RESONANT INJECTION AND EJECTION FOR SECTOR-FIELD ACCELERATORS

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I. INTRODUCTION

The process of beam injection into, or ejection from, a circular accelerator by means of resonantly exciting or damping the betatron oscillation with magnetic or electric field bumps is, by now, a well-known art. Recently the effect of field bumps on ion motions in a sector-field accelerator has been the subject of many intensive numerical studies.[1,2,3] Analytically, a general treatment[4] has been given for the idealized problem of a linear betatron oscillation influenced by small but arbitrary excitation forces all in exact resonance with the free oscillation. Here, we shall present a general analytical treatment for studying the motion of ions in a sector-field accelerator (non-linear free oscillation) under the influence of small but arbitrary field bumps which are approximately in resonance with the free oscillation. The general method of analysis is further illustrated by a specific example to show how handy approximate formulas can be derived to give the essential near-resonant behaviors of the ion motions.

II. REDUCTION OF THE GENERAL HAMILTONIAN

We assume that in the absence of the bump field the magnetic field (intrinsic field) possesses perfect sector and median-plane symmetries. To simplify the formulation we shall admit only bump fields which preserve the median-plane symmetry and shall study only small oscillations in the median plane (no vertical oscillation). The method of treatment and the results obtained under these restrictions exhibit all the essential features, and extensions of the present work to include more general cases are quite straightforward. The equilibrium orbit is defined as the closed orbit in the intrinsic field and, hence, possesses sector symmetry. The Hamiltonian for the horizontal betatron oscillation of an ion expanded about its equilibrium orbit is[5]

$$H\ (x, p_x;\ \theta) = H^{(1)} + H^{(2)} + H^{(3)} + H^{(4)} + \cdots \quad (1)$$

where

$$\begin{cases} H^{(1)} = (\Delta\mu)\ x & (2) \\ H^{(2)} = \frac{1}{2} p_x^2 + \frac{1}{2}(\mu^2 + \nu + \mu\Delta\mu + \Delta\nu)x^2 \\ H^{(3)} = \frac{1}{2}\mu x p_x^2 + \frac{1}{6}(2\mu\nu + \lambda + 2\mu\Delta\nu + \Delta\lambda)x^3 \\ H^{(4)} = \frac{1}{8} p_x^4 + \frac{1}{24}(3\mu\lambda + \sigma + 3\mu\Delta\lambda + \Delta\sigma)\ x^4 \end{cases}$$

and

$$\begin{cases} \mu(\theta) = \frac{eR}{cp} B_0, & \Delta\mu(\theta) = \frac{eR}{cp}(\Delta B)_0 \quad (3) \\ \nu(\theta) = \frac{eR}{cp}\left(\frac{\partial B}{\partial x}\right)_0, & \Delta\nu(\theta) = \frac{eR}{cp}\left(\frac{\partial \Delta B}{\partial x}\right)_0 \\ \lambda(\theta) = \frac{eR}{cp}\left(\frac{\partial^2 B}{\partial x^2}\right)_0, & \Delta\lambda(\theta) = \frac{eR}{cp}\left(\frac{\partial^2 \Delta B}{\partial x^2}\right)_0 \\ \sigma(\theta) = \frac{eR}{cp}\left(\frac{\partial^3 B}{\partial x^3}\right), & \Delta\sigma(\theta) = \frac{eR}{cp}\left(\frac{\partial^3 \Delta B}{\partial x^3}\right)_0 \end{cases}$$

The quantities appearing in (1), (2) and (3) are defined below:

x = perpendicular displacement from the equilibrium orbit in units of R = canonical coordinate variable
p_x = canonical momentum variable conjugate to x
$R = 1/2\pi$ (length of equilibrium orbit)
$\theta = 1/R$ (distance along equilibrium orbit) = independent variable
B = magnitude of intrinsic field
ΔB = magnitude of bump field
subscript $_0$ = value on equilibrium orbit

e = charge of ion
p = momentum of ion
c = velocity of light

The quantities μ, ν, λ, σ giving the normal (perpendicular to equilibrium orbit) derivatives of the intrinsic field have sector periodicity $2\pi/N$ (N = number of sectors), and are much larger than the bump quantities $\Delta\mu$, $\Delta\nu$, $\Delta\lambda$, $\Delta\sigma$ which may have only the minimum periodicity 2π. Terms containing factors of $\Delta\mu$, $\Delta\nu$, $\Delta\lambda$, $\Delta\sigma$ etc. are small bump terms and vanish when the bump field is turned off, whereas terms containing only factors pertaining to the intrinsic field are large intrinsic terms.

The first part of this treatment follows closely that for the study of non-linear betatron oscillations.[6] Because of the expansion about the equilibrium orbit in the intrinsic field $H^{(1)}$ contains only a small bump term. The existing largest term in H, namely the intrinsic terms in $H^{(2)}$, can be eliminated by the following series of transformations.

A. *Floquet Transformation*

It is well-known that the coefficient of x^2 in the intrinsic terms of $H^{(2)}$ may be transformed to a constant, say, Q^2 (Q = betatron oscillation frequency) by the Floquet transformation[7] which consists of a transformation of θ followed by a canonical transformation of x and p_x. All other terms in H are, of course, also affected by the Floquet transformation. However, these are either bump terms or terms of higher degrees in x and p_x and, hence, much smaller than the intrinsic terms in $H^{(2)}$. For intrinsic fields with relatively small flutter (true for all medium energy sector-field cyclotrons) the modifications of these small terms by the Floquet transformation are even smaller and shall be neglected. Under this "smooth approximation" then, we shall leave all terms in the Hamiltonian (1) and (2) unchanged, except replacing the coefficient of x^2 in the intrinsic terms of $H^{(2)}$ by the constant Q^2, namely substituting for $H^{(2)}$ in (2)

$$H^{(2)} = \tfrac{1}{2}(p_x^2 + Q^2 x^2) + \tfrac{1}{2}(\mu\Delta\mu + \Delta\nu)x^2 \qquad (4)$$

with the understanding that x, p_x, and θ now denote their corresponding quantities after the Floquet transformation.

The intrinsic terms in $H^{(2)}$, now, produce only a rotation in the $x\ p_x$ phase plane and can, hence, be eliminated by a transformation to a rotating coordinate frame in the phase plane. This can best be done using complex canonical variables.

B. *Transformation to Complex Variables z and z**

$$\begin{cases} z = \sqrt{Q}\,x + i\dfrac{p_x}{\sqrt{Q}} \\ z^* = \sqrt{Q}\,x - i\dfrac{p_x}{\sqrt{Q}} \end{cases} \qquad (5)$$

C. *Canonical Tranformation to Rotating Frame with Generating Function*

$$G_1(z, Z^*; \theta) = zZ^* e^{iQ\theta}, \qquad (6)$$

which gives the relation

$$z = Ze^{-iQ\theta} \qquad (7)$$

The transformed complex Hamiltonian is, now,

$$\Phi(Z, Z^*; \theta) = \Phi^{(1)} + \Phi^{(2)} + \Phi^{(3)} + \Phi^{(4)} + \cdots \qquad (8)$$

where

$$\begin{cases} \dfrac{\Phi^{(1)}}{-2i} = A^{(1)} Z e^{-iQ\theta} + \text{c. c.} \qquad (9) \\ \dfrac{\Phi^{(2)}}{-2i} = A^{(2)} Z^2 e^{-2iQ\theta} + B^{(2)} ZZ^* + \text{c. c.} \\ \dfrac{\Phi^{(3)}}{-2i} = A^{(3)} Z^3 e^{-3iQ\theta} + B^{(3)} Z^2 Z^* e^{-iQ\theta} + \text{c. c.} \\ \dfrac{\Phi^{(4)}}{-2i} = A^{(4)} Z^4 e^{-4iQ\theta} + B^{(4)} Z^3 Z^* e^{-2iQ\theta} + C^{(4)} Z^2 Z^{*2} + \text{c. c.} \end{cases}$$

and

$$\begin{cases} A^{(1)} = \dfrac{1}{2(1!)Q^{1/2}}(\Delta\mu) \qquad (10) \\ A^{(2)} = B^{(2)} = \dfrac{1}{2^2(2!)Q}(\mu\Delta\mu + \Delta\nu) \\ A^{(3)} = \dfrac{1}{2^3(3!)Q^{3/2}}(2\mu\nu + \lambda + 2\mu\Delta\nu + \Delta\lambda - 3\mu Q^2) \\ B^{(3)} = \dfrac{3}{2^3(3!)Q^{3/2}}(2\mu\nu + \lambda + 2\mu\Delta\nu + \Delta\lambda + \mu Q^2) \\ A^{(4)} = \dfrac{1}{3}C^{(4)} = \dfrac{1}{2^4(4!)Q^2}(3\mu\lambda + \sigma + 3\mu\Delta\lambda + \Delta\sigma + 3Q^4) \\ B^{(4)} = \dfrac{4}{2^4(4!)Q^2}(3\mu\lambda + \sigma + 3\mu\Delta\lambda + \Delta\sigma - 3Q^4) \end{cases}$$

with c.c. denoting the complex conjugate terms. After these tranformations $\Phi^{(1)}$ and $\Phi^{(2)}$ both contain only small bump terms, and intrinsic terms enter only in the small third and higher degree terms in Z and Z^*. The coefficients $A^{(k)}$, $B^{(k)}$, $C^{(k)}$ are functions of θ with at least the periodicity of 2π. To exhibit this property we shall write

$$\begin{cases} A^{(k)} = \sum_{n=-\infty}^{\infty} a_n^{(k)} e^{in\theta}, \quad a_{-n}^{(k)} = a_n^{(k)*} \\ B^{(k)} = \sum_{n=-\infty}^{\infty} b_n^{(k)} e^{in\theta}, \quad b_{-n}^{(k)} = b_n^{(k)*} \\ C^{(k)} = \sum_{n=-\infty}^{\infty} c_n^{(k)} e^{in\theta}, \quad c_{-n}^{(k)} = c_n^{(k)*} \end{cases} \tag{11}$$

where $a_n^{(k)}$, $b_n^{(k)}$, $c_n^{(k)}$ are complex constants with respect to θ. When (11) is substituted in (9) we get

$$\begin{cases} \dfrac{\Phi^{(1)}}{-2i} = Z \sum_{n=-\infty}^{\infty\prime} a_n^{(1)} e^{i(n-Q)\theta} + \text{c. c.} \\ \dfrac{\Phi^{(2)}}{-2i} = Z^2 \sum_{n=-\infty}^{\infty\prime} a_n^{(2)} e^{i(n-2Q)\theta} + \\ \qquad ZZ^* \sum_{n=-\infty}^{\infty\prime} b_n^{(2)} e^{in\theta} + \text{c. c.} \\ \dfrac{\Phi^{(3)}}{-2i} = Z^3 \sum_{n=-\infty}^{\infty} a_n^{(3)} e^{i(n-3Q)\theta} + \\ \qquad Z^2Z^* \sum_{n=-\infty}^{\infty} b_n^{(3)} e^{i(n-Q)\theta} + \text{c. c.} \\ \dfrac{\Phi^{(4)}}{-2i} = Z^4 \sum_{n=-\infty}^{\infty} a_n^{(4)} e^{i(n-4Q)\theta} + \\ \qquad Z^3Z^* \sum_{n=-\infty}^{\infty} b_n^{(4)} e^{i(n-2Q)\theta} + \\ \qquad Z^2Z^{*2} \sum_{n=-\infty}^{\infty} c_n^{(4)} e^{in\theta} + \text{c. c.} \end{cases} \tag{12}$$

where the primes on the summations in $\Phi^{(1)}$ and $\Phi^{(2)}$ indicate that the harmonics of the intrinsic field are missing in these summations.

For small oscillations, namely small Z, since the terms appearing in Φ are either bump terms or terms of the third or higher degree in Z, Φ is small showing that Z is a slowly varying function of θ. Those terms in Φ with exponential $e^{i\omega\theta}$ where ω is of the order of unity or larger are too rapidly oscillating to produce appreciable secular effect on Z. Only terms with very small or vanishing ω (resonant terms) can produce large secular (resonant) effects on Z. In terms of first-order perturbation theory, where we average Φ over θ to get the first-order approximate Hamiltonian, we see that terms with large ω will average to zero and that only terms with small or vanishing ω remain in the approximate Hamiltonian. Setting the exponent of each of the terms in (12) equal to zero, we obtain the resonance-condition:

$$kQ = n \qquad k,\, n = 0,\, 1,\, 2,\, \ldots \tag{13}$$

For any specific problem, therefore, the first step is to determine the relevant resonant terms and to simplify the Hamiltonian (8) and (12) by neglecting all other terms. We shall illustrate this method of treatment by a specific example.

III. EXTRACTION FROM A 3-SECTOR CYCLOTRON OPERATING NEAR Q = 1

A. *Further Reduction of the Hamiltonian*

For a 3-sector cyclotron the intrinisc field has only harmonics of orders $n = 0, 3, 6, \ldots$ Therefore, for operation near $Q = 1$ the nearby intrinsic resonances (those derived from the intrinsic field) are $0Q = 0$, $3Q = 3$, $6Q = 6$, etc. To decide on the necessary bump field, since, as we shall see later, the stability of motion in the immediate neighborhood of the equilibrium orbit (small Z) is caused by a term of the second degree in Z and Z^* in the Hamiltonian, to destroy this stability to effect extraction the bump term will have to be of either the first or the second degree in Z. This means that we will have to supply either a first-harmonic or a second-harmonic bump field. We shall study only the case of a first-harmonic bump field. The same treatment applies quite straightforwardly also to the case of a second-harmonic bump field. With a first-harmonic bump field ($n = 1$) and an operation near $Q = 1$, the nearby bump resonance (that derived from the bump field) is clearly $1Q = 1$. Writing $Q = 1 + \epsilon$; keeping in (12) the bump terms for the bump resonance $1Q = 1$, the intrinsic term for the lowest order (hence, strongest) intrinsic resonance $3Q = 3$, and the "intrinsic" term for the neutral resonance $0Q = 0$; and dropping all other terms we get

$$\frac{\Phi}{-2i} = a_1^{(1)} Z e^{-i\epsilon\theta} + a_3^{(3)} Z^3 e^{-3i\epsilon\theta} + b_1^{(3)} Z^2 Z^* e^{-i\epsilon\theta} + c_0^{(4)} Z^2 Z^{*2} + \text{c. c.} \tag{14}$$

Now we make the canonical transformation to the coordinate frame rotating in the Z phase plane with angular speed ϵ to follow the precessional motion of the phase point caused by not being exactly on the resonance ($\epsilon \neq 0$). The generating function is:

$$G_2(Z, \zeta^*; \theta) = Z\zeta^* e^{-i\epsilon\theta}, \tag{15}$$

which gives the relation

$$Z = \zeta e^{i\epsilon\theta}. \tag{16}$$

The transformed Hamiltonian is

$$\frac{1}{-2i}\Psi(\zeta, \zeta^*, \theta) = a_1^{(1)}\zeta + \frac{\epsilon}{4}\zeta\zeta^* + a_3^{(3)}\zeta^3 + b_1^{(3)}\zeta^2\zeta^* + c_0^{(4)}\zeta^2\zeta^{*2} + \text{c. c.} \tag{17}$$

The coefficient $c_0^{(4)}$ is real and the coefficient $a_3^{(3)}$ can be made real (phase angle $=0$) by choosing the origin of θ to be where the third harmonic of $A^{(3)}$ (i.e. the third harmonic of $2\mu\nu + \lambda - 3\mu Q^2$, since the bump terms in $A^{(3)}$ are assumed not to contain the third harmonic) is a maximum. To simplify the formulation further, we shall assume that the bump field is so arranged that the coefficients $a_1^{(1)}$ and $b_1^{(3)}$ have the same phase angle α, and to simplify the notation we shall rewrite

$$a_3^{(3)} \equiv A,\ c_0^{(4)} \equiv B,\ a_1^{(1)} \equiv Ce^{i\alpha},\ b_1^{(3)} \equiv De^{i\alpha}. \tag{18}$$

This gives

$$\frac{\Psi}{-2i} = \frac{\epsilon}{2}\zeta\zeta^* + A(\zeta^3 + \zeta^{*3}) + 2B\zeta^2\zeta^{*2} + C(\zeta e^{i\alpha} + \zeta^* e^{-i\alpha}) + D(\zeta^2\zeta^* e^{i\alpha} + \zeta^{*2}\zeta e^{-i\alpha}), \tag{19}$$

where the A and B terms are the intrinsic terms and the C and D terms are the bump terms.

To facilitate physical interpretation, we transform the complex Hamiltonian (19) back to real canonical variables. This can be done in two ways. If we choose the real Cartesian canonical variables X and P defined by

$$\zeta = X + iP, \tag{20}$$

we get the real Cartesian Hamiltonian

$$K_C(X, P; \theta) = \frac{\epsilon}{2}(X^2 + P^2) + 2A(X^3 - 3XP^2) + 2B(X^2 + P^2)^2 + 2C(X\cos\alpha - P\sin\alpha) + 2D(X^2 + P^2)(X\cos\alpha - P\sin\alpha) \tag{21}$$

Or, else, we can transform (19) to the real polar canonical variables ϕ and ρ defined by

$$\zeta = \sqrt{\rho}\, e^{i\phi} \tag{22}$$

(Note here that although $\sqrt{\rho}$ is the physical radial length in the phase plane, ρ is the canonical momentum variable conjugate to ϕ.) We get for the real polar Hamiltonian:

$$-\frac{1}{2}K_P(\phi, \rho; \theta) = \frac{\epsilon}{2}\rho + 2A\rho^{3/2}\cos 3\phi + 2B\rho^2 + 2C\rho^{1/2}\cos(\phi + \alpha) + 2D\rho^{3/2}\cos(\phi + \alpha). \tag{23}$$

For easy reference we shall list below the relationships between the coefficients A, B, C, D, and ϵ and the magnetic field and linear horizontal betatron oscillation parameters; and those between the new real canonical variables X, P, ϕ, ρ and the original canonical variables x, p_x.

$$\left\{\begin{aligned} A\cos 3\theta &= \text{third (intrinsic) harmonic of } \frac{2\mu\nu + \lambda - 3\mu Q^2}{2^4(3\,!)Q^{3/2}} \\ &\cong \text{third (intrinsic) harmonic of } \frac{1}{96}(2\mu\nu + \lambda - 3\mu) \\ B &= \theta - \text{average of } \frac{3(3\mu\lambda + \sigma + 3Q^4)}{2^4(4\,!)Q^2} \\ &\cong \theta - \text{average of } \frac{1}{128}(3\mu\lambda + \sigma + 3) \\ C\cos(\theta + \alpha) &= \text{first (bump) harmonic of } \frac{\Delta\mu}{4(1\,!)Q^{1/2}} \\ &\cong \text{first (bump) harmonic of } \frac{1}{4}(\Delta\mu) \\ D\cos(\theta + \alpha) &= \text{first (bump) harmonic of } \frac{3(2\mu\Delta\nu + \Delta\lambda)}{2^4(3\,!)Q^{3/2}} \\ &\cong \text{first (bump) harmonic of } \frac{1}{32}(2\Delta\nu + \Delta\lambda) \\ \epsilon &= Q - 1 \end{aligned}\right. \tag{24}$$

and

$$\begin{cases} x = \dfrac{1}{\sqrt{Q}} \sqrt{\rho} \cos(\theta - \phi) = \dfrac{1}{\sqrt{Q}} (X \cos\theta + P \sin\theta) \\ P_x = -\sqrt{Q}\sqrt{\rho} \sin(\theta - \phi) = \sqrt{Q}(-X \sin\theta + P \cos\theta) \end{cases} \tag{25}$$

Equation (25) shows that if we look at x and p_x only once every revolution they simply coincide respectively with $\frac{1}{\sqrt{Q}} X \cong X$ and $\sqrt{Q} P \cong P$.

B. *Study of Properties of Resonant Extraction*

The Hamiltonians (21) and (23) being explicitly independent of θ give immediately the trajectories of the phase points as K_C = constant or K_P = constant. Without the bump terms ($C = D = 0$) the phase point trajectories are the well-known three sided figures (Fig. 1). The fixed points are given by the conditions $\frac{d\phi}{d\theta} = \frac{\partial H_\rho}{\partial \rho} = 0$, and $\frac{d\rho}{d\theta} = \frac{\partial H_\rho}{\partial \phi} = 0$ to be

$$\sqrt{\rho} = 0 \tag{26}$$

and

$$\begin{cases} \sqrt{\rho} = \dfrac{-3A \pm \sqrt{9A^2 - 8\epsilon B}}{8B} \\ \phi = 0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3} \end{cases} \qquad \begin{cases} \sqrt{\rho} = \dfrac{3A \pm \sqrt{9A^2 - 8\epsilon B}}{8B} \\ \phi = \dfrac{\pi}{3}, \dfrac{3\pi}{3}, \dfrac{5\pi}{3} \end{cases} \tag{27}$$

where it is assumed, of course, that $9A^2 - 8\epsilon B \geqslant 0$ and where the signs in front of the radicals are to be chosen so as to make $\sqrt{\rho} > 0$. Three of the six fixed points given in (27) are stable and three are unstable. The central stable phase area is that bounded by the separatrices passing through the three unstable fixed points.

Because of the appearance of $\cos 3\phi$ in K_P there are two typical sets of values for the phase angle of the bump terms; the set of $\alpha = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$ and the set of $\alpha = \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}$. Phase plots for values of α belonging to the same set differ only by a rotation of the phase plane. Moreover, the plots for $\alpha = \pi$ are the same as those for $\alpha = 0$ with the signs of C and D reversed. Therefore, it is sufficient to study only the case of $\alpha = 0$ with unrestricted signs for C and D. The qualitative effects of the addition of these bump terms are shown in Fig. 2.

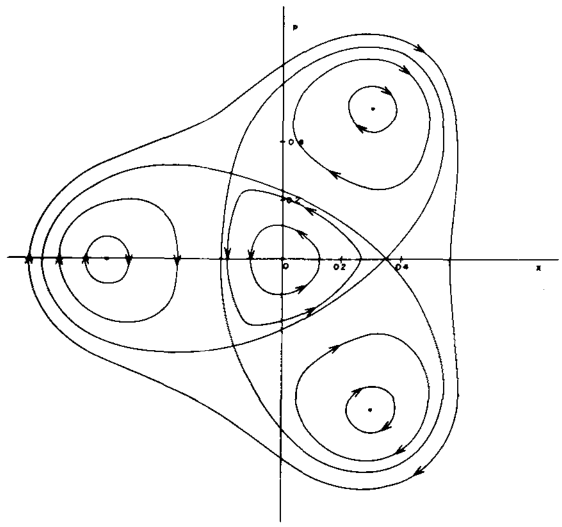

Fig. 1 Phase plot with no bump term ($\epsilon = -0.4$, $A = \frac{1}{12}$, $B = \frac{1}{4}$, $C = D = 0$)

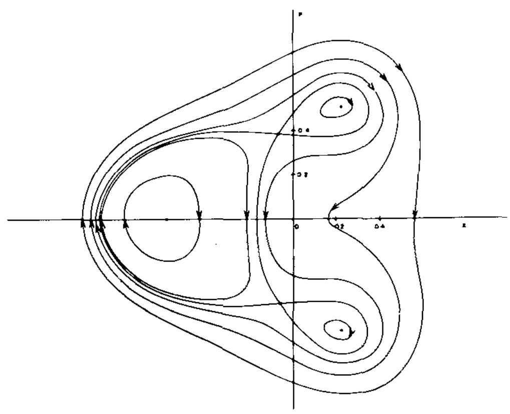

Fig. 2a Phase plot with C bump term ($\epsilon = -0.4$, $A = \frac{1}{12}$, $B = \frac{1}{4}$, $C = 0.03$, $D = 0$)

For practical cases of medium energy sector-field cyclotrons the coefficient B is very small, and the three stable fixed points on the outside are quite far out. Since for beam extraction we are only interested in the behavior of the central region of the phase plane, as a good approximation we can put $B=0$. With both α and B put equal to zero, the Hamiltonians (21) and (23) reduce to

$$K_C = 2CX + \frac{\epsilon}{2}(X^2 + P^2) + 2A(X^3 - 3XP^2) + 2DX(X^2 + P^2) \tag{28}$$

and

$$K_P = -2\left[2C\rho^{1/2}\cos\phi + \frac{\epsilon}{2}\rho + 2\rho^{3/2}(A\cos 3\phi + D\cos\phi)\right] \tag{29}$$

Without the bump terms ($C=D=0$) the three outside stable fixed points, now, recede to ∞ leaving the phase plots open on the outside (Fig. 3), and the three unstable fixed points are, now, given by

$$\sqrt{\rho} = -\frac{\epsilon}{6A}, \quad \phi = 0, \frac{2\pi}{3}, \frac{4\pi}{3} \quad \text{if } \frac{\epsilon}{6A} < 0 \tag{30}$$

or

$$\sqrt{\rho} = \frac{\epsilon}{6A}, \quad \phi = \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3} \quad \text{if } \frac{\epsilon}{6A} > 0 \tag{31}$$

The phase plots of these two cases are just the reflections of each other about the P-axis. Henceforth, we shall concentrate only on the case when $\frac{\epsilon}{6A} < 0$. The separatrices are, now, straight lines passing through the three unstable fixed points bounding an equilateral triangular shaped stable-phase region of area,

$$S_0 = \frac{1}{16\sqrt{3}}\frac{\epsilon^2}{A^2}. \tag{32}$$

With the bump terms the fixed points are given by $\frac{dX}{d\theta} = \frac{\partial H_C}{\partial P} = 0$ and $\frac{dP}{d\theta} = -\frac{\partial H_P}{\partial X} = 0$ to be

$$X = \frac{-\epsilon \pm \sqrt{\epsilon^2 - 48C(A+D)}}{12(A+D)}, \quad P = 0 \tag{33}$$

and

$$X = \frac{\epsilon}{4(3A-D)}, \quad P = \pm\frac{1}{4}\sqrt{\frac{16C}{3A-D} + \frac{\epsilon^2(9A+D)}{(3A-D)^3}} \tag{34}$$

One of the two fixed points given in (33) is stable and the remaining three are unstable. When

$$C(A+D) \geqslant \frac{\epsilon^2}{48} \tag{35}$$

the first pair of fixed points (33) (one stable, one unstable) vanishes, and the stable phase area shrinks down to zero leaving the central region of the phase plane opened up at the corner on the X-axis. (Fig. 4a.) When

$$C\frac{(3A-D)^2}{9A+D} \leqslant -\frac{\epsilon^2}{16} \tag{36}$$

the second pair of fixed points (34) (both un-

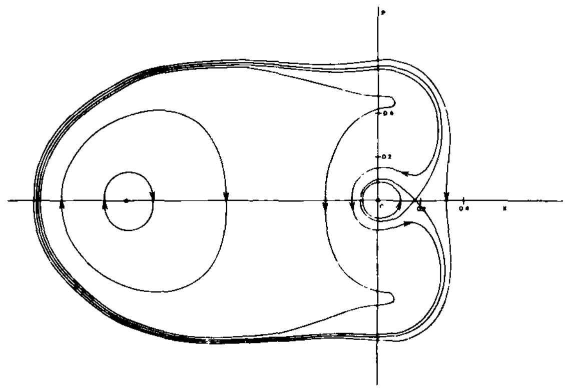

Fig. 2b Phase plot with D bump term ($\epsilon = -0.4$, $A = \frac{1}{12}$, $B = \frac{1}{4}$, $C = 0$, $D = \frac{1}{4}$)

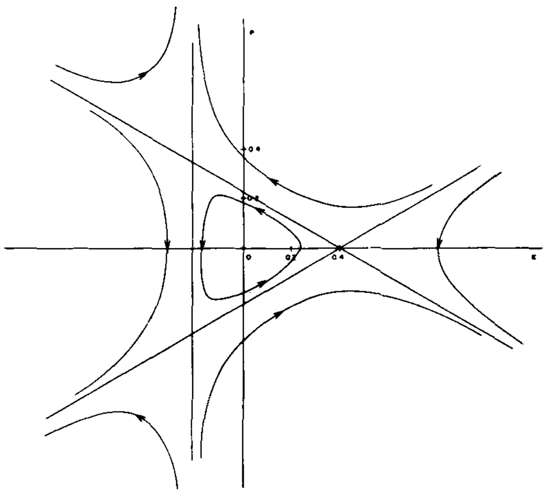

Fig. 3 Phase plot for $B = 0$ with no bump term ($\epsilon = -0.4$, $A = \frac{1}{6}$, $B = C = D = 0$)

stable) vanishes, and the central region of the phase plane is opened up at the other two corners (Fig. 4b). In either case, the destruction of the stable phase area will cause the originally stable-phase points to stream away from the central region, and extraction of the beam is, thus, accomplished.

To study the asymptotic behaviors of these phase points after leaving the central region we derive, first, the asymptotic streaming direction ϕ_a of the phase points at large $\sqrt{\rho}$ (meaning that $\sqrt{\rho}$ is large enough so that only its highest degree terms in the Hamiltonians (28) and (29) are important, but not so large that

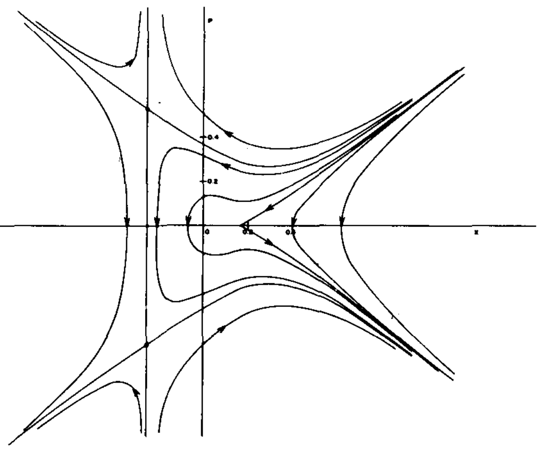

Fig. 4a Phase plot for $B=0$ with bump terms—one corner opening case
($\epsilon=-0.4$, $A=\frac{1}{6}$, $B=0$, $C=0.0125$, $D=0.1$)

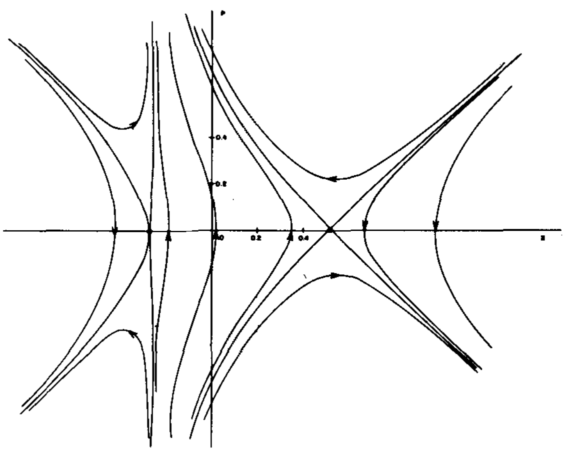

Fig. 4b Phase plot for $B=0$ with bump terms—two corner opening case
($\epsilon=-0.4$, $A=\frac{1}{6}$, $B=0$, $C=-0.11$, $D=0.1$)

the B terms which are dropped in these Hamiltonians would become important), by equating the $\rho^{1/2}$ term in $\frac{d\phi}{d\theta}=\frac{\partial H_P}{\partial \rho}$ to zero. This gives

$$A\cos 3\phi_a + D\cos\phi_a = [4A\cos^2\phi_a - (3A-D)]\cos\phi_a = 0 \tag{37}$$

or

$$\phi_a = \pi\pm\phi_0, \qquad \pi\pm\frac{\pi}{2}, \qquad \pm\phi_0 \tag{38}$$

where

$$\phi_0 \equiv \cos^{-1}\sqrt{\frac{3A-D}{4A}} \;\begin{matrix} > 0 \\ < \frac{\pi}{2}\end{matrix} \quad \text{if } 0<\frac{3A-D}{4A}<1$$

The asymptotic radial "velocities" $\left(\frac{d\sqrt{\rho}}{d\theta}\right)_a$ of the motion of phase points along these directions are obtained by substituting (38) in the equation $\frac{d\sqrt{\rho}}{d\theta}=\frac{1}{2\sqrt{\rho}}\frac{d\rho}{d\theta}=-\frac{1}{2\sqrt{\rho}}\frac{\partial H_P}{\partial \phi}$, keeping only the ρ term on the right-hand side.

This gives

$$\left(\frac{d\sqrt{\rho}}{d\theta}\right)_a = \begin{cases} \pm 2\rho(3A-D)\sqrt{\frac{A+D}{A}} & \text{for } \phi_a=\pi\pm\phi_0 \\ \mp 2\rho(3A-D) & \text{for } \phi_a=\pi\pm\frac{\pi}{2} \\ \mp 2\rho(3A-D)\sqrt{\frac{A+D}{A}} & \text{for } \phi_a=\pm\phi_0 \end{cases} \tag{39}$$

Thus, for the one-corner opening case, the phase points stream into the central region of the phase plane in the asymptotic direction $\phi_a=\phi_0$ and stream out of the central region in the asymptotic direction $\phi_a=-\phi_0$ both with the asymptotic speed $2\rho(3A-D)\sqrt{\frac{A+D}{A}}$; and for the two-corner opening case, the phase points stream into the central region in the asymptotic direction $\phi_a=3\pi/2$ and stream out of the central region in the asymptotic direction $\phi_a=\pi/2$ both with the asymptotic speed $2\rho(3A-D)$ (see Fig. 4). It should be noted, here, that the asymptotic streaming direction has no real physical significance since, at different azimuths around the cyclotron, the corresponding phase plots are rotated from one another. It is always possible to find an azimuth where the asymptotic outstreaming direction is along the X-axis to give the maximum increment per revolution in the displacement of the ions from the equilibrium orbit to facilitate entrance into a magnetic or an electrostatic channel.

Next, we would like to know asymptotically what region of the phase plane is occupied by the phase points which were originally in the stable-phase area before the bump field was turned on (or before the ion orbits were displaced into the physical space where bump field exists). Qualitatively we know that this asymptotic occupied region must be an elongated oval with the long dimension along the outgoing asymptotic direction. The extremities of the width w of this area at a given $\sqrt{\rho}$ lie on trajectories with values of the Hamiltonian which differ by

$$\Delta K_P(\sqrt{\rho}) \cong \frac{\partial K_P}{\partial \phi}\Delta\phi = -\frac{d\rho}{d\theta}\Delta\phi$$
$$= -2\frac{d\sqrt{\rho}}{d\theta}(\sqrt{\rho}\Delta\phi) \cong -2\overline{\left(\frac{d\sqrt{\rho}}{d\theta}\right)}_a w \tag{40}$$

Now, since the area S of the occupied region is an invariant (Liouville theorem) the length ℓ of this region may be approximately given by

$$\ell \cong \frac{4}{\pi}\frac{S}{w_{\max}} = -\frac{8}{\pi}\frac{S}{(\Delta K_P)_{\max}}\overline{\left(\frac{d\sqrt{\rho}}{d\theta}\right)}_a \tag{41}$$

where S is the invariant area of the oval, $\overline{\left(\frac{d\sqrt{\rho}}{d\theta}\right)}_a$ is the asymptotic radial streaming velocity averaged over the length of the oval, and $(\Delta K_P)_{\max}$ is the maximum range of the values of K_P for all phase points in the oval; and where for lack of more detailed information we have assumed the oval to be an ellipse with major and minor axes ℓ and w respectively. For any other "ovalish" shape the factor $\frac{4}{\pi}$ should be modified. To calculate $(\Delta K_P)_{\max}$ we assume the bump field to be turned on adiabatically. (Adiabaticity, here, means that the rate of change of the bump field is small compared to that of the original variables x and p_x. However, since the transformed variables ϕ and ρ or X and P are slowly varying themselves we can still assume that these variables do not change appreciably during the turn-on period.) Thus, the Hamiltonians during the turn-on period have the same forms as (28) and (26) except, now, C and D are explicit functions of θ having the asymptotic values $C(\theta)=D(\theta)=0$ before the turn-on period and $C(\theta)=C=$ constant, $D(\theta)=D=$constant after the turn-on period. The motion before and after the turn-on period when C and D are asymptotically independent of θ is still given by $K_P=$constant; but the constant values of the Hamiltonian of the same ion before $(K_P{}^0)$ and after (K_P) the turn-on period are different. From the "equation of motion" of K_P, namely,

$$\frac{dK_P}{d\theta} = [K_P, K_P] + \frac{\partial K_P}{\partial \theta} = \frac{\partial K_P}{\partial \theta} \tag{42}$$

where the square brackets denote the Poisson Bracket, and expression (29) for K_P we see that these two values of the Hamiltonian are related by

$$K_P = K_P{}^0 - 4\ (C+D\rho)\ \sqrt{\rho}\cos\phi, \tag{43}$$

where ϕ and ρ are some kind of average values over the turn-on period; but since these variables do not change appreciably during this time we can approximate them by their values at the beginning of the turn-on period. Assuming that before the turn-on period the phase points occupy a small (considerably smaller than the total stable phase area), hence, almost circular area with radius $\sqrt{\rho_0}$ about the origin we see from (43) that the maximum range of K_P is the difference between its values at the phase points $(\phi=0,\ \sqrt{\rho}=\sqrt{\rho_0})$ and $(\phi=\pi,\ \sqrt{\rho}=\sqrt{\rho_0})$. Although these two points have the same $K_P{}^0 \cong -\epsilon\rho_0$, they produce the largest ΔK_P, namely $(\Delta K_P)_{\max} = -8\ (C+D)\rho_0\ \sqrt{\rho_0}$, from the second term on the right-hand side of (43). Substituting this and $S=\pi\rho_0$ in (41) we get

$$\ell = \frac{\sqrt{\rho_0}}{C+D\rho_0}\overline{\left(\frac{d\sqrt{\rho}}{d\theta}\right)}_a \tag{44}$$

or

$$n \equiv \frac{\ell}{\delta\sqrt{\rho}} = \frac{1}{2\pi}\frac{\sqrt{\rho_0}}{C+D\rho_0} \tag{45}$$

where $\delta\sqrt{\rho} \equiv 2\pi\overline{\left(\frac{d\sqrt{\rho}}{d\theta}\right)}_a$ is the radial distance the phase points traverse per revolution of the ions averaged over the length of the oval, and n is the number of revolutions it takes for the occupied phase area to "move out of itself" or, in other words, the number of revolutions it takes to extract all the ions. As a general guide in designing the bump field we can remark that, of the two bump terms, C is primarily responsible for the destruction or opening up of the stable phase area, and D serves primarily as a control of the asymptotic behaviors of the phase points after streaming out of the central region.

We have, so far, centered our discussion on the extraction problem. It is clear, however, that all formulas derived above apply equally well to the reverse process, namely that of the resonant injection. However, since many approximations have been employed in the derivation of these handy formulas, they can not be expected to give quantitatively exact results. Nevertheless, these approximate results are fairly good approximations; and can serve as a valuable guide for the design and the understanding of the effects of the field bumps, and for performing the exact computing machine studies.

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