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PAPER

Coherent states for a system of an electron moving in a plane

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Abstract

In this paper, we construct the coherent states for a system of an electron moving in a plane under uniform external magnetic and electric fields. These coherent states are built in the context of both discrete and continuous spectra and satisfy the Gazeau-Klauder coherent state properties Gazeau and Klauder (1999 *J. Phys. A: Math. Gen.* **32**, 123–132).

1. Introduction

The system of charged quantum particles interacting with a constant magnetic field continues to attract intensive studies and is without a doubt one of the most investigated quantum systems, mainly motivated by condensed matter physics and quantum optics. A review article devoted to this quantum system and its related different kind of coherent states (CSs) was recently elaborated by Dodonov (see [1] and the complete reference list therein).

The concept of what is now called coherent states has been of great interest to the scientific community since the work of Schrödinger in 1926 [2] on the quantum harmonic oscillator (HO), where he introduced a specific quantum state that has dynamical behavior that is most similar to that of the classical HO. The conditions any family of states must fulfill to be coherent were elaborated by Klauder as follows: continuity in complex label, normalization, non orthogonality, unity operator resolution with unique positive weight function of the integration measure, temporal stability and action identity [3]. More details on the CSs and their different generalizations can be found in the literature [4–10], the list is not of course exhaustive.

In his study [11], Landau found that the system of electronic motion in a static uniform magnetic field can be assimilated in two dimensions to a harmonic oscillator, with an energy structure of equidistant discrete levels, with a distance $\hbar\omega_c$ (ω_c is the cyclotron frequency), each level being highly degenerate. Such a system, more often named Landau model, also provides a natural description for other well known significant phenomena, the so-called integer and fractional quantum Hall effects. In these last years, in the search of understanding the main features of the fractional quantum Hall effect (FQHE) [12, 13], many efforts have been done in the literature to find a wave function which minimizes the energy of a two-dimensional system of electrons subjected to a strong constant magnetic field applied perpendicularly to the sample, independently of the electron density. In [14], a system of electrons, essentially a two-dimensional crystal, has been considered. Besides, the wave function introduced has been modified to lower the energy in order to explain the experimental data. From an appropriate quantization of the classical variables of the system Hamiltonian, Bagarello *et al* (see [14, 15] and references therein), have modified the single electron wave function in view of the study of localization properties. The similar quantization has been also used to investigate the Bohm-Aharonov effect ([16, 17] and references therein) emphasizing the fact that it is not the electric and the magnetic fields but the electromagnetic potentials which are the fundamental quantities in quantum mechanics.

In a previous work [18], a connection has been established between quantum Hall effect and vector coherent states (VCSs) [19, 20] by applying the various construction methods developed in the literature. In the same way, the motion of an electron in a noncommutative xy plane, in a constant magnetic field background coupled with a harmonic potential was examined with the relevant VCSs constructed and discussed [21]. The Barut-Girardello CSs have been built for Landau levels of a gas of spinless charged particles, subject to a perpendicular magnetic field confined in a harmonic potential where the thermodynamical and the statistical properties have been investigated [22]. See also [1] and references quoted therein. Recently [23], from a matrix (operator) formulation of the Landau problem and the corresponding Hilbert space, an analysis of various VCSs extended to diagonal matrix domains has been performed on the basis of Landau levels.

The construction of CSs for continuous spectrum was first proposed for the Gazeau-Klauder CSs in [24] and later in [25–27]. In the present work, we follow the method developed in [24], by considering Landau levels, to built various classes of CSs as in [18, 28, 29] arising from physical Hamiltonian describing a charged particle in an electromagnetic field, by introducing additional parameters useful for handling discrete and continuous spectra of the Hamiltonian. The eigenvalue problem is presented and the quantum Hamiltonian spectra provided in the two possible orientations of the magnetic field by considering the infinite degeneracies of the Landau levels. The CSs are constructed with relevant properties discussed for both continuous and discrete spectra, and for purely discrete spectrum.

The paper is organized as follows. In section 2, we revisit the model of electron moving on plane where the eigenvalue problems are explicitely set and solved. The position and momentum operators, satisfying canonical commutation relations, established for the considered Hamiltonians are also defined. Section 3 is devoted to the construction of CSs for the quantum Hamiltonian possessing both continuous and discrete spectra by following the method developped in [18, 24]. Concluding remarks are given in section 4.

2. Electron moving in a plane revisited

In this section, we revisit the system of an electron moving in a plane as in [17], where we consider different scenarios for the symmetric gauge and the scalar potential.

Consider an electron moving in the plane xy under the uniform external electric field $\overrightarrow{E} = -\overrightarrow{\nabla}\Phi(x, y)$ and the uniform external magnetic field $\overrightarrow{B}$ which is perpendicular to the plane described by the Hamiltonian

$$H = \frac{1}{2m}\left(\overrightarrow{p} + \frac{e}{c}\overrightarrow{A}\right)^2 - e\Phi. \tag{1}$$

2.1. Case of the symmetric gauge

Let's consider the symmetric gauge

$$\overrightarrow{A} = \left(\frac{B}{2}y, -\frac{B}{2}x\right). \tag{2}$$

Experimentally, the electric field $\overrightarrow{E}$ is oriented according to one of the two possible directions of the plane. Suppose the scalar potential is defined as

$$\Phi(x, y) = -Ey. \tag{3}$$

Substituting the relations (2) and (3) in (1), the corresponding classical Hamiltonian, denoted by H_1, reads

$$H_1(x, y, p_x, p_y) = \frac{1}{2m}\left[\left(p_x + \frac{eB}{2c}y\right)^2 + \left(p_y - \frac{eB}{2c}x\right)^2\right] + eEy. \tag{4}$$

A canonical quantization of this system is obtained by promoting the classical variables x, y, p_x, p_y, to the operators X, Y, P_x, P_y which satisfy the nonvanishing canonical commutation relations

$$[X, P_x] = i\hbar = [Y, P_y]. \tag{5}$$

The Hamiltonian operator is derived from (4) as follows

$$\hat{H}_1(X, Y, P_x, P_y) = \frac{1}{2m}\left[\left(P_x + \frac{eB}{2c}Y\right)^2 + \left(P_y - \frac{eB}{2c}X\right)^2\right] + eEY. \tag{6}$$

In order to solve the eigenvalue problem

$$\hat{H}_1\Psi = \mathcal{E}\Psi, \tag{7}$$

it is convenient to perform the change of variables as below

$$Z = X + iY, \quad P_z = \frac{1}{2}(P_x - iP_y), \tag{8}$$

satisfying the nonvanishing commutations relations

$$[Z, P_z] = i\hbar = [\bar{Z}, P_{\bar{z}}], \tag{9}$$

and to define two sets of annihilation and creation operators $b,\ b^\dagger$ and $d,\ d^\dagger$ given by

$$b = 2P_{\bar{z}} - i\frac{eB}{2c}Z + \lambda, \quad b^\dagger = 2P_z + i\frac{eB}{2c}\bar{Z} + \lambda, \tag{10}$$

$$d = 2P_{\bar{z}} + i\frac{eB}{2c}Z, \quad d^\dagger = 2P_z - i\frac{eB}{2c}\bar{Z}, \tag{11}$$

with $\lambda = \frac{mcE}{B}$. These two sets of operators commute each other and satisfy the following commutation relations

$$[b, b^\dagger] = 2m\hbar\omega_c\mathbf{I}, \quad [d^\dagger, d] = 2m\hbar\omega_c\mathbf{I}, \tag{12}$$

where $\omega_c = \frac{eB}{mc}$ is known as the cyclotron frequency and $\mathbf{I}$ is the unit operator. The Hamiltonian $\hat{H}_1$ can be then re-expressed as follows:

$$\hat{H}_1 = \frac{1}{4m}(b^\dagger b + bb^\dagger) - \frac{\lambda}{2m}(d^\dagger + d) - \frac{\lambda^2}{2m}, \tag{13}$$

where

$$d + d^\dagger = 2P_x - \frac{eB}{c}Y. \tag{14}$$

In order to compute the eigenvalues $\mathcal{E}$ and eigenvectors Ψ, we split $\hat{H}_1$ in (13) into two commuting parts in the following manner:

$$\hat{H}_1 = \hat{H}_{1_{OSC}} - \hat{T}_1, \tag{15}$$

where $\hat{H}_{1_{OSC}}$ denotes the harmonic oscillator part

$$\hat{H}_{1_{OSC}} = \frac{1}{4m}(b^\dagger b + bb^\dagger), \tag{16}$$

while the part linear in d and $d^\dagger$ is given by

$$\hat{T}_1 = \frac{\lambda}{2m}(d^\dagger + d) + \frac{\lambda^2}{2m}. \tag{17}$$

The annihilation and creation operators b and $b^\dagger$ can be also rewritten as follows:

$$b = \sqrt{2m\hbar\omega_c}\,b', \quad b^\dagger = \sqrt{2m\hbar\omega_c}\,b'^\dagger, \quad [b', b'^\dagger] = \mathbf{I}, \tag{18}$$

with

$$b' = \sqrt{\frac{2m\omega_c}{\hbar}}\left(\frac{P_{\bar{z}}}{m\omega_c} - i\frac{Z}{4} + \frac{\lambda}{2m\omega_c}\right), \quad b'^\dagger = \sqrt{\frac{2m\omega_c}{\hbar}}\left(\frac{P_z}{m\omega_c} + i\frac{\bar{Z}}{4} + \frac{\lambda}{2m\omega_c}\right). \tag{19}$$

Then, one has

$$b|0\rangle = 0, \quad b|n\rangle = \sqrt{n}\sqrt{2m\omega_c\hbar}\,|n-1\rangle, \quad b^\dagger|n\rangle = \sqrt{n+1}\sqrt{2m\omega_c\hbar}\,|n+1\rangle \tag{20}$$

leading to

$$|n+1\rangle = \frac{1}{\sqrt{2m\omega_c\hbar(n+1)}}b^\dagger|n\rangle, \tag{21}$$

and, recurrently, to

$$\Phi_n \equiv |n\rangle = \frac{1}{\sqrt{(2m\omega_c\hbar)^n n!}}(b^\dagger)^n|0\rangle. \tag{22}$$

The harmonic oscillator Hamiltonian $\hat{H}_{1_{OSC}}$ reduces to

$$\hat{H}_{1_{OSC}} = \frac{\hbar\omega_c}{2}(2N' + \mathbf{I}), \quad N' = b'^\dagger b' \tag{23}$$

with eigenvalues $\mathcal{E}_{n_{1_{OSC}}}$ given by

$$\mathcal{E}_{n_{1_{OSC}}} = \hbar\omega_c\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots, \tag{24}$$

corresponding to the eigenvectors defined by (22).

The eigenvalue equation $\hat{T}_1\phi = \mathcal{E}\phi$ can be reduced to

$$\left(-i\frac{\partial}{\partial x} - \frac{m\omega_c}{2\hbar}y\right)\phi - \frac{m\mathcal{E}}{\hbar\lambda}\phi = 0. \tag{25}$$

Setting $\alpha = \frac{m\mathcal{E}}{\hbar\lambda}$, it becomes

$$-i\frac{\partial}{\partial x}\phi = \left(\frac{m\omega_c}{2\hbar}y + \alpha\right)\phi, \tag{26}$$

whose solution is readily found to be

$$\phi_\alpha \equiv \phi_\alpha(x, y) = e^{i(\alpha x + \frac{m\omega_c}{2\hbar}xy)}, \quad \alpha \in \mathbb{R}. \tag{27}$$

Then, the eigenvalues of the operator $\hat{T}_1$, corresponding to eigenfuctions (27), are given by

$$\mathcal{E}_\alpha = \frac{\hbar\lambda}{m}\alpha + \frac{\lambda^2}{2m}, \quad \alpha \in \mathbb{R}, \tag{28}$$

indicating that this spectrum, labeled by α, is continuous. Therefore, to sum up, the eigenvectors and the energy spectrum of the Hamiltonian $\hat{H}_1$ are determined by the following formulas:

$$\Psi_{(n,\alpha)} = \Phi_n \otimes \phi_\alpha \equiv |n, \alpha\rangle,$$

$$\mathcal{E}_{(n,\alpha)} = \frac{\hbar\omega_c}{2}(2n + 1) - \frac{\hbar\lambda}{m}\alpha - \frac{\lambda^2}{2m}, \quad \alpha \in \mathbb{R},\ n = 0, 1, 2,\ldots \tag{29}$$

2.2. Case of the second possible symmetric gauge

We consider now the symmetric gauge

$$\vec{A} = \left(-\frac{B}{2}y, \frac{B}{2}x\right), \tag{30}$$

with the scalar potential given by

$$\Phi(x, y) = -Ex. \tag{31}$$

The classical Hamiltonian H in equation (1) becomes

$$H_2(x, y, p_x, p_y) = \frac{1}{2m}\left[\left(p_x - \frac{eB}{2c}y\right)^2 + \left(p_y + \frac{eB}{2c}x\right)^2\right] + eEx. \tag{32}$$

By mean of canonical quantization and proceeding like in the previous section, we define the two sets of annihilation and creation operators defined by

$$\mathfrak{b}^\dagger = -2iP_{\bar{z}} + \frac{eB}{2c}Z + \lambda, \quad \mathfrak{b} = 2iP_z + \frac{eB}{2c}\bar{Z} + \lambda, \tag{33}$$

$$\mathfrak{d} = 2iP_z - \frac{eB}{2c}\bar{Z}, \quad \mathfrak{d}^\dagger = -2iP_{\bar{z}} - \frac{eB}{2c}Z, \tag{34}$$

with λ defined as in (10) and (11). They also commute each with other and satisfy the commutation relations (12). The corresponding Hamiltonian operator $\hat{H}_2$ can be then written as

$$\hat{H}_2 = \frac{1}{4m}(\mathfrak{b}^\dagger\mathfrak{b} + \mathfrak{b}\mathfrak{b}^\dagger) - \frac{\lambda}{2m}(\mathfrak{d}^\dagger + \mathfrak{d}) - \frac{\lambda^2}{2m}, \tag{35}$$

where the following relation

$$\mathfrak{d}^\dagger + \mathfrak{d} = 2P_y - \frac{eB}{c}X, \tag{36}$$

is obtained. Here, the harmonic oscillator part is given by

$$\hat{H}_{2_{OSC}} = \frac{1}{4m}(\mathfrak{b}^\dagger\mathfrak{b} + \mathfrak{b}\mathfrak{b}^\dagger) \tag{37}$$

and the linear part by

$$\hat{T}_2 = \frac{\lambda}{2m}(\mathfrak{d}^\dagger + \mathfrak{d}) + \frac{\lambda^2}{2m}. \tag{38}$$

The annihilation and creation operators $\mathfrak{b}$ and $\mathfrak{b}^\dagger$ become here

$$\mathfrak{b} = \sqrt{2m\hbar\omega_c}\,\mathfrak{b}', \quad \mathfrak{b}^\dagger = \sqrt{2m\hbar\omega_c}\,\mathfrak{b}'^\dagger, \quad [\mathfrak{b}', \mathfrak{b}'^\dagger] = \mathbf{I}, \tag{39}$$

with

$$\mathfrak{b}' = \sqrt{\frac{2m\omega_c}{\hbar}}\left(\frac{iP_z}{m\omega_c} + \frac{\bar{Z}}{4} + \frac{\lambda}{2m\omega_c}\right), \quad \mathfrak{b}'^{\dagger} = \sqrt{\frac{2m\omega_c}{\hbar}}\left(-\frac{iP_{\bar{z}}}{m\omega_c} + \frac{Z}{4} + \frac{\lambda}{2m\omega_c}\right). \tag{40}$$

From (36), it comes

$$\frac{\lambda}{2m}(\mathfrak{d}^{\dagger} + \mathfrak{d}) = \frac{\hbar\lambda}{m}\left(\frac{P_y}{\hbar} - \frac{1}{2}\frac{m\omega_c}{\hbar}X\right) = \frac{\hbar\lambda}{m}\left(-i\frac{\partial}{\partial y} - \frac{m\omega_c}{2\hbar}X\right). \tag{41}$$

Then, the eigenvalue equation $\hat{T}_2\phi = \mathcal{E}\phi$ is equivalent in this case to

$$\frac{\hbar\lambda}{m}\left(-i\frac{\partial}{\partial y} - \frac{m\omega_c}{2\hbar}X\right)\phi = \mathcal{E}\phi, \tag{42}$$

which leads to

$$\left(-i\frac{\partial}{\partial y} - \frac{m\omega_c}{2\hbar}x\right)\phi - \frac{m\mathcal{E}}{\hbar\lambda}\phi = 0. \tag{43}$$

Taking again $\alpha = \frac{m\mathcal{E}}{\hbar\lambda}$, it follows the equation

$$-i\frac{\partial}{\partial y}\phi = \left(\frac{m\omega_c}{2\hbar}x + \alpha\right)\phi, \tag{44}$$

which can be solved to give the eigenfunctions

$$\phi_\alpha \equiv \phi_\alpha(x, y) = e^{i(\alpha y + \frac{m\omega_c}{2\hbar}xy)}, \quad \alpha \in \mathbb{R}, \tag{45}$$

of the operator $\hat{T}_2$ corresponding to eigenvalues expressed as in (28). Therefore, the eigenvectors and eigenvalues of the Hamiltonian $\hat{H}_2$, as previously determined for $\hat{H}_1$, are obtained as

$$\begin{aligned}\Psi_{(l,\alpha)} &= \Phi_l \otimes \phi_\alpha \equiv |l, \alpha\rangle,\\ \mathcal{E}_{(l,\alpha)} &= \frac{\hbar\omega_c}{2}(2l+1) - \frac{\hbar\lambda}{m}\alpha - \frac{\lambda^2}{2m}, \quad \alpha \in \mathbb{R}, \; l = 0, 1, 2, \dots\end{aligned} \tag{46}$$

Let us introduce the position and momentum operators obtained from the annihilation and creation operators (10) and (33) as

$$\begin{aligned}\hat{Q}_1 &= \frac{1}{2\sqrt{m\omega_c\hbar}}(b^{\dagger} + b), \quad \hat{P}_1 = \frac{i}{2\sqrt{m\omega_c\hbar}}(b^{\dagger} - b),\\ \hat{Q}_2 &= \frac{1}{2\sqrt{m\omega_c\hbar}}(\mathfrak{b}^{\dagger} + \mathfrak{b}), \quad \hat{P}_2 = \frac{i}{2\sqrt{m\omega_c\hbar}}(\mathfrak{b}^{\dagger} - \mathfrak{b}),\end{aligned} \tag{47}$$

respectively, where the following commutation relations

$$\begin{aligned}&[b, \mathfrak{b}^{\dagger}] = 0 = [b^{\dagger}, \mathfrak{b}], \quad [b, \mathfrak{b}] = 0 = [b^{\dagger}, \mathfrak{b}^{\dagger}],\\ &[\hat{Q}_1, \hat{P}_2] = 0 = [\hat{Q}_2, \hat{P}_1], \quad [\hat{Q}_1, \hat{Q}_2] = 0 = [\hat{P}_1, \hat{P}_2]\end{aligned} \tag{48}$$

are satisfied. Then, we respectively have in the gauges $\vec{A} = \left(\frac{B}{2}y, -\frac{B}{2}x\right)$ and $\vec{A} = \left(-\frac{B}{2}y, \frac{B}{2}x\right)$

$$\hat{H}_{1osc} = \frac{\hbar\omega_c}{2}[Q_1^2 + P_1^2], \quad \hat{H}_{2osc} = \frac{\hbar\omega_c}{2}[Q_2^2 + P_2^2], \quad [\hat{H}_{1osc}, \hat{H}_{2osc}] = 0. \tag{49}$$

Thus, from (29), (46) and (49), the eigenvectors denoted $|\Psi_{nl}\rangle := |n, l\rangle = |n\rangle \otimes |l\rangle$ of $\hat{H}_{1osc}$ can be so chosen that they are also the eigenvectors of $\hat{H}_{2osc}$ as follows:

$$\hat{H}_{1osc}|\Psi_{nl}\rangle = \hbar\omega_c\left(n + \frac{1}{2}\right)|\Psi_{nl}\rangle, \; \hat{H}_{2osc}|\Psi_{nl}\rangle = \hbar\omega_c\left(l + \frac{1}{2}\right)|\Psi_{nl}\rangle, \; n, l = 0, 1, 2, \dots, \infty \tag{50}$$

so that $\hat{H}_{2osc}$ lifts the degeneracy of $\hat{H}_{1osc}$ and vice versa.

From (28), consider the shifted eigenvalues

$$\mathcal{E}'_\alpha := \mathcal{E}_\alpha - \frac{\lambda^2}{2m} = \frac{\hbar\lambda}{m}\alpha, \tag{51}$$

where the states $|\epsilon_\alpha\rangle$ are delta-normalized states and form the orthonormal basis $\{|\epsilon_\alpha\rangle, \alpha \in \mathbb{R}\}$. They satisfy the eigenvalue equation

$$\left(\hat{T}_1 - \frac{\lambda^2}{2m}I_{\mathfrak{H}_C}\right)|\epsilon_\alpha\rangle = \mathcal{E}'_\alpha|\epsilon_\alpha\rangle, \tag{52}$$

which is the same equation for the operator $\hat{T}_2$.

3. Construction of coherent states

In this section, CSs are constructed, considering the two possible orientations of the magnetic field as in [18] as well as additional parameters, originated from discrete and continuous aspects of the Hamiltonian spectrum in line with [24]. As a matter of comparison, we first replace the original Hamiltonian operators by their corresponding shifted counterparts, as done in [24]. Then, we investigate the full operators and analyze the results.

3.1. Case of the shifted quantum Hamiltonian

Let $\mathfrak{H}_{D+C} := \mathfrak{H}_D \oplus \mathfrak{H}_C$ be the Hilbert space associated to the operator $\mathcal{H}_D \oplus \mathcal{H}_C$, where $\mathfrak{H}_D$ and $\mathfrak{H}_C$ are associated to discrete and continuous spectra, respectively. Let consider the discrete shifted Hamiltonian $\mathcal{H}_D := H_{1_{osc}} - \frac{\hbar\omega_c}{2}\mathbf{I}_{\mathfrak{H}_D}$ and the continuous shifted Hamiltonian $\mathcal{H}_C := T_1 - \frac{\lambda^2}{2m}\mathbf{I}_{\mathfrak{H}_C}$, where $\mathbf{I}_{\mathfrak{H}_D}$ and $\mathbf{I}_{\mathfrak{H}_C}$ denote the identity operators on $\mathfrak{H}_D$ and $\mathfrak{H}_C$, respectively. Let $\mathfrak{H}_D$ be spanned by the eigenvectors $|\Psi_{nl}\rangle \equiv |n, l\rangle$ of $H_{1_{OSC}}$ and $H_{2_{OSC}}$ provided by (50). Besides, let $\mathfrak{H}_C$ be the Hilbert space associated to the continuous spectrum spanned by the eigenvectors of the operator T_1 denoted $|\epsilon_\alpha\rangle$ in equation(52).

The shifted Hamiltonian $\left(H_{1_{osc}} - \frac{\hbar\omega_c}{2}\mathbf{I}_{\mathfrak{H}_D}\right) - \left(T_1 - \frac{\lambda^2}{2m}\mathbf{I}_{\mathfrak{H}_C}\right)$ possesses a spectrum which is discrete and degenerate according to (29); the Landau levels are infinitely degenerate and given by $\{\mathcal{E}'_{n_{osc}} = \hbar\omega_c n,\ n = 0, 1, 2,\ldots\}$ while the continuous spectrum is furnished by $\{\mathcal{E}'_\alpha,\ \alpha \in \mathbb{R}\}$. So, from (17) and (23), the positive eigenvalues are

$$\mathcal{E}'_{n,\alpha} = \mathcal{E}'_n - \mathcal{E}'_\alpha = \hbar\omega_c\left(n - \frac{\lambda}{m\omega_c}\alpha\right) = \hbar\omega_c(n - \epsilon_\alpha),\ \ \epsilon_\alpha = \frac{\lambda}{m\omega_c}\alpha, \tag{53}$$

such that, for all $n \in \mathbb{N}^*$, $\alpha \leqslant \frac{m\omega_c}{\lambda}$. For the continuous spectrum, one also requires the condition $\mathcal{E}'_\alpha = -\hbar\omega_c\epsilon_\alpha \geqslant 0$, which implies $\alpha \leqslant 0$. Therefore, the energy positivity condition should be: $\alpha \leqslant 0$.

Provided the positivity of the eigenvalues, as required [18] for the operator T_1 (respectively T_2), the CSs related to $\mathcal{H}_D \oplus \mathcal{H}_C$, are given by the *unnormalized* states [24]

$$\begin{aligned}|J, \gamma; J', \gamma'; l; K, \theta; \beta\rangle &= f(K, \theta)|J, \gamma; J', \gamma'; l\rangle + e^{-i\beta}g(J, \gamma, J', \gamma')|K, \theta\rangle\\ &= f(K, \theta)[\mathcal{N}(J)\mathcal{N}(J')]^{-1/2}J'^{l/2}e^{il\gamma'}\sum_{n=0}^{\infty}\frac{J^{n/2}e^{-in\gamma}}{\sqrt{n!l!}}|\Psi_{nl}\rangle\\ &\quad + e^{-i\beta}g(J, \gamma, J', \gamma')\mathcal{N}_\rho(K)^{-1/2}\int_0^\infty\frac{K^{\epsilon_\alpha^-/2}e^{i\epsilon_\alpha\theta}}{\sqrt{\rho(\epsilon_\alpha^-)}}|\epsilon_\alpha^-\rangle\, d\epsilon_\alpha^-,\end{aligned} \tag{54}$$

with $\epsilon_\alpha^- =: -\epsilon_\alpha \geqslant 0$. The labeling parameters are chosen such that: $0 \leqslant J, J', K \leqslant \infty, -\infty < \gamma, \gamma', \theta < \infty$ and $0 \leqslant \beta < 2\pi$, f and g are scalar functions. The normalization constants for the states $|J, \gamma; J', \gamma'; l\rangle \in \mathfrak{H}_D$ are given by

$$\sum_{l=0}^{\infty}\langle J, \gamma; J', \gamma'; l|J, \gamma; J', \gamma'; l\rangle = \frac{1}{\mathcal{N}(J)}\sum_{n=0}^{\infty}\frac{J^n}{n!}\frac{1}{\mathcal{N}(J')}\sum_{l=0}^{\infty}\frac{J'^l}{l!} = 1. \tag{55}$$

Besides, $|K, \theta\rangle \in \mathfrak{H}_C$ and

$$\begin{aligned}\langle K, \theta|K, \theta\rangle = 1 &\Rightarrow \mathcal{N}_\rho(K)^{-1}\int_0^\infty\frac{K^{\epsilon_\alpha^-}}{\rho(\epsilon_\alpha^-)}d\epsilon_\alpha^-\langle\epsilon_\alpha^-|\epsilon_\alpha^-\rangle = \mathcal{N}_\rho(K)^{-1}\int_0^\infty\frac{K^{\epsilon_\alpha^-}}{\rho(\epsilon_\alpha^-)}d\epsilon_\alpha^-\\ &\Rightarrow \mathcal{N}_\rho(K) = \int_0^\infty\frac{K^{\epsilon_\alpha^-}}{\rho(\epsilon_\alpha^-)}d\epsilon_\alpha^-.\end{aligned} \tag{56}$$

The continuity of the combined CSs follows from the continuity of the separate states and of the functions f and g, which are assumed. Indeed, from the definition, we have

$$\begin{aligned}&\||J, \gamma; J', \gamma'; l; K, \theta; \beta\rangle - |\tilde{J}, \tilde{\gamma}; \tilde{J}', \tilde{\gamma}'; l; \tilde{K}, \tilde{\theta}; \beta\rangle\|^2\\ &= |f(K, \theta)|^2\langle J, \gamma; J', \gamma'; l|J, \gamma; J', \gamma'; l\rangle + |g(J, \gamma; J', \gamma')|^2\langle K, \theta|K, \theta\rangle\\ &+ |f(\tilde{K}, \tilde{\theta})|^2\langle\tilde{J}, \tilde{\gamma}; J', \tilde{\gamma}'; l|\tilde{J}, \tilde{\gamma}; \tilde{J}', \tilde{\gamma}'; l\rangle + |g(J, \gamma; J', \gamma')|^2\langle\tilde{K}, \tilde{\theta}|\tilde{K}, \tilde{\theta}\rangle\\ &+ f(K, \theta)f(\tilde{K}, \tilde{\theta})^*\langle\tilde{J}, \tilde{\gamma}; \tilde{J}', \tilde{\gamma}'; l|J, \gamma; J', \gamma'; l\rangle + f(K, \theta)^*f(\tilde{K}, \tilde{\theta})\langle J, \gamma; J', \gamma'; l|\tilde{J}, \tilde{\gamma}; \tilde{J}', \tilde{\gamma}'; l\rangle\\ &+ g(J, \gamma; J', \gamma')g(\tilde{J}, \tilde{\gamma}; \tilde{J}', \tilde{\gamma}')^*\langle\tilde{K}, \tilde{\theta}|K, \theta\rangle + g(J, \gamma; J', \gamma')^*g(\tilde{J}, \tilde{\gamma}; \tilde{J}', \tilde{\gamma}')\langle K, \theta|\tilde{K}, \tilde{\theta}\rangle\end{aligned} \tag{57}$$

such that

$$\lim_{(J,\gamma;J',\gamma';K,\theta)\to(\tilde{J},\tilde{\gamma};\tilde{J}',\tilde{\gamma}';\tilde{K},\tilde{\theta})}\||J, \gamma; J', \gamma'; l; K, \theta; \beta\rangle - |\tilde{J}, \tilde{\gamma}; \tilde{J}', \tilde{\gamma}'; l; \tilde{K}, \tilde{\theta}; \beta\rangle\|^2 = 0. \tag{58}$$

Now, let us investigate the resolution of the identity or the completeness relation which is expressed in terms of the projectors onto the states $|J,\gamma;J',\gamma';l;K,\theta;\beta\rangle$.

Proposition 3.1. *The CSs (54) satisfy, on $\mathfrak{H}_{D+C}$, the resolution of the identity*

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi}|J,\gamma;J',\gamma';l;K,\theta;\beta\rangle\langle J,\gamma;J',\gamma';l;K,\theta;\beta| \\ d\mu_B(\gamma)d\mu_B(\gamma')\frac{d\theta}{2\pi}\frac{d\beta}{2\pi}\mathcal{N}(J)\mathcal{N}(J')\mathcal{N}_\rho(K)d\nu(J)d\nu(J')d\lambda(K)=\mathbf{I}_{\mathfrak{H}_D^l}+\mathbf{I}_{\mathfrak{H}_C} \tag{59}$$

where $\mathbf{I}_{\mathfrak{H}_D^l}$, $\mathbf{I}_{\mathfrak{H}_D^n}$ are the identity operators on the subspaces $\mathfrak{H}_D^n$, $\mathfrak{H}_D^l$ of $\mathfrak{H}_D$ such that

$$\sum_{n=0}^\infty|\Psi_{nl}\rangle\langle\Psi_{nl}|=\mathbf{I}_{\mathfrak{H}_D^l},\quad \sum_{l=0}^\infty|\Psi_{nl}\rangle\langle\Psi_{nl}|=\mathbf{I}_{\mathfrak{H}_D^n}. \tag{60}$$

$d\mu_B$ *refers to the* Bohr measure *[18] provided as follows*

$$\langle f|g\rangle_{ns}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\overline{f(\gamma)}g(\gamma)d\gamma:=\int_{\mathbb{R}}\overline{f(\gamma)}g(\gamma)d\mu_B(\gamma) \tag{61}$$

given on the Hilbert space $\mathfrak{H}_{ns}$ of functions $f:\mathbb{R}\to\mathbb{C}$, which is complete with respect to the scalar product $\langle .|.\rangle_{ns}$. $d\lambda(K)=\sigma(K)dK$, *and* $\sigma(K)$ *is a non-negative weight function $\sigma(K)\geqslant 0$ such that*

$$\int_0^\infty K^{\epsilon_\alpha^-}\sigma(K)dK\equiv\rho(\epsilon_\alpha^-). \tag{62}$$

On the Hilbert spaces $\mathfrak{H}_D$, $\mathfrak{H}_C$ and $\mathfrak{H}_{D+C}$, we have the following essential relations

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^{2\pi}|f(K,\theta)|^2|J,\gamma;J',\gamma';l\rangle\langle J,\gamma;J',\gamma';l| \\ d\mu_D(J,\gamma,J',\gamma')d\mu_C(K,\theta)\frac{d\beta}{2\pi}=\mathbf{I}_{\mathfrak{H}_D^l},$$
$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^{2\pi}|g(J,\gamma,J',\gamma')|^2|K,\theta\rangle\langle K,\theta| \\ d\mu_D(J,\gamma,J',\gamma')d\mu_C(K,\theta)\frac{d\beta}{2\pi}=\mathbf{I}_{\mathfrak{H}_C},$$
$$\int_0^{2\pi}e^{i\beta}\frac{d\beta}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty\int_0^\infty\int_0^\infty g(J,\gamma,J',\gamma')^*f(K,\theta)|J,\gamma;J',\gamma';l\rangle\langle K,\theta| \\ d\mu_D(J,\gamma,J',\gamma')d\mu_C(K,\theta)=0. \tag{63}$$

that need to be satisfied, where $d\mu_D$ *and* $d\mu_C$ *are the measures associated to the discrete-spectrum CSs $\{J,\gamma,J',\gamma'\}$ and continuous-spectrum CSs $\{K,\theta\}$ labeling parameters, respectively. The identity operator $\mathbf{I}_{\mathfrak{H}_{D+C}}$ is the direct sum of the identity operators $\mathbf{I}_{\mathfrak{H}_D}$ and $\mathbf{I}_{\mathfrak{H}_C}$ which act on the complementary subspaces $\mathfrak{H}_D$ and $\mathfrak{H}_C$, respectively, corresponding to discrete and continuous spectra.*

Noting that the integration over β, $0\leqslant\beta<2\pi$ eliminates the third relation above, which is related to the off-diagonal terms, the three conditions (63) are reduced to

$$\int_{\mathbb{R}}\int_0^\infty|f(K,\theta)|^2d\mu_C(K,\theta)=1,$$
$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty\int_0^\infty|g(J,\gamma,J',\gamma')|^2d\mu_D(J,\gamma,J',\gamma')=1. \tag{64}$$

In view of getting the resolution of the identity, let us take the functions f and g as in [24], such that

$$f(K,\theta)=\mathcal{N}_f\,e^{-\frac{K^2+\theta^2}{2}},\quad g(J,\gamma,J',\gamma')=\mathcal{N}_g\,e^{-\frac{J^2+J'^2}{2}}, \tag{65}$$

where the factors $\mathcal{N}_g$ and $\mathcal{N}_f$ are chosen so that

$$\mathcal{N}_f^2\int_{\mathbb{R}}\int_0^\infty e^{-(K^2+\theta^2)}d\mu_C(K,\theta)=1,$$
$$\mathcal{N}_g^2\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty\int_0^\infty e^{-(J^2+J'^2)}d\mu_D(J,\gamma,J',\gamma')=1. \tag{66}$$

Proof. See in the appendix. □

Proposition 3.2. *The property of temporal stability can be obtained here by postulating similar assumptions as in [24], such that $0 \leqslant \mathcal{H}_D \leqslant \Omega$ and $\Omega < \mathcal{H}_C$, i.e. the Hamiltonians are adjusted so that $0 < \mathcal{H}_C - \Omega$. By taking into account the phase factor $e^{-i\beta}$, it comes the following relation*

$$\begin{aligned} e^{-i\mathcal{H}t}|J, \gamma; J', \gamma'; l; K, \theta; \beta\rangle &= f(K, \theta)|J, \gamma + \omega_c t; J', \gamma'; l\rangle \\ &\quad + e^{-i(\beta+\Omega t)}g(J, \gamma, J', \gamma')|K, \theta + \omega_c t\rangle \\ &= |J, \gamma + \omega_c t; J', \gamma'; l; K, \theta + \omega_c t; \beta + \Omega t\rangle, \end{aligned} \tag{67}$$

with $\mathcal{H} = \mathcal{H}_D + (\mathcal{H}_C - \Omega)$.

Proof. See in the appendix. □

The action identity as noticed in [24] is difficult to obtain with the combined CSs given in (54).

3.2. Case of the Hamiltonian $H_{2_{osc}} - T_2$

By analogy of the setting in section 3.1, we study the shifted Hamiltonian $\left(H_{2_{osc}} - \frac{\hbar\omega_c}{2}\mathbf{1}_{\mathfrak{H}_D}\right) - \left(T_2 - \frac{\lambda^2}{2m}\mathbf{1}_{\mathfrak{H}_C}\right)$.

The related CSs are here given on $\mathfrak{H}_{D+C}$ by

$$\begin{aligned} |J, \gamma; J', \gamma'; n; K, \theta; \beta\rangle &= f(K, \theta)|J, \gamma; J', \gamma'; n\rangle + e^{-i\beta}g(J, \gamma, J', \gamma')|K, \theta\rangle \\ &= f(K, \theta)[\mathcal{N}(J)\mathcal{N}(J')]^{-1/2}J^{n/2}e^{-in\gamma}\sum_{l=0}^{\infty}\frac{J'^{l/2}e^{il\gamma'}}{\sqrt{n!l!}}|\Psi_{nl}\rangle \\ &\quad + e^{-i\beta}g(J, \gamma, J', \gamma')\mathcal{N}_\rho(K)^{-1/2}\int_0^\infty \frac{K^{\epsilon_\alpha^-/2}e^{i\epsilon_\alpha\theta}}{\sqrt{\rho(\epsilon_\alpha^-)}}|\epsilon_\alpha^-\rangle d\epsilon_\alpha^-, \end{aligned} \tag{68}$$

where the normalization constants are given as in (55) with the relation (56) also satisfied.

Proposition 3.3. *The CSs satisfy, on $\mathfrak{H}_{D+C}$, the resolution of the identity*

$$\begin{aligned} &\int_0^\infty\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi}|J, \gamma; J', \gamma'; n; K, \theta; \beta\rangle\langle J, \gamma; J', \gamma'; n; K, \theta; \beta| \\ &d\mu_B(\gamma)d\mu_B(\gamma')\frac{d\theta}{2\pi}\frac{d\beta}{2\pi}\mathcal{N}(J)\mathcal{N}(J')\mathcal{N}_\rho(K)d\nu(J)d\nu(J')d\lambda(K) = \mathbf{1}_{\mathfrak{H}_D^n} + \mathbf{1}_{\mathfrak{H}_C}. \end{aligned} \tag{69}$$

Proof. See that of proposition 3.1. □

Proposition 3.4. *The temporal stability property is given by*

$$\begin{aligned} e^{-i\mathcal{H}t}|J, \gamma; J', \gamma'; n; K, \theta; \beta\rangle &= f(K, \theta)|J, \gamma; J', \gamma' + \omega_c t; n\rangle \\ &\quad + e^{-i(\beta+\Omega t)}g(J, \gamma, J', \gamma')|K, \theta + \omega_c t\rangle \\ &= |J, \gamma; J', \gamma' + \omega_c t; n; K, \theta + \omega_c t; \beta + \Omega t\rangle. \end{aligned} \tag{70}$$

Proof. See that of proposition 3.2. □

3.3. Case of the unshifted Hamiltonians H_1 and H_2

The eigenvalues $\mathcal{E}_{n,\alpha}$ of the Hamiltonian operators H_1 and H_2, given respectively in equations (29) and (46), can be rewritten as $\mathcal{E}_{n,\alpha} = \mathcal{E}_n + \mathcal{E}_\alpha$, where

$$\mathcal{E}_n = \hbar\omega_c\left(n + \frac{1}{2}\right),\quad \mathcal{E}_\alpha = -\hbar\omega_c\epsilon_\alpha,\quad \epsilon_\alpha = \frac{\lambda}{m\omega_c}\alpha + \frac{\lambda^2}{2m\hbar\omega_c}. \tag{71}$$

The required conditions $\mathcal{E}_{n,\alpha} > 0$ for all $n \in \mathbb{N}$ and hence $\mathcal{E}_\alpha \geqslant 0$ lead to the relations

$$\alpha \leqslant -\frac{\lambda}{2\hbar}. \tag{72}$$

Setting

$$\rho(n) = \mathcal{E}_1\mathcal{E}_2 \ldots \mathcal{E}_n \tag{73}$$

such that

$$\rho(n) = \prod_{k=1}^{n} \hbar\omega_c\left(k + \frac{1}{2}\right) = (\kappa)^n\left(\frac{3}{2}\right)_n, \quad \kappa = \hbar\omega_c \tag{74}$$

where $\left(\frac{3}{2}\right)_n$ stands for the Pochhammer symbol [30]. The CSs, related to H_1, defined in line with (54), are now given, where the function $\rho(n) = n!$ is replaced by the one in (74), by

$$\begin{aligned} |J, \gamma; J', \gamma'; l; K, \theta; \beta\rangle &= f(K, \theta)[\mathcal{N}(J)\mathcal{N}(J')]^{-1/2} J'^{l/2} e^{i\mathcal{E}_l\gamma'} \sum_{n=0}^{\infty} \frac{J^{n/2} e^{-i\mathcal{E}_n\gamma}}{\sqrt{\rho(n)\rho(l)}} |\Psi_{nl}\rangle \\ &+ e^{-i\beta} g(J, \gamma, J', \gamma')\mathcal{N}_\rho(K)^{-1/2} \int_0^\infty \frac{K^{\epsilon_\alpha^-/2} e^{i\epsilon_\alpha\theta}}{\sqrt{\rho(\epsilon_\alpha^-)}} |\epsilon_\alpha^-\rangle d\epsilon_\alpha^- \end{aligned} \tag{75}$$

yields

$$\begin{aligned} \mathcal{N}(J) &= \sum_{n=0}^{\infty} \frac{J^n}{\rho(n)} = \sum_{n=0}^{\infty} \frac{J^n}{\kappa^n \left(\frac{3}{2}\right)_n} = {}_1F_1\left(1; \frac{3}{2}; \frac{J}{\kappa}\right), \\ \mathcal{N}(J') &= \sum_{l=0}^{\infty} \frac{J'^l}{\rho(l)} = \sum_{l=0}^{\infty} \frac{J'^l}{\kappa^l \left(\frac{3}{2}\right)_l} = {}_1F_1\left(1; \frac{3}{2}; \frac{J'}{\kappa}\right), \end{aligned} \tag{76}$$

with the relation (56) also remaining here valid.

Proposition 3.5. *The CSs (75) satisfy, on $\mathfrak{H}_{D+C}$, a resolution of the identity given in (59), where the measures $d\nu(J)$ and $d\nu(J')$ are now given by*

$$\begin{aligned} d\nu(J) &= \frac{\Gamma^2(n+1)}{\kappa^\eta \Gamma(\eta)} {}_1F_1\left(1; \eta; \frac{J}{\kappa}\right) \frac{e^{-J/\kappa} J^{-n+\eta-1}}{\left(\frac{1}{\kappa} + \mu - \sigma\right)^n} L_n^{\eta-1}[(\mu - \sigma)J] dJ, \\ d\nu(J') &= \frac{\Gamma^2(l+1)}{\kappa^\eta \Gamma(\eta)} {}_1F_1\left(1; \eta; \frac{J'}{\kappa}\right) \frac{e^{-J'/\kappa} J'^{-l+\eta-1}}{\left(\frac{1}{\kappa} + \mu - \sigma\right)^l} L_l^{\eta-1}[(\mu - \sigma)J'] dJ', \ \eta = \frac{3}{2}, \end{aligned} \tag{77}$$

where the quantities $L_n^{\eta-1}[(\mu - \sigma)J]$, $L_l^{\eta-1}[(\mu - \sigma)J']$ are the Laguerre polynomials, and lead to the identities [30]

$$\begin{aligned} n! \int_0^\infty t^{\nu-n} e^{\mu t} L_n^{\nu-n}[(\mu - \sigma)t] e^{-st} dt &= \Gamma(\nu+1)(s-\sigma)^n (s-\mu)^{-\nu-1}, \ \Re(\nu) > n-1 \\ l! \int_0^\infty t^{\nu-l} e^{\mu t} L_l^{\nu-l}[(\mu - \sigma)t] e^{-st} dt &= \Gamma(\nu+1)(s-\sigma)^l (s-\mu)^{-\nu-1}, \ \Re(\nu) > l-1 \end{aligned} \tag{78}$$

with $\nu = n + \eta - 1$ (resp. $\nu = l + \eta - 1$) and $\frac{1}{\kappa} = s - \mu$.

The CSs for the Hamiltonian H_2, similar to the ones in (75), can be constructed in the same way, with the labeling parameters J, γ playing the role of J', γ' and vice versa.

4. Concluding remarks

Coherent states have been constructed for Hamiltonians with both discrete and continuous spectra, in the context of the motion of an electron in an electromagnetic field, arising in the quantum Hall effect by considering shifted and unshifted spectra, respectively. These coherent states satisfy the Gazeau-Klauder coherent states criteria that are the continuity in the labels, the resolution of the identity and the temporal stability. The action identity property remains difficult to obtain in the combined coherent states as noticed in [24].

An extension of this work that is currently under investigation is the construction of coherent states for a Hamiltonian in the case of an electric field depending simultaneously on both x and y directions, and for Hamiltonian operators admitting discrete eigenvalues and eigenfuctions in appropriate Hilbert space [31].

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix

Proof of proposition 3.1. From (54), we get

$$
\begin{aligned}
&|J,\gamma;J',\gamma';l;K,\theta;\beta\rangle\langle J,\gamma;J',\gamma';l;K,\theta;\beta|\\
&=|f(K,\theta)|^2\frac{J'^{l}}{[\mathcal{N}(J)\mathcal{N}(J')]}\sum_{n,p=0}^{\infty}\frac{J^{\frac{n+p}{2}}e^{i(p-n)\gamma}}{\sqrt{n!l!p!l!}}|\Psi_{nl}\rangle\langle\Psi_{pl}|\\
&\quad+|g(J,\gamma,J',\gamma')|^2\mathcal{N}_\rho(K)^{-1}\int_0^\infty\int_0^\infty\frac{K^{\frac{\epsilon_\alpha^-+\epsilon_{\alpha'}^-}{2}}e^{i(\epsilon_\alpha-\epsilon_{\alpha'})\theta}}{\sqrt{\rho(\epsilon_\alpha^-)\rho(\epsilon_{\alpha'}^-)}}|\epsilon_\alpha^-\rangle\langle\epsilon_{\alpha'}^-|d\epsilon_\alpha^-d\epsilon_{\alpha'}^-\\
&\quad+f(K,\theta)^*e^{-i\beta}g(J,\gamma,J',\gamma')|K,\theta\rangle\langle J;\gamma,J';\gamma'|\\
&\quad+f(K,\theta)e^{i\beta}g(J,\gamma,J',\gamma')^*|J,\gamma;J',\gamma'\rangle\langle K,\theta|.
\end{aligned}
\tag{A.1}
$$

Then, the following equalities are valid:

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi}|J,\gamma;J',\gamma';l;K,\theta;\beta\rangle\langle J,\gamma;J',\gamma';l;K,\theta;\beta|\\
&d\mu_B(\gamma)d\mu_B(\gamma')\mathcal{N}(J)\mathcal{N}(J')\mathcal{N}_\rho(K)\frac{d\theta}{2\pi}\frac{d\beta}{2\pi}\\
&=\sum_{n,p=0}^{\infty}\frac{1}{\sqrt{n!l!p!l!}}\left[\int_{\mathbb{R}}|f(K,\theta)|^2\mathcal{N}_\rho(K)\frac{d\theta}{2\pi}\right]\int_0^{2\pi}\int_0^{2\pi}J^{\frac{n+p}{2}}J'^{l}\\
&\frac{d\gamma'}{2\pi}\frac{d\beta}{2\pi}\delta_{np}|\Psi_{nl}\rangle\langle\Psi_{pl}|+\left[\int_{\mathbb{R}}\int_{\mathbb{R}}|g(J,\gamma,J',\gamma')|^2d\mu_B(\gamma)d\mu_B(\gamma')\mathcal{N}(J)\mathcal{N}(J')\right]\\
&\int_0^\infty\int_0^\infty\int_0^{2\pi}\frac{K^{\frac{\epsilon_\alpha^-+\epsilon_{\alpha'}^-}{2}}}{\sqrt{\rho(\epsilon_\alpha^-)\rho(\epsilon_{\alpha'}^-)}}\delta(\epsilon_\alpha-\epsilon_{\alpha'})\frac{d\beta}{2\pi}d\epsilon_\alpha^-d\epsilon_{\alpha'}^-|\epsilon_\alpha^-\rangle\langle\epsilon_{\alpha'}^-|\\
&=\sum_{n=0}^{\infty}\frac{1}{n!l!}\left[\int_{\mathbb{R}}|f(K,\theta)|^2\mathcal{N}_\rho(K)\frac{d\theta}{2\pi}\right]J^nJ'^{l}|\Psi_{nl}\rangle\langle\Psi_{pl}|\\
&\quad+\left[\int_{\mathbb{R}}\int_{\mathbb{R}}|g(J,\gamma,J',\gamma')|^2d\mu_B(\gamma)d\mu_B(\gamma')\mathcal{N}(J)\mathcal{N}(J')\right]\int_0^\infty\frac{K^{\epsilon_\alpha^-}}{\rho(\epsilon_\alpha^-)}d\epsilon_\alpha^-|\epsilon_\alpha^-\rangle\langle\epsilon_\alpha^-|.
\end{aligned}
\tag{A.2}
$$

The conditions (64) implies

$$
\int_{\mathbb{R}}\int_0^\infty|f(K,\theta)|^2\mathcal{N}_\rho(K)d\lambda(K)\frac{d\theta}{2\pi}=\int_{\mathbb{R}}\int_0^\infty|f(K,\theta)|^2d\mu_C(K,\theta)=1,
\tag{A.3}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty\int_0^\infty|g(J,\gamma,J',\gamma')|^2d\mu_B(\gamma)d\mu_B(\gamma')\mathcal{N}(J)\mathcal{N}(J')d\nu(J)d\nu(J')\\
&=\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty\int_0^\infty|g(J,\gamma,J',\gamma')|^2d\mu_D(J,\gamma,J',\gamma')=1.
\end{aligned}
\tag{A.4}
$$

The measures $d\nu(J)=e^{-J}dJ$ and $d\nu(J')=e^{-J'}dJ'$ are such that the moment problems given by

$$
\int_0^\infty\frac{J^n}{n!}d\nu(J)=1,\qquad\int_0^\infty\frac{J'^{l}}{l!}d\nu(J')=1
\tag{A.5}
$$

are satisfied. Therefore,

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^{2\pi}|J,\gamma;J',\gamma';l;K,\theta;\beta\rangle\langle J,\gamma;J',\gamma';l;K,\theta;\beta|\\
&\times d\mu_B(\gamma)d\mu_B(\gamma')\frac{d\theta}{2\pi}\frac{d\beta}{2\pi}\mathcal{N}(J)\mathcal{N}(J')\mathcal{N}_\rho(K)d\nu(J)d\nu(J')d\lambda(K)\\
&=\sum_{n=0}^{\infty}|\Psi_{nl}\rangle\langle\Psi_{nl}|+\int_0^\infty|\epsilon_\alpha^-\rangle\langle\epsilon_\alpha^-|d\epsilon_\alpha^-=\mathbf{I}_{\mathfrak{H}_D^l}+\mathbf{I}_{\mathfrak{H}_C}.
\end{aligned}
\tag{A.6}
$$

□

Proof of proposition 3.2. By definition, we have

$$
\begin{aligned}
& e^{-i\mathcal{H}t}|J,\,\gamma;\,J',\,\gamma';\,l;\,K,\,\theta;\,\beta\rangle \\
& \equiv f(K,\,\theta)[\mathcal{N}(J)\mathcal{N}(J')]^{-1/2}J'^{l/2}e^{il\gamma'}\sum_{n=0}^{\infty}\frac{J^{n/2}e^{-in\gamma}}{\sqrt{n!l!}}e^{-i\mathcal{H}_D t}|\Psi_{nl}\rangle \\
& + e^{-i(\mathcal{H}_C-\Omega)t}e^{-i\beta}g(J,\,\gamma,\,J',\,\gamma')\mathcal{N}_\rho(K)^{-1/2}\int_0^\infty \frac{K^{\epsilon_\alpha^-/2}e^{i\epsilon_\alpha\theta}}{\sqrt{\rho(\epsilon_\alpha^-)}}|\epsilon_\alpha^-\rangle\, d\epsilon_\alpha^- \\
& = f(K,\,\theta)[\mathcal{N}(J)\mathcal{N}(J')]^{-1/2}J'^{l/2}e^{il\gamma'}\sum_{n=0}^{\infty}\frac{J^{n/2}e^{-in(\gamma+\omega_c t)}}{\sqrt{n!l!}}|\Psi_{nl}\rangle \\
& + e^{-i(\beta+\Omega t)}g(J,\,\gamma,\,J',\,\gamma')\mathcal{N}_\rho(K)^{-1/2}\int_0^\infty \frac{K^{\epsilon_\alpha^-/2}e^{i\epsilon_\alpha(\theta+\omega_c t)}}{\sqrt{\rho(\epsilon_\alpha^-)}}|\epsilon_\alpha^-\rangle\, d\epsilon_\alpha^- \\
& = f(K,\,\theta)|J,\,\gamma+\omega_c t;\,J',\,\gamma';\,l\rangle + e^{-i(\beta+\Omega t)}g(J,\,\gamma,\,J',\,\gamma')|K,\,\theta+\omega_c t\rangle \\
& = |J,\,\gamma+\omega_c t;\,J',\,\gamma';\,l;\,K,\,\theta+\omega_c t;\,\beta+\Omega t\rangle.
\end{aligned}
\tag{A.7}
$$

□

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