RESEARCH ARTICLE | JUNE 11 2024

Eigenvalues of quantum Gelfand invariants

Naihuan Jing ; Ming Liu; Alexander Molev

J. Math. Phys. 65, 061703 (2024)
https://doi.org/10.1063/5.0211081

Journal of Mathematical Physics | ARTICLE | pubs.aip.org/aip/jmp

Eigenvalues of quantum Gelfand invariants

Cite as: J. Math. Phys. **65**, 061703 (2024); doi: 10.1063/5.0211081
Submitted: 28 March 2024 • Accepted: 20 May 2024 •
Published Online: 11 June 2024

View Online Export Citation CrossMark

Naihuan Jing,[1,a)] Ming Liu,[2,b)] and Alexander Molev[3,c)]

AFFILIATIONS

[1] Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695, USA
[2] School of Mathematical Sciences, South China University of Technology, Guangzhou, Guangdong 510640, China
[3] School of Mathematics and Statistics, University of Sydney, Sydney, NSW 2006, Australia

[a)] jing@math.ncsu.edu
[b)] mamliu@scut.edu.cn
[c)] **Author to whom correspondence should be addressed:** alexander.molev@sydney.edu.au

ABSTRACT

We consider the quantum Gelfand invariants which first appeared in a landmark paper by Reshetikhin *et al.* [Algebra Anal. **1**(1), 178–206 (1989)]. We calculate the eigenvalues of the invariants acting in irreducible highest weight representations of the quantized enveloping algebra for $\mathfrak{gl}_n$. The calculation is based on Liouville-type formulas relating two families of central elements in the quantum affine algebras of type A.

I. INTRODUCTION

The *quantized enveloping algebras* and *quantum affine algebras* associated with simple Lie algebras comprise remarkable families of quantum groups, as introduced by Drinfeld[5] and Jimbo.[13] These algebras and their representations have since found numerous connections with many areas in mathematics and physics.

In this paper we will be concerned with those families associated with the general linear Lie algebras $\mathfrak{gl}_n$. Both the quantized enveloping algebra $\mathrm{U}_q(\mathfrak{gl}_n)$ and the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ admit *R-matrix* (or *RTT*) *presentations* going back to the work of the Leningrad school headed by Faddeev; see e.g., Refs. 19 and 27 for reviews of the foundations of the R-matrix approach originated in the quantum inverse scattering method.

Central elements in both $\mathrm{U}_q(\mathfrak{gl}_n)$ and $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ are constructed with the use of the R-matrix presentations and found as coefficients of the respective *quantum determinants*; see Refs. 3, 14, and 19 and also Ref. 6 for more general constructions of central elements in the quantized enveloping algebras and quantum affine algebras. As pointed out in Ref. 27, the quantum traces of powers of generator matrices are central in $\mathrm{U}_q(\mathfrak{gl}_n)$; see also Ref. 1. By taking the limit $q \to 1$ one recovers the central elements of $\mathrm{U}(\mathfrak{gl}_n)$ going back to Ref. 8, which are known as the *Gelfand invariants*. Note that a different generalization of the Gelfand invariants for $\mathfrak{gl}_n$ as central elements in $\mathrm{U}_q(\mathfrak{gl}_n)$, was given in Ref. 9, where their eigenvalues in irreducible highest weight representations were calculated. A new family of central elements in $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ was given in Ref. 2 and they were related to the quantum determinants by Liouville-type formulas, although they were not accompanied by proofs. This result is quite analogous to the corresponding quantum Liouville formulas for the Yangians originated in Ref. 22 and we give a complete proof in this paper.

By taking the images of the new central elements of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ under the evaluation homomorphism, we recover the quantum Gelfand invariants of Ref. 27. The Liouville formulas will then allow us to calculate the eigenvalues of the quantum Gelfand invariants in irreducible highest weight representations of $\mathrm{U}_q(\mathfrak{gl}_n)$. We thus obtain q-analogues of the Perelomov–Popov formulas.[24] To recall the eigenvalue formulas from Ref. 24, consider the irreducible highest weight representation $L(\lambda)$ of $\mathfrak{gl}_n$ with the highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$ and combine the standard basis elements E_{ij} into the matrix $E = [E_{ij}]$. Then the eigenvalue of the Gelfand invariant $\mathrm{tr}\, E^m$ in $L(\lambda)$ is found by

$$\mathrm{tr}\, E^m \mapsto \sum_{k=1}^{n} \ell_k^m \,\frac{(\ell_1 - \ell_k + 1)\cdots(\ell_n - \ell_k + 1)}{(\ell_1 - \ell_k)\cdots \wedge \cdots (\ell_n - \ell_k)}, \tag{1.1}$$

where $\ell_i = \lambda_i + n - i$ and the symbol $\wedge$ indicates that the zero factor is skipped. Formula (1.1) can be derived with the use of R-matrix calculations in the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ or in the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$; see Ref. 20, Sec. 7.1 and Ref. 21, Sec. 4.8, respectively.

Our main result concerning the quantum Gelfand invariants is the following theorem, where we calculate the eigenvalues of the quantum traces $\mathrm{tr}_q\, M^m$ of the powers of the generator matrix $M = L^-(L^+)^{-1}$ in the representation $L_q(\lambda)$ of $\mathrm{U}_q(\mathfrak{gl}_n)$ (see Sec. III for the definitions). We use a standard notation for the q-numbers

$$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad k \in \mathbb{Z}\,.$$

Theorem 1.1. *The eigenvalue of the quantum Gelfand invariant* $\mathrm{tr}_q\, M^m$ *in* $L_q(\lambda)$ *is found by*

$$\mathrm{tr}_q M^m \mapsto \sum_{k=1}^{n} q^{2\ell_k m} \frac{[\ell_1 - \ell_k + 1]_q \cdots [\ell_n - \ell_k + 1]_q}{[\ell_1 - \ell_k]_q \cdots \wedge \cdots [\ell_n - \ell_k]_q}. \tag{1.2}$$

We will prove Theorem 1.1 in Sec. III by deriving it from the Liouville formula given in Theorem 2.4 in a way similar to Ref. 20, Sec. 7.1. We also consider three more families of central elements of $\mathrm{U}_q(\mathfrak{gl}_n)$ which, however, turn out to coincide with the quantum Gelfand invariants, up to a possible replacement $q \mapsto q^{-1}$.

In the limit $q \to 1$ we have

$$\frac{M - 1}{q - q^{-1}} \to E \tag{1.3}$$

by (3.1) and (3.2) below. Therefore, the Gelfand invariants in $\mathrm{U}(\mathfrak{gl}_n)$ are recovered from the elements $\mathrm{tr}_q\, M^m$ in the limit, while the Perelomov–Popov formulas (1.1) follow from (1.2); see Remark 3.1 below.

We point out a related recent work,[18] where explicit formulas for certain central elements in the reflection equation algebras were given and their relation with the quantum Gelfand invariants were reviewed. This includes the connection with an earlier construction of central elements in Ref. 25 and with the Cayley–Hamilton theorem and Newton identities of Refs. 10, 12, and 23.

This work was completed during the first and third named authors' visits to the South China University of Technology and to the Shanghai University. They are grateful to the Departments of Mathematics in both universities for the warm hospitality.

II. LIOUVILLE FORMULAS

We will regard q as a nonzero complex number which is not a root of unity. Recall the R-matrix presentation of the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ as introduced in Ref. 26. We follow[4] and use the same settings as in our earlier work.[17] Let $e_{ij} \in \mathrm{End}\,\mathbb{C}^n$ denote the standard matrix units. Consider the R-matrix

$$R(x) = \frac{f(x)}{q - q^{-1}x}(R - x\,\widetilde{R}), \tag{2.1}$$

where

$$R = q\sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1})\sum_{i<j} e_{ij} \otimes e_{ji} \tag{2.2}$$

and

$$\widetilde{R} = q^{-1}\sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} - (q - q^{-1})\sum_{i>j} e_{ij} \otimes e_{ji},$$

while the formal power series

$$f(x) = 1 + \sum_{k=1}^{\infty} f_k x^k, \qquad f_k = f_k(q),$$

is uniquely determined by the relation

$$f(xq^{2n}) = f(x)\,\frac{(1 - xq^2)\,(1 - xq^{2n-2})}{(1 - x)\,(1 - xq^{2n})}.$$

The *quantum affine algebra* $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ is generated by elements

$$l^+_{ij}[-r], \qquad l^-_{ij}[r] \qquad \text{with} \quad 1 \leqslant i,j \leqslant n, \qquad r = 0, 1, \ldots,$$

and the invertible central element q^c, subject to the defining relations

$$l^+_{ji}[0] = l^-_{ij}[0] = 0 \qquad \text{for} \qquad 1 \leqslant i < j \leqslant n,$$
$$l^+_{ii}[0]\, l^-_{ii}[0] = l^-_{ii}[0]\, l^+_{ii}[0] = 1 \qquad \text{for} \qquad i = 1, \dots, n,$$

and

$$R(u/v)L^\pm_1(u)L^\pm_2(v) = L^\pm_2(v)L^\pm_1(u)R(u/v), \tag{2.3}$$
$$R(uq^{-c}/v)L^+_1(u)L^-_2(v) = L^-_2(v)L^+_1(u)R(uq^c/v). \tag{2.4}$$

In the last two formulas we consider the matrices $L^\pm(u) = \big[\, l^\pm_{ij}(u)\big]$, whose entries are formal power series in u and u^{-1},

$$l^+_{ij}(u) = \sum_{r=0}^{\infty} l^+_{ij}[-r]\; u^r, \qquad l^-_{ij}(u) = \sum_{r=0}^{\infty} l^-_{ij}[r]\; u^{-r}.$$

Here and below we regard the matrices as elements

$$L^\pm(u) = \sum_{i,j=1}^{n} e_{ij} \otimes l^\pm_{ij}(u) \in \operatorname{End}\mathbb{C}^n \otimes \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)[[u^{\pm 1}]]$$

and use a subscript to indicate a copy of the matrix in the multiple tensor product algebra

$$\underbrace{\operatorname{End}\mathbb{C}^n \otimes \dots \otimes \operatorname{End}\mathbb{C}^n}_{k} \otimes \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)[[u^{\pm 1}]] \tag{2.5}$$

so that

$$L^\pm_a(u) = \sum_{i,j=1}^{n} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(k-a)} \otimes l^\pm_{ij}(u).$$

We regard the usual matrix transposition also as the linear map

$$t : \operatorname{End}\mathbb{C}^n \to \operatorname{End}\mathbb{C}^n, \qquad e_{ij} \mapsto e_{ji}.$$

For any $a \in \{1, \dots, k\}$ we will denote by t_a the corresponding partial transposition on the algebra (2.5) which acts as t on the a-th copy of $\operatorname{End}\mathbb{C}^n$ and as the identity map on all the other tensor factors.

The scalar factor in (2.1) is necessary for the R-matrix to satisfy the *crossing symmetry relations*.[7] We will use one of them given by

$$\big(R_{12}(x)^{-1}\big)^{t_2} D_2 R_{12}(xq^{2n})^{t_2} = D_2, \tag{2.6}$$

where D denotes the diagonal $n \times n$ matrix

$$D = \operatorname{diag}\big[q^{n-1}, q^{n-3}, \dots, q^{-n+1}\big].$$

The following proposition was stated in Ref. 2, Eq. (4.28) without proof.

Proposition 2.1. There exist a series $z^+(u)$ in u and a series $z^-(u)$ in u^{-1} with coefficients in the algebra $U_q(\widehat{\mathfrak{gl}}_n)$ such that

$$L^\pm(uq^{2n})^t D (L^\pm(u)^{-1})^t = z^\pm(u) D \tag{2.7}$$

and

$$(L^\pm(u)^{-1})^t D^{-1} L^\pm(uq^{2n})^t = z^\pm(u) D^{-1}. \tag{2.8}$$

Moreover, the coefficients of the series $z^\pm(u)$ belong to the center of the algebra $U_q(\widehat{\mathfrak{gl}}_n)$.

Proof. Multiply both sides of (2.3) by $L^\pm_2(v)^{-1}$ from the left and the right and apply the transposition t_2 to get

$$R(u/v)^{t_2}\big(L^\pm_2(v)^{-1}\big)^t L^\pm_1(u) = L^\pm_1(u)\big(L^\pm_2(v)^{-1}\big)^t R(u/v)^{t_2}$$

and hence

$$\big(R(u/v)^{t_2}\big)^{-1} L^\pm_1(u)\big(L^\pm_2(v)^{-1}\big)^t = \big(L^\pm_2(v)^{-1}\big)^t L^\pm_1(u)\big(R(u/v)^{t_2}\big)^{-1}. \tag{2.9}$$

Journal of Mathematical Physics | ARTICLE | pubs.aip.org/aip/jmp

Use the crossing symmetry relation (2.6) to replace the R-matrix by

$$\big(R(u/v)^{t_2}\big)^{-1}=D_2^{-1}\big(R(u/vq^{2n})^{-1}\big)^{t_2}D_2$$

and get

$$\big(R(u/vq^{2n})^{-1}\big)^{t_2}D_2L_1^{\pm}(u)\big(L_2^{\pm}(v)^{-1}\big)^{t}D_2^{-1}=D_2\big(L_2^{\pm}(v)^{-1}\big)^{t}L_1^{\pm}(u)D_2^{-1}\big(R(u/vq^{2n})^{-1}\big)^{t_2}.$$

Now cancel the scalar factors appearing in (2.1) on both sides of this relation and observe that the R-matrix $R-x\,\widetilde{R}$ evaluated at $x=1$ equals $(q-q^{-1})P$, where P is the permutation operator. Therefore,

$$\big((R-\widetilde{R})^{-1}\big)^{t_2}=\frac{1}{q-q^{-1}}\,Q\qquad\text{with}\qquad Q=\sum_{i,j=1}^{n}e_{ij}\otimes e_{ij}.$$

Hence, by taking $u=vq^{2n}$ we get

$$QD_2L_1^{\pm}(vq^{2n})\big(L_2^{\pm}(v)^{-1}\big)^{t}D_2^{-1}=D_2\big(L_2^{\pm}(v)^{-1}\big)^{t}L_1^{\pm}(vq^{2n})D_2^{-1}Q.$$

Since Q is an operator in $\operatorname{End}\mathbb{C}^n\otimes\operatorname{End}\mathbb{C}^n$ with a one-dimensional image, both sides must be equal to $Qz^{\pm}(v)$ for series $z^{\pm}(v)$ with coefficients in the quantum affine algebra. Using the relations $QX_1=QX_2^t$ and $X_1Q=X_2^tQ$ which hold for an arbitrary matrix X, we can write the definition of $z^{\pm}(v)$ as

$$QL_2^{\pm}(vq^{2n})^{t}D_2\big(L_2^{\pm}(v)^{-1}\big)^{t}=Q\,D_2\,z^{\pm}(v)\tag{2.10}$$

and

$$\big(L_2^{\pm}(v)^{-1}\big)^{t}D_2^{-1}L_2^{\pm}(vq^{2n})^{t}Q=D_2^{-1}\,Q\,z^{\pm}(v).\tag{2.11}$$

By taking trace over the first copy of $\operatorname{End}\mathbb{C}^n$ on both sides of (2.10) and (2.11) we arrive at (2.7) and (2.8), respectively.

We will now use (2.8) to show that the series $z^{-}(v)$ commutes with $L^{+}(u)$. We have

$$L_1^{+}(u)z^{-}(v)=L_1^{+}(u)D_2\big(L_2^{-}(v)^{-1}\big)^{t}D_2^{-1}L_2^{-}(vq^{2n})^{t}.\tag{2.12}$$

Transform relation (2.4) in the same way as we did for (2.3) in the beginning of the proof to get the following counterpart of (2.9):

$$\big(R(uq^{-c}/v)^{t_2}\big)^{-1}L_1^{+}(u)\big(L_2^{-}(v)^{-1}\big)^{t}=\big(L_2^{-}(v)^{-1}\big)^{t}L_1^{+}(u)\big(R(uq^{c}/v)^{t_2}\big)^{-1}.$$

Hence, the right hand side of (2.12) equals

$$D_2R_{12}(uq^{-c}/v)^{t_2}\big(L_2^{-}(v)^{-1}\big)^{t}L_1^{+}(u)\big(R_{12}(uq^{c}/v)^{t_2}\big)^{-1}D_2^{-1}L_2^{-}(vq^{2n})^{t}.\tag{2.13}$$

Applying again (2.6), we can write

$$\big(R_{12}(uq^{c}/v)^{t_2}\big)^{-1}D_2^{-1}=D_2^{-1}\big(R_{12}(uq^{c}/vq^{2n})^{-1}\big)^{t_2}.$$

Continue transforming (2.13) by using the following consequence of (2.4):

$$L_1^{+}(u)\big(R_{12}(uq^{c}/vq^{2n})^{-1}\big)^{t_2}L_2^{-}(vq^{2n})^{t}=L_2^{-}(vq^{2n})^{t}\big(R_{12}(uq^{-c}/vq^{2n})^{-1}\big)^{t_2}L_2^{-}(vq^{2n})^{t},$$

so that (2.13) becomes

$$D_2R_{12}(uq^{-c}/v)^{t_2}\big(L_2^{-}(v)^{-1}\big)^{t}D_2^{-1}L_2^{-}(vq^{2n})^{t}\big(R_{12}(uq^{-c}/vq^{2n})^{-1}\big)^{t_2}L_1^{+}(u).$$

By (2.8) this simplifies to

$$D_2R_{12}(uq^{-c}/v)^{t_2}D_2^{-1}\big(R_{12}(uq^{-c}/vq^{2n})^{-1}\big)^{t_2}z^{-}(v)L_1^{+}(u)$$

which equals $z^{-}(v)L_1^{+}(u)$ by (2.6). This proves that $L_1^{+}(u)z^{-}(v)=z^{-}(v)L_1^{+}(u)$. The relation $L_1^{-}(u)z^{-}(v)=z^{-}(v)L_1^{-}(u)$ and the centrality of $z^{+}(v)$ are verified in the same way. □

Remark 2.2. The counterparts of the series $z^{\pm}(u)$ for the quantum affine algebras of types B, C and D appear in Ref. 15, Proposition 3.3 and Ref. 16, Proposition 3.3, where they were introduced by relations analogous to (2.7) and (2.8).

Corollary 2.3. We have the formulas

$$z^{\pm}(u) = \frac{1}{[n]_q} \operatorname{tr} D L^{\pm}(uq^{2n}) L^{\pm}(u)^{-1} \tag{2.14}$$

and

$$z^{\pm}(u) = \frac{1}{[n]_q} \operatorname{tr} D^{-1} L^{\pm}(u)^{-1} L^{\pm}(uq^{2n}). \tag{2.15}$$

Proof. The formulas follow by taking trace on both sides of the respective matrix relations (2.7) and (2.8). □

Recall that the *quantum determinants* $\operatorname{qdet} L^{+}(u)$ and $\operatorname{qdet} L^{-}(u)$ are series in u and u^{-1}, respectively, whose coefficients belong to the center of the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$:

$$\operatorname{qdet} L^{\pm}(u) = \sum_{\sigma \in \mathfrak{S}_n} (-q)^{-l(\sigma)}\, l^{\pm}_{\sigma(1)1}(uq^{2n-2}) \cdots l^{\pm}_{\sigma(n)n}(u),$$

where $l(\sigma)$ denotes the length of the permutation σ.

The following is a q-analogue of the quantum Liouville formula of Ref. 22. It was stated in Ref. 2, Eq. (4.32).

Theorem 2.4. *We have the relations*

$$z^{\pm}(u) = \frac{\operatorname{qdet} L^{\pm}(uq^{2})}{\operatorname{qdet} L^{\pm}(u)}. \tag{2.16}$$

Proof. We follow (Ref. 20, Sec. 1.9) and use the *quantum comatrices* $\widehat{L}^{\pm}(u)$ introduced in Refs. 23 and 28. They are defined by the relations

$$\widehat{L}^{\pm}(uq^{2})\; L^{\pm}(u) = \operatorname{qdet} L^{\pm}(u)\, 1, \tag{2.17}$$

where 1 denotes the identity matrix. We will derive formulas for the entries of the matrices $\widehat{L}^{\pm}(u)$ by using the quantum minor relations reviewed e.g., in Ref. 11 which we outline below.

Recall that the q-permutation operator $P^{\,q} \in \operatorname{End}\mathbb{C}^n \otimes \operatorname{End}\mathbb{C}^n$ is defined by

$$P^{\,q} = \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + q \sum_{i>j} e_{ij} \otimes e_{ji} + q^{-1} \sum_{i<j} e_{ij} \otimes e_{ji}.$$

The action of the symmetric group $\mathfrak{S}_k$ on the space $(\mathbb{C}^n)^{\otimes k}$ can be defined by setting $s_i \mapsto P^{\,q}_{s_i} := P^{\,q}_{i,i+1}$ for $i = 1, \dots, k-1$, where s_i denotes the transposition $(i, i+1)$. If $\sigma = s_{i_1} \cdots s_{i_l}$ is a reduced decomposition of an element $\sigma \in \mathfrak{S}_k$ then we set $P^{\,q}_{\sigma} = P^{\,q}_{s_{i_1}} \cdots P^{\,q}_{s_{i_l}}$. Denote by $e_1, \dots, e_n$ the canonical basis vectors of $\mathbb{C}^n$. Then for any indices $a_1 < \cdots < a_k$ and any $\tau \in \mathfrak{S}_k$ we have

$$P^{\,q}_{\sigma}(e_{a_{\tau(1)}} \otimes \cdots \otimes e_{a_{\tau(k)}}) = q^{l(\sigma\tau^{-1}) - l(\tau)}\, e_{a_{\tau\sigma^{-1}(1)}} \otimes \cdots \otimes e_{a_{\tau\sigma^{-1}(k)}}. \tag{2.18}$$

We denote by $A^{(k)}_q$ the q-antisymmetrizer

$$A^{(k)}_q = \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn} \sigma \cdot P^{q}_{\sigma}. \tag{2.19}$$

The defining relations (2.3) and the *fusion procedure*[3] for the quantum affine algebra imply the relations

$$A^{(k)}_q L^{\pm}_1(uq^{2k-2}) \cdots L^{\pm}_k(u) = L^{\pm}_k(u) \cdots L^{\pm}_1(uq^{2k-2})\; A^{(k)}_q. \tag{2.20}$$

The quantum minors are the series $l^{\pm}{}^{a_1 \cdots a_k}_{b_1 \cdots b_k}(u)$ with coefficients in $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ defined by the expansion of the elements (2.20) as

$$\sum_{a_i, b_i} e_{a_1 b_1} \otimes \cdots \otimes e_{a_k b_k} \otimes l^{\pm}{}^{a_1 \cdots a_k}_{b_1 \cdots b_k}(u).$$

Explicit formulas for the quantum minors have the form: if $a_1 < \cdots < a_k$ then

$$l^{\pm}{}^{a_1 \cdots a_k}_{b_1 \cdots b_k}(u) = \sum_{\sigma \in \mathfrak{S}_k} (-q)^{-l(\sigma)} \cdot l^{\pm}_{a_{\sigma(1)} b_1}(uq^{2k-2}) \cdots l^{\pm}_{a_{\sigma(k)} b_k}(u). \tag{2.21}$$

If $b_1 < \cdots < b_k$ (and the a_i are arbitrary) then

$$l^{\pm}{}^{a_1\cdots a_k}_{b_1\cdots b_k}(u) = \sum_{\sigma\in\mathfrak{S}_k} (-q)^{l(\sigma)} \cdot l^{\pm}_{a_k b_{\sigma(k)}}(u)\cdots l^{\pm}_{a_1 b_{\sigma(1)}}(uq^{2k-2})$$

and for any $\tau \in \mathfrak{S}_k$ we have

$$l^{\pm}{}^{a_1\cdots a_k}_{b_{\tau(1)}\cdots b_{\tau(k)}}(u) = (-q)^{-l(\tau)} l^{\pm}{}^{a_1\cdots a_k}_{b_1\cdots b_k}(u).$$

The following lemma was pointed out in Refs. 23 and 28.

Lemma 2.5. The (i,j) entry of the matrix $\widehat{L}^{\pm}(u)$ is given by

$$\widehat{l}^{\pm}_{ij}(u) = (-q)^{j-i}\, l^{\pm}{}^{1\ldots\widehat{j}\ldots n}_{1\ldots\widehat{i}\ldots n}(u), \tag{2.22}$$

where the hats on the right-hand side indicate the indices to be omitted.

Proof. According to (2.21), the quantum determinants are defined by the relations

$$A_q^{(n)} L_1^{\pm}(uq^{2n-2})\cdots L_n^{\pm}(u) = A_q^{(n)}\ \mathrm{qdet}\ L^{\pm}(u).$$

Hence, the definition (2.17) of the quantum comatrices implies

$$A_q^{(n)} L_1^{\pm}(uq^{2n-4})\cdots L_{n-1}^{\pm}(u) = A_q^{(n)}\ \widehat{L}_n^{\pm}(u).$$

Apply both sides to the vector $e_1\otimes\cdots\otimes\widehat{e_i}\otimes\cdots\otimes e_n\otimes e_j$ and use (2.18) and (2.19) to equate the coefficients of the vector $A_q^{(n)}(e_1\otimes\cdots\otimes e_n)$ to obtain (2.22). □

We will also use the relations for the transposed comatrices; see Refs. 23 and 28.

Lemma 2.6. We have the relations

$$D\widehat{L}^{\pm}(u)^t D^{-1} L^{\pm}(uq^{2n-2})^t = \mathrm{qdet}\ L^{\pm}(u)\, 1.$$

Proof. Since each relation only involves generators belonging to the subalgebra of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ generated by the coefficients of the series $l^{+}_{ij}(u)$ or $l^{-}_{ij}(u)$, we may assume that the central charge is specialized to zero, $c = 0$, and derive the desired relations in the quotient algebra $\mathrm{U}^{\circ}_q(\widehat{\mathfrak{gl}}_n)$. The mapping

$$\theta : L^{\pm}(u) \to L^{\mp}(u^{-1})^t$$

defines an automorphism of the algebra $\mathrm{U}^{\circ}_q(\widehat{\mathfrak{gl}}_n)$. By Lemma 2.5 and the quantum minor formulas, for the images under θ we have

$$\theta : \widehat{L}^{\pm}(u) \mapsto D\widehat{L}^{\mp}(u^{-1}q^{-2n+4})^t D^{-1}$$

and

$$\theta : \mathrm{qdet}\ L^{\pm}(u) \mapsto \mathrm{qdet}\ L^{\mp}(u^{-1}q^{-2n+2}).$$

It remains to apply θ to both sides of relations (2.17) and then replace u by $u^{-1}q^{-2n+2}$. □

Now, using (2.17) and the centrality of qdet $L^{\pm}(u)$, we can write (2.7) as

$$z^{\pm}(u) = L^{\pm}(uq^{2n})^t D (L^{\pm}(u)^{-1})^t D^{-1} = (\mathrm{qdet}\ L^{\pm}(u))^{-1} L^{\pm}(uq^{2n})^t D\widehat{L}^{\pm}(uq^2)^t D^{-1}.$$

By applying Lemma 2.6 with u replaced by uq^2 we find that this expression coincides with the right hand side of (2.16).

Remark 2.7. Since the second part of Proposition 2.1 was not used in the Proof of Theorem 2.4, the fact that the coefficients of both series $z^{+}(u)$ and $z^{-}(u)$ belong to the center of the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ also follows from (2.16) due to the respective properties of the quantum determinants qdet $L^{+}(u)$ and qdet $L^{-}(u)$.

III. QUANTUM GELFAND INVARIANTS

We will follow[4] to define the *quantized enveloping algebra* $\mathrm{U}_q(\mathfrak{gl}_n)$ in its R-matrix presentation[27] as the algebra generated by elements l_{ij}^+ and l_{ij}^- with $1 \leqslant i,j \leqslant n$ subject to the relations

$$l_{ij}^- = l_{ji}^+ = 0, \qquad 1 \leqslant i < j \leqslant n,$$
$$l_{ii}^- \, l_{ii}^+ = l_{ii}^+ \, l_{ii}^- = 1, \qquad 1 \leqslant i \leqslant n,$$
$$R\, L_1^{\pm} L_2^{\pm} = L_2^{\pm} L_1^{\pm} R, \qquad R\, L_1^{+} L_2^{-} = L_2^{-} L_1^{+} R,$$

where the R-matrix R is defined in (2.2), while L^+ and L^- are the matrices

$$L^{\pm} = \sum_{i,j} e_{ij} \otimes l_{ij}^{\pm} \in \operatorname{End} \mathbb{C}^n \otimes \mathrm{U}_q(\mathfrak{gl}_n).$$

The universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ is recovered from $\mathrm{U}_q(\mathfrak{gl}_n)$ in the limit $q \to 1$ by the formulas

$$\frac{l_{ij}^-}{q - q^{-1}} \to E_{ij}, \qquad \frac{l_{ji}^+}{q - q^{-1}} \to -E_{ji} \qquad \text{for} \quad i > j, \tag{3.1}$$

and

$$\frac{l_{ii}^- - 1}{q - 1} \to E_{ii}, \qquad \frac{l_{ii}^+ - 1}{q - 1} \to -E_{ii} \qquad \text{for} \quad i = 1, \dots, n. \tag{3.2}$$

Set $M = L^-(L^+)^{-1}$. By Ref. 27, the quantum traces defined by $\operatorname{tr}_q M^m = \operatorname{tr} D M^m$ belong to the center of the algebra $\mathrm{U}_q(\mathfrak{gl}_n)$. The elements $\operatorname{tr}_q M^m$ act by multiplication by scalars in the irreducible highest weight representations $L_q(\lambda)$. The representation $L_q(\lambda)$ of $\mathrm{U}_q(\mathfrak{gl}_n)$ is generated by a nonzero vector ξ such that

$$l_{ij}^+ \, \xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n,$$
$$l_{ii}^- \, \xi = q^{\lambda_i} \, \xi \qquad \text{for} \quad 1 \leqslant i \leqslant n,$$

for an n-tuple $\lambda = (\lambda_1, \dots, \lambda_n)$ of integers (or real numbers). This is a q-deformation of the irreducible $\mathfrak{gl}_n$-module $L(\lambda)$ with the highest weight λ.

Proof of Theorem 1.1. We will show that the eigenvalue of the quantum Gelfand invariant $\operatorname{tr}_q M^m$ in $L_q(\lambda)$ is given by formula (1.2).

Recall the evaluation homomorphism $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n) \to \mathrm{U}_q(\mathfrak{gl}_n)$ defined by

$$L^+(u) \mapsto L^+ - L^- u, \qquad L^-(u) \mapsto L^- - L^+ u^{-1}, \qquad q^c \mapsto 1, \tag{3.3}$$

and apply it to both sides of the Liouville formula

$$z^+(u) = \frac{\operatorname{qdet} L^+(uq^2)}{\operatorname{qdet} L^+(u)} \tag{3.4}$$

proved in Theorem 2.4. The image of the quantum determinant $\operatorname{qdet} L^+(u)$ is found by

$$\sum_{\sigma \in \mathfrak{S}_n} (-q)^{-l(\sigma)} \left(l_{\sigma(1)1}^+ - l_{\sigma(1)1}^- u q^{2n-2} \right) \cdots \left(l_{\sigma(n)n}^+ - l_{\sigma(n)n}^- u \right).$$

By applying this element to the highest vector ξ of $L_q(\lambda)$, we find that its eigenvalue is given by

$$\left(q^{-\lambda_1} - q^{\lambda_1 + 2n - 2} u \right) \cdots \left(q^{-\lambda_n} - q^{\lambda_n} u \right) = q^{n(n-1)/2} (q^{-\ell_1} - q^{\ell_1} u) \cdots (q^{-\ell_n} - q^{\ell_n} u)$$

with $\ell_i = \lambda_i + n - i$. Therefore, the eigenvalue of the image of the right hand side of (3.4) in $L_q(\lambda)$ is given by

$$\frac{(q^{-\ell_1} - q^{\ell_1 + 2} u) \cdots (q^{-\ell_n} - q^{\ell_n + 2} u)}{(q^{-\ell_1} - q^{\ell_1} u) \cdots (q^{-\ell_n} - q^{\ell_n} u)}.$$

To expand this rational function into a series in u write it as

$$C + \frac{a_1}{1 - q^{2\ell_1} u} + \cdots + \frac{a_n}{1 - q^{2\ell_n} u}$$

to find that the constants a_k are given by

$$a_k = (q^{n-1} - q^{n+1})\,\frac{[\ell_1 - \ell_k + 1]_q \cdots [\ell_n - \ell_k + 1]_q}{[\ell_1 - \ell_k]_q \cdots \wedge \cdots [\ell_n - \ell_k]_q}.$$

Then write

$$\frac{1}{1 - q^{2\ell_k} u} = \sum_{m=0}^{\infty} q^{2\ell_k m}\, u^m.$$

On the other hand, using (2.14), for the image of $z^+(u)$ under the evaluation homomorphism (3.3) we get

$$\frac{1}{[n]_q}\,\mathrm{tr}\, D(L^+ - L^- u q^{2n})(L^+ - L^- u)^{-1} = \frac{1}{[n]_q}\,\mathrm{tr}\, D(1 - M u q^{2n})(1 - M u)^{-1}$$

which equals

$$1 + (q^{n-1} - q^{n+1}) \sum_{m=1}^{\infty} \mathrm{tr}_q M^m\, u^m.$$

Formula (1.2) now follows by equating the coefficients of the powers u^m on both sides of the power expansions. Note that the formula is also valid for $m = 0$. □

Remark 3.1. Due to (1.3), the Gelfand invariant $\mathrm{tr}\, E^m \in \mathrm{U}(\mathfrak{gl}_n)$ is obtained as the limit value as $q \to 1$ of the expression

$$\frac{1}{(q - q^{-1})^m}\,\mathrm{tr}\, D(M - 1)^m = \frac{1}{(q - q^{-1})^m} \sum_{r=0}^{m} \binom{m}{r} (-1)^{m-r}\, \mathrm{tr}_q M^r.$$

We will look at the limit value of the corresponding linear combination of the eigenvalues given by (1.2) and note that the q-numbers specialize by the rule $[r]_q \to r$ as $q \to 1$. Hence, for each $k = 1, \dots, n$ it suffices to find the limit value of the expression

$$\frac{1}{(q - q^{-1})^m} \sum_{r=0}^{m} \binom{m}{r} (-1)^{m-r}\, q^{2\ell_k r} = \frac{(q^{2\ell_k} - 1)^m}{(q - q^{-1})^m},$$

which equals $\ell_k{}^m$, thus yielding formula (1.1). □

Since the evaluation homomorphism (3.3) is surjective, the centrality property of the elements $\mathrm{tr}_q\, M^m$ also follows from the above calculations due to Proposition 2.1. Similarly, by using the three remaining formulas in (2.14) and (2.15), we get three more families of central elements in $\mathrm{U}_q(\mathfrak{gl}_n)$ together with the relations between them given by

$$\mathrm{tr}\, D^{-1}((L^+)^{-1} L^-)^m = \mathrm{tr}_q M^m \qquad \text{and} \qquad \mathrm{tr}\, D^{-1}((L^-)^{-1} L^+)^m = \mathrm{tr}\, D(L^+ (L^-)^{-1})^m.$$

Moreover, additional relations between the families are provided by the isomorphism

$$\mathrm{U}_q(\mathfrak{gl}_n) \to \mathrm{U}_{q^{-1}}(\mathfrak{gl}_n), \qquad L^{\pm} \mapsto (L^{\pm})^{-1}.$$

This implies that the formulas for the eigenvalues of the other two central elements of $\mathrm{U}_q(\mathfrak{gl}_n)$ in $L_q(\lambda)$ are obtained by the replacement $q \mapsto q^{-1}$ in (1.2):

$$\mathrm{tr}\, D(L^+ (L^-)^{-1})^m \mapsto \sum_{k=1}^{n} q^{-2\ell_k m}\, \frac{[\ell_1 - \ell_k + 1]_q \cdots [\ell_n - \ell_k + 1]_q}{[\ell_1 - \ell_k]_q \cdots \wedge \cdots [\ell_n - \ell_k]_q}.$$

Alternatively, these eigenvalue formulas can be derived in the same way as in the Proof of Theorem 1.1 by working with the other Liouville formula in (2.16) instead of (3.4).

ACKNOWLEDGMENTS

This work was supported by the Australian Research Council, Grant No. DP240101572.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Naihuan Jing: Methodology (equal). **Ming Liu**: Methodology (equal). **Alexander Molev**: Methodology (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES

[1] Baumann, P., "On the center of quantized enveloping algebras," J. Algebra **203**, 244–260 (1998).

[2] Belliard, S. and Ragoucy, E., "The nested Bethe ansatz for 'all' open spin chains with diagonal boundary conditions," J. Phys. A: Math. Theor. **42**(20), 205203 (2009).

[3] Cherednik, I. V., "A new interpretation of Gelfand–Tzetlin bases," Duke Math. J. **54**, 563–577 (1987).

[4] Ding, J. and Frenkel, I. B., "Isomorphism of two realizations of quantum affine algebra $U_q(\widehat{\mathfrak{gl}}(n))$," Commun. Math. Phys. **156**, 277–300 (1993).

[5] Drinfeld, V. G., "Hopf algebras and the quantum Yang–Baxter equation," Sov. Math. Dokl. **32**, 254–258 (1985).

[6] Etingof, P. I., "Central elements for quantum affine algebras and affine MacDonald's operators," Math. Res. Lett. **2**, 611–628 (1995).

[7] Frenkel, I. B. and Reshetikhin, N. Y., "Quantum affine algebras and holonomic difference equations," Commun. Math. Phys. **146**, 1–60 (1992).

[8] Gelfand, I. M., "Center of the infinitesimal group ring," Mat. Sb. **26**, 103–112 (1950).

[9] Gould, M. D., Zhang, R. B., and Bracken, A. J., "Generalized Gel'fand invariants and characteristic identities for quantum groups," J. Math. Phys. **32**, 2298–2303 (1991).

[10] Gurevich, D. I., Pyatov, P. N., and Saponov, P. A., "Hecke symmetries and characteristic relations on reflection equation algebras," Lett. Math. Phys. **41**, 255–264 (1997).

[11] Hopkins, M. J. and Molev, A. I., "A q-analogue of the centralizer construction and skew representations of the quantum affine algebra," SIGMA **2**, 092 (2006).

[12] Isaev, A., Ogievetsky, O., and Pyatov, P., "On quantum matrix algebras satisfying the Cayley–Hamilton–Newton identities," J. Phys. A: Math. Gen. **32**, L115–L121 (1999).

[13] Jimbo, M., "A q-difference analogue of $\mathrm{U}(\mathfrak{g})$ and the Yang–Baxter equation," Lett. Math. Phys. **10**, 63–69 (1985).

[14] Jimbo, M., "A q-analogue of $U_q(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang–Baxter equation," Lett. Math. Phys. **11**, 247–252 (1986).

[15] Jing, N., Liu, M., and Molev, A., "Isomorphism between the R-matrix and Drinfeld presentations of quantum affine algebra: Type C," J. Math. Phys. **61**(3), 031701 (2020).

[16] Jing, N., Liu, M., and Molev, A., "Isomorphism between the R-matrix and Drinfeld presentations of quantum affine algebra: Types B and D," SIGMA **16**, 043 (2020).

[17] Jing, N., Liu, M., and Molev, A., "Quantum Sugawara operators in type A," arXiv:2212.11435.

[18] Jordan, D. and White, N., "The center of the reflection equation algebra via quantum minors," J. Algebra **542**, 308–342 (2020).

[19] Kulish, P. P. and Sklyanin, E. K., "Quantum spectral transform method: Recent developments," in *Integrable Quantum Field Theories*, *Lecture Notes in Physics Vol. 151* (Springer, Berlin, 1982), pp. 61–119.

[20] Molev, A., *Yangians and Classical Lie Algebras*, *Mathematical Surveys and Monographs Vol. 143* (American Mathematical Society, Providence, RI, 2007).

[21] Molev, A., *Sugawara Operators for Classical Lie Algebras*, *Mathematical Surveys and Monographs Vol. 229* (American Mathematical Society, Providence, RI, 2018).

[22] Nazarov, M. L., "Quantum Berezinian and the classical Capelli identity," Lett. Math. Phys. **21**, 123–131 (1991).

[23] Nazarov, M. and Tarasov, V., "Yangians and Gelfand–Zetlin bases," Publ. Res. Inst. Math. Sci. **30**, 459–478 (1994).

[24] Perelomov, A. M. and Popov, V. S., "Casimir operators for semisimple Lie algebras," Izv. Akad. Nauk SSSR Ser. Mat. **32**, 1368–1390 (1968).

[25] Reshetikhin, N. Y., "Quasitriangular Hopf algebras and invariants of links," Algebra Anal. **1**(2), 169–188 (1989).

[26] Reshetikhin, N. Y. and Semenov-Tian-Shansky, M. A., "Central extensions of quantum current groups," Lett. Math. Phys. **19**, 133–142 (1990).

[27] Reshetikhin, N. Y., Takhtadzhyan, L. A., and Faddeev, L. D., "Quantization of Lie groups and Lie algebras," Algebra Anal. **1**(1), 178–206 (1989).

[28] Tarasov, V. O., "Cyclic monodromy matrices for $sl(n)$ trigonometric R-matrices," Commun. Math. Phys. **158**, 459–484 (1993).