Novel Representation of an Integrated Correlator in $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory

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An integrated correlator of four superconformal stress-tensor primaries of $\mathcal{N}=4$ supersymmetric $SU(N)$ Yang-Mills theory (SYM), originally obtained by localization, is reexpressed as a two-dimensional lattice sum that is manifestly invariant under $SL(2,\mathbb{Z})$ S duality. This expression is shown to satisfy a novel Laplace equation in the complex coupling constant τ that relates the $SU(N)$ integrated correlator to those of the $SU(N+1)$ and $SU(N-1)$ theories. The lattice sum is shown to precisely reproduce known perturbative and nonperturbative properties of $\mathcal{N}=4$ SYM for any finite N, as well as extending previously conjectured properties of the large-N expansion.

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The $\mathcal{N}=4$ supersymmetric Yang-Mills ($\mathcal{N}=4$ SYM) theory [1] is a highly nontrivial four-dimensional conformal field theory that is of exceptional interest for a variety of reasons. It possesses maximal supersymmetry, which enables many of its properties to be determined analytically. Furthermore, its relation to string theory in $\mathrm{AdS}_5\times S^5$ via the AdS/CFT correspondence provides a model for more general examples of holography.

Of particular significance to this Letter is the analysis of the integrated correlation function of four superconformal primaries that was formulated in terms of an N-dimensional matrix model in Ref. [2], and further developed in Refs. [3–6]. This integrated correlator was defined in terms of the partition function of $\mathcal{N}=2^*$ SYM theory, which is a mass deformation of the superconformal $\mathcal{N}=4$ SYM theory with mass parameter m. The suitably normalized $\mathcal{N}=2^*$ $SU(N)$ partition function, on a round S^4, $Z_N(m,\tau,\bar\tau)$, was determined by Pestun using supersymmetric localization [7]. Our notation follows usual conventions where the complex Yang-Mills coupling constant is defined by $\tau=\tau_1+i\tau_2 := \theta/2\pi + i4\pi/g_{\mathrm{YM}}^2$.

In Ref. [2] the integrated correlator of four superconformal primaries was identified with the $m\to 0$ limit of four derivatives acting on $\log Z_N$ that has the form

$$\mathcal{G}_N(\tau,\bar\tau) := \frac{1}{4}\Delta_\tau\partial_m^2 \log Z_N(m,\tau,\bar\tau)\big|_{m=0} = -\frac{8}{\pi}\int_0^\infty dr\int_0^\pi d\theta \frac{r^3\sin^2(\theta)}{U^2}\mathcal{T}_N(U,V), \tag{1}$$

where $\Delta_\tau = 4\tau_2^2\partial_\tau\partial_{\bar\tau}$ is the hyperbolic Laplacian and the cross ratios U, V are defined by

$$U=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\qquad V=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}, \tag{2}$$

and are related to r and θ by $U=1+r^2-2r\cos(\theta)$ and $V=r^2$ [8]. The function $\mathcal{T}_N(U,V)$ is related to the four-point correlator by

$$\langle\mathcal{O}_2(x_1,Y_1)\ldots\mathcal{O}_2(x_4,Y_4)\rangle = \frac{1}{x_{12}^4x_{34}^4}[\mathcal{T}_{N,\mathrm{free}}(U,V;Y_i)+\mathcal{I}_4(U,V;Y_i)\mathcal{T}_N(U,V)], \tag{3}$$

where $\mathcal{O}_2(x_i,Y_i)$ is a superconformal primary in the $\mathbf{20'}$ of the $SU(4)$ R symmetry, which is encoded in the dependence on the null vectors Y_i. $\mathcal{T}_{N,\mathrm{free}}(U,V;Y_i)$ is the free field correlator and the prefactor $\mathcal{I}_4(U,V;Y_i)$ is determined by superconformal symmetry [9,10]. So we only focus on the nontrivial part, $\mathcal{T}_N(U,V)$.

As pointed out in Ref. [2], the relation [Eq. (1)] between the mass derivatives of the localized partition function and the integrated four-point correlator may lead to mixing with long operators, such as the Konishi operator. Not only do such effects decouple in the large-N strong coupling limit, as argued in Ref. [2], but they also do not appear at finite N

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and finite coupling [11]. We will see direct evidence of this statement in our results later in this Letter.

The results in this Letter follow from a reformulation of $\mathcal{G}_N(\tau,\bar{\tau})$, as a two-dimensional lattice sum that makes manifest many of its properties for all values of N and τ [12]. These results, which are based on a wealth of evidence concerning the structure of $\mathcal{G}_N(\tau,\bar{\tau})$ in various limits, take the form of a conjecture rather than a mathematical theorem:

Conjecture.—The integrated correlation function of four superconformal primary operators in the stress tensor multiplet of $\mathcal{N}=4$ $SU(N)$ supersymmetric Yang-Mills theory is given by the lattice sum

$$\mathcal{G}_N(\tau,\bar{\tau}) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty e^{-t\pi[(|m+n\tau|^2)/\tau_2]} B_N(t)dt, \quad (4)$$

where $B_N(t)$ has the form

$$B_N(t) = \frac{\mathcal{Q}_N(t)}{(t+1)^{2N+1}}, \quad (5)$$

and where $\mathcal{Q}_N(t)$ is a polynomial of degree $2N-1$ defined by

$$\mathcal{Q}_N(t) = -\frac{N(N-1)(1-t^2)^{N+1}}{2(1-t)^2}\left\{[3+(8N+3t-6)t]P_N^{(1,-2)}\left(\frac{1+t^2}{1-t^2}\right) + \frac{3t^2-8Nt-3}{1+t}P_N^{(1,-1)}\left(\frac{1+t^2}{1-t^2}\right)\right\}, \quad (6)$$

and $P_N^{(\alpha,\beta)}(z)$ is a Jacobi polynomial.

The following general properties of $B_N(t)$ are of importance in the following:

$$B_N(t) = \frac{1}{t}B_N\left(\frac{1}{t}\right), \quad (7)$$

and

$$\int_0^\infty B_N(t)dt = \frac{N(N-1)}{4}, \qquad \int_0^\infty B_N(t)\frac{1}{\sqrt{t}}dt = 0. \quad (8)$$

Using relationships between derivatives of Jacobi polynomials leads to the recurrence relation

$$t\frac{d^2}{dt^2}[tB_N(t)] = N(N-1)B_{N+1}(t) - 2(N^2-1)B_N(t) + N(N+1)B_{N-1}(t). \quad (9)$$

The lattice sum Eq. (4) is convergent for τ in the upper half plane $\tau_2 = \mathrm{Im}\tau > 0$ and it is manifestly invariant under the $SL(2,\mathbb{Z})$ transformations

$$\tau \to \gamma\cdot\tau = \frac{a\tau+b}{c\tau+d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \quad (10)$$

which is in accord with the expectations of Montonen-Olive duality [14–16].

An important consequence of Eq. (4) together with Eq. (9) is that $\mathcal{G}_N(\tau,\bar{\tau})$ satisfies the following corollary:

Corollary.—The integrated correlator satisfies a Laplace-difference equation of the form

$$(\Delta_\tau - 2)\mathcal{G}_N(\tau,\bar{\tau}) = N(N-1)\mathcal{G}_{N+1}(\tau,\bar{\tau}) - 2N^2\mathcal{G}_N(\tau,\bar{\tau}) + N(N+1)\mathcal{G}_{N-1}(\tau,\bar{\tau}). \quad (11)$$

This follows by applying the Laplacian Δ_τ to Eq. (4) and using Eq. (9). Equation (11) provides powerful constraints on $\mathcal{G}_N$ that relate the dependence on the coupling τ and the dependence on N in a manner that will be discussed later. For now we note that as $N\to\infty$, assuming $\mathcal{G}_N$ is a differentiable function of N, Eq. (11) becomes a Laplace equation in both τ and N, taking the form

$$(\Delta_\tau - 2)\mathcal{G}_N(\tau,\bar{\tau}) \underset{N\to\infty}{=} (N^2\partial_N^2 - 2N\partial_N)\mathcal{G}_N(\tau,\bar{\tau}), \quad (12)$$

where terms of higher order in $1/N$ have been suppressed.

The structure of the integrated correlator.—The $\mathcal{N}=2^*$ partition function appearing in Eq. (1) has the form [7]

$$Z_N(m,\tau,\bar{\tau}) = \int d^N a_i \delta\left(\sum_i a_i\right)\left(\prod_{i<j} a_{ij}^2\right) e^{-(8\pi^2/g_{\mathrm{YM}}^2)\sum_k a_k^2} \times \hat{Z}_N^{\mathrm{pert}}(m,a_{ij})|\hat{Z}_N^{\mathrm{inst}}(m,\tau,a_{ij})|^2. \quad (13)$$

The perturbative factor in Eq. (13) is given by

$$\hat{Z}_N^{\mathrm{pert}}(m,a_{ij}) = H(m)\prod_{i,j}\frac{H(a_{ij})}{H(a_{ij}+m)}, \quad (14)$$

where $H(z) = e^{-(1+\gamma)z^2}G(1+iz)G(1-iz)$, and $G(z)$ is a Barnes G function (and γ is the Euler constant). The factor $|\hat{Z}_N^{\mathrm{inst}}|^2$ is the product of Nekrasov partition functions that describes contributions from instantons and anti-instantons localized at the poles of S^4 [17].

In the following we will consider the Fourier expansion of the integrated correlator, using the notation

$$\mathcal{G}_N(\tau,\bar{\tau}) \coloneqq \sum_{k\in\mathbb{Z}}\mathcal{G}_{N,k}(\tau,\bar{\tau}) \coloneqq \sum_{k\in\mathbb{Z}} e^{2\pi i k\tau_1}\mathcal{F}_{N,|k|}(\tau_2). \quad (15)$$

Details of the derivation of the following results are presented in Ref. [13].

(i) Gauge group $SU(2)$: When $N=2$, the perturbative contribution arises solely from $\hat{Z}_2^{\rm pert}$ and it is straightforward to show that it can be expressed as an asymptotic series in $g_{\rm YM}^2$ of the form

$$\mathcal{G}_{2,0}^{(i)}(y) \sim \sum_{s=2}^{\infty} \frac{(2s-1)\Gamma(2s+1)(-1)^s}{2^{2s-1}\Gamma(s-1)} \zeta(2s-1) y^{1-s}, \qquad (16)$$

where $y = \pi\tau_2 = 4\pi^2/g_{\rm YM}^2$. This can be resummed by expressing it as a convergent Borel integral

$$\mathcal{G}_{2,0}(y) = y \int_0^\infty \frac{e^{-t}(6t - 9t^2 + 2t^3)}{2\sinh^2(\sqrt{yt})} dt. \qquad (17)$$

In this form it can also be reexpanded at strong coupling in positive powers of y,

$$\mathcal{G}_{2,0}^{(ii)}(y) \sim \frac{1}{2} + \sum_{s=2}^{\infty} (s-1)(2s-1)^2 \Gamma(s+1) \zeta(2s) \frac{(-y)^s}{\pi^{2s}}. \qquad (18)$$

It will be useful to formally identify the $k=0$ mode of $\mathcal{G}_{2,0}$ in Eq. (17) with the average of the y^s and y^{1-s} terms,

$$\mathcal{G}_{2,0}(y) = \frac{1}{2}\left(\mathcal{G}_{2,0}^{(i)}(y) + \mathcal{G}_{2,0}^{(ii)}(y)\right). \qquad (19)$$

The nonzero modes corresponding to instanton and anti-instanton contributions can be extracted from the $|\hat{Z}_2^{\rm inst}|^2$ factor in Eq. (13) by extending the analysis in Ref. [5]. This involves a systematic decomposition of $\Delta_\tau \partial_m^2 \hat{Z}_2^{\rm inst}(m,\tau,a_{ij})|_{m=0}$ in terms of a sum over rectangular Young diagrams with $\hat{m}$ rows and n columns, where $k = \hat{m}n$ is the instanton number. The resulting k-instanton contribution is

$$\mathcal{G}_{2,k}(\tau,\bar{\tau}) = \frac{e^{2\pi i k \tau_1}}{2} \sum_{\substack{\hat{m}\neq 0, n\neq 0 \\ \hat{m}n = k}} \int_0^\infty e^{-\pi\tau_2(\hat{m}^2/t + n^2 t)} \sqrt{\frac{\tau_2}{t}} B_2(t) dt, \qquad (20)$$

with $B_2(t)$ given in Eq. (5) for $N=2$. This integral can be expanded as an infinite sum of K-Bessel functions using the integral representation

$$\int_0^\infty e^{-a^2/t - b^2 t} t^{\nu-1} dt = 2\left(\frac{a}{b}\right)^\nu K_\nu(2ab), \qquad (21)$$

with $a = \sqrt{\pi\tau_2}\hat{m}$ and $b = \sqrt{\pi/\tau_2} n$.

We now recognize that the total integrated correlator, $\mathcal{G}_2 = \mathcal{G}_{2,0} + \sum_{k\neq 0} \mathcal{G}_{2,k}$, is an infinite sum of nonholomorphic Eisenstein series with integer index and with rational coefficients

$$\mathcal{G}_2(\tau,\bar{\tau}) = \frac{1}{4} + \frac{1}{2}\sum_{s=2}^{\infty} c_s^{(2)} E(s;\tau,\bar{\tau}), \qquad (22)$$

where

$$c_s^{(2)} = \frac{(-1)^s}{2}(s-1)(1-2s)^2 \Gamma(s+1). \qquad (23)$$

In making this identification we recall that a nonholomorphic Eisenstein series has a Fourier expansion of the form

$$E(s;\tau,\bar{\tau}) := \frac{1}{\pi^s} \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}} = \frac{2\zeta(2s)}{\pi^s}\tau_2^s + \frac{2\sqrt{\pi}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\pi^s\Gamma(s)}\tau_2^{1-s} + \sum_{k\neq 0} e^{2\pi i k\tau_1} \frac{4}{\Gamma(s)} |k|^{s-1/2} \sigma_{1-2s}(|k|) \sqrt{\tau_2} K_{s-\frac{1}{2}}(2\pi|k|\tau_2). \qquad (24)$$

We further note that $E(s,\tau,\bar{\tau})$ satisfies the Laplace equation

$$[\Delta_\tau - s(s-1)]E(s,\tau,\bar{\tau}) = 0. \qquad (25)$$

Upon substituting the integral representation

$$E(s;\tau,\bar{\tau}) = \sum_{(m,n)\neq(0,0)} \int_0^\infty e^{-t\pi[(|m+n\tau|^2)/\tau_2]} \frac{t^{s-1}}{\Gamma(s)} dt, \qquad (26)$$

into Eq. (22) it takes the form given in Eq. (4) with $N=2$.

(ii) Gauge groups $SU(N)$ with $N>2$: Here the direct analysis of Eq. (1) is considerably more complicated and is presented in more detail in Ref. [13], where the expression for $B_N(t)$ in Eq. (5) is motivated. However, for the purposes of this Letter it is more efficient to use the Laplace-difference equation, Eq (11), to generate the expression for the integrated correlator when $N>2$. Once we input the boundary conditions $\mathcal{G}_1 = 0$ and $\mathcal{G}_2$ given by Eq. (22), the correlators for theories with higher N are generated recursively. They may be expressed as

$$\mathcal{G}_N(\tau,\bar{\tau}) = \frac{N(N-1)}{8} + \frac{1}{2}\sum_{s=2}^{\infty} c_s^{(N)} E(s;\tau,\bar{\tau}), \quad (27)$$

where the coefficients $c_s^{(N)}$ are rational numbers that depend on N and are generated by the expansion of $B_N(t)$ in the form

$$B_N(t) = \sum_{s=2}^{\infty} \frac{c_s^{(N)}}{\Gamma(s)} t^{s-1}. \quad (28)$$

The coefficients $c_s^{(N)}$ can also be determined up to any desired order by substituting the series Eq. (27) into Eq. (11) and solving iteratively in terms of the coefficients $c_s^{(2)}$ given in Eq. (23).

Properties of the integrated correlator.—The integrated correlator has interesting behavior when expanded in various domains of the parameters, N and τ. We will here discuss three of these domains.

(a) Finite N, small $\lambda = g_{\rm YM}^2 N$: This is the domain of standard Yang-Mills perturbation theory. The expansion of the perturbative part of the expression Eq. (4) has the form

$$\mathcal{G}_{N,0}(\tau_2) = (N^2-1)\left[\frac{3\zeta(3)a}{2} - \frac{75\zeta(5)a^2}{8} + \frac{735\zeta(7)a^3}{16} - \frac{6615\zeta(9)(1+\frac{2}{7}N^{-2})a^4}{32} + \frac{114345\zeta(11)(1+N^{-2})a^5}{128} - \frac{3864861\zeta(13)(1+\frac{25}{11}N^{-2}+\frac{4}{11}N^{-4})a^6}{1024} + \mathcal{O}(a^7)\right], \quad (29)$$

where $a = g_{\rm YM}^2 N/(4\pi^2)$ and arbitrary $N \geq 2$. When $N=2$, this reduces to Eq. (16). Although the perturbative expansion of the unintegrated four-point correlator has a very complicated dependence on the cross ratios U, V, the above expression is remarkably simple, consisting of a power series in a with coefficients that are rational multiples of odd Riemann zeta values.

This expansion is in rather impressive agreement with known facts concerning the perturbative expansion of the four-point correlator of superconformal primaries of $\mathcal{N}=4$ SYM. The expressions for the unintegrated correlator up to three loops (up to order a^3) are given in Ref. [18]. In Ref. [13] we have verified the integrals of the one-loop and two-loop contributions agree with the coefficients proportional to a and to a^2 in Eq. (29). In performing these integrals we make use of the all-order results for ladder diagrams [19]. The one-loop and two-loop contributions are special cases of such ladder diagrams. However, at higher loops the correlator contains more general diagrams that we have not evaluated.

A further property that is apparent from the perturbative expansion Eq. (29) is the dependence on N. We see that up to order a^3 the coefficients do not depend on N, apart from the overall factor of (N^2-1). This is in accord with known perturbative properties of the correlator, which develop a dependence on N^{-2} at order a^4 [20,21]. From Eq. (29) we anticipate that an extra power of N^{-2} will appear at every subsequent even power of a, which is in agreement with the observations in Ref. [22].

All these precise agreements would be spoiled if there were any additional contribution such as mixing with a three-point correlator involving the Konishi operator.

(b) Large N, with fixed $\lambda = g_{\rm YM}^2 N$: In this limit instantons are of order $e^{-8\pi^2 kN/\lambda}$ and are therefore suppressed. The large-N expansion of the correlator is 't Hooft's topological expansion,

$$\mathcal{G}_N(\tau,\bar{\tau}) \sim \sum_{g=0}^{\infty} N^{2-2g}\mathcal{G}^{(g)}(\lambda), \quad (30)$$

where the leading term is of order N^2 and is given by the sum of planar Feynman diagrams in Yang-Mills perturbation theory. Given our knowledge of the coefficients $c_s^{(N)}$ in Eq. (28) we are able to determine the λ dependence of $\mathcal{G}^{(g)}$ order by order in N. For small λ the leading term is given by the series

$$\mathcal{G}^{(0)}(\lambda) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}\zeta(2n+1)\Gamma(n+\frac{3}{2})^2}{\pi^{2n+1}\Gamma(n)\Gamma(n+3)}\lambda^n, \quad (31)$$

which converges for $|\lambda| < \pi^2$. It can be resummed to give

$$\mathcal{G}^{(0)}(\lambda) = \lambda \int_0^{\infty} dw w^3 \frac{{}_1F_2(\frac{5}{2};2,4;-\frac{w^2\lambda}{\pi^2})}{4\pi^2\sinh^2(w)}, \quad (32)$$

which is well defined for $\lambda \geq \pi^2$ and coincides with the result of Ref. [2] after using an identity that relates ${}_1F_2$ and Bessel functions J_α.

However, following an analysis similar to Ref. [23], it is easy to see that the large-λ expansion of Eq. (32) is divergent and not Borel summable since the Borel integral is obstructed by a branch cut along the positive axis. This signifies that in order to reproduce the exact result [Eq. (32)] one needs a resurgent nonperturbative completion $\Delta\mathcal{G}^{(0)}(\lambda)$, which is determined in Ref. [13] to be of the form

$$\Delta\mathcal{G}^{(0)}(\lambda) = i\bigg(8\mathrm{Li}_0(e^{-2\sqrt{\lambda}}) + \frac{18\mathrm{Li}_1(e^{-2\sqrt{\lambda}})}{\lambda^{1/2}} + \frac{117\mathrm{Li}_2(e^{-2\sqrt{\lambda}})}{4\lambda} + \frac{489\mathrm{Li}_3(e^{-2\sqrt{\lambda}})}{16\lambda^{3/2}} + \cdots\bigg). \tag{33}$$

This expression is a sum of "instantonic" terms that are nonperturbative in $1/\sqrt{\lambda}$, with coefficients $O(e^{-2\sqrt{\lambda}})$ that are similar to those found in Refs. [24–26] for the cusp anomalous dimension and other quantities in $\mathcal{N}=4$ SYM [27]. Similar arguments lead to nonperturbative completions of $\mathcal{G}^{(g)}$. For example, the expression for $\Delta\mathcal{G}^{(1)}(\lambda)$ is also determined in Ref. [13] and takes an analogous form as Eq. (33). We believe that such nonperturbative effects in the large-λ expansion have a holographic interpretation in terms of string world sheet instantons.

(c) Large N, with fixed g_{YM}^2: This is the large-N limit in which Yang-Mills instantons contribute in a manner that ensures that $SL(2,\mathbb{Z})$ S duality is manifest. The form of $\mathcal{G}_N$ can be obtained (as in Ref. [13]) by a large-N expansion of $B_N(t)$ [defined in Eq. (5)], which is an expansion in half-integer powers of N. It is easy to check that this leads to a solution of Eq. (11) of the form

$$\mathcal{G}_N(\tau,\bar\tau) \sim \frac{N^2}{4} + \sum_{\ell=0}^{\infty} N^{\frac{1}{2}-\ell} \sum_{s=3/2}^{\ell+3/2} d_\ell^s E(s;\tau,\bar\tau), \tag{34}$$

which is a series of Eisenstein series with $s\in\mathbb{Z}+1/2$. The terms with $s=\ell+\frac{3}{2}$ satisfy the limiting large-N Laplace equation, Eq. (12), but this does not determine their coefficients, which have to be input from the expansion of $B_N(t)$, giving

$$d_\ell^{\ell+\frac{3}{2}} = \frac{(\ell+1)\Gamma(\ell-\frac{1}{2})\Gamma(\ell+\frac{3}{2})\Gamma(\ell+\frac{5}{2})}{2^{2\ell+2}\pi^{3/2}\ell!}. \tag{35}$$

Once $d_\ell^{\ell+3/2}$ is input the Laplace-difference equation determines the rest of the solution. This reproduces and extends the results of Ref. [5], where the first few coefficients were obtained. For example, terms with $s=\ell-\frac{1}{2}>0$ and $s=\ell-\frac{5}{2}>0$ are given by

$$d_\ell^{\ell-1/2} = -\frac{(\ell-1)^2(2\ell+9)\Gamma(\ell-\frac{1}{2})\Gamma(\ell+\frac{1}{2})^2}{32\,2^{2\ell+3}\pi^{3/2}\ell!},$$
$$d_\ell^{\ell-5/2} = \frac{(\ell-3)^2(20\ell^2+48\ell-293)\Gamma(\ell-\frac{5}{2})}{45\,2^{2\ell+5}\pi^{3/2}\Gamma(\ell)} \Gamma\left(\ell-\frac{3}{2}\right)\Gamma\left(\ell+\frac{3}{2}\right). \tag{36}$$

Finally, we believe that the considerations of this Letter generalize to a second integrated correlator that was considered in Ref. [4] and further explored in Ref. [6]. This is obtained from the $\mathcal{N}=2^*$ partition function by applying four derivatives with respect to mass, $\mathcal{G}'_N(\tau,\bar\tau) \coloneqq \partial_m^4 \log Z_N(m,\tau,\bar\tau)|_{m=0}$, which again generates a supersymmetric integrated correlator of four superconformal primaries, but with a different integration measure.

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