Lorentz-Covariant Theories of Higher-Spin Fields

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Abstract

On the basis of our recent modifications of the Dirac formalism we generalize the Bargmann-Wigner formalism for higher spins to be compatible with other formalisms for bosons. Relations with dual electrodynamics, with the Ogievetskii-Polubarinov notoph and the Weinberg 2(2J+1) theory are found. Next, we introduce the dual analogues of the Riemann tensor and derive corresponding dynamical equations in the Minkowski space. Relations with the Marques-Spehler chiral gravity theory are discussed. The problem of indefinite metrics, particularly, in quantization of 4-vector fields is clarified. We also try to provide some mathematical foundations to the modern non-commutative theories.

1. Introduction.

Recent advances in astrophysics [1] suggest the existence of fundamental scalar fields [2, 3]. On the other hand, the $(1/2, 1/2)$ representation of the Lorentz group provides suitable frameworks for introduction of the $S = 0$ field, Ref. [4]. In this paper, starting from the very beginning we propose a generalized theory in the 4-vector representation, for the antisymmetric tensor field of the second rank as well. The results can be useful in any theory dealing with the light phenomena.

The plan of my talk is following:

- Antecedents. Mapping between the Weinberg-Tucker-Hammer (WTH) formulation and antisymmetric tensor (AST) fields of the 2nd rank. Modified Bargmann-Wigner (BW) formalism. Pseudovector potential. Parity.
- Matrix form of the general equation in the $(1/2, 1/2)$ representation. Lagrangian in the matrix form. Masses.
- Standard Basis and Helicity Basis.

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- Dynamical invariants. Field operators. Propagators. The indefinite metric.
- Spin-2 Framework.
- Non-commutativity.

2. Preliminaries.

I am going to give an overview of my previous works in order you to understand motivations better. In Ref. [2, 3] I derived the Maxwell-like equations with the additional gradient of a scalar field χ from the first principles. Here they are:

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} + \nabla Im\chi\,, \tag{1}$$

$$\nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \nabla Re\chi\,, \tag{2}$$

$$\nabla\cdot\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}Re\chi\,, \tag{3}$$

$$\nabla\cdot\mathbf{B} = \frac{1}{c}\frac{\partial}{\partial t}Im\chi\,. \tag{4}$$

The χ may depend on the $\mathbf{E}, \mathbf{B}$, so we can have the non-linear electrodynamics. Of course, similar equations can be obtained in the massive case $m \neq 0$, i.e., within the Proca-like theory. We should then consider

$$(E^2 - c^2\mathbf{p}^2 - m^2c^4)\Psi^{(3)} = 0\,. \tag{5}$$

In the spin-1/2 case the analogous equation can be written for the two-component spinor ($c = \hbar = 1$)

$$(EI^{(2)} - \sigma\cdot\mathbf{p})(EI^{(2)} + \sigma\cdot\mathbf{p})\Psi^{(2)} = m^2\Psi^{(2)}\,, \tag{6}$$

or, in the 4-component form [1]

$$[i\gamma_\mu\partial_\mu + m_1 + m_2\gamma^5]\Psi^{(4)} = 0\,. \tag{9}$$

[1] There exist various generalizations of the Dirac formalism. For instance, the Barut generalization is based on

$$[i\gamma_\mu\partial_\mu + a(\partial_\mu\partial_\mu)/m - \kappa]\Psi = 0\,, \tag{7}$$

which can describe states of different masses. If one fixes the parameter a by the requirement that the equation gives the state with the classical anomalous magnetic moment, then $m_2 = m_1(1 + \frac{3}{2\alpha})$, i.e., it gives the muon mass. Of course, one can propose a generalized equation:

$$[i\gamma_\mu\partial_\mu + a + b\partial_\mu\partial_\mu + \gamma_5(c + d\partial_\mu\partial_\mu)]\Psi = 0\,; \tag{8}$$

and, perhaps, even that of higher orders in derivatives.

In the spin-1 case we have

$$(EI^{(3)} - \mathbf{S}\cdot\mathbf{p})(EI^{(3)} + \mathbf{S}\cdot\mathbf{p})\Psi^{(3)} - \mathbf{p}(\mathbf{p}\cdot\Psi^{(3)}) = m^2\Psi^{(3)}\,, \qquad (10)$$

that lead to (1-4), when $m = 0$. We can continue writing down equations for higher spins in a similar fashion.

On this basis we are ready to generalize the BW formalism [5, 6]. Why is that convenient? In Ref. [11, 7] I presented the mapping between the WTH equation, Ref. [8, 9], and the equations for AST fields. The equation for a 6-component field function is [2]

$$[\gamma_{\alpha\beta}p_\alpha p_\beta + Ap_\alpha p_\alpha + Bm^2]\Psi^{(6)} = 0\,. \qquad (11)$$

Corresponding equations for the AST fields are:

$$\partial_\alpha\partial_\mu F^{(1)}_{\mu\beta} - \partial_\beta\partial_\mu F^{(I1)}_{\mu\alpha} + \frac{A-1}{2}\partial_\mu\partial_\mu F^{(1)}_{\alpha\beta} - \frac{B}{2}m^2 F^{(1)}_{\alpha\beta} = 0 \qquad (12)$$

$$\partial_\alpha\partial_\mu F^{(2)}_{\mu\beta} - \partial_\beta\partial_\mu F^{(2)}_{\mu\alpha} - \frac{A+1}{2}\partial_\mu\partial_\mu F^{(2)}_{\alpha\beta} + \frac{B}{2}m^2 F^{(2)}_{\alpha\beta} = 0 \qquad (13)$$

depending on the parity properties of $\Psi^{(6)}$ (the first case corresponds to the eigenvalue $P = -1$; the second one, to $P = +1$).

We noted:

- One can derive equations for the dual tensor $\tilde{F}_{\alpha\beta}$, which are similar to (12,13), Ref. [10, 11].
- In the Tucker-Hammer case ($A = 1$, $B = 2$), the first equation gives the Proca theory $\partial_\alpha\partial_\mu F_{\mu\beta} - \partial_\beta\partial_\mu F_{\mu\alpha} = m^2 F_{\alpha\beta}$. In the second case one finds something different, $\partial_\alpha\partial_\mu F_{\mu\beta} - \partial_\beta\partial_\mu F_{\mu\alpha} = (\partial_\mu\partial_\mu - m^2)F_{\alpha\beta}$
- If $\Psi^{(6)}$ has no definite parity, e. g., $\Psi^{(6)} = \text{column}(\mathbf{E} + i\mathbf{B}\;\; \mathbf{B} + i\mathbf{E}\,)$, the equation for the AST field will contain both the tensor and the dual tensor:

$$\partial_\alpha\partial_\mu F_{\mu\beta} - \partial_\beta\partial_\mu F_{\mu\alpha} = \frac{1}{2}\partial^2 F_{\alpha\beta} + [-\frac{A}{2}\partial^2 + \frac{B}{2}m^2]\tilde{F}_{\alpha\beta}. \qquad (14)$$

- Depending on the relation between A and B and on which parity solution do we consider, the WTH equations may describe different mass states. For instance, when $A = 7$ and $B = 8$ we have the second mass state $(m')^2 = 4m^2/3$.

We tried to find relations between the generalized WTH theory and other spin-1 formalisms. Therefore, we were forced to modify the Bargmann-Wigner formalism [10, 12]. For instance, we introduced the sign operator

[2] In order to have solutions satisfying the Einstein dispersion relations $E^2 - \mathbf{p}^2 = m^2$ we have to assume $B/(A+1) = 1$, or $B/(A-1) = 1$.

in the Dirac equations which are the input for the formalism for symmetric 2-rank spinor:

$$[i\gamma_\mu\partial_\mu + \epsilon_1 m_1 + \epsilon_2 m_2\gamma_5]_{\alpha\beta}\,\Psi_{\beta\gamma} = 0\,, \tag{15}$$

$$[i\gamma_\mu\partial_\mu + \epsilon_3 m_1 + \epsilon_4 m_2\gamma_5]_{\gamma\beta}\,\Psi_{\alpha\beta} = 0\,, \tag{16}$$

In general we have 16 possible combinations, but 4 of them give the same sets of the Proca-like equations. We obtain [10]:

$$\partial_\mu A_\lambda - \partial_\lambda A_\mu + 2m_1 A_1 F_{\mu\lambda} + im_2 A_2 \epsilon_{\alpha\beta\mu\lambda} F_{\alpha\beta} = 0\,, \tag{17}$$

$$\partial_\lambda F_{\mu\lambda} - \frac{m_1}{2} A_1 A_\mu - \frac{m_2}{2} B_2 \tilde{A}_\mu = 0\,, \tag{18}$$

with $A_1 = (\epsilon_1 + \epsilon_3)/2$, $A_2 = (\epsilon_2 + \epsilon_4)/2$, $B_1 = (\epsilon_1 - \epsilon_3)/2$, and $B_2 = (\epsilon_2 - \epsilon_4)/2$. See the additional constraints in the cited paper [10]. So, we have the dual tensor and the pseudovector potential in the Proca-like sets. The pseudovector potential is the same as that which enters in the Duffin-Kemmer set for the spin 0.

Moreover, it appears that the properties of the polarization vectors with respect to parity operation depend on the choice of the spin basis. For instance, in Ref. [10, 13] the momentum-space polarization vectors have been listed in the helicity basis. Berestetskiĭ, Lifshitz and Pitaevskiĭ claimed too, Ref. [14], that the helicity states cannot be the parity states. If one applies common-used relations between fields and potentials it appears that the **E** and **B** fields have no usual properties with respect to space inversions.

Thus, the conclusions of the previous works are:

- The mapping exists between the WTH formalism for $S = 1$ and the AST fields of four kinds (provided that the solutions of the WTH equations are of the definite parity).
- Their massless limits contain additional solutions comparing with the Maxwell equations. This was related to the possible theoretical existence of the Ogievetskiĭ-Polubarinov-Kalb-Ramond notoph, Ref. [15, 16, 17].
- In some particular cases $(A = 0, B = 1)$ massive solutions of different parities are naturally divided into the classes of causal and tachyonic solutions.
- If we want to take into account the solutions of the WTH equations of different parity properties, this induces us to generalize the BW, Proca and Duffin-Kemmer formalisms.
- In the $(1/2, 0) \oplus (0, 1/2)$, $(1, 0) \oplus (0, 1)$ etc. representations it is possible to introduce the parity-violating frameworks. The corresponding solutions are the mixing of various polarization states.
- The sum of the Klein-Gordon equation with the $(S, 0) \oplus (0, S)$ equations may change the theoretical content even on the free level. For instance, the higher-spin equations may actually describe various spin and mass states.

- The mappings exists between the WTH solutions of undefined parity and the AST fields, which contain both tensor and dual tensor. They are eight.
- The 4-potentials and electromagnetic fields [10, 13] in the helicity basis have different parity properties comparing with the standard basis of the polarization vectors.
- In the previous talk [18] I presented a theory in the $(1/2,0)\oplus(0,1/2)$ representation in the helicity basis. Under the space inversion operation, different helicity states transform each other, $Pu_h(-\mathbf{p}) = -iu_{-h}(\mathbf{p})$, $Pv_h(-\mathbf{p}) = +iv_{-h}(\mathbf{p})$.

3. The theory of 4-vector field.

First of all, we show that the equation for the 4-vector field can be presented in a matrix form. Recently, S. I. Kruglov proposed, Refs. [19], a general form of the Lagrangian for 4-potential field B_μ, which also contains the spin-0 state. Initially, we have

$$\alpha\partial_\mu\partial_\nu B_\nu + \beta\partial_\nu^2 B_\mu + \gamma m^2 B_\mu = 0\,, \tag{19}$$

provided that derivatives commute. When $\partial_\nu B_\nu = 0$ (the Lorentz gauge) we obtain spin-1 states only. However, if it is not equal to zero we have a scalar field and an axial-vector potential. We can also verify this statement by consideration of the dispersion relations of the equation (19). One obtains 4+4 states (two of them may differ in mass from others).

Next, one can fix one of the constants α, β, γ without loosing any physical content. For instance, when $\alpha = -2$ one gets the equation

$$[\delta_{\mu\nu}\delta_{\alpha\beta} - \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}]\,\partial_\alpha\partial_\beta B_\nu + A\partial_\alpha^2\delta_{\mu\nu}B_\nu - Bm^2 B_\mu = 0\,, \tag{20}$$

where $\beta = A+1$ and $\gamma = -B$. In the matrix form the equation (20) reads:

$$\left[\gamma_{\alpha\beta}\partial_\alpha\partial_\beta + A\partial_\alpha^2 - Bm^2\right]_{\mu\nu} B_\nu = 0\,, \tag{21}$$

with

$$[\gamma_{\alpha\beta}]_{\mu\nu} = \delta_{\mu\nu}\delta_{\alpha\beta} - \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\,. \tag{22}$$

They are the analogs of the Barut-Muzinich-Williams (BMW) γ-matrices for bivector fields.[3] It is easy to prove by the textbook method [21] that γ_{44} can serve as the parity matrix.

[3] One can also define the analogs of the BMW $\gamma_{5,\alpha\beta}$ matrices

$$\gamma_{5,\alpha\beta} = \frac{i}{6}[\gamma_{\alpha\kappa}, \gamma_{\beta\kappa}]_{-,\mu\nu} = i[\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}]\,. \tag{23}$$

As opposed to $\gamma_{\alpha\beta}$ matrices they are totally antisymmetric. They are related to boost and rotation generators of this representation. The γ-matrices are pure real; γ_5-matrices are pure imaginary. In the $(1/2,1/2)$ representation, we need 16 matrices to form the complete set.

Lagrangian and the equations of motion. Let us try

$$\mathcal{L} = (\partial_\alpha B^*_\mu)[\gamma_{\alpha\beta}]_{\mu\nu}(\partial_\beta B_\nu) + A(\partial_\alpha B^*_\mu)(\partial_\alpha B_\mu) + Bm^2 B^*_\mu B_\mu \,. \qquad (24)$$

On using the Lagrange-Euler equation we have

$$[\gamma_{\nu\beta}]_{\kappa\tau}\partial_\nu\partial_\beta B_\tau + A\partial_\nu^2 B_\kappa - Bm^2 B_\kappa = 0\,. \qquad (25)$$

It may be presented in the form of (19).

Masses. We are convinced that in the case of spin 0, we have $B_\mu \to \partial_\mu \chi$; in the case of spin 1 we have $\partial_\mu B_\mu = 0$.

$$(\delta_{\mu\nu}\delta_{\alpha\beta} - \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})\partial_\alpha\partial_\beta\partial_\nu\chi = -\partial^2\partial_\mu\chi\,. \qquad (26)$$

Hence, from (25) we have

$$[(A-1)\partial_\nu^2 - Bm^2]\partial_\mu\chi = 0\,. \qquad (27)$$

If $A-1=B$ we have the spin-0 particles with masses $\pm m$ with the correct relativistic dispersion.

In another case

$$[\delta_{\mu\nu}\delta_{\alpha\beta} - \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}]\partial_\alpha\partial_\beta B_\nu = \partial^2 B_\mu\,. \qquad (28)$$

Hence,

$$[(A+1)\partial_\nu^2 - Bm^2]B_\mu = 0\,. \qquad (29)$$

If $A+1=B$ we have the spin-1 particles with masses $\pm m$ with the correct relativistic dispersion.

The equation (25) can be transformed in two equations:

$$\left[\gamma_{\alpha\beta}\partial_\alpha\partial_\beta + (B+1)\partial_\alpha^2 - Bm^2\right]_{\mu\nu} B_\nu = 0, \text{ spin 0 with} \pm m, \qquad (30)$$

$$\left[\gamma_{\alpha\beta}\partial_\alpha\partial_\beta + (B-1)\partial_\alpha^2 - Bm^2\right]_{\mu\nu} B_\nu = 0, \text{ spin 1 with} \pm m. \qquad (31)$$

The first one has the solution with spin 0 and masses $\pm m$. However, it has also the spin-1 solution with the *different* masses, $[\partial_\nu^2 + (B+1)\partial_\nu^2 - Bm^2]B_\mu = 0$:

$$\tilde{m} = \pm\sqrt{\frac{B}{B+2}}m\,. \qquad (32)$$

The second one has the solution with spin 1 and masses $\pm m$. But, it also has the *spin-0* solution with the *different* masses, $[-\partial_\nu^2 + (B-1)\partial_\nu^2 - Bm^2]\partial_\mu\chi = 0$. So, $\tilde{m} = \pm\sqrt{\frac{B}{B-2}}m$. One can come to the same conclusion by checking the dispersion relations from $\mathrm{Det}[\gamma_{\alpha\beta}p_\alpha p_\beta - Ap_\alpha p_\alpha + Bm^2] = 0$. When

$\tilde{m}^2 = \frac{4}{3}m^2$, we have $B = -8, A = -7$, that is compatible with our consideration of bi-vector fields, Ref. [7]. Thus, one can form the Lagrangian with the particles of spines 1, masses $\pm m$, the particle with the mass $\sqrt{\frac{4}{3}}m$, spin 1, for which the particle is equal to the antiparticle, by choosing the appropriate creation/annihilation operators; and the particles with spines 0 with masses $\pm m$ and $\pm\sqrt{\frac{4}{5}}m$ (some of them may be neutral).

Energy-momentum tensor. According to Ref. [6], it is defined as

$$T_{\mu\nu} = -\sum_{\alpha}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu B_\alpha)}\partial_\nu B_\alpha + \partial_\nu B^*_\alpha \frac{\partial\mathcal{L}}{\partial(\partial_\mu B^*_\alpha)}\right] + \mathcal{L}\delta_{\mu\nu} \tag{33}$$

$$P_\mu = -i\int T_{4\mu} d^3\mathbf{x}\,. \tag{34}$$

$$\begin{aligned} T_{\mu\nu} &= -(\partial_\kappa B^*_\tau)[\gamma_{\kappa\mu}]_{\tau\alpha}(\partial_\nu B_\alpha) - (\partial_\nu B^*_\alpha)[\gamma_{\mu\kappa}]_{\alpha\tau}(\partial_\kappa B_\tau) \\ &\quad - A[(\partial_\mu B^*_\alpha)(\partial_\nu B_\alpha) + (\partial_\nu B^*_\alpha)(\partial_\mu B_\alpha)] + \mathcal{L}\delta_{\mu\nu} \\ &= -(A+1)[(\partial_\mu B^*_\alpha)(\partial_\nu B_\alpha) + (\partial_\nu B^*_\alpha)(\partial_\mu B_\alpha)] + \big[(\partial_\alpha B^*_\mu)(\partial_\nu B_\alpha) \\ &\quad + (\partial_\nu B^*_\alpha)(\partial_\alpha B_\mu)] + [(\partial_\alpha B^*_\alpha)(\partial_\nu B_\mu) + (\partial_\nu B^*_\mu)(\partial_\alpha B_\alpha)] + \mathcal{L}\delta_{\mu\nu}. \end{aligned} \tag{35}$$

Remember that after substitutions of the explicit forms of the γ's, the Lagrangian is

$$\begin{aligned} \mathcal{L} &= (A+1)(\partial_\alpha B^*_\mu)(\partial_\alpha B_\mu) - (\partial_\nu B^*_\mu)(\partial_\mu B_\nu) - (\partial_\mu B^*_\mu)(\partial_\nu B_\nu) \\ &\quad + Bm^2 B^*_\mu B_\mu\,, \end{aligned} \tag{36}$$

and the third term cannot be removed by the standard substitution $\mathcal{L} \to \mathcal{L}' + \partial_\mu \Gamma_\mu\,, \Gamma_\mu = B^*_\nu \partial_\nu B_\mu - B^*_\mu \partial_\nu B_\nu$ to get the textbook Lagrangian $\mathcal{L}' = (\partial_\alpha B^*_\mu)(\partial_\alpha B_\mu) + m^2 B^*_\mu B_\mu\,.$

The current vector is defined

$$J_\mu = -i\sum_\alpha\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu B_\alpha)}B_\alpha - B^*_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\mu B^*_\alpha)}\right], \tag{37}$$

$$Q = -i\int J_4 d^3\mathbf{x}\,. \tag{38}$$

$$\begin{aligned} J_\lambda &= -i\left\{(\partial_\alpha B^*_\mu)[\gamma_{\alpha\lambda}]_{\mu\kappa}B_\kappa - B^*_\kappa[\gamma_{\lambda\alpha}]_{\kappa\mu}(\partial_\alpha B_\mu)\right. \\ &\quad \left. + A(\partial_\lambda B^*_\kappa)B_\kappa - AB^*_\kappa(\partial_\lambda B_\kappa)\right\} \\ &= -i\left\{(A+1)[(\partial_\lambda B^*_\kappa)B_\kappa - B^*_\kappa(\partial_\lambda B_\kappa)] + [B^*_\kappa(\partial_\kappa B_\lambda) - (\partial_\kappa B^*_\lambda)B_\kappa]\right. \\ &\quad \left. + [B^*_\lambda(\partial_\kappa B_\kappa) - (\partial_\kappa B^*_\kappa)B_\lambda]\right\}\,. \end{aligned} \tag{39}$$

Again, the second term and the last term cannot be removed at the same time by adding the total derivative to the Lagrangian. These terms correspond to the contribution of the scalar (spin-0) portion.

Angular momentum. Finally,

$$\mathcal{M}_{\mu\alpha,\lambda} = x_\mu T_{\{\alpha\lambda\}} - x_\alpha T_{\{\mu\lambda\}} + \mathcal{S}_{\mu\alpha,\lambda} = x_\mu T_{\{\alpha\lambda\}} - x_\alpha T_{\{\mu\lambda\}} - i\left\{\sum_{\kappa\tau}\frac{\partial\mathcal{L}}{\partial(\partial_\lambda B_\kappa)}\mathcal{T}_{\mu\alpha,\kappa\tau}B_\tau + B^*_\tau\mathcal{T}_{\mu\alpha,\kappa\tau}\frac{\partial\mathcal{L}}{\partial(\partial_\lambda B^*_\kappa)}\right\} \quad (40)$$

$$\mathcal{M}_{\mu\nu} = -i\int\mathcal{M}_{\mu\nu,4}d^3\mathbf{x}\,, \quad (41)$$

where $\mathcal{T}_{\mu\alpha,\kappa\tau} \sim [\gamma_{5,\mu\alpha}]_{\kappa\tau}$.

The field operator. Various-type field operators are possible in this representation. Let us remind the textbook procedure to get them. During the calculations below we have to present $1 = \theta(k_0) + \theta(-k_0)$ in order to get positive- and negative-frequency parts.

$$\begin{aligned} A_\mu(x) &= \frac{1}{(2\pi)^3}\int d^4k\,\delta(k^2-m^2)e^{+ik\cdot x}A_\mu(k) \quad (42)\\ &= \frac{1}{(2\pi)^3}\int\frac{d^3\mathbf{k}}{2E_k}\theta(k_0)[A_\mu(k)e^{+ik\cdot x} + A_\mu(-k)e^{-ik\cdot x}]\\ &= \frac{1}{(2\pi)^3}\sum_\lambda\int\frac{d^3\mathbf{k}}{2E_k}[\epsilon_\mu(k,\lambda)a_\lambda(k)e^{+ik\cdot x} + \epsilon_\mu(-k,\lambda)a_\lambda(-k)e^{-ik\cdot x}]\,. \end{aligned}$$

Moreover, we should transform the second part to $\epsilon^*_\mu(k,\lambda)b^\dagger_\lambda(k)$ as usual. In such a way we obtain the charge-conjugate states. Of course, one can try to get P-conjugates or CP-conjugate states too. We set

$$\sum_\lambda\epsilon_\mu(-k,\lambda)a_\lambda(-k) = \sum_\lambda\epsilon^*_\mu(k,\lambda)b^\dagger_\lambda(k)\,, \quad (43)$$

multiply both parts by $\epsilon_\nu[\gamma_{44}]_{\nu\mu}$, and use the normalization conditions for polarization vectors.

In the $(\frac{1}{2},\frac{1}{2})$ representation we can also expand the second term in the different way:

$$\sum_\lambda\epsilon_\mu(-k,\lambda)a_\lambda(-k) = \sum_\lambda\epsilon_\mu(k,\lambda)a_\lambda(k)\,. \quad (44)$$

From the first definition we obtain (the signs $\mp$ depends on the value of σ):

$$b^\dagger_\sigma(k) = \mp\sum_{\mu\nu\lambda}\epsilon_\nu(k,\sigma)[\gamma_{44}]_{\nu\mu}\epsilon_\mu(-k,\lambda)a_\lambda(-k)\,, \quad (45)$$

The second definition is $\Lambda^2_{\sigma\lambda} = \mp \sum_{\nu\mu} \epsilon^*_\nu(k,\sigma)[\gamma_{44}]_{\nu\mu}\epsilon_\mu(-k,\lambda)$. The field operator will only destroy particles.

Propagators. From Ref. [21] it is known for the real vector field:

$$< 0|T(B_\mu(x)B_\nu(y)|0 > = \\ -i\int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} (\frac{\delta_{\mu\nu} + k_\mu k_\nu/\mu^2}{k^2+\mu^2+i\epsilon} - \frac{k_\mu k_\nu/\mu^2}{k^2+m^2+i\epsilon}) \,. \quad (46)$$

If $\mu = m$ (this depends on relations between A and B) we have the cancelation of divergent parts. Thus, we can overcome the well-known difficulty of the Proca theory with the massless limit.

If $\mu \neq m$ we can still have a *causal* theory, but in this case we need more than one equation, and should apply the method proposed in Ref. [11]. The reasons were that the Weinberg equation propagates both causal and tachyonic solutions.

Indefinite metrics. Usually, one considers the hermitian field operator in the pseudo-Euclidean metric for the electromagnetic potential:

$$A_\mu = \sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3 2E_k} [\epsilon_\mu(k,\lambda)a_\lambda(\mathbf{k}) + \epsilon^*_\mu(k,\lambda)a^\dagger_\lambda(\mathbf{k})] \quad (47)$$

with *all* four polarizations to be independent ones. Next, one introduces the Lorentz condition in the weak form

$$[a_{0_t}(\mathbf{k}) - a_0(\mathbf{k})]|\phi >= 0 \quad (48)$$

and the indefinite metrics in the Fock space, Ref. [20]: $a^*_{0_t} = -a_{0_t}$ and $\eta a_\lambda = -a^\lambda\eta$, $\eta^2 = 1$, in order to get the correct sign in the energy-momentum vector and to not have the problem with the vacuum average.

We observe:

1) that the indefinite metric problems may appear even on the massive level in the Stueckelberg formalism;
2) The Stueckelberg theory has a good massless limit for propagators, and it reproduces the handling of the indefinite metric in the massless limit (the electromagnetic 4-potential case);
3) we generalized the Stueckelberg formalism (considering, at least, two equations); instead of charge-conjugate solutions we may consider the $P-$ or $CP-$ conjugates. The potential field becomes to be the complex-valued field, that may justify the introduction of the anti-hermitian amplitudes.

4. The Spin-2 Case

The general scheme for derivation of higher-spin equations was given in [5]. A field of rest mass m and spin $j \geq \frac{1}{2}$ is represented by a completely symmetric multispinor of rank $2j$. The particular cases $j = 1$ and $j = \frac{3}{2}$ were given in the textbooks, e. g., ref. [6]. The spin-2 case can also be of some interest because it is generally believed that the essential features of the gravitational field are obtained from transverse components of the $(2,0) \oplus (0,2)$ representation of the Lorentz group. Nevertheless, questions of the redandant components of the higher-spin relativistic equations are not yet understood in detail.

In this section we use the commonly-accepted procedure for the derivation of higher-spin equations. We begin with the equations for the 4-rank symmetric spinor:

$$[i\gamma^\mu \partial_\mu - m]_{\alpha\alpha'} \Psi_{\alpha'\beta\gamma\delta} = 0\, , [i\gamma^\mu \partial_\mu - m]_{\beta\beta'} \Psi_{\alpha\beta'\gamma\delta} = 0 \tag{49}$$

$$[i\gamma^\mu \partial_\mu - m]_{\gamma\gamma'} \Psi_{\alpha\beta\gamma'\delta} = 0\, , [i\gamma^\mu \partial_\mu - m]_{\delta\delta'} \Psi_{\alpha\beta\gamma\delta'} = 0. \tag{50}$$

The massless limit (if one needs) should be taken in the end of all calculations.

We proceed expanding the field function in the set of symmetric matrices (as in the spin-1 case). The total function is

$$\Psi_{\{\alpha\beta\}\{\gamma\delta\}} = (\gamma_\mu R)_{\alpha\beta} (\gamma^\kappa R)_{\gamma\delta} G_\kappa{}^{\;\mu} + (\gamma_\mu R)_{\alpha\beta} (\sigma^{\kappa\tau} R)_{\gamma\delta} F_{\kappa\tau}{}^{\mu} + (\sigma_{\mu\nu} R)_{\alpha\beta} (\gamma^\kappa R)_{\gamma\delta} T_\kappa{}^{\;\mu\nu} + (\sigma_{\mu\nu} R)_{\alpha\beta} (\sigma^{\kappa\tau} R)_{\gamma\delta} R_{\kappa\tau}{}^{\mu\nu}; \tag{51}$$

and the resulting tensor equations are:

$$\frac{2}{m}\partial_\mu T_\kappa{}^{\;\mu\nu} = -G_\kappa{}^{\;\nu}\, , \frac{2}{m}\partial_\mu R_{\kappa\tau}{}^{\mu\nu} = -F_{\kappa\tau}{}^{\nu}\, , \tag{52}$$

$$T_\kappa{}^{\;\mu\nu} = \frac{1}{2m}\left[\partial^\mu G_\kappa{}^{\;\nu} - \partial^\nu G_\kappa{}^{\;\mu}\right]\, , \tag{53}$$

$$R_{\kappa\tau}{}^{\mu\nu} = \frac{1}{2m}\left[\partial^\mu F_{\kappa\tau}{}^{\nu} - \partial^\nu F_{\kappa\tau}{}^{\mu}\right]\, . \tag{54}$$

The constraints are re-written to

$$\frac{1}{m}\partial_\mu G_\kappa{}^{\;\mu} = 0\, , \quad \frac{1}{m}\partial_\mu F_{\kappa\tau}{}^{\mu} = 0\, , \tag{55}$$

$$\frac{1}{m}\epsilon_{\alpha\beta\nu\mu}\partial^\alpha T_\kappa{}^{\;\beta\nu} = 0\, , \quad \frac{1}{m}\epsilon_{\alpha\beta\nu\mu}\partial^\alpha R_{\kappa\tau}{}^{\beta\nu} = 0\, . \tag{56}$$

However, we need to make symmetrization over these two sets of indices $\{\alpha\beta\}$ and $\{\gamma\delta\}$. The total symmetry can be ensured if one contracts the function $\Psi_{\{\alpha\beta\}\{\gamma\delta\}}$ with *antisymmetric* matrices $R^{-1}_{\beta\gamma}$, $(R^{-1}\gamma^5)_{\beta\gamma}$ and $(R^{-1}\gamma^5\gamma^\lambda)_{\beta\gamma}$ and equate all these contractions to zero (similar to the $j = 3/2$ case considered in ref. [6, p. 44]. We encountered with the known difficulty of the theory for spin-2 particles in the Minkowski space. We explicitly showed that all field functions become to be equal to zero. Such a situation cannot be considered as a satisfactory one (because it does not give us any physical information) and can be corrected in several ways. We modified the formalism [12]. The field function is now presented as

$$\Psi_{\{\alpha\beta\}\gamma\delta} = \alpha_1(\gamma_\mu R)_{\alpha\beta}\Psi^{\mu}_{\gamma\delta} + \alpha_2(\sigma_{\mu\nu} R)_{\alpha\beta}\Psi^{\mu\nu}_{\gamma\delta} + \alpha_3(\gamma^5\sigma_{\mu\nu} R)_{\alpha\beta}\widetilde{\Psi}^{\mu\nu}_{\gamma\delta}\,, \quad (57)$$

with

$$\Psi^{\mu}_{\{\gamma\delta\}} = \beta_1(\gamma^\kappa R)_{\gamma\delta} G_\kappa{}^{\ \mu} + \beta_2(\sigma^{\kappa\tau} R)_{\gamma\delta} F_{\kappa\tau}{}^{\mu} + \beta_3(\gamma^5\sigma^{\kappa\tau} R)_{\gamma\delta}\widetilde{F}_{\kappa\tau}{}^{\mu}, \quad (58)$$

$$\Psi^{\mu\nu}_{\{\gamma\delta\}} = \beta_4(\gamma^\kappa R)_{\gamma\delta} T_\kappa{}^{\ \mu\nu} + \beta_5(\sigma^{\kappa\tau} R)_{\gamma\delta} R_{\kappa\tau}{}^{\mu\nu} + \beta_6(\gamma^5\sigma^{\kappa\tau} R)_{\gamma\delta}\widetilde{R}_{\kappa\tau}{}^{\mu\nu}, \quad (59)$$

$$\widetilde{\Psi}^{\mu\nu}_{\{\gamma\delta\}} = \beta_7(\gamma^\kappa R)_{\gamma\delta}\widetilde{T}_\kappa{}^{\ \mu\nu} + \beta_8(\sigma^{\kappa\tau} R)_{\gamma\delta}\widetilde{D}_{\kappa\tau}{}^{\mu\nu} + \beta_9(\gamma^5\sigma^{\kappa\tau} R)_{\gamma\delta} D_{\kappa\tau}{}^{\mu\nu}. \quad (60)$$

Hence, the function $\Psi_{\{\alpha\beta\}\{\gamma\delta\}}$ can be expressed as a sum of nine terms:

$$\begin{aligned}\Psi_{\{\alpha\beta\}\{\gamma\delta\}} &= \alpha_1\beta_1(\gamma_\mu R)_{\alpha\beta}(\gamma^\kappa R)_{\gamma\delta} G_\kappa{}^{\ \mu} + \alpha_1\beta_2(\gamma_\mu R)_{\alpha\beta}(\sigma^{\kappa\tau} R)_{\gamma\delta} F_{\kappa\tau}{}^{\mu}\\ &+ \alpha_1\beta_3(\gamma_\mu R)_{\alpha\beta}(\gamma^5\sigma^{\kappa\tau} R)_{\gamma\delta}\widetilde{F}_{\kappa\tau}{}^{\mu} + \alpha_2\beta_4(\sigma_{\mu\nu} R)_{\alpha\beta}(\gamma^\kappa R)_{\gamma\delta} T_\kappa{}^{\ \mu\nu}\\ &+ \alpha_2\beta_5(\sigma_{\mu\nu} R)_{\alpha\beta}(\sigma^{\kappa\tau} R)_{\gamma\delta} R_{\kappa\tau}{}^{\mu\nu} + \alpha_2\beta_6(\sigma_{\mu\nu} R)_{\alpha\beta}(\gamma^5\sigma^{\kappa\tau} R)_{\gamma\delta}\widetilde{R}_{\kappa\tau}{}^{\mu\nu}\\ &+ \alpha_3\beta_7(\gamma^5\sigma_{\mu\nu} R)_{\alpha\beta}(\gamma^\kappa R)_{\gamma\delta}\widetilde{T}_\kappa{}^{\ \mu\nu} + \alpha_3\beta_8(\gamma^5\sigma_{\mu\nu} R)_{\alpha\beta}(\sigma^{\kappa\tau} R)_{\gamma\delta}\widetilde{D}_{\kappa\tau}{}^{\mu\nu}\\ &+ \alpha_3\beta_9(\gamma^5\sigma_{\mu\nu} R)_{\alpha\beta}(\gamma^5\sigma^{\kappa\tau} R)_{\gamma\delta} D_{\kappa\tau}{}^{\mu\nu}\,. \end{aligned} \quad (61)$$

The corresponding dynamical equations are given by the set

$$\frac{2\alpha_2\beta_4}{m}\partial_\nu T_\kappa{}^{\ \mu\nu} + \frac{i\alpha_3\beta_7}{m}\epsilon^{\mu\nu\alpha\beta}\partial_\nu \widetilde{T}_{\kappa,\alpha\beta} = \alpha_1\beta_1 G_\kappa{}^{\ \mu}\,; \quad (62)$$

$$\begin{aligned}&\frac{2\alpha_2\beta_5}{m}\partial_\nu R_{\kappa\tau}{}^{\mu\nu} + \frac{i\alpha_2\beta_6}{m}\epsilon_{\alpha\beta\kappa\tau}\partial_\nu \widetilde{R}^{\alpha\beta,\mu\nu} + \frac{i\alpha_3\beta_8}{m}\epsilon^{\mu\nu\alpha\beta}\partial_\nu \widetilde{D}_{\kappa\tau,\alpha\beta} -\\ &- \frac{\alpha_3\beta_9}{2}\epsilon^{\mu\nu\alpha\beta}\epsilon_{\lambda\delta\kappa\tau} D^{\lambda\delta}{}_{\alpha\beta} = \alpha_1\beta_2 F_{\kappa\tau}{}^{\mu} + \frac{i\alpha_1\beta_3}{2}\epsilon_{\alpha\beta\kappa\tau}\widetilde{F}^{\alpha\beta,\mu}\,; \end{aligned} \quad (63)$$

$$2\alpha_2\beta_4 T_\kappa{}^{\ \mu\nu} + i\alpha_3\beta_7\epsilon^{\alpha\beta\mu\nu}\widetilde{T}_{\kappa,\alpha\beta} = \frac{\alpha_1\beta_1}{m}(\partial^\mu G_\kappa{}^{\ \nu} - \partial^\nu G_\kappa{}^{\ \mu})\,; \quad (64)$$

$$\begin{aligned}&2\alpha_2\beta_5 R_{\kappa\tau}{}^{\mu\nu} + i\alpha_3\beta_8\epsilon^{\alpha\beta\mu\nu}\widetilde{D}_{\kappa\tau,\alpha\beta} + i\alpha_2\beta_6\epsilon_{\alpha\beta\kappa\tau}\widetilde{R}^{\alpha\beta,\mu\nu} -\\ &- \frac{\alpha_3\beta_9}{2}\epsilon^{\alpha\beta\mu\nu}\epsilon_{\lambda\delta\kappa\tau} D^{\lambda\delta}{}_{\alpha\beta} =\\ &= \frac{\alpha_1\beta_2}{m}(\partial^\mu F_{\kappa\tau}{}^{\nu} - \partial^\nu F_{\kappa\tau}{}^{\mu}) + \frac{i\alpha_1\beta_3}{2m}\epsilon_{\alpha\beta\kappa\tau}(\partial^\mu \widetilde{F}^{\alpha\beta,\nu} - \partial^\nu \widetilde{F}^{\alpha\beta,\mu})\,. \end{aligned} \quad (65)$$

The essential constraints can be found in Ref. [24]. They are the results of contractions of the field function (61) with three antisymmetric matrices, as above. Furthermore, one should recover the above relations in the particular case when $\alpha_3 = \beta_3 = \beta_6 = \beta_9 = 0$ and $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \beta_4 = \beta_5 = \beta_7 = \beta_8 = 1$.

As a discussion we note that in such a framework we already have physical content because only certain combinations of field functions would be equal to zero. In general, the fields $F_{\kappa\tau}{}^{\mu}$, $\widetilde{F}_{\kappa\tau}{}^{\mu}$, $T_{\kappa}{}^{\mu\nu}$, $\widetilde{T}_{\kappa}{}^{\mu\nu}$, and $R_{\kappa\tau}{}^{\mu\nu}$, $\widetilde{R}_{\kappa\tau}{}^{\mu\nu}$, $D_{\kappa\tau}{}^{\mu\nu}$, $\widetilde{D}_{\kappa\tau}{}^{\mu\nu}$ can correspond to different physical states and the equations above describe oscillations one state to another. Furthermore, from the set of equations (62-65) one obtains the *second*-order equation for symmetric traceless tensor of the second rank ($\alpha_1 \neq 0$, $\beta_1 \neq 0$):

$$\frac{1}{m^2}\left[\partial_\nu \partial^\mu G_\kappa{}^{\nu} - \partial_\nu \partial^\nu G_\kappa{}^{\mu}\right] = G_\kappa{}^{\mu} \,. \tag{66}$$

After the contraction in indices κ and μ this equation is reduced to the set

$$\partial_\mu G^\mu{}_\kappa = F_\kappa \,, \tag{67}$$

$$\frac{1}{m^2}\partial_\kappa F^\kappa = 0 \,, \tag{68}$$

i. e., to the equations connecting the analogue of the energy-momentum tensor and the analogue of the 4-vector potential. Further investigations may provide additional foundations to "surprising" similarities of gravitational and electromagnetic equations in the low-velocity limit.

The questions of "non-commutativity" see in Ref. [23].

5. Conclusions

- The $(1/2, 1/2)$ representation contains both the spin-1 and spin-0 states (cf. with the Stueckelberg formalism).
- Unless we take into account the fourth state (the "time-like" state, or the spin-0 state) the set of 4-vectors is *not* a complete set in a mathematical sense.
- We cannot remove terms like $(\partial_\mu B^*_\mu)(\partial_\nu B_\nu)$ terms from the Lagrangian and dynamical invariants unless apply the Fermi method, i. e., manually. The Lorentz condition applies only to the spin 1 states.
- We have some additional terms in the expressions of the energy-momentum vector (and, accordingly, of the 4-current and the Pauli-Lunbanski vectors), which are the consequence of the impossibility to apply the Lorentz condition for spin-0 states.
- Helicity vectors are not eigenvectors of the parity operator. Meanwhile, the parity is a "good" quantum number, $[\mathcal{P}, \mathcal{H}]_- = 0$ in the Fock space.
- We are able to describe the states of different masses in this representation from the beginning.
- Various-type field operators can be constructed in the $(1/2, 1/2)$ representation space. For instance, they can contain C, P and CP conjugate states. Even if $b^\dagger_\lambda = a^\dagger_\lambda$ we can have complex 4-vector fields. We found the relations between creation, annihilation operators for different types of the field operators B_μ.

- Propagators have good behavior in the massless limit as opposed to those of the Proca theory.
- The spin-2 case can be considered on an equal footing with the spin-1 case.

Acknowledgments. I am grateful to Profs. Y. S. Kim, S. I. Kruglov, W. Rodrigues, R. Yamaleev and participants of the recent conferences for useful discussions.

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