv \mathcal{A}_i(x, y)$ ($\mathbb{C} \equiv y^a \partial_{y^a}$ is Liouville's) and the problem is reduced to the case of symmetric N. That is, as a first result (Th. 2.2, Cor. 2.1):

> **Theorem A**. Given a pseudo-Finsler metric L, the solutions of the affine equation have a fibered structure on the symmetric solutions with fiber isomorphic to the space of anisotropic (0-homogeneous) 1-forms $\mathcal{A}$, so that, for each solution N, there is a unique symmetric one $\Pi^{\mathcal{S}\mathrm{ym}}(\mathrm{N})$ such that $\mathrm{N} = \Pi^{\mathcal{S}\mathrm{ym}}(\mathrm{N}) + \mathcal{A} \otimes \mathbb{C}$ for some $\mathcal{A}$.

However, the symmetric case is not trivial, as N is governed by a PDE at each $p \in M$. Even more, the following subtlety appears for global uniqueness at p: when L is indefinite, its domain $A \subseteq \mathrm{T}M \setminus \mathbf{0}$ is naturally conic, being $L_{\partial A} = 0$, as the indicatrix (and some homogeneous elements) becomes ill-defined at ∂A. Notice also that, in Lorentzian signature, A would correspond to the future-directed timelike directions, and the restriction to these (including the future-directed lightlike directions as a limit) is well motivated by physical interpretations [8]. However, we will develop (fiberwise) global techniques which work for *proper* solutions, i.e., smoothly extendible to ∂A (defns. 2.8, 2.14). The fibered structure in Theorem A is naturally transferred to the proper solutions (Prop. 2.14) and we prove the existence of a unique fibre in relevant general cases such as the following (see Th. 2.3):

Theorem B. Any analytic proper indefinite pseudo-Finsler metric L admits at most one analytic proper symmetric solution N of the affine variational equation (2.19).

The proof relies on an original divisibility argument which is developed in full detail (Lem. 2.5). Moreover, we emphasize that the essential property at this point is just *fiberwise analyticity* (Def. 2.15, Rems. 2.20, 2.22). This is much weaker than analyticity and, indeed, it holds trivially for all the smooth (non-analytic) affine and pseudo-Riemannian elements.

We also give other arguments, based on the maximum principle and the eigenvalues of the Laplacian, which yield some extensions of Th. A without fiberwise analyticity (Th. 2.5, Cor 2.3), as well as applications to the positive definite case (Th. 2.6). These arguments provide also the proof of the following result (Th. 2.7), which is relevant for the metric Palatini equation.

Theorem C. Let L be a (properly) Lorentz-Finsler metric and N any nonlinear connection smoothly extendible to ∂A with Ricci scalar Ric. If the Einstein-type scalar $(n+2)\,\mathrm{Ric} - L\, g^{ab}\,\mathrm{Ric}_{\cdot a \cdot b}$ vanishes, then Ric vanishes too.

Indeed, when the mean Landsberg tensor Lan_i vanishes, as it occurs in the classical case, this equation agrees with the one obtained by the Hilbert approach (i.e., the aforementioned in [21]). So, the result above is relevant for the consistency of the vacuum Einstein equations. In comparison with the elementary pseudo-Riemannian case (Rem. 2.24), where it is valid in any signature, our result is technically more complicated and has a properly Finslerian applicability. As the aforementioned results, it relies on Lem. 2.6, also proven in full detail.

To complete the approach, one should check at what extent the natural (Berwald) nonlinear connection N^L associated with L plays a role similar to that which ∇^g plays in the classical Palatini setting. Notice that N^L is naturally associated with the geodesic spray of L, so this issue is related to the Palatini physical interpretations about free falling observers. The solution involves the *Landsberg* tensor Lan or, more precisely, the mean Landsberg $\mathrm{Lan}_i = \mathrm{Lan}^a_{ai}$ (see Cor. 2.1, Rem. 2.15, Prop. 2.11, Rem. 2.16):

Theorem D. Given a pseudo-Finsler L, its nonlinear Berwald connection N^L is a solution of the affine variational equation (2.19) iff $\mathrm{Lan}_i = 0$.

In this case, any other solution N shares its pregeodesics with N^L iff it lies in the same fiber, i.e., $\mathrm{N} = \mathrm{N}^L + \mathcal{A} \otimes \mathbb{C}$ for some $\mathcal{A}$; then, it shares geodesics iff $\mathcal{A}_a\, y^a = 0$.

Otherwise, when Lan_i does not vanish identically, neither N^L is a solution nor any solution N can share pregeodesics with N^L.

In any case, when L and N are proper, any N-geodesic γ has constant sign of $L(\dot{\gamma})$. Moreover, in the Lorentz-Finsler case (no matter how Lan_i is), the causal character (timelike, lightlike) of the N-geodesics does not change, the lightlike N-geodesics coincide with the corresponding L-geodesics and, hence, the lightlike N-pregeodesics are the cone (pre-)geodesics inherent to the L-cone structure.

It is worth pointing out that the properties about sharing geodesics and pregeodesics hold not only for the fiber of N^L but also for any other fiber of solutions (with independence of Lan_i). Moreover, further compatibility conditions of ∇ and L appear for connections differing only in some $\mathcal{A} \otimes \mathbb{C}$ from a symmetric one (not necessarily solutions), see Prop. 2.10. As a summary of all these results:

When $\mathrm{Lan}_i = 0$*, the fibered structure of the affine solutions, the fact that* N^L *determines one of such fibers, the uniqueness of this fiber under mild conditions (properness, fiberwise analyticity), the subsequent status of* N^L *as the unique symmetric solution, and the fact that all these solutions share pregeodesics (those of* L*), recover and extend naturally all the conclusions of the classical Palatini formalism for the connection (apart from those for the metric, at least in the vacuum case). However, no such extension is possible when* $\mathrm{Lan}_i \neq 0$.

As commented above in Theorem D, when $\mathrm{Lan}_i \neq 0$, the solutions N of the affine equation do not share pregeodesics with L. This fact can have several interpretations. Taking into account that the main goal of the Hilbert functional is to obtain the Einstein field equations, one could infer that the solutions N are very suitable for computing them. Nevertheless, it is not clear which is the best connection to compute the trajectories of the Finsler spacetime. The connections N relate more closely the Jacobi equation to our field equation, whereas the geodesics of L satisfy a variational principle.

From the technical viewpoint, we introduce detailedly all the elements we need, which are spread in the literature under different viewpoints and implicit frameworks.

Full proofs of the results are also provided (including straightforward but lengthy computations) to permit traceability.

With this spirit, in §2.2 the required ingredients on Finsler Geometry and anisotropic calculus are introduced. The so-called *Finslerian connections* [12, 39], i.e., pairs (N, ∇^*) composed by a nonlinear N and a linear connection ∇^*, the latter for the vertical bundle $\mathrm{V}A \longrightarrow A$, do not really enter into our work; instead, anisotropic connections [24, 25] will suffice and will introduce a simple and intuitive Koszul derivative directly on M. Anyway, any anisotropic connection ∇ can be identified canonically with a vertically trivial ∇^* (see [28] for this and other results linking both approaches), so the readers tied to this classical framework can rewrite our computations in the way they prefer. In §2.3, the metric-affine (Palatini) variational calculus is developed. Here, independently, L yields the indicatrix $\{L = 1\}$ and a volume element, while N yields the Ricci scalar (Remark 2.9). Full details of the proofs of the affine and metric equations, as well as of the crucial divergence formula in the suitably projectivized space, are provided in the Appendices. In §2.4, the study of the solutions for N is reduced to the symmetric case, including the fibered structure of the space of solutions and the properties shared by the elements of each fiber (Cor. 2.1). Moreover, a detailed study of the different types of metric and geodesic compatibility for the solutions is carried out (Props. 2.10, 2.11, 2.12). Finally, in §2.5, the main results on proper solutions are distributed into two subsections, the first one on techniques related to divisibility by L (eventually using fiberwise analyticity), and the second one related to the maximum principle. Using both types of results, the classical solutions are revisited in the last subsection.

2.2 Standard geometric objects

The main aim of this section is to fix notation and conventions.

Let M be a connected[3] smooth[4] manifold of dimension[5] $n \geq 2$. The Einstein convention is employed, the indices $a, b, c, d, e, i, j, k, l$ run in the set $\{1, ..., n\}$, and for clarity, we use i, j, k as free indices and a, b, c, d, e as summation indices. Charts

[3]Only for simplicity. In general, all of our developments are valid on each connected component of M.

[4]This will mean C^∞ and all the objects will be smooth. Nevertheless, some results may not need so much regularity. For instance, those of §2.5.2 only require a finite number of vertical derivatives existing with continuity at each $p \in M$.

[5]In dimension 1 our action functional would trivialize.

$(U, x = (x^1, ..., x^n))$ for M induce natural charts $(\mathrm{T}U, (x, y) = (x^1, ..., x^n, y^1, ..., y^n))$ for $\mathrm{T}M$. Putting $\partial_i := \partial/\partial x^i$ and $\dot{\partial}_i := \partial/\partial y^i$, under a change $(U, x) \rightsquigarrow (\bar{U}, \bar{x})$,

$$\bar{\partial}_i = \frac{\partial x^a}{\partial \bar{x}^i}\,\partial_a + \bar{y}^b\,\frac{\partial^2 x^a}{\partial \bar{x}^b\,\partial \bar{x}^i}\,\dot{\partial}_a, \qquad \dot{\bar{\partial}}_i = \frac{\partial x^a}{\partial \bar{x}^i}\,\dot{\partial}_a$$

as local vector fields on $\mathrm{T}M$. Let $A \subseteq \mathrm{T}M$ be open with $\pi(A) = M$ for π the natural projection. The restriction $\pi_A : A \longrightarrow M$ defines a fibered manifold with *fibers* $A_p := A \cap \mathrm{T}_pM$ $(p \in M)$ and *vertical distribution* $\mathrm{V}A \longrightarrow A$,

$$\mathrm{V}_vA := \operatorname{Ker} \mathrm{T}_v\pi_A = \mathrm{T}_v(A_{\pi(v)}) = \operatorname{Span}\left\{\dot{\partial}_i|_v\right\} \subseteq \mathrm{T}_vA$$

($v \in A$, where $\mathrm{T}_v\pi_A$ is the *tangent map* or *differential* of π_A). The reader is referred to [33] for the general theory of fibered manifolds. We shall employ the framework of the anisotropic tensors [24, 25]; especially, the viewpoint and conventions of [28] can be helpful for the reader. An *r-contravariant s-covariant A-anisotropic tensor* is a section T of the pullback bundle

$$\pi_A^*(\overset{r)}{\bigotimes}\mathrm{T}M \otimes \overset{s)}{\bigotimes}\mathrm{T}^*M) \longrightarrow A;$$

we denote by $\mathcal{T}^r_s(M_A)$ the space of such sections. They have locally the form

$$T_v = T^{a_1,\dots,a_r}_{b_1,\dots,b_s}(v)\,\partial_{a_1}\Big|_{\pi(v)} \otimes \dots \otimes \partial_{a_r}\Big|_{\pi(v)} \otimes \mathrm{d}x^{b_1}_{\pi(v)} \otimes \dots \otimes \mathrm{d}x^{b_s}_{\pi(v)}$$

for certain $T^{i_1,\dots,i_r}_{j_1,\dots,j_s}(x, y)$'s defined on $A \cap \mathrm{T}U$ that transform tensorially under $(U, x) \rightsquigarrow (\bar{U}, \bar{x})$. There is a *vertical isomorphism* identifying anisotropic with vertical vector fields on A:

$$X_v = X^a(v)\,\partial_a|_{\pi(v)} \in \mathrm{T}_{\pi(v)}M \longleftrightarrow X^{\mathrm{V}}_v = X^a(v)\,\dot{\partial}_a|_v \in \mathrm{V}_vA \tag{2.1}$$

(notice that when the X^i's are constant on a fiber A_p, this formula makes explicit the identification between the vertical spaces at the different $v \in A_p$). In particular, the canonical anisotropic vector $\mathbb{C} \in \mathcal{T}^1_0(M_A)$ defined by

$$\mathbb{C}_v = v = y^a(v)\,\partial_a|_{\pi(v)} \tag{2.2}$$

corresponds to the *Liouville vector field* $\mathbb{C}^{\mathrm{V}}$ [21, 39, 43] (note that in the last two references $\mathbb{C}$ is used for what we denote $\mathbb{C}^{\mathrm{V}}$). The vertical derivatives

$$T^{i_1,\dots,i_r}_{j_1,\dots,j_s\cdot j_{s+1}}(x, y) := \dot{\partial}_{j_{s+1}} T^{i_1,\dots,i_r}_{j_1,\dots,j_s}(x, y) = \frac{\partial T^{i_1,\dots,i_r}_{j_1,\dots,j_s}}{\partial y^{j_{s+1}}}(x, y)$$

define a new anisotropic tensor: the *vertical differential* of T; we denote it by $\dot\partial T \in \mathcal{T}^r_{s+1}(M_A)$ and by $\dot\partial_X T \in \mathcal{T}^r_s(M_A)$ its contraction with X in the new index. For instance,

$$\dot\partial_{\mathbb{C}} T = y^{b_{s+1}}\, T^{a_1,\dots,a_r}_{b_1,\dots,b_s\cdot b_{s+1}}\, \partial_{a_1} \otimes \dots \otimes \partial_{a_r} \otimes \mathrm{d}x^{b_1} \otimes \dots \otimes \mathrm{d}x^{b_s}.$$

An anisotropic tensor T can actually be *isotropic*, in that $T^{i_1,\dots,i_r}_{j_1,\dots,j_s}(x,y) = T^{i_1,\dots,i_r}_{j_1,\dots,j_s}(x)$. This is equivalent to the constancy of the restriction T_p to each fiber A_p ($p \in M$). Hence, it means that T reduces to a tensor field on M, which we will not distinguish notationally from T itself.

2.2.1 Homogeneous tensors

The following three notions of (positive) homogeneity are extracted from [24] and [43, Defs. 1.5.2 and 1.5.3] respectively.

Definition 2.1. *A is* conic *if $A \subseteq \mathrm{T}M \setminus \mathbf{0}$ and $\lambda v \in A$ for all $v \in A$, $\lambda \in \mathbb{R}^+$. In such a case, let $\alpha \in \mathbb{R}$.*

(i) $T \in \mathcal{T}^r_s(M_A)$ is α-homogeneous *if $T_{\lambda v} = \lambda^\alpha T_v$. That is, its coordinates are α-homogeneous (in y): $T^{i_1,\dots,i_r}_{j_1,\dots,j_s}(x,\lambda y) = \lambda^\alpha T^{i_1,\dots,i_r}_{j_1,\dots,j_s}(x,y)$.*
(ii) A vector field $\mathcal{X}$ on A is α-homogeneous *if $\mathcal{X}_{\lambda v} = \lambda^{\alpha-1}\left(\mathrm{T}h_\lambda\right)_v(\mathcal{X}_v)$, where $h_\lambda : A \longrightarrow A$, $h_\lambda(v) = \lambda v$. That is, if $\mathcal{X} = \mathcal{X}^a\partial_a + \mathcal{X}^{n+a}\dot\partial_a$, then $\mathcal{X}^i(x,y)$ and $\mathcal{X}^{n+i}(x,y)$ are, resp., $(\alpha-1)$- and α-homogeneous.*
(iii) An s-form ω on A is α-homogeneous *if $\left(\mathrm{T}h_\lambda\right)^*_v(\omega_{\lambda v}) = \lambda^\alpha \omega_v$, ($*$ means pullback). That is, if $\omega_{i_1,\dots,i_\mu|j_1,\dots,j_\nu}$ is the component of ω on $\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_\mu}\wedge\mathrm{d}y^{j_1}\wedge\dots\wedge\mathrm{d}y^{j_\nu}$ ($\mu+\nu=s$), then $\omega_{i_1,\dots,i_\mu|j_1,\dots,j_\nu}(x,y)$ is $(\alpha-\nu)$-homogeneous.*

Moreover, $\mathrm{h}^\alpha\mathcal{T}^r_s(M_A)$ and $\mathrm{h}^\alpha\mathcal{F}(A) := \mathrm{h}^\alpha\mathcal{T}^0_0(M_A)$ will denote the space of α-homogeneous anisotropic tensors and functions, resp.

Clearly, $\dot\partial : \mathrm{h}^\alpha\mathcal{T}^r_s(M_A) \longrightarrow \mathrm{h}^{\alpha-1}\mathcal{T}^r_{s+1}(M_A)$ is a well-defined linear morphism. The items (i) and (ii) are consistent with the identification of anisotropic and vertical vector fields in (2.1). In particular, both $\mathbb{C}$ and $\mathbb{C}^{\mathrm{V}}$ are 1-homogeneous, whereas any isotropic tensor field ($T^{i_1,\dots,i_r}_{j_1,\dots,j_s}(x,y) = T^{i_1,\dots,i_r}_{j_1,\dots,j_s}(x)$) is 0-homogeneous. The homogeneities of the coordinates of a 1-form $\omega = \omega_{a|}\,\mathrm{d}x^a + \omega_{|a}\,\mathrm{d}y^a$ are switched with respect to those of $\mathcal{X} = \mathcal{X}^a\partial_a + \mathcal{X}^{n+a}\dot\partial_a$ in concordance with the intrinsic meanings of $\mathcal{X}^i = 0$ and

$\omega_{|i} = 0$. The above expressions in coordinates and Euler's Theorem yield directly the following characterizations (consistently with [24, (6)] and [43, Ths. 1.5.2 and 1.5.3]).

Proposition 2.1. Assume that A is conic. Then:

(i) $T \in \mathcal{T}^r_s(M_A)$ is in $\mathrm{h}^\alpha \mathcal{T}^r_s(M_A)$ if and only if $\dot\partial_{\mathbb{C}} T = \alpha T$, i.e.,

$$y^{b_{s+1}} T^{a_1,\dots,a_r}_{b_1,\dots,b_s \cdot b_{s+1}}(x, y) = \alpha T^{a_1,\dots,a_r}_{b_1,\dots,b_s}(x, y).$$

(ii) A vector field $\mathcal{X}$ on A is α-homogeneous if and only if its Lie derivative along the Liouville field satisfies $\mathscr{L}_{\mathbb{C}^{\mathrm{V}}}(\mathcal{X}) = (\alpha - 1)\,\mathcal{X}$.

(iii) An s-form ω on A is α-homogeneous if and only if $\mathscr{L}_{\mathbb{C}^{\mathrm{V}}}(\omega) = \alpha\,\omega$.

The *positive projectivization* of the conic A plays the same role in our variational calculus as in [21]. We denote it by $\mathbb{P}^+A$, so that $\mathbb{P}^+ : A \longrightarrow \mathbb{P}^+A$, $v \longmapsto \mathbb{P}^+v$, is the natural projection. The 0-homogeneous s-forms on A induce $(s-1)$-forms on $\mathbb{P}^+A$. This correspondence was implicitly taken into account in the notation of [21], but we state it in ours for the reader's convenience.

Proposition 2.2. Assume that A is conic, and let ω be a 0-homogeneous s-form and $\mathcal{X}$ a 1-homogeneous vector field there. Then:

(i) The interior product $\mathcal{X} \lrcorner \omega$ is 0-homogeneous as well.

(ii) In the case $\mathcal{X} = \mathbb{C}^{\mathrm{V}}$, this interior product is the pullback of a unique $(s-1)$-form on $\mathbb{P}^+A$. We denote this one by $\underline{\omega}$, so that

$$\mathbb{C}^{\mathrm{V}} \lrcorner \omega = \left(\mathbb{P}^+\right)^* \underline{\omega}. \tag{2.3}$$

Moreover, $\underline{\omega}$ vanishes at $\mathbb{P}^+v \in \mathbb{P}^+A$ if and only if $\mathbb{C}^{\mathrm{V}} \lrcorner \omega$ vanishes at one, and hence all, representatives v of $\mathbb{P}^+v$.

(iii) The exterior differential $\mathrm{d}\omega$ is 0-homogeneous too with

$$\underline{\mathrm{d}\omega} = -\mathrm{d}\underline{\omega}.$$

Proof. (i) This is clear from the expression in coordinates of $\mathcal{X} \lrcorner \omega$ and Def. 2.1 (iii).

(ii) In order to define $\underline{\omega}$ at $\mathbb{P}^+v \in \mathbb{P}^+A$, one has to specify how it acts on s vectors in $\mathrm{T}_{\mathbb{P}^+v}\mathbb{P}^+A$. As $\mathrm{T}_v\mathbb{P}^+ : \mathrm{T}_vA \longrightarrow \mathrm{T}_{\mathbb{P}^+v}\mathbb{P}^+A$ is onto, those are always of the form $\mathrm{T}_v\mathbb{P}^+u_1$, ..., $\mathrm{T}_v\mathbb{P}^+u_s$ for some $u_1, \dots, u_s \in \mathrm{T}_vA$. And as (2.3) must be satisfied, the only possibility is to define

$$\begin{aligned} \underline{\omega}_{\mathbb{P}^+v}(\mathrm{T}_v\mathbb{P}^+u_1, \dots, \mathrm{T}_v\mathbb{P}^+u_s)(=: \left\{\left(\mathbb{P}^+\right)^* \underline{\omega}\right\}_v(u_1, \dots, u_s)) &= \left(\mathbb{C}^{\mathrm{V}} \lrcorner \omega\right)_v(u_1, \dots, u_s) \\ &= \omega_v(\mathbb{C}^{\mathrm{V}}_v, u_1, \dots, u_s) \end{aligned}$$

(where $\mathbb{C}^{\mathrm{V}}_{v}$ is just v under the natural identification $\mathrm{T}_{\pi(v)}M \equiv \mathrm{V}_{v}A \subsetneq \mathrm{T}_{v}A$, recall (2.2)). Finally, it is straightforward to see that this definition is consistent: the property $\mathrm{Ker}\,\mathrm{T}_{v}\mathbb{P}^{+} = \mathrm{Span}\left\{\mathbb{C}^{\mathrm{V}}_{v}\right\}$ allows one to check that it is independent of the representatives u_{μ} of $\mathrm{T}_{v}\mathbb{P}^{+}u_{\mu}$, whereas the properties $\left(\mathrm{T}h_{\lambda}\right)^{*}_{v}(\omega_{\lambda v}) = \omega_{v}$ and $\mathbb{C}^{\mathrm{V}}_{\lambda v} = \left(\mathrm{T}h_{\lambda}\right)_{v}(\mathbb{C}^{\mathrm{V}}_{v})$ allow one to check that it is independent of the representative v of $\mathbb{P}^{+}v$. Finally, from the construction with arbitrary $\left\{u_{1}, ..., u_{s}\right\}$, it is clear that $\underline{\omega}_{\mathbb{P}^{+}v} = 0$ if and only if $\omega_{v}(\mathbb{C}^{\mathrm{V}}_{v}, -, ..., -) = 0$.

(iii) Prop. 2.1 (iii), Cartan's formula for the Lie derivative and $\mathscr{L}_{\mathbb{C}^{\mathrm{V}}}(\omega) = 0$ give the 0-homogeneity of $\mathrm{d}\omega$:

$$\mathscr{L}_{\mathbb{C}^{\mathrm{V}}}(\mathrm{d}\omega) = \mathbb{C}^{\mathrm{V}} \lrcorner \mathrm{dd}\omega + \mathrm{d}(\mathbb{C}^{\mathrm{V}} \lrcorner \mathrm{d}\omega) = \mathrm{d}(\mathbb{C}^{\mathrm{V}} \lrcorner \mathrm{d}\omega) = \mathrm{d}(\mathscr{L}_{\mathbb{C}^{\mathrm{V}}}(\omega)) - \mathrm{dd}(\mathbb{C}^{\mathrm{V}} \lrcorner \omega) = 0.$$

For the last assertion, it suffices to see that $-\mathrm{d}\underline{\omega}$ satisfies the property that defines $\underline{\mathrm{d}\omega}$. Using the same properties as above,

$$\left(\mathbb{P}^{+}\right)^{*}(-\mathrm{d}\underline{\omega}) = -\mathrm{d}\left(\mathbb{P}^{+}\right)^{*}\underline{\omega} = -\mathrm{d}(\mathbb{C}^{\mathrm{V}} \lrcorner \omega) = -\mathscr{L}_{\mathbb{C}^{\mathrm{V}}}(\omega) + \mathbb{C}^{\mathrm{V}} \lrcorner \mathrm{d}\omega = \mathbb{C}^{\mathrm{V}} \lrcorner \mathrm{d}\omega,$$

so indeed $-\mathrm{d}\underline{\omega} = \underline{\mathrm{d}\omega}$. ∎

2.2.2 Homogeneous connections

There are a number of equivalent ways of defining the connections that we work with; most of them were discussed in [28]. Here, motivated by the spirit of the variational calculus, we choose alternative definitions that present the connections as sections of certain affine bundles over A. Then we pass to their coordinates, to ensure that we indeed are working with the same objects as in [28, (5) and (12)]. This conveys notational differences: for instance, when anisotropic connections are regarded as sections, we denote them by Γ, and when they are regarded as *Koszul covariant derivations*, we denote them by ∇. As a last comment, we will always work with homogeneous objects (even if we keep mentioning their homogeneity), so from now onward we assume that A is conic.

Consider affine connections on M (i.e., linear connections for $\mathrm{T}M \longrightarrow M$). Their Christoffel symbols $\Gamma^{k}_{ij}(x)$ have the transformation cocycle

$$\bar{\Gamma}^{k}_{ij}(x) = \frac{\partial \bar{x}^{k}}{\partial x^{c}}(x)\,\frac{\partial^{2} x^{c}}{\partial \bar{x}^{i}\,\partial \bar{x}^{j}}(x) + \frac{\partial \bar{x}^{k}}{\partial x^{c}}(x)\,\frac{\partial x^{a}}{\partial \bar{x}^{i}}(x)\,\frac{\partial x^{b}}{\partial \bar{x}^{j}}\,\Gamma^{c}_{ab}(x) \tag{2.4}$$

under changes of charts. Using an analogous of [29, §6.4], one can check that this cocycle determines an affine bundle $\mathbf{C}M \longrightarrow M$, which is so that its sections are precisely the affine connections on M.[6]

Definition 2.2. *A* homogeneous A-anisotropic connection *is a section* Γ *of the pullback affine bundle* $\pi_A^*(\mathbf{C}M) \longrightarrow A$ *(hence a map* $v \in A \mapsto \Gamma_v \in \mathbf{C}_{\pi(v)}M$*) subject to* $\Gamma_{\lambda v} = \Gamma_v$.

Remark 2.1. The construction of $\mathbf{C}M \longrightarrow M$ guarantees that such a Γ has natural coordinates $\Gamma^k_{ij}(x, y)$, while the condition $\Gamma_{\lambda v} = \Gamma_v$ translates into the 0-homogeneity of those. This means that a (homogeneous) anisotropic connection in the sense above is equivalent to a collection of (0-homogeneous) functions Γ^k_{ij} on $A \cap \mathrm{T}U$ associated with each chart such that, under changes $(U, x) \rightsquigarrow (\bar{U}, \bar{x})$, (2.4) is satisfied with $\bar{\Gamma}^k_{ij}(x, y)$, $\Gamma^c_{ab}(x, y)$ in place of $\bar{\Gamma}^k_{ij}(x)$, $\Gamma^c_{ab}(x)$. By [28, Prop. 1 (2)], it is also equivalent to a (homogeneous) anisotropic connection ∇ in the sense of [28, Def. 4], [24, Def. 3.1]. Hence, as announced, the viewpoint here is unified with the one of those references and all the developments in [24, 28] can be applied.

Consider now the 1-jet prolongation $\mathbf{J}^1 A \longrightarrow A \longrightarrow M$; one is referred to [29, §12] for a systematic treatment of jets. Recall that for $p \in M$, two local A-valued vector fields V, V' on M *determine the same* 1*-jet at* p if they and their first order partial derivatives (on any chart) coincide at p. These 1*-jets* (equivalence classes) $j^1_p V$ are the elements of the fiber $\mathbf{J}^1_p A$ of $\mathbf{J}^1 A \longrightarrow M$, but also $j^1_p V \mapsto V_p$ is a well-defined projection and one obtains $\mathbf{J}^1 A \longrightarrow A$, which is an affine bundle. The following definition is standard in the theory of fibered manifolds, see [29, §17.1] for instance.

Definition 2.3. *A* homogeneous nonlinear *(or* Ehresmann*)* connection *for* $A \longrightarrow M$ *is a section* N *of* $\mathbf{J}^1 A \longrightarrow A$ *(hence a choice of* 1*-jet* $\mathrm{N}_v = j^1_{\pi(v)} V$ *with* $V_{\pi(v)} = v$ *at each* $v \in A$*) with the requirement that if* $\mathrm{N}_v = j^1_{\pi(v)} V$*, then* $\mathrm{N}_{\lambda v} = j^1_{\pi(\lambda v)} (\lambda V)$.

Remark 2.2. (A) Knowing that $V_{\pi(v)} = v$, the 1-jet $\mathrm{N}_v = j^1_{\pi(v)} V$ is determined by the partial derivatives $\mathrm{N}^k_i(v) = -\partial_i V^k(\pi(v))$; these are functions $\mathrm{N}^k_i(x, y)$, while the condition $\mathrm{N}_{\lambda v} = j^1_{\pi(\lambda v)} (\lambda V)$ translates into their 1-homogeneity. This means that a (homogeneous) nonlinear connection is equivalent to a collection of (1-homogeneous) functions N^k_i on $A \cap \mathrm{T}U$ associated with each chart such that, under changes $(U, x) \rightsquigarrow$

[6] A more specific presentation of this affine bundle is given as follows. Given $p \in M$, say that two affine connections on M are *equivalent at* p if when they act on any vector fields on M, the results coincide at p for both connections. Then the equivalence classes are the elements of the fiber $\mathbf{C}_p M$. Hence, it is clear that an affine connection yields such an element at each p.

$(\bar{U}, \bar{x})$, the transformation cocycle

$$\bar{\mathrm{N}}_i^k(x,y) = \frac{\partial \bar{x}^k}{\partial x^c}(x)\,\frac{\partial^2 x^c}{\partial \bar{x}^i\,\partial \bar{x}^b}(x)\,\bar{y}^b + \frac{\partial \bar{x}^k}{\partial x^c}(x)\,\frac{\partial x^a}{\partial \bar{x}^i}(x)\,\mathrm{N}_a^c(x,y) \tag{2.5}$$

is satisfied. By [28, Rem. 3], it is also equivalent to a (homogeneous) nonlinear connection in any of the usual senses; for instance, that of an *(invariant by homotheties) horizontal distribution* $\mathrm{H}A \longrightarrow A$, where

$$\mathrm{H}_v A := \mathrm{Span}\left\{ \delta_i\big|_v \right\} \subseteq \mathrm{T}_v A, \qquad \delta_i\big|_v := \partial_i\big|_v - \mathrm{N}_i^a(v)\ \dot{\partial}_a\big|_v . \tag{2.6}$$

Hence, the perspective here is unified with the one of references such as [28, §4], [39, §3], [12, §4] and [43, Ch. 2][7]. The N-horizontal distribution provides the N-*horizontal isomorphism*

$$X_v = X^a(v)\,\partial_a\big|_{\pi(v)} \in \mathrm{T}_{\pi(v)}M \longleftrightarrow X_v^{\mathrm{H}} := X^a(v)\,\delta_a\big|_v \in \mathrm{H}_v A, \tag{2.7}$$

which identifies $\mathrm{h}^\alpha \mathcal{T}_0^1(M_A)$ with the space of $(\alpha+1)$-homogeneous horizontal vector fields on A.

(B) From the cocycles (2.4) (for $\Gamma_{ik}^k(x,y)$) and (2.5), the affine structures of the spaces of homogeneous anisotropic and nonlinear connections are given respectively as follows. For a fixed Γ_0 and $Q \in \mathrm{h}^0\mathcal{T}_2^1(M_A)$, $\Gamma := \Gamma_0 + Q$ has coordinates $\left(\Gamma_0\right)_{ij}^k + Q_{ij}^k$, while for a fixed N_0 and $J \in \mathrm{h}^1\mathcal{T}_1^1(M_A)$, $\mathrm{N} := \mathrm{N}_0 + J$ has coordinates $\left(\mathrm{N}_0\right)_i^k + J_i^k$.

Definition 2.4.

(i) By [28, Th. 2 (1)], any homogeneous anisotropic connection Γ *induces canonically a homogeneous nonlinear connection of coordinates* $\mathrm{N}_i^k = \Gamma_{i\,a}^{\,k} y^a$. *We call it the* underlying nonlinear connection of Γ.

(ii) By [28, Th. 2 (2)], any homogeneous nonlinear connection N *induces canonically a homogeneous anisotropic connection of coordinates* $\Gamma_{ij}^k = \mathrm{N}_{i\cdot j}^k = \dot{\partial}_j \mathrm{N}_i^k$. *We call it the* vertical differential *or* Berwald anisotropic connection of N *and denote it by* $\dot{\partial}\mathrm{N}$.

Given any homogeneous anisotropic connection Γ, the corresponding covariant derivative ∇ maps $\mathrm{h}^\alpha\mathcal{T}_s^r(M_A)$ to $\mathrm{h}^\alpha\mathcal{T}_{s+1}^r(M_A)$. For $T \in \mathrm{h}^\alpha\mathcal{T}_s^r(M_A)$, ∇T is given in coordinates by

$$\nabla_{j_{s+1}} T_{j_1,\dots,j_s}^{i_1,\dots,i_r} := \delta_{j_{s+1}} T_{j_1,\dots,j_s}^{i_1,\dots,i_r} + \sum_\mu \Gamma_{j_{s+1}a}^{i_\mu} T_{j_1,\dots,j_s}^{i_1,\dots,\overset{(\mu)}{a},\dots,i_r} - \sum_\mu \Gamma_{j_{s+1}j_\mu}^{a} T_{j_1,\dots,\underset{(\mu)}{a},\dots,j_s}^{i_1,\dots,i_r}, \tag{2.8}$$

[7]Even though the N_i^k's in this reference are not the same as ours (see the different cocycle [43, (2.8)]), they necessarily are in correspondence with ours.

where the δ_j are those of (2.6) for the underlying nonlinear connection (and thus underlying horizontal distribution) N of Γ. In particular, for $f \in \mathrm{h}^\alpha\mathcal{F}(A)$ and $X \in \mathrm{h}^\alpha\mathcal{T}^1_0(M_A)$, $\nabla_X f = X^{\mathrm{H}}(f)$ only depends on that underlying nonlinear connection.

Proposition 2.3. For any anisotropic connection, $\nabla\mathbb{C} = 0$, i.e., $\nabla_j y^i = 0$.

Proof. $\mathbb{C} = y^a\,\partial_a \in \mathrm{h}^1\mathcal{T}^1_0(M_A)$, so by (2.8), $\nabla\mathbb{C}$ has coordinates

$$\nabla_j y^i = \delta_j y^i + \Gamma^i_{ja}\, y^a = \partial_j y^i - \mathrm{N}^a_j\,\dot\partial_a\, y^i + \Gamma^i_{ja}\, y^a = -\mathrm{N}^a_j\,\delta^i_a + \Gamma^i_{ja}\, y^a = 0,$$

where δ^i_a is the usual Kronecker's and only the fact that N is the underlying nonlinear connection of Γ was used for the last equality. ∎

The *curvature*, the *(Finslerian) Ricci scalar* and the *torsion*[8] *of a homogeneous nonlinear connection* N can be regarded as homogeneous anisotropic tensors $\mathcal{R} \in \mathrm{h}^1\mathcal{T}^1_2(M_A)$, $\mathrm{Ric} \in \mathrm{h}^2\mathcal{F}(A)$ and $\mathrm{Tor} \in \mathrm{h}^0\mathcal{T}^1_2(M_A)$ respectively, with coordinates

$$\mathcal{R}^k_{ij} = \delta_j\mathrm{N}^k_i - \delta_i\mathrm{N}^k_j, \qquad \mathrm{Ric} = y^b\,\mathcal{R}^a_{ba}, \qquad \mathrm{Tor}^k_{ij} = \mathrm{N}^k_{i\cdot j} - \mathrm{N}^k_{j\cdot i} \tag{2.9}$$

(recall (2.7)). We say that N is *symmetric* when $\mathrm{Tor} = 0$. By direct computation, one has the following commutation formulas:

$$\left[\delta_i, \delta_j\right] = \mathcal{R}^k_{ij}\,\dot\partial_k, \qquad \left[\delta_i, \dot\partial_j\right] = \mathrm{N}^k_{i\cdot j}\,\dot\partial_k, \qquad \left[\dot\partial_i, \dot\partial_j\right] = 0. \tag{2.10}$$

Remark 2.3. Anisotropic connections Γ can actually be *isotropic*, in the sense that $\Gamma^k_{ij}(x,y) = \Gamma^k_{ij}(x)$, while nonlinear connections N can actually be *linear*, in the sense that $\mathrm{N}^k_i(x,y) = \Gamma^k_{ia}(x)\,y^a$. In either case, the $\Gamma^k_{ij}(x)$'s are some functions that necessarily define an affine connection (as a section of $\mathbf{C}M \longrightarrow M$, see (2.4) and (2.5)) and Γ or N is homogeneous. Hence, there is a natural identification between affine connections on M, isotropic Γ's and linear N's. Under this identification, each isotropic Γ gets identified with its underlying N, which turns out to be linear, and then $\Gamma = \dot\partial\mathrm{N}$. This is consistent with [28, Th. 2 (4)].

Remark 2.4. Let $\nabla_{\partial_k}\nabla_{\partial_j}\partial_i - \nabla_{\partial_j}\nabla_{\partial_k}\partial_i = \mathrm{R}^l_{ijk}(x)\,\partial_l$ define the classical curvature of an affine connection $\Gamma : M \longrightarrow \mathbf{C}M$ with the convention of [45]. If, as above, one identifies this with a connection N of curvature $\mathcal{R}$, then it is straightforward to prove that

$$y^a\,\mathrm{R}^l_{ajk}(x) = \mathcal{R}^l_{jk}(x,y), \qquad y^a\,y^b\,\mathrm{R}^c_{abc}(x) = \mathrm{Ric}(x,y), \tag{2.11}$$

[8] Note that when defining, as in [28, Def. 5], the *torsion* of any homogeneous anisotropic connection Γ by $\Gamma^k_{ij} - \Gamma^k_{ji}$, the torsion of N turns out to be just that of $\dot\partial\mathrm{N}$. However, in this work we will reserve the notation Tor for the torsion of a nonlinear connection. Compare with more abstract references such as [39, §3.3], [44, §7].

so the symmetric part of the classical Ricci tensor is

$$\frac{1}{2}\left(\mathrm{R}^c_{ijc}(x)+\mathrm{R}^c_{jic}(x)\right)=\frac{1}{2}\left(y^a\,y^b\,\mathrm{R}^c_{abc}(x)\right)_{\cdot i\cdot j}=\frac{1}{2}\,\mathrm{Ric}_{\cdot i\cdot j}(x,y)$$

and the scalar curvature constructed with any pseudo-Riemannian metric g on M is

$$\mathrm{Scal}(x)=\frac{1}{2}\,g^{ab}(x)\,\left(\mathrm{R}^c_{abc}(x)+\mathrm{R}^c_{bac}(x)\right)=\frac{1}{2}\,g^{ab}(x)\,\mathrm{Ric}_{\cdot a\cdot b}(x). \tag{2.12}$$

Observe that we follow the same sign convention for $\mathcal{R}$ as in [46, §II A], [21, §II B] but our sign for Ric is the standard one in Riemannian Geometry and thus opposite to that of the cited references.

2.2.3 Sprays

In this subsection, we will present the sprays as sections of an affine bundle, unifying later this viewpoint with the more classical one discussed in [28, §6.1].

TA has natural coordinates (x,y,z,w), where (x,y) are the natural coordinates of any $v\in A$ and then we write $z^a\,\partial_a+w^a\,\dot\partial_a$ for the elements of T_vA. The vertical distribution VA is described on them by $\left\{z^i=0\right\}$, which implies that it is a vector subbundle of T$A\longrightarrow A$. Analogously, it follows that the set SA described by $\left\{z^i=y^i\right\}$ is an affine subbundle of T$A\longrightarrow A$. In [39, §2], this is referred to as the *symmetrized bundle*.

Definition 2.5. *A* spray *on* A *is a section* G *of* S$A\longrightarrow A$*, 2-homogeneous as a vector field on* A *(see Def. 2.1 (ii) and Prop. 2.1 (ii)).*

Remark 2.5. (A) These are exactly the fields of the form

$$\mathrm{G}=y^a\,\partial_a-2\,\mathrm{G}^a\,\dot\partial_a$$

for certain 2-homogeneous coefficients $\mathrm{G}^k(x,y)$. This means that a spray is equivalent to a collection of 2-homogeneous functions G^k on $A\cap\mathrm{T}U$ associated with each chart such that, under changes $(U,x)\rightsquigarrow(\bar U,\bar x)$,

$$\bar{\mathrm{G}}^k=\frac{1}{2}\,\frac{\partial\bar x^k}{\partial x^c}\,\frac{\partial^2x^c}{\partial\bar x^a\,\partial\bar x^b}\,\bar y^a\,\bar y^b+\frac{\partial\bar x^k}{\partial x^c}\,\mathrm{G}^c. \tag{2.13}$$

(B) From the cocycle (2.13), the affine structure of the space of sprays is given as follows: for a fixed spray G_0 and $Z:=Z^a\,\partial_a\in\mathrm{h}^2\mathcal{T}^1_0(M_A)$, $\mathrm{G}=\mathrm{G}_0-2\,Z$ has coordinates $\mathrm{G}^k_0+Z^k$. The cause of this discrepancy is that we have decided to maintain the standard convention that G (and not $-2\,\mathrm{G}$) equals $y^a\,\partial_a-2\,\mathrm{G}^a\,\dot\partial_a$, whereas the anisotropic vector with coordinates $-2\,Z^i$ is $-2\,Z$ (and not Z).

Definition 2.6.

(i) *By [28, Prop. 3 (1)], any homogeneous nonlinear connection* N *induces canonically a spray of coordinates* $\mathrm{G}^i = \mathrm{N}^i_a\, y^a/2$. *We call it the* underlying spray of N.
(ii) *By [28, Prop. 3 (2)], any spray* G *induces canonically a symmetric homogeneous nonlinear connection of coordinates* $\mathrm{N}^k_i = \mathrm{G}^k_{\cdot i} = \dot\partial_i \mathrm{G}^k$. *We call it the* vertical differential *or* Berwald nonlinear connection of G *and denote it by* $\dot\partial\mathrm{G}$.

The (projections to M of the) integral curves of a spray G are its *geodesics*. Its *pregeodesics* are those curves in M that can be (positively) reparametrized to be geodesics.

Proposition 2.4. A spray $\mathrm{G} = \mathrm{G}_0 - 2\,Z$ shares pregeodesics with G_0 if and only if $Z = \rho\,\mathbb{C}$ for some $\rho \in \mathrm{h}^1\mathcal{F}(A)$.

For a proof see [49, Lem. 12.1.1].

2.2.4 Pseudo-Finsler metrics

Definition 2.7. *A* (conic) pseudo-Finsler metric *defined on the open and conic* $A \subseteq \mathrm{T}M \setminus \mathbf{0}$ *with* $\pi(A) = M$ *is an* $L \in \mathrm{h}^2\mathcal{F}(A)$ *whose* fundamental tensor $g = \dot\partial^2 L/2 \in \mathrm{h}^0\mathcal{T}^0_2(M_A)$ *is non-degenerate at every* $v \in A$.

Remark 2.6. Taking into account the nature of the variational problem that we will pose, we shall assume that our pseudo-Finsler metrics do not have *lightlike* directions in the fixed A, namely $L(v) \neq 0$ for all $v \in A$.

We always denote $F := \sqrt{|L|} \in \mathrm{h}^1\mathcal{F}(A)$; indices of tensors are lowered and raised with g_{ij} and g^{ij} respectively. By direct computation, one has the following identites:

$$L_{\cdot i} = 2\,y_i (:= 2\,g_{ia}\,y^a), \qquad y_{i\cdot j} = g_{ij},$$
$$F_{\cdot i} = \frac{\mathrm{sgn}(L)}{F}\,y_i, \qquad \left(\frac{y_i}{L}\right)_{\cdot j} = \frac{g_{ij}}{L} - 2\,\frac{y_i}{L}\,\frac{y_j}{L} = \left(\frac{y_j}{L}\right)_{\cdot i}.$$

From these and the 2-homogeneity of L, it follows that

$$L = \frac{1}{2}\,L_{\cdot a\cdot b}\,y^a\,y^b = g_{ab}\,y^a\,y^b = y_b\,y^b.$$

Definition 2.8. *(A) We say that a pseudo-Finsler metric* L *defined on* A *is* proper *if*

(i) *Each fiber* A_p *(*$p \in M$*) is connected with* $L > 0$ *on* A*,*

(ii) L *extends smoothly to* $\overline{A} \subseteq \mathrm{T}M \setminus \mathbf{0}$ *with* $L(v) = 0$ *and* g_v *non-degenerate for* $v \in \partial A := \overline{A} \setminus A$.

Then g *has a constant signature on* $\overline{A}$.

(B) When that signature is Lorentzian $(+, -, ..., -)$*,* L *is* (properly) Lorentz-Finsler. *A* Finsler spacetime *is any triple* (M, A, L) *with* L *Lorentz-Finsler.*

(C) When the signature is positive definite, necessarily $A = \mathrm{T}M \setminus \mathbf{0}$ *and* L *is* Finsler.

Remark 2.7. Let us comment the parts of the last definition:

(A) g has constant signature on $\overline{A}$ because the connectedness of M together with (i) implies that A is connected. Moreover, the *indicatrix* $\{L = 1\}$ and (thanks to (ii)) the *lightcone* $\partial A = \{L = 0\}$ are smooth hypersurfaces:

$$\mathrm{d}L_v(u^{\mathrm{V}}) = u^a\, L_{\cdot a}(v) = 2\, u^a\, y_a(v) = 2\, u^a\, g_{ab}(v)\, v^b = 2\, g_v(u, v)$$

for $u \in \mathrm{T}_{\pi(v)}M$, so $\mathrm{d}L_v$ never vanishes identically for $v \in \overline{A} \subseteq \mathrm{T}M \setminus \mathbf{0}$.

(B) We want such an L to be defined only on *future causal vectors* (so $L \geq 0$ together with $(+, -, ..., -)$ as the Lorentzian signature is a choice of convention). There is a Physics motivation for this assumption [8, §1], but it also has interesting mathematical implications. For instance, $\overline{A_p} \subseteq \mathrm{T}_pM \setminus 0$ is contained in an open half-space: there is a vector hyperplane Π_p that does not intersect $\overline{A_p}$; thus, A already determines a *time orientation.* For this and other geometric consequences (such as convexity) for $\overline{A}$ of L being Lorentz-Finsler, see [27, Props. 2.6 and 3.4]. [9]

(C) The positive definiteness of g together with (ii) implies that actually $\partial A = \emptyset$, so necessarily $A = \mathrm{T}M \setminus \mathbf{0}$. The converse is also true, namely, if the signature is non-definite, then each A_p must be a proper subset of $\mathrm{T}_pM \setminus 0$. Indeed, otherwise, $\{L = 1\} \cap \mathrm{T}_pM$ would enclose the origin, contradicting its curvature properties (determined by g_{A_p}). So, in the non-definite case, $\partial A_p \neq \emptyset$, which we will use in Th. 2.3.

A key geometric object associated with a pseudo-Finsler metric L defined on A is its *metric spray* G^L,

$$\left(\mathrm{G}^L\right)^i := \frac{1}{4}\, g^{ia} \left(2\, \partial_c g_{ab} - \partial_a g_{bc}\right) y^b\, y^c. \tag{2.14}$$

[9]Additionally, in [40] it is proven that one can actually extend L to a pseudo-Finsler metric with Lorentzian fundamental tensor on the whole $\mathrm{T}M \setminus \mathbf{0}$ (in a highly non-unique way in contrast to the extension to $\overline{A}$).

The Berwald $\mathrm{N}^L := \dot\partial \mathrm{G}^L$ is the *metric nonlinear connection.* From now on, given any anisotropic connection Γ, it will be convenient to write ∇^Γ instead of just ∇ for its corresponding covariant derivative, ∇^{N} in case that $\Gamma = \dot\partial \mathrm{N}$ for a nonlinear connection N, and ∇^L in case that $\Gamma = \dot\partial \mathrm{N}^L$ (this is the *Berwald anisotropic connection of* L [24, §4.3], [49, Ch. 7]). Due to Defs. 2.6 (ii) and 2.4 (ii), the notions of Γ*-(pre)geodesics* and N*-(pre)geodesics* make sense, and due to (2.14), so does that of L*-(pre)geodesics.* When using N^L, which is always symmetric, the curvature and the Ricci scalar in (2.9) will be denoted $\mathcal{R}^L$ and Ric^L resp., as they can be associated with[10] L.

The *Cartan tensor* is

$$\mathrm{C} := \frac{1}{2}\dot\partial g \in \mathrm{h}^{-1}\mathcal{T}_3^0(M_A).$$

It is symmetric, so it makes sense to define the *mean Cartan tensor* as its metric trace, with components

$$\mathrm{C}_i := g^{ab}\,\mathrm{C}_{abi}.$$

By vertically differentiating $g_{ia}\,g^{ak} = \delta_i^k$, one obtains the following identities:

$$\mathrm{C}_i^{jk} = -\frac{1}{2}\,g^{jk}_{\cdot i}, \qquad \mathrm{C}^j = -\frac{1}{2}\,g^{ja}_{\cdot a}.$$

The *Landsberg tensor* is

$$\mathrm{Lan} := \frac{1}{2}\,\nabla^L g \in \mathrm{h}^0\mathcal{T}_3^0(M_A)$$

(it can also be defined in terms of the *Berwald tensor* [24, (37)], however, $\mathrm{Lan} = \nabla^L g/2$ is the way in which it will arise in this work). Note that here it has the same sign as in [24, 25, 46] and the opposite in [5, 21, 49]. The Landsberg tensor is symmetric too, so it makes sense to define the *mean Landsberg tensor*, with components

$$\mathrm{Lan}_i := g^{ab}\,\mathrm{Lan}_{abi}.$$

Remark 2.8. A pseudo-Finsler L is equivalent to a symmetric and non-degenerate $g \in \mathrm{h}^0\mathcal{T}_2^0(M_A)$ with totally symmetric Cartan tensor [2, Th. 3.4.2.1]. This justifies being able to identify L with g whenever it is needed. For instance, L can be *pseudo-Riemannian*, in the sense that g is such kind of metric. This is equivalent to g being isotropic and to L being *quadratic*, namely $L(x,y) = \Psi_{ab}(x)\,y^a\,y^b/2$ for some isotropic and symmetric tensor $\Psi/2$ that then necessarily equals g.

[10]For a Finsler L (g is positive definite), Ric^L coincides on $\{L=1\}$ with the Ricci scalar defined as a sum of $n-1$ *flag curvatures* as in [5, (7.6.2a)].

2.3 Metric-affine variational calculus

For the remainder of the manuscript, N and L are, respectively, a homogeneous non-linear connection and a pseudo-Finsler metric defined on the open and conic A with $L > 0$ there. Our metric-affine formalism is akin to the metric formalism of [21]. Its steps are: determination of a volume form on A, divergence formulas, choice of a Lagrangian function, induction (according to Prop. 2.2) of forms on[11] $\mathbb{P}^+ A$ to construct an action there, and variation of this with respect to N and with respect to L.

Given (N, L), there is a natural way of constructing a 0-homogeneous volume form on A. The N-horizontal and vertical isomorphisms allow us to define scalar products on $\mathrm{H}_v A$ and $\mathrm{V}_v A$:

$$g_v^{\mathrm{H}}(X_v^{\mathrm{H}}, Y_v^{\mathrm{H}}) := g_v(X_v, Y_v), \qquad g_v^{\mathrm{V}}(X_v^{\mathrm{V}}, Y_v^{\mathrm{V}}) := g_v(\frac{X_v}{F(v)}, \frac{X_v}{F(v)}) = \frac{g_v(X_v, Y_v)}{L(v)} \tag{2.15}$$

for $X, Y \in \mathcal{T}_0^1(M_A)$. Each one has its own volume form:

$$d\mu_v^{\mathrm{H}} := \sqrt{\left|\det g_v^{\mathrm{H}}\left(\delta_i\big|_v, \delta_j\big|_v\right)\right|}\, \mathrm{d}x_v^1 \wedge \ldots \wedge \mathrm{d}x_v^n =: \sqrt{\left|\det g_{ij}(v)\right|}\, \mathrm{d}x_v,$$

$$d\mu_v^{\mathrm{V}} := \sqrt{\left|\det g_v^{\mathrm{V}}\left(\dot\partial_i\big|_v, \dot\partial_j\big|_v\right)\right|}\, \delta y_v^1 \wedge \ldots \wedge \delta y_v^n =: \frac{\sqrt{\left|\det g_{ij}(v)\right|}}{F(v)^n}\, \delta y_v,$$

where the $\mathrm{d}x_v^i$ and $\delta y_v^i := \mathrm{d}y_v^i + \mathrm{N}_a^i(v)\,\mathrm{d}x_v^a$ are restricted to the horizontal and vertical subspaces respectively. A $2n$-form is induced on $\mathrm{T}_v A = \mathrm{H}_v A \oplus \mathrm{V}_v A$:

$$d\mu_v := d\mu_v^{\mathrm{H}} \wedge d\mu_v^{\mathrm{V}} = \frac{\left|\det g_{ij}(v)\right|}{F(v)^n}\, \mathrm{d}x_v \wedge \delta y_v. \tag{2.16}$$

Remark 2.9. Even though we used N and L to construct $d\mu$, this turns out to depend on L alone, as

$$\begin{aligned}\mathrm{d}x \wedge \delta y &= \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \wedge \left(\mathrm{d}y^1 + \mathrm{N}_{a_1}^1\, \mathrm{d}x^{a_1}\right) \wedge \ldots \wedge \left(\mathrm{d}y^n + \mathrm{N}_{a_n}^n\, \mathrm{d}x^{a_n}\right)\\ &= \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \wedge \mathrm{d}y^1 \wedge \ldots \wedge \mathrm{d}y^n\\ &= \mathrm{d}x \wedge \mathrm{d}y.\end{aligned}$$

[11] Integrating on this projectivization as in [21], instead of the indicatrix $\{L = 1\}$, solves the technical issue of the integration domain depending on the variable L, present in [46].

Taking the nature of our variational approach into account, it was of the most theoretical importance to define our volume form a priori in terms of both the connection and the metric. On the other hand, by (2.16), $d\mu$ is the volume form of the *Sasaki-type metric* $g_v^{\mathrm{H}} \overset{\perp}{\oplus} g_v^{\mathrm{V}}$, and by the previous observation, it also coincides with the volume form of the *Sasaki metric of* g (that is, $g_v^{\mathrm{H}} \overset{\perp}{\oplus} g_v^{\mathrm{V}}$ for $\mathrm{N} = \mathrm{N}^L$). Note that the definition of g_v^{V} dividing by F as in (2.15) is what guarantees the 0-homogeneity of $d\mu$.

This $d\mu$ allows us to define the divergence of any vector field $\mathcal{X}$ on A as

$$\mathrm{div}(\mathcal{X})\, d\mu := \mathcal{L}_{\mathcal{X}}(d\mu) = \mathrm{d}(\mathcal{X} \lrcorner d\mu).$$

In the case of a 1-homogeneous $\mathcal{X}$, by Prop. 2.2 (iii), one has the property that justifies discarding the divergence terms in the variational calculus:

$$\underline{\mathrm{div}(\mathcal{X})\, d\mu} = -\mathrm{d}(\underline{\mathcal{X} \lrcorner d\mu}).$$

The following divergence formulas, generalizing [21, (24) and (25)], are the key to the derivation of our equations. Their proof is in Appendix A.

Proposition 2.5. For $X \in \mathcal{T}_0^1(M_A)$,

$$\mathrm{div}(X^{\mathrm{H}}) = X^c \left\{ \left(g^{ab} - \frac{n}{2}\frac{1}{L}\, y^a y^b \right) \nabla_c^{\mathrm{N}} g_{ab} + \mathrm{Tor}^a_{\ ca} \right\} + \nabla_a^{\mathrm{N}} X^a, \tag{2.17}$$

$$\mathrm{div}(X^{\mathrm{V}}) = \left(2\,\mathrm{C}_a - n\frac{y_a}{L} \right) X^a + X^a_{\cdot a}. \tag{2.18}$$

If $X \in \mathrm{h}^0\mathcal{T}_0^1(M_A)$, then $\underline{\mathrm{div}(X^{\mathrm{H}})\, d\mu} = -\mathrm{d}(\underline{X^{\mathrm{H}} \lrcorner d\mu})$ on $\mathbb{P}^+A$, and if $X \in \mathrm{h}^1\mathcal{T}_0^1(M_A)$, then $\underline{\mathrm{div}(X^{\mathrm{V}})\, d\mu} = -\mathrm{d}(\underline{X^{\mathrm{V}} \lrcorner d\mu})$.

Definition 2.9. *Let $D \subseteq \mathbb{P}^+A$[12] be a relatively compact subset. Along this article and relative to D, the* action functional *will be*

$$\mathcal{S}^D[\mathrm{N}, L] := \int_D \underline{L^{-1}\,\mathrm{Ric}\, d\mu}$$

and the alternative action functional *will be*

$$\mathcal{S}_\star^D[\mathrm{N}, L] := \int_D \underline{g^{ab}\,\mathrm{Ric}_{\cdot a \cdot b}\, d\mu}.$$

[12] $d\mu$ defines a global orientation on A, the one making $(\partial_1, \dots, \partial_n, \dot{\partial}_1, \dots, \dot{\partial}_n)$ positive, regardless of the ones that we chose for $d\mu^{\mathrm{H}}$, $d\mu^{\mathrm{V}}$ and without requiring M to be orientable. As $\underline{d\mu}$ is again a volume form (see the comment at the end of Prop. 2.2 (ii)), an orientation on $\mathbb{P}^+A$ is inherited.

The relation between these two is due to [21, Lem. 3]. We state it in our notation.

Proposition 2.6. For $f \in \mathrm{h}^0\mathcal{F}(A)$, one has

$$\left\{g^{ab}\,(Lf)_{\cdot a \cdot b} - 2nf\right\} d\mu = \operatorname{div}(X^{\mathrm{V}})\, d\mu,$$

where X^{V} is the vertical field corresponding to $X := L\, g^{ab}\, f_{\cdot b}\, \partial_a \in \mathrm{h}^1\mathcal{T}^1_0(M_A)$. As a consequence, the functionals that we are considering are equal up to a factor of $2n$ and a boundary term:

$$\mathcal{S}^D_\star[\mathrm{N}, L] - 2n\, \mathcal{S}^D[\mathrm{N}, L] = -\int_{\partial D} \underline{X^{\mathrm{V}} \lrcorner d\mu}.$$

Proof. As in the proof of [21, Lem. 3], using the 0-homogeneity of f, one directly computes

$$g^{ab}\,(Lf)_{\cdot a \cdot b} = 2nf + L\, g^{ab}\, f_{\cdot a \cdot b}.$$

On the other hand, by (2.18),

$$\begin{aligned}
\operatorname{div}(X^{\mathrm{V}}) &= \left(2\,\mathrm{C}_a - n\,\frac{y_a}{L}\right) L\, g^{ab}\, f_{\cdot b} + \left(L\, g^{ab}\, f_{\cdot b}\right)_{\cdot a} \\
&= 2L\,\mathrm{C}^b\, f_{\cdot b} + \left(2\, y_a\, g^{ab}\, f_{\cdot b} + L\, g^{ab}_{\cdot a}\, f_{\cdot b} + L\, g^{ab}\, f_{\cdot a \cdot b}\right) \\
&= L\, g^{ab}\, f_{\cdot a \cdot b};
\end{aligned}$$

the 0-homogeneity of f was used twice and $g^{ab}_{\cdot a} = -2\,\mathrm{C}^b$ (§2.2.4) was used once. ∎

We shall work with $\mathcal{S}^D$, as it is of first order on N and second order on L while $\mathcal{S}^D_\star$ is of third order on N. The advantage of the latter, on the other hand, is that it is closer to the Einstein-Hilbert-Palatini action, the functional of the classical metric-affine formalism [7] (compare with [21, Prop. 6]).

Proposition 2.7. Suppose that N is linear, L is (positive definite) Riemannian and $D = \bigcup_{p \in D_0} \mathbb{P}^+(\mathrm{T}M \setminus \mathbf{0})_p$ for a relatively compact $D_0 \subseteq M$. Then

$$\mathcal{S}^D_\star[\mathrm{N}, L] = 2\,\mathrm{Vol}(\mathbb{S}^{n-1}) \int_{D_0} \mathrm{Scal}\, d\mathrm{V},$$

where Scal is the scalar curvature constructed with N (regarded as an affine connection) and g, $d\mathrm{V}$ is the g-volume element on M, and $\mathrm{Vol}(\mathbb{S}^{n-1})$ is a universal constant.

Proof. A standard argument with a partition of the unity on $\mathbb{P}^+(\mathrm{T}M \setminus \mathbf{0})$ induced by one on M allows us to use Fubini's Theorem to obtain the following:

$$\begin{aligned}
\mathcal{S}_\star^D[\mathrm{N}, L] &= \int_{\mathbb{P}^+ v \in D} g^{ab} \operatorname{Ric}_{\cdot a \cdot b} \, \underline{d\mu}_{\mathbb{P}^+ v} \\
&= \int_{\mathbb{P}^+ v \in D} g^{ab}(\pi(v)) \operatorname{Ric}_{\cdot a \cdot b}(\pi(v)) \, \underline{d\mu}_{\mathbb{P}^+ v} \\
&= \int_{p \in D_0} g^{ab}(p) \operatorname{Ric}_{\cdot a \cdot b}(p) \left(\int_{\mathbb{P}^+ v \in D_p} \underline{d\mu}_{D_p} \right) d\mathrm{V}_p \\
&= \operatorname{Vol}(\mathbb{S}^{n-1}) \int_{p \in D_0} g^{ab}(p) \operatorname{Ric}_{\cdot a \cdot b}(p) \, d\mathrm{V}_p \\
&= 2 \operatorname{Vol}(\mathbb{S}^{n-1}) \int_{p \in D_0} \operatorname{Scal}(p) \, d\mathrm{V}_p,
\end{aligned}$$

where we used (2.12) and the fact that each fiber $D_p = \mathbb{P}^+(\mathrm{T}M \setminus \mathbf{0})_p$ inherits a metric that makes it isometric to the round sphere $\mathbb{S}^{n-1}$. Indeed, $\mathbb{P}^+(\mathrm{T}M \setminus \mathbf{0})$ is naturally identified with the sphere bundle $\{L = 1\}$, where the metric is induced by $g^{\mathrm{H}} \overset{\perp}{\oplus} g^{\mathrm{V}}$, the Sasaki metric of g. Moreover, the induced $\underline{d\mu}_{D_p}$ is the volume form of the round metric on D_p because $d\mu$ is the volume form of $g^{\mathrm{H}} \overset{\perp}{\oplus} g^{\mathrm{V}}$ (see Rem. 2.9). ∎

In the non-definite case, it is not possible to integrate on a compact fiber with universal volume at each $p \in M$. Hence, one does not seem to be able to actually recover the Einstein-Hilbert-Palatini action in general. Nonetheless, the positive definiteness of g and the compactness of the fibers are superfluous when it comes to our variational calculus, for all of it is local on $\mathbb{P}^+ A$ and formally the same in every signature. Thus, Prop. 2.7 indeed guarantees a priori the consistency of our equations with the (vacuum) EHP ones.

Remark 2.10. Let us sum up the reasons for choosing $L^{-1} \operatorname{Ric}$ as our metric-affine Lagrangian function.

(i) It is the first and most natural (0-homogeneous) curvature scalar that is derived from N.
(ii) The second most natural scalar, $g^{ab} \operatorname{Ric}_{\cdot a \cdot b}$, turns out to be variationally equivalent to it.
(iii) Moreover, $g^{ab} \operatorname{Ric}_{\cdot a \cdot b}$ reduces to the EHP Lagrangian in the classical case.
(iv) The metric Lagrangian of [21, 46] is $L^{-1} \operatorname{Ric}^L$.

Definition 2.10. *(A) A* variation of N *is a smooth one-parameter family of homogeneous nonlinear connections* $\mathrm{N}(\tau)$ *with* $\mathrm{N}(0)=\mathrm{N}$. *Its* variational field *is*

$$\mathrm{N}' = \left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \mathrm{N}(\tau) \in \mathrm{h}^1\mathcal{T}^1_1(M_A)$$

(see (2.5)*). Analogously for a* variation of L, *whose* variational field *is*

$$L' = \left.\frac{\partial}{\partial\tau}\right|_{\tau=0} L(\tau) \in \mathrm{h}^2\mathcal{F}(A).$$

(B) Given a relatively compact subset $D \subseteq \mathbb{P}^+A$, we say that a variation $\mathrm{N}(\tau)$ *is D-*admissible *if the projectivized support of its variational field,* $\mathbb{P}^+(\overline{\{v \in A : \mathrm{N}'_v \neq 0\}}^A)$, *is contained in D. In such a case, without loss of generality, we shall assume that D is open with smooth boundary $\partial D \subseteq \mathbb{P}^+A$. We say that* $\mathrm{N}(\tau)$ *is* admissible *if it is D-admissible for some D. Analogously for $L(\tau)$.*

In terms of the metric connection, we write

$$\mathrm{N} = \mathrm{N}^L + \mathcal{J}, \qquad \mathcal{J} \in \mathrm{h}^1\mathcal{T}^1_1(M_A).$$

The computations needed to derive our equations are in Appendices B and C.

Theorem 2.1 (Metric-affine Finslerian Einstein equations).

(i) (AFFINE EQUATION) The equality

$$\left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \mathcal{S}^D[\mathrm{N}(\tau), L] = 0$$

is fulfilled for all admissible variations $\mathrm{N}(\tau)$ *of* N *if and only if the equality of homogeneous anisotropic tensors*

$$\left\{2\,\mathrm{Lan}_b + (n+2)\,\frac{y_a}{L}\,\mathcal{J}^a_b - 2\,\mathrm{C}_a\,\mathcal{J}^a_b - \left(\mathcal{J}^a_{b\cdot a} + \mathcal{J}^a_{a\cdot b}\right)\right\}\left(\delta^b_i\, y^j - y^b\,\delta^j_i\right) - \left(\mathcal{J}^j_{i\cdot a} - \mathcal{J}^j_{a\cdot i}\right) y^a = 0 \tag{2.19}$$

is fulfilled on A.

(ii) (METRIC EQUATION) The equality

$$\left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \mathcal{S}^D[\mathrm{N}, L(\tau)] = 0$$

is fulfilled for all admissible variations $L(\tau)$ of L if and only if the equality of homogeneous anisotropic scalars

$$(n+2)\,\mathrm{Ric} - L\, g^{ab}\, \mathrm{Ric}_{\cdot a \cdot b} = 0 \tag{2.20}$$

is fulfilled on A.

2.4 The affine equation

Along this section, L (and thus its associated N^L) is fixed.

Definition 2.11. $\mathrm{Sol}_L(A)$ *will be the* space of solutions of the affine equation (2.19). *That is, the set of those* N*'s such that* $\mathcal{J} := \mathrm{N} - \mathrm{N}^L \in \mathrm{h}^1\mathcal{T}_1^1(M_A)$ *solves*

$$\left(2\,\mathrm{Lan}_a + 2\,\mathcal{B}^{\mathcal{J}}_a\right)\left(\delta^a_i\, y^j - y^a\, \delta^j_i\right) - \left(\mathcal{J}^j_{i\cdot a} - \mathcal{J}^j_{a\cdot i}\right) y^a = 0 \tag{2.21}$$

on A (but not necessarily the metric equation (2.20)*); here,*

$$\mathcal{B}^{\mathcal{J}}_i := \frac{n+2}{2}\frac{y_a}{L}\,\mathcal{J}^a_i - \mathrm{C}_a\,\mathcal{J}^a_i - \frac{1}{2}\left(\mathcal{J}^a_{i\cdot a} + \mathcal{J}^a_{a\cdot i}\right), \qquad \mathcal{B}^{\mathcal{J}} \in \mathrm{h}^0\mathcal{T}^0_1(M_A). \tag{2.22}$$

$\mathrm{Sol}^{\mathcal{S}\mathrm{ym}}_L(A)$ *will be the* space of symmetric solutions of the affine equation.

Remark 2.11. When nonempty, $\mathrm{Sol}_L(A)$ is an affine space directed by the space of solutions of

$$2\,\mathcal{B}^{\mathcal{J}_*}_a\left(\delta^a_i\, y^j - y^a\,\delta^j_i\right) - \left\{\left(\mathcal{J}_*\right)^j_{i\cdot a} - \left(\mathcal{J}_*\right)^j_{a\cdot i}\right\} y^a = 0, \tag{2.23}$$

while $\mathrm{Sol}^{\mathcal{S}\mathrm{ym}}_L(A)$ is an affine subspace of $\mathrm{Sol}_L(A)$. N^L is in $\mathrm{Sol}_L(A)$ (and thus in $\mathrm{Sol}^{\mathcal{S}\mathrm{ym}}_L(A)$) when $\mathcal{J} = 0$ solves (2.21), i.e., precisely when the mean Landsberg tensor vanishes ($\mathrm{Lan}_i = 0$). Notice that the vanishing of this tensor does not imply the vanishing of the whole Lan, see [36].

Remark 2.12. Recall that the affine connections solving the classical metric-affine formalism (see [7, (17)] and references therein) are a Levi-Civita ∇^g (with Christoffel symbols $(\Gamma^g)^k_{ij}(x)$) plus any tensor of the form $\mathcal{A} \otimes \mathrm{Id}$ with $\mathcal{A}$ an isotropic 1-form. These affine connections can be regarded either as isotropic Γ's or linear N's (Rem. 2.3); from the latter viewpoint, they are of the form $\mathrm{N}^L + \mathcal{A} \otimes \mathbb{C}$. In other words, the isotropic connection $(\Gamma^g)^k_{ij}(x) + \mathcal{A}_i(x)\,\delta^k_j$ is identified with its underlying linear connection $(\Gamma^g)^k_{ib}(x)\, y^b + \mathcal{A}_i(x)\, y^k$). Thus, the map $\mathrm{N} \mapsto \mathrm{N} + \mathcal{A} \otimes \mathbb{C}$ is a translation on the space of solutions of the classical formalism whenever $\mathcal{A}$ is isotropic. Here we shall prove the extension of this result to our formalism stating a previous lemma for further referencing.

Lemma 2.1. Let $\mathrm{N} = \mathrm{N}^L + \mathcal{J}$ with $\mathcal{J} \in \mathrm{h}^1\mathcal{T}_1^1(M_A)$. Then:

(i) The torsion of N is given by

$$\mathrm{Tor}_{ij}^k = \mathcal{J}_{i\cdot j}^k - \mathcal{J}_{j\cdot i}^k. \tag{2.24}$$

(ii) The curvature of N is given in terms of that of N^L by

$$\mathcal{R}_{ij}^k = \left(\mathcal{R}^L\right)_{ij}^k + \left(\nabla_j^L \mathcal{J}_i^k - \mathcal{J}_{i\cdot a}^k \mathcal{J}_j^a\right) - \left(\nabla_i^L \mathcal{J}_j^k - \mathcal{J}_{j\cdot a}^k \mathcal{J}_i^a\right). \tag{2.25}$$

(iii) The Ricci scalar of N is given in terms of that of N^L by

$$\mathrm{Ric} = \mathrm{Ric}^L - y^b\, \nabla_b^L\, \mathcal{J}_a^a + \nabla_a^L\left(\mathcal{J}_b^a\, y^b\right) + y^b\, \mathcal{J}_{c\cdot a}^c\, \mathcal{J}_b^a - y^b\, \mathcal{J}_{b\cdot a}^c\, \mathcal{J}_c^a. \tag{2.26}$$

(iv) The N-covariant derivative of g is given by

$$\nabla_k^{\mathrm{N}} g_{ij} = 2\,\mathrm{Lan}_{ijk} - 2\,\mathrm{C}_{ija}\, \mathcal{J}_k^a - \mathcal{J}_{k\cdot i}^a\, g_{aj} - \mathcal{J}_{k\cdot j}^a\, g_{ia}. \tag{2.27}$$

Proof. (i) This comes from the definition (2.9) together with the symmetry of N^L.

(ii) Using (2.6),

$$\delta_j \mathrm{N}_i^k = \left(\delta_j^L - \mathcal{J}_j^a\, \dot\partial_a\right)\left\{\left(\mathrm{N}^L\right)_i^k + \mathcal{J}_i^k\right\} = \delta_j^L\left(\mathrm{N}^L\right)_i^k + \delta_j^L \mathcal{J}_i^k - \left(\mathrm{N}^L\right)_{i\cdot a}^k\, \mathcal{J}_j^a - \mathcal{J}_{i\cdot a}^k \mathcal{J}_j^a,$$

and completing $\delta_j^L \mathcal{J}_i^k$ to $\nabla_j^L \mathcal{J}_i^k$ (see (2.8)),

$$\delta_j \mathrm{N}_i^k = \delta_j^L\left(\mathrm{N}^L\right)_i^k + \nabla_j^L \mathcal{J}_i^k + \left(\mathrm{N}^L\right)_{j\cdot i}^a\, \mathcal{J}_a^k - \left(\mathrm{N}^L\right)_{j\cdot a}^k\, \mathcal{J}_i^a - \left(\mathrm{N}^L\right)_{i\cdot a}^k\, \mathcal{J}_j^a - \mathcal{J}_{i\cdot a}^k \mathcal{J}_j^a.$$

Hence, again by the symmetry of N^L, (2.9) yields (2.25).

(iii) This also comes from the definition (2.9), this time together with (2.25) and the fact that $\nabla_i^L y^j = 0$ (Prop. 2.3).

(iv) Again using (2.8) and (2.6),

$$\begin{aligned}\nabla_k^{\mathrm{N}} g_{ij} &= \delta_k g_{ij} - \mathrm{N}_{k\cdot i}^a\, g_{aj} - \mathrm{N}_{k\cdot j}^a\, g_{ia}\\ &= \delta_k^L g_{ij} - \mathcal{J}_k^a\, \dot\partial_a g_{ij} - \left(\mathrm{N}^L\right)_{k\cdot i}^a\, g_{aj} - \mathcal{J}_{k\cdot i}^a\, g_{aj} - \left(\mathrm{N}^L\right)_{k\cdot j}^a\, g_{ia} - \mathcal{J}_{k\cdot j}^a\, g_{ia},\end{aligned}$$

from where the definitions $\mathrm{C} = \dot\partial g/2$ and $\mathrm{Lan} = \nabla^L g/2$ yield (2.27). ∎

Lemma 2.2. For any $\mathcal{A} \in \mathrm{h}^0\mathcal{T}_1^0(M_A)$, the map $\mathrm{N} \mapsto \mathrm{N} + \mathcal{A} \otimes \mathbb{C}$ preserves the Ricci scalar of all homogeneous nonlinear connections. As a consequence, such a map is a translation on $\mathrm{Sol}_L(A)$, i.e., $\left(\mathcal{J}_*\right)_i^k := \mathcal{A}_i\, y^k$ solves (2.23).

Proof. For $\mathrm{N} =: \mathrm{N}^L + \mathcal{J}$, the Ricci scalar of $\mathrm{N}_* := \mathrm{N} + \mathcal{A} \otimes \mathbb{C}$ can be computed with (2.26) by putting $\mathcal{J}_* := \mathcal{J} + \mathcal{A} \otimes \mathbb{C}$ in place of $\mathcal{J}$. Using $\nabla^L_i y^j = 0$ (Prop. 2.3), the 1-homogeneity of $\mathcal{J}$ and the 0-homogeneity of $\mathcal{A}$,

$$y^b \nabla^L_b \left(\mathcal{J}_*\right)^a_a = y^b \nabla^L_b \mathcal{J}^a_a + y^b \nabla^L_b \left(\mathcal{A}_a y^a\right),$$

$$\nabla^L_a \left(\mathcal{J}_*\right)^a_b y^b = \nabla^L_a \left(\mathcal{J}^a_b y^b\right) + y^a \nabla^L_a \left(\mathcal{A}_b y^b\right),$$

$$\begin{aligned} y^b \left(\mathcal{J}_*\right)^c_{c\cdot a} \left(\mathcal{J}_*\right)^a_b &= y^b \left(\mathcal{J}^c_{c\cdot a} + \mathcal{A}_{c\cdot a} y^c + \mathcal{A}_c \delta^c_a\right) \left(\mathcal{J}^a_b + \mathcal{A}_b y^a\right) \\ &= y^b \left(\mathcal{J}^c_{c\cdot a} \mathcal{J}^a_b + \mathcal{A}_{c\cdot a} y^c \mathcal{J}^a_b + \mathcal{A}_c \mathcal{J}^c_b + \mathcal{J}^c_c \mathcal{A}_b + \mathcal{A}_a y^a \mathcal{A}_b\right), \\ y^b \left(\mathcal{J}_*\right)^c_{b\cdot a} \left(\mathcal{J}_*\right)^a_c &= y^b \left(\mathcal{J}^c_{b\cdot a} + \mathcal{A}_{b\cdot a} y^c + \mathcal{A}_b \delta^c_a\right) \left(\mathcal{J}^a_c + \mathcal{A}_c y^a\right) \\ &= y^b \left(\mathcal{J}^c_{b\cdot a} \mathcal{J}^a_c + \mathcal{A}_{b\cdot a} y^c \mathcal{J}^a_c + \mathcal{A}_b \mathcal{J}^c_c + \mathcal{J}^c_b \mathcal{A}_c + \mathcal{A}_a y^a \mathcal{A}_b\right). \end{aligned}$$

Putting these together,

$$\begin{aligned} \mathrm{Ric}_* = \mathrm{Ric}^L &- y^b \nabla^L_b \mathcal{J}^a_a + \nabla^L_a \left(\mathcal{J}^a_b y^b\right) + y^b \mathcal{J}^c_{c\cdot a} \mathcal{J}^a_b - y^b \mathcal{J}^c_{b\cdot a} \mathcal{J}^a_c \\ &- y^b \nabla^L_b \left(\mathcal{A}_a y^a\right) + y^a \nabla^L_a \left(\mathcal{A}_b y^b\right) \\ &+ y^b \left(\mathcal{A}_{c\cdot a} y^c \mathcal{J}^a_b + \mathcal{A}_c \mathcal{J}^c_b + \mathcal{J}^c_c \mathcal{A}_b + \mathcal{A}_a y^a \mathcal{A}_b\right) \\ &- y^b \left(\mathcal{A}_{b\cdot a} y^c \mathcal{J}^a_c + \mathcal{A}_b \mathcal{J}^c_c + \mathcal{J}^c_b \mathcal{A}_c + \mathcal{A}_a y^a \mathcal{A}_b\right) \\ &= \mathrm{Ric}^L - y^b \nabla^L_b \mathcal{J}^a_a + \nabla^L_a \left(\mathcal{J}^a_b y^b\right) + y^b \mathcal{J}^c_{c\cdot a} \mathcal{J}^a_b - y^b \mathcal{J}^c_{b\cdot a} \mathcal{J}^a_c \\ &= \mathrm{Ric}. \end{aligned}$$

Having established that the translation by $\mathcal{A} \otimes \mathbb{C}$ preserves the Ricci scalar, recall Th. 2.1 (ii) and Def. 2.9. Clearly, $\mathcal{S}^D[\mathrm{N} + \mathcal{A} \otimes \mathbb{C}, L] = \mathcal{S}^D[\mathrm{N}, L]$ for any nonlinear connection N, so, as it is standard in Variational Calculus, the translation maps critical points of the action to critical points. Indeed, if $\mathrm{N} \in \mathrm{Sol}_L(A)$, then every ($D$-admissible) variation of $\mathrm{N} + \mathcal{A} \otimes \mathbb{C}$ is of the form $\mathrm{N}(\tau) + \mathcal{A} \otimes \mathbb{C}$ for a (D-admissible) variation of N, so

$$\left.\frac{\partial}{\partial \tau}\right|_{\tau=0} \mathcal{S}^D[\mathrm{N}(\tau) + \mathcal{A} \otimes \mathbb{C}, L] = \left.\frac{\partial}{\partial \tau}\right|_{\tau=0} \mathcal{S}^D[\mathrm{N}(\tau), L] = 0;$$

by Th. 2.1 (ii), $\mathrm{N} + \mathcal{A} \otimes \mathbb{C}$ solves (2.21) too and so $\mathcal{A} \otimes \mathbb{C}$ solves (2.23). ∎

2.4.1 Reduction to the symmetric case

Keep in mind that a homogeneous nonlinear connection is symmetric if and only if it is the vertical differential (also called *Berwald nonlinear connection*) of a spray, see

[28, Prop. 3 (4)]. This is the case for N^L, so a homogeneous nonlinear connection is symmetric if and only if it is of the form $\mathrm{N}^L + \dot\partial\mathcal{Z}$ for some $\mathcal{Z} \in \mathrm{h}^2\mathcal{T}_0^1(M_A)$. The next result provides the geometric invariants of the type of non-symmetric connections that will be relevant when reducing the affine equation to the symmetric case.

Proposition 2.8. Suppose that $\mathrm{N} = \mathrm{N}^L + \dot\partial\mathcal{Z} + \mathcal{A} \otimes \mathbb{C}$ for some $\mathcal{Z} \in \mathrm{h}^2\mathcal{T}_0^1(M_A)$ and $\mathcal{A} \in \mathrm{h}^0\mathcal{T}_1^0(M_A)$. Then:

(i) Its torsion, underlying spray and covariant derivative of g are given respectively by

$$\mathrm{Tor}^k_{ij} = \left(\mathcal{A}_{i\cdot j} - \mathcal{A}_{j\cdot i}\right) y^k + \mathcal{A}_i\, \delta^k_j - \mathcal{A}_j\, \delta^k_i, \tag{2.28}$$

$$\mathrm{G}^i = \left(\mathrm{G}^L\right)^i + \mathcal{Z}^i + \frac{1}{2}\mathcal{A}_a\, y^a\, y^i, \tag{2.29}$$

$$\begin{aligned}\nabla^{\mathrm{N}}_k g_{ij} &= 2\,\mathrm{Lan}_{ijk} - 2\,\mathrm{C}_{ija}\,\mathcal{Z}^a_{\cdot k} - \left(\mathcal{Z}^a_{\cdot k\cdot i}\, g_{aj} + \mathcal{Z}^a_{\cdot k\cdot j}\, g_{ai}\right)\\ &\quad - \left(\mathcal{A}_{k\cdot i}\, y_j + \mathcal{A}_{k\cdot j}\, y_i\right) - 2\, g_{ij}\,\mathcal{A}_k.\end{aligned} \tag{2.30}$$

(ii) The torsion of N determines $\mathcal{A}$ as

$$2\,(n-1)\,\mathcal{A}_i\, y^k = (n-1)\,\mathrm{Tor}^k_{ib}\, y^b - \left(\mathrm{Tor}^a_{ab}\, y^b\right)_{\cdot i} y^k - \mathrm{Tor}^a_{ab}\, y^b\,\delta^k_i. \tag{2.31}$$

(iii) N shares pregeodesics with another $\mathrm{N}_0 = \mathrm{N}^L + \dot\partial\mathcal{Z}_0 + \mathcal{A}_0 \otimes \mathbb{C}$ if and only if $\mathcal{Z} = \mathcal{Z}_0 + \varrho\,\mathbb{C}$ for some $\varrho \in \mathrm{h}^1\mathcal{F}(A)$.

Proof. (i) Formula (2.28) is obtained by substituting $\mathcal{J}^k_i = \mathcal{Z}^k_{\cdot i} + \mathcal{A}_i\, y^k$ in Lem. 2.1 (i) and using that $\mathcal{Z}^k_{\cdot i\cdot j} = \mathcal{Z}^k_{\cdot j\cdot i}$. Formula (2.29) follows from Def. 2.6 (i) and the 2-homogeneity of $\mathcal{Z}$ (the underlying spray of N^L is G^L). Finally, formula (2.30) is obtained by substitution in Lem. 2.1 (iii) of the term

$$\begin{aligned}&2\,\mathrm{Lan}_{ijk} - 2\,\mathrm{C}_{ija}\,\mathcal{J}^a_k - \mathcal{J}^a_{k\cdot i}\, g_{aj} - \mathcal{J}^a_{k\cdot j}\, g_{ia}\\ &= 2\,\mathrm{Lan}_{ijk} - 2\,\mathrm{C}_{ija}\left(\mathcal{Z}^a_{\cdot k} + \mathcal{A}_k\, y^a\right)\\ &\quad - \left(\mathcal{Z}^a_{\cdot k\cdot i} + \mathcal{A}_{k\cdot i}\, y^a + \mathcal{A}_k\,\delta^a_i\right) g_{aj} - \left(\mathcal{Z}^a_{\cdot k\cdot j} + \mathcal{A}_{k\cdot j}\, y^a + \mathcal{A}_k\,\delta^a_j\right) g_{ia};\end{aligned}$$

using $\mathrm{C}_{ija}\, y^a = 0$ yields the result.

(ii) From (2.28), one computes

$$\mathrm{Tor}^k_{ib}\, y^b = -\mathcal{A}_{b\cdot i}\, y^b\, y^k + \mathcal{A}_i\, y^k - \mathcal{A}_b\, y^b\,\delta^k_i = -\left(\mathcal{A}_b\, y^b\right)_{\cdot i} y^k + 2\,\mathcal{A}_i\, y^k - \mathcal{A}_b\, y^b\,\delta^k_i, \tag{2.32}$$

$$\mathrm{Tor}^a_{ab}\, y^b = -(n-1)\,\mathcal{A}_b\, y^b$$

(the 0-homogeneity of $\mathcal{A}$ and the 1-homogeneity of $\mathcal{A}_b\, y^b$ were used). Substituting $\mathcal{A}_b\, y^b$ back in (2.32), multiplying everything by $(n-1)$ and rearranging produces (2.31).

(iii) This follows from applying 2.4 to sprays G and G_0 of the form (2.29). ∎

Theorem 2.2. $\mathrm{N} \in \mathrm{Sol}_L(A)$ *if and only if it is of the form* $\mathrm{N} = \mathrm{N}^L + \dot{\partial}\mathcal{Z} + \mathcal{A} \otimes \mathbb{C}$ *for some* $\mathcal{Z} \in \mathrm{h}^2\mathcal{T}_0^1(M_A)$ *such that* $\mathrm{N}^L + \dot{\partial}\mathcal{Z} \in \mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A)$ *and* $\mathcal{A} \in \mathrm{h}^0\mathcal{T}_1^0(M_A)$. *In such a case,* $(\mathcal{Z}, \mathcal{A})$ *is unequivocally determined by* N *as*

$$\mathcal{Z}^j = \frac{1}{2}\mathcal{J}_a^j\, y^a - \mathcal{B}_a^{\mathcal{J}}\, y^a\, y^j, \qquad \mathcal{A}_i = \mathrm{Lan}_i + \mathcal{B}_i^{\mathcal{J}} + \left(\mathcal{B}_a^{\mathcal{J}}\, y^a\right)_{\cdot i}, \tag{2.33}$$

where $\mathcal{J} := \mathrm{N} - \mathrm{N}^L$ *and* $\mathcal{B}^{\mathcal{J}}$ *is defined by* (2.22).

Proof. We observe that, using the 1-homogeneity of $\mathcal{J}$, the affine equation (2.21) can be rewritten as

$$\mathcal{J}_i^j = \left(\mathrm{Lan}_a + \mathcal{B}_a^{\mathcal{J}}\right)\left(\delta_i^a\, y^j - y^a\, \delta_i^j\right) + \frac{1}{2}\left(\mathcal{J}_a^j\, y^a\right)_{\cdot i}$$

and that this allows one to derive the form of the general solution. Indeed, using that $\mathrm{Lan}_a\, y^a = 0$,

$$\begin{aligned}
\mathcal{J}_i^j &= \mathrm{Lan}_i\, y^j + \mathcal{B}_i^{\mathcal{J}}\, y^j - \mathcal{B}_a^{\mathcal{J}}\, y^a\, \delta_i^j + \frac{1}{2}\left(\mathcal{J}_a^j\, y^a\right)_{\cdot i} \\
&= \mathrm{Lan}_i\, y^j + \mathcal{B}_i^{\mathcal{J}}\, y^j - \left(\mathcal{B}_a^{\mathcal{J}}\, y^a\, y^j\right)_{\cdot i} + \left(\mathcal{B}_a^{\mathcal{J}}\, y^a\right)_{\cdot i} y^j + \frac{1}{2}\left(\mathcal{J}_a^j\, y^a\right)_{\cdot i} \\
&= \left(\frac{1}{2}\mathcal{J}_a^j\, y^a - \mathcal{B}_a^{\mathcal{J}}\, y^a\, y^j\right)_{\cdot i} + \left\{\mathrm{Lan}_i + \mathcal{B}_i^{\mathcal{J}} + \left(\mathcal{B}_a^{\mathcal{J}}\, y^a\right)_{\cdot i}\right\} y^j,
\end{aligned}$$

which tells us that $\mathcal{J} = (\mathrm{N} - \mathrm{N}^L) = \dot{\partial}\mathcal{Z} + \mathcal{A} \otimes \mathbb{C}$ together with (2.33). Lemma 2.2 ensures that $\mathrm{N} = \mathrm{N}^L + \dot{\partial}\mathcal{Z} + \mathcal{A} \otimes \mathbb{C}$ is in $\mathrm{Sol}_L(A)$ if and only if the symmetric part $\mathrm{N}^L + \dot{\partial}\mathcal{Z}$ is.

We derive the uniqueness of the pair $(\mathcal{Z}, \mathcal{A})$ from Prop. 2.8 (ii): the torsion of $\mathrm{N} = \mathrm{N}^L + \dot{\partial}\mathcal{Z} + \mathcal{A} \otimes \mathbb{C}$ determines $\mathcal{A}$, which in turn determines $\dot{\partial}\mathcal{Z}$, and from here $\mathcal{Z}$ is determined due to its 2-homogeneity. ∎

Now we characterize the elements of $\mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A)$.

Proposition 2.9. $\mathrm{N}^L + \dot{\partial}\mathcal{Z} \in \mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A)$ if and only if $\mathcal{Z}$ solves

$$\mathrm{Lan}_i + \frac{n+2}{2}\frac{y_a}{L}\mathcal{Z}_{\cdot i}^a - \mathrm{C}_a\, \mathcal{Z}_{\cdot i}^a - \left\{(n+2)\frac{y_a}{L}\mathcal{Z}^a - 2\,\mathrm{C}_a\, \mathcal{Z}^a\right\}_{\cdot i} = 0, \tag{2.34}$$

$$(n+2)\frac{y_a}{L}\mathcal{Z}^a - 2\,\mathrm{C}_a\, \mathcal{Z}^a - \mathcal{Z}_{\cdot a}^a = 0. \tag{2.35}$$

Proof. We restrict the affine equation (2.21) to symmetric connections (see Lem. 2.1 (i)). As for these connections $\mathcal{J}^{j}_{i\cdot k} - \mathcal{J}^{j}_{k\cdot i} = \mathrm{Tor}^{j}_{ik} = 0$, using also $\mathrm{Lan}_a\, y^a = 0$, the equation reads

$$0 = \left(\mathrm{Lan}_a + \mathcal{B}^{\mathcal{J}}_a\right)\left(\delta^a_i\, y^j - y^a\,\delta^j_i\right) = \left(\mathrm{Lan}_i + \mathcal{B}^{\mathcal{J}}_i\right) y^j - \mathcal{B}^{\mathcal{J}}_a\, y^a\,\delta^j_i. \tag{2.36}$$

This is trivially implied by $\mathrm{Lan}_i + \mathcal{B}^{\mathcal{J}}_i = 0$, but the converse is also true, for taking the trace of (2.36) yields $-(n-1)\,\mathcal{B}^{\mathcal{J}}_a\, y^a = 0$. Thus, recalling (2.22) and writing $\mathcal{J}^k_i = \mathcal{Z}^k_{\cdot i}$, $\mathcal{Z}^a_{\cdot i\cdot a} + \mathcal{Z}^a_{\cdot a\cdot i} = 2\,\mathcal{Z}^a_{\cdot a\cdot i}$, the equation describing $\mathrm{Sol}^{\mathcal{S}\mathrm{ym}}_L(A)$ is

$$\mathrm{Lan}_i + \frac{n+2}{2}\frac{y_a}{L}\,\mathcal{Z}^a_{\cdot i} - \mathrm{C}_a\,\mathcal{Z}^a_{\cdot i} - \mathcal{Z}^a_{\cdot a\cdot i}\left(= \mathrm{Lan}_i + \mathcal{B}^{\mathcal{J}}_i\right) = 0. \tag{2.37}$$

Clearly, (2.34)+(2.35) are sufficient for this. However, they are also necessary: (2.35) is obtained by contracting (2.37) with y^i and using $\mathrm{Lan}_a\, y^a = 0$, the 2-homogeneity of $\mathcal{Z}$, and the 1-homogeneity of $\mathcal{Z}^a_{\cdot a}$. ∎

In Prop. 2.9, we have obtained two *torsion-free affine equations* with somewhat complicated expressions. Next, we are going to formulate them in a way that it is much more convenient for our main results (those of §2.5).

Definition 2.12. *For $\mathcal{Z} \in \mathrm{h}^2\mathcal{T}^1_0(M_A)$, we denote*

$$\sigma^{\mathcal{Z}} := \frac{y_a}{L}\,\mathcal{Z}^a = \frac{g(\mathcal{Z},\mathbb{C})}{L} \in \mathrm{h}^1\mathcal{F}(A) \tag{2.38}$$

and

$$\mathcal{K}^{\mathcal{Z}}_i := -\frac{2}{n+2}\left(2\,\mathrm{C}_{a\cdot i}\,\mathcal{Z}^a + \mathrm{C}_a\,\mathcal{Z}^a_{\cdot i}\right), \qquad \mathcal{K}^{\mathcal{Z}} \in \mathrm{h}^0\mathcal{T}^0_1(M_A). \tag{2.39}$$

Remark 2.13. Thanks to the (-1)-homogeneity of the mean Cartan tensor and the 2-homogeneity of $\mathcal{Z}$, one has the important property

$$\mathcal{K}^{\mathcal{Z}}_a\, y^a = 0,$$

exactly the same as for the mean Landsberg tensor.

Lemma 2.3. $\mathrm{N}^L + \dot\partial\mathcal{Z} \in \mathrm{Sol}^{\mathcal{S}\mathrm{ym}}_L(A)$ if and only if $\mathcal{Z}$ solves

$$\mathcal{Z}^i = 2\sigma^{\mathcal{Z}}\, y^i - L\, g^{ia}\left(\sigma^{\mathcal{Z}}_{\cdot a} + \mathcal{K}^{\mathcal{Z}}_a\right) + \frac{2}{n+2} L\,\mathrm{Lan}^i, \tag{2.40}$$

$$(n+2)\,\sigma^{\mathcal{Z}} - 2\,\mathrm{C}_a\,\mathcal{Z}^a - \mathcal{Z}^a_{\cdot a} = 0. \tag{2.41}$$

Moreover, when assuming the form (2.40) for $\mathcal{Z}$, (2.41) reads

$$(n-2)\,\sigma^{\mathcal{Z}} - L\, g^{ab}\left(\sigma^{\mathcal{Z}}_{\cdot a\cdot b} + \mathcal{K}^{\mathcal{Z}}_{a\cdot b} - \frac{2}{n+2}\mathrm{Lan}_{a\cdot b}\right) = 0. \tag{2.42}$$

Proof. In the notation introduced in Def. 2.12, (2.35) becomes (2.41). For the reexpression of (2.34) as (2.40), recall from §2.2.4 that

$$\left(\frac{y_j}{L}\right)_{\cdot i} = \frac{g_{ij}}{L} - 2\,\frac{y_i}{L}\,\frac{y_j}{L}.$$

By completing $L^{-1}\, y_a\, \mathcal{Z}^a_{\cdot i}$ to a derivative of $\sigma^{\mathcal{Z}} = L^{-1}\, y_a\, \mathcal{Z}^a$ and simplifying, the left hand side of (2.34) becomes

$$\begin{aligned}
&\mathrm{Lan}_i + \frac{n+2}{2}\,\frac{y_a}{L}\,\mathcal{Z}^a_{\cdot i} - \mathrm{C}_a\,\mathcal{Z}^a_{\cdot i} - \left\{(n+2)\,\frac{y_a}{L}\,\mathcal{Z}^a - 2\,\mathrm{C}_a\,\mathcal{Z}^a\right\}_{\cdot i} \\
&= \mathrm{Lan}_i + \frac{n+2}{2}\,\sigma^{\mathcal{Z}}_{\cdot i} - \frac{n+2}{2}\left(\frac{y_a}{L}\right)_{\cdot i}\mathcal{Z}^a - \mathrm{C}_a\,\mathcal{Z}^a_{\cdot i} - (n+2)\,\sigma^{\mathcal{Z}}_{\cdot i} + 2\left(\mathrm{C}_a\,\mathcal{Z}^a\right)_{\cdot i} \\
&= -\frac{n+2}{2}\,\frac{g_{ia}}{L}\,\mathcal{Z}^a + (n+2)\,\frac{y_a}{L}\,\mathcal{Z}^a\,\frac{y_i}{L} - \frac{n+2}{2}\,\sigma^{\mathcal{Z}}_{\cdot i} \\
&\quad + 2\,\mathrm{C}_{a\cdot i}\,\mathcal{Z}^a + \mathrm{C}_a\,\mathcal{Z}^a_{\cdot i} + \mathrm{Lan}_i, \\
&= -\frac{n+2}{2}\,\frac{g_{ia}}{L}\,\mathcal{Z}^a + (n+2)\,\sigma^{\mathcal{Z}}\,\frac{y_i}{L} - \frac{n+2}{2}\,\sigma^{\mathcal{Z}}_{\cdot i} \\
&\quad - \frac{n+2}{2}\,\mathcal{K}^{\mathcal{Z}}_i + \mathrm{Lan}_i.
\end{aligned}$$

Thus, after multiplying by $2\,(n+2)^{-1}\,L$ and raising the index, (2.34) becomes (2.40).

Let us reexpress (2.41) as (2.42). For $\mathcal{Z}$ of the form

$$\mathcal{Z}_i = 2\sigma^{\mathcal{Z}}\,y_i - L\left(\sigma^{\mathcal{Z}}_{\cdot i} + \mathcal{K}^{\mathcal{Z}}_i\right) + \frac{2}{n+2} L\,\mathrm{Lan}_i,$$

using $y_{i\cdot j} = g_{ij}$ and $L_{\cdot j} = 2\,y_j$, one has

$$\begin{aligned}
\mathcal{Z}_{i\cdot j} =& 2\,y_i\,\sigma^{\mathcal{Z}}_{\cdot j} + 2\sigma^{\mathcal{Z}}\,g_{ij} - 2\left(\sigma^{\mathcal{Z}}_{\cdot i} + \mathcal{K}^{\mathcal{Z}}_i\right) y_j - L\left(\sigma^{\mathcal{Z}}_{\cdot i\cdot j} + \mathcal{K}^{\mathcal{Z}}_{i\cdot j}\right) \\
&+ \frac{4}{n+2}\mathrm{Lan}_i\,y_j + \frac{2}{n+2} L\,\mathrm{Lan}_{i\cdot j}
\end{aligned}$$

Using now the 1-homogeneity of $\sigma^{\mathcal{Z}}$, $\mathcal{K}^{\mathcal{Z}}_a\,y^a = 0$ (see Rem. 2.13) and $\mathrm{Lan}_a\,y^a = 0$,

$$\begin{aligned}
g^{ab}\,\mathcal{Z}_{a\cdot b} &= 2\sigma^{\mathcal{Z}} + 2n\,\sigma^{\mathcal{Z}} - 2\sigma^{\mathcal{Z}} - L\,g^{ab}\left(\sigma^{\mathcal{Z}}_{\cdot a\cdot b} + \mathcal{K}^{\mathcal{Z}}_{a\cdot b}\right) + \frac{2}{n+2}\,L\,g^{ab}\,\mathrm{Lan}_{a\cdot b} \\
&= 2n\,\sigma^{\mathcal{Z}} - L\,g^{ab}\left(\sigma^{\mathcal{Z}}_{\cdot a\cdot b} + \mathcal{K}^{\mathcal{Z}}_{a\cdot b} - \frac{2}{n+2}\,\mathrm{Lan}_{a\cdot b}\right).
\end{aligned}$$

On the other hand, it is also true that

$$g^{ab}\,\mathcal{Z}_{a\cdot b} = g^{ab}\left(g_{ac}\,\mathcal{Z}^c\right)_{\cdot b} = g^{ab}\left(2\,\mathrm{C}_{abc}\,\mathcal{Z}^c + g_{ac}\,\mathcal{Z}^c_{\cdot b}\right) = 2\,\mathrm{C}_a\,\mathcal{Z}^a + \mathcal{Z}^a_{\cdot a}.$$

Taking into account the last two formulas, the left hand side of (2.41) becomes

$$\begin{aligned}&(n+2)\,\sigma^{\mathcal{Z}} - 2\,\mathrm{C}_a\,\mathcal{Z}^a - \mathcal{Z}^a_{\cdot a}\\ &= (n+2)\,\sigma^{\mathcal{Z}} - \left\{2n\,\sigma^{\mathcal{Z}} - L\,g^{ab}\left(\sigma^{\mathcal{Z}}_{\cdot a\cdot b} + \mathcal{K}^{\mathcal{Z}}_{a\cdot b} - \frac{2}{n+2}\,\mathrm{Lan}_{a\cdot b}\right)\right\}.\end{aligned}$$

Thus, after simplifying and rearranging, (2.41) becomes (2.42). ∎

2.4.2 Pregeodesics and Ricci scalar of solutions

Corollary 2.1. There is a well-defined projection

$$\begin{aligned}\Pi^{\mathcal{S}\mathrm{ym}} : \mathrm{Sol}_L(A) \longrightarrow \mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A),\\ \mathrm{N} = \mathrm{N}^L + \dot\partial\mathcal{Z} + \mathcal{A}\otimes\mathbb{C} \mapsto \mathrm{N}^L + \dot\partial\mathcal{Z},\end{aligned}$$

with the following properties:

(i) For $\mathrm{N}^L + \dot\partial\mathcal{Z} \in \mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A)$, the only symmetric representative of the fiber $\left(\Pi^{\mathcal{S}\mathrm{ym}}\right)^{-1}(\mathrm{N}^L + \dot\partial\mathcal{Z})$ is $\mathrm{N}^L + \dot\partial\mathcal{Z}$ itself.

(ii) Two elements $\mathrm{N}, \mathrm{N}_0 \in \mathrm{Sol}_L(A)$ share pregeodesics if and only if they are on the same fiber.

(iii) The pregeodesics of $\mathrm{N} \in \mathrm{Sol}_L(A)$ are those of L only in case that $\Pi^{\mathcal{S}\mathrm{ym}}(\mathrm{N}) = \mathrm{N}^L$.

(iv) All the representatives of a fiber share Ricci scalar.

Proof. $\Pi^{\mathcal{S}\mathrm{ym}}$ is well-defined due to Th. 2.2. [13]

(i) By Prop. 2.8 (ii), if $\mathrm{N} = \mathrm{N}^L + \dot\partial\mathcal{Z} + \mathcal{A}\otimes\mathbb{C}$ is symmetric, then $\mathcal{A} = 0$.

(ii) $\mathrm{N} = \mathrm{N}^L + \dot\partial\mathcal{Z} + \mathcal{A}\otimes\mathbb{C}$ and $\mathrm{N}_0 = \mathrm{N}^L + \dot\partial\mathcal{Z}_0 + \mathcal{A}_0\otimes\mathbb{C}$ being on the same fiber of $\Pi^{\mathcal{S}\mathrm{ym}}$ means that $\mathcal{Z} = \mathcal{Z}_0$, from where Prop. 2.8 (iii) tells us that they share pregeodesics. Conversely, if this happens, then $\mathcal{Z} = \mathcal{Z}_0 + \varrho\,\mathbb{C}$ with $\mathrm{N}^L + \dot\partial\mathcal{Z}, \mathrm{N}^L + \dot\partial\mathcal{Z}_0 \in \mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A)$ and $\varrho \in \mathrm{h}^1\mathcal{F}(A)$. By Lem. 2.3, both $\mathcal{Z}$ and $\mathcal{Z}_0$ solve (2.41), so

$$\begin{aligned}0 &= (n+2)\,\sigma^{\mathcal{Z}} - 2\,\mathrm{C}_a\,\mathcal{Z}^a - \mathcal{Z}^a_{\cdot a}\\ &= (n+2)\,\sigma^{\mathcal{Z}_0} + (n+2)\,\varrho - 2\,\mathrm{C}_a\left(\mathcal{Z}_0\right)^a - \left(\mathcal{Z}_0\right)^a_{\cdot a} - \left(\varrho_{\cdot a}\,y^a + \varrho\,\delta^a_a\right)\\ &= (n+2)\,\sigma^{\mathcal{Z}_0} - 2\,\mathrm{C}_a\left(\mathcal{Z}_0\right)^a - \left(\mathcal{Z}_0\right)^a_{\cdot a} + \varrho\\ &= \varrho\end{aligned}$$

[13] It could be defined on any connection of the form $\mathrm{N} = \mathrm{N}^L + \dot\partial\mathcal{Z} + \mathcal{A}\otimes\mathbb{C}$ with $\mathcal{Z} \in \mathrm{h}^2\mathcal{T}^1_0(M_A)$ and $\mathcal{A} \in \mathrm{h}^0\mathcal{T}^0_1(M_A)$, for the argument that we used to prove the uniqueness of $(\mathcal{Z}, \mathcal{A})$ is independent of N being in $\mathrm{Sol}_L(A)$ (see the proof of the mentioned theorem).

(the definition (2.38) of $\sigma^{\mathcal{Z}}$, $\mathrm{C}_a\, y^a = 0$ and the 1-homogeneity of ϱ were used). Thus, $\mathcal{Z} = \mathcal{Z}_0$, which means that N and N_0 are on the same fiber.

(iii) Suppose that $\mathrm{N} = \mathrm{N}^L + \dot\partial\mathcal{Z} + \mathcal{A}\otimes\mathbb{C}$ shares pregeodesics with N^L. This time, Prop. 2.8 (iii) gives us $\mathcal{Z} = \varrho\,\mathbb{C}$ and analogous computations to the previous item yield $\varrho = 0$. From here, $\Pi^{\mathcal{S}\mathrm{ym}}(\mathrm{N}) = \Pi^{\mathcal{S}\mathrm{ym}}(\mathrm{N}^L + \mathcal{A}\otimes\mathbb{C}) = \mathrm{N}^L$.

(iv) This is due to Lem. 2.2.

Remark 2.14. Despite the notation, this projection $\Pi^{\mathcal{S}\mathrm{ym}}$ is not the same as the canonical one of (always homogeneous) nonlinear connections onto symmetric nonlinear connections; the latter is $\mathrm{N} = \dot\partial\mathrm{G} + J \mapsto \dot\partial\mathrm{G}$ with G the underlying spray of N. While N and $\dot\partial\mathrm{G}$ actually share geodesics, they do not necessarily share Ricci scalar.

Let us focus briefly on those $\mathrm{N} \in \mathrm{Sol}_L(A)$ with $\Pi^{\mathcal{S}\mathrm{ym}}(\mathrm{N}) = \mathrm{N}^L$ (i.e., $\dot\partial\mathcal{Z} = 0$ and, by homogeneity, $\mathcal{Z} = 0$).

Definition 2.13. *We refer to the elements of*

$$\left(\Pi^{\mathcal{S}\mathrm{ym}}\right)^{-1}(\mathrm{N}^L) = \begin{cases} \left\{\mathrm{N}^L + \mathcal{A}\otimes\mathbb{C} : \ \mathcal{A} \in \mathrm{h}^0\mathcal{T}^0_1(M_A)\right\} & \text{if } \ \mathrm{Lan}_i = 0, \\ \emptyset & \text{otherwise}, \end{cases}$$

as formally classical solutions of the affine equation (2.19). *Consistently, in case that L is pseudo-Riemannian, we refer to those elements of $\left(\Pi^{\mathcal{S}\mathrm{ym}}\right)^{-1}(\mathrm{N}^L)$ with $\mathcal{A}$ isotropic as* classical solutions.

Remark 2.15. $\left(\Pi^{\mathcal{S}\mathrm{ym}}\right)^{-1}(\mathrm{N}^L)$ being nonempty is equivalent to N^L being in $\mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A)$ and, in turn, to $\mathrm{Lan}_i = 0$ (see Rem. 2.11), which in particular happens in case that L is pseudo-Riemannian. When $\left(\Pi^{\mathcal{S}\mathrm{ym}}\right)^{-1}(\mathrm{N}^L) \neq \emptyset$, its elements have the form of the (underlying linear connections of the) solutions of the classical Palatini formalism (see Rem. 2.12). The difference is that our formalism allows for a non–pseudo-Riemannian L and an anisotropic $\mathcal{A}$, hence the distintion between *formally classical* and *classical* solutions.

In Cor. 2.1, we have seen that the formally classical solutions are exactly those that share pregeodesics with L. Their Ricci scalar is the metric one Ric^L and, when they do exist, the only symmetric one among them is N^L itself. Their importance can be recognized also from the Physics viewpoint. If one wants to model the free fall of particles in a Finsler spacetime equipped with N, in principle they could choose between two different postulates: either particles follow N-geodesics or they follow L-geodesics. When N is formally classical, at least the trajectories and measured proper times coincide for both options.

For these reasons, in the case $\mathrm{Lan}_i = 0$ it is natural to ask whether actually all solutions are formally classical. In general, one can ask if there is only one fiber (equiv., only one symmetric solution). This is studied in §2.5, where a positive answer is provided in many cases of interest.

2.4.3 Metric compatibility conditions

When g and Γ are isotropic, the compatibility of the connection with the metric just means $\nabla^{\Gamma}_k g_{ij} = 0$. When one further restricts to solutions of the classical metric-affine formalism, either one of the conditions of vanishing torsion or $\nabla^{\Gamma}_k g_{ij} = 0$ suffices to select the Levi-Civita connection; moreover, $g^{ab}\,\nabla^{\Gamma}_k g_{ab} = 0$ also suffices [7, (18)].

In the general Finslerian setting, vanishing torsion together with $\nabla^{\Gamma}_k g_{ij} = 0$ determines Γ as the Levi-Civita–Chern anisotropic connection of g [24, 26, 28, 49]. Nevertheless, there are at least seven nonequivalent concepts of metric compatibility that one could think of. Each one is given by the vanishing of one of the following tensors, where we assume that N is the underlying nonlinear connection of Γ:

$$\nabla^{\Gamma}_k g_{ij}, \qquad \nabla^{\mathrm{N}}_k g_{ij}, \qquad \nabla^{\Gamma}_k y_j = \nabla^{\Gamma}_k g_{aj}\, y^a, \qquad \nabla^{\mathrm{N}}_k y_j = \nabla^{\mathrm{N}}_k g_{aj}\, y^a,$$

$$\nabla^{\Gamma}_k g_{ab}\, y^a\, y^b = \nabla^{\Gamma}_k L = \nabla^{\mathrm{N}}_k L = \nabla^{\mathrm{N}}_k g_{ab}\, y^a\, y^b, \qquad y^c\, \nabla^{\Gamma}_c g_{ij}, \qquad y^c\, \nabla^{\mathrm{N}}_c g_{ij};$$

keep in mind that always $\nabla_k y^j = 0$ (Prop. 2.3), but $\nabla_k y_j := \nabla_k \left(g_{ja}\, y^a\right) \neq g_{ja}\, \nabla_k y^a$. When restricting to solutions of our affine equation, some metric compatibility conditions select a single element of each fiber $\left(\Pi^{\mathcal{S}\mathrm{ym}}\right)^{-1}(\mathrm{N}^L + \dot\partial \mathcal{Z})$, much like $\mathrm{Tor}^k_{ij} = 0$ selects $\mathrm{N}^L + \dot\partial \mathcal{Z}$. This, in turn, has important consequences.

Until the end of this section, we use that N is of the form $\mathrm{N}^L + \dot\partial\mathcal{Z} + \mathcal{A} \otimes \mathbb{C}$ for some $\mathcal{Z} \in \mathrm{h}^2\mathcal{T}^1_0(M_A)$ and $\mathcal{A} \in \mathrm{h}^0\mathcal{T}^0_1(M_A)$, which in particular holds true whenever $\mathrm{N} \in \mathrm{Sol}_L(A)$.

Lemma 2.4. For $\mathrm{N} = \mathrm{N}^L + \dot\partial\mathcal{Z} + \mathcal{A} \otimes \mathbb{C}$, one has

$$\nabla^{\mathrm{N}}_i y_k (= \nabla^{\mathrm{N}}_i g_{bk}\, y^b) = -\left(\mathcal{Z}^a_{\cdot i}\, g_{ak} + y_a\, \mathcal{Z}^a_{\cdot i \cdot k}\right) - L\, \mathcal{A}_{i\cdot k} - 2\, \mathcal{A}_i\, y_k, \tag{2.43}$$

$$\nabla^{\mathrm{N}}_i L (= \nabla^{\mathrm{N}}_i y_c\, y^c) = -2\, y_a\, \mathcal{Z}^a_{\cdot i} - 2L\, \mathcal{A}_i, \tag{2.44}$$

$$\left(\nabla^{\mathrm{N}}_i L\right)_{\cdot k} = 2\, \nabla^{\mathrm{N}}_i y_k. \tag{2.45}$$

Proof. In Prop. 2.8 we showed formula (2.30), from where (2.43) follows by contracting with y^j and using $\mathrm{Lan}_{ibk}\, y^b = 0$, $\mathrm{C}_{bki}\, y^b = 0$, the 1-homogeneity of $\mathcal{Z}^j_{\cdot i}$, and the

0-homogeneity of $\mathcal{A}$. Formula (2.44) follows from (2.43) by doing the same. Finally, from comparing the vertical differential of (2.44) with (2.43), and using $y_{j\cdot k} = g_{jk}$ and $L_{\cdot k} = 2\, y_k$, formula (2.45) follows. ∎

Proposition 2.10. The following are equivalent:

(i) $\nabla^{\mathrm{N}}_i L = 0$;
(ii) N is the underlying nonlinear connection of some anisotropic connection Γ for which $\nabla^{\Gamma}_k g_{ij} = 0$. In this case, one can choose $\Gamma^k_{ij} = \mathrm{N}^k_{i\cdot j} + Q^k_{ij}$ with $Q^k_{ij} := g^{ka}\, \nabla^{\mathrm{N}}_i g_{ja}/2$;
(iii) $\nabla^{\mathrm{N}}_i y_k = 0$;
(iv) $\mathcal{A}_i = -y_a\, \mathcal{Z}^a_{\cdot i}/L$.

Proof. (i)⟹(iii) By (2.45), $2\, \nabla^{\mathrm{N}}_i y_k = \left(\nabla^{\mathrm{N}}_i L\right)_{\cdot k} = 0$.

(iii)⟹(ii) The condition $\nabla^{\mathrm{N}}_i y_k = 0$ implies that the chosen Q above fulfills $Q^k_{ib}\, y^b = 0$, so the underlying nonlinear connection of $\Gamma = \dot{\partial}\mathrm{N} + Q$ is N. Then, $\nabla^{\Gamma}_k g_{ij} = 0$ is obtained just by substituting our choice in the general expression

$$\nabla^{\Gamma}_k g_{ij} = \delta_k g_{ij} - \Gamma^a_{ki}\, g_{aj} - \Gamma^a_{kj}\, g_{ia} = \nabla^{\mathrm{N}}_k g_{ij} - Q^a_{ki}\, g_{aj} - Q^a_{kj}\, g_{ia}.$$

(see (2.8)).

(ii)⟹(i) Note that for any Γ, such as the one above, the covariant derivative of a function only depends on the underlying nonlinear connection N. Together with $L = g_{ab}\, y^a\, y^b$ and $\nabla_i y^j = 0$, this provides $\nabla^{\mathrm{N}}_i L = \nabla^{\Gamma}_i L = \nabla^{\Gamma}_i g_{bc}\, y^b\, y^c = 0$. [14]

(i)⟺(iv) This is clear from (2.44).

∎

Proposition 2.11. L is constant along N-geodesics if and only if $\mathcal{A}_a\, y^a = -2\, y_a\, \mathcal{Z}^a/L$. In particular, this is the case if $\nabla^{\mathrm{N}}_i L = 0$.

Proof. Let $\gamma(t)$ be an N-geodesic, so that it solves

$$0 = \frac{\mathrm{d}\dot{\gamma}^k}{\mathrm{d}t} + 2\,\mathrm{G}^k(\gamma,\dot{\gamma}) = \frac{\mathrm{d}\dot{\gamma}^k}{\mathrm{d}t} + \mathrm{N}^k_c(\gamma,\dot{\gamma})\,\dot{\gamma}^c,$$

G being the underlying spray of N. Then, using that γ solves the above equation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L(\gamma,\dot{\gamma}) &= \dot{\gamma}^a\partial_a L(\gamma,\dot{\gamma}) + \frac{\mathrm{d}\dot{\gamma}^a}{\mathrm{d}t}\dot{\partial}_a L(\gamma,\dot{\gamma})\\ &= \dot{\gamma}^a\partial_a L(\gamma,\dot{\gamma}) - \mathrm{N}^a_c(\gamma,\dot{\gamma})\dot{\gamma}^c\dot{\partial}_a L(\gamma,\dot{\gamma}) = \dot{\gamma}^a\nabla^{\mathrm{N}}_a L.\end{aligned}$$

[14] Notice, thus, that (iii)⟹(ii)⟹(i) is true for connections of arbitrary form.

Moreover, from (2.44) and the 2-homogeneity of $\mathcal{Z}$,

$$y^c \nabla^{\mathrm{N}}_c L = -4\, y_a\, \mathcal{Z}^a - 2L\, \mathcal{A}_a\, y^a,$$

which concludes the first equivalence. In case that $\nabla^{\mathrm{N}}_i L = 0$, by Prop. 2.10, one has $\mathcal{A}_i = -y_a\, \mathcal{Z}^a_{\cdot i}/L$, and by the 2-homogeneity of $\mathcal{Z}$, also $\mathcal{A}_a\, y^a = -2\, y_a\, \mathcal{Z}^a/L$. ∎

Remark 2.16. From the beginning we assumed that the connections are defined on A, where L does not vanish; however, L and N could be defined further, on some set with vanishing L (as in the case of Def. 2.8). Then Prop. 2.11 still applies to it. The conclusion is that the tangent vectors to the N-geodesics starting at $\{L = 0\}$ remain in $\{L = 0\}$ (and so the N-geodesics starting at $\{L > 0\}$ or $\{L < 0\}$ remain in these sets as well). In fact, this is true for the pregeodesics of $\mathrm{N} = \mathrm{N}^L + \dot{\partial}\mathcal{Z} + \mathcal{A} \otimes \mathbb{C}$ with arbitrary $\mathcal{A}$, for all of these N's share pregeodesics with another one that is of the form of Prop. 2.10 (see Cor. 2.1 (i)). In the case of proper solutions, this result will be improved by Th. 2.4.

Next, we will not only use the form of N, but also that it is a solution of the affine equation (2.19) (so $\Pi^{\mathcal{S}\mathrm{ym}}(\mathrm{N}) := \mathrm{N}^L + \dot{\partial}\mathcal{Z} \in \mathrm{Sol}^{\mathcal{S}\mathrm{ym}}_L(A)$ and $\mathcal{Z}$ solves (2.34)+(2.35), see Cor. 2.1 and Prop. 2.9 respectively).

Proposition 2.12. For any $\mathrm{N} = \mathrm{N}^L + \dot{\partial}\mathcal{Z} + \mathcal{A} \otimes \mathbb{C} \in \mathrm{Sol}_L(A)$, the following are equivalent:

(i) $g^{ab}\, \nabla^{\mathrm{N}}_i g_{ab} = 0$,
(ii) $\mathcal{A}_i = -(n+2)\, y_a\, \mathcal{Z}^a_{\cdot i}/(2n\, L)$.

Proof. Contracting (2.30) with g^{ij},

$$g^{ab}\, \nabla^{\mathrm{N}}_i g_{ab} = 2\,\mathrm{Lan}_i - 2\,\mathrm{C}_a\, \mathcal{Z}^a_{\cdot i} - 2\, \mathcal{Z}^a_{\cdot a \cdot i} - 2n\, \mathcal{A}_i = -(n+2)\, \frac{y_a}{L}\, \mathcal{Z}^a_{\cdot i} - 2n\, \mathcal{A}_i$$

(the 0-homogeneity of $\mathcal{A}$ and the fact that $\mathcal{Z}$ solves (2.37) were used). ∎

Proposition 2.13. Let $n \geq 3$ and, for any $\mathrm{N} = \mathrm{N}^L + \dot{\partial}\mathcal{Z} + \mathcal{A} \otimes \mathbb{C} \in \mathrm{Sol}_L(A)$, consider the following conditions: $\mathrm{Tor}^k_{ij} = 0$, $\nabla^{\mathrm{N}}_k L = 0$, $g^{ab}\, \nabla^{\mathrm{N}}_k g_{ab} = 0$. If two of them hold, then actually $\mathrm{N} = \mathrm{N}^L$ and the three of them hold. In particular, this is the case when $\nabla^{\mathrm{N}}_k g_{ij} = 0$.

Proof. Due to Props. 2.8, 2.10 and 2.12, the conditions are equivalent to

$$\mathcal{A}_i = 0, \qquad \mathcal{A}_i = -\frac{y_a}{L}\, \mathcal{Z}^a_{\cdot i}, \qquad \mathcal{A}_i = -\frac{n+2}{2n}\, \frac{y_a}{L}\, \mathcal{Z}^a_{\cdot i}$$

respectively, so combining any two of them results in

$$0 = \mathcal{A}_i = \frac{y_a}{L}\,\mathcal{Z}^a_{\cdot i},$$

and, by the 2-homogenity of $\mathcal{Z}$,

$$y_a\,\mathcal{Z}^a_{\cdot b}\,y^b = 2\,y_a\,\mathcal{Z}^a.$$

With this, recall form §2.2.4 that

$$\left(\frac{y_j}{L}\right)_{\cdot i} = \frac{g_{ij}}{L} - 2\,\frac{y_i}{L}\,\frac{y_j}{L}$$

so

$$0 = \frac{y_a}{L}\,\mathcal{Z}^a_{\cdot i} = \left(\frac{y_a}{L}\,\mathcal{Z}^a\right)_{\cdot i} - \left(\frac{y_a}{L}\right)_{\cdot i}\mathcal{Z}^a = -\left(\frac{g_{ia}}{L} - 2\,\frac{y_a}{L}\,\frac{y_i}{L}\right)\mathcal{Z}^a = -\frac{g_{ia}}{L}\,\mathcal{Z}^a.$$

As both $\mathcal{Z}$ and $\mathcal{A}$ vanish, N is the metric connection N^L.

Remark 2.17. Imposing two conditions is required to select N^L among $\mathrm{Sol}_L(A)$, whereas in the classical Palatini formalism only one suffices. While $\nabla^{\mathrm{N}}_k g_{ij} = 0$ is enough to select the metric connection, in the Finslerian setting this should be viewed as a fairly strong requirement, for not even N^L always fulfills it ($\nabla^L_k g_{ij} = 2\,\mathrm{Lan}_{ijk}$).

2.5 General results on proper solutions

The standard theory on differential equations is applicable to the local existence of solutions of our affine and metric equations (Theorem 2.1), see for example [52] in the analytic case. So, generically, one would expect a high multiplicity of solutions, but these solutions would be defined only on a neighborhood of some directions in the tangent bundle. However, a more interesting behaviour occurs if one focuses on the global problem which arises when all the elements can be properly extended at ∂A. Notice also that, apart from its mathematical interest, this assumption will be relevant from the Physics standpoint in order to consider lightlike geodesics.

We will use two different types of techniques for these uniqueness results. The first one relies on a weak hypothesis of analyticity and the second one in the maximum principle. In both cases, the behavior of L at ∂A (or the fact that $\partial A = \emptyset$ in the positive definite case) becomes crucial.

Along this section, we will work essentially in dimension $n \geq 3$, which will be required for different reasons, and we will assume the existence of a prescribed proper L (recall Def. 2.8 and Rem. 2.7). So, N^L and the other metric objects, such as G^L, Ric^L and Lan, are also smooth at the boundary[15]. Accordingly, we work with the solutions $\mathrm{N} = \mathrm{N}^L + \mathcal{J}$ of the affine equation (2.19) that extend smoothly to ∂A (that is, such that $\mathcal{J}$ does).

Definition 2.14. *Given the proper pseudo-Finsler metric L, we say that* N *is a* proper solution of (2.19) *if* $\mathrm{N} \in \mathrm{Sol}_L(A)$ *and it smoothly extends to all of* $\overline{A}$*. The set of these solutions will be denoted* $\mathrm{Sol}_L(\overline{A})$*.*

As a synthesis of §2.4, keep in mind that the elements of $\mathrm{Sol}_L(A)$ are of the form $\mathrm{N} = \mathrm{N}^L + \dot{\partial}\mathcal{Z} + \mathcal{A} \otimes \mathbb{C}$ for some $\mathcal{Z} \in \mathrm{h}^2\mathcal{T}_0^1(M_A)$, $\mathcal{A} \in \mathrm{h}^0\mathcal{T}_1^0(M_A)$ and that then $\Pi^{\mathcal{S}\mathrm{ym}}(\mathrm{N}) := \mathrm{N}^L + \dot{\partial}\mathcal{Z}$ is in $\mathrm{Sol}_L(A)$ as well. In case that $\mathcal{Z}$ and $\mathcal{A}$ extend smoothly to $\overline{A}$, we will write $\mathcal{Z} \in \mathrm{h}^2\mathcal{T}_0^1(M_{\overline{A}})$, $\mathcal{A} \in \mathrm{h}^0\mathcal{T}_1^0(M_{\overline{A}})$, and analogously for anisotropic tensors of all types. The following result justifies restricting further our study to symmetric ($\mathcal{A} = 0$) proper solutions.

Proposition 2.14. Given $\mathrm{N} = \mathrm{N}^L + \dot{\partial}\mathcal{Z} + \mathcal{A} \otimes \mathbb{C} \in \mathrm{Sol}_L(A)$, it is in $\mathrm{Sol}_L(\overline{A})$ if and only if $\mathcal{Z} \in \mathrm{h}^2\mathcal{T}_0^1(M_{\overline{A}})$ and $\mathcal{A} \in \mathrm{h}^0\mathcal{T}_1^0(M_{\overline{A}})$. Consequently, $\Pi^{\mathcal{S}\mathrm{ym}} : \mathrm{Sol}_L(A) \longrightarrow \mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A)$ maps $\mathrm{Sol}_L(\overline{A})$ onto $\mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A) \cap \mathrm{Sol}_L(\overline{A})$.

Proof. Trivially, the smoothness at ∂A of $\mathcal{Z}$ and $\mathcal{A}$ suffices for that of N. Conversely, if N is smooth on $\overline{A}$, then so is its torsion, from where (2.31) shows that so is $\mathcal{A}$ (this uses that the canonical $\mathbb{C} = y^a\,\partial_a$ never vanishes on $\overline{A}$). As now N, N^L and $\mathcal{A}$ are smooth on $\overline{A}$, so must be $\dot{\partial}\mathcal{Z} = \mathrm{N} - \mathrm{N}^L - \mathcal{A} \otimes \mathbb{C}$; by homogeneity, the smoothness of $\dot{\partial}\mathcal{Z}$ anywhere is equivalent to that of $\mathcal{Z}$ (because $2\,\mathcal{Z}^i = \mathcal{Z}^i_{\cdot a}\,y^a$). For the last assertion, if $\mathrm{N} = \mathrm{N}^L + \dot{\partial}\mathcal{Z} + \mathcal{A} \otimes \mathbb{C} \in \mathrm{Sol}_L(\overline{A})$, we have seen that the symmetric solution $\Pi^{\mathcal{S}\mathrm{ym}}(\mathrm{N}) = \mathrm{N}^L + \dot{\partial}\mathcal{Z}$ is smooth on $\overline{A}$ as well. ∎

Remark 2.18. The space of proper solutions of the affine equation is the affine space $\mathrm{Sol}_L(\overline{A})$, which is equal to the proper solutions of (2.21). Its associated vector space given by the proper solutions of (2.23), that is, the equation obtained from (2.21) *dropping the Landsberg term* (recall Def. 2.11 and Rem. 2.11). From Prop. 2.14 only the space $\mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A) \cap \mathrm{Sol}_L(\overline{A})$ will be relevant for the issues of uniqueness. As this is also an affine space, our aim will be to prove that $\mathcal{W} := \mathcal{Z} - \mathcal{Z}_0$ will vanish whenever $\mathrm{N}^L + \mathcal{Z}, \mathrm{N}^L + \mathcal{Z}_0 \in \mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A) \cap \mathrm{Sol}_L(\overline{A})$. Taking into account Lem. 2.3, the problem is

[15] This is checked just by looking at the coordinate expression (2.14) of G^L and recalling that N^L, Ric^L or Lan are constructed with derivatives of it). Note, however, that the assumption of non-degeneracy of g at ∂A becomes essential.

reduced to the uniqueness of $\mathcal{W}=0$ as a solution of both eqn. (2.40) *setting* $\mathrm{Lan}_i=0$ and either (2.41) or (2.42).

2.5.1 Fiberwise analytic solutions

Taking into account Rem. 2.18, let us study the uniqueness of $\mathcal{W}$ on each fiber $A_p \subseteq \mathrm{T}_pM$, $p\in M$. Let $\mathcal{W}\in \mathrm{h}^2\mathcal{T}^1_0(M_A)$ and define $\sigma^{\mathcal{W}}\in \mathrm{h}^1\mathcal{F}(A)$, $\mathcal{K}^{\mathcal{W}}\in \mathrm{h}^0\mathcal{T}^0_1(M_A)$ exactly as in (2.38), (2.39) recalling $\mathcal{K}^{\mathcal{W}}_a\, y^a=0$ (Rem. 2.13), so that $\mathcal{W}$ satisfies:

$$\mathcal{W}^i = 2\sigma^{\mathcal{W}}\, y^i - L\, g^{ia}\left(\sigma^{\mathcal{W}}_{\cdot a}+\mathcal{K}^{\mathcal{W}}_a\right), \tag{2.46}$$

$$(n+2)\,\sigma^{\mathcal{W}} - 2\,\mathrm{C}_a\,\mathcal{W}^a - \mathcal{W}^a_{\cdot a}=0, \tag{2.47}$$

the latter interchangeable with

$$(n-2)\,\sigma^{\mathcal{W}} - L\, g^{ab}\left(\sigma^{\mathcal{W}}_{\cdot a\cdot b}+\mathcal{K}^{\mathcal{W}}_{a\cdot b}\right)=0. \tag{2.48}$$

Lemma 2.5. Suppose that $\mathcal{W}$ solves (2.46), (2.47) on A, it extends smoothly to $\overline{A}$ and $n\geq 3$. Then, $\mathcal{W}$ is divisible up to the boundary by all the powers of L, that is, $\mathcal{W}=L^\nu\,\widetilde{\mathcal{W}}$ for all $\nu\in\mathbb{N}$ with $\widetilde{\mathcal{W}}$ smooth on[16] $\overline{A}$.

Proof. Reasoning by induction, let $\nu=1$. As the metric and $\mathcal{W}$ are smooth on $\overline{A}$, so are $\mathcal{K}^{\mathcal{W}}$ (because of its definition (2.39)) and $\sigma^{\mathcal{W}}$ (because of (2.47)). Using this and $n\geq 3$, (2.48) shows that $\sigma^{\mathcal{W}}$ is divisible by L: $\sigma^{\mathcal{W}}=L\,\widetilde{\sigma^{\mathcal{W}}}$ with $\widetilde{\sigma^{\mathcal{W}}}$ smooth on $\overline{A}$. Substituting this in (2.46):

$$\mathcal{W}^i = L\left\{2\widetilde{\sigma^{\mathcal{W}}}\, y^i - g^{ia}\left(\sigma^{\mathcal{W}}_{\cdot a}+\mathcal{K}^{\mathcal{W}}_a\right)\right\} = L\,\widetilde{\mathcal{W}}^i$$

with $\widetilde{\mathcal{W}}$ smooth on $\overline{A}$. Let us suppose that $\mathcal{W}$ is divisible by L^ν and prove that $\mathcal{W}$ is actually divisible by $L^{\nu+1}$. We do this in five steps.

Step 1 : $\mathcal{K}^{\mathcal{W}}$ is divisible by $L^{\nu-1}$. Indeed, if we substitute $\mathcal{W}=L^\nu\,\widetilde{\mathcal{W}}$ on the definition of $\mathcal{K}^{\mathcal{W}}$ and use that $L_{\cdot i}=2\,y_i$,

$$\begin{aligned}\mathcal{K}^{\mathcal{W}}_i &= -\frac{2}{n+2}\left\{2L^\nu\,\mathrm{C}_{a\cdot i}\,\widetilde{\mathcal{W}}^a+\mathrm{C}_a\left(L^\nu\,\widetilde{\mathcal{W}}^a\right)_{\cdot i}\right\}\\ &= -\frac{2}{n+2}\left\{2L^\nu\,\mathrm{C}_{a\cdot i}\,\widetilde{\mathcal{W}}^a+\mathrm{C}_a\left(2\nu L^{\nu-1}\,\widetilde{\mathcal{W}}^a\,y_i+L^\nu\,\widetilde{\mathcal{W}}^a_{\cdot i}\right)\right\}\\ &= -\frac{2}{n+2}L^{\nu-1}\left(2L\,\mathrm{C}_{a\cdot i}\,\widetilde{\mathcal{W}}^a+2\nu\,\mathrm{C}_a\,\widetilde{\mathcal{W}}^a\,y_i+L\,\mathrm{C}_a\,\widetilde{\mathcal{W}}^a_{\cdot i}\right)\\ &= L^{\nu-1}\,\widetilde{\mathcal{K}^{\mathcal{W}}}_i\end{aligned} \tag{2.49}$$

[16]Whenever an anisotropic tensor is said to be "divisible by L^ν", we mean that the quotient by this is a tensor that extends smoothly to $\partial A=\{L=0\}$, as it is trivially smooth on $A=\{L>0\}$.

with $\widetilde{\mathcal{K}^{\mathcal{W}}}$ smooth on $\overline{A}$. From $\mathcal{K}^{\mathcal{W}}_a\, y^a = 0$ (Rem. 2.13), it follows that

$$\widetilde{\mathcal{K}^{\mathcal{W}}}_a\, y^a = 0. \tag{2.50}$$

Step 2 : $\sigma^{\mathcal{W}}$ is divisible by L^{ν}. First, it is divisible by $L^{\nu-1}$:

$$\sigma^{\mathcal{W}} = \frac{y_a}{L}\,\mathcal{W}^a = \frac{y_a}{L}\,L^{\nu}\,\widetilde{\mathcal{W}} = L^{\nu-1}\,\widetilde{\sigma^{\mathcal{W}}}$$

(by the definition (2.38) and the induction hypothesis). It follows that $\widetilde{\sigma^{\mathcal{W}}}$ is smooth on $\overline{A}$ and $(3-2\nu)$-homogeneous. Now, rewrite the terms appearing in (2.48), first $L\, g^{ab}\, \mathcal{K}^{\mathcal{W}}_{a\cdot b}$ and then $L\, g^{ab}\, \sigma^{\mathcal{W}}_{\cdot a\cdot b}$. For the former, we use (2.50) in the form $g^{ab}\,\widetilde{\mathcal{K}^{\mathcal{W}}}_a\, y_b = 0$ and again $L_{\cdot i} = 2\, y_i$:

$$\begin{aligned} L\, g^{ab}\, \mathcal{K}^{\mathcal{W}}_{a\cdot b} = L\, g^{ab} \left(L^{\nu-1}\,\widetilde{\mathcal{K}^{\mathcal{W}}}_a\right)_{\cdot b} &= L\, g^{ab}\left\{2\,(\nu-1)\,L^{\nu-2}\,\widetilde{\mathcal{K}^{\mathcal{W}}}_a\, y_b + L^{\nu-1}\,\widetilde{\mathcal{K}^{\mathcal{W}}}_{a\cdot b}\right\} \\ &= L^{\nu}\, g^{ab}\,\widetilde{\mathcal{K}^{\mathcal{W}}}_{a\cdot b}. \end{aligned} \tag{2.51}$$

For the latter,

$$\sigma^{\mathcal{W}}_{\cdot i} = \left(L^{\nu-1}\,\widetilde{\sigma^{\mathcal{W}}}\right)_{\cdot i} = 2\,(\nu-1)\,L^{\nu-2}\widetilde{\sigma^{\mathcal{W}}}\, y_i + L^{\nu-1}\,\widetilde{\sigma^{\mathcal{W}}}_{\cdot i},$$

$$\begin{aligned} \sigma^{\mathcal{W}}_{\cdot i\cdot j} &= 2\,(\nu-1)\left(L^{\nu-2}\widetilde{\sigma^{\mathcal{W}}}\, y_i\right)_{\cdot j} + \left(L^{\nu-1}\,\widetilde{\sigma^{\mathcal{W}}}_{\cdot i}\right)_{\cdot j} \\ &= 2\,(\nu-1)\left\{2\,(\nu-2)\,L^{\nu-3}\widetilde{\sigma^{\mathcal{W}}}\, y_i\, y_j + L^{\nu-2}\, y_i\,\widetilde{\sigma^{\mathcal{W}}}_{\cdot j} + L^{\nu-2}\widetilde{\sigma^{\mathcal{W}}}\, g_{ij}\right\} \\ &\quad + 2\,(\nu-1)\,L^{\nu-2}\,\widetilde{\sigma^{\mathcal{W}}}_{\cdot i}\, y_j + L^{\nu-1}\,\widetilde{\sigma^{\mathcal{W}}}_{\cdot i\cdot j}, \end{aligned}$$

and using that $g^{ab}\, y_a\, y_b = L$ and the $(3-2\nu)$-homogeneity of $\widetilde{\sigma^{\mathcal{W}}}$,

$$\begin{aligned} L\, g^{ab}\,\sigma^{\mathcal{W}}_{\cdot a\cdot b} &= 2\,(\nu-1)\,L\left\{2\,(\nu-2)\,L^{\nu-2}\widetilde{\sigma^{\mathcal{W}}} + (3-2\nu)\,L^{\nu-2}\widetilde{\sigma^{\mathcal{W}}} + nL^{\nu-2}\widetilde{\sigma^{\mathcal{W}}}\right\} \\ &\quad + 2\,(\nu-1)\,(3-2\nu)\,L^{\nu-1}\widetilde{\sigma^{\mathcal{W}}} + L^{\nu}\, g^{ab}\,\widetilde{\sigma^{\mathcal{W}}}_{\cdot a\cdot b}, \\ &= -4\,(\nu-1)^2\, L^{\nu-1}\widetilde{\sigma^{\mathcal{W}}} + 2n\,(\nu-1)\,L^{\nu-1}\widetilde{\sigma^{\mathcal{W}}} + L^{\nu}\, g^{ab}\,\widetilde{\sigma^{\mathcal{W}}}_{\cdot a\cdot b}. \\ &= -4\,(\nu-1)^2\,\sigma^{\mathcal{W}} + 2n\,(\nu-1)\,\sigma^{\mathcal{W}} + L^{\nu}\, g^{ab}\,\widetilde{\sigma^{\mathcal{W}}}_{\cdot a\cdot b}. \end{aligned} \tag{2.52}$$

Substituting (2.51) and (2.52) in (2.48) and rearranging yields

$$\left\{4\,(\nu-1)^2 - 2n\,(\nu-1) + (n-2)\right\}\sigma^{\mathcal{W}} = L^{\nu}\, g^{ab}\left(\widetilde{\sigma^{\mathcal{W}}}_{\cdot a\cdot b} + \widetilde{\mathcal{K}^{\mathcal{W}}}_{a\cdot b}\right).$$

The polynomial $4\mathbf{X}^2 - 2n\mathbf{X} + (n-2)$ on $\mathbf{X}$ has no integer roots whenever $n \neq 2$.[17] Thus, as required,

$$\sigma^{\mathcal{W}} = L^{\nu} \widetilde{\widetilde{\sigma^{\mathcal{W}}}} \tag{2.53}$$

with $\widetilde{\widetilde{\sigma^{\mathcal{W}}}}$ smooth on $\overline{A}$. It also follows that $\widetilde{\widetilde{\sigma^{\mathcal{W}}}}$ is $(1-2\nu)$-homogeneous.

Step 3 : $\mathcal{K}^{\mathcal{W}}$ is divisible by L^{ν}. From the penultimate equality on (2.49),

$$\mathcal{K}^{\mathcal{W}}_{i} = -\frac{2}{n+2} L^{\nu-1} \left(2L\, \mathrm{C}_{b\cdot i}\, \widetilde{\mathcal{W}}^{b} + 2\nu\, \mathrm{C}_{b}\, \widetilde{\mathcal{W}}^{b}\, y_{i} + L\, \mathrm{C}_{b}\, \widetilde{\mathcal{W}}^{b}_{\cdot i} \right). \tag{2.54}$$

So, it suffices to show that $\mathrm{C}_{a}\, \widetilde{\mathcal{W}}^{a}$ is divisible by L. Rewriting (2.46) using induction,

$$\widetilde{\mathcal{W}}^{i} = \frac{\mathcal{W}^{i}}{L^{\nu}} = 2\sigma^{\mathcal{W}} \frac{y^{i}}{L^{\nu}} - \frac{1}{L^{\nu-1}} g^{ia} \left(\sigma^{\mathcal{W}}_{\cdot a} + \mathcal{K}^{\mathcal{W}}_{a} \right).$$

As $\mathrm{C}_{a}\, y^{a} = 0$,

$$\mathrm{C}_{a}\, \widetilde{\mathcal{W}}^{a} = -\frac{1}{L^{\nu-1}} \mathrm{C}^{a}\, \sigma^{\mathcal{W}}_{\cdot a} - \frac{1}{L^{\nu-1}} \mathrm{C}^{a}\, \mathcal{K}^{\mathcal{W}}_{a}.$$

Now we need to check that both $\mathrm{C}^{a}\, \sigma^{\mathcal{W}}_{\cdot a}$ and $\mathrm{C}^{a}\, \mathcal{K}^{\mathcal{W}}_{a}$ are divisible ν times. For the former, we use (2.53) and $\mathrm{C}^{a}\, y_{a} = 0$:

$$\mathrm{C}^{a}\, \sigma^{\mathcal{W}}_{\cdot a} = \mathrm{C}^{a} \left(L^{\nu}\, \widetilde{\widetilde{\sigma^{\mathcal{W}}}} \right)_{\cdot a} = \mathrm{C}^{a} \left(2\nu L^{\nu-1} \widetilde{\widetilde{\sigma^{\mathcal{W}}}}\, y_{a} + L^{\nu}\, \widetilde{\widetilde{\sigma^{\mathcal{W}}}}_{\cdot a} \right) = L^{\nu}\, \mathrm{C}^{a}\, \widetilde{\widetilde{\sigma^{\mathcal{W}}}}_{\cdot a}.$$

For the latter, again we use (2.54) and $\mathrm{C}^{a}\, y_{a} = 0$:

$$\begin{aligned} \mathrm{C}^{a}\, \mathcal{K}^{\mathcal{W}}_{a} &= -\frac{2}{n+2} L^{\nu-1} \left(2L\, \mathrm{C}^{a}\, \mathrm{C}_{b\cdot a}\, \widetilde{\mathcal{W}}^{b} + 2\nu\, \mathrm{C}_{b}\, \widetilde{\mathcal{W}}^{b}\, \mathrm{C}^{a}\, y_{a} + L\, \mathrm{C}^{a}\, \mathrm{C}_{b}\, \widetilde{\mathcal{W}}^{b}_{\cdot a} \right) \\ &= -\frac{2}{n+2} L^{\nu} \left(2\, \mathrm{C}^{a}\, \mathrm{C}_{b\cdot a}\, \widetilde{\mathcal{W}}^{b} + \mathrm{C}^{a}\, \mathrm{C}_{b}\, \widetilde{\mathcal{W}}^{b}_{\cdot a} \right). \end{aligned}$$

Going back, these substeps and Rem. 2.13 prove the divisibility

$$\mathcal{K}^{\mathcal{W}} = L^{\nu}\, \widetilde{\widetilde{\mathcal{K}^{\mathcal{W}}}} \qquad \text{with} \qquad \widetilde{\widetilde{\mathcal{K}^{\mathcal{W}}}}_{a}\, y^{a} = 0. \tag{2.55}$$

[17] Its roots are $\mathbf{X} = \frac{n \pm \sqrt{n^2 - 4n + 8}}{4}$, so if either of them was an integer, then $n^2 - 4n + 8$ would be a perfect square, say $n^2 - 4n + \left(8 - m^2\right) = 0$ with m integer. This would mean that $n = 2 \pm \sqrt{m^2 - 4}$, so $m^2 - 4$ and m^2 would be two perfect squares differing by 4. This is impossible unless $m^2 = 4$, which corresponds to $n = 2$.

Step 4 : $\sigma^{\mathcal{W}}$ is divisible by $L^{\nu+1}$. Now that we know that $\sigma^{\mathcal{W}} = L^{\nu}\widetilde{\widetilde{\sigma^{\mathcal{W}}}}$ and $\mathcal{K}^{\mathcal{W}} = L^{\nu}\,\widetilde{\widetilde{\mathcal{K}^{\mathcal{W}}}}$, we turn our attention back to (2.48). The analogous computation to that on (2.51), this time using (2.55), shows that

$$L\, g^{ab}\, \mathcal{K}^{\mathcal{W}}_{a\cdot b} = L^{\nu+1}\, g^{ab}\, \widetilde{\widetilde{\mathcal{K}^{\mathcal{W}}}}_{a\cdot b}.$$

The analogous computations to those leading to (2.52), this time using the $(1-2\nu)$-homogeneity of $\widetilde{\widetilde{\sigma^{\mathcal{W}}}}$, shows that

$$L\, g^{ab}\, \sigma^{\mathcal{W}}_{\cdot a\cdot b} = -4\nu^2\sigma^{\mathcal{W}} + 2n\,\nu\sigma^{\mathcal{W}} + L^{\nu+1}\, g^{ab}\, \widetilde{\widetilde{\sigma^{\mathcal{W}}}}_{\cdot a\cdot b}.$$

Substituting these in (2.48) and rearranging yields

$$\left\{4\nu^2 - 2n\nu + (n-2)\right\}\sigma^{\mathcal{W}} = L^{\nu+1}\, g^{ab}\left(\widetilde{\widetilde{\sigma^{\mathcal{W}}}}_{\cdot a\cdot b} + \widetilde{\widetilde{\mathcal{K}^{\mathcal{W}}}}_{a\cdot b}\right),$$

and the inexistence of integer roots of $4\mathbf{X}^2 - 2n\mathbf{X} + (n-2)$ yields the divisibility

$$\sigma^{\mathcal{W}} = L^{\nu+1}\widetilde{\widetilde{\widetilde{\sigma^{\mathcal{W}}}}}.$$

Step 5 : $\mathcal{W}$ is divisible by $L^{\nu+1}$. Substituting $\sigma^{\mathcal{W}} = L^{\nu+1}\widetilde{\widetilde{\widetilde{\sigma^{\mathcal{W}}}}}$, $\mathcal{K}^{\mathcal{W}} = L^{\nu}\,\widetilde{\widetilde{\mathcal{K}^{\mathcal{W}}}}$ in (2.46) and computing, one gets $\mathcal{W}^i = L^{\nu+1}\,\widetilde{\widetilde{\mathcal{W}}}^i$ with $\widetilde{\widetilde{\mathcal{W}}}$ smooth on $\overline{A}$, which completes the proof.

Remark 2.19. Assume that $\mathrm{N}^L + \dot{\partial}\mathcal{Z} \in \mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A) \cap \mathrm{Sol}_L(\overline{A})$ (so that $\mathcal{Z} \in \mathrm{h}^2\mathcal{T}^1_0(M_{\overline{A}})$ solves (2.40), (2.41)) and that Lan_i is divisible up to ∂A by L^{ν}, where $\nu \in \mathbb{N} \cup \{0\}$. Then the argument above proves that $\mathcal{Z}$ is divisible by $L^{\nu+1}$. In particular, $\mathcal{Z}$ always is divisible by L.

Definition 2.15. *We say that an anisotropic tensor $T \in \mathrm{h}^{\alpha}\mathcal{T}^r_s(M_{\overline{A}})$ is* fiberwise analytic on $\overline{A}$ *if it is analytic when restricted to every $\overline{A}_p \subseteq \mathrm{T}_pM$.*

Remark 2.20. In coordinates, T is fiberwise analytic when all $T^{i_1,\dots,i_r}_{j_1,\dots,j_s}(x,y)$ are analytic in y. In particular, this property holds for most explicit pseudo-Finsler metrics, $L \equiv L(x,y)$, such as pseudo-Riemannian or Randers ones. This notion does not require of any additional analytic structure to be well-defined: each T_pM has a canonical one as a vector space. By contrast, the notion of being *analytic on* $\overline{A}$ does. Anyway, obviously, “analytic” implies “fiberwise analytic”.

Theorem 2.3. *Assume that the proper pseudo-Finsler metric L is of non-definite signature and $n \geq 3$. Then there exists at most one* $\mathrm{N} = \mathrm{N}^L + \dot\partial \mathcal{Z} \in \mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A) \cap \mathrm{Sol}_L(\overline{A})$ *such that the spray difference* $-2\,\mathcal{Z} = \mathrm{G} - \mathrm{G}^L$ *(equiv., the connection difference* $\dot\partial \mathcal{Z} = \mathrm{N} - \mathrm{N}^L$*) is fiberwise analytic on* $\overline{A}$.

Proof. The analyticity (resp., fiberwise analyticity) of $-2\,\mathcal{Z}$ is equivalent to that of $\dot\partial\mathcal{Z}$ because this is constructed with fiber derivatives of $\mathcal{Z}$ but also $-2\,\mathcal{Z}^i = -\mathcal{Z}^i_{\cdot a}\, y^a$.

Let $\mathrm{N}_0 = \mathrm{N}^L + \dot\partial\mathcal{Z}_0$ be another solution with the same properties. Then $\mathcal{W} := \mathcal{Z} - \mathcal{Z}_0$ is fiberwise analytic on $\overline{A}$ too. By Prop. 2.14, $\mathcal{W}$ is smooth there, and by Lem. 2.3, it solves (2.46)+(2.47). For all $\nu \in \mathbb{N}$, Lem. 2.5 allows us to write $\mathcal{W} = L^\nu\, \widetilde{\mathcal{W}}$ with $\widetilde{\mathcal{W}}$ smooth on $\overline{A}$. After restricting this to each $\overline{A_p}$, when one computes the vertical derivatives of the functions $\mathcal{W}^i$ by induction, it becomes clear that $\mathcal{W}^i_{\cdot j_1 \cdot j_2 \cdots \cdot j_{\nu-1}} = L\, T^i_{j_1 \cdots j_{\nu-1}}$ with $T^i_{j_1\cdots j_{\nu-1}}$ a smooth function on $\overline{A_p}$. This shows that all derivatives of all orders vanish on $\partial A_p = \left\{ v \in \overline{A_p} \;:\; L(v) = 0 \right\}$. Now we develop $\mathcal{W}^i$ in Taylor series on an open subset of $\overline{A_p}$ around some $v \in \partial A_p$ (this exists due to the signature being non-definite, recall Rem 2.7 (C)). Clearly the analytic $\mathcal{W}^i$ vanishes on that open set and, as A_p is connected, it vanishes on all of A_p. Thus, $\mathcal{Z}_p = \mathcal{Z}_0\big|_p + \mathcal{W}_p = \mathcal{Z}_0\big|_p$. ∎

Corollary 2.2. With the hypotheses of Th. 2.3, in case that L (equiv., g) is analytic on $\overline{A}$, there exists at most one symmetric and proper solution $\mathrm{N} = \mathrm{N}^L + \dot\partial\mathcal{Z}$ of the affine equation (2.19) analytic on $\overline{A}$.

Proof. The analyticity of L is equivalent to that of g by the analogous reasoning as in the theorem above. In case that L is analytic, so are G^L and $\mathrm{N}^L = \dot\partial \mathrm{G}^L$ (recall the coordinate expression (2.14)), so the analyticity of $\mathrm{N} = \mathrm{N}^L + \dot\partial\mathcal{Z}$ becomes equivalent to that of $\dot\partial\mathcal{Z}$ and implies its fiberwise analyticity. Thus, Th. 2.3 applies. ∎

Remark 2.21. The techniques above can be used to obtain nonexistence results for fiberwise analytic solutions in some cases. Namely, if Lan_i is not 0 but it is divisible by all the powers of L (what implies that Lan_i is not fiberwise analytic on $\overline{A}$), then no proper solution $\mathrm{N} = \mathrm{N}^L + \dot\partial\mathcal{Z}$ with $\mathcal{Z}$ fiberwise analytic can exist (indeed, by Rem. 2.19 such a $\mathcal{Z}$ would be divisible by all the powers of L too and the same argument of Th. 2.3 would prove that $\mathcal{Z} = 0$, contradicting $\mathrm{Lan}_i \neq 0$).

A relevant issue is whether the N-geodesics will be defined on all the L-lightlike directions, which becomes obviously important for physical interpretations in Lorentzian signature. We will take advantage of the fact that $\mathcal{Z}$ is always divisible by L (Rem. 2.19) to prove that every symmetric and proper solution of the affine equation (2.19) shares its lightlike geodesics with L, notably with their parametrizations included. In the Lorentz-Finsler case, they are the cone geodesics of the cone structure naturally associated with L [27, Th. 6.6] with distinguished parametrizations. Recall that the tangent vectors to the L-geodesics starting at $\partial A = \{L = 0\}$ remain in ∂A (this, for instance, follows from Prop. 2.11 by taking $\mathcal{Z} = 0$ and $\mathcal{A} = 0$).

Theorem 2.4. *Let* $\mathrm{N} = \mathrm{N}^L + \dot{\partial}\mathcal{Z} \in \mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A) \cap \mathrm{Sol}_L(\overline{A})$. *Then the unique* N*-geodesic starting at each* $v \in \partial A$ *coincides with the corresponding (lightlike)* L*-geodesic.*

Proof. We saw that $\mathcal{Z} = L\,\widetilde{\mathcal{Z}}$ with $\widetilde{\mathcal{Z}}$ smooth on $\overline{A}$. Let $\gamma(t)$ be the unique L-geodesic with initial condition $\dot{\gamma}(0) = v$, so that it solves

$$\frac{\mathrm{d}\dot{\gamma}^i}{\mathrm{d}t} + 2\left(\mathrm{G}^L\right)^i(\dot{\gamma}(t)) = 0.$$

Then $L(\dot{\gamma}(t)) = L(v) = 0$ and $\mathcal{Z}_{\dot{\gamma}(t)} = L(\dot{\gamma}(t))\,\widetilde{\mathcal{Z}}_{\dot{\gamma}(t)} = 0$, allowing us to write

$$0 = \frac{\mathrm{d}\dot{\gamma}^i}{\mathrm{d}t} + 2\left(\mathrm{G}^L\right)^i(\dot{\gamma}(t)) + 2\,\mathcal{Z}^i(\dot{\gamma}(t)) = \frac{\mathrm{d}\dot{\gamma}^i}{\mathrm{d}t} + 2\,\mathrm{G}^i(\dot{\gamma}(t)).$$

Recall that G is the underlying spray of N, so $\gamma(t)$ turns out to be the N-geodesic with initial condition v.

Remark 2.22. Although we have been working with proper metrics, as far as the results of this section 2.5.1 are concerned, this assumption can be somewhat weakened. Indeed, assume only: (i) each fiber A_p ($p \in M$) is connected and $L \neq 0$ on it; (ii) L extends smoothly to some conic B with $A \subseteq B \subseteq \overline{A} \subseteq \mathrm{T}M \setminus \mathbf{0}$ and g is non-degenerate therein; (iii) each $B_p \setminus A_p$ is nonempty and formed by L-lightlike directions. Accordingly, consider those $\mathrm{N} = \mathrm{N}^L + \dot{\partial}\mathcal{Z} \in \mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A)$ that extend smoothly to B. Then Ths. 2.3 and 2.4, as well as Rem. 2.21, still hold true. Moreover, Lem. 2.5 and Th. 2.3 could straightforwardly be stated for a single fiber B_p. Summing up, the point here is that the techniques of this subsection do not really require of any global hypothesis at the boundary of each A_p, but only the existence at each point of a lightlike direction to which L and N can be smoothly extended. By contrast, those of the next subsection will actually require of solutions defined on the whole $\overline{A_p}$.

2.5.2 Results from scalar elliptic PDEs

Inspired by (2.20) and (2.42), we consider the equation

$$\kappa f - L\,g^{ab} f_{\cdot a \cdot b} = 0 \tag{2.56}$$

with parameter $\kappa \in \mathbb{R}$. This time we emphasize its study on each single fiber A_p ($p \in M$) and we work in coordinates adapted to its homogeneity. Thus, regard (by restriction) f as an α-homogeneous smooth function on A_p and take another positive 1-homogeneous function $\mathbf{r}$ there (in particular, we will take $\mathbf{r} = F_p = \sqrt{L_p}$ later). Consider the smooth[18] hypersurface $\Sigma^{\mathbf{r}} = \{\mathbf{r} = 1\}$, so that

$$A_p \equiv \mathbb{R}^+ \times \Sigma^{\mathbf{r}}, \qquad v \equiv (\mathbf{r}(v), \frac{v}{\mathbf{r}(v)}).$$

The indices $\bar{c}, \bar{d}$ will run in the set $\{1, ..., n-1\}$. Take coordinates $(z_\Sigma^1, ..., z_\Sigma^{n-1})$ on $\Sigma^{\mathbf{r}}$. Together with the natural coordinate on $\mathbb{R}^+$, they induce coordinates on A_p. These turn out to be $(\mathbf{r}, z_A^1, ..., z_A^{n-1})$, where the $z_A^{\bar{c}}$'s are the $z_\Sigma^{\bar{c}}$'s extended by 0-homogeneity:

$$z_A^{\bar{c}}(v) = z_\Sigma^{\bar{c}}(\frac{v}{\mathbf{r}(v)}).$$

We refer to $(\mathbf{r}, z_A^1, ..., z_A^{n-1})$ as *generalized polar coordinates.*

By the 1-homogeneity of $\mathbf{r}$ and the 0-homogeneity of the $z_A^{\bar{c}}$'s,

$$\mathbb{C}^{\mathrm{V}} = y^a \, \partial_{y^a} = y^a \left(\frac{\partial \mathbf{r}}{\partial y^a} \, \partial_{\mathbf{r}} + \frac{\partial z_A^{\bar{c}}}{\partial y^a} \, \partial_{z_A^{\bar{c}}} \right) = \mathbf{r} \, \partial_{\mathbf{r}}$$

on A_p. For $v_0 \in \Sigma^{\mathbf{r}}$, one straightforwardly checks that $(v_0, \partial_{z_\Sigma^1}\big|_{v_0}, ..., \partial_{z_\Sigma^{n-1}}\big|_{v_0})$ is the dual basis of $(\mathrm{d}\mathbf{r}_{v_0}, \left(\mathrm{d}z_A^1\right)_{v_0}, ..., \left(\mathrm{d}z_A^{n-1}\right)_{v_0})$, so $\partial_{z_\Sigma^{\bar{c}}}\big|_{v_0} = \partial_{z_A^{\bar{c}}}\big|_{v_0}$. From now on we will not distinguish between the z_Σ and the z_A, denoting either of them by z. For f, being α-homogeneous means that

$$f(\mathbf{r}, z^1, ..., z^{n-1}) = f_{\Sigma^{\mathbf{r}}}(z^1, ..., z^{n-1}) \, \mathbf{r}^\alpha,$$

so $\partial_{z^{\bar{c}}} f$ is α-homogeneous as well.

Lemma 2.6. Let $n \geq 2$. Any α-homogeneous solution f of (2.56) on A_p must be $f = 0$ in any of the following two cases:

(A) L is Lorentz-Finsler, f extends smoothly to $\overline{A_p}$, $\kappa \neq 0$, $\alpha \leq 2$, and $\kappa \leq \alpha(\alpha + n - 2)$ with one of these inequalities being strict.

(B) L is Finsler (thus $A_p = \overline{A_p} = \mathrm{T}_p M \setminus 0$) and $\kappa > \alpha(\alpha + n - 2)$.

[18] Regarding (also by restriction) $(y^1, ..., y^n)$ as linear coordinates on $T_p M \supseteq A_p$, by homogeneity one has $\mathrm{d}\mathbf{r}_v(\mathbb{C}_v^{\mathrm{V}}) = y^a(v) \, \mathbf{r}_{\cdot a}(v) = \mathbf{r}(v) = 1 \neq 0$ for $v \in \Sigma^{\mathbf{r}} \subseteq A_p$.

Proof. Case (A) . First, rewrite (2.56) on A_p in terms of $F = \sqrt{L}(> 0)$,

$$\kappa \frac{f}{F^\alpha} - F^{2-\alpha} g^{ab} f_{\cdot a \cdot b} = 0, \tag{2.57}$$

and this expression in terms of

$$\widetilde{f} = \frac{f}{F^\alpha}.$$

Using $F_{\cdot i} = y_i/F$, $g^{ab} y_a y_b = F^2$ (§2.2.4) and the 0-homogeneity of $\widetilde{f}$,

$$f_{\cdot i} = \left(F^\alpha \widetilde{f}\right)_{\cdot i} = \alpha F^{\alpha-2} \widetilde{f} y_i + F^\alpha \widetilde{f}_{\cdot i},$$

$$\begin{aligned} f_{\cdot i \cdot j} &= \alpha \left\{ (\alpha - 2) F^{\alpha-4} \widetilde{f} y_i y_j + F^{\alpha-2} y_i \widetilde{f}_{\cdot j} + F^{\alpha-2} \widetilde{f} g_{ij} \right\} \\ &\quad + \alpha F^{\alpha-2} \widetilde{f}_{\cdot i} y_j + F^\alpha \widetilde{f}_{\cdot i \cdot j}, \\ F^{2-\alpha} g^{ab} f_{\cdot a \cdot b} &= \alpha (\alpha + n - 2) \widetilde{f} + F^2 g^{ab} \widetilde{f}_{\cdot a \cdot b}. \end{aligned}$$

Substituting this and rearranging, (2.57) reads

$$-L g^{ab} \widetilde{f}_{\cdot a \cdot b} - \{\alpha (\alpha + n - 2) - \kappa\} \widetilde{f} = 0. \tag{2.58}$$

Now, rewrite (2.58) in generalized polar coordinates $(\mathbf{r}, z^1, ..., z^{n-1})$ with $\mathbf{r} = F_p$, so that $\Sigma^{\mathbf{r}}$ is the indicatrix of L at p and $(z^1, ..., z^{n-1})$ are global coordinates on $\Sigma^{\mathbf{r}}$ with values in a relatively compact domain[19] $D \subsetneq \mathbb{R}^{n-1}$ which then are extended to A_p by 0-homogeneity. Using $\partial_{\mathbf{r}} = \mathbf{r}^{-1} \mathbb{C}^{\mathrm{V}}$ and $\mathbb{C}^{\mathrm{V}}(\widetilde{f}) = 0$ (0-homogeneity of $\widetilde{f}$),

$$\widetilde{f}_{\cdot i} = \partial_{y^i} \widetilde{f} = \frac{\partial \mathbf{r}}{\partial y^i} \partial_{\mathbf{r}} \widetilde{f} + \frac{\partial z^{\bar{c}}}{\partial y^i} \partial_{z^{\bar{c}}} \widetilde{f} = \frac{\partial z^{\bar{c}}}{\partial y^i} \partial_{z^{\bar{c}}} \widetilde{f}.$$

Using that $\partial_{z^{\bar{c}}} \widetilde{f}$ is 0-homogeneous too,

$$\begin{aligned} \widetilde{f}_{\cdot i \cdot j} = \partial_{y^j} \left(\frac{\partial z^{\bar{c}}}{\partial y^i} \partial_{z^{\bar{c}}} \widetilde{f}\right) &= \frac{\partial^2 z^{\bar{c}}}{\partial y^i \partial y^j} \partial_{z^{\bar{c}}} \widetilde{f} + \frac{\partial z^{\bar{c}}}{\partial y^i} \partial_{y^j} (\partial_{z^{\bar{c}}} \widetilde{f}) \\ &= \frac{\partial^2 z^{\bar{c}}}{\partial y^i \partial y^j} \partial_{z^{\bar{c}}} \widetilde{f} + \frac{\partial z^{\bar{c}}}{\partial y^i} \frac{\partial z^{\bar{d}}}{\partial y^j} \partial^2_{z^{\bar{c}} z^{\bar{d}}} \widetilde{f}. \end{aligned}$$

[19] As $\overline{A_p}$ is contained in an open half-space determined by some vector hyperplane $\Pi_p \subsetneq \mathrm{T}_p M$ (Rem. 2.7 (B)), any hyperplane Ξ_p contained in that half-space and parallel to Π_p will be intersected exactly once by each ray in $\overline{A_p}$. These points give $D \subsetneq \Xi_p$ and its boundary ∂D, which is the intersection of the cone ∂A_p with Ξ_p.

From these,

$$
\begin{aligned}
L\, g^{ab}\, \widetilde{f}_{\cdot a \cdot b} &= L\, g^{ab}\, \frac{\partial^2 z^{\bar{c}}}{\partial y^a\, \partial y^b}\, \partial_{z^{\bar{c}}} \widetilde{f} + L\, g^{ab}\, \frac{\partial z^{\bar{c}}}{\partial y^a}\, \frac{\partial z^{\bar{d}}}{\partial y^b}\, \partial^2_{z^{\bar{c}} z^{\bar{d}}} \widetilde{f} \\
&= L\, g^{ab}\, \frac{\partial^2 z^{\bar{c}}}{\partial y^a\, \partial y^b}\, \partial_{z^{\bar{c}}} \widetilde{f} + L\, g^{-1}(\mathrm{d}y^a, \mathrm{d}y^b)\, \frac{\partial z^{\bar{c}}}{\partial y^a}\, \frac{\partial z^{\bar{d}}}{\partial y^b}\, \partial^2_{z^{\bar{c}} z^{\bar{d}}} \widetilde{f} \\
&= L\, g^{ab}\, \frac{\partial^2 z^{\bar{c}}}{\partial y^a\, \partial y^b}\, \partial_{z^{\bar{c}}} \widetilde{f} + L\, g^{-1}(\mathrm{d}z^{\bar{c}}, \mathrm{d}z^{\bar{d}})\, \partial^2_{z^{\bar{c}} z^{\bar{d}}} \widetilde{f}.
\end{aligned}
$$

Substituting this, (2.58) reads

$$
\begin{aligned}
-L\, g^{-1}(\mathrm{d}z^{\bar{c}}, \mathrm{d}z^{\bar{d}})\, \partial^2_{z^{\bar{c}} z^{\bar{d}}} \widetilde{f} - L\, g^{ab}\, \frac{\partial^2 z^{\bar{c}}}{\partial y^a\, \partial y^b}\, \partial_{z^{\bar{c}}} \widetilde{f} \\
- \{\alpha\, (\alpha + n - 2) - \kappa\}\, \widetilde{f} = 0.
\end{aligned}
\tag{2.59}
$$

To check that the matrix $g^{-1}(\mathrm{d}z^{\bar{c}}, \mathrm{d}z^{\bar{d}})_{\Sigma^{\mathrm{r}}}$ is negative definite, notice that, for each $v_0 \in \Sigma^{\mathrm{r}}$, g_{v_0} is of signature $(+, -, \dots, -)$, the radial direction v_0 is positive definite and g_{v_0}-orthogonal to $\mathrm{T}_{v_0}\Sigma_p = \mathrm{Span}\left\{\partial_{z^1}|_{v_0}, \dots, \partial_{z^{n-1}}|_{v_0}\right\}$ and the g_{v_0}-flat isomorphism maps $\mathrm{T}_{v_0}\Sigma_p$ into $\mathrm{Span}\left\{\mathrm{d}z^1_{v_0}, \dots, \mathrm{d}z^{n-1}_{v_0}\right\}$.

The restriction $\widetilde{f}_{\Sigma^{\mathrm{r}}}$ satisfies (2.59) on its domain D with $L = 1$:

$$
\begin{aligned}
-g^{-1}(\mathrm{d}z^{\bar{c}}, \mathrm{d}z^{\bar{d}})_{\Sigma^{\mathrm{r}}}\, \partial^2_{z^{\bar{c}} z^{\bar{d}}} \widetilde{f}_{\Sigma^{\mathrm{r}}} - \left(g^{ab}\, \frac{\partial^2 z^{\bar{c}}}{\partial y^a\, \partial y^b}\right)_{\Sigma^{\mathrm{r}}} \partial_{z^{\bar{c}}} \widetilde{f}_{\Sigma^{\mathrm{r}}} \\
- \{\alpha\, (\alpha + n - 2) - \kappa\}\, \widetilde{f}_{\Sigma^{\mathrm{r}}} = 0.
\end{aligned}
\tag{2.60}
$$

This equation is uniformly elliptic on compact subsets, as $-g^{-1}(\mathrm{d}z^{\bar{c}}, \mathrm{d}z^{\bar{d}})_{\Sigma^{\mathrm{r}}}$ is continuous and positive definite (see [18, Ch. 3]). Moreover, one of our hypothesis is $-\{\alpha\, (\alpha + n - 2) - \kappa\} \leq 0$, thus, the classical maximum principles [18, §3.1 and 3.2] will be applicable to its solutions. In particular, a standard application of the weak maximum principle [18, Th. 3.3] shows that $\widetilde{f}_{\Sigma^{\mathrm{r}}}$ and $f_0 = 0$ are equal if $\widetilde{f}_{\Sigma^{\mathrm{r}}}$ is continuous and vanishes on ∂D. These conditions follow from (2.57) when $\alpha < 2$ (recall that $F^{2-\alpha}$ vanishes on ∂A_p and f is smooth therein by hypothesis), while if $\alpha = 2$, (2.57) still implies that $\widetilde{f}$ is smooth on $\overline{A_p}$ and the result follows from (2.59) using the hypothesis of strict inequality for κ.

Case (B). Now, the coordinates $(z^1, \dots, z^{n-1})$ cannot cover the whole indicatrix Σ^{r} (which is compact) but, if $\widetilde{f}_{\Sigma^{\mathrm{r}}}$ is not constant, we can take them around any maximum $v_m \in \Sigma^{\mathrm{r}}$ where $\widetilde{f}_{\Sigma^{\mathrm{r}}}$ is not locally equal to $c_m := \widetilde{f}_{\Sigma^{\mathrm{r}}}(v_m)$. Reasoning as in the case (A) , one arrives at (2.59) and (say, after an overall change of sign) strict uniform ellipticity follows from the new hypothesis on κ. If $c_m \geq 0$, a direct application of the strong maximum principle [18, Th. 3.5] shows that $\widetilde{f}_{\Sigma r}$ has to be locally equal to c_m. So, $\widetilde{f}_{\Sigma r}$ must be constant and, by (2.59), equal to 0. If $c_m \leq 0$, reason with $-\widetilde{f}_{\Sigma r}$.

In Th. 2.3, we obtained a general uniqueness result for solutions of the torsion-free affine equations (2.40), (2.41) under the hypothesis of fiberwise-analyticity. As a first application of Lemma 5.1, this hypothesis is dropped in some particular cases.

Theorem 2.5. *Assume that L is Lorentz-Finsler and $n \geq 3$. If $\mathrm{N} = \mathrm{N}^L + \dot{\partial}\mathcal{Z} \in \mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A) \cap \mathrm{Sol}_L(\overline{A})$ and*

$$2\,\mathrm{C}_{a\cdot i}\,\mathcal{Z}^a + \mathrm{C}_a\,\mathcal{Z}^a_{\cdot i} + \mathrm{Lan}_i = 0, \tag{2.61}$$

then actually $\mathcal{Z} = 0$ and thus $\mathrm{Lan}_i = 0$.

Proof. Using the notation (2.39), the hypothesis (2.61) means

$$\mathcal{K}_i^{\mathcal{Z}} = \frac{2}{n+2}\,\mathrm{Lan}_i.$$

Thus, the equations (2.40), (2.41), (2.42) read, respectively,

$$\mathcal{Z}^i = 2\sigma^{\mathcal{Z}}\,y^i - L\,g^{ia}\,\sigma^{\mathcal{Z}}_{\cdot a}, \tag{2.62}$$

$$(n+2)\,\sigma^{\mathcal{Z}} = -\mathrm{Lan}_i, \tag{2.63}$$

$$(n-2)\,\sigma^{\mathcal{Z}} - L\,g^{ab}\,\sigma^{\mathcal{Z}}_{\cdot a\cdot b} = 0. \tag{2.64}$$

The function $f := \sigma^{\mathcal{Z}}_p$, which is smooth on $\overline{A_p}$ by (2.63), solves (2.56) on A_p with parameters $\alpha = 1$, $\kappa = n-2$ (by (2.64)). Applying Lem. 2.6 (recall $\kappa \neq 0$ as $n \geq 3$) yields $\sigma^{\mathcal{Z}}_p = 0$, for all $p \in M$. Thus, (2.62) yields $\mathcal{Z} = 0$. Finally, recall Rem. 2.11: N^L being in $\mathrm{Sol}_L(A)$ implies $\mathrm{Lan}_i = 0$.

Corollary 2.3. If L is Lorentz-Finsler with vanishing mean Cartan tensor ($\mathrm{C}_i = 0$) and $n \geq 3$, then its associated nonlinear connection N^L is the unique element of $\mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A) \cap \mathrm{Sol}_L(\overline{A})$.

Proof. As the mean Landsberg tensor can be written as a derivative of C_i (see [49, (6.37)]), the hypothesis (2.61) follows trivially and Th. 2.5 applies.

Remark 2.23. In [39, Remark 5.3], the relevance of the condition $\mathrm{C}_i = 0$ in the study of alternative Finslerian Einstein equations is stressed, namely, it guarantees the symmetry of certain Ricci tensors. In the positive definite case, Deicke's Theorem [5, Th. 14.4.1] establishes that the only Finsler metrics with $\mathrm{C}_i = 0$ are the Riemannian ones. The Berwald-Moor metrics [4] are improper Lorentz-Finsler counterexamples, as they cannot be properly extended to ∂A; as far as we know, no proper Lorentz-Finsler counterexamples appears in the literature.

In Lem. 2.6, the case (B) provided a positive definite version of the case (A). However, it did so for $\kappa > \alpha(\alpha + n - 2)$, which is the opposite inequality arising in the proof Th. 2.5; this prevents a result for Finsler instead of Lorentz-Finsler metrics. However, we are going to prove that the uniqueness of solutions in the Riemannian case can be obtained by means of a further study of the Laplacian of f, that is, the solutions in the Riemannian Palatini approach agree with those in the Finslerian Palatini one. For the following result, recall that in the case of Finsler metrics, $A = \mathrm{T}M \setminus \mathbf{0}$; hence, all the corresponding solutions of the affine equation (2.19) are trivially proper.

Theorem 2.6. *Assume that L is (positive definite) Riemannian and $n \geq 3$. Then* N^L *is the only element of* $\mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A) = \mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(\mathrm{T}M \setminus \mathbf{0})$.

Proof. Let $\mathrm{N} = \mathrm{N}^L + \dot{\partial}\mathcal{Z} \in \mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(\mathrm{T}M \setminus \mathbf{0})$. By using, in Lem. 2.3, the vanishing of the mean Cartan and Landsberg tensors, $\mathcal{Z}$ solves

$$\mathcal{Z}^i = 2\sigma^{\mathcal{Z}}\, y^i - L\, g^{ia}\, \sigma^{\mathcal{Z}}_{\cdot a}, \tag{2.65}$$

$$(n-2)\,\sigma^{\mathcal{Z}} - L\, g^{ab}\, \sigma^{\mathcal{Z}}_{\cdot a \cdot b} = 0. \tag{2.66}$$

When rewritting (2.66) in terms of

$$\widetilde{\sigma^{\mathcal{Z}}} = \frac{\sigma^{\mathcal{Z}}}{F} \in \mathrm{h}^0\mathcal{F}(\mathrm{T}M \setminus \mathbf{0})$$

(put $\alpha = 1$ and $\kappa = n - 2$ in (2.58)), one gets

$$L\, g^{ab}\, \widetilde{\sigma^{\mathcal{Z}}}_{\cdot a \cdot b} + \widetilde{\sigma^{\mathcal{Z}}} = 0, \tag{2.67}$$

which in turn can be restricted to each $\mathrm{T}_pM \setminus 0$. This time, g_p is just a positive definite scalar product on T_pM, its indicatrix being a round sphere: $\Sigma^{F_p} = \left\{v \in \mathrm{T}_pM \setminus 0 : L(v) = 1\right\} \equiv \mathbb{S}^{n-1}$. Thus, $g^{ab}\,\partial^2_{y^a y^b}$ is the Laplacian of the Euclidean $\mathbb{R}^n$ and, as $\widetilde{\sigma^{\mathcal{Z}}}_p$ is 0-homogeneous, it is well-known [50, Prop. 22.1] that

$$\left(g^{ab}\, \widetilde{\sigma^{\mathcal{Z}}}_{\cdot a \cdot b}\right)_{\mathbb{S}^{n-1}} = \Delta_{\mathbb{S}^{n-1}} \widetilde{\sigma^{\mathcal{Z}}}.$$

Because of this, (2.67) restricted to $\mathbb{S}^{n-1}$ becomes

$$-\Delta_{\mathbb{S}^{n-1}} \widetilde{\sigma^{\mathcal{Z}}} = \widetilde{\sigma^{\mathcal{Z}}}. \tag{2.68}$$

The set of eigenvalues of $-\Delta_{\mathbb{S}^{n-1}}$ is

$$\mathrm{Spec}(-\Delta_{\mathbb{S}^{n-1}}) = \{\nu\,(\nu + n - 2) : \ \nu \in \mathbb{N} \cup \{0\}\}$$

([50, Th. 22.1], we follow the conventions of this reference). As $n \geq 3$, then $1 \notin \mathrm{Spec}(-\Delta_{\mathbb{S}^{n-1}})$ and $\widetilde{\sigma^{\mathcal{Z}}} = 0$, as it solves (2.68). Thus, $\mathcal{Z} = 0$ from (2.65), as required. ∎

The following last consequence of Lem. 2.6 is relevant for the consistency of the metric equation (2.20).

Theorem 2.7. *Let L be Lorentz-Finsler and* N *any nonlinear connection (non-necessarily in* $\mathrm{Sol}_L(A)$*) which extends smoothly to* $\overline{A}$*. If the Ricci scalar* Ric *of* N *satisfies, for some* $\kappa < 2n$,

$$\kappa \,\mathrm{Ric} - L\, g^{ab}\, \mathrm{Ric}_{\cdot a \cdot b} = 0,$$

then actually $\mathrm{Ric} = 0$*. In particular, if* $n \geq 3$ *then the variational metric eqn.* (2.20), $(n+2)\,\mathrm{Ric} - L\, g^{ab}\, \mathrm{Ric}_{\cdot a \cdot b} = 0$*, implies* $\mathrm{Ric} = 0$.

Proof. $f := \mathrm{Ric}_p$ is α-homogeneous for $\alpha = 2$, smooth on $\overline{A_p}$ (due to the hypothesis on N) and solves (2.56) on A_p for κ. Thus, Lem. 2.6 applies for the chosen κ. ∎

Remark 2.24. (A) This result can be applied to pairs (N, L) which solve the variational equations. Recall that the Ricci scalar is equal for the solutions obtained starting at one N and making an $\mathcal{A}$-translation in the space of solutions $\mathrm{N} + \mathcal{A} \otimes \mathbb{C}$ (Prop. 2.8). This ensures the consistency of such solutions as in the classical Palatini case [7]. In particular, when N^L is a solution (i.e., when $\mathrm{Lan}_i = 0$), Ric becomes Ric^L.

(B) In any dimension $n \geq 3$, the classical vacuum Einstein equation for pseudo-Riemannian metrics $L(x, y) = g_{ab}(x)\, y^a\, y^b$ can be expressed as

$$4\,\mathrm{Ric}^L - L\, g^{ab}\, \mathrm{Ric}^L_{\cdot a \cdot b} = 0$$

(contract both of its indices with $\mathbb{C}$, and use (2.11) and (2.12) with the Levi-Civita connection). Thus, when interpreted as an equation for pseudo-Finsler metrics, this one would be the most direct extension of the Einstein equation. Notice that Th. 2.7 also applies to it, so for any proper Lorentz-Finsler metric it is equivalent to $\mathrm{Ric}^L = 0$ as well. From a technical viewpoint, it is quite remarkable that this is a nontrivial Finslerian result which requires Lorentzian signature, while in the classical pseudo-Riemannian case an elementary algebraic argument suffices in any signature.

(C) The variational equation studied by Hohmann, Pfeifer, Voicu and Wohlfarth [21, 46] agrees with our metric equation when $\mathrm{Lan}_i = 0$ (in any dimension)[20]. The discrepancy when $\mathrm{Lan}_i \neq 0$ may be interesting, at least from a mathematical viewpoint. As we have seen, in this case no solution N of our affine equation can have the same pregeodesics as N^L and it is not clear the role of N^L then. However, no matter

[20]Formulas (77) and (79) in [21] are immediately generalized from dimension 4, yielding the terms $-(n+2)\,\mathrm{Ric}^L$ and $L\, g^{ab}\, \mathrm{Ric}^L_{\cdot a \cdot b}$ respectively, while it can be checked that (78) there still yields only terms that vanish when the mean Landsberg tensor does.

the affine solution one chooses, our metric equation is the vanishing of its Ric. For the cited authors, however, it is a more complicated one which involves L and Lan.

(D) Th. 2.7 also complements previous results obtained for the metric nonlinear connection of certain Berwald metrics [17, Th. 3], [20, Prop. 4]. The conclusion of our theorem holds even though the metrics there cannot be extended to ∂A as properly Lorentz-Finsler.

(E) Previous comments strongly support that the natural generalization of Einstein vacuum equations must be the vanishing of the Ricci scalar for some solution N of the affine equation. When $\mathrm{Lan}_i = 0$, N^L would be a distinguished solution which, in fact, it would be the unique symmetric one under the mild conditions studied before. Let us point that $\mathrm{Ric}^L = 0$ as a vacuum equation was first proposed by Rutz [48] and has been further studied in some cases [38].

2.5.3 Recovery of the classical solutions

Finally, let us restrict our attention to pseudo-Riemannian metrics and affine connections (or, equivalently, linear N's, $\mathrm{N}_i^k(x, y) = \Gamma_{ib}^k(x)\, y^b$). Then the solutions of the Finslerian metric-affine formalism (described by (2.19), (2.20)) are exactly those of the classical one. This fact will be proved directly, even though we will give some hints to regard it as a corollary of our results in §2.5.1 and §2.5.2, which go way beyond the classical case. Keep in mind that the isotropic Γ's solving the classical metric-affine formalism [7, (17)] can be identified with their underlying linear N's, so in Def. 2.13 we refer as *classical solutions* to those $\mathrm{N} = \mathrm{N}^L + \mathcal{A} \otimes \mathbb{C}$ with L pseudo-Riemannian and $\mathcal{A}$ isotropic.

Theorem 2.8. *Assume that L is pseudo-Riemannian,* N *is linear and $n \geq 3$. Then one has* $\mathrm{N} \in \mathrm{Sol}_L(A)$ *if and only if*

$$\mathrm{N} = \mathrm{N}^L + \mathcal{A} \otimes \mathbb{C}$$

for some isotropic $\mathcal{A}$. For these connections, $\mathrm{Ric} = \mathrm{Ric}^L$ *and* (N, L) *solves also the metric equation* (2.20) *if and only if L solves the classical (vacuum) Einstein equation*

$$\mathrm{Ric}^L = 0.$$

Proof. L being pseudo-Riemannian, $\mathrm{Lan}_i = 0$, so $\mathrm{N}^L \in \mathrm{Sol}_L(A)$ (Rem. 2.11) and $\mathrm{N}^L + \mathcal{A} \otimes \mathbb{C} \in \mathrm{Sol}_L(A)$ (Lem. 2.2). Let us establish that these, with $\mathcal{A}$ isotropic, are all the linear elements of $\mathrm{Sol}_L(A)$.

Again because L is pseudo-Riemannian, N^L is linear ($\left(\mathrm{N}^L\right)_i^k(x,y) = (\Gamma^g)_{ib}^k(x)\,y^b$ with Γ^g the isotropic Levi-Civita connection), and because $\mathrm{N} = \mathrm{N}^L + \dot\partial\mathcal{Z} + \mathcal{A}\otimes\mathbb{C} \in \mathrm{Sol}_L(A)$ is assumed linear too, $\mathcal{A}$ must be isotropic. Indeed, from the definition it is clear that the torsion of the linear N is isotropic, and from (2.31),

$$
\begin{aligned}
2\,(n-1)\,\mathcal{A}_i\,y^k &= (n-1)\,\mathrm{Tor}_{ib}^k\,y^b - \mathrm{Tor}_{ai}^a\,y^k - \mathrm{Tor}_{ab}^a\,y^b\,\delta_i^k,\\
2\,(n-1)\left(\mathcal{A}_{i\cdot j}\,y^k + \mathcal{A}_i\,\delta_j^k\right) &= (n-1)\,\mathrm{Tor}_{ij}^k - \mathrm{Tor}_{ai}^a\,\delta_j^k - \mathrm{Tor}_{aj}^a\,\delta_i^k,\\
2n\,(n-1)\,\mathcal{A}_i &= (n-1)\,\mathrm{Tor}_{ia}^a - n\,\mathrm{Tor}_{ai}^a - \mathrm{Tor}_{ai}^a = -2n\,\mathrm{Tor}_{ai}^a
\end{aligned}
$$

(we vertically differentiated, contracted the indices k with j, and used the 0-homogeneity of $\mathcal{A}$ and the antisymmetry of Tor).

As $\mathcal{A}$ is isotropic, it follows that $\mathcal{Z}$ is quadratic: $\mathcal{Z}^i(x,y) = \boldsymbol{\Phi}_{ab}^i(x)\,y^a\,y^b/2$ for some isotropic and symmetric $(1,2)$ tensor $\boldsymbol{\Phi}$. Indeed, formula (2.29) for the underlying spray G of $\mathrm{N} = \mathrm{N}^L + \dot\partial\mathcal{Z} + \mathcal{A}\otimes\mathbb{C}$ can be written as

$$
\frac{1}{2}\,\Gamma_{ab}^i(x)\,y^a\,y^b = \frac{1}{2}\,(\Gamma^g)_{ab}^i\,(x)\,y^a\,y^b + \mathcal{Z}^i(x,y) + \frac{1}{2}\,\mathcal{A}_a(x)\,\delta_b^i\,y^a\,y^b
$$

and the symmetric part of $\Gamma_{jk}^i - (\Gamma^g)_{jk}^i - \mathcal{A}_j\,\delta_k^i$ is an isotropic tensor.

Now, recalling that $\mathrm{N}^L + \dot\partial\mathcal{Z} \in \mathrm{Sol}_L^{\mathcal{S}\mathrm{ym}}(A)$, one has two options. In a direct manner, using that $\mathcal{Z}$ solves (2.40), (2.41), (2.42) and the vanishing of the mean Cartan and Landsberg tensors,

$$
\begin{aligned}
(n+2)\,\sigma^{\mathcal{Z}} &= \mathcal{Z}_{\cdot a}^a = \boldsymbol{\Phi}_{ab}^a\,y^b,\\
0 &= (n-2)\,\sigma^{\mathcal{Z}} - L\,g^{ab}\,\sigma_{\cdot a\cdot b}^{\mathcal{Z}}\\
&= (n-2)\,\sigma^{\mathcal{Z}} - L\,g^{ab}\left(\frac{1}{n+2}\,\boldsymbol{\Phi}_{cd}^c\,y^d\right)_{\cdot a\cdot b}\\
&= (n-2)\,\sigma^{\mathcal{Z}},\\
\mathcal{Z}^i &= 2\,\sigma^{\mathcal{Z}}\,y^i - L\,g^{ia}\,\sigma_{\cdot a\cdot b}^{\mathcal{Z}} = 0
\end{aligned}
$$

(as $n \geq 3$). Alternatively, one can use that, as N^L is linear and $\mathcal{Z}$ quadratic, also $\mathrm{N}^L + \dot\partial\mathcal{Z} \in \mathrm{Sol}_L(\overline{A})$ and $\mathcal{Z}$ is fiberwise analytic on $\overline{A}$, so either Th. 2.3 or Th. 2.6 (depending on the signature and again becuase $n \geq 3$) can be applied [21] to conclude that $\mathrm{N}^L + \dot\partial\mathcal{Z} = \mathrm{N}^L$.

[21] There would be the technical issue that in non-definite signature, one can regard a pseudo-Riemannian g as a proper pseudo-Finsler L only locally in general. Namely, under Def. 2.8 one chooses a certain connected A_p at each point, but the usual pseudo-Riemannian setting includes cases (i.e. non time-orientable Lorentzian metrics) where such a choice cannot carried out. Anyway, the former approach of direct computations avoids this issue altogether.

We have proven that if $\mathrm{N} \in \mathrm{Sol}_L(A)$, then $\mathrm{N} = \mathrm{N}^L + \mathcal{A} \otimes \mathbb{C}$ with $\mathcal{A}$ isotropic. As this N shares fiber in $\mathrm{Sol}_L(A)$ with N^L, Cor. 2.1 iv) gives $\mathrm{Ric} = \mathrm{Ric}^L$. The metric equation (2.20) for (N, L) thus reads

$$(n+2)\,\mathrm{Ric}^L - L\, g^{ab}\, \mathrm{Ric}^L_{\cdot a \cdot b} = 0. \tag{2.69}$$

However, once again as L is pseudo-Riemannian, Ric^L is quadratic too. Indeed, $\mathrm{Ric}^L = \Psi_{ab}\, y^a\, y^b/2$ with $\Psi/2$ being the (isotropic and symmetric) classical Ricci tensor of L (use (2.11) with the Levi-Civita connection). Thus, (2.69) becomes

$$\begin{aligned} 0 &= \frac{n+2}{2}\,\Psi_{ab}(x)\, y^a\, y^b - L(x,y)\, g^{ab}(x) \left(\frac{1}{2}\,\Psi_{cd}(x)\, y^c\, y^d\right)_{\cdot a \cdot b} \\ &= \frac{n+2}{2}\,\Psi_{ab}(x)\, y^a\, y^b - L(x,y)\, g^{ab}(x)\,\Psi_{ab}(x) \\ &= \left(\frac{n+2}{2}\,\Psi_{cd}(x) - g^{ab}(x)\,\Psi_{ab}(x)\, g_{cd}(x)\right) y^c\, y^d, \end{aligned}$$

which is clearly equivalent to

$$\frac{n+2}{2}\,\Psi_{ij} - g^{ab}\,\Psi_{ab}\, g_{ij} = 0.$$

By taking metric trace (and once again as $n \geq 3$), one sees that this one is equivalent to $\Psi = 0$, but this is also true for the classical Einstein equation $\mathrm{Ric}^L = 0$. This completes the proof. ∎

Remark 2.25. As a last remark, recall that, apart from the classical solutions, a pseudo-Riemannian L admits also the formally classical ones, $\mathrm{N} = \mathrm{N}^L + \mathcal{A} \otimes \mathbb{C}$ with $\mathcal{A}$ anisotropic and 0-homogeneous. No other proper solutions can appear in the Lorentzian and Riemannian cases, by Cor. 2.3 and Th. 2.6 resp. For general non-definite signature, Th. 2.3 establishes that there cannot appear other proper solutions with fiberwise analytic symmetric part $\Pi^{\mathcal{S}\mathrm{ym}}(\mathrm{N})$.

Appendix A: Proof of Prop. 2.5 (Divergence formulas)

In order to prove (2.17), we will lift the anisotropic connection[22] $\dot\partial\mathrm{N}$ to a linear (Koszul) connection $\widehat{\nabla}^{\mathrm{N}}$ for $\mathrm{T}A \longrightarrow A$. For this, recall [28, Th. 3], [24, §4.4], and the N-horizontal

[22] This construction works for any anisotropic connection Γ in place of $\dot\partial\mathrm{N}$. In particular, taking Γ as the Levi-Civita–Chern anisotropic connection of the metric [24, 26, 28, 49], this justifies regarding Chern-Rund's as a connection for $\mathrm{T}A \longrightarrow A$.

and vertical isomorphisms (2.7) and (2.1) respectively. One can regard the anisotropic $\dot{\partial}$N as a vertically trivial linear connection for $\mathrm{V}A \longrightarrow A$ as in [28, Th. 3], resulting in

$$\widehat{\nabla}^{\mathrm{N}}_{X^{\mathrm{H}}}\left(Y^{\mathrm{V}}\right) := \left(\nabla^{\mathrm{N}}_{X} Y\right)^{\mathrm{V}}$$

for $X, Y \in \mathcal{T}^1_0(M_A)$. Imposing also

$$\widehat{\nabla}^{\mathrm{N}}_{X^{\mathrm{H}}}\left(Y^{\mathrm{H}}\right) := \left(\nabla^{\mathrm{N}}_{X} Y\right)^{\mathrm{H}}$$

and maintaining the vertical triviality, $\widehat{\nabla}^{\mathrm{N}}$ extends unequivocally (by linearity) to act on any vector fields on A. Then, by construction,

$$\widehat{\nabla}^{\mathrm{N}}_{\delta_i}\delta_j = \mathrm{N}^a_{i\cdot j}\,\delta_a, \qquad \widehat{\nabla}^{\mathrm{N}}_{\delta_i}\dot{\partial}_j = \mathrm{N}^a_{i\cdot j}\,\dot{\partial}_a, \qquad \widehat{\nabla}^{\mathrm{N}}_{\dot{\partial}_i}\delta_j = 0, \qquad \widehat{\nabla}^{\mathrm{N}}_{\dot{\partial}_i}\dot{\partial}_j = 0. \tag{2.70}$$

The *torsion of* $\widehat{\nabla}^{\mathrm{N}}$ is defined, for vector fields $\mathcal{X}$, $\mathcal{Y}$ on A, by

$$\widehat{\mathrm{Tor}}(\mathcal{X},\mathcal{Y}) = \widehat{\nabla}^{\mathrm{N}}_{\mathcal{X}}\mathcal{Y} - \widehat{\nabla}^{\mathrm{N}}_{\mathcal{Y}}\mathcal{X} - \left[\mathcal{X},\mathcal{Y}\right].$$

Along the proof, the indices $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ will run in the set $\{1, \dots, 2n\}$ (i, j, k remain in $\{1, \dots, n\}$) and the local frame $(\delta_1, \dots, \delta_n, \dot{\partial}_1, \dots, \dot{\partial}_n)$ is denoted by $(E_1, \dots, E_{2n})$ with the dual coframe $(\mathrm{d}x^1, \dots, \mathrm{d}x^n, \delta y^1, \dots, \delta y^n)$ being denoted by $(E^1, \dots, E^{2n})$. Putting, accordingly, $\widehat{\nabla}^{\mathrm{N}}_{E_{\hat{\imath}}} E_{\hat{\jmath}} =: \widehat{\Gamma}^{\hat{k}}_{\hat{\imath}\hat{\jmath}}\, E_{\hat{k}}$ and taking (2.70) into account, it follows that

$$\widehat{\Gamma}^{\hat{k}}_{\hat{\imath}\hat{\jmath}} = \begin{cases} \mathrm{N}^k_{i\cdot j} & \text{if} \quad (\hat{\imath},\hat{\jmath},\hat{k}) = (i,j,k) \quad \text{or} \quad (\hat{\imath},\hat{\jmath},\hat{k}) = (i, n+j, n+k), \\ 0 & \text{otherwise,} \end{cases} \tag{2.71}$$

while putting $\widehat{\mathrm{Tor}}(\mathcal{X},\mathcal{Y}) =: \mathcal{X}^{\hat{\imath}}\,\mathcal{Y}^{\hat{\jmath}}\,\widehat{\mathrm{Tor}}^{\hat{k}}_{\hat{\imath}\hat{\jmath}}\, E_{\hat{k}}$, it follows that

$$\widehat{\mathrm{Tor}}^{\hat{k}}_{\hat{\imath}\hat{\jmath}} = \widehat{\Gamma}^{\hat{k}}_{\hat{\imath}\hat{\jmath}} - \widehat{\Gamma}^{\hat{k}}_{\hat{\jmath}\hat{\imath}} - E^{\hat{k}}(\left[E_{\hat{\imath}}, E_{\hat{\jmath}}\right]). \tag{2.72}$$

In a standard manner, we can express any Lie derivative

$$\mathcal{L}_{\mathcal{X}}(d\mu) = \mathcal{L}_{\mathcal{X}}(d\mu)(E_1, \dots, E_{2n})\, E^1 \wedge \dots \wedge E^{2n} =: \mathcal{L}_{\mathcal{X}}(d\mu)_E\, E^1 \wedge \dots \wedge E^{2n}$$

where

$$d\mu = \frac{\left|\det g_{ij}(v)\right|}{F(v)^n}\, E^1 \wedge \dots \wedge E^{2n} =: d\mu_E\, E^1 \wedge \dots \wedge E^{2n} \tag{2.73}$$

in terms of $\widehat{\nabla}^{\mathrm{N}}$. Indeed,

$$\begin{aligned}
\mathscr{L}_{\mathscr{X}}(d\mu)_E &= \mathscr{L}_{\mathscr{X}}(d\mu(E_1, ..., E_{2n})) - \sum_{\hat{j}=1}^{2n} d\mu(E_1, ..., \mathscr{L}_{\mathscr{X}} E_{\hat{j}}, ..., E_{2n}) \\
&= \mathscr{X}(d\mu_E) - \sum_{\hat{j}=1}^{2n} d\mu(E_1, ..., \left[\mathscr{X}, E_{\hat{j}}\right], ..., E_{2n}) \\
&= \mathscr{X}(d\mu_E) - \sum_{\hat{j}=1}^{2n} d\mu(E_1, ..., \widehat{\nabla}^{\mathrm{N}}_{\mathscr{X}} E_{\hat{j}} - \widehat{\nabla}^{\mathrm{N}}_{E_{\hat{j}}} \mathscr{X} - \widehat{\mathrm{Tor}}(\mathscr{X}, E_{\hat{j}}), ..., E_{2n}) \\
&= \mathscr{X}(\log d\mu_E)\, d\mu_E - \sum_{\hat{j}=1}^{2n} d\mu(..., \mathscr{X}^{\hat{i}}\, \widehat{\Gamma}^{\hat{k}}_{\hat{i}\hat{j}}\, E_{\hat{k}}, ...) \\
&\quad + \sum_{\hat{j}=1}^{2n} d\mu(..., E_{\hat{j}}(\mathscr{X}^{\hat{i}})\, E_{\hat{i}} + \widehat{\Gamma}^{\hat{k}}_{\hat{j}\hat{i}}\, \mathscr{X}^{\hat{i}}\, E_{\hat{k}}, ...) + \sum_{\hat{j}=1}^{2n} d\mu(..., \mathscr{X}^{\hat{i}}\, \widehat{\mathrm{Tor}}^{\hat{k}}_{\hat{i}\hat{j}}\, E_{\hat{k}}, ...) \\
&= \left\{ \mathscr{X}(\log d\mu_E) - \mathscr{X}^{\hat{i}}\, \widehat{\Gamma}^{\hat{j}}_{\hat{i}\hat{j}} + \left(E_{\hat{j}}(\mathscr{X}^{\hat{j}}) + \widehat{\Gamma}^{\hat{j}}_{\hat{j}\hat{i}}\, \mathscr{X}^{\hat{i}} \right) + \mathscr{X}^{\hat{i}}\, \widehat{\mathrm{Tor}}^{\hat{j}}_{\hat{i}\hat{j}} \right\} d\mu_E,
\end{aligned}$$

so

$$\begin{aligned}
&\mathrm{div}(\mathscr{X})\, d\mu \\
&= \mathscr{L}_{\mathscr{X}}(d\mu)_E\, E^1 \wedge ... \wedge E^{2n} \\
&= \left\{ \mathscr{X}(\log d\mu_E) - \mathscr{X}^{\hat{i}}\, \widehat{\Gamma}^{\hat{j}}_{\hat{i}\hat{j}} + \left(E_{\hat{j}}(\mathscr{X}^{\hat{j}}) + \widehat{\Gamma}^{\hat{j}}_{\hat{j}\hat{i}}\, \mathscr{X}^{\hat{i}} \right) + \mathscr{X}^{\hat{i}}\, \widehat{\mathrm{Tor}}^{\hat{j}}_{\hat{i}\hat{j}} \right\} d\mu_E\, E^1 \wedge ... \wedge E^{2n} \\
&= \left\{ \mathscr{X}(\log d\mu_E) - \mathscr{X}^{\hat{i}}\, \widehat{\Gamma}^{\hat{j}}_{\hat{i}\hat{j}} + \left(E_{\hat{j}}(\mathscr{X}^{\hat{j}}) + \widehat{\Gamma}^{\hat{j}}_{\hat{j}\hat{i}}\, \mathscr{X}^{\hat{i}} \right) + \mathscr{X}^{\hat{i}}\, \widehat{\mathrm{Tor}}^{\hat{j}}_{\hat{i}\hat{j}} \right\} d\mu
\end{aligned} \tag{2.74}$$

(and note that $\left(\mathscr{X}(\log d\mu_E) - \mathscr{X}^{\hat{i}}\, \widehat{\Gamma}^{\hat{j}}_{\hat{i}\hat{j}} \right) d\mu = \widehat{\nabla}^{\mathrm{N}}_{\mathscr{X}} d\mu$).

One has the identities

$$E_i(\det g) = \det(g)\, g^{ab}\, \delta_i g_{ab} = \det(g) \left(g^{ab}\, \nabla^{\mathrm{N}}_i g_{ab} + 2\, \mathrm{N}^a_{i\cdot a} \right)$$

(using Jacobi's formula for the derivative of a determinant and (2.8)),

$$E_i(F) = \frac{\mathrm{sgn}(L)}{2F}\, \delta_i L = \frac{\mathrm{sgn}(L)}{2F}\, \nabla^{\mathrm{N}}_i L = \frac{\mathrm{sgn}(L)}{2F}\, \nabla^{\mathrm{N}}_i g_{ab}\, y^a\, y^b$$

(using $F = \sqrt{|L|}$, $L = g_{ab}\, y^a\, y^b$ and $\nabla^{\mathrm{N}}_i y^j = 0$),

$$E_{n+i}(\det g) = 2 \det(g)\, \mathrm{C}_i,$$

(using again Jacobi and the definition of the mean Cartan tensor), and

$$E_{n+i}(F) = \frac{\operatorname{sgn}(L)}{F}\, y_i$$

(using again $F = \sqrt{|L|}$ and $L_{\cdot i} = 2\, y_i$). From them and (2.73), it follows that

$$E_i(\log d\mu_E) = \frac{E_i(d\mu_E)}{d\mu_E} = \left(g^{ab} - \frac{n}{2}\frac{1}{L}\, y^a\, y^b\right) \nabla^{\mathrm{N}}_i g_{ab} + 2\, \mathrm{N}^a_{i\cdot a}, \tag{2.75}$$

$$E_{n+i}(\log d\mu_E) = \frac{E_{n+i}(d\mu_E)}{d\mu_E} = 2\, \mathrm{C}_i - n\, \frac{y_i}{L}. \tag{2.76}$$

We take $\mathcal{X} = X^{\mathrm{H}} = X^a\, E_a$. Using (2.75), (2.71), (2.72) and the commutation formulas (2.10), we have

$$\mathcal{X}(\log d\mu_E) = X^c \left(g^{ab} - \frac{n}{2}\frac{1}{L}\, y^a\, y^b\right) \nabla^{\mathrm{N}}_c g_{ab} + 2\, X^c\, \mathrm{N}^a_{c\cdot a},$$

$$-\mathcal{X}^{\hat\imath}\, \widehat\Gamma^{\hat\jmath}_{\hat\imath\hat\jmath} = -\mathcal{X}^{\hat\imath}\, \widehat\Gamma^{a}_{\hat\imath a} - \mathcal{X}^{\hat\imath}\, \widehat\Gamma^{n+a}_{\hat\imath\, n+a} = -X^c\, \mathrm{N}^a_{c\cdot a} - X^c\, \mathrm{N}^a_{c\cdot a} = -2\, X^c\, \mathrm{N}^a_{c\cdot a},$$

$$\begin{aligned} E_{\hat\jmath}(\mathcal{X}^{\hat\jmath}) + \widehat\Gamma^{\hat\jmath}_{\hat\jmath\hat\imath}\, \mathcal{X}^{\hat\imath} &= E_a(\mathcal{X}^a) + E_{n+a}(\mathcal{X}^{n+a}) + \widehat\Gamma^{a}_{a\hat\imath}\, \mathcal{X}^{\hat\imath} + \widehat\Gamma^{n+a}_{n+a\,\hat\imath}\, \mathcal{X}^{\hat\imath} \\ &= \delta_a X^a + \mathrm{N}^a_{a\cdot c}\, X^c \\ &= \nabla^{\mathrm{N}}_a X^a, \end{aligned}$$

$$\begin{aligned} \mathcal{X}^{\hat\imath}\, \widehat{\mathrm{Tor}}^{\hat\imath}_{\hat\imath\hat\jmath} &= \mathcal{X}^{\hat\imath} \left(\widehat\Gamma^{\hat\jmath}_{\hat\imath\hat\jmath} - \widehat\Gamma^{\hat\jmath}_{\hat\jmath\hat\imath} - E^{\hat\jmath}([E_{\hat\imath}, E_{\hat\jmath}])\right) \\ &= \mathcal{X}^{\hat\imath} \left(\widehat\Gamma^{a}_{\hat\imath a} + \widehat\Gamma^{n+a}_{\hat\imath\, n+a} - \widehat\Gamma^{a}_{a\hat\imath} - \widehat\Gamma^{n+a}_{n+a\,\hat\imath} - E^a([E_{\hat\imath}, E_a]) - E^{n+a}([E_{\hat\imath}, E_{n+a}])\right) \\ &= X^c \left(\mathrm{N}^a_{c\cdot a} + \mathrm{N}^a_{c\cdot a} - \mathrm{N}^a_{a\cdot c} - \mathrm{d}x^a([\delta_c, \delta_a]) - \delta y^a([\delta_c, \dot\partial_a])\right) \\ &= X^c \left(2\, \mathrm{N}^a_{c\cdot a} - \mathrm{N}^a_{a\cdot c} - \mathrm{N}^a_{c\cdot a}\right) \\ &= X^c\, \mathrm{Tor}^a_{ca}. \end{aligned}$$

Putting these together, (2.74) proves (2.17).

Now we take $\mathcal{X} = X^{\mathrm{V}} = X^a\, E_{n+a}$. Using (2.76), and again (2.71), (2.72) and the commutation formulas (2.10), we have

$$\mathcal{X}(\log d\mu_E) = \left(2\, \mathrm{C}_c - n\, \frac{y_c}{L}\right) X^c,$$

$$-\mathcal{X}^{\hat\imath}\, \widehat\Gamma^{\hat\jmath}_{\hat\imath\hat\jmath} = -\mathcal{X}^{\hat\imath}\, \widehat\Gamma^{a}_{\hat\imath a} - \mathcal{X}^{\hat\imath}\, \widehat\Gamma^{n+a}_{\hat\imath\, n+a} = -X^c\, \widehat\Gamma^{a}_{n+c\, a} - X^c\, \widehat\Gamma^{n+a}_{n+c\, n+a} = 0,$$

$$E_{\hat\jmath}(\mathcal{X}^{\hat\jmath}) + \widehat\Gamma^{\hat\jmath}_{\hat\jmath\hat\imath}\, \mathcal{X}^{\hat\imath} = E_a(\mathcal{X}^a) + E_{n+a}(\mathcal{X}^{n+a}) + \widehat\Gamma^{a}_{a\hat\imath}\, \mathcal{X}^{\hat\imath} + \widehat\Gamma^{n+a}_{n+a\,\hat\imath}\, \mathcal{X}^{\hat\imath} = \dot\partial_a X^a = X^a_{\cdot a},$$

$$
\begin{aligned}
\mathcal{X}^{\hat\imath}\, \widehat{\mathrm{Tor}}^{\hat\imath}_{\hat\imath\hat\jmath} =& \mathcal{X}^{\hat\imath}\left(\widehat{\Gamma}^{\hat\jmath}_{\hat\imath\hat\jmath} - \widehat{\Gamma}^{\hat\jmath}_{\hat\jmath\hat\imath} - E^{\hat\jmath}([E_{\hat\imath}, E_{\hat\jmath}])\right)\\
=& \mathcal{X}^{\hat\imath}\left(\widehat{\Gamma}^{a}_{\hat\imath a} + \widehat{\Gamma}^{n+a}_{\hat\imath\, n+a} - \widehat{\Gamma}^{a}_{a\hat\imath} - \widehat{\Gamma}^{n+a}_{n+a\,\hat\imath} - E^{a}([E_{\hat\imath}, E_{a}]) - E^{n+a}([E_{\hat\imath}, E_{n+a}])\right)\\
=& X^{c}\left(-\mathrm{d}x^{a}([\dot\partial_c, \delta_a]) - \delta y^{a}([\dot\partial_c, \dot\partial_a])\right)\\
=& \, 0.
\end{aligned}
$$

Putting these together, (2.74) proves (2.18),[23] and yields the proposition.

Appendix B: Proof of Th. 2.1 (Affine equation)

When varying N by $\mathrm{N}(\tau)$, taking Rem. 2.9 into account, it is immediate to check that

$$
\begin{aligned}
\left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \mathcal{S}^{D}[\mathrm{N}(\tau), L] &= \int_D \underline{\left.\frac{\partial}{\partial\tau}\right|_{\tau=0} L^{-1}\,\mathrm{Ric}(\tau)\, d\mu}\\
&= \int_D \underline{L^{-1} \left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \mathrm{Ric}(\tau)\, d\mu}.
\end{aligned}
\tag{2.77}
$$

Using (2.9) and (2.6),

$$
\mathrm{Ric}(\tau) = \delta_b(\tau)\mathrm{N}^c_a(\tau)\left(\delta^b_c\, y^a - y^b\, \delta^a_c\right), \qquad \delta_j(\tau)\mathrm{N}^k_i(\tau) = \partial_j \mathrm{N}^k_i(\tau) - \mathrm{N}^d_j(\tau)\,\dot\partial_d \mathrm{N}^k_i(\tau);
$$

here, δ^a_c is Kronecker's, in contrast to $\delta_j(\tau)$, which comes from $\mathrm{N}(\tau)$.

Let us express the derivative of $\delta_j(\tau)\mathrm{N}^k_i(\tau)$ in terms of ∇^{N} and $\mathrm{Tor}^k_{ib}\, y^b = \left(\mathrm{N}^k_{i\cdot b} - \mathrm{N}^k_{b\cdot i}\right) y^b = \mathrm{N}^k_{a\cdot b}\left(\delta^a_i\, y^b - y^a\, \delta^b_i\right)$. We do this by commuting $\partial_\tau|_0$ with ∂_j and $\dot\partial_d$,

$$
\left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \left\{\delta_j(\tau)\mathrm{N}^k_i(\tau)\right\} = \partial_j \left(\mathrm{N}'\right)^k_i - \left(\mathrm{N}'\right)^d_j \mathrm{N}^k_{i\cdot d} - \mathrm{N}^d_j \left(\mathrm{N}'\right)^k_{i\cdot d} = \delta_j \left(\mathrm{N}'\right)^k_i - \left(\mathrm{N}'\right)^d_j \mathrm{N}^k_{i\cdot d}
$$

and then adding and substracting $-\mathrm{N}^d_{j\cdot i}\left(\mathrm{N}'\right)^k_d + \mathrm{N}^k_{j\cdot d}\left(\mathrm{N}'\right)^d_i$ so as to obtain the same terms as in (2.8),

$$
\left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \left\{\delta_j(\tau)\mathrm{N}^k_i(\tau)\right\} = \nabla^{\mathrm{N}}_j \left(\mathrm{N}'\right)^k_i + \mathrm{N}^d_{j\cdot i}\left(\mathrm{N}'\right)^k_d - \mathrm{N}^k_{j\cdot d}\left(\mathrm{N}'\right)^d_i - \mathrm{N}^k_{i\cdot d}\left(\mathrm{N}'\right)^d_j.
$$

[23] Notice, however, that (2.18) is a purely vertical identity independent of N. So, it could also have been proven by direct computation without any connection for $\mathrm{T}A \longrightarrow A$.

With this,

$$
\begin{aligned}
&L^{-1} \left.\frac{\partial}{\partial \tau}\right|_{\tau=0} \mathrm{Ric}(\tau) \\
&= L^{-1}\left\{\nabla^{\mathrm{N}}_{b}\,(\mathrm{N}')^{c}_{a} + \mathrm{N}^{d}_{b\cdot a}\,(\mathrm{N}')^{c}_{d} - \mathrm{N}^{c}_{b\cdot d}\,(\mathrm{N}')^{d}_{a} - \mathrm{N}^{c}_{a\cdot d}\,(\mathrm{N}')^{d}_{b}\right\}\left(\delta^{b}_{c}\,y^{a} - y^{b}\,\delta^{a}_{c}\right) \\
&= L^{-1}\left\{\nabla^{\mathrm{N}}_{b}\,(\mathrm{N}')^{c}_{a}\left(\delta^{b}_{c}\,y^{a} - y^{b}\,\delta^{a}_{c}\right) + \mathrm{N}^{d}_{b\cdot a}\left(\delta^{b}_{c}\,y^{a} - y^{b}\,\delta^{a}_{c}\right)(\mathrm{N}')^{c}_{d}\right\} \\
&= -\left\{L^{-1}\,\nabla^{\mathrm{N}}_{c}\,(\mathrm{N}')^{d}_{d}\,y^{c} - L^{-1}\,\nabla^{\mathrm{N}}_{c}\,(\mathrm{N}')^{c}_{d}\,y^{d} - L^{-1}\,\mathrm{Tor}^{d}_{ca}\,y^{a}\,(\mathrm{N}')^{c}_{d}\right\}.
\end{aligned}
\tag{2.78}
$$

Recall that, by Prop. 2.3, $\nabla^{\mathrm{N}}_{i} L = \nabla^{\mathrm{N}}_{i} g_{ab}\, y^{a}\, y^{b}$. Calling $X := L^{-1}\,(\mathrm{N}')^{d}_{d}\, y^{c}\,\partial_{c} \in \mathrm{h}^{0}\mathcal{T}^{1}_{0}(M_{A})$ and using (2.17),

$$
\begin{aligned}
&L^{-1}\,\nabla^{\mathrm{N}}_{c}\,(\mathrm{N}')^{d}_{d}\,y^{c} \\
&= \nabla^{\mathrm{N}}_{c}(L^{-1}\,(\mathrm{N}')^{d}_{d}\,y^{c}) - \nabla^{\mathrm{N}}_{c}(L^{-1})\,(\mathrm{N}')^{d}_{d}\,y^{c} \\
&= \mathrm{div}(X^{\mathrm{H}}) - L^{-1}\left\{\left(g^{ab} - \frac{n}{2}\frac{1}{L}\,y^{a}\,y^{b}\right)\nabla^{\mathrm{N}}_{c} g_{ab} + \mathrm{Tor}^{a}_{ca}\right\} y^{c}\,(\mathrm{N}')^{d}_{d} \\
&\quad + L^{-2}\,y^{c}\,\nabla^{\mathrm{N}}_{c} g_{ab}\,y^{a}\,y^{b}\,(\mathrm{N}')^{d}_{d} \\
&= \mathrm{div}(X^{\mathrm{H}}) - L^{-1}\left\{\left(g^{ab} - \frac{n+2}{2}\frac{1}{L}\,y^{a}\,y^{b}\right)\nabla^{\mathrm{N}}_{c} g_{ab} + \mathrm{Tor}^{a}_{ca}\right\} y^{c}\,(\mathrm{N}')^{d}_{d}.
\end{aligned}
\tag{2.79}
$$

Analogously, calling $Y := L^{-1}\,(\mathrm{N}')^{c}_{d}\, y^{d}\,\partial_{c} \in \mathrm{h}^{0}\mathcal{T}^{1}_{0}(M_{A})$,

$$
\begin{aligned}
&L^{-1}\,\nabla^{\mathrm{N}}_{c}\,(\mathrm{N}')^{c}_{d}\,y^{d} \\
&= \mathrm{div}(Y^{\mathrm{H}}) - L^{-1}\left\{\left(g^{ab} - \frac{n+2}{2}\frac{1}{L}\,y^{a}\,y^{b}\right)\nabla^{\mathrm{N}}_{c} g_{ab} + \mathrm{Tor}^{a}_{ca}\right\} y^{d}\,(\mathrm{N}')^{c}_{d}.
\end{aligned}
\tag{2.80}
$$

Substituting (2.79) and (2.80) in (2.78),

$$
\begin{aligned}
&L^{-1} \left.\frac{\partial}{\partial \tau}\right|_{\tau=0} \mathrm{Ric}(\tau) \\
&= -\mathrm{div}(X^{H}) + \mathrm{div}(Y^{H}) + L^{-1}\left\{\left(g^{ab} - \frac{n+2}{2}\frac{1}{L}\,y^{a}\,y^{b}\right)\nabla^{\mathrm{N}}_{c} g_{ab} + \mathrm{Tor}^{a}_{ca}\right\} y^{c}\,(\mathrm{N}')^{d}_{d} \\
&\quad - L^{-1}\left\{\left(g^{ab} - \frac{n+2}{2}\frac{1}{L}\,y^{a}\,y^{b}\right)\nabla^{\mathrm{N}}_{c} g_{ab} + \mathrm{Tor}^{a}_{ca}\right\} y^{d}\,(\mathrm{N}')^{c}_{d} + L^{-1}\,\mathrm{Tor}^{d}_{ca}\,y^{a}\,(\mathrm{N}')^{c}_{d}.
\end{aligned}
$$

Prop. 2.5 also guarantees that, upon integration on $\mathbb{P}^{+}A$, the divergence terms can be discarded. Indeed:

$$
\int_{D} \underline{\mathrm{div}(X^{\mathrm{H}})\,d\mu} = -\int_{D} \mathrm{d}(\underline{X^{\mathrm{H}} \lrcorner\, d\mu}) = -\int_{\partial D} \underline{X^{\mathrm{H}} \lrcorner\, d\mu}
$$

(analogously for $\underline{\mathrm{div}(Y^{\mathrm{H}})\, d\mu}$) and, by the fact that $\mathrm{N}(\tau)$ is D-admissible (Def. 2.10), X and Y vanish on $(\mathbb{P}^+)^{-1}(\partial D)$, so $\underline{X^{\mathrm{H}} \lrcorner d\mu}$ and $\underline{Y^{\mathrm{H}} \lrcorner d\mu}$ vanish on ∂D (see the comment at the end of Prop. 2.2 (ii)). The remaining terms, substituting back in (2.77), can be expressed as

$$\begin{aligned}
&\left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \mathcal{S}^D[\mathrm{N}(\tau), L] \\
&= \underline{\int_D L^{-1} \left\{ \left(g^{ab} - \frac{n+2}{2} \frac{1}{L} y^a y^b \right) \nabla^{\mathrm{N}}_c g_{ab} + \mathrm{Tor}^a_{\ ca} \right\} \left(\delta^d_e y^c - y^d \delta^c_e \right) \left(\mathrm{N}' \right)^e_{\ d}\, d\mu} \\
&\quad + \underline{\int_D L^{-1} \mathrm{Tor}^d_{\ ea} y^a \left(\mathrm{N}' \right)^e_{\ d}\, d\mu}.
\end{aligned}$$

The field $\mathrm{N}' \in \mathrm{h}^1\mathcal{T}^1_1(M_A)$ with $\mathbb{P}^+(\mathrm{Supp}\,\mathrm{N}')$ relatively compact in $\mathbb{P}^+A$ is arbitrary: for any such N', there exists a variation $\mathrm{N}(\tau)$ that has it as its variational field (for instance, $\mathrm{N}(\tau) = \mathrm{N} + \tau\, \mathrm{N}'$). Thanks to this, the standard argument of the calculus of variations can be applied (on a D around each $\mathbb{P}^+v \in \mathbb{P}^+A$). We conclude that the vanishing of all the $\left.\partial_\tau\right|_0 \mathcal{S}^D[\mathrm{N}(\tau), L]$'s is equivalent to

$$\left\{ \left(g^{ab} - \frac{n+2}{2} \frac{1}{L} y^a y^b \right) \nabla^{\mathrm{N}}_c g_{ab} + \mathrm{Tor}^a_{\ ca} \right\} \left(\delta^c_i y^j - y^c \delta^j_i \right) - \mathrm{Tor}^j_{\ ia} y^a = 0 \tag{2.81}$$

on A.

The only thing that remains is to reexpress this in terms of $\mathcal{J} := \mathrm{N} - \mathrm{N}^L$. Substituting (2.24) and (2.27) in (2.81) yields the required equation (2.19).

Appendix C: Proof of Th. 2.1 (Metric equation)

When varying L by $L(\tau)$, it is immediate that

$$\begin{aligned}
\left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \mathcal{S}^D[\mathrm{N}, L(\tau)] &= \int_D \left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \underline{L(\tau)^{-1} \mathrm{Ric}\, d\mu(\tau)} \\
&= \int_D \underline{\left.\frac{\partial}{\partial\tau}\right|_{\tau=0} \left\{ L(\tau)^{-1} \mathrm{Ric}\, d\mu(\tau) \right\}} \\
&= -\int_D \underline{L^{-1} \frac{\mathrm{Ric}}{L} L'\, d\mu} + \int_D \underline{L^{-1} \mathrm{Ric} \left.\frac{\partial}{\partial\tau}\right|_{\tau=0} d\mu(\tau)}.
\end{aligned} \tag{2.82}$$

By (2.16),

$$d\mu(\tau) = \frac{\left|\det g_{ij}(\tau)\right|}{L(\tau)^{\frac{n}{2}}} \mathrm{d}x \wedge \mathrm{d}y.$$

We compute the derivative of this taking into account that

$$\frac{\partial}{\partial \tau}\bigg|_{\tau=0} g_{ij}(\tau) = \frac{1}{2}\, L'_{\cdot i \cdot j}, \qquad \frac{\partial}{\partial \tau}\bigg|_{\tau=0} L(\tau)^{\frac{n}{2}} = \frac{n}{2}\, L^{\frac{n}{2}-1}\, L' \; :$$

by Jacobi's formula for the derivative of a determinant,

$$\begin{aligned}\frac{\partial}{\partial \tau}\bigg|_{\tau=0} d\mu(\tau) &= \left(\frac{1}{2}\,\frac{\left|\det g_{ij}\right|}{L^{\frac{n}{2}}}\, g^{ab}\, L'_{\cdot a \cdot b} - \frac{n}{2}\,\frac{\left|\det g_{ij}\right|}{L^{n}}\, L^{\frac{n}{2}-1}\, L'\right) \mathrm{d}x \wedge \mathrm{d}y \\ &= \left(\frac{1}{2}\, g^{ab}\, L'_{\cdot a \cdot b} - \frac{n}{2}\,\frac{1}{L}\, L'\right) d\mu.\end{aligned}$$

Substituting in (2.82) and putting $\widetilde{\mathrm{Ric}} := L^{-1}\,\mathrm{Ric} \in \mathrm{h}^0\mathcal{F}(A)$,

$$\frac{\partial}{\partial \tau}\bigg|_{\tau=0} \mathcal{S}^D[\mathrm{N}, L(\tau)] = -\frac{n+2}{2}\int_D \underline{L^{-1}\,\widetilde{\mathrm{Ric}}\, L'\, d\mu} + \frac{1}{2}\int_D \underline{\widetilde{\mathrm{Ric}}\, g^{ab}\, L'_{\cdot a \cdot b}\, d\mu}. \quad (2.83)$$

Calling $X := \widetilde{\mathrm{Ric}}\, g^{ab}\, L'_{\cdot a}\,\partial_b \in \mathrm{h}^1\mathcal{T}^1_0(M_A)$ and using (2.18), $g^{ib}_{\cdot b} = -2\,\mathrm{C}^i$, and the 2-homogeneity of L',

$$\begin{aligned}\widetilde{\mathrm{Ric}}\, g^{ab}\, L'_{\cdot a \cdot b} &= X^b_{\cdot b} - g^{ab}\,\widetilde{\mathrm{Ric}}_{\cdot b}\, L'_{\cdot a} - \widetilde{\mathrm{Ric}}\, g^{ab}_{\cdot b}\, L'_{\cdot a} \\ &= \mathrm{div}(X^{\mathrm{V}}) - \widetilde{\mathrm{Ric}}\, g^{ab}\left(2\,\mathrm{C}_b - n\,\frac{y_b}{L}\right) L'_{\cdot a} - g^{ab}\,\widetilde{\mathrm{Ric}}_{\cdot b}\, L'_{\cdot a} + 2\,\widetilde{\mathrm{Ric}}\,\mathrm{C}^a\, L'_{\cdot a} \\ &= \mathrm{div}(X^{\mathrm{V}}) + 2nL^{-1}\,\widetilde{\mathrm{Ric}}\, L' - g^{ab}\,\widetilde{\mathrm{Ric}}_{\cdot b}\, L'_{\cdot a}.\end{aligned}$$

Calling $Y := L'\, g^{ab}\,\widetilde{\mathrm{Ric}}_{\cdot b}\,\partial_a \in \mathrm{h}^1\mathcal{T}^1_0(M_A)$ and again using (2.18), $g^{ia}_{\cdot a} = -2\,\mathrm{C}^i$, and the 0-homogeneity of $\widetilde{\mathrm{Ric}}$,

$$\begin{aligned}\widetilde{\mathrm{Ric}}\, g^{ab}\, L'_{\cdot a \cdot b} &= \mathrm{div}(X^{\mathrm{V}}) + 2nL^{-1}\,\widetilde{\mathrm{Ric}}\, L' - Y^a_{\cdot a} + g^{ab}_{\cdot a}\,\widetilde{\mathrm{Ric}}_{\cdot b}\, L' + g^{ab}\,\widetilde{\mathrm{Ric}}_{\cdot a \cdot b}\, L' \\ &= \mathrm{div}(X^{\mathrm{V}}) + 2nL^{-1}\,\widetilde{\mathrm{Ric}}\, L' - \mathrm{div}(Y^{\mathrm{V}}) \\ &\quad + g^{ab}\left(2\,\mathrm{C}_a - n\,\frac{y_a}{L}\right)\widetilde{\mathrm{Ric}}_{\cdot b}\, L' - 2\,\mathrm{C}^b\,\widetilde{\mathrm{Ric}}_{\cdot b}\, L' + g^{ab}\,\widetilde{\mathrm{Ric}}_{\cdot a \cdot b}\, L' \\ &= \mathrm{div}(X^{\mathrm{V}}) - \mathrm{div}(Y^{\mathrm{V}}) + 2nL^{-1}\,\widetilde{\mathrm{Ric}}\, L' + g^{ab}\,\widetilde{\mathrm{Ric}}_{\cdot a \cdot b}\, L'.\end{aligned}$$

Substituting this back in (2.83) and dropping the divergence terms (by the analogous reasoning as in Appendix B),

$$\begin{aligned}&\frac{\partial}{\partial \tau}\bigg|_{\tau=0} \mathcal{S}^D[\mathrm{N}, L(\tau)] \\ &= -\frac{n+2}{2}\int_D \underline{L^{-1}\,\widetilde{\mathrm{Ric}}\, L'\, d\mu} + n\int_D \underline{L^{-1}\,\widetilde{\mathrm{Ric}}\, L'\, d\mu} + \frac{1}{2}\int_D \underline{g^{ab}\,\widetilde{\mathrm{Ric}}_{\cdot a \cdot b}\, L'\, d\mu} \\ &= \frac{n-2}{2}\int_D \underline{L^{-1}\,\widetilde{\mathrm{Ric}}\, L'\, d\mu} + \frac{1}{2}\int_D \underline{g^{ab}\,\widetilde{\mathrm{Ric}}_{\cdot a \cdot b}\, L'\, d\mu}.\end{aligned}$$

The field $L' \in \mathrm{h}^2\mathcal{F}(A)$ with $\mathbb{P}^+(\operatorname{Supp} L')$ relatively compact and small enough in $\mathbb{P}^+A$ is arbitrary: for any such L', there exists a variation $L(\tau)$ that has it as its variational field (for instance, $L(\tau) = L + \tau\, L'$). Again, the standard argument of the calculus of variations can be applied around each $\mathbb{P}^+v \in \mathbb{P}^+A$, concluding that the vanishing of all the $\partial_\tau\big|_0\, \mathcal{S}^D[\mathrm{N}, L(\tau)]$'s is equivalent to

$$(n-2)\, L^{-1}\, \widetilde{\mathrm{Ric}} + g^{ab}\, \widetilde{\mathrm{Ric}}_{\cdot a \cdot b} = 0.$$

Finally, one straightforwardly rewrites

$$(n-2)\, L^{-1}\, \widetilde{\mathrm{Ric}} + g^{ab}\, \widetilde{\mathrm{Ric}}_{\cdot a \cdot b} = -\,(n+2)\, L^{-2}\, \mathrm{Ric} + L^{-1}\, g^{ab}\, \mathrm{Ric}_{\cdot a \cdot b};$$

indeed, the right hand side of this becomes the left hand side by the same computations as in the beginning of the proof of Lem. 2.6, yielding the required equation (2.20).

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3 | Schur theorem for the Ricci curvature of any weakly Landsberg Finsler metric

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Abstract

The Ricci version of the Schur theorem is shown to hold for a wide class of Finsler metrics. What is more, let F be any Finsler metric whose Ricci curvature is a function $\rho\colon\ M^n \to \mathbb{R}$ (i.e., (M^n, F) is Einstein), where $n \geq 3$. For $x \in M$, we express $\mathrm{d}\rho_x$ as an average over the indicatrix in $\mathrm{T}_x M$ of the Hilbert form weighted by a combination of derivatives of the Landsberg tensor. As a consequence of this general expression, if the metric is weakly Landsberg, then ρ must be constant. The proof is based on the invariance of certain functionals under $\mathrm{Diff}(M)$.

Furthermore, the Schur theorem holds for the class of pseudo-Finsler metrics with quadratic Ricci scalar. We extend previous arguments in the literature that proved this in particular cases. Lastly, we analyze the status of the general problem.

Keywords — *Finsler geometry, Pseudo-Finsler geometry, Ricci curvature, Schur theorem, weakly Landsberg metric, variational calculus, diffeomorphism invariance, Finsler indicatrix.*

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3.1 Introduction

In 1886, Friedrich Schur showed that if a connected Riemannian manifold M^n with $n \geq 3$ has sectional curvature that varies only as a function of the manifold's points, then the sectional curvature must be constant [48]. Later, it was realized that this could be strengthened by using the second Bianchi identity.[1] Specifically, if a manifold's Ricci curvature is a function only of its points, then it is constant. This refined result, by closeness to the original one, will be referred to as *Schur's theorem* also.

Being valid in any metric signature (see e.g. [43]), it constraints the possible pseudo-Riemannian geometries of dimension greater than 2. From a different viewpoint, it states that the only metrics with isotropic Ricci curvature are the solutions to Einstein's vacuum equation with a cosmological constant. Given the theorem's importance, the problem of its extension to Finsler or Lorentz-Finsler[2] geometries has become a well-known open question [7].

(Pseudo-)Finsler geometry generalizes the (pseudo-)Riemannian one by allowing for arbitrary (pseudo-)norms instead of just scalar products. In it, there is a notion of curvature tensor [8, (6.29)] which depends only on the geodesic spray of the pseudo-Finsler function (and not on any of the amount of connections that one can associate with it). The flag curvature and Ricci curvature can both be derived from this curvature tensor. Thus, there appear two Finslerian Schur theorems that one could conceive of: the flag curvature version and the Ricci curvature version. The former has been proven for any pseudo-Finsler metric [11, 20, 39, 53], while the latter is the open problem we are concerned with, mainly in the standard Finsler case.

A milestone was Robles', who proved the Ricci version for Randers metrics [46], after which many researchers have established it for metrics of other specific forms [56, 19, 50, 57]. In [47, Th. 1.1], it was proved for a Finsler manifold which is Landsberg, compact and of so-called *SCR type*. In [21, Th. 1], it was proved in the Berwald case. (See also the discussion in §3.5.) For completeness, let us mention that recently Schur-type results have been obtained also for the mean Berwald curvature [54, 38].

[1] Even though Bianchi published his identities in 1902 [13], the contracted ones had already appeared in 1880 by the hand of Aurel Voss [55], and these are enough to derive the result, as we comment on below.

[2] The reader is referred to [26, 34, 30, 44, 9] for a sample of the growing interest in generalizing the relativistic spacetimes by means of a Finslerian geometry, from the viewpoints of both mathematics and physics.

In this article, we present a proof of the Ricci curvature version of the Schur theorem for weakly Landsberg Finsler manifolds, Cor. 3.1. In particular, we improve upon [47, Th. 1.1] and [21, Th. 1], so, to the best of our knowledge, [46, Lem 3.4] and our result provide the only classes of Finsler metrics containing the Riemannian ones for which the Ricci-Schur problem is solved. We use our main theorem, Th. 3.2, to prove Cor. 3.1, so our findings may be synthesized as follows.

Theorem. On an Einstein Finsler manifold (M^n, F) of Ricci curvature $\rho\colon\ M^n \to \mathbb{R}$, there holds a pointwise relation

$$(n-2)\,\partial_i\rho(x) = -2n\frac{\int_{(\mathrm{T}_xM)^+} F(x,y)^{-2}\,\mathfrak{P}_{|0}(x,y)\,y_i\,d\Sigma_x^+(y)}{\int_{(\mathrm{T}_xM)^+} d\Sigma_x^+(y)}, \tag{3.1}$$

where $d\Sigma_x^+$ is the Sasaki-induced volume form on the (positively) projectivized tangent space $\left(\mathrm{T}_xM\right)^+$ and $F^{-2}\mathfrak{P}_{|0}\,y_i\,\mathrm{d}x^i$ is a sum of linear and quadratic terms on the mean Landsberg tensor $P_i\,\mathrm{d}x^i$ and its derivatives (see (3.26) and (3.6)). In particular, if $P_i = 0$ and $n \geq 3$, then ρ is constant.

Let us lay out the philosophy underlying our approach. If one were to attempt to prove the Finslerian Schur theorem, a first idea could be to use the Bianchi identities [6, Ch. 3], but this is not fruitful [46, App. B 2]. Still, in the classical proof one only uses the contracted second identity or, equiv., that the Einstein tensor is divergence-free. Noticeably, this fact can be traced back to the diffeomorphism invariance of the Einstein-Hilbert action in a derivation standard within the foundations of general relativity [51, Ch. 3.3.3], [14, Ch. 9.2]. This strategy offers a clear generalization inspired by recent developments. Indeed, the research on Finslerian gravity has extended the Hilbert functional to pseudo-Finsler geometry[3] [45, 27, 32], and some of the consequences of the extension's invariance have already been explored [28]. It is by continuing this path that we reach the above theorem. There is a noticeable difference with the classical result, however. While the new functional is defined on the indicatrix (equiv., projectivized tangent) bundle, its symmetry group is still $\mathrm{Diff}(M)$, leading to integral identities such as (3.1). The need for compact indicatrices is the reason for restricting our work to metrics that are positive definite on all of $\mathrm{T}M \setminus \mathbf{0}$.

Our study is supplemented with a revision of those partial Ricci-Schur theorems in the literature which hold for pseudo-Finsler metrics [21, Th. 1], [47, Lem. 3.2], [4,

[3] In fact, this was done for the first time in [18], even if implicitly and in the positive definite case.

Th. 3.1]. We clarify how they can be unified into a single statement, Th. 3.3, whose only hypothesis is that of the 2-homogeneous Ricci scalar being quadratic.

The article is structured as follows. In §3.2, we discuss all the notions that we require of, with emphasis on Einstein pseudo-Finsler metrics and on the volume form on the positive projectivization of $\mathrm{T}M$. In §3.3, we define two Finslerian functionals and establish their $\mathrm{Diff}(M)$-invariance and the Noether identities that follow from it. In §3.4, we employ them to prove Th. 3.2 and Cor. 3.1; additionally we state the mentioned pseudo-Finsler result, Th. 3.3. Finally, in §3.5, we analyze our findings in relation to the status of the general problem.

3.2 Notation and terminology

M will always be a smooth manifold of dimension $n \geq 2$ and a, b, i, j, k, l will be indices in $\{1, \ldots, n\}$ for which the Einstein summation convention holds. Let $\pi : \mathrm{T}M \to M$ denote the projection of the tangent bundle, whose elements we regard as pairs (x, y) where $x \in M$ and $y \in \mathrm{T}_x M$. We will also consider a subset $A \subseteq \mathrm{T}M \setminus \mathbf{0}$ which is open, *conic* (i.e., if $(x, y) \in A$, then $(x, \mu y) \in A$ for all $\mu \in]0, +\infty[$) and with $\pi(A) = M$. In the same way as in [6, Ch. 1.1], we denote by $(U, (x^i))$ an arbitrary coordinate chart for M, inducing a natural chart $(\mathrm{T}U, (x^i, y^i))$ for $\mathrm{T}M$ and, then, for A. Whenever convenient, we abbreviate

$$\partial_i := \frac{\partial}{\partial x^i}, \qquad \dot{\partial}_i := \frac{\partial}{\partial y^i}.$$

We shall adopt the formalism of *(A-)anisotropic tensors*[4] [29]. These contain the same information as the d-tensors [15, Ch. 2.5], but without the need to keep track of wether they live on the horizontal or vertical distributions of A, see [28, §II C]. Anyway, typically we will work only with their components in natural coordinates, e.g., an anisotropic 1-form θ will be characterized by the local functions θ_i such that $\theta_{(x,y)} = \theta_i(x, y)\, \mathrm{d}x^i$ for $(x, y) \in A$. The prime example is the *Liouville*, or *canonical, anisotropic vector field* $\mathbb{C}$ (cf. [27, 28, 32]):

$$\mathbb{C} = \mathbb{C}^i\, \partial_i, \qquad \mathbb{C}^i(x, y) := y^i \text{ for all } (x, y) \in \mathrm{T}M.$$

As a particular case, the anisotropic functions are just the smooth functions on A; we write $\mathcal{F}(A)$ for the set of all of these, which in a natural manner contains $\mathcal{F}(M)$,

[4]Which will always be taken to be smooth, i.e. C^∞ or just as differentiable as necessary, the same as M and A.

that of smooth functions on M.[5] A function $f \in \mathcal{F}(A)$ is *(positively) r-homogeneous* if $f(x, \mu y) = \mu^r f(x, y)$ for all $(x, y) \in A$ and $\mu \in\,]0, +\infty[$; analogous definition for an anisotropic tensor of arbitrary covariance and contravariance degrees in terms of its components. On the other hand, the *vertical derivative* of an anisotropic tensor is another one of one covariant degree more and whose components we denote by

$$f_{\cdot j} := \dot{\partial}_j f = \frac{\partial f}{\partial y^j}, \qquad \theta_{i \cdot j} := \dot{\partial}_j \theta_i = \frac{\partial \theta_i}{\partial y^j}, \qquad \dots \tag{3.2}$$

Euler's theorem lays at the core of the Finslerian theory, relating homogeneity with vertical derivatives and the Liouville field (cf. [8, 6, 15, 32, 28]):

Theorem 3.1. *For $r \in \mathbb{R}$, a function $f \in \mathcal{F}(A)$ is r-homogeneous if and only if the contraction of its vertical derivative with $\mathbb{C}$ equals rf, that is,*

$$y^i f_{\cdot i} = rf.$$

Then the vertical derivative of f is $(r-1)$-homogeneous. The same holds for the r-homogeneity of A-anisotropic tensors of arbitrary type.

3.2.1 Pseudo-Finsler geometry and Einstein metrics

We refer to [8, 49] for a systematic treatment of pseudo-Finsler manifolds. Many distinguishig features of the Lorentz-Finsler ones are analyzed in [31]. However, our main case of interest is the standard Finsler one, covered by [6].

Definition 3.1. *Let $A \subseteq \mathrm{T}M \setminus \mathbf{0}$ be as above. A* pseudo-Finsler metric (defined on A) *is a 2-homogeneous function $L \in \mathcal{F}(A)$ whose* fundamental tensor g*, of components*

$$g_{ij}(x, y) := \frac{1}{2} L_{\cdot i \cdot j}(x, y) = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}(x, y)$$

is non-degenerate at all $(x, y) \in A$. The pseudo-Finsler metric L is a (standard) Finsler metric *if it is defined on $\mathrm{T}M \setminus \mathbf{0}$ and g is positive definite there. Then it follows that L is positive everywhere, so we shall identify it with its (1-homogeneous) square root*

$$F := \sqrt{L} \in \mathcal{F}(\mathrm{T}M \setminus \mathbf{0})$$

and denote by $\mathrm{Fins}(M)$ *the set of all of these F's.*

[5] In fact, as is standard, for any manifold N we will understand that $\mathcal{F}(N)$ is the $\mathbb{R}$-algebra of its smooth functions, $\mathfrak{X}(N)$ the Lie algebra of its tangent vector fields, and $\Omega(N)$ the graded algebra of its differential forms.

Remark 3.1. The main point of this definition is to make apparent that metrics which are positive definite but defined only on a proper $A \subset \mathrm{T}M \setminus \mathbf{0}$ (such as Kropina's [35]) do not qualify as *Finsler* for us. Most results in §3.3 and §3.4 will not apply to them.

As is typical, we lower and raise indices of anisotropic tensors, resp., with the components g_{ij} and with those of the inverse fundamental tensor, g^{ij}, as in the following. The *Cartan tensor* $C = C_{ijk}\, \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k$ of a pseudo-Finsler metric L has as its components

$$C_{ijk} := \frac{1}{2} g_{ij\cdot k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

As these are symmetric, the *mean Cartan tensor* $\mathrm{tr}_g C = C_i\, \mathrm{d}x^i$ is well-defined, with components

$$C_i := C^a_{ia} = g^{ab} C_{iab}.$$

In the Finsler case, we will also need to consider the *Hilbert* 1*-form* ω,

$$\omega_i := F_{\cdot i} = \frac{y_i}{F} = g_{ia}\frac{y^a}{F}. \tag{3.3}$$

The geodesic equation of the pseudo-Finsler metric L provides its *spray coefficients*, namely

$$G^i = \frac{1}{4} g^{ic} \left(\frac{\partial g_{cb}}{\partial x^a} + \frac{\partial g_{ac}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^c} \right) y^a y^b. \tag{3.4}$$

For the next definitions, we mostly follow [27, 28], particularly §II B of each of them. We use (3.4) to define the *horizontal subbundle* $\mathrm{H}A \subset \mathrm{T}A$ associated with the pseudo-Finsler metric L. Namely, for $(x, y) \in A$, we put

$$\mathrm{H}_{(x,y)}A := \mathrm{Span}\left\{ \delta_i\big|_{(x,y)} : i \in \{1, \ldots, n\} \right\}, \qquad \delta_i := \partial_i - G^a_{\cdot i}\dot{\partial}_a; \tag{3.5}$$

this way, a local basis of $\mathrm{T}A = \mathrm{H}A \oplus \mathrm{V}A$ is $\left\{\delta_i, \dot{\partial}_i : i \in \{1, \ldots, n\}\right\}$. Moreover, the horizontal subbundle allows one to define Christoffel symbols Γ^i_{jk} of the (horizontal part of the) *Chern-Rund connection* (formula below (12) in [27]). We also denote with a vertical bar $_|$ the index introduced by this covariant derivative to any anisotropic tensor, i.e., by analogy with (3.2),

$$f_{|j} := \delta_j f, \qquad \theta_{i|j} := \delta_j \theta_i - \Gamma^k_{ji}\theta_k, \qquad \ldots$$

Now, adopting the notation of [19, 50, 46] for convenience, a $_{|0}$ will represent the contraction with the Liouville field $\mathbb{C}$ of the new Chern index, i.e.,

$$f_{|0} := f_{|j} y^j = y^j \delta_j f, \qquad \theta_{i|0} := \theta_{i|j} y^j = y^j \delta_j \theta_i - y^j \Gamma^k_{ji}\theta_k, \qquad \ldots \tag{3.6}$$

(In [15, 27, 28], the $_{|0}$ is called *dynamical covariant derivative* and denoted by ∇). Finally, we define the *Landsberg tensor* $P = P_{ijk}\,\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k$ by

$$P_{ijk} := g_{ia}\left(G^a_{\cdot j \cdot k} - \Gamma^a_{jk}\right),$$

which are symmetric, and the *mean Landsberg tensor* $\mathrm{tr}_g P = P_i\,\mathrm{d}x^i$ by

$$P_i := P^a_{ia} = g^{ab} P_{iab} = G^a_{\cdot a \cdot i} - \Gamma^a_{ai}.$$

The pseudo-Finsler metric L is said to be *weakly Landsberg* precisely when its mean Landsberg tensor vanishes, i.e., $P_i = 0$ in any natural coordinates.

Remark 3.2. Here we list some elementary properties of the Finslerian objects which we will use without further mention. The Chern derivative is Leibnizian for tensor products and commutes with contractions; consequently, the same is true of $_{|0}$. The $_{|i}$ preserves the homogeneity degree of anisotropic tensors, while the $_{|0}$ increases it by one. They satisfy $g_{ij|k} = 0$, $g^{ij}_{|k} = 0$ and, then, $g_{ij|0} = 0$, $g^{ij}_{|0} = 0$, $L_{|k} = 0$, $L_{|0} = 0$. On the other hand, $g^{ij}_{\cdot i} = -2C^j$ and $y_i P^i = 0$. The canonical field $\mathbb{C}$ is 1-hom.; C and $\mathrm{tr}_g C$ are (-1)-hom; and g, ω, P, and $\mathrm{tr}_g P$ are 0-hom.

The *(2-homogeneous) Ricci scalar* of L can be defined from its spray coefficients (3.4):

$$\mathrm{Ric} = 2\frac{\partial G^i}{\partial x^i} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^i} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^i} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^i} \in \mathcal{F}(A). \tag{3.7}$$

Remark 3.3. Note that in this article the Ricci scalar has the same sign as in [49] and [32], and it is the opposite of the scalar of [27], there denoted R. Meanwhile, we have mantained the sign convention of the (mean) Landsberg tensor of [27], entailing that P_{ijk} (resp. P_i) is the opposite of the Lan_{ijk} (resp. Lan_i) of [32].

Definition 3.2. *A pseudo-Finsler manifold* (M, L) *is* Einstein *if its* Ricci curvature*, namely the function defined on* $A_* := \{(x, y) \in A : L(x, y) \neq 0\}$ *by* $\frac{\mathrm{Ric}(x,y)}{L(x,y)}$*, is actually independent of* y*. Equivalently,*[6] *if there exists some* $\rho \in \mathcal{F}(M)$ *such that* $\mathrm{Ric} = \rho L$.

Remark 3.4. Going beyond the Riemannian theory [12], Akbar-Zadeh named *generalized Einstein* those Finsler metrics whose Ricci curvature is isotropic [1]. Later, it became standard to call them just *Einstein* [46, 7, 4, 19, 21, 47, 50, 56, 57]. Here, merely for convenience in the exposition, we employ this terminology also in the pseudo-Finsler case. In view of the different studies on Finslerian gravity, [45, 27, 44, 32, 28],

[6] This is since each $A_* \cap \mathrm{T}_x M$ is dense in $A \cap \mathrm{T}_x M$ (otherwise the non-degeneracy of g would be contradicted). Anyway, when the metric is Finsler, $A_* = A = \mathrm{T}M \setminus \mathbf{0}$ and the Einstein condition is just $\mathrm{Ric} = \rho F^2$ there.

one should keep in mind that this is in principle independent of the metric solving any proposed Finslerian Einstein equation.

3.2.2 Volume forms, divergences and fiber integrals

The *(positively) projectivized tangent bundle*[7] is one of the spaces which clasically have been used as base manifold for the Finslerian geometric objects (see [17, Ch. 8] or [6, Ch. 2], where it is called the *projective sphere bundle*). In [27], it was recognized as a convenient setting for variational Finslerian gravity theories, which inspire our developments, and a detailed construction was given. Here we shall employ the notation of [32], so the canonical projection to the projectivized bundle is

$$\begin{aligned} \mathbb{P}^+ \colon \mathrm{T}M \setminus \mathbf{0} &\longrightarrow \mathbb{P}^+\mathrm{T}M, \\ (x,y) &\longmapsto \mathbb{P}^+(x,y) = (x, \mathbb{P}^+ y) := (x, \{\mu y \colon \mu \in]0,+\infty[\}). \end{aligned}$$

Moreover, in this article we abbreviate

$$(\mathrm{T}M)^+ := \mathbb{P}^+\mathrm{T}M$$

and its fiber at $x \in M$ by

$$\left(\mathrm{T}_x M\right)^+ := \mathbb{P}^+\mathrm{T}_x M \subset (\mathrm{T}M)^+ .$$

Now we follow the studies in [28, 32] of how differential forms and divergences on $(\mathrm{T}M)^+$ are induced by homogeneous forms and divergences on $\mathrm{T}M \setminus \mathbf{0}$. Given $F \in \mathrm{Fins}(M^n)$, the $(\mathrm{T}M)^+$*-volume form associated with* F can be defined as the projectivized volume form provided by the Sasaki metric. More precisely, it is the unique $(2n-1)$-form Ξ such that

$$\left(\mathbb{P}^+\right)^* \Xi = \frac{\det\left(g_{ab}\right)}{F^n}\, \mathrm{d}^{(n)}x \wedge \iota_{\mathbb{C}^{\mathrm{V}}}(\mathrm{d}^{(n)}y) \tag{3.8}$$

(as elements of $\Omega(\mathrm{T}M \setminus \mathbf{0})$; the * represents a pullback and

$$\mathrm{d}^{(n)}x := \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n, \qquad \iota_{\mathbb{C}^{\mathrm{V}}}(\mathrm{d}^{(n)}y) = \sum_{i=1}^{n} (-1)^{i-1}\, y^i \bigwedge_{j\neq i} \mathrm{d}y^j).$$

[7]Consequently with our terminology on homogeneity, we tipically omit the word *positively* since we will not work with the *absolutely projectivized bundle*.

We denote[8]

$$\Xi =: d\Sigma^+ \in \Omega_{2n-1}((\mathrm{T}M)^+).$$

With respect to it, one defines the *divergence* of any $\mathcal{X} = X^i\delta_i + Y^i\dot{\partial}_i \in \mathfrak{X}(\mathrm{T}M \setminus \mathbf{0})$ such that the X^i are 0-hom. and the Y^i are 1-hom.:

$$\operatorname{div}(\mathcal{X}) \in \mathcal{F}((\mathrm{T}M)^+), \qquad \operatorname{div}(\mathcal{X})\, d\Sigma^+ := -\mathfrak{L}_{\mathcal{X}}(d\Sigma^+).$$

(where $\mathfrak{L}_{\mathcal{X}}$ is the Lie derivative along $\mathcal{X}$ regarded as a vector field on $(\mathrm{T}M)^+$, see [28, §III] and [32, Prop. 3]). The following divergence formulas have appeared repeatedly in the literature [18, 45, 27, 28, 32]:

$$\operatorname{div}(X^i\delta_i) = X^i_{|i} - P_i X^i, \qquad \operatorname{div}(uy^i\delta_i) = u_{|0}, \tag{3.9}$$

$$\operatorname{div}(Y^i\dot{\partial}_i) = Y^i_{\cdot i} + 2C_iY^i - n\frac{y_i}{F^2}Y^i, \tag{3.10}$$

where the homogeneity degrees of the components X^i and Y^i are 0 and 1 resp. and that of the function u is -1.

Our last aim for this section is to be able to fiberwise integrate and average anisotropic tensor fields on $(\mathrm{T}M)^+$. For this, given $F \in \operatorname{Fins}(M^n)$ and a chart $(U, (x^i))$ for M, one locally splits $d\Sigma^+$: there exists a unique $(n-1)$-form Θ on $(\mathrm{T}U)^+$ such that[9]

$$\left(\mathbb{P}^+\right)^*\Theta = \frac{\det\left(g_{ij}\right)}{F^n}\iota_{\mathbb{C}^\mathrm{V}}(\mathrm{d}^{(n)}y), \qquad d\Sigma^+\big|_{(\mathrm{T}U)^+} = \mathrm{d}^{(n)}x \wedge \Theta.$$

It will be convenient for us to abuse notation and put

$$\Theta =: d\Sigma^+_x \in \Omega_{n-1}((\mathrm{T}U)^+).$$

Now let $f \in \mathcal{F}(\mathrm{T}U \setminus \mathbf{0})$ be a 0-homogeneous local function; equiv., $f \in \mathcal{F}((\mathrm{T}U)^+)$. For each fixed $x \in U$, the restriction $f|_{(\mathrm{T}_xM)^+}$ can be integrated against (the pullback to $\left(\mathrm{T}_xM\right)^+$ of) $d\Sigma^+_x$; such *fiber integral* will be represented by

$$\int_{(\mathrm{T}_xM)^+} f(x,y)\, d\Sigma^+_x(y).$$

This way, the use of Fubini's theorem in coordinates allows one to write

$$\int_{(\mathrm{T}U)^+} f\, d\Sigma^+ = \int_U \left(\int_{(\mathrm{T}_xM)^+} f(x,y)\, d\Sigma^+_x(y)\right)\mathrm{d}^{(n)}x. \tag{3.11}$$

[8]Here we find the notation $d\Sigma^+$ more convenient than the dV_0^+ and $d\underline{\mu}$ appearing in [27, 32] resp.

[9]This follows from [28, Prop. 15], as did the existence and uniqueness of $d\Sigma^+$.

However, when the chart is changed, $(U, (x^i)) \rightsquigarrow (\widetilde{U}, (\widetilde{x}^i))$, the fiberwise integral does not provide a well-defined function on $U \cap \widetilde{U}$, but rather a density. Indeed, the transformation law for the fiber volume form is

$$d\Sigma_x^+ \rightsquigarrow \widetilde{d\Sigma_x^+} = \det\left(\frac{\partial x^i}{\partial \widetilde{x}^j}\right) d\Sigma_x^+,$$

and hence

$$\int_{(\mathrm{T}_x M)^+} f(x,y)\,\widetilde{d\Sigma_x^+}(y) = \det\left(\frac{\partial x^i}{\partial \widetilde{x}^j}(x)\right) \int_{(\mathrm{T}_x M)^+} f(x,y)\,d\Sigma_x^+(y).$$

This is fixed by instead considering the *fiberwise average of f*, namely

$$\langle f \rangle \in \mathcal{F}(U), \qquad \langle f \rangle(x) := \frac{\int_{(\mathrm{T}_x M)^+} f(x,y)\,d\Sigma_x^+(y)}{\int_{(\mathrm{T}_x M)^+} d\Sigma_x^+(y)},$$

so that if $f \in \mathcal{F}((\mathrm{T}M)^+)$, then clearly $\langle f \rangle \in \mathcal{F}(M)$. In particular, all this applies to the local components of any 0-hom. anisotropic tensor defined on $\mathrm{T}M \setminus \mathbf{0}$, whose fiberwise averages will turn out to transform as a tensor on M. For example, if $\theta = \theta_i\,\mathrm{d}x^i$ is a 0-homogeneous 1-form defined on $\mathrm{T}M \setminus \mathbf{0}$, then the *fiberwise average of θ* is $\langle \theta \rangle \in \Omega_1(M)$ defined by

$$\langle \theta \rangle = \langle \theta \rangle_i\,\mathrm{d}x^i, \qquad \langle \theta \rangle_i(x) := \langle \theta_i \rangle(x) = \frac{\int_{(\mathrm{T}_x M)^+} \theta_i(x,y)\,d\Sigma_x^+(y)}{\int_{(\mathrm{T}_x M)^+} d\Sigma_x^+(y)}.$$

Remark 3.5. Keep in mind that each $F \in \mathrm{Fins}(M)$ provides a natural isomorphism of its indicatrix bundle $\{(x,y) \in \mathrm{T}M \setminus \mathbf{0} : F(x,y) = 1\}$ with $(\mathrm{T}M)^+$ and this is compatible with (fiber) integration [28, §III B 2]. Thus, even though we work on $(\mathrm{T}M)^+$ for convenience, it is sensible to regard $\langle \theta \rangle$ as the *average of θ over indicatrices of F*. It also becomes natural to extend the notion of *fiberwise average* to the case in which θ is homogeneous of an arbitrary degree $r \in \mathbb{R}$, by defining it just as $\langle F^{-r}\theta \rangle$.

Remark 3.6. Despite the notation, our averaging procedure is distinct from others in the literature [25, 29, 52].

3.3 Diff(M)-invariance of Finslerian functionals

As our main result will be a consequence of the diffeomorphism invariance of certain functionals, we find illustrative to start with a brief account of the analogous classical development for the Einstein-Hilbert action. For this, we assume that M is oriented[10] and follow [10, §4.2], though with our own notation. Let λ be the *pseudo-Riemannian Hilbert Lagrangian*, defined by $\lambda[g] := g^{ij}\mathrm{ric}_{ij}\, d\mathrm{Vol}$. Let $U \subseteq M$ be a precompact open subset and denote $\mathcal{F}^U[g] := \int_U \lambda[g]$. For any variation of g with variational field $h = h_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$, it is well known that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}^U[g_t]\Big|_{t=0} = -\int_U \left(\mathrm{ric}^{ij} - \frac{1}{2}g^{ab}\mathrm{ric}_{ab}\, g^{ij}\right) h_{ij}\, d\mathrm{Vol}.$$

At no point will we require g to be critical for $\mathcal{F}^U$, but, regardless, there is a subset of variations for which this derivative is always 0. Indeed, for an orientation-preserving $\varphi \in \mathrm{Diff}(M)$, the transformation laws associated with the pullback metric $\varphi^* g$ imply that λ is diffeomorphism *equivariant*: $\lambda[\varphi^* g] = \varphi^* \lambda[g]$. In order to see that $\mathcal{F}^U$ is diffeomorphism *invariant*, one restricts to those φ's whose support,

$$\mathrm{Supp}\,\varphi := \overline{\{x \in M : \ \varphi(x) \neq x\}},$$

is contained in U, for then $\varphi : U \to U$ and the change of variables theorem gives $\mathcal{F}^U[\varphi^* g] = \mathcal{F}^U[g]$. Now one only needs to take a $\xi \in \mathfrak{X}(M)$ with $\mathrm{Supp}\,\xi \subset U$ and consider the variation given by its flow φ_t, for which $g_t := \varphi_t^* g$ and $h_{ij} = \nabla_j \xi_i + \nabla_i \xi_j$, obtaining with an integration by parts

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}^U[\varphi_t^* g]\Big|_{t=0} &= -2\int_U \left(\mathrm{ric}^{ij} - \frac{1}{2}g^{ab}\mathrm{ric}_{ab}\, g^{ij}\right) \nabla_j \xi_i\, d\mathrm{Vol} \\ &= 2\int_U \nabla_j \left(\mathrm{ric}^{ij} - \frac{1}{2}g^{ab}\mathrm{ric}_{ab}\, g^{ij}\right) \xi_i\, d\mathrm{Vol}. \end{aligned} \tag{3.12}$$

The arbitrariness of ξ and U allows to conclude that, whatever the pseudo-Riemannian metric g may be,

$$\nabla_j \left(\mathrm{ric}^{ji} - \frac{1}{2}g^{ab}\mathrm{ric}_{ab}\, g^{ji}\right) = 0. \tag{3.13}$$

Remark 3.7. This derivation and its Finslerian counterpart, which we develop in this section, can also be viewed as an instance of Noether's second theorem [41]. In relation to the $\mathrm{Diff}(M)$ symmetry enabling it, a number of nomenclatures appear in the literature, such as *naturalness* ([28, App. 3 a], [36],[33]), *general invariance* in the sense of [37], or some of the notions of *general covariance* [16, 2, 42] in physics. Furthermore, distinctions are made between the (ultimately equivalent) *passive* and *active* covariances, see [23, Ch. 2.5]. In our treatment, for simplicity, we choose to stick to the terms *diff-equivariance/invariance* and to an active viewpoint.

[10] Only momentarily and just for simplicity. The manifolds which we will work with later, $\mathrm{T}M$ and $(\mathrm{T}M)^+$, are always orientable.

We shall prove two lemmas (Lem. 3.1 and Lem. 3.2). It must be stressed that the first of them is already implicit in [27, 28]: it could be obtained by formally applying [28, Th. 31 1.] to the natural Lagrangian (3.14). Nevertheless, we have opted for a self-contained development with the intention of focusing on how the statement and proof must be adapted in the standard Finsler case (Rem. 3.8) and arbitrary dimension (Rem. 3.9).

3.3.1 First functional

Some preliminaries are in order. We define the *Finslerian Hilbert Lagrangian* as

$$\Lambda\colon \operatorname{Fins}(M) \longrightarrow \Omega_{2n-1}((\mathrm{T}M)^+), \qquad \Lambda[F] = \frac{\mathrm{Ric}}{F^2}\, d\Sigma^+, \tag{3.14}$$

and, relative to a precompact open subset $D^+ \subseteq (\mathrm{T}M)^+$, the *Finslerian Hilbert functional* as

$$\mathcal{S}^{D^+}\colon \operatorname{Fins}(M) \longrightarrow \mathbb{R}, \qquad \mathcal{S}^{D^+}[F] = \int_{D^+} \Lambda[F]. \tag{3.15}$$

Its variational calculus has already been studied in[11] [18, 45, 27, 28, 32]. Here we focus on those variations induced by the action of $\operatorname{Diff}(M)$, which can be regarded as a subgroup of $\operatorname{Diff}((\mathrm{T}M)^+)$ due to the naturalness of $(\mathrm{T}M)^+$ over M. To be precise, for $\varphi \in \operatorname{Diff}(M)$, we define, respectively, its *lift to* $\mathrm{T}M$ and its *lift to* $(\mathrm{T}M)^+$ as the maps $\Phi \in \operatorname{Diff}(\mathrm{T}M)$ and $\Phi^+ \in \operatorname{Diff}((\mathrm{T}M)^+)$ given by

$$\Phi(x, y) := (\varphi(x), \mathrm{d}\varphi_x(y)), \qquad \Phi^+(x, \mathbb{P}^+ y) := (\varphi(x), \mathbb{P}^+(\mathrm{d}\varphi_x(y))),$$

for $x \in M$ and $y \in \mathrm{T}_x M$. This way, one has natural commutative diagrams

$$\begin{array}{ccc} \mathrm{T}M & \xrightarrow{\ \Phi\ } & \mathrm{T}M \\ {\scriptstyle \mathbb{P}^+}\downarrow & & \downarrow{\scriptstyle \mathbb{P}^+} \\ (\mathrm{T}M)^+ & \xrightarrow{\ \Phi^+\ } & (\mathrm{T}M)^+ \\ {\scriptstyle \pi^+}\downarrow & & \downarrow{\scriptstyle \pi^+} \\ M & \xrightarrow{\ \varphi\ } & M \end{array}$$

in which the horizontal arrows are all diffeomorphisms.

[11] There, some arguments for considering $\mathcal{S}$ as an appropriate generalization of the pseudo-Riemannian functional $\mathcal{F}$ are outlined. One is that (up to a multiplicative constant) $\mathcal{S}^{D^+}$ is variationally equivalent to $\int_{D^+} g^{ij}\mathrm{Ric}_{\cdot i \cdot j}\, d\Sigma^+$, which, for Riemannian metrics and $D^+ = (\mathrm{T}U)^+$, in fact reduces to $\mathcal{F}^U$ upon fiberwise integration [27, Prop. 6], [32, Props. 3.4 and 3.5].

This provides pullbacks of all differential forms on $\mathrm{T}M$ and $(\mathrm{T}M)^+$. In particular, we can understand the pullback by φ of any $f \in \mathcal{F}(\mathrm{T}M \setminus \mathbf{0})$ to be $\Phi^* f = f \circ \Phi$, and in the event that f is 0-hom. and identified with an $f \in \mathcal{F}((\mathrm{T}M)^+)$, this coincides with $(\Phi^+)^* f = f \circ \Phi^+$. If $f = F \in \mathrm{Fins}(M)$, then $\Phi^* F$ also is in $\mathrm{Fins}(M)$ and φ is an *isometry* between the Finsler manifolds $(M, \Phi^* F)$ and (M, F). Naturally, the geometric objects of two the isometric manifolds are related by the lifts Φ and Φ^+ in a consistent manner, implying that the Finslerian Hilbert Lagrangian (3.14) is $\mathrm{Diff}(M)$-*equivariant*:

$$\Lambda[\Phi^* F] = \left(\Phi^+\right)^* \Lambda[F]. \tag{3.16}$$

On the other hand, given the flow φ_t generated by some $\xi \in \mathfrak{X}(M)$, the corresponding variational field is[12]

$$v := \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (\Phi_t^* F^2) \bigg|_{t=0} = \xi_{i|j} y^i y^j = \left(y_i \xi^i\right)_{|0}, \tag{3.17}$$

which induces

$$\frac{v}{F^2} = \left(\frac{y_i \xi^i}{F^2}\right)_{|0} \in \mathcal{F}((\mathrm{T}M)^+). \tag{3.18}$$

The variational formulas of [18, 27, 28] are valid as long as one works on an integration domain D^+ containing $\mathrm{Supp}(\frac{v}{F^2})$ in its interior, all while the Lagrangian being smooth on the compact $\overline{D^+} \subset (\mathrm{T}M)^+$.

Remark 3.8. The only domains that we shall use are those of the form $D^+ = (\mathrm{T}U)^+$ with $U \subseteq M$ precompact, open and with smooth boundary; these are well adapted to the above requirements. Indeed, if $\mathrm{Supp}\,\xi \subset U$, then $\mathrm{Supp}(\frac{v}{F^2}) \subset (\mathrm{T}U)^+$ and any boundary terms (recall (3.9) and (3.10)) depending on ξ will automatically vanish:

$$\int_{(\mathrm{T}U)^+} \mathrm{div}(Y^a[F,\xi]\dot{\partial}_a)\, d\Sigma^+ = 0, \qquad \int_{(\mathrm{T}U)^+} u[F,\xi]_{|0}\, d\Sigma^+ = 0, \qquad \ldots \tag{3.19}$$

Note that these integration domains are not available for general pseudo-Finsler metrics, which is the reason for having restricted ourselves to the standard Finsler case.[13]

Remark 3.9. Let us assume the conditions of Rem. 3.8 and take into account our conventions for Ric, P and v, as well as $n = \dim M$. From [18, (2.19)], one straightforwardly checks that the variation formula for (3.15) under variational fields (3.18)

[12] We follow [27, Prop. 7], adopting its notation for the variational field, and use the facts that $y^i_{|j} = 0$ and $F_{|j} = 0$.

[13] The formula [27, (80)] was derived for $\mathrm{Supp}(\frac{v}{L})$ arbitrarily small, which is incompatible with (3.18). Similarly, [28, Th. 31] needs of a hypothesis on the support of the Lagrangian on each Lorentz-Finsler indicatrix which would trivialize the Schur theorem that we seek.

reads

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}^{(\mathrm{T}U)^+}[F_t]\Big|_{t=0}\\&=\int_{(\mathrm{T}U)^+}\left\{g^{ab}\mathrm{Ric}_{\cdot a\cdot b}-(n+2)\frac{\mathrm{Ric}}{F^2}+2g^{ab}\left(P_{a|b}-P_aP_b+P_{a|0\cdot b}\right)\right\}\frac{v}{F^2}\,d\Sigma^+.\end{aligned}\tag{3.20}$$

The equivariance (3.16) only holds for diffeomorphisms coming from the base manifold M; therefore, (3.20) is only informative when applied with vector fields of components $\xi^i(x)$. It will lead to an equality of the form

$$0=\int_U\left(\int_{(\mathrm{T}_xM)^+}\cdots\right)_i\xi^i(x)\,\mathrm{d}^{(n)}x,$$

which, in contrast to (3.12) and (3.13), will produce an integral identity on each fiber of $(\mathrm{T}M)^+$ (cf. [28, Rem. 32]).

Lemma 3.1. Given $D^+=(\mathrm{T}U)^+$ as in Rem. 3.8, the Finslerian Hilbert functional (3.15) is $\mathrm{Diff}(M)$*-invariant*: for $F\in\mathrm{Fins}(M)$ and $\varphi\in\mathrm{Diff}(M)$ with lift $\Phi\in\mathrm{Diff}(\mathrm{T}M)$ and $\mathrm{Supp}\,\varphi\subset U$, one has

$$\mathcal{S}^{(\mathrm{T}U)^+}[\Phi^*F]=\mathcal{S}^{(\mathrm{T}U)^+}[F].$$

Consequently, every Finsler metric has the pointwise property that

$$\begin{aligned}&\int_{(\mathrm{T}_xM)^+}\left\{g^{ab}\mathrm{Ric}_{\cdot a\cdot b}-(n+2)\frac{\mathrm{Ric}}{F^2}\right\}_{|0}(x,y)\,\frac{y_i}{F(x,y)^2}\,d\Sigma_x^+(y)\\&=-2\int_{(\mathrm{T}_xM)^+}g^{ab}(x,y)\left(P_{a|b}-P_aP_b+P_{a|0\cdot b}\right)_{|0}(x,y)\,\frac{y_i}{F(x,y)^2}\,d\Sigma_x^+(y)\end{aligned}\tag{3.21}$$

in coordinates around any $x\in M$.

Proof. Let us parallel the classical derivation with which we started §3.3. The first claim follows from the $\mathrm{Diff}(M)$-equivariance (3.16) of the Finslerian Hilbert Lagrangian by using the change of variables theorem with the lift of φ to $(\mathrm{T}M)^+$, which satisfies that $\Phi^+:\ (\mathrm{T}U)^+\to(\mathrm{T}U)^+$ due to the hypothesis on the support.

For the second claim, take any $\xi\in\mathfrak{X}(M)$ with $\mathrm{Supp}\,\xi\subset U$ and represent by φ_t its associated one-parameter group. Then $\varphi_t\in\mathrm{Diff}(M)$ with $\mathrm{Supp}\,\varphi_t\subset U$, so for the corresponding variation $F_t:=\Phi_t^*F$ the left hand side of (3.20) is 0. The variational field is (3.18), so integrating by parts in the resulting formula (3.20) (taking (3.19) into account) yields

$$\begin{aligned} &\int_{(\mathrm{T}U)^+} \left\{ g^{ab}\mathrm{Ric}_{\cdot a \cdot b} - (n+2)\frac{\mathrm{Ric}}{L} \right\}_{|0} \frac{y_i \xi^i}{F^2}\, d\Sigma^+ \\ &= -2\int_{(\mathrm{T}U)^+} g^{ab}\left(P_{a|b} - P_a P_b + P_{a|0\cdot b}\right)_{|0} \frac{y_i\xi^i}{F^2}\, d\Sigma^+. \end{aligned} \tag{3.22}$$

The only thing that remains in order to get (3.21) is to express (3.22) in terms of fiber integrals (3.11) (assuming that U is a coordinate domain) and then invoke the arbitrariness of ξ and U. ∎

3.3.2 Second functional

Now, if we just applied (3.21) to the Einstein case, $\mathrm{Ric} = \rho F^2$ with $\rho \in \mathcal{F}(M)$, we would get $y^a \partial_a \rho \frac{y_i}{F^2}$ as the integrand on the left hand side, but we are interested in $\partial_i \rho$ itself. Our second lemma provides an expression of the differential of a function which might be of independent interest, and it will be used to rewrite said left hand side.

Lemma 3.2. The functional $\mathcal{I} : \mathcal{F}(M) \times \mathrm{Fins}(M) \to \mathbb{R}$ defined by $\mathcal{I}[\rho, F] := \int \rho\, d\Sigma^+$ (the integration domain being as in Rem. 3.8) is $\mathrm{Diff}(M)$-invariant: in the conditions of Lem. 3.1,

$$\mathcal{I}[\varphi^*\rho, \Phi^* F] = \mathcal{I}[\rho, F].$$

Consequently, on any Finsler manifold (M, F) of dimension n, the differential of any function $\rho \in \mathcal{F}(M)$ satisfies the identity

$$\partial_i \rho(x) \int_{(\mathrm{T}_x M)^+} d\Sigma_x^+(y) = n \int_{(\mathrm{T}_x M)^+} y^a \partial_a \rho(x) \frac{y_i}{F(x,y)^2}\, d\Sigma_x^+(y) \tag{3.23}$$

in coordinates around any $x \in M$.

Proof. The proof proceeds analogously to that of Lem. 3.1, so let $\xi \in \mathfrak{X}(M)$ be the corresponding generator of the one-parameter group φ_t with lift Φ_t and variational field v given by (3.17). Denote by $\partial_t|_0$ the derivative $\frac{\partial}{\partial t}(\dots)\Big|_{t=0}$ with respect to the flow parameter, and by $d\Sigma_t^+$ the form defined by (3.8) with $\Phi_t^* F$ in place of F. One then has

$$
\begin{aligned}
0 &= \partial_t\big|_0 \int \varphi_t^*(\rho)\, d\Sigma_t^+ \\
&= \int \partial_t\big|_0 \left(\rho\circ\varphi_t\right) d\Sigma^+ \\
&\quad + \int \rho \frac{\partial_t\big|_0 \det\left(g_{ab}\circ\Phi_t\right)}{\det\left(g_{ab}\right)}\, d\Sigma^+ + \int \rho \frac{\partial_t\big|_0 \left(F^{-n}\circ\Phi_t\right)}{F^{-n}}\, d\Sigma^+ \\
&= \int \xi(\rho)\, d\Sigma^+ + \int \rho g^{ab} v_{\cdot a\cdot b}\, d\Sigma^+ - n\int \rho \frac{v}{F^2}\, d\Sigma^+.
\end{aligned}
\tag{3.24}
$$

By a simple computation (using (3.10) and the 2-hom. of v in the third line),

$$
\begin{aligned}
\rho g^{ab} v_{\cdot a\cdot b} &= \left(\rho g^{ab} v_{\cdot a}\right)_{\cdot b} - \rho g^{ab}_{\cdot b} v_{\cdot a} \\
&= \left(\rho g^{ab} v_{\cdot a}\right)_{\cdot b} + 2\rho C^a v_{\cdot a} \\
&= \operatorname{div}(\rho g^{ba} v_{\cdot a}\dot\partial_b) + n\frac{y_b}{F^2}\rho g^{ba} v_{\cdot a} = \operatorname{div}(\rho g^{ba} v_{\cdot a}\dot\partial_b) + 2n\rho\frac{v}{F^2}.
\end{aligned}
$$

Substituting this back into (3.24) and neglecting all the boundary terms (recall (3.9), (3.6) and (3.11)),

$$
\begin{aligned}
0 &= \int \xi(\rho)\, d\Sigma^+ + 2n\int \rho\frac{v}{F^2}\, d\Sigma^+ - n\int \rho\frac{v}{F^2}\, d\Sigma^+ \\
&= \int \xi^i\partial_i\rho\, d\Sigma^+ + n\int \rho\frac{\left(y_i\xi^i\right)_{|0}}{F^2}\, d\Sigma^+ \\
&= \int \xi^i\partial_i\rho\, d\Sigma^+ - n\int \rho_{|0}\frac{y_i\xi^i}{F^2}\, d\Sigma^+ \\
&= \int \left\{\int \left(\partial_i\rho(x) - n y^a\partial_a\rho(x)\frac{y_i}{F(x,y)^2}\right) d\Sigma_x^+(y)\right\} \xi^i(x)\, \mathrm{d}^{(n)}x.
\end{aligned}
$$

The arbitrariness of ξ and of the coordinate domain allows one to conclude. ∎

3.4 Schur-type theorems

3.4.1 For weakly Landsberg Finsler metrics

Our main theorem is a representation of the differential of the Ricci curvature for any Einstein Finsler manifold (Th. 3.2). Since it will follow by particularizing the identities obtained in §3.3, let us compute the Ric-terms of (3.21) in the Einstein case. By substituting $\mathrm{Ric} = \rho F^2$ there ($\rho \in \mathcal{F}(M)$), one gets

$$\begin{aligned} g^{ab}\mathrm{Ric}_{\cdot a \cdot b} - (n+2)\frac{\mathrm{Ric}}{F^2} = g^{ab}\left(\rho F^2\right)_{\cdot a \cdot b} - (n+2)\,\rho &= g^{ab}\left\{2\rho g_{ab}\right\} - (n+2)\,\rho \\ &= 2n\rho - (n+2)\,\rho \\ &= (n-2)\,\rho, \end{aligned}$$

from where (3.6) and (3.5) yield

$$\left\{g^{ab}\mathrm{Ric}_{\cdot a \cdot b} - (n+2)\frac{\mathrm{Ric}}{F^2}\right\}_{|0} = (n-2)\,\rho_{|0} = (n-2)\,y^a\partial_a\rho. \tag{3.25}$$

As a second step, let us express the integrand in the right hand side of (3.21) in a more concise way. We denote

$$\mathfrak{P} := g^{ij}\left(P_{i|j} - P_iP_j + P_{i|0\cdot j}\right) \in \mathcal{F}(\mathrm{T}M \setminus \mathbf{0}), \tag{3.26}$$

so that said integrand is $\mathfrak{P}_{|0}\,\frac{y_i}{F^2}$.

Finally, recall from §3.2 the invariant notation for fiberwise averaging associated with the $(\mathrm{T}M)^+$-volume form $d\Sigma^+$. Again, consider the splitting induced by a coordinate chart $(U,(x^i))$: omitting natural pullbacks, $d\Sigma^+ = \mathrm{d}^{(n)}x \wedge d\Sigma^+_x$ with $d\Sigma^+_x = \frac{\det\left(g_{ij}\right)}{F^n}\,\iota_{\mathbb{C}^{\mathrm{V}}}(\mathrm{d}^{(n)}y)$. Let θ be an r-homogeneous anisotropic 1-form defined on $\mathrm{T}M\setminus\mathbf{0}$. Then the transformation laws of the components θ_i and of $d\Sigma^+_x$ under changes $\left(x^i\right) \rightsquigarrow \left(\tilde{x}^i\right)$, imply that the fiberwise average

$$\langle\theta\rangle = \langle\theta\rangle_i\,\mathrm{d}x^i \in \Omega(M),$$

given (for any $x \in M$) by

$$\langle\theta\rangle_i(x) := \langle\theta_i\rangle(x) = \frac{\int_{(\mathrm{T}_xM)^+} F(x,y)^{-r}\theta_i(x,y)\,d\Sigma^+_x(y)}{\int_{(\mathrm{T}_xM)^+} d\Sigma^+_x(y)}, \tag{3.27}$$

is globally well-defined.

Theorem 3.2. *Let (M,F) be a Finsler manifold of dimension $n \geq 2$. Assume that $\mathrm{Ric} = \rho F^2$ with $\rho \in \mathcal{F}(M)$. Then the Hilbert form ω and the function $\mathfrak{P}$ (defined by* (3.3) *and* (3.26) *resp.) satisfy that*

$$(n-2)\,\mathrm{d}\rho = -2n\langle\mathfrak{P}_{|0}\,\omega\rangle;$$

equally, in coordinates around each $x \in M$,

$$(n-2)\,\partial_i\rho(x) = -2n\frac{\int_{(\mathrm{T}_xM)^+} F(x,y)^{-2}\,\mathfrak{P}_{|0}(x,y)\,y_i\,d\Sigma^+_x(y)}{\int_{(\mathrm{T}_xM)^+} d\Sigma^+_x(y)}. \tag{3.28}$$

Proof. In this situation, the conclusions of Lems. 3.1 and 3.2 are valid. Taking (3.25) and (3.26) into account, the identity (3.21) becomes

$$(n-2)\int_{(\mathrm{T}_x M)^+} y^a \partial_a \rho(x) \frac{y_i}{F(x,y)^2}\, d\Sigma_x^+(y)$$
$$= -2\int_{(\mathrm{T}_x M)^+} \mathfrak{P}_{|0}(x,y) \frac{y_i}{F(x,y)^2}\, d\Sigma_x^+(y).$$

Putting this together with (3.23) directly yields (3.28), whose right hand side is exactly $-2n\langle \mathfrak{P}_{|0}\,\omega\rangle_i(x)$ (see (3.27) and recall that the 1-form of components $\mathfrak{P}_{|0}\,\omega_i = \mathfrak{P}_{|0}\,\frac{y_i}{F}$ is homogeneous of degree $r = 1$). ∎

Thus, on an Einstein Finsler manifold of dimension 3 or greater, the differential of the Ricci curvature is proportional to $\langle \mathfrak{P}_{|0}\,\omega\rangle$, whereas in dimension 2 this averaged invariant vanishes. By computing the explicit expression of $\mathfrak{P}_{|0}$ from (3.26), one obtains the announced Schur theorem under the hypothesis that a certain combination of derivatives of the P_i's is 0.

Corollary 3.1. Let (M^n, F) be a connected Finsler manifold with vanishing mean Landsberg tensor $P_i\, \mathrm{d}x^i$ or, with more generality,

$$g^{ij}\left(P_{i|j|0} - 2P_i P_{j|0} - P_{i|0\cdot j|0}\right) = 0.$$

If $\mathrm{Ric} = \rho F^2$ with $\rho \in \mathcal{F}(M)$ and $n \geq 3$, then ρ is constant.

3.4.2 For Ric-quadratic pseudo-Finsler metrics

The method employed for deriving Cor. 3.1 cannot be extended to the Lorentz-Finsler or non-standard Finsler cases, not even if the metric is Landsberg or Berwald, see Rem. 3.8. Independently, in [21, 47], the Ricci-Schur theorem was proved for Berwald standard Finsler manifolds. The authors of [21] did so by means of Szabó's theorem [52, Th. 1], but this is not actually essential (cf. [47, p. 318]) and the proof works in any signature.[14] Despite this, the corresponding general statement is not explicitly present in the literature as far as we are aware. We end this article by clarifying the relevant extension of [47, Lem. 3.2] and [21, Th. 1], and observing that it turns out to also extend [4, Th. 3.1].

[14] The importance of this point is accentuated by some Lorentz-Finsler metrics being known to violate Szabo's theorem [24].

For $A \subseteq \mathrm{T}M \setminus \mathbf{0}$ as in §3.2, let $f \in \mathcal{F}(A)$ be 2-homogeneous. We will consider f to be *quadratic* if it is the restriction to A of $h(\mathbb{C}, \mathbb{C})$, where h is a 2-covariant tensor field on M. Then, requiring h to be symmetric, we have that for any $(x, y) \in A$,

$$f(x, y) = h_{ij}(x) y^i y^j, \qquad h_{ij}(x) = \frac{1}{2} \frac{\partial^2 f}{\partial y^i \partial y^j}(x, y).$$

This vertical Hessian being independent of y is the well-known necessary and sufficient condition for f to be quadratic. A careful use of it will be key for Th. 3.3, in particular when $f = L$ is a pseudo-Finsler metric.

Remark 3.10. Whenever L is Berwald,[15] its Ricci scalar Ric is quadratic, without any need of L being affinely equivalent to any quadratic metric as in the conclusion of Szabó's theorem. One sees this in [47, p. 318] or by directly plugging $G^i(x, y) = \frac{1}{2}\gamma^i_{ab}(x) y^a y^b$ into (3.7). Moreover, any pseudo-Finsler metric $L \in \mathcal{F}(A)$ which is R-*quadratic* in the sense of [4] also has a quadratic Ric.

Theorem 3.3. *Let (M, L) be a connected pseudo-Finsler manifold with quadratic Ricci scalar. If (M^n, L) is Einstein with $n \geq 3$, then it is either (globally) Ricci-flat or (globally) pseudo-Riemannian with constant $\frac{\mathrm{Ric}}{L}$.*

Proof. As above, write $\mathrm{Ric} = \rho L$ with $\rho \in \mathcal{F}(M)$. We will go over the proof in [21, §3] with the idea of [47, p. 318], so let us decompose the open set $\{x \in M : \rho(x) \neq 0\}$ into its connected components U_α and fix one of the indices $\alpha \in \{1, 2, \dots\}$.[16] By taking the vertical Hessian of the Einstein condition, one has, for $(x, y) \in A \cap \mathrm{T}U_\alpha$,

$$\frac{1}{2\rho(x)} \mathrm{Ric}_{\cdot i \cdot j}(x, y) = g_{ij}(x, y), \tag{3.29}$$

but under our hypotheses the left hand side of this is actually independent of y. This way, as in [47], our equality (3.29) already implies that L is quadratic on U_α: there exists a pseudo-Riemannian metric h^α there such that $L|_{A \cap \mathrm{T}U_\alpha} = (h^\alpha)_{ij}\, y^i y^j$. Denote by ric^α the Ricci tensor of h^α, so that $\mathrm{Ric}|_{A \cap \mathrm{T}U_\alpha} = \left(\mathrm{ric}^\alpha\right)_{ij} y^i y^j$. All that remains is to note that then for $x \in U_\alpha$,

$$\left(\mathrm{ric}^\alpha\right)_{ij}(x) = \frac{1}{2} \mathrm{Ric}_{\cdot i \cdot j}(x, y) = \rho(x) g_{ij}(x, y) = \rho(x) \left(h^\alpha\right)_{ij}(x),$$

(by taking any $y \in A \cap \mathrm{T}_x M$), so the pseudo-Riemannian Schur theorem [43, Ex. 21 (a)] proves that $\rho|_{U_\alpha}$ is constant. We have shown that ρ takes a countable amount of

[15] We take L being *Berwald* to mean that its spray coefficients are quadratic expressions $2G^i(x, y) = \gamma^i_{ab}(x) y^a y^b$. Notwithstanding, if the fibers $A \cap \mathrm{T}_x M$ are not connected, Th. 3.3 obviously extends to the case in which merely $\frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l} = 0$ or $\frac{\partial^3 \mathrm{Ric}}{\partial y^i \partial y^j \partial y^k} = 0$.

[16] Notice that in [21, §3], A is not the domain of L, but a subset of M: one of our U_α's.

values on $M = \{\rho = 0\} \cup U_1 \cup U_2 \cup \ldots$, thus reaching the conclusion the same way as in [21]. ∎

3.5 Discussion of results and outlook

We end the article by discussing research directions which could extend our results, as well as comparing Th. 3.2 (entailing Cor. 3.1) and Th. 3.3 with all the other partial theorems that we are aware of in the literature on this problem [47, 46, 57, 19, 50, 56, 21, 4]. We take the opportunity to speculate on some possible outcomes of the question of a general Finslerian Ricci-Schur theorem.

3.5.1 About Th. 3.2

A first point to highlight is that (3.28) not only yields the Schur theorem under a hypothesis weaker than the metric being weakly Landsberg. In principle, it characterizes exactly when an Einstein Finsler n-manifold with $n \geq 3$ is of constant Ricci curvature ρ. The practicality of this characterization may be improved with a better understanding of the function $\mathfrak{P} = g^{ij}\left(P_{i|j} - P_i P_j - P_{i|0\cdot j}\right)$, and in particular of the fiber integrals of $\mathfrak{P}_{|0}\frac{y_i}{F^2}$. Secondly, notice the nontrivial $n = 2$ instance of (3.28) (the Riemannian version of our approach produces no information in this dimension). We believe that our findings together with the key role of $\mathfrak{P}$ in the Finslerian Einstein equations [27] may motivate further study of this term,[17] which could turn out a significant geometric quantity.

As Th. 3.2, other Schur-type results that seem to need of a Finsler metric defined on all of $\mathrm{T}M \setminus 0$ are [47, Th. 1.1] (implied by ours), [46, Lem 3.4] and [57, Th. 1.1]. By contrast, [19, Th. 1.1], [50, Th. 1.1] and [56, Th. 1.1] appear to be straightforwardly extensible to the local case. One can observe that the proofs in the latter references are of a distinct nature, strongly relying on the specified form of their metrics.[18] In view of this and our result, a noteworthy possibility to explore could be that of the Schur

[17] It is known that $\mathfrak{P} = \mathrm{div}(P^i \delta_i + P^i_{|0}\dot{\partial}_i)$ (communicated to the author by Voicu, see also [18]), which might be exploited in order to improve upon Th. 3.2. Regarding the Einstein equations, it is remarkable that $\mathfrak{P}$ and P_i encapsulate all the differences between the Finslerian Hilbert and Finslerian Palatini [32] formalisms.

[18] This ends up reflecting in the fact that [57, 56, 19, 50] prove that an Einstein metric of the respective class must be Ricci-flat, even in dimension 2, with some exceptions in the last reference.

theorem being true for any standard Finsler metric with some nonstandard counterexamples existing. This might be analogous to other situations in Finsler geometry, such as those of the Landsberg unicorns [5] or the Palatini formalism.[19]

3.5.2 About Th. 3.3

We have clarified that the statement of this result contains [21, Th. 1], [47, Lem. 3.2] and [4, Th. 3.1]. As a first comment, this means that we have generalized [21] in two different directions, as Cor. 3.1 also does so. Moreover, a second way of deducing [47, Th. 1.1] from our findings emerges here. Indeed, what this reference actually proves is that an Einstein Finsler manifold which is compact, Landsberg and of SCR type must be Berwald. Then one may use Th. 3.3 to get [47, Th. 1.1] as a corollary.

Taking all this into account, Th. 3.3 becomes essentially the only Finslerian Schur theorem which is known for indefinite metrics. In our opinion, another possibility worth considering might be that of the Schur theorem's truth being dependent on the signature, contrary to the classical situation. Proving or disproving it would be an important problem particularly in the Lorentzian case due to its relation with Finslerian gravity [45].

[19] Examples of the former with singularities at some directions are known [3, 22], while regular ones can be suspected not to exist. As for the Finslerian Palatini formalism, multiplicity of local solutions of the PDEs is to be expected, while uniqueness theorems are obtained from globality assumptions on the appropriate subset of $(\mathrm{T}M)^+$, see [32, §5].

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4 On the significance of the stress-energy tensor in Finsler spacetimes

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Abstract

We revisit the physical arguments which lead to the definition of the stress-energy tensor T in the Lorentz-Finsler setting (M, L) starting at classical Relativity. Both the standard heuristic approach using fluids and the Lagrangian one are taken into account. In particular, we argue that the Finslerian breaking of Lorentz symmetry makes T an anisotropic 2-tensor (i.e., a tensor for each L-timelike direction), in contrast with the energy-momentum vectors defined on M. Such a tensor is compared with different ones obtained by using a Lagrangian approach. The notion of divergence is revised from a geometric viewpoint and, then, the conservation laws of T for each observer field are revisited. We introduce a natural *anisotropic Lie bracket derivation*, which leads to a divergence obtained from the volume element and the non-linear connection associated with L alone. The computation of this divergence selects the Chern anisotropic connection, thus giving a geometric interpretation to previous choices in the literature.

Keywords — *divergence in Finsler manifolds, stress-energy tensor, Finsler spacetime, Lorentz symmetry breaking, Very Special Relativity.*

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4.1 Introduction

This article has a double aim in Lorentz-Finsler Geometry. The first one is to revisit the physical grounds of the stress-energy tensor T §4.3. The possible extensions of the relativistic T are discussed from the viewpoint of both fluids mechanics and Lagrangian systems. The second one is to revise geometrically the notion of divergence §4.4, yielding consequences about the conservation of T §4.5. With this aim, we introduce new notions of Lie bracket and derivative associated with a nonlinear connection and applicable to anisotropic tensors fields, which appear naturally in Finsler Geometry.

Finslerian modifications of General Relativity aim to find a tensor T collecting the possible anisotropies in the distribution of energy, momentum and stress, which will serve as a source for the (now Lorentz-Finsler) geometry of the spacetime [14, 15, 24, 28, 40]. Some of these proposals may be waiting for experimental evidence, postponing then how the basic relativistic notions would be affected. However, such a discussion is relevant to understand the scope and implications of the introduced Finslerian elements. In a previous reference [1], the fundamentals of observers in the Finslerian setting were extensively studied, including its compatibility with the Ehlers-Pirani-Schild approach. Now we focus on the stress-energy tensor T.

The difficulty to study such a T is apparent. Recall that, using the principle of equivalence, General Relativity is reduced infinitesimally into the Special one, which provides a background for interpretations. However, in the Lorentz-Finsler case, the infinitesimal model is changed into a Lorentz *norm* (instead of scalar product), implying a breaking of Lorentz invariance. This is a substantial issue in its own right which has been studied in the context of *Very Special Relativity* and others [3, 5, 10, 8, 23]. As an additional difficulty, the infinitesimal model changes with the point.[1]

Two noticeable pre-requisites are the following: (a) only the value of the Lorentz-Finsler metric on causal directions is relevant [1, 19] (this is briefly commented in the setup §4.2.3), and (b) there is a big variety of possible extensions of the relativistic kinematic objects to the Finsler case, at least from the geometric viewpont (see the appendix §4.6). Taking into account these issues, the extension of the notion of stress-energy tensor to the Finslerian setting is discussed in §4.3.

[1] Berwald spaces [7, 9] are an exception, as the parallel transport becomes an isometry between the Lorentz norms. Thus, in some sense, these spaces would admit a principle of equivalence with respect to a Lorentz normed space (non-necessarily to Lorentz-Minkowski spacetime).

We start at the *fluids approach*. As a preliminary question, energy-momentum is discussed, §4.3.1. We emphasize that, even though this is well-defined as a tangent vector in each tangent space T_pM, $p \in M$, different observers u, u' at p will use coordinates related by non-trivial linear transformations. Indeed, the latter will depend on both L and the chosen way to measure relative velocities. Moreover, when the stress-energy T is considered §4.3.2, the arguments in Classical Mechanics and Relativity which support its status as a tensor hold only partially in the Lorentz-Finsler setting. Indeed, T acquires a nonlinear nature which is codified in an (observer-dependent) anisotropic tensor, rather than in a tensor on M.

The Lagrangian approach is discussed in §4.3.3. This approach has been developed recently by Hohmann, Pfeifer and Voicu [13, 16], who introduced an *energy-momentum scalar function*. Here, we discuss the analogies and differences of this function with the canonical relativistic stress-energy tensor $\delta S_{matter}/\delta g^{\mu\nu}$ and the 2-tensor T obtained from the fluids approach above. Relevant issues are the existence of different ways to obtain a 2-tensor starting at a scalar function, the recovery of this function from a matter Lagrangian and the possibility to consider the Palatini Lagrangian as the background one (rather than Einstein-Hilbert type Lagrangians used by the cited authors; recall that Palatini's becomes especially meaningful in the Finslerian case [22]). The important case of kinetic gases is considered explicitly (Ex. 4.3.2).

Once the definition of T has been discussed, we focus on its conservation §4.5, revisiting first the divergence theorem §4.4. This is crucial in the Finslerian setting because, as discussed before, the Lagrangian approach above does not guarantee a conservation law as the relativistic $\mathrm{div}(G) = 0$.

§4.4 analyzes the divergence from a purely mathematical viewpoint. Now, L is regarded as pseudo-Finsler (the results will be useful not only in any indefinite signature but also in the classical positive definite case) and T will not be assumed to be symmetric a priori. Classically, the divergence of a vector field Z is defined with the derivation associated with the Lie bracket $[Z, X] = \mathfrak{L}_Z X$, applied to the volume element. In the Finslerian case, however, the Lie derivative and bracket do not make sense for arbitrary anisotropic vector fields. This difficulty was circumvented by Rund [36], who redefined $\mathrm{div}(Z)$ in such a way that a type of divergence theorem held. However, the Lie viewpoint is restored here.

§4.4.1 Once a nonlinear connection HA (seen as a horizontal distribution on A) is prescribed, we can define a Lie bracket $\mathfrak{l}^{\mathrm{H}}_Z X$ and, then, a Lie derivative $\mathfrak{L}^{\mathrm{H}}_Z X$ (Defs. 4.1 and 4.2; Th. 4.1 (C)). Noticeably, the former $\mathfrak{l}^{\mathrm{H}}_Z$ is expressible in terms of the infinitesimal flow of Z (Prop. 4.1).

§4.4.2 The divergence of Z is naturally defined by using this Lie bracket (Def. 4.3). For the computation of $\mathrm{div}(Z)$, however, one can use an *anisotropic connection* ∇ (this can be seen as a Finsler connection dropping its vertical part, see §4.2) and a priori Chern's one is not especially privileged (Prop. 4.2).

§4.4.3. We give a general Finslerian version of the divergence theorem for any anisotropic vector field Z, emphasizing the role of the choice of an (admissible) vector field $V : M \to A$, which in the Lorentzian case can be interpreted as an observer field; this is expressed in terms of integration of forms in the spirit of Cartan's formula (Th. 4.2, Rem. 4.5). We also explain how the boundary term can be expressed in different ways by using a normal either with respect to the pseudo-Riemannian metric g_V or to the fundamental tensor, which were the choices of Rund [36] and Minguzzi [30] resp.

§4.5 gives some applications to conservation laws.

§4.5.1. First, we discuss the definition of divergence for the case of T. Our definition for vector fields was not biased to the Chern anisotropic connection, but this will be used for $\mathrm{div}(T)$ (Def. 4.4). The reason is that $\mathrm{div}(T)$ should behave under contraction in a similar way as in the isotropic case (namely, as in formula (4.14)), which privileges Chern's connection (Prop. 4.3).

§4.5.2. As an interlude about the appeareance of Chern's ∇, a comparison with the possible use of Berwald's and previous approaches in the literature is done.

§4.5.3. A conservation law for the flow of $T_V(X_V)$ is obtained (Cor. 4.2), stressing three hypotheses on the vanishing for V of elements related to the stress-energy T ($\mathrm{div}(T) = 0$), the anisotropic vector X ($\mathfrak{L}^{\mathrm{H}}_X g = 0$, generalizing the isotropic case) and a derivative of V. The latter hypothesis is genuinely Finslerian and it means that some terms related to the nonlinear covariant derivative $\mathrm{D}V$ must vanish globally (V can always be chosen such that they vanish at some point). It is worth pointing out that our general formula for the integral of the divergence (4.39) recovers the classical interpretation of the divergence as an infinitesimal growth of the flow (now observer-dependent). So, $\mathrm{div}(T) = 0$ is equivalent to the conservation of energy-momentum in the instantaneous restspace of each observer, see Rem. 4.10.

We finish by applying this general result to two examples. First to Lorentz norms, showing that the conservation laws of Special Relativity still hold even though, now, the conserved quantity may be different for different observers. As a second example, we give natural conditions so that the flow of $T_V(X_V)$ (whenever it exists as a Lebesgue integral, eventually equal to $\pm\infty$) is equal in any two Cauchy hypersurfaces of a globally hyperbolic Finsler spacetime. Indeed, we refine a previous result by Minguzzi [30], who assumed that L was defined on the whole $\mathrm{T}M$ and $T_V(X_V)$ was compactly supported. We show that a combination of Rund's and Minguzzi's ways to compute the boundary terms allows one to obtain appropriate decay rates (namely, the properly Finslerian hypothesis (4.52)) which ensure the conservation.

4.2 Preliminaries and setup

First, let us set up some notation. In all the present text, M is a connected smooth (C^∞) manifold of dimension $n \geq 2$. As in previous references [21, 22], any coordinate chart $(U, (x^1, ..., x^n))$ of M naturally induces a chart $(\mathrm{T}U, (x^1, ..., x^n, y^1, ..., y^n))$ of $\mathrm{T}M$ defined by the fact that

$$v = y^i(v) \left.\frac{\partial}{\partial x^i}\right|_{\pi(v)}$$

for $v \in \mathrm{T}U$, where $\pi : \mathrm{T}M \to M$ is the canonical projection. We abbreviate

$$\frac{\partial}{\partial x^i} =: \partial_i, \qquad \frac{\partial}{\partial y^i} =: \dot{\partial}_i;$$

these are vector fields on $\mathrm{T}U$. At any rate, we will express our results in coordinate-free and geometric terms.

4.2.1 Anisotropic tensors

We shall employ the framework of anisotropic tensors, following [17, 18, 21], as it is simpler than previous ones. An open subset $A \subseteq \mathrm{T}M$ with $\pi(A) = M$ is fixed; the elements $v \in A$ are called *observers*. We will denote by $\mathcal{T}^r_s(M_A)$ the space of (smooth) r-contravariant s-covariant A-anisotropic tensor fields ($r, s \in \mathbb{N} \cup \{0\}$), and by $\mathcal{T}(M_A) := \bigoplus_{r,s} \mathcal{T}^r_s(M_A)$ the full anisotropic tensor algebra. $\mathcal{F}(A) = \mathcal{T}^0_0(M_A)$ will be the space of functions on A. This time we will also put $\mathfrak{X}(M_A) := \mathcal{T}^1_0(M_A)$ for the space of anisotropic vector fields and $\Omega_s(M_A)$ for the space of anisotropic s-forms (alternating anisotropic tensors, so that $\Omega_1(M_A) := \mathcal{T}^0_1(M_A)$). The space $\mathcal{T}(M)$ of classical tensor fields will be seen as a subspace of $\mathcal{T}(M_A)$, formed by the *isotropic* elements, namely those which depend only on the point $p \in M$ and not on the observer at it. In particular, $\mathfrak{X}(M) \subseteq \mathfrak{X}(M_A)$. There is a distinguished element of $\mathfrak{X}(M_A)$: the *canonical (or Liouville) anisotropic vector field*,

$$\mathbb{C} = y^i\,\partial_i, \qquad \mathbb{C}_v := v.$$

For an open set $U \subseteq M$, we will put $\mathfrak{X}^A(U)$ for the set of (local) *observer fields*, that is, those $V \in \mathfrak{X}(U)$ such that $V_p \in A \cap \mathrm{T}_pM$ for all $p \in U$. Given one of these and $T \in \mathcal{T}^r_s(M_A)$, their composition, denoted by $T_V \in \mathcal{T}^r_s(U)$, makes sense. Finally, for $X \in \mathfrak{X}(M_A)$, there is also a canonical derivation $\dot\partial_X : \mathcal{T}^r_s(M_A) \to \mathcal{T}^r_s(M_A)$: the *vertical derivative along* X,

$$\left(\dot\partial_X T\right)_v := \lim_{t\to 0}\frac{T_{v+tX_v} - T_v}{t}, \qquad \left(\dot\partial_X T\right)^{i_1,\dots,i_r}_{j_1,\dots,j_s} = X^{j_{s+1}}\dot\partial_{j_{s+1}} T^{i_1,\dots,i_r}_{j_1,\dots,j_s}.$$

4.2.2 Nonlinear and anisotropic connections

In this article, a *nonlinear connection on* $A \to M$ is defined as a (horizontal) subbundle $\mathrm{H}A \subseteq \mathrm{T}A$ such that $\mathrm{T}A = \mathrm{H}A \oplus \mathrm{V}A$, where $\mathrm{V}A := \mathrm{Ker}(\mathrm{d}\pi)|_A$ is the vertical subbundle. For other options and the rudiments, see [21]. Nonlinear connections are characterized by their *nonlinear coefficients* N^i_j,

$$\mathrm{H}_vA = \mathrm{Span}\left\{\left.\delta_i\right|_v\right\}, \qquad \delta_i := \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i\frac{\partial}{\partial y^j}, \tag{4.1}$$

and also by their *nonlinear covariant derivative* $\mathrm{D}_X : \mathfrak{X}^A(U) \to \mathfrak{X}(U)$,

$$\mathrm{D}_X V := X^j\left(\frac{\partial V^i}{\partial x^j} + N^i_j(V)\right)\partial_i, \tag{4.2}$$

for $X \in \mathfrak{X}(U)$. They also provide (at least locally) a *nonlinear parallel transport* of observers $v \in A \cap \mathrm{T}_{\gamma(0)}M$ along curves $\gamma : [0,t] \to M$. Namely, a map $P_t : A_{\gamma(0)} \to A_{\gamma(t)}$ defined as $P_t(v) = V(t)$, being V the only vector field along γ such that $V(0) = v$ and $D_{\dot\gamma}V = 0$ (see [21, Def. 12] and the comment below).

An A*-anisotropic connection* is an operator $\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M_A)$ satisfying the usual Koszul derivation properties, see [17, 18, 22]. In a chart domain U, they are characterized by their Christoffel symbols $\Gamma^i_{jk} : A \cap TU \to \mathbb{R}$,

$$\nabla_{\partial_j}\partial_k =: \Gamma^i_{jk}\partial_i.$$

They can be seen as vertically trivial linear connections on the vector bundle $\mathrm{V}A \to A$ [21, Th. 3]. On the other hand, every anisotropic connection has an *underlying nonlinear connection*, the only one with nonlinear coefficients

$$N^i_j := \Gamma^i_{jk} y^k.$$

As a consequence, they define the *covariant derivative* $\nabla\colon \mathcal{T}^r_s(M_A) \to \mathcal{T}^r_{s+1}(M_A)$ for any anisotropic tensor:

$$\nabla_{j_{s+1}} T^{i_1,\dots,i_r}_{j_1,\dots,j_s} = \delta_{j_{s+1}} T^{i_1,\dots,i_r}_{j_1,\dots,j_s} + \sum_{\mu=1}^{r} \Gamma^{i_\mu}_{j_{s+1}k} T^{i_1,\dots,k,\dots,i_r}_{j_1,\dots,j_s} - \sum_{\nu=1}^{s} \Gamma^k_{j_{s+1}j_\mu} T^{i_1,\dots,i_r}_{j_1,\dots,k,\dots,j_s}.$$

4.2.3 Lorentz-Finsler metrics

From now on, we will always assume that A is conic ($\lambda v \in A$ for $v \in A$ and $\lambda \in (0,\infty)$). We shall follow the definitions and conventions in [20, 21]. In particular, a *Finsler spacetime* (M, L) is a (connected) manifold M endowed with a *(properly) Lorentz-Finsler metric* $L\colon \overline{A} \subseteq \mathrm{T}M \setminus \mathbf{0} \to [0,\infty)$. L is required to be smooth, positive homogeneous and, when restricted to each $A_p := \mathrm{T}_pM \cap A$ ($p \in M$), its vertical Hessian g is non-degenerate with signature $(+,-,\dots,-)$; A_p must be connected and salient, and its boundary in $\mathrm{T}M \setminus \mathbf{0}$, which must be equal to $L^{-1}(0)$, is a (strong) *cone structure* $\mathcal{C}$. In particular, at each point p, L is a *Lorentz norm*. By positive homogeneity, L is determined by its *indicatrix* $L^{-1}(1)$.

Notice that the cone $\mathcal{C}$ yields a natural notion of timelike, lightlike and spacelike tangent vectors but L is not defined on the latter. Indeed, we are not interested in the value of L on spacelike vectors by physical reasons which are analyzed in [1]. Roughly, only particles (massive, massless) can be measured and, so, experimental evidences only can affect Σ and $\mathcal{C}$. Even though this also happens in classical Relativity, the value of the Lorentz metric on the (future-directed) timelike vectors is enough to extend it to all the directions. Indeed, the anisotropies in Finsler spacetimes should be regarded as originated by the distribution of matter and energy in the causal directions rather than by (unobservable) spacelike anisotropies.

Even though it is the Lorentz-Finsler case which has a physical interpretation, in all other aspects the theory carries on if L is just *pseudo-Finsler*, namely positively 2-homogeneous with non-degenerate g on A. In fact, this is the context in which we will develop §4.4 and 4.5, as they are of a more mathematical character.

The *Cartan tensor* of L is

$$\mathrm{C} := \frac{1}{2}\dot{\partial} g, \qquad \mathrm{C}_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

It is actually symmetric, so one can define the *mean Cartan tensor* as

$$\mathrm{C}^{\mathrm{m}}(X) := \mathrm{trace}_g\left\{\mathrm{C}(X,-,-)\right\}, \qquad (\mathrm{C}^{\mathrm{m}})_j = g^{ik}\mathrm{C}_{ijk} =: \mathrm{C}_j, \tag{4.3}$$

for $X \in \mathfrak{X}(M_A)$. L has also a canonically associated connection: the *metric nonlinear connection*, HA, of nonlinear coefficients

$$N^i_j := \gamma^i_{jk} y^k - \mathrm{C}^i_{jk}\gamma^k_{ab} y^a y^b, \qquad \gamma^i_{jk} := \frac{1}{2} g^{ic}\left(\frac{\partial g_{cj}}{\partial x^k} + \frac{\partial g_{ck}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^c}\right). \tag{4.4}$$

This is the underlying nonlinear connection of several anisotropic connections. One is the (Levi-Civita)–Chern ∇, the only symmetric anisotropic connection that parallelizes g. It is the horizontal part of Chern-Rund's and Cartan's classical connections and it has Christoffel symbols

$$\Gamma^i_{jk} := \frac{1}{2} g^{il}\left(\frac{\delta g_{lj}}{\delta x^k} + \frac{\delta g_{lk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^l}\right), \tag{4.5}$$

where the δ_i are those associated with (4.4). Another one is the Berwald $\widehat{\nabla}$. This is the horizontal part of Berwald's and Hashiguchi's classical connections and it has Christoffel symbols

$$\widehat{\Gamma}^i_{jk} := \frac{1}{2} g^{il}\left(\frac{\delta g_{lj}}{\delta x^k} + \frac{\delta g_{lk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^l}\right) + \mathrm{Lan}^i_{jk}. \tag{4.6}$$

Here, Lan^i_{jk} are the components of a tensor metrically equivalent to the *Landsberg tensor of L*, which, among many other ways, can be defined as

$$\mathrm{Lan}_{ijk} := \frac{1}{2} g_{lm}\dot{\partial}_i\dot{\partial}_j N^l_k y^m$$

for the N^l_k of (4.4) (see [17, (37)]). The Landsberg tensor is actually symmetric too, so one can define the *mean Landsberg tensor of L* as

$$\mathrm{Lan}^{\mathrm{m}}(X) := \mathrm{trace}_g\left\{\mathrm{Lan}(X,-,-)\right\}, \qquad (\mathrm{Lan}^{\mathrm{m}})_j = g^{ik}\mathrm{Lan}_{ijk} =: \mathrm{Lan}_j. \tag{4.7}$$

4.3 Basic interpretations on the stress-energy tensor T

Let us start with a discussion at each event $p \in M$ of a Finsler spacetime (M, L). We can consider T_pM endowed with the Lorentz norm $L|_{\mathrm{T}_pM}$. In most of this section, the discussion relies essentially on the particular case when M is a real affine n-space with associated vector space V (which plays the role of T_pM in the general case) and L is a Lorentz-Finsler norm on V with indicatrix Σ and cone C included in V. Given $u, u' \in \Sigma$, consider the corresponding fundamental tensors g_u and $g_{u'}$ and take orthonormal bases B_u, $B_{u'}$, obtained extending u, u'. In a natural way, these bases live in $\mathrm{T}_uV, \mathrm{T}_{u'}V$ and they can be identified with bases in V itself. Assuming this, the change of coordinates between B_u, $B_{u'}$ is linear but not a Lorentz transformation, in general.

Extending the interpretations in Relativity, $p \in M$ is an *event*, the affine simplification includes the case of Very Special Relativity [3, 5, 10], $u \in \Sigma$ can be regarded as an *observer*, the tangent space to the indicatrix $\mathrm{T}_u\Sigma$ (i.e., the subspace g_u-orthogonal to u in $\mathrm{T}_uV \equiv V$) becomes the *restspace* of the observer u, and B_u is an *inertial reference frame* for this observer. The *Lorentz invariance breaking* corresponds to the fact that the bases B_u and $B_{u'}$ are orthonormal for the different metrics $g_u, g_{u'}$ and, thus, the linear transformation between the coordinates of B_u and $B_{u'}$ (when regarded as elements of the same vector space $\mathrm{T}_uV \equiv V \equiv \mathrm{T}_{u'}V$) is not a Lorentz one. If the affine simplification is dropped, such elements (observers, restspaces) must be regarded as instantaneous at $p \in M$.

It is worth emphasizing that, according to the viewpoint introduced in [19] and discussed extensively in [1], the spacelike directions are not physically relevant for the Lorentz-Finsler metric. However, each (instantaneous) observer does have a restspace with a Euclidean scalar product. In the case of classical Relativity, Lorentz-invariance permits natural identifications between these restspaces, and they become consistent with the value of the scalar product on spacelike directions. Certainly, a Lorentz norm L could be extended outside these directions (maintaining the Lorentz signature for its fundamental tensor) but this can be done in many different ways, and no relation with the scalar products $g_u, u \in \Sigma$ would hold.

The dropping of natural identifications associated with the Lorentz invariance implies that many notions which are unambiguously defined in classical Relativity admit many different alternatives now. In the Appendix we analyze some of them for the *relative velocity* between observers as well as other kinematical concepts. This is taken into account in the following discussion about how the Finslerian setting affects the notion of *energy-momentum-stress* tensor.

4.3.1 Particles and dusts: anisotropic picture of isotropic elements

In principle, there is no reason to modify the classical relativistic interpretation of $p = mu$ as the *(energy-) momentum vector* of a particle of *(rest) mass* $m > 0$ moving in the observer's direction $u \in \Sigma$. Moreover, if the particle moves in such a way that m is constant, it will be represented by a unit timelike curve $\gamma(\tau)$ such that $p(\tau) = m\gamma'(\tau)$ will be its instantaneous momentum at each *proper time* τ. The (covariant) derivative $p' = m\gamma''$ would be the force F acting on the particle, which is necessarily $g_{\gamma'}$-orthogonal to γ' (i.e., the force lies in the instantaneous restspace of the particle). Then, the relativistic conservation of the momentum in the absence of external forces would retain its natural meaning, namely, if the particle represented by (m, γ) splits into two (m_1, γ_1) and (m_2, γ_2) at some τ_0 then $m\gamma'(\tau_0) = m_1\gamma_1'(\tau_0) + m_2\gamma_2'(\tau_0)$.

The Appendix suggests that the way how an observer u may measure the energy-momentum and conservation may be non-trivial. In particular, if one assumes that an observer u measures $m\gamma' \in \mathrm{T}_pM$ by using a g_u-orthonormal basis B_u in general, $g_u(m\gamma', m\gamma') \neq m^2(= L(m\gamma'))$. Moreover, as we have already commented, the coordinates for other observer u' will not transform by means of Lorentz transformation. However, as the transformation of their coordinates is still linear, and both of them will write consistently $m\gamma'(\tau_0) = m_1\gamma_1'(\tau_0) + m_2\gamma_2'(\tau_0)$ in their coordinates.

Particles are also the basis to model dusts, which constitute the simplest class of relativistic fluids. A dust is represented by a number-flux vector field $N = nU$, where U represents the intrinsic velocity of the particle in the dust, i.e. a comoving observer, and n is the density of the dust for each momentaneously comoving reference frame. Comparing with the case of energy momentum, N is also an intrinsic object which lives at the tangent space of each point and U gives the privileged observer who measures n. However, the measures of n by different observers involve different measures of the volume. As explained in the Appendix, the length contraction may be fairly unrelated to the relative velocities of the observers. This implies a more complicated transformation of the coordinates by different observers. Anyway, the transformations between these coordinates would remain linear and, so, they could still agree in the fact that they are measuring the same intrinsic vector field.

Summing up, in the case of both particles and dusts, one assumes that the physical property lives in V (or, more properly, in each tangent space T_pM of the affine space) and there is a privileged (comoving) observer u. The transformation of coordinates for other observer u' may be complicated but, at the end, it is a linear transformation which can be determined by specifying the geometric quantities which are being measured as well as the geometry of Σ. Thus, by using the coordinates measured by each observer one could construct and anisotropic vector field at each $p \in M$, which will fulfill some constraints, as the measurement by one of the observers (in particular, the privileged one) would determine the measurements by all the others.

4.3.2 Emergence of an anisotropic stress-energy tensor

The situation, however, is subtler for more general fluids, which are modelled classically by a 2-tensor on the underlying manifold.

Let us start recalling the Newtonian and Lorentzian cases. In Classical Mechanics one starts working in an orthonormal basis of Euclidean space to obtain the components T_{ij} of the Cauchy stress tensor, which give the flux of i-momentum (or force) across the j-surface in the background[2]. The laws of conservation of linear momentum and static equilibrium of forces imply that these components give truly a 2-tensor (linear in each variable) and the conservation of linear momentum implies that this tensor is symmmetric.

In the relativistic setting, each observer will determine some symmetric components T^{ij} in its restspace by essentially the same procedure as above. Additionally, it constructs T^{00}, T^{0i} and T^{i0} as the density energy, energy flux across i-surface and i-momentum density, resp. The interpretation of these magnitudes completes the symmetry[3] $T^{0i} = T^{i0}$ as well as the linearity in the 0-component. However, the bilinearity in the components $T^{\mu\nu}$ has been only ensured for vectors in the restspace of the observer. In Relativity, one can claim Lorentz invariance in order to complete the reasons justifying that, finally, the components $T^{\mu\nu}$ will transform as a tensor[4].

Nevertheless, it is not clear in Lorentz-Finsler geometry why the transformation

[2]In this section, $i, j = 1, 2, 3$ and $\mu, \nu = 0, 1, 2, 3$, but in the others they will run freely from 1 to n ($= \dim M$).

[3]The symmetry of T is dropped for the case of theories with high spin because of its contribution to angular momentum.

[4]See for example [37, §4.5], [26, §35].

of the components T_{ij} from an observer u to a second one u' must be linear, taking into account that they apply to spacelike coordinates in distinct Euclidean subspaces and no Lorentz-invariance is assumed. Indeed, the following simple academic example shows that this is not the case.

Example 4.3.1. Assume that (M, L) is an affine space with a Lorentz norm with domain A and consider the anisotropic tensor[5] $\mathbf{T} = L^{-1}\phi\ \mathbb{C} \otimes \mathbb{C}$, where $\mathbb{C}$ is the canonical (Liouville) vector field and $\phi : \Sigma \to \mathbb{R}$ is a smooth function which is extended as a 0-homogeneous function on A. Then, for each $u \in \Sigma$ and $w \in \mathrm{T}_u\Sigma$ one has $\mathbf{T}_u(u, u) = \phi(u)$, $\mathbf{T}_u(w, w) = 0$, $\mathbf{T}_u(u, w) = 0$. In this case, each $\mathbf{T}_u$ is a symmetric 2-tensor, but the information on $\mathbf{T}$ requires the knowledge of $\phi(u)$ for all possible $u \in \Sigma$. Recall that this example holds even if (M, L) is the Lorentz-Minkowski spacetime regarded as a Finsler spacetime (but no Lorentz-invariance is assumed for $\mathbf{T}$).

Therefore, the following issues about T appear:

(a) Observer dependence: even if we assume that the components $T^{\mu\nu}$ measured by any observer u are bilinear and then, it is a standard tensor, the components measured by a second observer u' may transform by a linear map which depends on Σ as well as the experimental way of measuring (as in the case of the energy-momentum vector).
(b) Nonlinearity: it is not clear even why such a linear transformation must exist, as bilinearity is only ensured in the direction of u and of its restspace. Thus, the tensor T_u measured by a single observer u would not be enough to grasp the physics of the fluid at each event $p \in M$, as in the example above.
(c) Contribution of the anisotropies of Σ: as an additional possibility, the local geometry of Σ at u underlies the measurements of this observer and might provide a contribution for the stress-energy tensor itself.

Summing up, Lorentz-Finsler geometry leads to assume that the measurements by u are not enough to determine the state of the fluid and the stress-energy tensor should be regarded as a non-isotropic tensor field, determined by the measurements of all the observers.

[5]The division by L is so that $\mathbf{T}$ is 0-homogeneous overall, as anisotropic stress-energy tensors should be in order to correctly generalize the classical case.

Formally, this means an *anisotropic tensor* $T \in \mathcal{T}^2_0(M_A)$ (see [21] for a summary of the formal approach), which can be expressed locally as

$$T_v = T^{\mu\nu}(v)\, \partial_\mu\big|_x \otimes \partial_\nu\big|_x, \qquad v = y^\mu \left.\frac{\partial}{\partial x^\mu}\right|_x \equiv (x, y) \in A \subset \mathrm{T}M,$$

where $T^{\mu\nu}(\lambda v) = T^{\mu\nu}(v)$ for all $\lambda > 0$ (i.e. T_v depends only on the direction of v). As a first approach (recall footnote 3), we can assume $T^{\mu\nu} = T^{\nu\mu}$. Consistently, we will assume that there exists a Lorentz-Finsler metric L on M with indicatrix $\Sigma \subset \mathrm{T}M$ and, so, indexes can be raised and lowered by using its fundamental tensor g. The fact that T has order 2 is important to establish classical analogies. However, other tensors might appear as more fundamental energy-momentum tensors and, then, one would try to derive a semi-classical 2-tensor as in §4.3.3.

In principle, the intuitive relativistic interpretations would be transplanted directly to each v, whenever $v \in \Sigma$. That is, given two g_v-unit vectors u, w, the value $T_v(u, w)$ of the 2-covariant stress-energy tensor perceived by the observer v (at $x = \pi(v)$) is obtained as the flux of w-energy-momentum per unit of g_v-volume orthogonal to u. More precisely, let $B(u)$ be a small coordinate 3-cube in a hypersurface g_v-orthogonal to u and P_B is the total flux of the energy-momentum of particles crossing $B(u)$ (being positive from the $-u$ side to the u side and negative the opposite direction), then the w-energy-momentum per unit of g_v-volume is

$$\epsilon\, T_v(u, w) := \lim_{Vol_{g_v}(B(u))\to 0} \frac{g_v(P_B, w)}{Vol_{g_v}(B(u))}.$$

where $\epsilon = g_v(w, w)$. As a Finslerian subtlety, recall that g_v is only defined in $\mathrm{T}_v(\mathrm{T}_x M)$ and then in $\mathrm{T}_x M$ (i.e., it is trivially extended to $B(u)$ in a coordinate depending way), but the above limit depends only on the value of g_v. Namely, if one considers two semi-Riemannian metrics g and $\tilde{g}$ in a neighborhood of p such that $g_p = \tilde{g}_p$ and B_n are open subsets with p in the interior of B_m for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} vol_g(B_m) = 0$, then

$$\lim_{m\to+\infty} \frac{vol_g(B_m)}{vol_{\tilde{g}}(B_m)} = 1.$$

In particular, we have the interpretations (recall signature $(+,-,-,-)$):

1. $T_v(v, v)$ is the energy density measured by $v \in \Sigma$,

$$T_v(v, v) := \lim_{Vol_{g_v}(B(v))\to 0} \frac{g_v(P_B, v)}{Vol_{g_v}(B(v))} = \lim_{Vol_{g_v}(B(v))\to 0} \frac{E_B}{Vol_{g_v}(B(v))},$$

being $E_B := g_v(P_B, v)$ the measured energy.

2. If w is g_v-orthogonal to v and g_v-unit, $T_v(w,v)$ measures the flow of energy per unit of g_v-volume in a surface g_v-orthogonal to v and w (i.e. some small surface of area A flowing a lapse Δt), while $T_v(v,u)$ measures the w-momentum density,

$$T_v(w,v) := \lim_{Vol_{g_v}(B(w))\to 0} \frac{g_v(P_B,v)}{Vol_{g_v}(B(w))} = \lim_{Vol_{g_v}(A)\to 0} \frac{1}{A}\left\{\lim_{\Delta t\to 0}\frac{E_B}{\Delta t}\right\}.$$

$$-T_v(v,w) := \lim_{Vol_{g_v}(B(v))\to 0} \frac{g_v(P_B,w)}{Vol_{g_v}(B(v))}.$$

3. If z,w are g_v-orthogonal to v and g_v-unit, $T_v(z,w)$ measures the flow of w-momentum per unit of g_v-volume in a surface g_v-orthogonal to v and z,

$$-T_v(z,w) := \lim_{Vol_{g_v}(B(z))\to 0} \frac{g_v(P_B,w)}{Vol_{g_v}(B(z))} = \lim_{Vol_{g_v}(A)\to 0} \frac{1}{A}\left\{\lim_{\Delta t\to 0}\frac{g_v(P_B,w)}{\Delta t}\right\}.$$

4.3.3 Lagrangian viewpoint

In the Lagrangian approach for Special Relativity, the background spacetime is assumed to be endowed with a flat metric η. So, the Lagrangian $\mathcal{L}$ is constructed by using the prescribed η and some matter fields ϕ_α. The stress-energy tensor coincides with the *canonical* energy-momentum tensor associated with the Lagrangian, in most cases (the exceptions include theories involving spin). This canonical tensor appears as the Noether current associated with the invariance by spacetime translations (i.e., when $\mathcal{L}(\phi_\alpha, \partial_\mu\phi_\alpha, x^\mu) \equiv \mathcal{L}(\phi_\alpha, \partial_\mu\phi_\alpha))$, namely[6]

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)}\partial^\nu\phi_\alpha - \eta^{\mu\nu}\mathcal{L}. \tag{4.8}$$

In principle, these interpretations would hold unaltered for the case of an affine space with a Lorentz norm, including the case of Very Special Relativity.

In General Relativity, however, the Lagrangian formulation introduces a background Lagrangian independent of matter fields (the Einstein-Hilbert one, eventually with a cosmological constant) and, then, a matter Lagrangian $\mathcal{L}_{matter}$ which includes a constant of coupling with the background. Then, the safest way to define the stress-energy is the canonical one obtained as the corresponding action term $\delta S_{matter}/\delta g^{\mu\nu}$

[6]See for example [41] (around formula (E.1.36)) or [26, §32].

in the Euler-Lagrange equations[7],

$$T_{\mu\nu} = -2\frac{\delta \mathcal{L}_{matter}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{matter}. \tag{4.9}$$

Any tensor obtained in this way will have some advantages to play the role of a stress-energy tensor, because it will be automatically symmetric (in contrast to (4.8)) and will have vanishing divergence.

In the Finslerian setting, the variational viewpoint has been systematically studied in a very recent paper by Hohmann, Pfeifer and Voicu [16]. Previously, the background Lagrangian closest to the Einstein-Hilbert functional in the Finslerian setting had been studied in [35, 13]. Such a functional is obtained as the integral of the Ricci scalar function on the indicatrix of the Lorentz-Finsler metric[8] L. Taking into account this background functional, they define the energy-momentum scalar function by taking the corresponding variational action term [16, formula (84)],

$$\mathfrak{T} = -2\frac{L^3}{|g|}\frac{\delta \mathcal{L}_{matter}}{\delta L}. \tag{4.10}$$

Notice that, here, the functional coordinate for the Lagrangian is L and, thus, an (anisotropic) function rather than a 2-tensor is obtained. However, starting at this function some tensors become useful [16, formulas (88), (91)], in particular a canonically associated (anisotropic Liouville) 2-tensor

$$\Theta^{\mu}_{\nu} = \frac{\mathfrak{T}}{L}\,\mathbb{C}^{\mu}\,\mathbb{C}_{\nu} \tag{4.11}$$

as in Example 4.3.1.

Notice that, essentially, the information of these tensors is codified in $\mathfrak{T}$. Even though such a tensor is justified by the procedure of Gotay-Mardsen in [11], some issues as the following ones might deserve interest for a further discussion:

1. This is not the unique natural possibility to construct an anisotropic 2-tensor

[7] See for example, [41, §E.1], [4, §4.3], [31, §21.2, §21.3].

[8] Some arguments which support strongly their choice are (see [14]): (a) the simplest analogous to the vacuum Einstein equation in the Finslerian approach Ricci = 0 (proposed by Rund [36], and satisfied by Finsler pp-waves [8]) is not a variational equation, (b) the Ricci scalar functional yields an Euler-Lagrange equation which agrees with Einstein's in the vacuum Lorentz case, and (c) this Euler-Lagrange equation is the variational completion of the Finslerian Ricci = 0.

starting at $\mathfrak{T}$. For example, an alternative would be the vertical Hessian[9],

$$T_{\mu\nu} = \dot{\partial}_{\mu,\nu}(\mathfrak{T}L) \equiv \frac{\partial^2(\mathfrak{T}L)}{\partial y^\mu \partial y^\nu}. \tag{4.12}$$

It is natural to wonder about the choice closer to the relativistic intuitions about the stress-energy.

2. Recently, the Palatini approach has also been studied for the Finslerian setting [22]. There, the dynamic variables are L and the components of an (independent) non-linear connection. Thus, a similar Lagrangian procedure would lead to a higher order tensor. In the relativistic setting this approach supports classical Relativity, as it recovers both equations and (in the symmetric case) the Levi-Civita connection. However, the Palatini approach is no longer equivalent in the Finslerian case, as it yields non-equivalent connections and it shows a variety of possibilities for the non-linear connections. So, it is natural to wonder about the most natural choice of a Lagrangian-based stress-energy tensor in this setting.

Finally, let us discuss an example analyzed from the Lagrangian viewpoint in [14, 16] taking into account also the observers' one in §4.3.2.

Example 4.3.2. The gravitational field sourced by a kinetic gas has been deeply studied in [14, 16]. In the relativistic setting, this is derived from the Einstein-Vlasov equations in terms of a 1 particle distribution function (1PDF) $\phi(x, \dot{x})$ which encodes how many gas particles at a given spacetime point x propagate on worldlines with normalized 4-velocity $\dot{x}$. Specifically, the stress energy tensor is:

$$T^{\mu\nu}(x) = \int_{\Sigma_x} \dot{x}^\mu \dot{x}^\nu \phi(x, \dot{x})\, dvol_{g_x}, \qquad x \in M,$$

being Σ_x the indicatrix (future-directed unit vectors of the Lorentz metric) and dVol_x the volume at each x. In [14], they propose to derive the gravitational field of a kinetic gas directly from the 1PDF without averaging, i.e., taking into account the full information on the velocity distribution. This leads to consider the function $\phi : \Sigma \to \mathbb{R}$, $u \equiv (x, \dot{x}) \mapsto \phi(u) \geq 0$ as an energy-momentum function which plays the role of a stress-energy tensor (even though it is a scalar rather than a 2-tensor). Moreover, the original Lorentz metric is naturally allowed to be Lorentz-Finsler, which permits to obtain more general cosmological models [14, §III].

[9] The multiplication by L is so that taking second vertical derivatives of the 2-homogeneous $\mathfrak{T}L$ produces a 0-homogeneous tensor, in the same way that the vertical Hessian of the 2-homogeneous function L is the 0-homogeneous fundamental tensor g.

Indeed, up to a coupling constant, ϕ is regarded directly as the matter source in the Finslerian Einstein-Hilbert equation (i.e., it is placed at the right-hand side of this equation, [14, eqn. (7)]). It is worth pointing out:

- ϕ can be reobtained as a Lagrangian energy-momentum by inserting it directly as a term in the background Lagrangian [16, eqn. (75)]. However, the Lagrangian is not natural then, as it depends on the variables of M (recall [16, Appendix 3, §(a)]).
- As discussed above, such a function allows one to construct several tensors, in particular the vertical Hessian $\partial^2\phi/\partial\dot{x}^\mu\partial\dot{x}^\nu$ (as in (4.12)), which also might play a role to compare with the relativistic $T^{\mu\nu}(x)$.

Anyway, starting at the 1PDF ϕ, another Finslerian interpretations would be possible. In particular, one can define the *energy momentum distribution* $\phi(u)u$. Then, given an observer $v \in \Sigma$ and a g_v-unit vector, the *w-energy momentum* might be defined as

$$g_v(u, w)\phi(u).$$

In particular, when $w = v$ this would be the *energy perceived by v* and when w is unit and g_v-orthogonal to v would be (minus) the *momentum in the direction w* (compare with the discussion at the end of §4.3.2). So, an alternative stress-energy tensor perceived by each observer $v \in \Sigma$ might be defined as the anisotropic tensor:

$$T_v(w, z) = \int_{\Sigma_{\pi(v)}} g_v(u, w)g_v(u, z)\phi(u)\, d\mathrm{vol}_{g_v}, \tag{4.13}$$

where the integration in u is carried out with the volume form of $(\Sigma_{\pi(v)}, g_v)$, denoted by $dvol_{g_v}$.

4.4 Divergence of anisotropic vector fields

After studying the basic properties of the Finslerian stress-energy tensor T, our next aim is to analyze the meaning and significance of the *infinitesimal conservation law* $\mathrm{div}(T) = 0$. Along this and the next section, we will always consider an anisotropic tensor $T \in \mathcal{T}^1_1(M_A)$ interpreted as an endomorphism of anisotropic vector fields. $T^\flat \in \mathcal{T}^0_2(M_A)$ and $T^\sharp \in \mathcal{T}^0_2(M_A)$ will be defined on vectors and 1-forms by $T^\flat(X,Y) := g(X,T(Y))$ and $T^\sharp(\theta,\eta) := g^*(T^*(\theta),\eta)$ resp., where g^* is the inverse fundamental tensor and T^* is the transpose of T. They will have components $\left(T^\flat\right)_{ij} = g_{il}T^l_j =: T_{ij}$ and $\left(T^\sharp\right)^{ij} = T^i_l g^{lj} =: T^{ij}$, and in principle we will not even assume that these are symmetric. We will be assuming that M is orientable an oriented. This is not restrictive: one could always reduce the theory to this case by pulling back all the objects (the fibered manifold $A \to M$ included) to the oriented double cover of M [27, Ch. 15].

Let us briefly recall the mathematically precise meaning of the conservation laws in classical General Relativity (g, T and X isotropic). One has

$$\mathrm{div}(T(X)) = \nabla_i(T^i_j X^j) = \nabla_i T^i_j X^j + T^i_j \nabla_i X^j = \mathrm{div}(T)(X) + \mathrm{trace}(T(\nabla X)) \quad (4.14)$$

with ∇ the Levi-Civita connection. The first contribution vanishes due to $\mathrm{div}(T) = 0$, and there are different situations in which the second one vanishes as well. For instance, if $T^\flat(-, \nabla\_X)$ is antisymmetric, then

$$\mathrm{trace}(T(\nabla X)) = T^i_j \nabla_i X^j = g^{il} T_{lj} \nabla_i X^j = \frac{1}{2} g^{il} \left(T_{lj}\nabla_i X^j + T_{ij}\nabla_l X^j\right) = 0, \quad (4.15)$$

and if $T^\flat$ is symmetric and $\nabla X^\sharp$ is antisymmetric (equiv., X is a Killing vector field), then also

$$\mathrm{trace}(T(\nabla X)) = g^{il} T_{lj} \nabla_i X^j = \frac{1}{2} T_{lj} \left(g^{li}\nabla_i X^j + g^{ji}\nabla_i X^l\right) = 0. \quad (4.16)$$

Anyway, whenever $\mathrm{trace}(T(\nabla X)) = 0$, one can integrate (4.14) and apply the pseudo-Riemannian divergence theorem to get the *integral conservation law*

$$\int_{\partial D} \iota_{T(X)}(d\mathrm{Vol}) = 0, \quad (4.17)$$

where $\overline{D}$ is a domain of appropriate regularity, ι is the interior product operator and $d\mathrm{Vol}$ is the metric volume form. In a sense that will be made more precise in §5, this is expressing that the total amount of X-momentum in a space region only changes along time as much as it flows across the spatial boundary of the region. In this way, there is no “creation" nor “destruction" of X-momentum in any space region.

Extending the infinitesimal or the integral conservation laws poses, first and foremost, the problem of appropriately defining the divergence of an anisotropic T. Observe that a priori it is not clear even how to define the divergence of a vector field Z, isotropic or not, as one could consider $\mathrm{trace}(\nabla Z)$ for different anisotropic connections ∇, mainly Chern’s and Berwald’s. An alternative is to seek for a more geometric, hence *unbiased*, definition. For instance, the *metric (anisotropic) volume form of L*,

$$d\mathrm{Vol} = \sqrt{\left|\det g_{ab}(x, y)\right|}\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \in \Omega_n(M_A) \quad (4.18)$$

for $(x^1, ..., x^n)$ positively oriented, is well-defined, and when $Z \in \mathfrak{X}(M)$ (i. e., Z is isotropic), so is the Lie derivative

$$\mathfrak{L}_Z : \mathcal{T}(M_A) \to \mathcal{T}(M_A)$$

(see [17, §5]). So, by analogy with the classical case, one could think of $\mathfrak{L}_Z(d\mathrm{Vol})$ for defining $\mathrm{div}(Z)$.

It turns out that the unbiased definition, including all $Z \in \mathfrak{X}(M_A)$, is achieved with a modification of this Lie derivative that we will regard as an extension of the classical Lie bracket. We devote the next subsection to the technical mathematical foundations of such an *anisotropic Lie bracket*, which needs of a nonlinear connection on $A \to M$ to be well-defined. All the maps $\mathcal{T}(M_A) \to \mathcal{T}(M_A)$ that will appear in §4.4.1 will be *(anisotropic) tensor derivations* in the sense of [17, Def. 2.6] and their local nature will be apparent, so we will not explicitly discuss it. For example, the Lie derivative along $Z \in \mathfrak{X}(M)$ is the only tensor derivation such that for $X \in \mathfrak{X}(M)$ and $f \in \mathcal{F}(A)$,

$$\mathfrak{L}_Z X = [Z, X], \qquad \mathfrak{L}_Z f = Z^{\mathrm{c}}(f) := Z^k \frac{\partial f}{\partial x^k} + y^k \frac{\partial Z^i}{\partial x^k} \frac{\partial f}{\partial y^i}. \tag{4.19}$$

4.4.1 Mathematical formalism of the anisotropic Lie bracket

During this subsection, we fix an arbitrary nonlinear connection given by $\mathrm{T}A = \mathrm{H}A \oplus \mathrm{V}A$ or by the nonlinear covariant derivative D (keep in mind (4.1) and (4.2)), and also an anisotropic vector field $Z \in \mathfrak{X}(M_A)$.

For $X \in \mathfrak{X}(M_A)$, it is very natural to consider the commutator of the horizontal lifts of Z and X:

$$\left[Z^{\mathrm{H}}, X^{\mathrm{H}}\right] = \left[Z^j \delta_j, X^k \delta_k\right] = \left(Z^j \delta_j X^i - X^j \delta_j Z^i\right) \delta_i + Z^j X^k \left[\delta_j, \delta_k\right] \in \mathfrak{X}(A).$$

We recall that $Z^j X^k \left[\delta_j, \delta_k\right]$ is always vertical. Indeed, $\left[\delta_j, \delta_k\right] = \mathcal{R}^i_{\ jk} \dot\partial_i$, where $\mathcal{R}$ is the curvature tensor of the nonlinear connection (see [22], where this curvature is regarded as an anisotropic tensor and the homogeneity of the connection is not really required). This means that the horizontal part of $\left[Z^{\mathrm{H}}, X^{\mathrm{H}}\right]$ has coordinates $Z^j \delta_j X^i - X^j \delta_j Z^i$, and this corresponds to a globally well-defined A-anisotropic vector field:

$$\mathfrak{l}^{\mathrm{H}}_Z X := \left(Z^j \delta_j X^i - X^j \delta_j Z^i\right) \partial_i \in \mathfrak{X}(M_A). \tag{4.20}$$

Definition 4.1. $\mathfrak{l}^{\mathrm{H}}_Z X$ *is the* anisotropic Lie bracket of Z and X with respect to the nonlinear connection HA.

Remark 4.1. The word "anisotropic" could be omited in the previous definition, in the sense that for $Z, X \in \mathfrak{X}(M_A)$, there is no other Lie bracket, isotropic or not, defined in general. Nonetheless, (4.20) makes apparent that when $Z, X \in \mathfrak{X}(M)$ (i. e., when Z and X are isotropic), $\mathfrak{l}^{\mathrm{H}}_Z X$ coincides with the standard Lie bracket $[Z, X]$ regardless of the connection.

Lemma 4.1. Given a nonlinear connection HA, $V \in \mathfrak{X}^A(U)$, $f \in \mathcal{F}(A)$ and anisotropic vector fields $X, Z \in \mathfrak{X}(M_A)$, it holds that

$$Z^{\mathrm{H}}(f) = Z(f(V)) - \dot{\partial}_{\mathrm{D}_Z V} f, \tag{4.21}$$

$$\left(\mathfrak{l}^{\mathrm{H}}_Z X\right)_V = \left[Z_V, X_V\right] - \left(\dot{\partial}_{\mathrm{D}_Z V} X\right)_V + \left(\dot{\partial}_{\mathrm{D}_X V} Z\right)_V. \tag{4.22}$$

Proof. Observe that

$$\begin{aligned} Z(f(V)) - \dot{\partial}_{\mathrm{D}_Z V} f &= Z^i \left(\frac{\partial f}{\partial x^i}(V) + \frac{\partial f}{\partial y^j}(V) \frac{\partial V^j}{\partial x^i} \right) \\ &\quad - \frac{\partial f}{\partial y^j}(V) Z^k \left(\frac{\partial V^j}{\partial x^k} - N^j_k(V) \right) \\ &= Z^i \left(\frac{\partial f}{\partial x^i}(V) - \frac{\partial f}{\partial y^j}(V) N^j_i(V) \right) \\ &= Z^{\mathrm{H}}(f), \end{aligned}$$

which concludes (4.21). In particular, $\delta_i f(V) = \partial_i(f(V)) - \left(\dot{\partial}_{\mathrm{D}_{\partial_i} V} f\right)(V)$, and using this in (4.20), (4.22) follows. ∎

We also recall that the *torsion* of an A-anisotropic connection ∇ [17, (18)], [21, Def. 5] is the anisotropic tensor $\mathrm{Tor} \in \mathcal{T}^1_2(M_A)$ defined on first on isotropic fields $Z, X \in \mathfrak{X}(M)$ by $\mathrm{Tor}(Z, X) = \nabla_Z X - \nabla_X Z - [Z, X]$ and then extended by $\mathcal{F}(A)$-bilinearity. Therefore, it can be regarded as and $\mathcal{F}(A)$-bilinear map $\mathrm{Tor} : \mathfrak{X}(M_A) \times \mathfrak{X}(M_A) \to \mathfrak{X}(M_A)$ and it has coordinates

$$\mathrm{Tor}^i_{\ jk} = \Gamma^i_{\ jk} - \Gamma^i_{\ kj}, \tag{4.23}$$

where the $\Gamma^i_{\ jk}$'s are the Christoffel symbols of ∇.[10]

[10]This is not to be mistaken by the torsion of the nonlinear connection HA, which would have coordinates $N^i_{\ j \cdot k} - N^i_{\ k \cdot j}$ (even though this can be seen as a particular case of the torsion of some ∇ and hence it is also denoted by Tor in [22]).

Theorem 4.1. *Let a nonlinear connection* $\mathrm{T}A = \mathrm{H}A \oplus \mathrm{V}A$ *and an anisotropic vector field* $Z \in \mathfrak{X}(M_A)$ *be fixed.*

(A) If ∇ *is any* A*-anisotropic connection whose underlying nonlinear connection is* $\mathrm{H}A$*, then for any* $X \in \mathfrak{X}(M_A)$*,*

$$\mathrm{Tor}(Z, X) = \nabla_Z X - \nabla_X Z - \mathfrak{l}^{\mathrm{H}}_Z X \tag{4.24}$$

(where Tor *is the torsion of* ∇*).*

(B) By imposing the Leibniz rule with respect to tensor products and the commutativity with contractions, the map $X \mapsto \mathfrak{l}^{\mathrm{H}}_Z X$ *extends unequivocally to an (anisotropic) tensor derivation* $\mathfrak{l}^{\mathrm{H}}_Z : \mathcal{T}^r_s(M_A) \to \mathcal{T}^r_s(M_A)$ *given by*

$$\begin{aligned}\mathfrak{l}^{\mathrm{H}}_Z T(\theta^1, \dots, \theta^r, X_1, \dots, X_s) = Z^{\mathrm{H}}(T(\theta^1, \dots, \theta^r, X_1, \dots, X_s)) \\ - \sum_{\mu=1}^{r} T(\theta^1, \dots, \mathfrak{l}^{\mathrm{H}}_Z \theta^\mu, \dots, \theta^r, X_1, \dots, X_s) \\ - \sum_{\nu=1}^{s} T(\theta^1, \dots, \theta^r, X_1, \dots, \mathfrak{l}^{\mathrm{H}}_Z X_\nu, \dots, X_s)\end{aligned} \tag{4.25}$$

for $\theta^\mu \in \Omega_1(M)$ *and* $X_\nu \in \mathfrak{X}(M)$*. In coordinates, if*

$$T = T^{i_1,\dots,i_r}_{j_1,\dots,j_s}(x,y)\, \partial_{i_1} \otimes \dots \otimes \partial_{i_r} \otimes \mathrm{d}x^{j_1} \otimes \dots \otimes \mathrm{d}x^{j_s},$$

then

$$\left(\mathfrak{l}^{\mathrm{H}}_Z T\right)^{i_1,\dots,i_r}_{j_1,\dots,j_s} = Z^k \frac{\delta T^{i_1,\dots,i_r}_{j_1,\dots,j_s}}{\delta x^k} - \sum_{\mu=1}^{r} \frac{\delta Z^{i_\mu}}{\delta x^k} T^{i_1,\dots,k,\dots,i_r}_{j_1,\dots,j_s} + \sum_{\nu=1}^{s} \frac{\delta Z^k}{\delta x^{j_\nu}} T^{i_1,\dots,i_r}_{j_1,\dots,k,\dots,j_s}. \tag{4.26}$$

(C) The map

$$\mathfrak{L}^{\mathrm{H}}_Z := \mathfrak{l}^{\mathrm{H}}_Z - \dot\partial_{\mathfrak{l}^{\mathrm{H}}_Z \mathbb{C}} : \mathcal{T}(M_A) \to \mathcal{T}(M_A)$$

is also a tensor derivation. When $Z \in \mathfrak{X}(M)$*,*

$$\mathfrak{L}^{\mathrm{H}}_Z T = \mathfrak{L}_Z T \tag{4.27}$$

for all $T \in \mathcal{T}(M_A)$*, where* $\mathfrak{L}_Z$ *is the Lie derivative* (4.19)*, regardless of the nonlinear connection.*

(D) Given $V \in \mathfrak{X}^A(U)$ *and* $\omega \in \Omega_n(M_A)$ *(*$n = \dim M$*), it holds that*

$$\left(\mathfrak{l}^{\mathrm{H}}_Z \omega\right)_V = \mathfrak{L}_{Z_V}(\omega_V) - \dot\partial_{\mathrm{D}_Z V}\omega - \mathrm{trace}(\dot\partial_{\mathrm{D}V} Z)\omega. \tag{4.28}$$

Proof. (A) It is straightforward to compute that the right hand side of (4.24) is $\mathcal{F}(A)$-multilinear. Moreover, the identity is trivial on isotropic vector fields $X, Z \in \mathfrak{X}(M)$, as $\mathfrak{l}^{\mathrm{H}}_Z X = [X, Z]$ in this case, which concludes.

(B) Given $f \in \mathcal{T}^0_0(M_A) = \mathcal{F}(A)$, for $X \in \mathcal{T}^1_0(M_A) = \mathfrak{X}(M_A)$ it follows from (4.20) that

$$\mathfrak{l}^{\mathrm{H}}_Z(fX) = Z^{\mathrm{H}}(f)X + f\,\mathfrak{l}^{\mathrm{H}}_Z X.$$

Thus, in order to respect the Leibniz rule, the only possibility is to define

$$\mathfrak{l}^{\mathrm{H}}_Z f = Z^{\mathrm{H}}(f) = Z^k \frac{\delta f}{\delta x^k}. \tag{4.29}$$

Now, given $\theta \in \mathcal{T}^0_1(M_A) = \Omega_1(M_A)$, in order to respect again the Leibniz rule and the commutativity with contractions, the only possibility is to define $\mathfrak{l}^{\mathrm{H}}_Z\theta$ on every $X \in \mathfrak{X}(M_A)$ by

$$\left(\mathfrak{l}^{\mathrm{H}}_Z\theta\right)(X) = Z^{\mathrm{H}}(\theta(X)) - \theta(\mathfrak{l}^{\mathrm{H}}_Z X) = \left(Z^k\frac{\delta\theta_j}{\delta x^k} + \frac{\delta Z^k}{\delta x^j}\theta_k\right)X^j. \tag{4.30}$$

(4.29), (4.20) and (4.30) make apparent that $\mathfrak{l}^{\mathrm{H}}_Z$ is already local on functions, vector fields and 1-forms, and they allow to compute

$$\mathfrak{l}^{\mathrm{H}}_Z(\partial_i) = -\frac{\delta Z^k}{\delta x^i}\partial_k, \qquad \mathfrak{l}^{\mathrm{H}}_Z(\mathrm{d}x^j) = \frac{\delta Z^j}{\delta x^k}\mathrm{d}x^k. \tag{4.31}$$

Finally, given $T \in \mathcal{T}^r_s(M_A)$, one is led to define $\mathfrak{l}^{\mathrm{H}}_Z T$ by (4.25). Clearly, this indeed provides a tensor derivation and (4.26) follows from the evaluation of (4.25) at $(\mathrm{d}x^{i_1}, ..., \mathrm{d}x^{i_r}, \partial_{j_1}, ..., \partial_{j_s})$ together with (4.29) and (4.31).

(C) $\dot\partial_X : \mathcal{T}(M_A) \to \mathcal{T}(M_A)$ is a tensor derivation for any $X \in \mathfrak{X}(M_A)$, in particular for

$$X = \mathfrak{l}^{\mathrm{H}}_Z\mathbb{C} = \left(Z^j\delta_j y^i - y^j\delta_j Z^i\right)\partial_i = -\left(Z^jN^i_j + y^j\delta_j Z^i\right)\partial_i \tag{4.32}$$

(see (4.20)). Thus, the difference $\mathfrak{L}^{\mathrm{H}}_Z = \mathfrak{l}^{\mathrm{H}}_Z - \dot\partial_{\mathfrak{l}^{\mathrm{H}}_Z\mathbb{C}}$ is again a derivation. As for the last assertion, where $Z \in \mathfrak{X}(M)$, we are going to use [17, Prop. 2.7]. For $X \in \mathfrak{X}(M)$, we have

$$\mathfrak{L}^{\mathrm{H}}_Z X = \mathfrak{l}^{\mathrm{H}}_Z X = [Z, X] = \mathfrak{L}_Z X \tag{4.33}$$

(recall Rem. 4.1). For $f \in \mathcal{F}(A)$, we have

$$\begin{aligned}\mathfrak{L}^{\mathrm{H}}_Z f = \mathfrak{l}^{\mathrm{H}}_Z f - \dot\partial_{\mathfrak{l}^{\mathrm{H}}_Z\mathbb{C}} f &= Z^j\delta_j f + \left(Z^jN^i_j + y^j\delta_j Z^i\right)\dot\partial_i f\\ &= Z^j\left(\partial_j f - N^i_j\dot\partial_i f\right) + \left(Z^jN^i_j + y^j\delta_j Z^i\right)\dot\partial_i f\\ &= Z^j\partial_j f + y^j\delta_j Z^i\dot\partial_i f\\ &= \mathfrak{L}_Z f\end{aligned}$$

(see (4.29), (4.32), (4.1) and (4.19)). As $\mathfrak{L}^{\mathrm{H}}_Z$ and $\mathfrak{L}_Z$ act the same on isotropic vector field and anisotropic functions, they are equal.

(D) Observe that for $X \in \mathfrak{X}(M)$, the term $\dot\partial_{\mathrm{D}_Z V} X$ vanishes in (4.22). Moreover, if $Z \in \mathfrak{X}(M_A)$ and $f \in \mathcal{F}(A)$, then $Z^{\mathrm{H}}(f)_V = Z_V(f(V)) - \left(\dot\partial_{\mathrm{D}_Z V} f\right)(V)$. Given a local reference frame $E_1, ..., E_n \in \mathfrak{X}(U)$, and taking into account the last two identities and the definitions of $\mathfrak{l}^{\mathrm{H}}$ and $\mathfrak{L}$, it follows that

$$\left(\mathfrak{l}^{\mathrm{H}}_Z\omega\right)_V(E_1, ..., E_n) - \mathfrak{L}_{Z_V}(\omega_V)(E_1, ..., E_n) = -\dot\partial_{D_Z V}\omega(E_1, ..., E_n) - \sum_{i=1}^{n} \omega(E_1, ..., \dot\partial_{D_{E_i} V} Z, ..., E_n).$$

As $\omega(E_1, ..., \dot\partial_{D_{E_i} V} Z, ..., E_n) = E_i^*(\dot\partial_{D_{E_i} V} Z)\omega_V(E_1, ..., E_n)$, (4.28) follows. ∎

Definition 4.2. *The tensor derivation* $\mathfrak{l}^{\mathrm{H}}_Z : \mathcal{T}(M_A) \to \mathcal{T}(M_A)$ *defined in Th. 4.1 (B) is the* (anisotropic) Lie bracket with Z, *while* $\mathfrak{L}^{\mathrm{H}}_Z : \mathcal{T}(M_A) \to \mathcal{T}(M_A)$ *is the* (anisotropic) Lie derivative along Z, *both of them* with respect to the connection HA.

Remark 4.2 (Anisotropic Lie bracket and Lie derivative). The derivation $\mathfrak{L}^{\mathrm{H}}_Z$ defined in Th. 4.1 (C) would be the *Lie derivative along* Z *with respect to* HA. Analogously to the discussion of Rem. 4.1, what makes this name consistent is (4.27): whenever the Lie derivative along Z was already defined, $\mathfrak{L}^{\mathrm{H}}_Z$ coincides with it. Even though the Lie bracket and the Lie derivative are equal in the classical regime, it is heuristically useful to regard $\mathfrak{l}^{\mathrm{H}}$ as the anisotropic generalization of the former and $\mathfrak{L}^{\mathrm{H}}$ as that of the latter, in order to distinguish them. It is actually $\mathfrak{l}^{\mathrm{H}}$, and not $\mathfrak{L}$, which will be relevant for the definition of divergence. The reason is that the former, as we will see below, has a clear geometric interpretation in terms of flows, while the latter would just add the term $\dot\partial_{\mathfrak{l}^{\mathrm{H}}_Z \mathbb{C}}$ to that interpretation. Moreover, Th. 4.1 (D) actually corresponds to a Cartan formula for $\mathfrak{L}_Z$ whose full development we postpone for a future work. Thus, $\mathfrak{L}_Z(d\mathrm{Vol}) = \mathfrak{L}^{\mathrm{H}}_Z(d\mathrm{Vol})$ can be regarded as an initial guess for the divergence of Z, but we will not employ $\mathfrak{L}^{\mathrm{H}}$ from now on.

Let us observe that given a diffeomorphism $\psi_t : M \to M$ that is the flow of an isotropic vector field Z, we can define the pullback $\psi_t^*(\omega)$ of an anisotropic differential form $\omega \in \Omega_s(M_A)$ as the anisotropic form given by $\psi_t^*(\omega)_v(u_1, ..., u_s) := \omega_{P_t(v)}(\mathrm{d}\psi_t(u_1), ..., \mathrm{d}\psi_t(u_s))$, where $P_t(v)$ is the HA-parallel transport of v along the integral curve of Z and $u_1, ..., u_s \in \mathrm{T}_{\pi(v)}M$.

Proposition 4.1. If $Z \in \mathfrak{X}(M)$ and $\omega \in \Omega_s(M_A)$, then

$$\mathfrak{l}^{\mathrm{H}}_Z\omega = \lim_{t\to 0} \frac{\psi_t^*(\omega) - \omega}{t}, \tag{4.34}$$

where ψ_t is the (possibly local) flow of Z.

Proof. Observe that $\psi_t^*(\omega)_v$ can be obtained as $\psi_t^*(\omega_V)$ with V an extension of v such that $\mathrm{D}_Z V = 0$. Then (4.28) and the classical formula for the Lie derivative in terms of the flow imply (4.34). ▮

Remark 4.3. Even though, for convenience, we stated the previous geometrical interpretation for an s-form ω, it should be clear that it holds true for any r-contravariant s-covariant A-anisotropic tensor.

4.4.2 Lie Bracket definition of divergence

Finally, in this and the next subsections a pseudo-Finsler metric L defined on A is fixed again. In its presence, and in view of the Riemannian case and Prop. 4.1, the most natural way of defining the divergence of an anisotropic vector field Z is by $\mathfrak{L}_Z^{\mathrm{H}}(d\mathrm{Vol})$. Here there is a canonical choice for $\mathrm{H}A$: the metric nonlinear connection of L. The definition obtained this way is unbiased, in that one does not choose any anisotropic connection *a priori*. Notwithstanding, it will turn out to be most conveniently expressed in terms of the Chern connection.

Definition 4.3. *For $Z \in \mathfrak{X}(M_A)$, its* divergence with respect to the pseudo-Finsler metric L *is the anisotropic function* $\mathrm{div}(Z) \in \mathcal{F}(A)$ *defined by*

$$\mathfrak{L}_Z^{\mathrm{H}}(d\mathrm{Vol}) =: \mathrm{div}(Z)\, d\mathrm{Vol},$$

where $\mathrm{H}A$ *and* $d\mathrm{Vol}$ *are, resp., the metric nonlinear connection* (4.4) *and the metric volume form* (4.18) *of* L.

Remark 4.4. Even though we will keep assuming it for simplicity, the hypothesis of M being orientable is not really needed for this definition. As in pseudo-Riemannian geometry, on small enough open sets $U \subseteq M$ it is always possible to choose an orientation, define $d\mathrm{Vol}_U \in \Omega_n(M_A)$ with respect to it and put $\mathrm{div}(Z)|_{A\cap TU}\, d\mathrm{Vol}_U := \mathfrak{L}_Z^{\mathrm{H}}(d\mathrm{Vol}_U)$. The different definitions will be coherent because when the orientation changes, $d\mathrm{Vol}_U$ changes to $-d\mathrm{Vol}_U$ and

$$\mathfrak{L}_Z^{\mathrm{H}}(-d\mathrm{Vol}_U) = -\mathfrak{L}_Z^{\mathrm{H}}(d\mathrm{Vol}_U) = -\mathrm{div}(Z)|_{A\cap TU}\, d\mathrm{Vol}_U = \mathrm{div}(Z)_{A\cap TU}\left(-d\mathrm{Vol}_U\right).$$

In particular, when M is orientable, $\mathrm{div}(Z)$ is independent of the orientation choice.

Proposition 4.2. Let L be a fixed pseudo-Finsler metric defined on A, and let $Z \in \mathfrak{X}(M_A)$. If ∇ is any symmetric A-anisotropic connection such that its underlying nonlinear connection is the metric one and $\nabla_Z(d\mathrm{Vol}) = 0$, then

$$\mathrm{div}(Z) = \mathrm{trace}(\nabla Z), \tag{4.35}$$

or in coordinates,

$$\operatorname{div}(Z) = \frac{\delta Z^i}{\delta x^i} + \Gamma^i_{ik} Z^k \tag{4.36}$$

This, in particular, is true for the (Levi-Civita)–Chern anisotropic connection of L, so one can take the Christoffel symbols to be those of (4.5).

Proof. One expresses the Z-Lie bracket of the volume form in terms of the anisotropic connection, analogously to the isotropic case. From (4.18) and the fact that $\mathfrak{L}^{\mathrm{H}}_Z$ is a tensor derivation, we obtain

$$\begin{aligned}
\operatorname{div}(Z)\sqrt{|\det g_{ab}|} &= \operatorname{div}(Z)\, d\mathrm{Vol}(\partial_1, \dots, \partial_n) \\
&= \mathfrak{L}^{\mathrm{H}}_Z(d\mathrm{Vol})(\partial_1, \dots, \partial_n) \\
&= \mathfrak{L}^{\mathrm{H}}_Z(d\mathrm{Vol}(\partial_1, \dots, \partial_n)) - \sum_{i=1}^{n} d\mathrm{Vol}(\partial_1, \dots, \mathfrak{L}^{\mathrm{H}}_Z \partial_i, \dots, \partial_n).
\end{aligned}$$

(4.29) and the fact that $\mathrm{H}A$ is the underlying nonlinear connection of ∇ give

$$\mathfrak{L}^{\mathrm{H}}_Z(d\mathrm{Vol}(\partial_1, \dots, \partial_n)) = Z^{\mathrm{H}}(d\mathrm{Vol}(\partial_1, \dots, \partial_n)) = \nabla_Z(d\mathrm{Vol}(\partial_1, \dots, \partial_n)).$$

(4.24) and $\mathrm{Tor} = 0$ give

$$d\mathrm{Vol}(\partial_1, \dots, \mathfrak{L}^{\mathrm{H}}_Z \partial_i, \dots, \partial_n) = d\mathrm{Vol}(\partial_1, \dots, \nabla_Z \partial_i, \dots, \partial_n) - d\mathrm{Vol}(\partial_1, \dots, \nabla_{\partial_i} Z, \dots, \partial_n).$$

From these and $\nabla_Z(d\mathrm{Vol}) = 0$,

$$\begin{aligned}
\operatorname{div}(Z)\sqrt{|\det g_{ab}|} &= \nabla_Z(d\mathrm{Vol}(\partial_1, \dots, \partial_n)) - \sum_{i=1}^{n} d\mathrm{Vol}(\partial_1, \dots, \nabla_Z \partial_i, \dots, \partial_n) \\
&\quad + \sum_{i=1}^{n} d\mathrm{Vol}(\partial_1, \dots, \nabla_{\partial_i} Z, \dots, \partial_n) \\
&= \nabla_Z(d\mathrm{Vol})(\partial_1, \dots, \partial_n) + \sum_{i=1}^{n} d\mathrm{Vol}(\partial_1, \dots, \nabla_{\partial_i} Z, \dots, \partial_n) \\
&= \sum_{i=1}^{n} d\mathrm{Vol}(\partial_1, \dots, \nabla_{\partial_i} Z, \dots, \partial_n) \\
&= \operatorname{trace}(\nabla Z)\sqrt{|\det g_{ab}|},
\end{aligned} \tag{4.37}$$

where the last equality is reasoned analogously as in the proof of (4.28).

For the Chern connection, it can be checked that $\nabla(d\mathrm{Vol}) = 0$ by considering a parallel orthonormal basis with respect to a parallel observer V along the integral curves of any vector field. The coordinate expression of $\operatorname{trace}(\nabla Z)$ in this case concludes (4.36). ∎

4.4.3 Divergence theorem and boundary term representations

Our Lie bracket derivation allows us to obtain a statement of the Finslerian divergence theorem that subsumes both Rund's [36, (3.17)] and Minguzzi's [30, Th. 2]. This way, it does not need of computations in coordinates from the beginning nor of the "pullback metric" (g_V in our notation). Naturally, our statement does not include Shen's [38, Th. 2.4.2], as this one is an independent generalization of the Riemannian theorem not dealing with anisotropic differential forms nor vector fields.

Lemma 4.2. For $X \in \mathfrak{X}(M_A)$, the vertical derivative of $d\mathrm{Vol}$ is given by

$$\dot{\partial}_X(d\mathrm{Vol}) = \mathrm{C}^{\mathrm{m}}(X)\, d\mathrm{Vol}, \tag{4.38}$$

where C^{m} is the mean Cartan tensor of L (see (4.3)).

Proof. Let $E_1(t), ..., E_n(t)$ be a positively oriented g_{v+tX}-orthonormal basis for every $t \in [0, \varepsilon]$ for a certain $\varepsilon > 0$. Then $d\mathrm{Vol}_{v+tX}(E_1(t), ..., E_n(t)) = 1$ for all $t \in [0, \varepsilon]$. This implies that

$$\dot{\partial}_X(d\mathrm{Vol})_v(E_1(0), ..., E_n(0)) + \sum_{i=1}^{n} d\mathrm{Vol}_v(E_1(0), ..., \dot{E}_i(0), ..., E_n(0)) = 0.$$

Moreover, as $g_{v+tX}(E_i(t), E_i(t)) = \pm 1$,

$$2\mathrm{C}_v(E_i(0), E_i(0), X) + 2g_v(\dot{E}_i(0), E_i(0)) = 0.$$

Using this relation above, we conclude (4.38). ∎

In the present article, by a *domain* $\overline{D}$ we understand a nonempty connected set which coincides with the closure of its interior D; then its boundary is $\partial\overline{D} = \partial D$. Physically, it is very important to include examples in which different parts of ∂D have different causal characters, and this tipically leads to the boundary not being totally smooth. Hence, we will make a weaker regularity assumption that still allows one to apply Stokes' theorem on $\overline{D}$. A subset of M has 0 *m-dimensional measure* if its intersection with any embedded m-dimensional submanifold $\sigma \subseteq M$ is of 0 measure in the smooth manifold σ. Finally, the interior product of an s-form ω with a vector field X will be

$$\imath_X\omega := \omega(X, -, ..., -).$$

Theorem 4.2. *Let L be a fixed pseudo-Finsler metric defined on A. If*

(i) $Z \in \mathfrak{X}(M_A)$ is an anisotropic vector field,

(ii) $V \in \mathfrak{X}^A(U)$ *is an* A*-admissible field with* $U \subseteq M$ *open, and*
(iii) $\overline{D} \subseteq U$ *is a domain with* ∂D *smooth up to subset of* 0 $(n-1)$*-dimensional measure on* M *and* $\mathrm{Supp}(Z_V) \cap \overline{D}$ *compact,*

then

$$\begin{aligned}&\int_D \mathrm{div}(Z)_V d\mathrm{Vol}_V + \int_D \left\{ \mathrm{C}^{\mathrm{m}}(\mathrm{D}_Z V) + \mathrm{trace}(\dot\partial_{\mathrm{D}V} Z) \right\} d\mathrm{Vol}_V \\ &= \int_{\partial D} \iota_{Z_V}(d\mathrm{Vol}_V),\end{aligned} \tag{4.39}$$

where C^{m} *is the mean Cartan tensor and* $\mathrm{D}V$ *is computed with the metric nonlinear connection* (4.4).

Proof. The idea is to apply Stokes' theorem to $\mathfrak{L}_{Z_V}(d\mathrm{Vol}_V)$. But taking into account (4.28) and Lem. 4.2, it follows that

$$\mathfrak{L}_{Z_V}(d\mathrm{Vol}_V) = \mathfrak{L}^{\mathrm{H}}_Z(d\mathrm{Vol})_V + \left\{ \mathrm{C}^{\mathrm{m}}(\mathrm{D}_Z V) + \mathrm{trace}(\dot\partial_{\mathrm{D}V} Z) \right\} d\mathrm{Vol}_V,$$

concluding (4.39). ∎

Remark 4.5 (Riemannian and Finslerian unit normals). Let $i : \Gamma \hookrightarrow M$ be the inclusion of a smooth open subset $\Gamma \subseteq \partial D$.

(i) Even though we do not use the pseudo-Riemannian metric g_V to derive Th. 4.2, from our physical viewpoint it is natural to use it to re-express the boundary term. If Γ is non-g_V-lightlike, then for a g_V-normal field $\widehat{N}_V$ and a transverse field X along i, the form

$$d\sigma_V := \mathrm{sgn}(g_V(\widehat{N}_V, \widehat{N}_V)) \frac{\sqrt{\left| g_V(\widehat{N}_V, \widehat{N}_V) \right|}}{g_V(\widehat{N}_V, X)} i^*(\iota_X(d\mathrm{Vol}_V)) \in \Omega_{n-1}(\Gamma) \tag{4.40}$$

is nonvanishing and independent of X. In particular,

$$d\sigma_V = \frac{1}{\sqrt{\left| g_V(\widehat{N}_V, \widehat{N}_V) \right|}} i^*(\iota_{\widehat{N}_V}(d\mathrm{Vol}_V))$$

is independent of the scale of $\widehat{N}_V$, which we will always assume to be g_V-unitary and D-salient, so

$$d\sigma_V = i^*(\iota_{\widehat{N}_V}(d\mathrm{Vol}_V))$$

coincides with the hypersurface g_V-volume form of Γ. Taking into account that $i^*(\iota_{Z_V}(d\mathrm{Vol}_V))$ vanishes wherever Z_V is tangent to Γ and that $g_V(\widehat{N}_V, \widehat{N}_V) = \pm 1$, (4.40) allows us to represent and the right hand side of (4.39) as

$$\int_\Gamma \iota_{Z_V}(d\mathrm{Vol}_V) = \int_\Gamma g_V(\widehat{N}_V, \widehat{N}_V) g_V(\widehat{N}_V, Z_V) d\sigma_V. \tag{4.41}$$

In fact, this is how Rund's divergence theorem follows from Th. 4.2.

(ii) There is another way that one can try to represent the boundary term. Namely, assume that there exists a smooth $\xi : p \in \Gamma \to \xi_p \in A \cap \mathrm{T}_p M$ with $\mathrm{T}_p\Gamma = \mathrm{Ker}\, g_{\xi_p}(\xi_p, -)$ and $L(\xi_p) = \pm 1$ (in the Lorentz-Finsler case, it will necessarily be $L(\xi) = 1$). This is called a *Finslerian unit normal along* Γ. Analogously as in (i), one can put

$$d\Sigma_V^\xi := L(\xi) \frac{1}{g_\xi(\xi, X)} i^*(\iota_X(d\mathrm{Vol}_V)) = i^*(\iota_\xi(d\mathrm{Vol}_V)),$$

$$\int_\Gamma \iota_{Z_V}(d\mathrm{Vol}_V) = \int_\Gamma \epsilon_\xi L(\xi) g_\xi(\xi, Z_V)\, d\Sigma_V^\xi; \tag{4.42}$$

here, due to the possible orientation difference between both sides,

$$\epsilon_\xi = \begin{cases} 1, & \text{where } \xi \text{ is } D\text{-salient}, \\ -1 & \text{where } \xi \text{ is } D\text{-entering}. \end{cases}$$

In fact, this is how Minguzzi deduces his divergence theorem [30, Th. 2]. Note, however, that he does it under the hypothesis of vanishing mean Cartan tensor ($\mathrm{C}^{\mathrm{m}} = 0$), which implies that $d\Sigma_V^\xi$ is independent of V. As we do not require this, Th. 4.2 is more general statement than Minguzzi's.

(iii) The Finslerian unit normal presents some issues in the general case, as we are not taking $A = \mathrm{T}M \setminus 0$. In our physical interpretation, with L Lorentz-Finsler, A consists of timelike vectors, so asking for a Finslerian unit normal is only reasonable when Γ is *L-spacelike*, that is, $\mathrm{T}_p\Gamma \cap (A \cap \partial A) = \emptyset$ for $p \in \Gamma$. In such a case, the strong concavity of the indicatrix $\{v \in A_p : L(v) = 1\}$ guarantees the existence and uniqueness of ξ: one defines ξ_p to be the unique vector such that $\mathrm{T}_p\Gamma + \xi_p$ and the indicatrix are tangent at ξ_p.

(iv) Of course, if L comes from a pseudo-Riemannian metric on M, then $\xi = \epsilon_\xi \widehat{N}_V = \epsilon_\xi \widehat{N}$ and $d\Sigma_V^\xi = \epsilon_\xi\, d\sigma_V = \epsilon_\xi\, d\sigma$.

(v) It should be clear from this discussion that the form that one integrates on the right hand side of (4.39) is always the same and that the only difference between Rund's and Minguzzi's divergence theorems is how each of them represents it. Notwithstanding, this is an important difference, for the boundary terms (4.41) and (4.42) could potentially have different physical interpretations.

4.5 Divergence of anisotropic tensor fields

Our developments of the previous section will allow us to obtain integral Finslerian conservation laws for a tensor T with $\operatorname{div}(T) = 0$. We obtain one for each $V \in \mathfrak{X}^A(U)$ satisfying certain hypotheses. Physically, T can be interpreted as an anisotropic stress-energy tensor and V as an observer field. We will also revisit two of the main examples with a clearer physical interpretation: Special Relativity and the conservation of the "total energy of the universe". In order to do all this, let us see how the Chern connection enters the Finslerian definition of $\operatorname{div}(T)$.

4.5.1 Definition of divergence with the Chern connection

Prop. 4.2 motivates the most natural definition of divergence of $T \in \mathcal{T}_1^1(M_A)$. Namely, by analogy with the classical case, we shall require (4.14) to hold for any anisotropic vector field $X \in \mathfrak{X}(M_A)$. This makes the Chern connection appear now: it is the only Finslerian connection ∇ for which one can assure that (4.35) holds independently of $Z := T(X)$. We shall also explore the conditions under which the term $\operatorname{trace}(\nabla Z)$ vanishes in the general Finslerian setting.

Proposition 4.3. Let L be a fixed pseudo-Finsler metric defined on A with metric nonlinear connection $\mathrm{H}A$ and Chern anisotropic connection ∇. Also, let $S \in \mathcal{T}_2^0(M_A)$ be symmetric, $v \in A$, $T \in \mathcal{T}_1^1(M_A)$ and $X \in \mathfrak{X}(M_A)$.

(A) The following are equivalent.

- (Ai) $S_v(-, \nabla^v_- X)$ is antisymmetric.
- (Aii) $\nabla^v X$ is anti-self-adjoint with respect to S_v, that is, $S_v(\nabla^v_- X, -) = -S_v(-, \nabla^v_- X)$.
- (Aiii) $\left(\mathfrak{L}^{\mathrm{H}}_X S\right)_v = \nabla^v_X S$.

(B) One has

$$\operatorname{div}(T(X)) - \operatorname{trace}(T(\nabla X)) = \mathbf{C}^1_2(\nabla T)(X),$$

where $\mathbf{C}^1_2$ is the operator that contracts the contravariant index with the covariant one introduced by ∇.

(C) One has $\operatorname{trace}(T(\nabla X))(v) = 0$ assuming any of the following conditions.

(Ci) $T_v^\flat(-,\nabla^v_- X)$ is antisymmetric.
(Cii) $T_v^\flat$ is symmetric and $\left(\mathfrak{L}^{\mathrm{H}}_X g\right)_v = 0$.

Proof. For (A), take $Y, W \in \mathfrak{X}(M)$. The antisymmetry of $S_v(-,\nabla^v_- X)$ reads

$$S_v(\nabla^v_Y X, W) = S_v(W, \nabla^v_Y X) = -S_v(Y, \nabla^v_W X),$$

which is exactly the anti-self-adjointness of $\nabla^v X$ with respect to S_v. Besides, (4.29) and (4.24) together with Tor $= 0$ for the Chern connection give

$$\begin{aligned}
&\mathfrak{L}^{\mathrm{H}}_X S(Y,W)\\
&= X^{\mathrm{H}}(S(Y,W)) - S(\mathfrak{L}^{\mathrm{H}}_X Y, W) - S(Y, \mathfrak{L}^{\mathrm{H}}_X W)\\
&= X^{\mathrm{H}}(S(Y,W)) - S(\nabla_X Y - \nabla_Y X, W) - S(Y, \nabla_X W - \nabla_W X)\\
&= \nabla_X S(Y,W) + S(\nabla_Y X, W) + S(Y, \nabla_W X),
\end{aligned} \tag{4.43}$$

which shows that $\left(\mathfrak{L}^{\mathrm{H}}_X S\right)_v = \nabla^v_X S$ also is equivalent to the anti-self-adjointness.

For (B), all the computations in (4.14) hold formally the same in the general Finslerian case due to Prop. 4.2.

As for the vanishing of $\mathrm{trace}(T(\nabla X))(v)$, it follows from (Ci) by the same computations as in (4.15). Indeed, the antisymmetry can be expressed as

$$T_{lj}(v)\nabla_i X^j(v) + T_{ij}(v)\nabla_l X^j(v) = 0.$$

It also follows from (Cii) by (4.16). Indeed, $\left(\mathfrak{L}^{\mathrm{H}}_X g\right)_v = 0$ is equivalent to $\nabla^v X$ being anti-self-adjoint with respect to g_v, and this can be expressed as

$$g^{li}(v)\nabla_i X^j(v) + g^{ji}(v)\nabla_i X^l(v) = 0.$$

∎

Remark 4.6 ($\mathfrak{L}^{\mathrm{H}}_X g$ and Finslerian Killing fields). In classical Relativity (g, T and X isotropic), the second condition in (C ii) above would read $\left(\mathfrak{L}_X g\right)_{\pi(v)} = 0$, and $\mathfrak{L}_X g = 0$ would be equivalent to X being a Killing vector field. In the general case, X being Killing can be defined by the conditions $X \in \mathfrak{X}(M)$ and $\mathfrak{L}_X L = 0$ [17, §5], but (using

Th. 4.1 (C), the facts that $\dot{\partial}\mathbb{C} = \mathrm{Id}$ and $\mathrm{C}(\mathbb{C}, -, -) = 0$, and also (4.43))

$$\begin{aligned}
\mathfrak{L}_X L &= \mathfrak{L}_X(g(\mathbb{C},\mathbb{C})) \\
&= \mathfrak{L}_X g(\mathbb{C},\mathbb{C}) + 2g(\mathfrak{L}_X\mathbb{C},\mathbb{C}) \\
&= \left(\mathfrak{l}^{\mathrm{H}}_X g - \dot{\partial}_{\mathfrak{l}^{\mathrm{H}}_X\mathbb{C}} g\right)(\mathbb{C},\mathbb{C}) + 2g(\mathfrak{l}^{\mathrm{H}}_X\mathbb{C} - \dot{\partial}_{\mathfrak{l}^{\mathrm{H}}_X\mathbb{C}}\mathbb{C},\mathbb{C}) \\
&= \mathfrak{l}^{\mathrm{H}}_X g(\mathbb{C},\mathbb{C}) - 2\mathrm{C}(\mathbb{C},\mathbb{C},\mathfrak{l}^{\mathrm{H}}_X\mathbb{C}) + 2g(\mathfrak{l}^{\mathrm{H}}_X\mathbb{C} - \mathfrak{l}^{\mathrm{H}}_X\mathbb{C},\mathbb{C}) \\
&= \mathfrak{l}^{\mathrm{H}}_X g(\mathbb{C},\mathbb{C}) \\
&= \nabla g(\mathbb{C},\mathbb{C}) + g(\nabla_{\mathbb{C}}X,\mathbb{C}) + g(\mathbb{C},\nabla_{\mathbb{C}}X) \\
&= 2g(\mathbb{C},\nabla_{\mathbb{C}}X)
\end{aligned}$$

This way, we see that neither of X being Killing or $\mathfrak{l}^{\mathrm{H}}_X g = 0$ implies the other, and additionally we recover the characterization of [12, Prop. 6.1 (i)].

Definition 4.4. *Let L be a fixed pseudo-Finsler metric defined on A with (Levi-Civita–)Chern anisotropic connection ∇. For $T \in \mathcal{T}^1_1(M_A)$, its* divergence *with respect to L is defined as*

$$\mathrm{div}(T) := \mathbf{C}^1_2(\nabla T) \in \mathcal{T}^0_1(M_A) = \Omega_1(M_A),$$

where $\mathbf{C}^1_2$ is the operator that contracts the contravariant index with the covariant one introduced by ∇. In coordinates,

$$\mathrm{div}(T)_j = \nabla_i T^i_j = \delta_i T^i_j + \Gamma^i_{ik} T^k_j - \Gamma^k_{ij} T^i_k \tag{4.44}$$

for the Christoffel symbols of (4.5).

Remark 4.7 (Divergence vs. raising and lowering indices).

(i) First and foremost, by construction, (4.14) indeed holds for any $X \in \mathfrak{X}(M_A)$. At this point, it is important that the connection with which one defines trace(∇X) is the Chern one.

(ii) Thanks to the fact that the Chern connection parallelizes g, namely $\nabla_k g_{ij} = 0$ and $\nabla_k g^{ij} = 0$, the following hold:

$$g^{ik}\nabla_k T_{ij} = g^{ik} g_{il} \nabla_k T^l_j = \nabla_k T^k_j = \mathrm{div}(T)_j, \tag{4.45}$$

$$\nabla_i T^{ij} = \nabla_i T^i_l g^{lj} = g^{jl}\mathrm{div}(T)_l. \tag{4.46}$$

This means that one could define the divergences of $S \in \mathcal{T}^0_2(M_A)$ and $R \in \mathcal{T}^2_0(M_A)$ straightforwardly,[11] $\mathrm{div}(S) = \mathbf{C}_{1,3}(\nabla S) \in \mathcal{T}^0_1(M_A) = \Omega_1(M_A)$ and

[11] Here, $\mathbf{C}_{1,3}$ is the operator that (metrically) contracts the first index of S with the one introduced by ∇, and $\mathbf{C}^1_1$ is the operator that (naturally) contracts the first index of R with the one introduced by ∇.

$\mathrm{div}(R) = \mathbf{C}^1_1(\nabla R) \in \mathcal{T}^1_0(M_A) = \mathfrak{X}(M_A)$, and then (4.45) and (4.46) would read respectively

$$\mathrm{div}(T^\flat) = \mathrm{div}(T),$$

$$\mathrm{div}(T^\sharp) = \mathrm{div}(T)^\sharp.$$

(iii) Regardless of this, in general we are not assuming the symmetry of $T^\flat$ or $T^\sharp$, we only did in Prop. 4.3 (Cii). Instead, at the beginning of §5 we fixed a convention for the order of the indices in T_{ij} and T^{ij} (for example, $T^\flat(X,Y) = g(X,T(Y)) \neq g(T(X),Y)$) —in the remainder of §4 and with said condition (Cii) only.

4.5.2 Chern vs. Berwald

One needs to keep in mind a discussion present in [21]. The metric connection HA is the underlying nonlinear connection of an infinite family of A-anisotropic connections ∇. One of them is the (Levi-Civita)–Chern connection of L, which is the horizontal part of Chern-Rund's and Cartan's classical connections and has Christoffel symbols (4.5). All the others are this one plus an anisotropic tensor $Q \in \mathcal{T}^1_2(M_A)$ with $Q(-,\mathbb{C}) = 0$ when viewed as an $\mathcal{F}(A)$-bilinear map $\mathfrak{X}(M_A)\times\mathfrak{X}(M_A) \to \mathfrak{X}(M_A)$. In particular, for $Q = -\mathrm{Lan}^\sharp$, one gets the Berwald anisotropic connection of L, which is the horizontal part of Berwald's and Hasiguchi's classical connections and has Christoffel symbols (4.6). We did not a priori select any of these ∇'s.

In some of the previous literature [6, 29, 32, 33], the Finslerian divergence of vector fields was chosen to be defined directly with the Chern connection. In [36, 30], the quantity trace(∇Z), with ∇ the Chern anisotropic connection, was referred to as the divergence of Z, though only after it had appeared in the divergence theorem. We have proven that the most natural definition leads to this characterization, hence clarifying why using Chern's covariant derivative is not arbitrary. Moreover, we have seen that said derivative fulfills the natural requisite (4.14) and is compatible with the lowering and raising of indices; these are key properties when it comes to the stress-energy tensor T. Still, it is important to compare this with what happens when one uses the other most natural covariant derivative: Berwald's.

Remark 4.8 (Divergence in terms of the Berwald connection). Let ∇ be the Chern anisotropic connection of L, with Christoffel symbols (4.5), and $\widehat{\nabla}$ be the Berwald one, with symbols (4.6).

(i) (4.36) and (4.44) read respectively

$$\mathrm{div}(Z) = \widehat{\nabla}_i Z^i + \mathrm{Lan}_k Z^k = \mathrm{trace}(\widehat{\nabla} Z) + \mathrm{Lan}^{\mathfrak{m}}(Z),$$

$$\begin{aligned}\mathrm{div}(T)_j &= \widehat{\nabla}_i T^i_j + \mathrm{Lan}_k T^k_j - \mathrm{Lan}^k_{ij} T^i_k \\ &= \mathbf{C}^1_2(\widehat{\nabla} T)_j + \mathrm{Lan}^{\mathfrak{m}}(T)_j - \mathbf{C}^1_1(\mathrm{Lan}^\sharp(T(-),-))_j,\end{aligned}$$

where $\mathrm{Lan}^{\mathfrak{m}}$ is the mean Landsberg tensor (see (4.7)) and the contraction operators have the obvious meanings. Moreover, for $X \in \mathfrak{X}(M_A)$

$$\begin{aligned}\mathrm{trace}(T(\nabla X)) = T^i_j \nabla_i X^j &= T^i_j \widehat{\nabla}_i X^j + T^i_j \mathrm{Lan}^j_{ik} X^k \\ &= \mathrm{trace}(T(\widehat{\nabla} X)) + \mathrm{trace}(\mathrm{Lan}^\sharp(T(-), X)),\end{aligned}$$

which makes (4.14) consistent with the previous formulas.

(ii) One sees that the vanishing of $\mathrm{Lan}^{\mathfrak{m}}$ (or of the mean Cartan $\mathrm{C}^{\mathfrak{m}}$, see [39, (6.37)]) implies that the divergence of elements of $\mathfrak{X}(M_A)$ coincides with the trace of their Berwald covariant derivative. However $\mathrm{Lan}^{\mathfrak{m}} = 0$ (or even $\mathrm{C}^{\mathfrak{m}} = 0$) is not enough if one wants to obtain the same characterization for elements of $\mathcal{T}^1_1(M_A)$.

Remark 4.9 (Sufficient conditions for $\mathfrak{l}^{\mathrm{H}}_X g = 0$ and being Finslerian Killing). In Rem. 4.16 one could see that $X \in \mathfrak{X}(M)$ together with $\nabla_{\mathbb{C}} X = 0$ is sufficient for X to be Killing. This condition does not privilege the Chern connection ∇ against the Berwald $\widehat{\nabla}$:

$$\nabla_{\mathbb{C}} X = \widehat{\nabla}_{\mathbb{C}} X + \mathrm{Lan}^\sharp(\mathbb{C}, X) = \widehat{\nabla}_{\mathbb{C}} X$$

(see [17, (38)], where $\mathfrak{L}^\flat$ is what here we would denote $\mathrm{Lan}^\sharp$). However, when it comes to the stress-energy tensor, we have seen that the relevant condition is not this, but rather $\mathfrak{l}^{\mathrm{H}}_X g = 0$. Prop. 4.3 (A) implies that $\nabla^v X = 0$ is sufficient for $\left(\mathfrak{l}^{\mathrm{H}}_X g\right)_v = 0$, and this does privilege ∇ against $\widehat{\nabla}$.

4.5.3 Finslerian conservation laws and main examples

Compare the results here with the classical case (4.17) and also with [30].

Corollary 4.1. Let L be a fixed pseudo-Finsler metric defined on A. If

(i) $X \in \mathfrak{X}(M_A)$ is an anisotropic vector field,
(ii) $V \in \mathfrak{X}^A(U)$ is an A-admissible field with $U \subseteq M$ open,

(iii) $T \in \mathcal{T}^1_1(M_A)$ is an anisotropic 2-tensor, and
(iv) $\overline{D} \subseteq U$ is a domain with ∂D smooth up to subset of 0 $(n-1)$-dimensional measure on M and $\mathrm{Supp}(X_V) \cap \overline{D}$ compact,

then

$$\begin{aligned} &\int_D \mathrm{div}(T)(X)\, d\mathrm{Vol}_V + \int_D \mathrm{trace}(T(\nabla X))_V\, d\mathrm{Vol}_V \\ &\quad + \int_D \{\mathrm{C^m}(\mathrm{D}_{T(X)}V) + \mathrm{trace}(\dot\partial_{\mathrm{D}V} T(X))\}\, d\mathrm{Vol}_V = \int_{\partial D} \iota_{T(X)_V}(d\mathrm{Vol}_V), \end{aligned} \tag{4.47}$$

where $\mathrm{C^m}$ is the mean Cartan tensor and $\mathrm{D}V$ is computed with the metric nonlinear connection (4.4).

Proof. Just take $Z = T(X)$ in Th. 4.2 and use part (B) or Prop. 4.3 . ∎

Remark 4.10. Observe that (4.47) allows for an interpretation of the divergence of T in terms of the flow in the boundary. Consider a sequence of domains D_m such that their volumes go to zero when $m \to +\infty$ and consider an observer V such that is infinitesimally parallel at $p \in M$, namely, $\mathrm{D}V = 0$ in $p \in M$ and X such that $\nabla^v X = 0$. Then (4.47) and the mean value theorem imply that

$$\mathrm{div}(T)_v(X) = \lim_{m\to+\infty} \frac{1}{\mathrm{Vol}_V(D_m)} \int_{\partial D_m} \iota_{T(X)_V}(d\mathrm{Vol}_V).$$

In particular, $\mathrm{div}(T)_v = 0$ can be interpreted as that the observer v measures conservation of energy in its restspace.

Corollary 4.2. In the ambient of the previous corollary, assume:

(i) $\mathrm{div}(T)_V = 0$.
(ii) Any of the conditions (Ci) or (Cii) of Prop. 4.3 holds for $T^\flat_V$.
(iii) $\mathrm{C^m}(\mathrm{D}_{T(X)}V) + \mathrm{trace}\left\{\dot\partial_{\mathrm{D}V}(T(X))\right\} = 0$.

Then

$$\int_{\partial D} \iota_{T_V(X_V)}(d\mathrm{Vol}_V) = 0. \tag{4.48}$$

Proof. It follows from Cor 4.1, taking into account that the hypotheses (i), (ii) and (iii) imply that the three first integrals in (4.47) vanish. ∎

Remark 4.11 (Sufficient conditions for the hypotheses (i), (ii) and (iii)).

(i) Obviously, $\mathrm{div}(T) = 0$ suffices, but we do not need to assume that the divergence vanishes for all observers.

(ii) $X = \mathbb{C}$ suffices. In fact, $\nabla \mathbb{C} = 0$ [22, Prop. 2.9], so (Ci) of Prop. 4.3 holds for $T_V^\flat$. Thus, assuming the other two hypotheses, we get

$$\int_{\partial D} \iota_{T_V(V)}(d\mathrm{Vol}_V) = 0.$$

(iii) Although the hypothesis may seem artificial as it stands, there are a number of natural situations in which it is guaranteed. First, in classical Relativity (g, T and X isotropic), because $\mathrm{C^m} = 0$ and $\dot\partial(T(X)) = 0$; the result is then independent of V. Second, when the observer field is parallel ($\mathrm{D}V = 0$), trivially. Third, when $\mathrm{D}V = \theta \otimes V$ for some 1-form V and $T(X)$ is 0-homogeneous, because of Euler's theorem. And fourth, in the situation described in [30, §5.1] (Z is our $T(X)$, s is our V and I is our $\mathrm{C^m}$).

Remark 4.12 (Representations of (4.48)). One needs to keep in mind Rem. 4.5. For a smooth part Γ of ∂D, one can use the (salient) Riemannian unit normal to represent

$$\begin{aligned}\int_\Gamma \iota_{T_V(X_V)}(d\mathrm{Vol}_V) &= \int_\Gamma g_V(\widehat{N}_V, \widehat{N}_V) g_V(\widehat{N}_V, T_V(X_V))\, d\sigma_V \\ &= \int_\Gamma g_V(\widehat{N}_V, \widehat{N}_V) T_V^\flat(\widehat{N}_V, X_V)\, d\sigma_V\end{aligned} \tag{4.49}$$

when Γ is non-g_V-lightlike, and the Finslerian unit normal to represent

$$\int_\Gamma \iota_{T_V(X_V)}(d\mathrm{Vol}_V) = \int_\Gamma \epsilon_\xi L(\xi) g_\xi(\xi, T_V(X_V))\, d\Sigma_V^\xi$$

when L is Lorentz-Finsler and Γ is L-spacelike. This makes it possible to have the very same conservation law (4.48) written in distinct ways, and in the examples below we will see that different expressions are preferable in different situations.

In the remainder of the section, we analyze the Finslerian conservation laws in two settings in which L is Lorentz-Finsler. In particular, g has signature $(+, -, \dots, -)$, A determines a time orientation, $L > 0$ on A, and (A, L) is maximal with these properties. We also have regularity conditions at ∂A, and in fact one sees that Th. 4.2 and Cor. 4.2 still hold when allowing that $Z, X \in \mathfrak{X}(M_{\overline{A}})$, $T \in \mathcal{T}^1_1(M_{\overline{A}})$ and $V \in \mathfrak{X}^{\overline{A}}(U)$. Despite this, in both settings it will be necessary to take V as L-timelike, so the regularity at ∂A will not be used.

4.5.3.1 Example: Lorentz norms on an affine space

In this example, we shall particularize Cor. 4.2 to the easiest Finslerian setting in which we can assure that its hypothesis (iii) holds. Namely, the structure of an affine space automatically provides an infinite number of parallel observer fields, $V \in \mathfrak{X}^A(M)$ with $\mathrm{D}V = 0$.

To be preicse, suppose that $M = E$ is an affine space equipped with a *Lorentz norm* on an open conic subset $A_* \subseteq \vec{E} \setminus 0$ (a positive pseudo-Minkowski norm with Lorentzian signature in [20, Def. 2.11]). Under the usual identifications, such a norm can be seen as a Lorentz-Finsler L on $A \subseteq \mathrm{T}E \setminus \mathbf{0} \equiv E \times \left(\vec{E} \setminus 0\right)$ that is independent of the first factor. Consequently, its fundamental tensor is nothing more than a Lorentzian scalar product g_v for each $v \in A_*$. The metric nonlinear connection of L coincides with the canonical connection of E, hence so do the Chern and Berwald anisotropic connections.[12] This is what implies that the parallel $V \in \mathfrak{X}^A(E)$ correspond exactly to the elements $v \in A_*$.

Let us introduce some notation. Given $(p_0, v) \in A$ with $L(v) = 1$, we can consider the Lorentzian scalar product g_v and the orthogonal hyperplane $\mathscr{R} := p_0 + \vec{\mathscr{R}} := p_0 + \left\{ w \in \vec{E} :\ g_v(v, w) = 0 \right\}$. We get an isometry $(t, p) \in \mathbb{R} \times \mathscr{R} \mapsto p + tv \in E$, where $\mathscr{R}$ is equipped with $-\left.g_v\right|_{\mathscr{R}}$ (a Euclidean scalar product), $\mathbb{R} \times \mathscr{R}$ with $\mathrm{d}t^2 + \left.g_v\right|_{\mathscr{R}}$ (a Lorentzian one) and E with g_v. Let $\overline{\Omega}$ be a compact domain of $\mathscr{R}$ with $\partial\Omega \subseteq \mathscr{R}$ smooth up to a null $(n-2)$-dimensional measure set, and let $\widehat{n}_v$ be its salient unit $\left(-\left.g_v\right|_{\mathscr{R}}\right)$-normal. Then for $t_0 < t_1$, the compact domain $\overline{D} \equiv \left[t_0, t_1\right] \times \overline{\Omega} \subseteq E$ has the required smoothness to apply Cor. 4.2, its boundary is $\partial D = \left\{t_1\right\} \times \overline{\Omega} \cup \left[t_0, t_1\right] \times \partial\Omega \cup \left\{t_0\right\} \times \overline{\Omega}$, and its salient g_v-normal is given by

$$\left.\widehat{N}_v\right|_{\{t_1\} \times \Omega} = v, \qquad \left.\widehat{N}_v\right|_{]t_0, t_1[\times \partial\Omega} = \widehat{n}_v, \qquad \left.\widehat{N}_v\right|_{\{t_0\} \times \Omega} = -v;$$

$$g_v(-v, -v) = g_v(v, v) = L(v) = 1,$$

$$g_v(\widehat{n}_v, \widehat{n}_v) = -\left(-\left.g_v\right|_{\mathscr{R}}\right)(\widehat{n}_v, \widehat{n}_v) = -1.$$

Remark 4.13. For a $V \in \mathfrak{X}^A(E)$ identifiable with $v \in A_*$, we know that the hypothesis (iii) of Cor. 4.2 holds automatically. If (i) and (ii) hold too, then we get (4.48), for which we can use the representation (4.49). However, given the nature of the metric “nonlinear" and Chern “anisotropic" connections, it is easy to convince oneself that

[12] For instance, it is clear that in affine coordinates the components of the metric spray vanish, so the geodesics are the straight lines of E.

evaluating the result of anisotropic computations on this V is the same as first evaluating on V and then computing with isotropic tensors. For instance $\operatorname{div}(T)_V = \operatorname{div}(T_V)$ and $\left(\mathfrak{L}^{\mathrm{H}}_X g\right)_V = \mathfrak{L}_{X_V}(g_V)$. As a consequence, mathematically we get exactly the same conservation laws as if we just were in the Lorentzian affine space (E, g_v). Physically, though, different observers will measure different momenta.

Corollary 4.3. Let $V \in \mathfrak{X}^A(E)$ parallely identifiable with an $v \in A_*$. If $T \in \mathcal{T}^1_1(E_A)$ is such that $\operatorname{div}(T_V) = 0$ and $X \in \mathfrak{X}(E_A)$ is such that $T^\flat_V(-, \nabla^V_- X)$ is antisymmetric, or $T^\flat_V$ is symmetric and $\mathfrak{L}_{X_V}(g_V) = 0$, then

$$\begin{aligned} 0 = &\int_{\{t_1\}\times\Omega} T^\flat_V(V, X_V)\, d\sigma_V - \int_{\{t_0\}\times\Omega} T^\flat_V(V, X_V)\, d\sigma_V \\ &- \int_{]t_0,t_1[\times\partial\Omega} T^\flat_V(\widehat{n}_V, X_V)\, d\sigma_V, \end{aligned} \tag{4.50}$$

where $d\sigma_V$ is identifiable with the volume form of $-\left.g_v\right|_\Omega$ on $\{t_\mu\}\times\Omega$ and coincides with the volume form of $\left.g_v\right|_{]t_0,t_1[\times\partial\Omega}$ on $]t_0, t_1[\times\partial\Omega$.

Physically, even though Lorentz norms generalize Very Special Relativity [3], the classical interpretations of Special Relativity are still valid; we list them for completeness: v is an instantaneous observer at an event p_0, $\vec{\mathscr{R}}$ is its restspace and $\mathscr{R}$ is the *simultaneity hyperplane of* v, namely the "universe at an instant, say $t = 0$, as seen by v". The affine space structure allows for a canonical propagation of v to all of the spacetime. Hence, if $\overline{\Omega}$ is a space region at $t = 0$, then $\overline{D}$ is the "evolution of $\overline{\Omega}$ along the time interval $[t_0, t_1]$ as witnessed by v". (4.50) expresses that *the variation after some time of the total amount of X_v-momentum in Ω is exactly equal to the amount of it that flowed across $\partial\Omega$.*

4.5.3.2 Example: Cauchy hypersurfaces in a Finsler spacetime

Here we present a construction which manifestly generalizes that of the previous example, again with straightforward physical interpretations, and we find an estimate that allows us to interpret (4.50) when $\partial\Omega$ is "at infinity". We will take $V \in \mathfrak{X}^A(U)$ with $U \subseteq M$ open, and we recall that we will assume the hypotheses of Cor. 4.2.

Suppose that the Finsler spacetime (M, L) is *globally hyperbolic*. By this, we mean that there is some (smooth, for simplicity) *L-Cauchy hypersurface* $\mathcal{S} \subseteq M$: every inextensible L-timelike curve $\gamma : I \to M$ (thus $\dot{\gamma}(t) \in A$) meets $\mathcal{S}$ exactly once. Let us assume that there are two L-spacelike Cauchy hypersurfaces $\mathcal{S}_0, \mathcal{S}_1 \subseteq U$ which do not intersect.[13] Then the results of [2] can be automatically transplanted: there exists a foliation by spacelike Cauchy hypersurfaces $M \equiv \mathbb{R} \times \mathcal{S}$ such that $\mathcal{S}_0 \equiv \{t_0\} \times \mathcal{S}$ and $\mathcal{S}_1 \equiv \{t_1\} \times \mathcal{S}$. Taking the Finslerian unit normal ξ to each level $\{t\} \times \mathcal{S}$ produces an L-timelike field $\xi \in \mathfrak{X}^A(M)$. We can take this ξ to be our V, but we will not do so for the most part of this example.

Suppose also that $\left\{\overline{\Omega_{0,m}}\right\}$ is an exhaustion by compact domains of $\mathcal{S}_0$, namely $\overline{\Omega_{0,m}} \subseteq \Omega_{0,m+1}$ and $\bigcup_{m\in\mathbb{N}} \Omega_{0,m} = \mathcal{S}_0$, such that $\partial\Omega_{0,m} \subseteq \mathcal{S}_0$ is smooth a. e. For $p \in \mathcal{S}_0$, let γ_p be the integral curve of V starting at p, which necessarily meets $\mathcal{S}_1$ at a unique instant $t_p \in \mathbb{R}$. Put

$$\Omega_{1,m} := \bigcup_{p\in\Omega_{0,m}} \gamma_p(\{t_p\}) \subseteq \mathcal{S}_1, \qquad \Gamma_p := \gamma_p\left[\min\{0,t_p\}, \max\{0,t_p\}\right],$$

$$D_m := \bigcup_{p\in\Omega_{0,m}} \Gamma_p \subseteq U, \qquad \Gamma_m := \bigcup_{p\in\partial\Omega_{0,m}} \Gamma_p.$$

Remark 4.14. By construction,

(i) $\left\{\overline{\Omega_{1,m}}\right\}$ is again an exhaustion by compact domains of $\mathcal{S}_1$ such that $\partial\Omega_{1,m} = \bigcup_{p\in\partial\Omega_{0,m}} \gamma_p(\{t_p\}) \subseteq \mathcal{S}_1$ is smooth a. e.

(ii) $\overline{D_m}$ is a compact domain of U with $\partial D_m = \overline{\Omega_{1,m}} \cup \Gamma_m \cup \overline{\Omega_{0,m}} \subseteq U$ smooth a. e. We do not really need to consider the union of all the D_m's.

Next, for $Z \in \mathfrak{X}(M_A)$, we shall give the quantitative decay condition on (some components of) Z_V so that the integral

$$\int_{\Gamma_m} \iota_{Z_V}(d\mathrm{Vol}_V)$$

vanishes in the limit. The key fact for it will be that V is everywhere tangent to Γ_m (this is composed of γ_p's). In particular, as V is g_V-timelike, so must be Γ_m.

Remark 4.15. The presence of V allows us to define an auxiliar Riemannian metric h_V on U with norm $\|-\|_V$, which gives a very natural way of quantifying. Namely, if

[13] The case when they interesect can be also considered by taking into account that, then, the open set $M \setminus J^+(\mathcal{S}_1 \cup \mathcal{S}_2)$ is still globally hyperbolic and a Cauchy hypersurface $\mathcal{S}_3$ of this open subset will be also Cauchy for M (and it will not intersect any of the previous ones).

$\{e_0 = V_p/F(V_p), e_1, ..., e_n\}$ is an orthonormal basis for g_{V_p}, then we prescribe it to be also h_{V_p}-orthonormal; equivalently,

$$h_{V_p}(u, w) = 2g_{V_p}(u, \frac{V_p}{F(V_p)})g_{V_p}(w, \frac{V_p}{F(V_p)}) - g_{V_p}(u, w).$$

Then, by construction:

(i) The volume form of h_V coincides with that of g_V, namely $d\mathrm{Vol}_V$.

(ii) The salient unit h_V-normal to Γ_m coincides with the corresponding g_V-normal. We denote it by $\widehat{N}_V$, as in 4.12.

(iii) The hypersurface volume form of Γ_m with respect to h_V coincides with the one computed with g_V, namely $d\sigma_V = i_m^*(\iota_{\widehat{N}_V}(d\mathrm{Vol}_V))$ with $i_m : \Gamma_m \hookrightarrow U$ the inclusion. Hence we speak just of the *hypersurface volume of* Γ_m, namely $\sigma_V(\Gamma_m)$. As $\widehat{N}_V$ is g_V-orthogonal to V, and hence g_V-spacelike, we can use the representation

$$\begin{aligned}\int_{\Gamma_m} \iota_{Z_V}(d\mathrm{Vol}_V) &= \int_{\Gamma_m} g_V(\widehat{N}_V, \widehat{N}_V)g_V(\widehat{N}_V, Z_V)\, d\sigma_V \\ &= -\int_{\Gamma_m} g_V(\widehat{N}_V, Z_V)\, d\sigma_V.\end{aligned} \tag{4.51}$$

Thanks to (4.51) and the fact that $g_V(\widehat{N}_V, V) = 0$, we intuitively see that if Z_V is proportional to V at infinity and the hypersurface volume does not grow too much, then the integral will be negligible. To be precise, we require that

$$K_m\sigma_V(\Gamma_m) \longrightarrow 0 \quad (m \longrightarrow \infty), \tag{4.52}$$

where

$$\begin{aligned}K_m :&= \max_{\Gamma_m} \left\| Z_V - g_V(Z_V, \frac{V}{F(V)})\frac{V}{F(V)} \right\|_V \\ &= \max_{\Gamma_m} \left\{ \sqrt{g_V(Z_V, \frac{V}{F(V)})^2 - g_V(Z_V, Z_V)} \right\}.\end{aligned}$$

Corollary 4.4. In the above set-up, let $T \in \mathcal{T}_1^1(M_A)$, $X \in \mathfrak{X}(M_A)$ and $V \in \mathfrak{X}^A(U)$ be such that the hypotheses of Cor. 4.2 hold on all the D_m's, and put $Z := T(X)$. If the decay condition (4.52) holds too, then

$$\int_{\Omega_{1,m}} \iota_{Z_V}(d\mathrm{Vol}_V) + \int_{\Omega_{0,m}} \iota_{Z_V}(d\mathrm{Vol}_V) \longrightarrow 0 \quad (m \longrightarrow \infty), \tag{4.53}$$

where $\Omega_{1,m}$ is constructed from $\Omega_{0,m}$ by intersecting the integral curves of V with $\mathcal{S}_1$.

Proof. Cor. 4.2 can be applied on $\overline{D_m}$, as $\mathrm{Supp}(Z_V)\cap\overline{D_m}$ is always compact. This and the representation (4.51) give

$$0=\int_{\Omega_{1,m}}\iota_{Z_V}(d\mathrm{Vol}_V)+\int_{\Omega_{0,m}}\iota_{Z_V}(d\mathrm{Vol}_V)-\int_{\Gamma_m}g_V(\hat{N}_V,Z_V)d\sigma_V. \tag{4.54}$$

Using the definition of h_V (Rem. 4.15) and the Cauchy-Schwarz inequality,

$$\begin{aligned}0&\leq\left|\int_{\Gamma_m}-g_V(\hat{N}_V,Z_V)d\sigma_V\right|\\&\leq\int_{\Gamma_m}\left|g_V(\hat{N}_V,Z_V)\right|d\sigma_V\\&=\int_{\Gamma_m}\left|g_V(\hat{N}_V,Z_V-g_V(Z_V,\frac{V}{F(V)})\frac{V}{F(V)})\right|d\sigma_V\\&=\int_{\Gamma_m}\left|-h_V(\hat{N}_V,Z_V-g_V(Z_V,\frac{V}{F(V)})\frac{V}{F(V)})\right|d\sigma_V\\&\leq\int_{\Gamma_m}\left\|\hat{N}_V\right\|_V\left\|Z_V-g_V(Z_V,\frac{V}{F(V)})\frac{V}{F(V)}\right\|_V d\sigma_V\\&=\int_{\Gamma_m}\left\|Z_V-g_V(Z_V,\frac{V}{F(V)})\frac{V}{F(V)}\right\|_V d\sigma_V\\&\leq\int_{\Gamma_m}K_m\,d\sigma_V\\&=K_m\sigma_V(\Gamma_m),\end{aligned}$$

so if $K_m\sigma_V(\Gamma_m)$ tends to 0, then so does the integral along Γ_m in (4.54). ∎

Remark 4.16. In Cor. 4.4, if one of the integrals of $\iota_{Z_V}(d\mathrm{Vol}_V)$ along $\mathcal{S}_0$ or $\mathcal{S}_1$ exists in the Lebesgue sense, then so does the other and (4.53) reads

$$\int_{\mathcal{S}_1}\iota_{Z_V}(d\mathrm{Vol}_V)+\int_{\mathcal{S}_0}\iota_{Z_V}(d\mathrm{Vol}_V)=0.$$

Note that they could be $\pm\infty$, as we have not assumed, for instance, that Z_V is compactly supported in the union of all the D_m's. Rather, we have assumed the decay condition (4.52) alone.

Remark 4.17 (Sufficient conditions for (4.52)). As for ensuring the decay condition, there are two possible scenarios.

(i) The hypersurface volume $\sigma_V(\Gamma_m)$ stays bounded. Then, it is enough for (4.52) that $K_m\to 0$, and one could instead postulate the stronger condition that the maximum outside D_m tends to 0, which is independent of the concrete compact exhaustion.

(ii) $\sigma_V(\Gamma_m)$ grows without bound. In this case, one can just postulate that the decay of K_m compensates the growth of $\sigma_V(\Gamma_m)$, but this does depend on the compact exhaustion.

Notice that this is a purely Finslerian difficulty. Indeed, suppose that g, T and X were isotropic and that $Z = T(X)$ was timelike. Then one could just set $V := Z$ and then carry out all the construction. Cor. 4.2 would be independent of the observer field (and its hypothesis (iii) would hold trivially), and $K_m = 0$ regardless of Γ_m. This is how we get the following statement of the classical law.

Corollary 4.5. In the above set-up, suppose that L comes from a Lorentzian metric on M. Let $T \in \mathcal{T}^1_1(M)$ and $X \in \mathfrak{X}(M)$ be such that $\mathrm{div}(T) = 0$ and $T^\flat(-, \nabla_- X)$ is antisymmetric, or $T^\flat$ is symmetric and $\mathfrak{L}_X g = 0$. If $Z := T(X)$ is timelike, then

$$\int_{\Omega_{1,m}} \iota_{Z_V}(d\mathrm{Vol}_V) + \int_{\Omega_{0,m}} \iota_{Z_V}(d\mathrm{Vol}_V) \longrightarrow 0 \quad (m \longrightarrow \infty),$$

where $\Omega_{1,m}$ is constructed from $\Omega_{0,m}$ by intersecting the integral curves of Z with $\mathcal{S}_1$.

Remark 4.18 (Conservation in terms of the Finslerian unit normal).

(i) One could try to represent also the integrals of (4.53) in terms of $d\sigma_V$, as in §4.5.3.1. However, according to Rem. 4.12, that would require assuming that $\mathcal{S}_\mu$ is non-g_V-lightlike, which is not very reasonable when all we know is that $\mathcal{S}_\mu$ L-spacelike and L-Cauchy.

(ii) On the other hand, in terms of the Finslerian unit normal ξ, (4.53) reads

$$\int_{\Omega_{1,m}} g_\xi(\xi, T_V(X_V))\, d\Sigma^\xi_V - \int_{\Omega_{0,m}} g_\xi(\xi, T_V(X_V))\, d\Sigma^\xi_V \longrightarrow 0 \tag{4.55}$$

when $m \to \infty$. The sign in front of the second integral is explained as follows (see Rem. 4.5 (ii)). $d\Sigma^\xi_V$ selects an orientation on each $\Omega_{\mu,m}$: the one for which $d\mathrm{Vol}_V(\xi, -, \dots, -)$ is positive. However, in (4.53) $\Omega_{1,m}$ already had an orientation $\mathfrak{O}_1$ and $\Omega_{0,m}$ had $\mathfrak{O}_0$: the D_m-salient ones. Necessarily,[14] exactly one of these

[14] Suppose, for instance, that $\mathcal{S}_1$ lays in the future of $\mathcal{S}_0$: the γ_p's departing from $\Omega_{0,m}$ reach points $\gamma_p(t_p) \in \Omega_{1,m}$ with $t_p > 0$. Take bases $(e_1, \dots, e_{n-1})$ for $\mathrm{T}_p\Omega_{0,m}$ and $(e'_1, \dots, e'_{n-1})$ for $\mathrm{T}_{\gamma_p(t_p)}\Omega_{1,m}$ such that $(V_p, e_1, \dots, e_{n-1})$ and $(V_{\gamma_p(t_p)}, e'_1, \dots, e'_{n-1})$ are $d\mathrm{Vol}$-positive. Then $(e_1, \dots, e_{n-1})$ and $(e'_1, \dots, e'_{n-1})$ are both $d\Sigma^\xi_V$-positive (ξ and V always lie in the same half-space), the former is $\mathfrak{O}_0$-negative (V is D_m-entering at $\mathcal{S}_0$) and the latter is $\mathfrak{O}_1$-positive (V is D_m-salient at $\mathcal{S}_1$).

agrees with the $d\Sigma_V^\xi$-orientation: $\mathfrak{D}_1$ if $\mathcal{S}_1$ lays in the future of $\mathcal{S}_0$ and $\mathfrak{D}_0$ if it is the opposite. Notice that this, and hence (4.55), would fail if the Cauchy hypersurfaces crossed.

(iii) In the case $V = \xi$, (4.55) becomes

$$\int_{\Omega_{1,m}} T_\xi^\flat(\xi, X_\xi)\, d\Sigma_\xi - \int_{\Omega_{0,m}} T_\xi^\flat(\xi, X_\xi)\, d\Sigma_\xi \longrightarrow 0,$$

a conservation law in which all the terms are purely Finslerian.

Summing up, in this example we have proven a Finslerian (observer-dependent) version of the classical law that *the total amount of X-momentum in the universe is conserved* (Cor. 4.4). Our formulation is asymptotic, so it is valid even for infinite total X_V-momentum (Rem. 4.16). We have recovered the classical law (Cor. 4.5), which always holds under hypotheses on T and X alone, while in the general Finslerian case nontrivial difficulties appear in the regime of big separation between the Cauchy hypersurfaces (high $\sigma(\Gamma_m)$, Rem. 4.17). Finally, we have expressed the law naturally in terms of the Finslerian unit normal (see (4.55)).

4.6 Conclusions

About the physical interpretation of T, §4.3:

1. *Heuristic interpretations from fluids*, §4.3.1 and 4.3.2 Possible breakings of Lorentz-invariance lead to non-trivial transformations of coordinates between observers. Such transformations are still linear and permit a well-defined energy-momentum vector at each tangent space T_pM, §4.3.1.
   However, the stress-energy-momentum T must not be regarded as a tensor on each T_pM, but as an anisotropic tensor. This depends intrinsically on each observer $u \in \Sigma$ and may vary with u in a nonlinear way. Indeed, the breaking of Lorentz invariance does not permit to fully replicate the relativistic arguments leading to (isotropic) tensors on M, even though classical interpretations of the anisotropic T in terms of fluxes can be maintained, §4.3.2.
2. *Lagrangian viewpoint*, §4.3.3. In principle, the interpretations of Special Relativity about the canonical energy-momentum tensor associated with the invariance by translations remain for Lorentz norms and, thus, in Very Special Relativity. In the case of Lorentz-Finsler metrics, some issues to be studied further appear:

(a) The canonical stress-energy tensor in Relativity $\delta S_{matter}/\delta g^{\mu\nu}$ leads to different types of (anisotropic) tensors in the Finslerian setting (a scalar function $\delta S_{matter}/\delta L$ on $A \subseteq \mathrm{T}M$ in the Einstein-Hilbert setting, higher order tensors in Palatini's). Starting at such tensors, different alternatives to recover the heuristic physical interpretations in terms of a 2-tensor appear.

(b) In the particularly interesting case of a kinetic gas [14, 16], the 1-PDF ϕ becomes naturally the matter source for the Euler-Lagrange equation of the Finslerian Einstein-Hilbert functional. However, the variational derivation of ϕ is obtained by means of a non-natural Lagrangian. This might be analyzed by sharpening the framework of variational completion for Finslerian Einstein equations [13].

About the divergence theorem for anisotropic vector fields Z, §4.4:

1. §4.4.1: For any Lorentz Finsler metric L, there is a natural definition of *anisotropic Lie bracket derivation along Z*, which depends only on the nonlinear connection $\mathrm{H}A$ and admits an interpretation by using flows.
2. §4.4.2: This bracket allows one to give a natural definition of $\mathrm{div}(Z)$ which depends exclusively on $\mathrm{H}A$ and the volume form of L. This provides a geometric interpretation for the definition of divergence introduced by Rund [36].
3. A general divergence theorem is obtained (Th. 4.2) so that §4.4.3:

   (a) It can be seen as a conservation law for Z measured by each observer field V, even if the conserved quantity depends on V.

   (b) The computation of the boundary term is intrinsically expressed in terms of forms. However, several metric elements can be used to re-express it, in particular the normal vector field for:
   (i) the pseudo-Riemannian metric g_V (Rund), or (ii) the pseudo-Finsler metric L, when L is defined on the whole $\mathrm{T}M$ (Minguzzi).

About the conservation of the stress-energy T §4.5:

1. §4.5.1 and 4.5.2: The computation of $\mathrm{div}(T)$ privileges the Levi-Civita–Chern anisotropic connection, showing explicit equivalence with Rund's approach.
2. Cors. 4.1 and 4.2: A vector field $T(X)_V$ on M is preserved assuming that some natural elements vanish on V for T, X and $\mathrm{D}V$.
3. §4.5.3: Natural laws of conservation on Cauchy hypersurfaces under general conditions (including rates of decay for unbounded domains) can be obtained by a combination of the techniques (i) and (ii) in the item 3. (b) above.

Appendix. Kinematics: observers and relative velocities

Here, we discusss a series of different possibilities for the notion of relative velocity between two observers, each one with a well-defined geometric construction. This is done as an academic exercise, because we do not discuss experimental issues (compare with [25, 34]). However, it is worth emphasizing that all the possibilities studied here are intrinsic to the geometry of a flat model and, thus to any Finsler spacetime.

Start at an affine space endowed with a Lorentz norm let $u, u' \in \Sigma$ be two distinct observers and consider the plane $\Pi := \mathrm{Span}\{u, u'\} \subset V$, which intersects transversally C and inherits a Lorentz Finsler norm with indicatrix $\Sigma_\Pi := \Pi \cap \Sigma$. Recall that both tangent spaces $\mathrm{T}_u\Pi$ and $\mathrm{T}_{u'}\Pi$ inherit naturally a Lorentz scalar product by restricting the fundamental tensors g_u and $g_{u'}$, resp. Moreover, their (1-dimensional) restspaces $l := \mathrm{T}_u\Sigma_\Pi$, $l' := \mathrm{T}_{u'}\Sigma_\Pi$ also inherit a positive definite metric. In what follows, only the geometry of Π will be relevant.

The Lorentz metric g_Π up to a constant

Notice that $\Pi \cap C_p$ is composed by two half-lines spanned by two C-lightlike directions $w_\pm$; we will consider the orientation Π provided by the choice (w_+, w_-). One can determine a scalar product g_Π in Π (which is unique up to a positive constant), regarding both w_+ and w_- as g_Π-lightlike in the same causal cone. It is easy to check that Σ must be a strongly convex curve which converges asymptotically to the vector lines spanned by $w_\pm$. This implies both $u \in \Sigma$ will be timelike for g_Π and its restspace l will be g_Π-spacelike; we can assume also that the orientation l_+ in l is induced by the chosen w_+.

Notice that $g_u(u, w_\pm) \geq 0$ by the fundamental inequality, but $w_\pm$ might be timelike or spacelike for g_u (although $g_u(u, w_\pm) \to 0$ as $u \to w_\pm$). This possibility might be regarded as a possible measurement of the speed of light with respect to u by the observers in Π, namely, this velocity is in the orientation l_+ when w_+ is g_u-spacelike and smaller than 1 when it is timelike. However, a priori it is not clear an operational way to carry out such a measurement. Moreover such a measurement might be regarded as something non-intrinsic to the speed of light but to the way of measuring it.

Nevertheless, as pointed out in [1, Section 6], there are several effects which might lead to a measurement of different speeds of light in different directions. So, we will consider that each Π has its own speeds of light $c_\Pi^\pm$ in each spacelike orientation $l_\pm$. Indeed, given u and an orientation l^+, the *speed of light* c_Π^+ will be defined as the the supremum of the relative velocities between u and all the observers u' such that $u'-u$ yields the orientation l^+. Next, we will explain several possible meanings of these velocities. To avoid cluttering, next we will write c_Π, assuming that the appropriate choice in $c_\Pi^\pm$ is done for each u'.

Simple relative velocity

As g_u determines naturally a Lorentz metric on V, we can define the *simple relative velocity* $v_u^s(u')$ of u' measured by u as the usual g_u-relativistic velocity between u, u' normalized to c_Π, i.e.

$$v_u^s(u') = c_\Pi \tanh(\theta) \quad \text{where} \quad \cosh\theta = -g_u(u,u') > 1,$$

(the latter by the reversed fundamental inequality). Clearly, $v_{u'}^s(u) \neq v_u^s(u')$ in general, but this does not seem a drawback in the Finslerian setting.

A support for the physical plausibility of this velocity is that one could expect that each observer u will work as in Special Relativity just choosing an orthonormal frame of g_u. The possibility $g_u(v,v) \neq 1$ might seem ackward from a dynamical viewpoint (see below), but it seems harmless as far as only kinematics is being considered. In principle, the comparison between the measurements of the two observers would be geometrically possible by using the unique isometry of $(T_u\Pi, g_u)$ to $(T_{u'}\Pi, g_{u'})$ which maps u into u' and is consistent with orientations induced from Π. What is more, this isometry can also be extended to a natural isometry from (T_uV, g_u) to $(T_{u'}V, g_{u'})$, namely, regard (Σ, g) as a Riemannian metric and use the parallel transport from u to u' along the segment of the curve $\Pi \cap \Sigma$ from u to u'. However, the following fact might suggest to explore further possibilities.

Remark 4.19. Assume that Σ is modified into the indicatrix $\bar\Sigma$ of another Lorentz-Finsler norm so that (i) $\bar\Sigma = \Sigma$ around u and (ii) $u' \in \bar\Sigma$ but its $\bar\Sigma$ restspace $\bar l'$ is different from l'. Then, the simple velocity would remain unaltered, i.e., $\bar v_u^s(u') = v_u^s(u')$.

Velocity as a distance between observers

Notice that Σ can be regarded as a Riemannian manifold with the restriction of the fundamental tensor g and, then, $\Sigma \cap \Pi$ can be regarded as a curve whose length can be computed. Then, the *observers' distance velocity* is defined as:

$$v^d(u, u') = c_\Pi \tanh \left(\text{length}_g \{\text{segment of } \Sigma \cap \Pi \text{ from } u \text{ to } u'\}\right).$$

Notice that this velocity is symmetric and it generalizes directly the one in Special Relativity providing a geometric interpretation for the addition of velocities. Recall that $v^d(u, u')$ has been defined essentially as a distance in $\Sigma \cap \Pi$, where Π depends of each pair of observers, thus, one might have $v^d(u, u') + v^d(u', u'') < v^d(u, u'')$ when $n > 2$. If one prefers to avoid such a possibility, it is enough to consider g-distance in the whole space of observers Σ (*observers' space distance velocity*), at least in the case that c_Π is regarded as independent of Π.

Remark 4.20. In the case studied in Remark 4.19, one would have $\bar{v}^d(u, u') \neq v^d(u, u')$ in general. However, the relative position of the restspaces l and l' does not play any special role.

Length-contraction and velocity

Consider a segment S of l with g_u-length ℓ and the strip of V obtained by translating S in the direction of u. Let S' be the intersection of this strip with l', which will be a new segment of $g_{u'}$-length ℓ'. Let $\lambda = \ell'/\ell$ be the *length-contraction parameter.* In the relativistic case, $\lambda < 1$ and $\lambda \to 0$ as $u' \to C_\Pi$. The former property does not hold for a general Lorentz norm but the latter does. So, whenever $\lambda < 1$ holds, we can define the *length-contractive velocity* $v^c_u(u')$ *of* u' *with respect to* u as:

$$v^c_u(u') = c_\Pi \sqrt{1 - \lambda^2}.$$

Again, this velocity is not symmetric. Because of the strong convexity of Σ, a different observer u' will have a different restspace l', but this does not imply a different length ℓ' nor velocity $v^c_u(u')$. However, this velocity gives a comparison between restspaces which was absent in the previous two velocities.

Symmetric Lorentz velocities in Π

Let us consider the Lorentzian scalar product g_Π en Π, unique up to a positive constant (which will be irrelevant for our purposes) introduced above. Recall that u and u' were timelike for g_Π and, moreover, both l and l' were spacelike. Now, we can define two velocities between u and u': the *simple Lorentz velocity*,

$$v^s(u,u') = c_\Pi \tanh(\theta) \quad \text{where} \quad \cosh\theta = -\frac{g_\Pi(u,u')}{\sqrt{g_\Pi(u,u)g_\Pi(u',u')}},$$

and the *length-contractive Lorentz velocity*,

$$v^c(u,u') = c_\Pi \tanh(\theta) \quad \text{where} \quad \cosh\theta = -\frac{|g_\Pi(n,n')|}{\sqrt{g_\Pi(n,n)g_\Pi(n',n')}},$$

where, in the latter, n, n' are g_Π-timelike vectors orthogonal to l, l', resp.

Clearly, both velocities are symmetric. Their appearance might be physically sound because the intrinsic Lorentz metric g_Π (up to a constant) can be regarded as an object available (or, at least, a compromise one) for all the observers, as it would depend directly on physical light rays.

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