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K-Essence Sources of Kerr–Schild Spacetimes

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Abstract: We extend a result by one of the authors, established for nonvacuum Einstein gravity, to minimally coupled k-essence scalar–tensor theories. First, we prove that in order to source a Kerr–Schild-type spacetime, the k-essence Lagrangian should be at most quadratic in the kinetic term. This is reduced to linear dependence when the Kerr–Schild null congruence is autoparallel. Finally, we show that solutions of the Einstein equations linearized in Kerr–Schild-type perturbations are also required to solve the full nonlinear system of Einstein equations, selecting once again k-essence scalar fields with linear Lagrangians in the kinetic term. The only other k-essence sharing the property of sourcing perturbative Kerr–Schild spacetimes, which are also exact, is the scalar field constant along the integral curves of the Kerr–Schild congruence, with the otherwise unrestricted Lagrangian.

Keywords: k-essence; Kerr–Schild maps; linearized and exact solutions

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1. Introduction

Most physically interesting metrics in Einstein gravity are of Kerr–Schild type. They include Schwarzschild and Kerr black holes for vacuums and the Kerr–Newman family and pp-waves for Einstein–Maxwell systems or the Vaidya radiating solution sourced by a null dust solution [1]. Such spacetimes are generated by a null congruence l^a through the map

$$\tilde{g}_{ab} = g_{ab} + \lambda l_a l_b \tag{1}$$

from the flat metric $g_{ab} = \eta_{ab}$, with λ an arbitrary parameter. The extension to a generic vacuum seed metric g_{ab} led to either a shearfree congruence l^a (containing all solutions with flat seed metric) or a unicity theorem for the shearing class (only containing one of the Kóta–Perjés metrics and its nontwisting limit, the Kasner metric) [2–4]. An important result was provided by Xanthopoulos [5], stating that all vacuum Kerr–Schild metrics arising as perturbations (with small λ) of vacuum seed spacetimes are also exact (hence solutions of the Einstein equations for arbitrary λ). This result was generalized for the nonvacuum case by one of us [6], proving that for any pair (g_{ab}, T_{ab}) of a seed metric and energy–momentum tensor, the pair $\left(\tilde{g}_{ab}, T_{ab} + \lambda T^{(1)}_{ab}\right)$ arising as a solution of the linearized solution (hence for small λ) generates an exact solution (with arbitrary λ) of the form $\left(\tilde{g}_{ab}, T_{ab} + \lambda T^{(1)}_{ab} + \lambda^2 l_{(a} T^{(1)}_{b)c} l^c\right)$, in the case when the null congruence is autoparallel (if not, a similar but more technical result holds).

While general relativity is precisely verified by Solar System tests, also all cosmological, astrophysical and gravitational wave observations are consistent with it, and modified gravitational theories are still of interest, provided they obey the observational constraints.

They are pursued due to the need to explain dark matter and dark energy, to properly model inflation, and to make room for quantum-gravity-motivated extensions in the low energy regime. A plethora of modifications relaxing one or more assumptions of the Lovelock theorem (gravity expressed solely by the metric tensor, obeying second-order dynamics in four spacetime dimensions with a divergence-free energy–momentum tensor) are still viable [7]. The relation of some of these modified theories with a subclass of Kerr–Schild metrics has been investigated for vector-tensor [8,9] theories.

A most natural modification arises by including a scalar field in the gravitational sector. By imposing second-order dynamics for both the metric and the scalar, such that Ostrogradsky instabilities are avoided, the Horndeski class emerges [10,11]. Additionally, requiring gravitational waves to propagate with the speed of light in a vacuum (in order to comply with observations of high frequency gravitational waves by LIGO and Virgo [12]) leads to [13–16] a restricted subclass of kinetic gravity braiding theories [17]. The dependence of the Lagrangian on the scalar ϕ of such theories is only through ϕ, $\Box\phi$ and the kinetic term

$$X = -\frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi \tag{2}$$

(with g^{ab} the inverse metric, ∇_a the Levi–Civita connection, and $\Box = \nabla_a \nabla^a$). Without the $\Box\phi$ dependence, they simplify to the k-essence class of scalar–tensor theories with the Lagrangian

$$\mathfrak{L}_\phi = \sqrt{-\mathfrak{g}} F(\phi, X) \tag{3}$$

(here, $\mathfrak{g}$ is the determinant of the metric and $F(\phi, X)$ an arbitrary function of ϕ and X). We will further assume minimal coupling to the metric, such that $\mathfrak{L}_\phi$ is supplemented with the Einstein–Hilbert action (in this case, the Einstein and Jordan frames coincide).

Such scalar dynamics was originally introduced for k-inflation models, with Lagrangians combining first- and second-order powers of X, and explored in the context of slow-roll and power law inflation scenarios [18]. A k-essence model with the Lagrangian consisting of a purely X-dependent function divided by the scalar squared was proposed to generate late time dark energy through the transformation of the scalar field into a negative pressure state [19]. A related k-essence Lagrangian consisting of a product of functions depending solely of ϕ and X, respectively, is suitable to accommodate for slow-roll, power-law, and pole-like inflation mechanisms, and it also appears in the effective action of string theory [20]. Such models are able to generate cosmic evolution, and they have the appealing feature that late time acceleration is not permanent.

Moreover, Lim, Sawicki, and Vikman proposed a unification of dark matter and dark energy in a single degree of freedom [21]. In their model, the k-essence field is supplemented by a second scalar acting as a Lagrange multiplier (hence, without a proper kinetic term), enforcing a relation between X and ϕ. As consequence of this constraint, the system retains a single dynamical degree of freedom, allowing for no wave-like modes, hence leading to a generalized k-essence with vanishing speed of sound, energy flowing along timelike geodesics (similarly as for dust), while possessing non-zero pressure. This model is able to reproduce the ΛCDM evolution, with the structure formation possibly affected. Similar techniques are explored for predicting the WIMP dark matter mass spectrum, while the nonvanishing pressure of matter mimics the cosmological constant [22]. Generalized k-essence was explored to heal the cosmological constant problem in Ref. [23], also to achieve a graceful exit from inflation through proper choices of the scalar potential [24].

The gravitational collapse of k-essence was investigated numerically in Ref. [25]. For strong fields yielding to black hole formation, in certain cases, the sound horizon may lie inside the light horizon, allowing for superluminal k-essence signals escaping the black hole. The evolution of a k-essence scalar field is governed by an effective metric (different from the spacetime metric). Its conformally related emergent gravity metric resembled a generalized Vaidya metric sourced by a superposition of perfect and null fluids, when the scalar was assumed to be driven by Dirac–Born–Infeld-type dynamics and assumed to depend only on one of the advanced and retarded null coordinates [26]. Gravitational collapse [27] and the evaporation of the emerging horizon [28] were also discussed.

In the present paper, we aim to investigate another feature related to k-essence, namely, under which conditions the linearity property of Kerr–Schild metrics proven in Ref. [6] would apply in this class of minimally coupled k-essence scalar fields, also dubbed Class (A) in the classification of Ref. [29]. In Section 2, we summarize the results of Ref. [6] on Kerr–Schild spacetimes with matter sources, necessary for our analysis on minimally coupled k-essence fields.

In Section 3, we impose the condition on the k-essence to source Kerr–Schild spacetimes. In Section 4, we analyze the requirements for lifting the solution of the linearized system to exact solution by increasing the Kerr–Schild parameter to arbitrary values. In Section 5, we repeat the analysis for a simpler case, left out from the previous discussion.

In Section 6, we address the question of how black hole properties are affected by Kerr–Schild maps. We also include an analysis of the scalar fields inside or outside the event horizons of black holes, also of cosmological scalar fields, in terms of equivalent fluids. We calculate the adiabatic speed of sound, which does not vanish for the types of k-essence scalar fields allowed by our requirements, a property already noted in Ref. [25], and we formally exclude the Laplacian instability regimes. Finally, we argue that k-essence Kerr–Schild seed spacetimes could be important in dynamical situations.

In Section 7, we summarize our results.

2. Kerr–Schild Spacetimes and K-Essence

In this section, we summarize the main results of Ref. [6] necessary for our forthcoming discussion and discuss how the k-essence fits into the generic scheme.

2.1. Nonvacuum Kerr–Schild Maps

The Ricci tensors of the Kerr–Schild and seed spacetimes are related as:

$$\tilde{R}_{ab} = R_{ab} + \lambda R^{(1)}_{ab} + \lambda^2 R^{(2)}_{ab} + \lambda^3 R^{(3)}_{ab}, \tag{4}$$

with the contributions

$$R^{(1)}_{ac} = \nabla_b \left(\nabla_{(a} \left(l_{c)} l^b \right) - \frac{1}{2} \nabla^b (l_a l_c) \right), \tag{5}$$

$$R^{(2)}_{ac} = \nabla_b l^b l_{(a} D l_{c)} + \frac{1}{2} D l_a D l_c + l_{(a} D D l_{c)} + l_a l_c \nabla_b l_d \nabla^{[b} l^{d]} - D l^b \nabla_b l_{(a} l_{c)} , \tag{6}$$

$$R^{(3)}_{ac} = -\frac{1}{2} l_a l_c D l^b D l_b \tag{7}$$

(here, $Dl^a = l^b \nabla_b l^a$ is the directional covariant derivative along the null congruence). Expressing the Ricci tensor contributions in Equation (4) through the Einstein equations written for both the seed and Kerr–Schild metrics, the condition

$$\lambda R^{(1)}_{ab} + \lambda^2 R^{(2)}_{ab} + \lambda^3 R^{(3)}_{ab} = \tilde{T}_{ab} - T_{ab} - \frac{1}{2} g_{ab} \left(\tilde{T} - T \right) - \frac{1}{2} \lambda l_a l_b \tilde{T} \tag{8}$$

emerges (we have absorbed the constants into a redefinition of the energy–momentum tensors). In Ref. [6] it was proven that when seeking the source of the Kerr–Schild spacetime in the form of the series

$$\tilde{T}_{ab} = T_{ab} + \lambda T_{ab}^{(1)} + \lambda^2 T_{ab}^{(2)} + \lambda^3 T_{ab}^{(3)} + \sum_{i=1}^{\infty} \lambda^{3+i} T_{ab}^{(3+i)} , \tag{9}$$

the terms of higher orders than three vanish. Furthermore, when l^a is autoparallel, $T_{ab}^{(3)} = 0$ also holds. Additionally requiring that the solution for small λ (the solution of the linear equation) solves the full set of Einstein equations leads to the condition

$$T_{ab}^{(2)} = l_{(a} T_{b)c}^{(1)} l^c . \tag{10}$$

This was announced as Theorem 2 in Ref. [6].

2.2. K-Essence

The dynamics of the k-essence is given by the action

$$S_\phi = \int \mathrm{d}^4 x \mathfrak{L}_\phi = \int \mathrm{d}^4 x \sqrt{-\mathfrak{g}} F(\phi, X) , \tag{11}$$

while its energy–momentum tensor emerges from its metric variation,

$$T_{ab} = \frac{-2}{\sqrt{-\mathfrak{g}}} \frac{\delta S_\phi}{\delta g^{ab}} \tag{12}$$

as

$$T_{ab} = F_X(\phi, X) \nabla_a \phi \nabla_b \phi + g_{ab} F(\phi, X) . \tag{13}$$

Here, the subscript X denotes the derivative with respect to X.

As we assume minimal coupling, the variation in the total action, the sum of the Einstein–Hilbert action and the k-essence contribution (11), with respect to the metric gives the Einstein equations sourced by the energy–momentum tensor (13). Hence, in this case, the results of Ref. [6] can be applied directly.

While the k-essence field is unaffected by the Kerr–Schild transformation, its energy–momentum tensor changes as it contains both the metric and its inverse (through X). Transforming them cf. Equation (1) and $\tilde{g}^{ab} = g^{ab} - \lambda l^a l^b$ leads to the Kerr–Schild transformed kinetic term

$$\tilde{X} = X + \lambda X^{(1)} , \tag{14}$$

with

$$X^{(1)} = \frac{1}{2} (D\phi)^2 . \tag{15}$$

The Kerr–Schild transformed energy–momentum tensor is

$$\tilde{T}_{ab} = F_{\tilde{X}}(\phi, \tilde{X}) \nabla_a \phi \nabla_b \phi + \tilde{g}_{ab} F(\phi, \tilde{X}) . \tag{16}$$

When $X^{(1)} = 0$, the sole change in the energy–momentum tensor appears through $\tilde{g}_{ab}$. This is possible if the k-essence is constant along the integral curves of the null congruence. We will discuss this special case at the end of the paper. In what follows, we concentrate on the generic case, when $X^{(1)} \neq 0$.

3. K-Essence Sourcing Kerr–Schild Spacetimes

3.1. Infinitesimal Kerr–Schild Maps

Until now, the parameter λ was arbitrary. In this subsection, we assume it is small; hence, both functions appearing in the Kerr–Schild transformed energy–momentum tensor of the k-essence can be expanded in the power series as

$$F(\phi,\tilde{X}) = \sum_{j=0}^{\infty} \frac{F_{X^j}(\phi,X)}{j!}\left(\lambda X^{(1)}\right)^j , \tag{17}$$

$$F_{\tilde{X}}(\phi,\tilde{X}) = \sum_{j=0}^{\infty} \frac{F_{X^{j+1}}(\phi,X)}{j!}\left(\lambda X^{(1)}\right)^j \tag{18}$$

where F_{X^j} denotes the j^{th} derivative of F with respect to X. Hence, for small λ and employing Equations (1), (14), (17), and (18), the leading order is given by the contribution of the seed spacetime:

$$T^{(0)}_{ab} = T_{ab} , \tag{19}$$

while

$$T^{(k)}_{ab} = \frac{1}{k!}\left(g_{ab} + \nabla_a\phi\nabla_b\phi\frac{\partial}{\partial X}\right)F_{X^k}(X^{(1)})^k + \frac{1}{(k-1)!}l_a l_b F_{X^{k-1}}(X^{(1)})^{k-1} \tag{20}$$

holds for any integer $k \geq 1$.

As proven in Ref. [6], the transformed energy–momentum tensor satisfies the field equations only when all contributions with $k \geq 4$ vanish (this is a generic statement, applying for any λ, including small values). Furthermore, if the congruence l^a is autoparallel, the $k = 3$ contribution is also zero. These conditions are expected to seriously constrain the functional form of the free function $F(\phi, X)$.

First, we prove the following

Theorem 1. *If $T^{(4)}_{ab} = 0$, then for all $k \geq 5$, the contributions $T^{(k)}_{ab}$ also vanish.*

Proof. We prove this by induction. Assume that the statement is true until $k - 1 \geq 4$; thus,

$$T^{(k-1)}_{ab} = \frac{1}{(k-1)!}\left(g_{ab} + \nabla_a\phi\nabla_b\phi\frac{\partial}{\partial X}\right)F_{X^{k-1}}(X^{(1)})^{k-1} + \frac{1}{(k-2)!}l_a l_b F_{X^{k-2}}\left(X^{(1)}\right)^{k-2} = 0 . \tag{21}$$

Contracting $T^{(k-1)}_{ab}$ with l^b and exploring the expression (15) for the nonvanishing $X^{(1)}$,

$$l_a F_{X^{k-1}} + \sqrt{2X^{(1)}}\nabla_a\phi F_{X^k} = 0 \tag{22}$$

emerges. Further contracting with l^a gives $F_{X^k} = 0$, which in turn implies $F_{X^{k-1}} = 0$ through Equation (22). (Hence, the vanishing of $T^{(k-1)}_{ab}$ also implies $F_{X^{k-2}} = 0$.) Then,

$$T^{(k)}_{ab} = \frac{1}{k!}\nabla_a\phi\nabla_b\phi F_{X^{k+1}}(X^{(1)})^k , \tag{23}$$

However, $F_{X^{k+1}}$ is also zero as it is the derivative of a function vanishing for any X. □

Next, we explore the conditions under which $T_{ab}^{(4)}$ would vanish. From the proof of Theorem 1, it can immediately be seen that it is equivalent to imposing $F_{X^3} = F_{X^4} = F_{X^5} = 0$. This is solved by quadratic functions of X:

$$F(\phi, X) = A(\phi)X^2 + B(\phi)X - V(\phi)\,, \tag{24}$$

with A, B, V arbitrary functions of ϕ.

In summary, Kerr–Schild spacetimes with infinitesimal parameter λ are solutions of the Einstein equations sourced by k-essence with the quadratic Lagrangian.

3.2. Finite Kerr–Schild Maps

Let us now ignore that the quadratic form of the k-essence Lagrangian was derived for infinitesimal Kerr–Schild maps and investigate such maps with finite parameters for spacetimes sourced by quadratic k-essence. The kinetic term transforms under such maps with arbitrary λ according to Equation (14), such that the function F in the Lagrangian becomes

$$F\big(\phi, \tilde{X}\big) = A(\phi)X^2 + B(\phi)X - V(\phi) + [2A(\phi)X + B(\phi)]\lambda X^{(1)} + A(\phi)\Big(\lambda X^{(1)}\Big)^2\,. \tag{25}$$

This agrees with the expression obtained from the expansion (17) for the F quadratic in X, confirming its validity for large λ. The same conclusion can also be reached by realizing that the convergence radius of the series expansion is infinite due to the vanishing derivatives.

Therefore, we reached the conclusion that k-essence models with the Lagrangian quadratic in the kinetic term source spacetimes are of Kerr–Schild type.

3.3. Autoparallel Null Congruence

The situation is further simplified by requiring the Kerr–Schild null congruence to be autoparallel, $Dl^a \propto l^a$. In this case, $T_{ab}^{(3)} = 0$ should also be imposed. We note that the proof of the Theorem 1 also holds for $k-1=3$, implying $F_{X^2} = 0$. Therefore, in the case of autoparallel Kerr–Schild congruences, the k-essence Lagrangian should be linear in the kinetic term,

$$F(\phi, X) = B(\phi)X - V(\phi)\,, \tag{26}$$

in order to source Kerr–Schild spacetimes.

4. Condition for the Solution of the Linearized System to Be Exact

In this section, we explore the requirement of the linearized solutions to also be exact. We start with the autoparallel case, then proceed to the generic case.

4.1. Autoparallel Kerr–Schild Congruences

Inserting the linear and quadratic contributions

$$T_{ab}^{(1)} = l_a l_b F + F_{X^2} X^{(1)} \nabla_a\phi \nabla_b\phi + g_{ab} F_X X^{(1)} \tag{27}$$

and

$$T_{ab}^{(2)} = \frac{1}{2}\left(g_{ab} + \nabla_a\phi\nabla_b\phi\frac{\partial}{\partial X}\right)F_{X^2}(X^{(1)})^2 + l_a l_b F_X(X^{(1)}) \tag{28}$$

of the energy–momentum tensor into the condition (10), we obtain

$$\left(g_{ab} + \nabla_a\phi\nabla_b\phi\frac{\partial}{\partial X}\right)F_{X^2}X^{(1)} = 2F_{X^2}(D\phi)l_{(a}\nabla_{b)}\phi\,. \tag{29}$$

This is automatically solved by the k-essence Lagrangian linear in X (implying $F_{X^2} = 0$).

4.2. Unicity

Theorem 2. *The solution of the linearized equation becomes exact only if the Kerr–Schild congruence is autoparallel (hence, the k-essence Lagrangian is linear in the kinetic term).*

Proof. Let us assume that the null congruence l^a is generic, rather than autoparallel. Then, the condition (10) for the linear solution to become exact is replaced by the statement of Theorem 1 of Ref. [6], giving

$$T_{ab}^{(3)} = -\frac{3}{4} l_a l_b (Dl^c Dl_c) \,, \tag{30}$$

$$\begin{aligned} 2T_{ab}^{(2)} &= 2l_{(a} T_{b)c}^{(1)} l^c - \frac{1}{2} g_{ab} (Dl^c Dl_c) + Dl_a Dl_b - l_a l_b (\nabla_c Dl^c) \\ &\quad + l_{(a} \Big[Dl_{b)} (\nabla_c l^c) + DDl_{b)} + \Big(\nabla_{b)} l_c - 2\nabla_{|c|} l_{b)} \Big) Dl^c \Big] \,. \end{aligned} \tag{31}$$

The k-essence with the quadratic Lagrangian in X, Equation (24) has all $T_{ab}^{(k\geq4)} = 0$ (and is therefore able to generate Kerr–Schild-type solutions) and

$$T_{ab}^{(3)} = \frac{1}{4} l_a l_b A(\phi) (D\phi)^4 \,. \tag{32}$$

Comparison with Equation (31) gives the coefficient of the quadratic contribution:

$$A(\phi) = -\frac{3(Dl^c Dl_c)}{(D\phi)^4} \,. \tag{33}$$

The first two expansion coefficients of the energy–momentum tensor, Equations (27) and (28), are

$$\begin{aligned} T_{ab}^{(1)} &= -\frac{3(Dl^c Dl_c)}{(D\phi)^4} \Big[l_a l_b X^2 + (D\phi)^2 (g_{ab} X + \nabla_a \phi \nabla_b \phi) \Big] \\ &\quad + B(\phi) \left(l_a l_b X + g_{ab} \frac{(D\phi)^2}{2} \right) - V(\phi) l_a l_b \end{aligned} \tag{34}$$

and

$$T_{ab}^{(2)} = -\frac{3(Dl^c Dl_c)}{4} \left(g_{ab} + l_a l_b \frac{4X}{(D\phi)^2} \right) + B(\phi) l_a l_b \frac{(D\phi)^2}{2} \,. \tag{35}$$

Inserting the latter and

$$2l_{(a} T_{b)c}^{(1)} l^c = -\frac{6(Dl^c Dl_c)}{(D\phi)^2} \Big(l_a l_b X + l_{(a} \nabla_{b)} \phi D\phi \Big) + B(\phi) l_a l_b (D\phi)^2 \tag{36}$$

into Equation (32) leads to the condition

$$\begin{aligned} 6\frac{l_{(a}\nabla_{b)}\phi}{D\phi} (Dl^c Dl_c) &= Dl_a Dl_b + g_{ab} (Dl^c Dl_c) - l_a l_b (\nabla_c Dl^c) \\ &\quad + l_{(a} \Big[Dl_{b)} (\nabla_c l^c) + DDl_{b)} + \Big(\nabla_{b)} l_c - 2\nabla_{|c|} l_{b)} \Big) Dl^c \Big] \,. \end{aligned} \tag{37}$$

It can then be seen that, in the autoparallel case, this becomes an identity, confirming our previous finding.

For generic null congruences, the trace of Equation (38) gives $l^b DDl_b = 0$, which can be rewritten as $Dl^b Dl_b = 0$, hence Dl^a null. Then, however, through Equation (33), $A = 0$.

Thus, we have proven that in order for the linearized Kerr–Schild solution to also be exact, the k-essence Lagrangian should be linear in the kinetic term.

We complete the proof by exploring the condition of Dl^a being null vector. Beside the autoparallel case $Dl^a = \alpha l^a$, already discussed, the other possibility would be $Dl^a = \beta k^a$, with k^a the second null vector of a pseudoorthonormal base (with property $k^a l_a = -1$). In this case, denoting $\delta = k^c \nabla_c$, Equation (38) reduces to

$$\begin{aligned} 0 &= \beta^2 k_a k_b - l_a l_b (\beta \nabla_c k^c + \delta\beta) \\ &\quad + l_{(a}\left[k_{b)}(\beta\nabla_c l^c + D\beta) + \beta D k_{b)} + \beta\left(\nabla_{b)} l_c - 2\nabla_{|c|} l_{b)}\right) k^c\right] . \end{aligned} \tag{38}$$

Its $l^a l^b$ projection shows $\beta = 0$, which renders Equation (39) an identity. Therefore, the only surviving possibility is l^a being autoparallel. □

5. Constant K-Essence Along the Integral Curves of the Kerr–Schild Null Congruence

For completeness, we also discuss the special case $D\phi = 0$, implying $X^{(1)} = 0$. In this case, the k-essence is constant along the integral curves of the Kerr–Schild null congruence and the energy–momentum tensor changes exclusively due to its dependence on $\tilde{g}_{ab}$:

$$\tilde{T}_{ab} = T_{ab} + \lambda l_a l_b F(\phi, X) . \tag{39}$$

With only $T^{(1)}_{ab} \neq 0$ in the expansion, Equation (31) gives Dl^c null, while Equation (32) simplifies to

$$\begin{aligned} 0 &= Dl_a Dl_b - l_a l_b (\nabla_c D l^c) \\ &\quad + l_{(a}\left[Dl_{b)}(\nabla_c l^c) + DDl_{b)} + \left(\nabla_{b)} l_c - 2\nabla_{|c|} l_{b)}\right) Dl^c\right] , \end{aligned} \tag{40}$$

a condition purely on the null congruence (F dropped out).

For autoparallel congruences $Dl^a = \alpha l^a$, the condition (41) becomes an identity. For the alternative case $Dl^a = \beta k^a$ it gives

$$\begin{aligned} 0 &= \beta^2 k_a k_b - l_a l_b (\beta \nabla_c k^c + \delta\beta) \\ &\quad + l_{(a}\left[\beta k_{b)}(\nabla_c l^c) + \beta D k_{b)} + k_{b)} D\beta + \beta\left(k^c \nabla_{b)} l_c - 2\delta l_{b)}\right)\right] , \end{aligned} \tag{41}$$

its $l^a l^b$ projection, implying that $\beta = 0$ leads to another identity.

With this, we have proven the following:

Theorem 3. *For k-essence fields constant along the integral curves of the autoparallel null Kerr–Schild congruence, the solutions of the linearized Einstein equations also solve the exact equations, with no further restriction on the functional form of the k-essence Lagrangian.*

6. On the Physical Interpretation of Kerr–Schild Maps

6.1. How Are Black Hole Properties Affected by Kerr–Schild Maps?

It is interesting to consider how the Kerr–Schild map transforms the characteristics of spacetime as it affects null geodesics, hence causality, horizon location, light deflection, gravitational lensing, and black hole shadows. The latter gained particular interest in light of the Event Horizon Telescope observations of the M87* and Sagittarius A* supermassive black holes [30,31].

The Kerr–Schild map changes the light cone in each point by modifying it everywhere except along one conserved direction generated by the Kerr–Schild congruence. For Kerr black holes with mass M and rotation parameter a, the deformation caused by a Kerr–Schild map can be easily visualized. The Kerr metric in Kerr–Schild coordinates (t', x, y, z) is

$$\tilde{g}_{ab} = \eta_{ab} + H l'_a l'_b \,, \tag{42}$$

with

$$H = \frac{2Mr^3}{r^4 + a^2 z^2} \,, \quad l'_a = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right) , \tag{43}$$

where the constant r surfaces are ellipsoidal, emerging from the null condition $l'^a l'_a = 0$ as

$$\frac{x^2}{r^2 + a^2} + \frac{y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1 \,. \tag{44}$$

Note that the form $H l'_a l'_b$ can be obtained from $\lambda l_a l_b$ by reparametrizing the null congruence. We also remark that for any decomposition $M = M_1 + M_2$ (and denoting $H_i = 2M_i r^3 / \left(r^4 + a^2 z^2\right)$, with $i = 1, 2$), the Kerr metric in the Kerr–Schild form can be decomposed in two equivalent ways:

$$\tilde{g}_{ab} = \eta_{ab} + \left(H_1 l'_a l'_b + H_2 l'_a l'_b\right) = \left(\eta_{ab} + H_1 l'_a l'_b\right) + H_2 l'_a l'_b \,. \tag{45}$$

The first decomposition is the initial interpretation of a Kerr–Schild map acting on the flat seed spacetime, transforming it into a Kerr spacetime with mass $M = M_1 + M_2$. The second decomposition represents a Kerr–Schild map from a Kerr seed spacetime with mass M_1 to another Kerr spacetime with mass M. Hence, one can interpret the Kerr–Schild map acting on a Kerr black hole as a simple increase in the mass. This means that the deflection of light increases, lensing is amplified, and the radius of the black hole shadow increases. A further remark concerns the null direction unaffected by the Kerr–Schild map. In the nonrotating case $a = 0$, the Kerr–Schild congruence from Equation (43) has purely radial spatial projection $(x/r, y/r, z/r)$. In the rotating case $a \neq 0$, these projections point perpendicularly to the ellipsoidal surface ($l'^a dl'_a = 0$ holds). Therefore, the Kerr–Schild map is conserving the symmetries and is not expected to change the shape of the black hole shadow.

6.2. The Scalar Field Inside and Outside a Black Hole

In this subsection, we qualitatively discuss the modifications induced by a k-essence scalar field of the type allowed by our Theorems on black holes. It is well known that a scalar with a timelike gradient is equivalent to a perfect fluid [32], at least at the nonperturbative level. By contrast, when the spacetime is static and spherically symmetric, the scalar field depends only on the radial coordinate; hence, its gradient is spacelike. In this case, the scalar energy–momentum tensor (13) is equivalent to an imperfect fluid with its tangential pressure equal to the negative of its energy density [33].

These properties can easily be seen by inserting the metric decomposition

$$g_{ab} = -n_a n_b + m_a m_b + h_{ab} \,, \tag{46}$$

(where n^a and m^a are an orthornormal pair, also normal to h_{ab}) into the energy–momentum tensor (13), yielding

$$T_{ab} = (2XF_X - F)n_a n_b + F(m_a m_b + h_{ab}) \,, \tag{47}$$

when the scalar gradient is timelike, $\nabla_a \phi \propto X^{1/2} n_a$ or

$$T_{ab} = (F - 2XF_X)m_a m_b + F(-n_a n_b + h_{ab}) \,, \tag{48}$$

when the scalar gradient is spacelike $\nabla_a \phi \propto (-X)^{1/2} m_a$.

In the spherically symmetric case, the scalar gradient points along $\partial/\partial r$. However, this direction transitions from spacelike outside the horizon to timelike inside the black hole.

6.2.1. Inside the Horizon

The scalar field trapped inside the horizon mimics a perfect fluid with energy density $\rho_{\rm in} = 2XF_X - F$ and isotropic pressure $p_{\rm in} = F$. For the quadratic case (24), the energy density and isotropic pressure become

$$\begin{aligned} \rho_{\rm in} &= 3A(\phi)X^2 + B(\phi)X + V(\phi) \,, \\ p_{\rm in} &= A(\phi)X^2 + B(\phi)X - V(\phi) \,. \end{aligned} \tag{49}$$

These contribute to the gravitational attraction of the black hole in the same manner as stellar matter (through the Tolman–Oppenheimer–Volkoff equation, which, however, in this case, appears as an integro-differential equation, with the mass function defined as an integral in terms of the functions A, B, V). The equation of state for the scalar results in

$$w_{\rm in} = \frac{p_{\rm in}}{\rho_{\rm in}} = 1 - \frac{2\left(AX^2 + V\right)}{3AX^2 + BX + V} \,. \tag{50}$$

When the potential dominates, $w_{\rm in} \approx -1$; thus, the scalar mimics dark energy.

The adiabatic sound speed (the propagation velocity of the scalar field perturbations) squared is

$$c_{s,\rm in}^2 = \left.\frac{dp_{\rm in}}{d\rho_{\rm in}}\right|_{s/n} = \left.\frac{p_{{\rm in},X}}{\rho_{{\rm in},X}}\right|_{s/n} = \frac{2AX + B}{6AX + B} = 1 - \frac{4AX}{6AX + B} \,, \tag{51}$$

where $s = S/V$ is the entropy density and $n = N/V$ the particle number density. We have explored that $d(s/n) = 0$ yields $d\phi = 0$ (a simple realization of this being $\phi = s/n$). The condition $d\phi = 0$ signifies that while the variations dp and $d\rho$ allow for arbitrary variations in X, they are such that ϕ should stay constant [34].

The pairs of functions A, B yielding Laplacian instability regimes with $c_{s,\rm in}^2 < 0$ should be excluded.

6.2.2. Outside the Horizon

In this case, the scalar is equivalent to an anisotropic fluid with energy density equal to the tangential pressures $\rho_{\rm out} = p_{\rm out}^t = F$ and radial pressure $p_{\rm out}^r = F - 2XF_X$, which in the quadratic case (24) become

$$\begin{aligned} \rho_{\rm out} &= p_{\rm out}^t = A(\phi)X^2 + B(\phi)X - V(\phi) \,, \\ p_{\rm out}^r &= -3A(\phi)X^2 - B(\phi)X - V(\phi) \,, \end{aligned} \tag{52}$$

resulting in

$$w^r_{\text{out}} = \frac{p^r_{\text{out}}}{\rho_{\text{out}}} = -1 - \frac{2(AX^2 + V)}{AX^2 + BX - V}\,, \quad w^t_{\text{out}} = \frac{p^t_{\text{out}}}{\rho_{\text{out}}} = 1\,. \tag{53}$$

When the potential dominates, $w^r_{\text{out}} \approx 1$; thus, the fluid approaches isotropy

$$\left(c^r_{s,\text{out}}\right)^2 = \left.\frac{dp^r_{\text{out}}}{d\rho_{\text{out}}}\right|_{s/n} = \left.\frac{p^r_{\text{out},X}}{\rho_{\text{out},X}}\right|_{s/n} = -1 - \frac{4AX}{2AX + B} \tag{54}$$

$$\left(c^t_{s,\text{out}}\right)^2 = \frac{dp^t_{\text{out}}}{d\rho_{\text{out}}} = 1\,. \tag{55}$$

Again, any pair of functions A, B yielding radial Laplacian instability regimes with $\left(c^r_{s,\text{out}}\right)^2 < 0$ should be excluded. Tangential Laplacian instability regimes do not arise.

6.3. Cosmological Scalar Field

In a cosmological setup, the scalar field is equivalent to a perfect fluid (47), with energy density and pressure already calculated as (50) and the equation of state as (50). The scalar field mimics dark energy (cosmological constant), when the potential dominates. The sole difference compared to the discussion on the black hole interior arises from the fact that, in this case, the scalar depends on time (instead of the radial coordinate). Hence, the adiabatic speed of sound squared reads

$$c_s^2 = \left.\frac{dp}{d\rho}\right|_{s/n} = \left.\frac{p_X}{\rho_X}\right|_{s/n} = 1 - \frac{4AX}{6AX + B}\,. \tag{56}$$

A physical requirement to impose on the set of functions A, B is to avoid the instability regime $c_s^2 < 0$.

6.4. On K-Essence Kerr–Schild Seed Spacetimes

In order to apply our Theorems, a seed spacetime generated by a nontrivial scalar field is needed. The simplest such solutions are expected to arise in highly symmetric situations. However, various unicity theorems forbid the scalar hair for k-essence black holes. Bekenstein ruled out the existence of stationary, asymptotically flat black holes with scalar hair for canonical scalar fields (quintessence) [35–37]. The generalized Brans–Dicke theories in the Einstein frame are also contained in this class; hence, its stationary and asymptotically flat black holes have no scalar hair either [38]. Another no-hair theorem for stationary, asymptotically flat black holes in a more generic class of k-essence models was provided by Graham and Jha [39], holding when F_X and ϕF_ϕ are of opposite and definite signs. No-hair theorems were also shown to hold for static, asymptotically flat black holes in Horndeski, Beyond Horndeski, Einstein scalar–Gauss–Bonnet, and Chern–Simons theories [40].

Due to the host of unicity theorems assuming stationarity and asymptotic flatness, we expect that our result would be useful in dynamical situations. Such a scenario could be a black hole with slowly growing scalar hair, arising either from cosmological evolution, or due to the slow motion of the black hole within an asymptotic spatial gradient in the scalar field [41]. Pairs of inspiralling black holes of this kind emit dipole radiation, first constrained by observations on the quasar OJ287 in Ref. [42]. Another model could be a Vaidya-type radiating solution with dynamical horizon, as discussed for a Dirac–Born–Infeld model in Ref. [28]. Perhaps the most interesting would be considering wavelike behaviors. For example, Einstein–Rosen cylindrical waves were derived for Brans–Dicke theories (which in the Einstein frame fit into our framework) in Ref. [43], including standing wave, solitonic wave, and particular pulse-wave-type solutions. For any such time evolving solution obtained from the k-essence Lagrangian linear in X, a Kerr–Schild spacetime would emerge as the solution of the Einstein equation linearized in λ instead of dealing with the full set of Einstein equations. Exploring such possibilities requires further investigations beyond the scope of this paper.

7. Concluding Remarks

Most of the physically interesting solutions of the Einstein equations, including black holes and radiation fields are of Kerr–Schild type. The generating technique of such solutions is quite elegant geometrically: the Kerr–Schild map modifies the metric through the addition of a diad of null vectors, therefore changing the light-cones in such a way that in each point a single null generator of the cone is left unmodified. One of the most fascinating properties of such Kerr–Schild spacetimes in a vacuum is that any solution of the linearized Einstein equation can be propelled into an exact solution of the full nonlinear Einstein equations by simply increasing the expansion parameter to finite values [5]. The conditions for generalizing this property to the case where matter sources are present are also known [6].

In the present paper, we investigated whether this property of Kerr–Schild maps holds for minimally coupled k-essence scalar–tensor theories. First, we proved that the Lagrangian of the k-essence should be at most quadratic in the kinetic term in order to source a Kerr–Schild-type spacetime. In a cosmological context, such models include dilatonic ghost condensate [44] and unified models of dark energy and dark matter [45].

This is reduced to a linear dependence $\mathfrak{L}_{\phi} = \sqrt{-\mathfrak{g}}[B(\phi)X - V(\phi)]$, when the Kerr–Schild null congruence is autoparallel. Generic junction conditions and the generalization of the Lanczos equation were derived [46] for such k-essence fields, which also include the quintessence models in the particular case $B = 1$.

Further, we proved a unicity theorem for k-essence. The solutions of the Einstein equations linearized in Kerr–Schild-type perturbations also solve the full nonlinear system of Einstein equations, when the k-essence is given by a linear Lagrangian in the kinetic term. The proof of the theorem omitted one special case of a k-essence everywhere constant along the integral curves of the Kerr–Schild congruence. We proved that such a k-essence field also shares the property of sourcing perturbative Kerr–Schild spacetimes which are exact, without the need to restrict in any way the functional form of its Lagrangian.

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