



A Schwarzian on the stretched horizon

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Abstract

It is well known that the Euclidean black hole action has a boundary term at the horizon proportional to the area. I show that if the horizon is replaced by a stretched horizon with appropriate boundary conditions, a new boundary term appears, described by a Schwarzian action similar to the recently discovered boundary actions in “nearly anti-de Sitter” gravity.

Keyword Quantum gravity, Black holes, Schwarzian action, Boundary actions, Near-horizon geometry

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1 Introduction

It has long been recognized that “boundary” degrees of freedom, either at infinity or at or near the horizon, may play a central role in black hole thermodynamics [1–5]. In the path integral, the entropy of the “Euclidean” black hole—the stationary black hole analytically continued to Riemannian signature—comes from a boundary term, either at infinity [6] or at the horizon [7, 8]; in the latter case, it is canonically conjugate to a horizon deficit angle [9]. At the microscopic level, the symmetries of the horizon suggest the existence of boundary degrees of freedom as “would-

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be diffeomorphisms,” deformations that would ordinarily be pure gauge but become physical because boundary conditions restrict the true gauge transformation [10–12]. In some cases, the argument for such degrees of freedom is quite strong. The (2+1)-dimensional BTZ black hole, for instance, has no local bulk degrees of freedom, but one can still explicitly construct an induced dynamical action at the boundary that may account for the Bekenstein–Hawking entropy [13, 14].

Recently, related boundary actions have proven important in a somewhat different context. In two-dimensional “nearly anti-de Sitter space” [15–17], the boundary of asymptotically AdS space at infinity is replaced by a finite boundary. A bulk action given by Jackiw–Teitelboim gravity [18, 19] or its variants then induces a boundary action that can be described by a Schwarzian,

$$I = C \int d\tau \phi\{f, \tau\}, \quad (1.1)$$

where the Schwarzian derivative $\{f, \tau\}$ is

$$\{f, \tau\} = \frac{\ddot{f}}{\dot{f}} - \frac{3}{2} \frac{\dot{f}^2}{f^2} \quad (1.2)$$

(a dot is a derivative with respect to τ). This is a powerful result: the Schwarzian action can be quantized exactly [20, 21], and has fascinating connections to a variety of conformal field theories and matrix models [22–24].

While most of the recent work on the Schwarzian action has taken place in the context of nearly anti-de Sitter space, the same action also appears elsewhere: in “nearly de Sitter space” [25], for instance, and as a corner term in asymptotically flat (2+1)-dimensional gravity [26]. The Schwarzian action is closely related to Liouville theory [21], which is ubiquitous in quantum gravity, and there are arguments from effective field theory that a Schwarzian at a one-dimensional boundary is generic [27]. So it may not be surprising to find similar actions elsewhere.

In particular, nearly anti-de Sitter space approximates the near horizon geometry of an extremal black hole, and it seems natural to ask whether there is an extension to more generic black holes. In this paper, I will show that a Schwarzian action does, in fact, describe the stretched horizon of an arbitrary nonextremal Euclidean black hole, albeit with slightly different boundary conditions. This opens up the possibility that some of the powerful results from nearly anti-de Sitter space may be applicable to this broader setting.

2 The boundary action

To understand the appearance of the Schwarzian action, we will first need two ingredients, the dimensional reduction of the near-horizon region and the form of the near-horizon metric. Both are fairly standard:

- **Near-horizon dimensional reduction**

It has been understood for some time that the near-horizon region of an extremal or

nearly extremal black hole can be described by a two-dimensional dilaton gravity model [28]. While it is not quite as well known, the same is true for the near-horizon region of an arbitrary stationary black hole [29, 30]. More precisely, write the metric near the horizon in the general form

$$ds^2 = g_{ab}dx^a dx^b + \phi_{\mu\nu}(dy^\mu + A_a^\mu dx^a)(dy^\nu + A_b^\nu dx^b), \quad (2.1)$$

where lower case Roman indices (a,b,...) run from 0 to 1 and label the “ r – t plane,” while lower case Greek indices (μ, ν, \dots) run from 2 to $D-1$ and label the transverse coordinates. Then as shown in [30], after a conformal rescaling of the metric, the near-horizon behavior is described by a two-dimensional action

$$I_2 = \frac{1}{16\pi G} \int_M d^2x \sqrt{g} \{ \varphi R + V[\varphi] \} + \dots, \quad (2.2)$$

where

$$\varphi = \sqrt{|\det \phi_{\mu\nu}|}$$

is the transverse volume element. The omitted terms are additional Kaluza-Klein-like matter fields, which couple to the two-dimensional metric and dilaton. But as shown in [30], these vanish at the horizon and are very small in the near-horizon region, making them irrelevant for the phenomena considered here.

As always, if the metric is fixed at a boundary, an extra Gibbons–Hawking boundary term is also required [6]. Its dimensionally reduced form is

$$I_{bdry} = \frac{1}{8\pi G} \int_{\partial M} dx \sqrt{h} \varphi K, \quad (2.3)$$

where h is the induced metric on the boundary ∂M and K is the trace of the extrinsic curvature of ∂M .

• The near-horizon metric

We will also need the near-horizon form of the metric, analytically continued to Riemannian signature. For a Schwarzschild black hole, it is well known that the dimensionally reduced metric near the horizon takes the form

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 \quad \text{with } N = \sqrt{2\kappa(r - r_+)}, \quad (2.4)$$

where κ is the surface gravity and the horizon is located at $r = r_+$. This turns out to be generic: the same form occurs for an arbitrary stationary nonextremal black hole, regardless of the presence of matter or even the detailed form of the field equations [4, 31]. If one now sets

$$r = r_+ + \frac{1}{2}\kappa\rho^2, \quad \tau = i\kappa t, \quad (2.5)$$

the near-horizon metric becomes¹

$$ds^2 = d\rho^2 + \rho^2 d\tau^2. \quad (2.6)$$

The horizon is at $\rho = 0$, and the coordinate ρ now has a clear physical meaning as the proper distance from the horizon, while the “Euclidean time” τ is periodic with a period of $2\pi - \Theta$, where the deficit angle Θ is conjugate to the transverse area φ and vanishes on shell [9].

We can now compute the boundary action (2.3) for a stretched horizon. We enlarge the horizon by considering a small closed path Δ encircling the origin. Such a path can be described as a parametrized curve $(\rho(\sigma), \tau(\sigma))$, where for later convenience we will take σ to be periodic with a fixed range $[0, 1]$. The length of Δ is then

$$\ell = \oint ds = \int_0^1 \varepsilon d\sigma \quad \text{with} \quad \varepsilon = \frac{ds}{d\sigma} = \left(\rho'^2 + \rho^2 \tau'^2\right)^{1/2}, \quad (2.7)$$

where a prime denotes a derivative with respect to σ . For ℓ to be small, ε must be small, which in turn implies a small ρ ; that is, Δ must remain close to the horizon.

Note that by the smooth Schoenflies theorem, a two-dimensional diffeomorphism can map any such curve into any other. But we will treat Δ as a boundary, and the gravitational action is not invariant under diffeomorphisms transverse to a boundary. In this sense, different choices of the path Δ represent the “would-be diffeomorphisms” of [10].

The unit tangent and normal vectors to Δ are

$$\begin{aligned} t^a &= \frac{1}{\varepsilon} (\rho', \tau') \\ n^a &= \frac{\rho}{\varepsilon} \left(\tau', -\frac{\rho'}{\rho^2} \right). \end{aligned} \quad (2.8)$$

The extrinsic curvature is thus

$$K = t^a t^b \nabla_a n_b = \frac{1}{\varepsilon^3} \left(\rho \rho' \tau'' - \rho \tau' \rho'' + 2\rho'^2 \tau' + \rho^2 \tau'^3 \right). \quad (2.9)$$

The induced metric on Δ is $\sqrt{h} = \varepsilon$, and with a little calculation, one can show that

$$\sqrt{h} K = \tau' - \frac{1}{\sqrt{\varepsilon^2 - \rho'^2}} \left(\rho'' - \rho' \frac{\varepsilon'}{\varepsilon} \right) = \tau' - \left(1 - \frac{\rho'^2}{\varepsilon^2} \right)^{-1/2} \left(\frac{\rho'}{\varepsilon} \right)'. \quad (2.10)$$

¹ Technically, the two-dimensional metric in (2.2) is conformally rescaled from the D -dimensional expression, but near the horizon the effect is higher order in ρ [30].

Hence the boundary action (2.3) takes the form

$$I_{bdry} = \frac{1}{8\pi G} \int_{\Delta} d\sigma \varphi \left(\tau' - \left(1 - \frac{\rho'^2}{\varepsilon^2} \right)^{-1/2} \left(\frac{\rho'}{\varepsilon} \right)' \right). \quad (2.11)$$

The second term in parentheses is a total derivative. If we choose φ to be constant on Δ , we therefore have

$$I_{bdry} = \frac{1}{8\pi G} \int_{\Delta} d\sigma \varphi \tau' = \frac{\varphi|_{\Delta}}{4G} \left(1 - \frac{\Theta}{2\pi} \right), \quad (2.12)$$

which is the standard boundary action; the only effect of enlarging the horizon is to slightly shift the value of the area $\varphi|_{\Delta}$. This is, of course, an approximation—the metric (2.6) has corrections away from the horizon—but any such corrections will be of order ρ^2 .

Now, observe that ε appears in (2.11) only in the combination $\varepsilon d\sigma = ds$. By a suitable choice of parametrization, we can take ε to be constant on Δ . In that case, from (2.7), ε is just the length of the stretched horizon, while σ is a rescaled proper length.

3 A Schwarzian at the horizon

Let us now drop the requirement that φ be constant on Δ . We will, however, assume that Δ is “not too irregular.” More precisely, while ρ is $\mathcal{O}(\varepsilon)$, we will now assume that ρ' is $\mathcal{O}(\varepsilon^2)$. Then from (2.7),

$$\frac{\rho}{\varepsilon} = \frac{1}{\tau'} \left(1 - \frac{\rho'^2}{\varepsilon^2} \right)^{1/2} \approx \frac{1}{\tau'} \left(1 - \frac{1}{2} \frac{\rho'^2}{\varepsilon^2} \right) \approx \frac{1}{\tau'} \left(1 - \frac{1}{2} \frac{\tau''^2}{\tau'^4} \right). \quad (3.1)$$

To lowest order, with ε held fixed, the boundary action (2.11) then becomes

$$I_{bdry} = \frac{1}{8\pi G} \int_{\Delta} d\sigma \varphi \left(\tau' + \left(\frac{\tau''}{\tau'^2} \right)' \right) = \frac{1}{8\pi G} \int_{\Delta} d\tau \varphi \left(1 - \frac{1}{2} \frac{\tau''^2}{\tau'^4} + \frac{1}{\tau'^2} \{\tau, \sigma\} \right). \quad (3.2)$$

(Note that this approximation only requires the lowest order term in (3.1); the next order term will be important below.)

We evidently have a problem: the piece we are interested in, the Schwarzian derivative, is a small correction to the standard area term (2.12), and can be modified by making small changes to the leading term. In particular, the Schwarzian derivative

transforms anomalously: under a reparametrization $\sigma \rightarrow \bar{\sigma}(\sigma)$,

$$\left(\frac{d\tau}{d\sigma}\right)^{-2} \{\tau, \sigma\} = \left(\frac{d\tau}{d\bar{\sigma}}\right)^{-2} \{\tau, \bar{\sigma}\} + \frac{1}{\tau'^2} \{\bar{\sigma}, \sigma\}. \quad (3.3)$$

Thus in the integrand in (3.2),

$$1 - \frac{1}{2} \frac{\tau''^2}{\tau'^4} \rightarrow 1 - \frac{1}{2} \frac{\tau''^2}{\tau'^4} + \frac{1}{\tau'^2} \{\bar{\sigma}, \sigma\},$$

and we can certainly find a new parametrization for which this term reduces to 1. Such a choice seems rather arbitrary, though. The parameters we are using are physically natural: τ is the background Killing time, and σ is the scaled proper length. So the question remains whether the particular structure in (3.2) has any deeper meaning.

For the particular case of the near-horizon black hole, it does. To see this, we will need two additional elements. First, as noted above, if φ is constant the boundary term reduces to the usual area factor, with no extra dynamics. This suggests that we might split off a constant part of φ to separate out the leading contribution. From (2.5), it is clear that the first nonconstant piece of φ appears at order ρ^2 , so to the order we are considering,

$$\varphi = \varphi_+ + \frac{1}{2} \rho \partial_\rho \varphi. \quad (3.4)$$

The boundary term (3.2) is thus

$$I_{bdry} = \frac{1}{8\pi G} \int_{\Delta} d\tau \varphi_+ + \frac{1}{16\pi G} \int_{\Delta} d\tau \rho \partial_\rho \varphi \left(1 - \frac{1}{2} \frac{\tau''^2}{\tau'^4} + \frac{1}{\tau'^2} \{\tau, \sigma\}\right). \quad (3.5)$$

Second, let us reconsider the boundary conditions at Δ . The full boundary term in the variation of the action $I_2 + I_{bdry}$ is [32]

$$\delta(I_2 + I_{bdry}) = \text{equations of motion} + \frac{1}{8\pi G} \int_{\Delta} d\sigma \left[K \sqrt{h} \delta\varphi + n^a \partial_a \varphi \delta(\sqrt{h}) \right]. \quad (3.6)$$

(The same result can be obtained in the Lorentzian setting from the symplectic form in [30], and is a special case of the results of [33].) The boundary conditions we have assumed so far are the standard Dirichlet conditions, in which φ and \sqrt{h} are fixed. This is certainly a reasonable choice, especially if one is taking the two-dimensional model to be fundamental.

For an action obtained by dimensional reduction, though, it seems equally natural to fix the transverse variables φ and $n^a \nabla_a \varphi$. Geometrically, this is a “free boundary” condition: the full geometry of the stretched horizon, intrinsic (given by h) and extrinsic (given by K), is unrestricted, while the transverse area profile is fixed. In the nearly

anti-de Sitter case, the metric h determines the inverse temperature, and fixing its conjugate $\partial_n \varphi$ leads to a microcanonical ensemble with fixed ADM mass [32]. For the nonextremal case this is trickier, since the ADM mass is much more complicated [34]. It is still true, however, that $\nabla_n \varphi$ remains conjugate to the length of the horizon in imaginary time, and thus conjugate to the inverse temperature β .

This new choice of boundary conditions requires an additional boundary term. From (3.6),

$$\begin{aligned} I_{bdry}^{(2)} &= -\frac{1}{8\pi G} \int_{\Delta} d\sigma \, n^a \nabla_a \varphi \sqrt{h} = -\frac{1}{8\pi G} \int_{\Delta} d\sigma \left[\rho \tau' \partial_{\rho} \varphi - \frac{\varepsilon \rho'}{\rho^2} \partial_{\tau} \varphi \right] \\ &= -\frac{1}{8\pi G} \int_{\Delta} d\tau \, \rho \partial_{\rho} \varphi, \end{aligned} \quad (3.7)$$

where I have used (2.8) and the fact that $\partial_{\tau} \varphi = 0$ for a stationary background.²

Now combine the two boundary actions (3.5) and (3.7). From (3.1), to the order we are considering,

$$\begin{aligned} I_{bdry} + I_{bdry}^{(2)} &= \frac{1}{8\pi G} \int_{\Delta} d\tau \, \varphi_+ - \frac{1}{16\pi G} \int_{\Delta} d\tau \, \rho \partial_{\rho} \varphi \left(1 + \frac{1}{2} \frac{\tau''^2}{\tau'^4} - \frac{1}{\tau'^2} \{\tau, \sigma\} \right) \\ &= \frac{\varphi|_+}{4G} \left(1 - \frac{\Theta}{2\pi} \right) - \frac{1}{16\pi G} \int_{\Delta} d\sigma \, \frac{\varepsilon}{\tau'^2} \partial_{\rho} \varphi \left(\tau'^2 - \{\tau, \sigma\} \right). \end{aligned} \quad (3.8)$$

The first term in (3.8) is the standard transverse area contribution. The second is essentially a Schwarzian action, with a coefficient that depends only on $\partial_{\rho} \varphi$. Thus by changing our boundary conditions and switching our background source from φ to $\partial_{\rho} \varphi$, we have isolated a distinct, and interesting, second order piece of the action.

4 Conclusion

Obtaining a horizon action is, of course, only a first step. The Schwarzian action has been extensively studied over the past few years in the nearly anti-de Sitter [15, 16] and nearly de Sitter [25] settings, and it seems plausible that some of those results can be adapted to this new case. There may be subtleties, though: the flat space metric (2.6) is not the metric anti-de Sitter space, and while the action (3.8) is almost the standard Schwarzian, it differs by an extra prefactor of $1/\tau'^2$. One might learn more by investigating the extremal limit of the action (3.8), where the expansion (3.4) breaks down; it should presumably be possible to reproduce known results.

It is tempting to try to extend this derivation to higher orders, to see whether one continues to obtain functions of the Schwarzian, as occurs in the nearly AdS case [35]. This is tricky, though: the near-horizon metric (2.6) has $\mathcal{O}(\rho^4)$ corrections that would

² The boundary value $\varphi|_{\Delta}$ can still depend on τ , of course, but only through the dependence of Δ on τ .

need to be accounted for, and universality would probably be lost. It should be possible to repeat this derivation in Lorentzian signature, although the definition of a stretched horizon becomes a bit delicate there [36]. In [32], a collection of alternative boundary conditions is described; the corresponding actions will probably not be Schwarzsians, but it could be worthwhile to understand the differences. Finally, it might be possible to generalize this approach to gain at least a bit of insight into the dynamics of black hole evaporation. The Euclidean continuation used here assumes a stationary metric, but one might artificially insert some time dependence into φ . From (3.7), this would add a new term to the action, with potentially interesting implications.

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Data Availability Data sharing is not applicable, as no data sets were generated or analyzed in this research.

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