PTEP

Prog. Theor. Exp. Phys. **2022** 113B01(23 pages)
DOI: 10.1093/ptep/ptac126

Holographic index calculation for Argyres–Douglas and Minahan–Nemeschansky theories

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Received July 29, 2022; Accepted August 31, 2022; Published September 17, 2022

We calculate the superconformal indices of the $\mathcal{N} = 2$ superconformal field theories realized on N coincident D3-branes in 7-brane backgrounds with constant axiodilaton via the anti-de Sitter/conformal field theory correspondence. We include the finite-N corrections as the contribution of D3-branes wrapped around 3-cycles in the internal space. We take only single-wrapping contributions into account for simplicity. We also determine the orders of the next-to-leading corrections that we do not calculate. The orders are relatively high, and we obtain many trustworthy terms. We give the results for $N = 1, 2, 3$ explicitly, and find nice agreement with known results.

Subject Index B10, B21, B23

1. Introduction

The recent development of quantum field theory owes a lot to brane realization in string theory and M-theory. Various theories are realized as theories on branes placed in appropriate backgrounds. Such theories are often strongly coupled, and many of them do not have known Lagrangian descriptions. The brane realization provides non-perturbative methods to analyze such theories. One of the most effective methods is the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1].

If a superconformal field theory (SCFT) is realized as the theory on N coincident branes, the brane system is well described by supergravity in the large-N limit, and we can extract physical information about the SCFT by studying the classical solution. For small N the quantum gravity correction is expected to be important, and in general qualitative analysis becomes difficult. Even so, it has been proposed in Ref. [2] that the finite-N correction to the superconformal index [3] can be calculated as the contribution of giant gravitons [4,5] without taking account of quantum gravity.

In this paper we investigate 4D $\mathcal{N} = 2$ supersymmetric theories realized on D3-branes put in 7-brane backgrounds with constant axiodilaton [6–9]. We denote a theory in this class by $G[N]$, where $G = H_0, H_1, H_2, D_4, E_6, E_7, E_8$ is the type of 7-brane and N, which is called the rank of the theory, is the number of D3-branes. $H_n[N]$ $(n = 0, 1, 2)$ are examples of a large class of

Funded by SCOAP³. SCOAP3 and OUP support the goals of the International Year of Basic Sciences for Sustainable Development

Table 1. The brane setup.

	0	1	2	3	X		Y		Z	
7-brane	○	○	○	○	○	○	○	○		
D3-branes	○	○	○	○						

Table 2. The deficit angle parameters α_G of the 7-brane and the dimensions Δ_G of Coulomb branch operators.

G	H_0	H_1	H_2	D_4	E_6	E_7	E_8
α_G	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$
Δ_G	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	2	3	4	6

theories called Argyres–Douglas theories [10,11] and $E_n[N]$ ($n = 6, 7, 8$) are called Minahan–Nemeschansky theories [12,13]. $D_4[N]$ is an SQCD with the gauge group $Sp(N)$.

Let X, Y, and Z be the three complex coordinates of $\mathbb{C}^3$ transverse to the N D3-branes. The 7-brane is placed at $Z = 0$ (Table 1). The global symmetry of the SCFT is $SU(2, 2|2) \times G \times SU(2)_F$, and its maximal compact bosonic subgroup is

$$U(1)_H \times SU(2)_J \times SU(2)_{\overline{J}} \times SU(2)_R \times U(1)_{R_Z} \times G \times SU(2)_F. \tag{1}$$

The subscripts of the F$U(1)$ and $SU(2)$ factors are generators of the factors. H is the Hamiltonian and J and $\overline{J}$ are the angular momenta. $U(1)_{R_Z}$ rotates the Z-plane, and $SU(2)_R \times SU(2)_F$ rotates $\mathbb{C}^2$ spanned by X and Y. We also define $R_X = R + F$ and $R_Y = R - F$ acting on the X- and Y-planes for later convenience. G is the gauge symmetry realized on the 7-brane, and $G = H_0$, H_1, and H_2 are regarded as the trivial group, $SU(2)$, and $SU(3)$, respectively, as symmetry groups. In addition, let $U(1)_A$ be the R-symmetry of the type IIB supergravity [14], which is broken to its discrete subgroup due to the flux quantization.

The presence of the 7-brane induces the deficit angle $\pi\alpha_G$ on the Z-plane shown in Table 2, and the Z-plane is restricted by

$$0 \leq \arg Z \leq \pi(2 - \alpha_G). \tag{2}$$

The two boundary rays, $Z = r$ and $Z = re^{\pi i(2-\alpha_G)}$ ($r \in \mathbb{R}_{\geq 0}$), are identified by the boundary condition

$$\mathcal{O}(r) = \mathcal{O}(e^{\pi i(2-\alpha_G)}r) \equiv U_{\alpha_G}\mathcal{O}(r)U_{\alpha_G}^{-1}, \quad U_{\alpha_G} = e^{\pi i(2-\alpha_G)(R_Z - \frac{1}{2}A)}, \tag{3}$$

for an arbitrary local operator $\mathcal{O}(Z)$. R_Z and A are normalized so that the supercharges carry $R_Z = \pm\frac{1}{2}$ and $A = \pm 1$,[1] and the $\mathcal{N} = 2$ supersymmetry is generated by supercharges with $R_Z - \frac{1}{2}A = 0$. The globally defined coordinate Z^{Δ_G} corresponds to the Coulomb branch operator with dimension $\Delta_G = 2/(2 - \alpha_G)$. If $G = D_4$, E_6, E_7, E_8 then Δ_G is an integer, and the identification (3) can be regarded as the orbifolding by $\mathbb{Z}_{\Delta_G}$ generated by U_{α_G}.

The system always contains a free hypermultiplet corresponding to the "center of mass" degrees of freedom along the X–Y directions. We exclude it from $G[N]$. Then, if $N = 1$, the $SU(2)_F$ becomes ineffective and the flavor symmetry $G \times SU(2)_F$ reduces to G. This fact will be useful in the analysis of rank-one theories in Sect. 3.

[1] Our normalization convention for generators can be read off from Eq. (6).

The BPS operator spectrum of $G[\infty]$ is studied in Refs. [15] and [16] by using the AdS/CFT correspondence. The geometry is $AdS_5 \times S^5_{\alpha_G}$ where $S^5_{\alpha_G}$ is the singular space defined from S^5 by the restriction (2) and the identification (3). The 7-brane worldvolume is $AdS_5 \times S^3$, where S^3 is the singular locus of $S^5_{\alpha_G}$. In the orbifold case considered in Ref. [15] the internal space is $S^5_{\alpha_G} = S^5/\mathbb{Z}_{\Delta_G}$, and the Kaluza–Klein modes in the orbifold are obtained from those in S^5 by picking up $\mathbb{Z}_{\Delta_G}$ invariant modes, whose quantum numbers satisfy

$$R_z - \frac{1}{2}A \in \Delta_G\mathbb{Z}. \tag{4}$$

The analysis in Ref. [15] was extended in Ref. [16] in two ways: The non-orbifold cases with $G = H_0, H_1, H_2$ were included by allowing fractional values of Δ_G in Eq. (4), and fields living on the 7-brane were taken into account to generate the spectrum of operators transformed non-trivially under G. The comparison with known results found nice agreement.

The purpose of this paper is to extend these analyses to $G[N]$ with finite N. We use the superconformal index [3]

$$\begin{aligned}\mathcal{I} &= \mathrm{tr}\left[e^{2\pi i(J+\bar{J})}q^{H+\bar{J}}y^{2J}u_x^{R_X}u_y^{R_Y}u_z^{R_Z}\prod_{i=1}^{\mathrm{rank}\,G}x_i^{p_i}\right] \quad (u_xu_yu_z=1)\\ &= \mathrm{tr}\left[e^{2\pi i(J+\bar{J})}q^{H+\bar{J}}y^{2J}u_z^{R_Z-R}u^{2F}\prod_{i=1}^{\mathrm{rank}\,G}x_i^{p_i}\right] \quad \left(u=\sqrt{\frac{u_x}{u_y}}\right)\end{aligned} \tag{5}$$

to express the BPS spectrum concisely, where p_i are Cartan generators of G. This index is associated with the supercharge $\mathcal{Q}$ carrying the quantum numbers

$$(H, J, \bar{J}, R_X, R_Y, R_Z, A) = \left(+\tfrac{1}{2}, 0, -\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2}, +1\right), \tag{6}$$

and is contributed by operators saturating the corresponding bound

$$\{\mathcal{Q}, \mathcal{Q}^\dagger\} = H - 2\bar{J} - R_X - R_Y - R_Z \geq 0. \tag{7}$$

The large-N index is given on the AdS side by

$$\mathcal{I}_G^{\mathrm{KK}} = \mathrm{Pexp}\left(i_G^{\mathrm{grav}} + i_G^{\mathrm{vec}} - i_{\mathrm{hyp}}\right). \tag{8}$$

The plethystic exponential Pexp is defined by

$$\mathrm{Pexp}\sum_s c_s f_s = \prod_s (1-f_s)^{-c_s}, \tag{9}$$

where f_s are monomials of fugacities and c_s are numerical coefficients.

i_G^{vec} is the single-particle index of the gauge multiplet on the 7-brane. The mode expansion of the vector multiplet on the 7-brane gives the representation [16]

$$\bigoplus_{l=0}^{\infty}\mathcal{B}_{\frac{l+2}{2},0(0,0)}\otimes[\tfrac{l}{2}]_F\otimes R_\theta^G. \tag{10}$$

We denote the irreducible G representation with the highest weight w by R_w^G, and R_θ^G is the adjoint representation. The corresponding single-particle index is

$$i_G^{\mathrm{vec}} = \frac{q^2u_xu_y - q^3}{(1-q^{\frac{3}{2}}y)(1-q^{\frac{3}{2}}y^{-1})(1-qu_x)(1-qu_y)}\chi_\theta^G, \tag{11}$$

where χ_θ^G is the character of R_θ^G. The explicit form of i_{hyp}, the contribution of the center-of-mass hypermultiplet, is

$$i_{\mathrm{hyp}} = \frac{q - q^2u_z}{(1-q^{\frac{3}{2}}y)(1-q^{\frac{3}{2}}y^{-1})}(u_x+u_y). \tag{12}$$

$i_G^{\rm grav}$, the contribution of the gravity multiplet in the bulk, will be calculated in the following sections.

For finite N, the index deviates from Eq. (8) at some order of the q expansion, and we are interested in this finite-N correction. We use the method used in Refs. [2,17,18], which reproduces the finite-N correction to the index of $\mathcal{N}=4$ SYM as the contribution of D3-branes wrapped on the internal space. It has also been applied to more general theories [19–22]. For a large class of 4D gauge theories with the dual geometry $AdS_5\times M$ the index is given by

$$\mathcal{I}=\mathcal{I}^{\rm KK}\sum_{\vec{n}}\mathcal{I}_{\vec{n}}, \tag{13}$$

where $\mathcal{I}^{\rm KK}$ is the index of massless fields in $AdS\times M$, and is given by Eq. (8) for the system of interest. $\mathcal{I}_{\vec{n}}$ is the contribution of wrapped D3-branes in the internal space M. $\vec{n}=(n_1,\dots,n_d)$ are wrapping numbers around appropriately chosen supersymmetric cycles in M. $\mathcal{I}_{\vec{n}}$ is the index of the brane system specified by $\vec{n}$, and is the product of two factors. One is the classical factor $\mathcal{I}_{\vec{n}}^{\rm cl}$ determined by the energy and charges of the wrapped branes without excitation of fields on them. The other factor is the index of the field theory realized on the brane system. For $\vec{n}=\vec{0}=(0,\dots,0)$ there are no wrapped branes and $\mathcal{I}_{\vec{0}}=1$. In the large-N limit the other contributions decouple and the formula reduces to the large-N relation $\mathcal{I}=\mathcal{I}^{\rm KK}$, while for finite N $\mathcal{I}_{\vec{n}}$ with $\sum_I n_I\geq 1$ give finite-N corrections.

In the case of $\mathcal{N}=4$ $U(N)$ SYM we use three three-cycles in S^5, defined by $X=0$, $Y=0$, and $Z=0$, and $\vec{n}=(n_x,n_y,n_z)$ [2,18]. Although S^5 is replaced by $S^5_{\alpha_G}$ in the system of interest, we assume that the same choice of the three-cycles still works.

The classical factor is related to the volumes of the cycles. In the case of $AdS_5\times S^5$ it is given as follows. Let $\Delta_{X=0}$ be the dimension of the operator corresponding to a D3-brane wrapped on the cycle $X=0$. $\Delta_{Y=0}$ and $\Delta_{Z=0}$ are also defined. The RR flux N and the AdS_5 radius L are related by $2\pi^2T_{\rm D3}L^4=N$. According to the dictionary of the AdS/CFT correspondence, this means $\Delta_{X=0}=\Delta_{Y=0}=\Delta_{Z=0}=N$, and a wrapped brane contributes $\sim q^N$ to the index. If there are n such branes, the corresponding term is $\sim q^{nN}$. By taking account of the R-charges carried by wrapped branes we obtain the classical factor $\mathcal{I}^{\rm cl}_{(n_x,n_y,n_z)}=(qu_x)^{Nn_x}(qu_y)^{Nn_y}(qu_z)^{Nn_z}$. If S^5 is replaced by $S^5_{\alpha_G}$ the volumes of the cycles are not all the same but are given by

$$\Delta_{X=0}=\Delta_{Y=0}=N,\quad \Delta_{Z=0}=\Delta_G N. \tag{14}$$

Correspondingly, the classical factor becomes

$$\mathcal{I}^{\rm cl}_{(n_x,n_y,n_z)}=(qu_x)^{n_xN}(qu_y)^{n_yN}(qu_z)^{n_z\Delta_GN}. \tag{15}$$

Because $\Delta_G>1$ for all G, the leading correction is given by $\mathcal{I}_{(1,0,0)}$ and $\mathcal{I}_{(0,1,0)}$, which are both of the order of $\sim q^N$. In this paper we focus only on these two corrections for simplicity, and calculate the approximate finite-N index by

$$\mathcal{I}^{\rm AdS}_{G[N]}=\mathcal{I}^{\rm KK}_G(1+\mathcal{I}_{(1,0,0)}+\mathcal{I}_{(0,1,0)}). \tag{16}$$

We use the superscripts "AdS" to indicate that the results are obtained by this equation.

Equation (16) is not exact but there must be an error due to other contributions neglected in Eq. (16):

$$\mathcal{I}_{G[N]}=\mathcal{I}^{\rm AdS}_{G[N]}+(\text{error}). \tag{17}$$

The source of the next-to-leading corrections determining the order of the error term in Eq. (17) depends on G. For $G=H_n$ the next-to-leading correction is $\mathcal{I}_{(0,0,1)}\sim q^{\Delta_GN}$, while for G

$= E_n$ the next-to-leading corrections are $\mathcal{I}_{(2,0,0)} \sim \mathcal{I}_{(1,1,0)} \sim \mathcal{I}_{(0,2,0)} \sim q^{2N}$. The $G = D_4$ case is marginal and all these contributions give corrections of the same order.

In the order estimation above we looked at only the classical factor (15) and neglected the contribution of the field theory on the branes. The latter often starts from a positive order term $\propto q^{\delta_G}$ ($\delta_G > 0$), and the order of $\mathcal{I}_{\vec{n}}$ is raised by δ_G. We will call this the "tachyonic shift" for reasons explained shortly. With the tachyonic shift the expected order of the error term is given by

$$H_n : \mathcal{O}(q^{\Delta_G N+\delta_G}), \quad D_4, E_n : \mathcal{O}(q^{2N+\delta_G}). \tag{18}$$

Let us briefly explain how to calculate $\mathcal{I}_{(1,0,0)}$ and $\mathcal{I}_{(0,1,0)}$. These two contributions are related by the Weyl reflection $u_x \leftrightarrow u_y$, and we consider $\mathcal{I}_{(1,0,0)}$, the contribution of a D3-brane wrapped on $X = 0$. The theory realized on the brane is the supersymmetric $U(1)$ gauge theory with the defect along the intersection with the 7-brane. The contribution of the vector multiplet is given by $\mathrm{Pexp}\, i_G^{\mathrm{D3}(X=0)}$, where the single-particle index $i_G^{\mathrm{D3}(X=0)}$ can be calculated by the mode analysis on the wrapped D3-brane. The first few terms of the q-expansion of $i_G^{\mathrm{D3}(X=0)}$ are

$$i_G^{\mathrm{D3}(X=0)} = \frac{1}{qu_x} + \frac{u_y}{u_x} + \cdots. \tag{19}$$

A detailed derivation will be given in the following sections; see also Appendix B. The first term has a negative exponent of q, and corresponds to the tachyonic mode with negative energy. Such a mode exists because the cycle is topologically trivial and the wrapped brane can be continuously unwrapped. According to the definition (9) the plethystic exponential is

$$\mathrm{Pexp}\, i_G^{\mathrm{D3}(X=0)} = \frac{1}{1-\frac{1}{qu_x}} \frac{1}{1-\frac{u_y}{u_x}} \cdots = -\frac{qu_x}{1-qu_x} \frac{1}{1-\frac{u_y}{u_x}} \cdots. \tag{20}$$

The first factor corresponding to the tachyonic mode gives the positive power of q. This is the origin of the tachyonic shift in $\mathcal{I}_{(1,0,0)}$. Similar shifts are expected in higher-order contributions, too, and the δ_G in Eq. (18) are the tachyonic shifts in the next-to-leading corrections.

The defect contribution can be treated perturbatively only in the $G = D_4$ case. Then, it turns out that chiral fermions live on the defect and the Fock space forms the basic representation of the affine D_4 algebra. We calculate the defect contribution for $G \neq D_4$ on the assumption that the Fock space of the defect degrees of freedom is the basic representation of the affine G algebra.

By combining the classical factor $\mathcal{I}_{(1,0,0)} = (qu_x)^N$, the vector multiplet contribution (20), and the defect contribution $\chi^{\widehat{G}}(qu_y)$ we obtain

$$\mathcal{I}_{(1,0,0)} = (qu_x)^N \chi^{\widehat{G}}(qu_y)\, \mathrm{Pexp}\, i_G^{\mathrm{D3}(X=0)}. \tag{21}$$

This paper is organized as follows. In the next section we confirm that the method works well for the D_4 case. Namely, we compare Eq. (16) with the results on the gauge theory side and confirm that the order of the error behaves like Eq. (18). Then, we move on to more interesting $G \neq D_4$ cases. We analyze the rank-one theories $G[1]$ in Sect. 3. We find a good agreement with the known results in the literature and determine δ_G, which will be used in the next section to determine the orders of errors for $N \geq 2$. In Sect. 4 we calculate the indices for $N \geq 2$. Again, the results are consistent with the known results. Section 5 is devoted to the conclusions and discussion. The appendices include technical details and some explicit results.

Table 3. Field contents of the $Sp(N)$ SQCD. The anti-symmetric tensor representation of $Sp(N)$ is not irreducible for $N \geq 2$ but contains a singlet. We exclude the singlet hypermultiplet from the definition of $D_4[N]$.

	$Sp(N)$	$SU(2)_F$	D_4
vector	adj	**1**	**1**
hyper	anti-sym	**2**	**1**
hyper	fund	**1**	$\mathbf{8}_v$

2. $D_4[N]$

2.1 *Large-N limit*

A D_4 7-brane is an O7-plane accompanied by four D7-branes [6]. The field contents of the $Sp(N)$ gauge theory realized on the D3-branes are shown in Table 3. We can calculate the superconformal index on the gauge theory side by using the localization formula for an arbitrary N. For example, the result in the large-N limit is

$$\mathcal{I}_{D_4[\infty]} = 1 + q^2 \left(u_z^2 + \chi_2^F u_z^{-1} + u_z^{-1} \chi_{\mathbf{28}}^{D_4} \right) + q^{\frac{5}{2}} \left(-u_z \chi_1^J \right) + \cdots , \tag{22}$$

where χ_n^F and χ_n^J are $SU(2)_F$ and $SU(2)_J$ characters

$$\chi_n^F = u^n + u^{n-2} + \cdots + u^{-n}, \quad \chi_n^J = y^n + y^{n-2} + \cdots + y^{-n}, \tag{23}$$

and $\chi_{\boldsymbol{n}}^G$ is the character of the n-dimensional irreducible G representation.

It is easy to reproduce the large-N index (22) by using Eq. (8) on the AdS side. $i_{D_4}^{\rm vect}$ and $i^{\rm hyp}$ are given in Eqs. (11) and (12), and $i_{D_4}^{\rm grav}$ is obtained as follows.

The supergravity Kaluza–Klein modes in $AdS_5 \times S^5$ form a series of $psu(2, 2|4)$ representations $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,n,0](0,0)}$ $(n = 1, 2, 3, \ldots)$ [23,24].[2] After the supersymmetry is broken to $\mathcal{N} = 2$ by the 7-brane, these are decomposed into irreducible $su(2, 2|2) \times SU(2)_F$ representations $\mathcal{B}_{\frac{m}{2},r(0,0)} \otimes [\frac{m}{2}]_F$ and $\mathcal{C}_{\frac{m}{2},r(0,0)} \otimes [\frac{m}{2}]_F$, where $[s]_F$ is the spin s $SU(2)_F$ representation and m and r are integers (see Appendix A). We need to pick up $\mathbb{Z}_2$ invariant ones from them. The components in these representations carry

$$R_z - \frac{1}{2} A = r, \tag{24}$$

and the orientifold projection leaves only representations with $r \in 2\mathbb{Z}$. By summing up their contributions we obtain

$$\begin{aligned} i_{D_4}^{\rm grav} &= \sum_{n=0}^{\infty} \sum_m i(\mathcal{B}_{\frac{m}{2},2n(0,0)}) \chi_m^F + \sum_{n=0}^{\infty} \sum_m i(\mathcal{C}_{\frac{m}{2},2n(0,0)}) \chi_m^F \\ &= \frac{1}{2} \left(\frac{qu_x}{1-qu_x} + \frac{qu_y}{1-qu_y} + \frac{qu_z}{1-qu_z} - \frac{q^{\frac{3}{2}} y}{1-q^{\frac{3}{2}} y} - \frac{q^{\frac{3}{2}} y^{-1}}{1-q^{\frac{3}{2}} y^{-1}} \right) \\ &\quad + \frac{1}{4} \left(\frac{(1+qu_x)(1+qu_y)(1-qu_z)(1+q^{\frac{3}{2}} y)(1+q^{\frac{3}{2}} y^{-1})}{(1-qu_x)(1-qu_y)(1+qu_z)(1-q^{\frac{3}{2}} y)(1-q^{\frac{3}{2}} y^{-1})} - 1 \right). \end{aligned} \tag{25}$$

$\mathcal{I}_{D_4}^{\rm KK}$ obtained by substituting Eqs. (25), (11), and (12) for Eq. (8) is shown in Eq. (D1d), and it agrees with Eq. (22).

[2] We use the notation in Ref. [25] for $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal representations.

2.2 *Finite-N corrections*

Let us consider the finite-N case. In the following we often set all fugacities except for q to 1 to save space. We will use "$\overset{\circ}{=}$" to express the unrefinement. For example, the Kaluza–Klein index (D1d), which is identical with Eq. (22), is expressed as follows:

$$\mathcal{I}_{D_4}^{\text{KK}} \overset{\circ}{=} 1 + 32q^2 - 2q^{\frac{5}{2}} + 31q^3 + 62q^{\frac{7}{2}} + 556q^4 - 4q^{\frac{9}{2}} + 1117q^5 + \cdots . \tag{26}$$

Let us compare this with the results for $N = 1, 2, 3$ calculated on the gauge theory side:

$$\begin{aligned}
\mathcal{I}_{D_4[1]} &\overset{\circ}{=} 1 + 29q^2 - 2q^{\frac{5}{2}} - 28q^3 + 60q^{\frac{7}{2}} + 298q^4 - 60q^{\frac{9}{2}} - 587q^5 + \cdots \\
\mathcal{I}_{D_4[2]} &\overset{\circ}{=} 1 + 32q^2 - 2q^{\frac{5}{2}} + 27q^3 + 62q^{\frac{7}{2}} + 467q^4 - 6q^{\frac{9}{2}} + 632q^5 + \cdots \\
\mathcal{I}_{D_4[3]} &\overset{\circ}{=} 1 + 32q^2 - 2q^{\frac{5}{2}} + 31q^3 + 62q^{\frac{7}{2}} + 551q^4 - 4q^{\frac{9}{2}} + 998q^5 + \cdots .
\end{aligned} \tag{27}$$

We find that the finite-N corrections start at q^{N+1}:

$$\begin{aligned}
\mathcal{I}_{D_4[1]} - \mathcal{I}_{D_4}^{\text{KK}} &\overset{\circ}{=} -3q^2 - 59q^3 - 2q^{\frac{7}{2}} - 258q^4 - 56q^{\frac{9}{2}} - 1704q^5 - 566q^{\frac{11}{2}} + \cdots , \\
\mathcal{I}_{D_4[2]} - \mathcal{I}_{D_4}^{\text{KK}} &\overset{\circ}{=} -4q^3 - 89q^4 - 2q^{\frac{9}{2}} - 485q^5 - 54q^{\frac{11}{2}} - 3671q^6 - 588q^{\frac{13}{2}} + \cdots , \\
\mathcal{I}_{D_4[3]} - \mathcal{I}_{D_4}^{\text{KK}} &\overset{\circ}{=} -5q^4 - 119q^5 - 2q^{\frac{11}{2}} - 712q^6 - 52q^{\frac{13}{2}} - 5648q^7 - 590q^{\frac{15}{2}} + \cdots .
\end{aligned} \tag{28}$$

The refined expression for the leading terms in Eq. (28) is

$$\mathcal{I}_{D_4[1]} - \mathcal{I}_{D_4}^{\text{KK}} = -q^{N+1} u_z^{-\frac{N+1}{2}} \chi_{N+1}^F + \cdots . \tag{29}$$

We want to reproduce this finite-N correction as the contribution of wrapped D3-branes. Let us first consider $\mathcal{I}_{(1,0,0)}$, the contribution of a D3-brane on $X = 0$. We have to consider fields arising from two kinds of strings: 3-3 strings and 3-7 strings.

3-3 strings give an $\mathcal{N} = 4$ vector multiplet on the wrapped D3-brane. Its fluctuation modes belong to two series of representations of unbroken supersymmetry. Due to a similarity to the bulk modes we use similar notation $\mathcal{B}_{m,r}^{\text{D3}(X=0)}$ and $\mathcal{C}_{m,r}^{\text{D3}(X=0)}$ for these representations; see Appendix B for details. The values of $R_Z - \frac{1}{2}A$ carried by the components of these representations are

$$R_Z - \frac{1}{2}A = r. \tag{30}$$

We obtain $i_{D_4}^{\text{D3}(X=0)}$ by summing up all contributions from $\mathcal{B}_{m,r}^{\text{D3}(X=0)}$ and $\mathcal{C}_{m,r}^{\text{D3}(X=0)}$ with $r \in 2\mathbb{Z}$:

$$\begin{aligned}
i_{D_4}^{\text{D3}(X=0)} &= \sum_{n=0}^{\infty} \sum_m i\left(\mathcal{B}_{m,2n}^{\text{D3}(X=0)}\right) + \sum_{n=0}^{\infty} \sum_m i\left(\mathcal{C}_{m,2n}^{\text{D3}(X=0)}\right) \\
&= \frac{1}{2}\left(\frac{(1+q^{-1}u_x^{-1})(1+q^{\frac{3}{2}}y)(1+q^{\frac{3}{2}}y^{-1})}{(1+qu_z)(1-qu_y)} - \frac{(1-q^{-1}u_x^{-1})(1-q^{\frac{3}{2}}y)(1-q^{\frac{3}{2}}y^{-1})}{(1-qu_z)(1-qu_y)}\right).
\end{aligned} \tag{31}$$

The q-expansion of Eq. (31) and its plethystic exponential have the forms (19) and (20), respectively.

We also have the contribution from 3-7 strings. Because there are eight DN directions only chiral fermions appear on the intersection. They couple to the gauge symmetry on the D3-brane and the D_4 symmetry on the 7-brane. An important fact is that the $U(1)$ gauge symmetry on the D3-brane is broken to $\mathbb{Z}_2$ along the intersection with the O7-plane just like the gauge symmetry on a type I D-string. The $\mathbb{Z}_2$ gauged fermion system is nothing but the free field realization of the $\widehat{D}_4$ current algebra, and the contribution to the index is the character of the basic representation:

$$\chi^{\widehat{D}_4}(qu_y) = \frac{1}{2}(Z_{\zeta=1} + Z_{\zeta=-1}), \quad Z_\zeta \equiv \mathrm{Pexp}\left(-\frac{(qu_y)^{\frac{1}{2}}}{1-qu_y}\zeta\,\chi^{D_4}_{\mathbf{8}_v}\right). \tag{32}$$

We note that we adopted the anti-periodic boundary condition for the fermions. This is necessary to obtain the triality invariant spectrum required by the S-duality invariance of the index [26,27].

Equation (21), with Eqs. (31) and (32), gives

$$\mathcal{I}_{(1,0,0)} = -\frac{(qu_x)^{N+1}}{1-\frac{u_y}{u_x}}\left(1 + q\left(u_x + u_z^2u_x^{-1} - u_z + u_y^2u_x^{-1} + u_y\chi^{D_4}_{\mathbf{28}}\right) + \cdots\right). \tag{33}$$

The contribution of a D3-brane around the $Y = 0$ cycle, $\mathcal{I}_{(0,1,0)}$, is obtained from this by the Weyl reflection $u_x \leftrightarrow u_y$. We can easily see that the leading term (29) is reproduced:

$$-\frac{(qu_x)^{N+1}}{1-\frac{u_y}{u_x}} - \frac{(qu_y)^{N+1}}{1-\frac{u_x}{u_y}} = -q^{N+1}u_z^{-\frac{N+1}{2}}\chi^F_{N+1}. \tag{34}$$

It will turn out that this term exists not only for $G = D_4$ but also for all G. Some higher-order terms are also correctly reproduced, and the results of the numerical calculation are

$$\begin{aligned}
\mathcal{I}_{D_4[1]} - \mathcal{I}^{\mathrm{AdS}}_{D_4[1]} &\overset{\circ}{=} -10q^5 + 20q^{\frac{11}{2}} - 2124q^6 + 2028q^{\frac{13}{2}} - 28\,273q^7 + 22\,214q^{\frac{15}{2}} + \cdots, \\
\mathcal{I}_{D_4[2]} - \mathcal{I}^{\mathrm{AdS}}_{D_4[2]} &\overset{\circ}{=} -20q^7 + 40q^{\frac{15}{2}} + \cdots, \\
\mathcal{I}_{D_4[3]} - \mathcal{I}^{\mathrm{AdS}}_{D_4[3]} &\overset{\circ}{=} -35q^9 + \cdots.
\end{aligned} \tag{35}$$

The orders of these errors agree with Eq. (18) with $\delta_{D_4} = 3$, and this suggests that the method works well for $D_4[N]$.

3. Rank-one theories

Now, let us apply our method to more interesting cases with $G \neq D_4$. The Kaluza–Klein contribution (8) is again calculated by using i^{vec}_G in Eq. (11), i_{hyp} in Eq. (12), and i^{grav}_G given by

$$i^{\mathrm{grav}}_G = \sum_{n=0}^{\infty}\sum_m i(\mathcal{B}_{\frac{m}{2},\Delta_G n(0,0)})\chi^F_m + \sum_{n=0}^{\infty}\sum_m i(\mathcal{C}_{\frac{m}{2},\Delta_G n(0,0)})\chi^F_m. \tag{36}$$

The results for $\mathcal{I}^{\mathrm{KK}}_G$ are shown explicitly in Appendix D.

$\mathcal{I}_{(1,0,0)}$ is calculated by Eq. (21) with the single-particle index

$$i^{\mathrm{D3}}_G = \sum_{n=0}^{\infty}\sum_m i\left(\mathcal{B}^{\mathrm{D3}(X=0)}_{\frac{m}{2},\Delta_G n}\right) + \sum_{n=0}^{\infty}\sum_m i\left(\mathcal{C}^{\mathrm{D3}(X=0)}_{\frac{m}{2},\Delta_G n}\right). \tag{37}$$

Concerning the defect contribution, we assume that it is the character $\chi^{\widehat{G}}$ of the basic representation of affine algebra $\widehat{G}$ based on the result in the D_4 case (see Appendix C for the explicit form of $\chi^{\widehat{G}}$). We obtain the following results:

$$\mathcal{I}^{\mathrm{AdS}}_{H_0[1]} = 1 + u_z^{\frac{6}{5}}q^{\frac{6}{5}} - u_z^{\frac{1}{5}}\chi^J_1q^{\frac{17}{10}} + u_z^{-\frac{4}{5}}q^{\frac{11}{5}} + u_z^{\frac{12}{5}}q^{\frac{12}{5}} + u_z^{\frac{6}{5}}\chi^J_1q^{\frac{27}{10}} \underline{+u_z^{\frac{29}{5}}q^{\frac{14}{5}}} + \cdots, \tag{38a}$$

$$\begin{aligned}
\mathcal{I}^{\mathrm{AdS}}_{H_1[1]} = {} & 1 + u_z^{\frac{4}{3}}q^{\frac{4}{3}} - u_z^{\frac{1}{3}}\chi^J_1q^{\frac{11}{6}} + u_z^{-1}\chi^{H_1}_{\mathbf{3}}q^2 + u_z^{-\frac{2}{3}}q^{\frac{7}{3}} + u_z^{\frac{8}{3}}q^{\frac{8}{3}} + u_z^{\frac{4}{3}}\chi^J_1q^{\frac{17}{6}} - \left(1 + \chi^{H_1}_{\mathbf{3}}\right)q^3 \\
& - u_z^{\frac{5}{3}}\chi^J_1q^{\frac{19}{6}} + \left(-u_z^{\frac{1}{3}} \underline{+u_z^{\frac{19}{3}}} - u_z^{\frac{1}{3}}\chi^J_2\right)q^{\frac{10}{3}} + \cdots,
\end{aligned} \tag{38b}$$

$$\begin{aligned}\mathcal{I}^{\rm AdS}_{H_2[1]} &= 1 + u_z^{\frac{3}{2}} q^{\frac{3}{2}} + \left(-u_z^{\frac{1}{2}}\chi_1^J + u_z^{-1}\chi_{\mathbf{8}}^{H_2}\right) q^2 + u_z^{-\frac{1}{2}} q^{\frac{5}{2}} + \left(-1 + u_z^3 + u_z^{\frac{3}{2}}\chi_1^J - \chi_{\mathbf{8}}^{H_2}\right) q^3 \\ &\quad + \left(-u_z^{\frac{1}{2}} + u_z^{-1}\chi_1^J - u_z^2\chi_1^J - u_z^{\frac{1}{2}}\chi_2^J + u_z^{-1}\chi_{\mathbf{8}}^{H_2}\chi_1^J\right) q^{\frac{7}{2}} \\ &\quad + \left(2u_z + u_z^4 \underline{+ u_z^7} + u_z^{-\frac{1}{2}}\chi_1^J + u_z^{-2}\chi_{\mathbf{27}}^{H_2}\right) q^4 + \cdots, \end{aligned} \tag{38c}$$

$$\begin{aligned}\mathcal{I}^{\rm AdS}_{D_4[1]} &= 1 + (u_z^2 + u_z^{-1}\chi_{\mathbf{28}}^{D_4}) q^2 - u_z\chi_1^J q^{\frac{5}{2}} - \chi_{\mathbf{28}}^{D_4} q^3 + (u_z^{-1} + u_z^2 + u_z^{-1}\chi_{\mathbf{28}}^{D_4})\chi_1^J q^{\frac{7}{2}} + (u_z^4 \\ &\quad - u_z\chi_2^J + u_z^{-2}\chi_{\mathbf{300}}^{D_4}) q^4 + (-1 - u_z^3 - \chi_{\mathbf{28}}^{D_4})\chi_1^J q^{\frac{9}{2}} + (2u_z^{-1} \underline{+ u_z^{-1}\chi_4^F + u_z^2\chi_2^F} + u_z^5 \\ &\quad + u_z^{-1}\chi_2^J + u_z^2\chi_2^J - u_z^{-1}\chi_{\mathbf{28}}^{D_4} + u_z^{-1}\chi_2^J\chi_{\mathbf{28}}^{D_4} - u_z^{-1}\chi_{\mathbf{300}}^{D_4} - u_z^{-1}\chi_{\mathbf{350}}^{D_4}) q^5 + \cdots, \end{aligned} \tag{38d}$$

$$\begin{aligned}\mathcal{I}^{\rm AdS}_{E_6[1]} &= 1 + u_z^{-1}\chi_{\mathbf{78}}^{E_6} q^2 + (-1 + u_z^3 - \chi_{\mathbf{78}}^{E_6}) q^3 + (u_z^{-1} - u_z^2 + u_z^{-1}\chi_{\mathbf{78}}^{E_6})\chi_1^J q^{\frac{7}{2}} + (2u_z \\ &\quad + u_z^{-2}\chi_{\mathbf{2430}}^{E_6}) q^4 + (-2 + u_z^3 - \chi_{\mathbf{78}}^{E_6})\chi_1^J q^{\frac{9}{2}} + (2u_z^{-1} \underline{+ u_z^{-1}\chi_4^F} - u_z^2 + u_z^{-1}\chi_2^J \\ &\quad - u_z^2\chi_2^J - u_z^{-1}\chi_{\mathbf{78}}^{E_6} + u_z^{-1}\chi_2^J\chi_{\mathbf{78}}^{E_6} - u_z^{-1}\chi_{\mathbf{2430}}^{E_6} - u_z^{-1}\chi_{\mathbf{2925}}^{E_6}) q^5 + \cdots, \end{aligned} \tag{38e}$$

$$\begin{aligned}\mathcal{I}^{\rm AdS}_{E_7[1]} &= 1 + u_z^{-1}\chi_{\mathbf{133}}^{E_7} q^2 + (-1 - \chi_{\mathbf{133}}^{E_7}) q^3 + (u_z^{-1} + u_z^{-1}\chi_{\mathbf{133}}^{E_7})\chi_1^J q^{\frac{7}{2}} + (u_z + u_z^4 \\ &\quad + u_z^{-2}\chi_{\mathbf{7371}}^{E_7}) q^4 + (-2 - u_z^3 - \chi_{\mathbf{133}}^{E_7})\chi_1^J q^{\frac{9}{2}} + (2u_z^{-1} \underline{+ u_z^{-1}\chi_4^F} + u_z^2 + u_z^{-1}\chi_2^J \\ &\quad - u_z^{-1}\chi_{\mathbf{133}}^{E_7} + u_z^{-1}\chi_2^J\chi_{\mathbf{133}}^{E_7} - u_z^{-1}\chi_{\mathbf{7371}}^{E_7} - u_z^{-1}\chi_{\mathbf{8645}}^{E_7}) q^5 + \cdots, \end{aligned} \tag{38f}$$

$$\begin{aligned}\mathcal{I}^{\rm AdS}_{E_8[1]} &= 1 + u_z^{-1}\chi_{\mathbf{248}}^{E_8} q^2 + (-1 - \chi_{\mathbf{248}}^{E_8}) q^3 + (u_z^{-1} + u_z^{-1}\chi_{\mathbf{248}}^{E_8})\chi_1^J q^{\frac{7}{2}} + (u_z + u_z^2\chi_{\mathbf{27\,000}}^{E_8}) q^4 \\ &\quad + (-2 - \chi_{\mathbf{248}}^{E_8})\chi_1^J q^{\frac{9}{2}} + (2u_z^{-1} \underline{+ u_z^{-1}\chi_4^F} + u_z^{-1}\chi_2^J - u_z^{-1}\chi_{\mathbf{248}}^{E_8} + u_z^{-1}\chi_2^J\chi_{\mathbf{248}}^{E_8} \\ &\quad - u_z^{-1}\chi_{\mathbf{27\,000}}^{E_8} - u_z^{-1}\chi_{\mathbf{30\,380}}^{E_8}) q^5 + \cdots. \end{aligned} \tag{38g}$$

Because we took only the leading corrections into account these results have errors, and it is important to determine their orders. Even without comparing these with the known results, we can find terms that cannot be correct. There are two types of such impossible terms. The first type includes terms depending on the $SU(2)_F$ fugacity u. Because $SU(2)_F$ symmetry decouples in the $N = 1$ case the index must be u-independent. Indeed, many u-dependent terms appearing in $\mathcal{I}_G^{\rm KK}$ shown in Appendix D are drastically canceled by the single-wrapping contributions. For example, the u-dependent term $q^2 u_z^{-1}\chi_2^F$ appearing in $\mathcal{I}_G^{\rm KK}$ for all G is canceled by the leading term of the finite-N correction (34) with $N = 1$. Even so, there still exist terms with non-trivial $SU(2)_F$ characters $\chi^F_{n>0}$, which must be canceled by higher-order corrections. The second type of impossible terms is terms diverging in the Coulomb branch limit:

$$q \to 0 \quad \text{with} \quad qu_x^{-2}, \quad qu_y^{-2}, \quad qu_z, \quad y \quad \text{fixed}. \tag{39}$$

In this limit the q-expansion of $\mathcal{I}_{(1,0,0)}$ includes diverging terms originating from the diverging factor $(qu_x)^{-1}$ in $i(\mathcal{B}^{{\rm D3}(X=0)}_{\frac{m}{2},r})$ (see Eq. (B2)). Such diverging terms appearing in Eq. (38) must also be canceled by higher-order corrections. The impossible terms in Eq. (38) are underlined.

The lowest-order impossible terms appearing in Eq. (38) are

$$u_z^3(u_z q)^{4\Delta_G - 2} \quad \text{for } G = H_0, H_1, H_2, \quad u_z^{-1}\chi_4^F q^5 \quad \text{for } G = D_4, E_6, E_7, E_8. \tag{40}$$

(In addition, we have another impossible term $u_z^2\chi_2^F q^5$ in the D_4 case.) If we assume that these are the lowest-order error terms, we can read off the tachyonic shifts for the next-to-leading

Table 4. The differences between results in Eq. (38) calculated by the approximate formula (16) and the previously known results in the references are shown.

Theory	$\mathcal{I}_{G[1]} - \mathcal{I}^{AdS}_{G[1]}$	Refs.
$H_0[1] = (A_1, A_2)$	$-u_z^{\frac{29}{5}} q^{\frac{14}{5}} + \cdots$	[28]
$H_1[1] = (A_1, A_3)$	$-u_z^{\frac{19}{3}} q^{\frac{10}{3}} + \cdots$	[29]
$H_2[1] = (A_1, D_4)$	$-(u_z^4 + u_z^7) q^4 + \cdots$	[30]
$D_4[1]$	$-(u_z^{-1} + u_z^{-1}\chi_4^F + u_z^2 \chi_2^F + u_z^5) q^5 + \cdots$	
$E_6[1]$	$-(u_z^{-1} + u_z^{-1}\chi_4^F) q^5 + \cdots$	[31]
$E_7[1]$	$-(u_z^{-1} + u_z^{-1}\chi_4^F) q^5 + \cdots$	[32]
$E_8[1]$	?	unknown

corrections

$$\delta_{H_n} = 3\Delta_G - 2, \quad \delta_{D_4} = \delta_{E_n} = 3 \tag{41}$$

by comparing Eqs. (18) and (40). Indeed, by comparing Eq. (38) and the known results in the literature, we can confirm that this is the case except for $G = E_8$, for which the superconformal index is not known; see Table 4.

It is also instructive to consider some limits simplifying the structure of the index [33]. The Hall–Littlewood index is defined by taking the limit

$$q \to 0 \quad \text{with} \quad qu_x, \quad qu_y, \quad \mathfrak{q} \equiv qu_z^{-\frac{1}{2}}, \quad y \quad \text{fixed}. \tag{42}$$

In this limit the Kaluza–Klein contribution becomes

$$\mathcal{I}_G^{\text{KK}}|_{\text{HL}} = \text{Pexp}\left(\sum_{k=2}^{\infty} \mathfrak{q}^k \left(\chi_k^F + \chi_{k-2}^F \chi_\theta^G\right)\right) = 1 + \mathfrak{q}^2\left(\chi_2^F + \chi_\theta^G\right) + \mathfrak{q}^3\left(\chi_3^F + \chi_1^F \chi_\theta^G\right) + \cdots. \tag{43}$$

By using $i_G^{\text{D3}(X=0)}|_{\text{HL}} = \frac{1}{qu}\frac{1}{1-qu^{-1}}$ and $\chi^{\widehat{G}}(qu^{-1}) = 1 + qu^{-1}\chi_\theta^G + \cdots$, we obtain

$$\mathcal{I}_{(1,0,0)} + \mathcal{I}_{(0,1,0)}|_{\text{HL}} = -\mathfrak{q}^2\chi_2^F - \mathfrak{q}^3\left(\chi_3^F + \chi_1^F\chi_\theta^G\right) + \cdots. \tag{44}$$

We can easily see that the u-dependent terms shown in Eqs. (43) and (44) at the order of $\mathfrak{q}^2$ and $\mathfrak{q}^3$ cancel. This cancellation also occurs at $\mathfrak{q}^4$ and $\mathfrak{q}^5$, and we find the first u-dependence at $\mathfrak{q}^6$. We obtain the following result:

$$\mathcal{I}_{G[1]}^{AdS}|_{\text{HL}} = 1 + \mathfrak{q}^2\chi_\theta^G + \mathfrak{q}^4\chi_{2\theta}^G \underline{+\mathfrak{q}^6(u - \text{dep.})} + \cdots. \tag{45}$$

We can also calculate the Schur index in a similar way. The Schur index is defined from the superconformal index by setting $y = q^{\frac{1}{2}}u_z^{-1}$, and is a function of $\mathfrak{q} = qu_z^{-\frac{1}{2}}$, u, and G fugacities. We obtain

$$\mathcal{I}_{G[1]}^{\text{AdS}}|_{\text{Sch}} = 1 + \mathfrak{q}^2\chi_\theta^G + \mathfrak{q}^4\left(1 + \chi_\theta^G + \chi_{2\theta}^G\right) \underline{+\mathfrak{q}^6(u - \text{dep.})} + \cdots. \tag{46}$$

Unlike the superconformal index these limits of the index do not acquire a contribution from the $Z = 0$ cycle, and the next-to-leading corrections should be of the order of $\mathfrak{q}^{2N+\delta_G}$. Equations (45) and (46) suggest that the higher-order corrections start at $\mathfrak{q}^6$. We can confirm this by comparing Eqs. (45) and (46) with known results [34–39]. This means that the tachyonic shift for the Hall–Littlewood index and the Schur index is

$$\delta_G = 4 \tag{47}$$

for all G.

4. Higher-rank theories

An advantage of the method using AdS/CFT is that we can deal with higher-rank theories in the same way as the rank-one theories. In this section we show the results for $N = 2$ and $N = 3$. Many of these have been calculated in the literature [40–43], and our results are consistent with them.

We determine the orders of the error terms, which are indicated by underlines in the following results, by using the tachyonic shifts (41) for the superconformal index and Eq. (47) for the Hall–Littlewood index and Schur index. Namely, the orders of the error terms for rank N are

$$\mathcal{O}(q^{\Delta_G(N+3)-2}) \quad (G = H_n), \quad \mathcal{O}(q^{2N+3}), \quad (G = D_4, E_n) \tag{48}$$

for the superconformal index and

$$\mathcal{O}(q^{2N+4}) \tag{49}$$

for the Hall–Littlewood and Schur indices.

Some of the following results in this section are consistent with the results of Refs. [40–43], and the others are newly obtained in this paper. We will use "$\overset{\circ,!}{=}$" to express the new results (as well as the unrefinement of the fugacities).

4.1 *Superconformal index*

$$\mathcal{I}^{\text{AdS}}_{H_0[2]} \overset{\circ,!}{=} 1 + q^{\frac{6}{5}} - 2q^{\frac{17}{10}} + 3q^2 + 3q^{\frac{11}{5}} + 2q^{\frac{12}{5}} - 2q^{\frac{27}{10}} - 4q^{\frac{29}{10}} - 4q^3 - q^{\frac{16}{5}} + 5q^{\frac{17}{5}} + 8q^{\frac{7}{2}} + 2q^{\frac{18}{5}} + 4q^{\frac{37}{10}} - 6q^{\frac{39}{10}} \underline{+5q^4} + \cdots, \tag{50a}$$

$$\mathcal{I}^{\text{AdS}}_{H_1[2]} \overset{\circ,!}{=} 1 + q^{\frac{4}{3}} - 2q^{\frac{11}{6}} + 6q^2 + 3q^{\frac{7}{3}} + 2q^{\frac{8}{3}} - 2q^{\frac{17}{6}} - q^3 - 4q^{\frac{19}{6}} + 2q^{\frac{10}{3}} + 14q^{\frac{7}{2}} + 5q^{\frac{11}{3}} - 2q^{\frac{23}{6}} + 15q^4 - 6q^{\frac{25}{6}} + q^{\frac{13}{3}} - 6q^{\frac{9}{2}} \underline{+7q^{\frac{14}{3}}} + \cdots, \tag{50b}$$

$$\mathcal{I}^{\text{AdS}}_{H_2[2]} \overset{\circ,!}{=} 1 + q^{\frac{3}{2}} + 9q^2 + 3q^{\frac{5}{2}} + 4q^3 + 27q^{\frac{7}{2}} + 41q^4 + 17q^{\frac{9}{2}} + 81q^5 \underline{+183q^{\frac{11}{2}}} + \cdots, \tag{50c}$$

$$\mathcal{I}^{\text{AdS}}_{D_4[2]} \overset{\circ}{=} 1 + 32q^2 - 2q^{\frac{5}{2}} + 27q^3 + 62q^{\frac{7}{2}} + 467q^4 - 6q^{\frac{9}{2}} + 632q^5 + 1924q^{\frac{11}{2}} + 3702q^6 + 2326q^{\frac{13}{2}} \underline{+8420q^7} + \cdots, \tag{50d}$$

$$\mathcal{I}^{\text{AdS}}_{E_6[2]} \overset{\circ,!}{=} 1 + 81q^2 + 75q^3 + 162q^{\frac{7}{2}} + 3166q^4 + 148q^{\frac{9}{2}} + 4863q^5 + 12\,812q^{\frac{11}{2}} + 78\,247q^6 + 21\,552q^{\frac{13}{2}} \underline{+158\,937q^7} + \cdots, \tag{50e}$$

$$\mathcal{I}^{\text{AdS}}_{E_7[2]} \overset{\circ,!}{=} 1 + 136q^2 + 129q^3 + 274q^{\frac{7}{2}} + 9049q^4 + 258q^{\frac{9}{2}} + 14\,616q^5 + 36\,722q^{\frac{11}{2}} + 389\,749q^6 + 63\,772q^{\frac{13}{2}} \underline{+825\,721q^7} + \cdots, \tag{50f}$$

$$\mathcal{I}^{\text{AdS}}_{E_8[2]} \overset{\circ,!}{=} 1 + 251q^2 + 244q^3 + 504q^{\frac{7}{2}} + 31128q^4 + 490q^{\frac{9}{2}} + 53756q^5 + 125504q^{\frac{11}{2}} + 2539245q^6 + 229488q^{\frac{13}{2}} \underline{+5896389q^7} + \cdots \tag{50g}$$

$$\begin{aligned}\mathcal{I}^{\rm AdS}_{H_0[3]} \overset{\circ,!}{=}\ & 1 + q^{\frac{6}{5}} - 2q^{\frac{17}{10}} + 3q^2 + 3q^{\frac{11}{5}} + 2q^{\frac{12}{5}} - 2q^{\frac{27}{10}} - 4q^{\frac{29}{10}} + 2q^{\frac{16}{5}} + 7q^{\frac{17}{5}} + 8q^{\frac{7}{2}} \\ & + 3q^{\frac{18}{5}} - 2q^{\frac{37}{10}} - 10q^{\frac{39}{10}} + q^4 - 8q^{\frac{41}{10}} + q^{\frac{21}{5}} + 11q^{\frac{22}{5}} + 2q^{\frac{9}{2}} + 15q^{\frac{23}{5}} + 10q^{\frac{47}{10}} \\ & + 4q^{\frac{24}{5}} - 2q^{\frac{49}{10}} + 10q^5 - 22q^{\frac{51}{10}} \underline{-21q^{\frac{26}{5}}} + \cdots ,\end{aligned} \tag{51a}$$

$$\begin{aligned}\mathcal{I}^{\rm AdS}_{H_1[3]} \overset{\circ,!}{=}\ & 1 + q^{\frac{4}{3}} - 2q^{\frac{11}{6}} + 6q^2 + 3q^{\frac{7}{3}} + 2q^{\frac{8}{3}} - 2q^{\frac{17}{6}} + 3q^3 - 4q^{\frac{19}{6}} + 5q^{\frac{10}{3}} + 14q^{\frac{7}{2}} \\ & + 7q^{\frac{11}{3}} - 8q^{\frac{23}{6}} + 22q^4 - 10q^{\frac{25}{6}} + 13q^{\frac{13}{3}} + 17q^{\frac{14}{3}} + 4q^{\frac{29}{6}} + 43q^5 - 14q^{\frac{31}{6}} \\ & - 15q^{\frac{16}{3}} + 60q^{\frac{11}{2}} + 7q^{\frac{17}{3}} + 32q^{\frac{35}{6}} \underline{+81q^6} + \cdots ,\end{aligned} \tag{51b}$$

$$\begin{aligned}\mathcal{I}^{\rm AdS}_{H_2[3]} \overset{\circ,!}{=}\ & 1 + q^{\frac{3}{2}} + 9q^2 + 3q^{\frac{5}{2}} + 8q^3 + 30q^{\frac{7}{2}} + 58q^4 + 44q^{\frac{9}{2}} + 111q^5 + 259q^{\frac{11}{2}} \\ & + 374q^6 + 462q^{\frac{13}{2}} \underline{+1000q^7} + \cdots ,\end{aligned} \tag{51c}$$

$$\begin{aligned}\mathcal{I}^{\rm AdS}_{D_4[3]} \overset{\circ}{=}\ & 1 + 32q^2 - 2q^{\frac{5}{2}} + 31q^3 + 62q^{\frac{7}{2}} + 551q^4 - 4q^{\frac{9}{2}} + 998q^5 + 1976q^{\frac{11}{2}} \\ & + 6661q^6 + 2862q^{\frac{13}{2}} + 17\,537q^7 + 35\,482q^{\frac{15}{2}} + 64\,679q^8 \\ & + 84\,630q^{\frac{17}{2}} \underline{+220\,412q^9} + \cdots ,\end{aligned} \tag{51d}$$

$$\begin{aligned}\mathcal{I}^{\rm AdS}_{E_6[3]} \overset{\circ,!}{=}\ & 1 + 81q^2 + 79q^3 + 162q^{\frac{7}{2}} + 3397q^4 + 156q^{\frac{9}{2}} + 6408q^5 + 13\,274q^{\frac{11}{2}} \\ & + 99\,165q^6 + 25\,290q^{\frac{13}{2}} + 273\,109q^7 + 570\,728q^{\frac{15}{2}} + 2283\,657q^8 \\ & + 1549\,838q^{\frac{17}{2}} \underline{+8097\,884q^9} + \cdots ,\end{aligned} \tag{51e}$$

$$\begin{aligned}\mathcal{I}^{\rm AdS}_{E_7[3]} \overset{\circ,!}{=}\ & 1 + 136q^2 + 133q^3 + 274q^{\frac{7}{2}} + 9445q^4 + 266q^{\frac{9}{2}} + 18\,101q^5 + 37\,520q^{\frac{11}{2}} \\ & + 450\,243q^6 + 72\,362q^{\frac{13}{2}} + 1271\,046q^7 + 2647\,564q^{\frac{15}{2}} + 16686\,266q^8 \\ & + 7415\,174q^{\frac{17}{2}} \underline{+61\,224\,202q^9} + \cdots ,\end{aligned} \tag{51f}$$

$$\begin{aligned}\mathcal{I}^{\rm AdS}_{E_8[3]} \overset{\circ,!}{=}\ & 1 + 251q^2 + 248q^3 + 504q^{\frac{7}{2}} + 31869q^4 + 498q^{\frac{9}{2}} + 62258q^5 + 126992q^{\frac{11}{2}} \\ & + 2747126q^6 + 249498q^{\frac{13}{2}} + 7961389q^7 + 16282232q^{\frac{15}{2}} + 181906110q^8 \\ & + 47084068q^{\frac{17}{2}} \underline{+691172658q^9} + \cdots .\end{aligned} \tag{51g}$$

4.2 *Hall–Littlewood index*

$$\mathcal{I}^{\rm AdS}_{H_0[2]}|_{\rm HL} \overset{\circ,!}{=} 1 + 3q^2 + 5q^4 + 7q^6 \underline{+30q^8} + \cdots , \tag{52a}$$

$$\mathcal{I}^{\rm AdS}_{H_1[2]}|_{\rm HL} \overset{\circ}{=} 1 + 6q^2 + 6q^3 + 20q^4 + 28q^5 + 65q^6 + 80q^7 \underline{+242q^8} + \cdots , \tag{52b}$$

$$\mathcal{I}^{\rm AdS}_{H_2[2]}|_{\rm HL} \overset{\circ}{=} 1 + 11q^2 + 16q^3 + 65q^4 + 142q^5 + 355q^6 + 700q^7 \underline{+1779q^8} + \cdots , \tag{52c}$$

$$\mathcal{I}^{\rm AdS}_{D_4[2]}|_{\rm HL} \overset{\circ}{=} 1 + 31q^2 + 56q^3 + 495q^4 + 1468q^5 + 6269q^6 + 19\,680q^7 \underline{+66\,611q^8} + \cdots , \tag{52d}$$

$$\mathcal{I}^{\rm AdS}_{E_6[2]}|_{\rm HL} \overset{\circ}{=} 1 + 81q^2 + 156q^3 + 3320q^4 + 11\,178q^5 + 98\,440q^6 + 401\,280q^7 \underline{+2356\,455q^8} + \cdots , \tag{52e}$$

$$\mathcal{I}^{\rm AdS}_{E_7[2]}|_{\rm HL} \overset{\circ}{=} 1 + 136q^2 + 266q^3 + 9315q^4 + 32\,830q^5 + 449\,050q^6 + 2026\,080q^7 \underline{+17\,206\,093q^8} + \cdots , \tag{52f}$$

$$\mathcal{I}^{\rm AdS}_{E_8[2]}|_{\rm HL} \overset{\circ,!}{=} 1 + 251q^2 + 496q^3 + 31\,625q^4 + 116\,248q^5 + 2747\,875q^6 + 13\,624\,000q^7 \underline{+187\,007\,628q^8} + \cdots . \tag{52g}$$

$$\mathcal{I}^{\rm AdS}_{H_0[3]}|_{\rm HL} \overset{\circ,!}{=} 1 + 3q^2 + 4q^3 + 6q^4 + 10q^5 + 17q^6 + 18q^7 + 31q^8 + 38q^9 \underline{+76q^{10}} + \cdots , \tag{53a}$$

$$\mathcal{I}^{\rm AdS}_{H_1[3]}|_{\rm HL} \overset{\circ}{=} 1 + 6q^2 + 10q^3 + 30q^4 + 58q^5 + 147q^6 + 258q^7 + 548q^8 + 952q^9 \underline{+1876q^{10}} + \cdots , \tag{53b}$$

$$\mathcal{I}^{\rm AdS}_{H_2[3]}|_{\rm HL} \overset{\circ}{=} 1 + 11q^2 + 20q^3 + 90q^4 + 218q^5 + 698q^6 + 1618q^7 + 4300q^8 + 9588q^9 \underline{+22\,634q^{10}} + \cdots , \tag{53c}$$

$$\mathcal{I}^{\rm AdS}_{D_4[3]}|_{\rm HL} \overset{\circ}{=} 1 + 31q^2 + 60q^3 + 580q^4 + 1858q^5 + 9457q^6 + 33\,066q^7 + 131\,755q^8 + 444\,502q^9 \underline{+1543\,882q^{10}} + \cdots , \tag{53d}$$

$$\mathcal{I}^{\rm AdS}_{E_6[3]}|_{\rm HL} \overset{\circ}{=} 1 + 81q^2 + 160q^3 + 3555q^4 + 12\,958q^5 + 121\,447q^6 + 556\,958q^7 + 3563\,694q^8 + 17\,126\,502q^9 \underline{+90\,513\,091q^{10}} + \cdots , \tag{53e}$$

$$\mathcal{I}^{\rm AdS}_{E_7[3]}|_{\rm HL} \overset{\circ}{=} 1 + 136q^2 + 270q^3 + 9715q^4 + 36\,718q^5 + 514\,230q^6 + 2592\,258q^7 + 22\,872\,825q^8 + 128\,145\,440q^9 \underline{+885\,685\,093q^{10}} + \cdots , \tag{53f}$$

$$\mathcal{I}^{\rm AdS}_{E_8[3]}|_{\rm HL} \overset{\circ,!}{=} 1 + 251q^2 + 500q^3 + 32370q^4 + 125498q^5 + 2966497q^6 + 16098498q^7 + 221148375q^8 + 1420026502q^9 \underline{+14229178180q^{10}} + \cdots . \tag{53g}$$

4.3 *Schur index*

$$\mathcal{I}^{\rm AdS}_{H_0[2]}|_{\rm Sch} \overset{\circ,!}{=} 1 + 3q^2 + 9q^4 + 2q^5 + 22q^6 + 6q^7 \underline{+62q^8} + \cdots , \tag{54a}$$

$$\mathcal{I}^{\rm AdS}_{H_1[2]}|_{\rm Sch} \overset{\circ}{=} 1 + 6q^2 + 6q^3 + 27q^4 + 36q^5 + 113q^6 + 162q^7 \underline{+471q^8} + \cdots , \tag{54b}$$

$$\mathcal{I}^{\rm AdS}_{H_2[2]}|_{\rm Sch} \overset{\circ}{=} 1 + 11q^2 + 16q^3 + 77q^4 + 160q^5 + 498q^6 + 1056q^7 \underline{+2950q^8} + \cdots , \tag{54c}$$

$$\mathcal{I}^{\rm AdS}_{D_4[2]}|_{\rm Sch} \overset{\circ}{=} 1 + 31q^2 + 56q^3 + 527q^4 + 1526q^5 + 7292q^6 + 23\,002q^7 \underline{+86\,239q^8} + \cdots , \tag{54d}$$

$$\mathcal{I}^{\rm AdS}_{E_6[2]}|_{\rm Sch} \overset{\circ}{=} 1 + 81q^2 + 156q^3 + 3402q^4 + 11\,336q^5 + 105\,163q^6 + 425\,412q^7 \underline{+2656\,809q^8} + \cdots , \tag{54e}$$

$$\mathcal{I}^{\rm AdS}_{E_7[2]}|_{\rm Sch} \overset{\circ}{=} 1 + 136q^2 + 266q^3 + 9452q^4 + 33\,098q^5 + 467\,818q^6 + 2095\,624q^7 \underline{+18\,564\,678q^8} + \cdots , \tag{54f}$$

$$\mathcal{I}^{\rm AdS}_{E_8[2]}|_{\rm Sch} \overset{\circ,!}{=} 1 + 251q^2 + 496q^3 + 31\,877q^4 + 116\,746q^5 + 2811\,378q^6 + 13\,865\,742q^7 \underline{+195\,272\,132q^8} + \cdots . \tag{54g}$$

$$\mathcal{I}^{\rm AdS}_{H_0[3]}|_{\rm Sch} \overset{\circ,!}{=} 1 + 3q^2 + 4q^3 + 10q^4 + 16q^5 + 36q^6 + 56q^7 + 110q^8 + 176q^9 \underline{+327q^{10}} + \cdots , \tag{55a}$$

$$\mathcal{I}^{\rm AdS}_{H_1[3]}|_{\rm Sch} \overset{\circ}{=} 1 + 6q^2 + 10q^3 + 37q^4 + 70q^5 + 208q^6 + 410q^7 + 1008q^8 + 2000q^9 \underline{+4501q^{10}} + \cdots , \tag{55b}$$

$$\mathcal{I}^{\rm AdS}_{H_2[3]}|_{\rm Sch} \overset{\circ}{=} 1 + 11q^2 + 20q^3 + 102q^4 + 240q^5 + 869q^6 + 2120q^7 + 6276q^8 + 15\,220q^9 + \underline{+40\,356q^{10}} + \cdots , \tag{55c}$$

$$\mathcal{I}^{\rm AdS}_{D_4[3]}|_{\rm Sch} \overset{\circ}{=} 1 + 31q^2 + 60q^3 + 612q^4 + 1920q^5 + 10\,568q^6 + 36\,968q^7 + 157\,850q^8 + 548\,848q^9 + \underline{+2039\,418q^{10}} + \cdots , \tag{55d}$$

$$\mathcal{I}^{\rm AdS}_{E_6[3]}|_{\rm Sch} \overset{\circ}{=} 1 + 81q^2 + 160q^3 + 3637q^4 + 13\,120q^5 + 128\,408q^6 + 583\,360q^7 + 3908\,179q^8 + 18\,828\,800q^9 + \underline{+103\,829\,612q^{10}} + \cdots , \tag{55e}$$

$$\mathcal{I}^{\rm AdS}_{E_7[3]}|_{\rm Sch} \overset{\circ}{=} 1 + 136q^2 + 270q^3 + 9852q^4 + 36\,990q^5 + 533\,401q^6 + 2666\,510q^7 + 24\,354\,958q^8 + 136\,003\,400q^9 + \underline{+972\,032\,920q^{10}} + \cdots , \tag{55f}$$

$$\mathcal{I}^{\rm AdS}_{E_8[3]}|_{\rm Sch} \overset{\circ,!}{=} 1 + 251q^2 + 500q^3 + 32622q^4 + 126000q^5 + 3030748q^6 + 16351000q^7 + 229826870q^8 + 1468558000q^9 + \underline{+15077246917q^{10}} + \cdots . \tag{55g}$$

5. Conclusions and discussion

In this paper we have calculated the superconformal indices of the $\mathcal{N} = 2$ theories realized by D3-7-brane systems. We first confirmed that the method works well in the D_4 case, for which we can perturbatively calculate the index both on the gauge theory side and on the AdS side, and then we applied the same method to more interesting cases with $G \neq D_4$. The results are consistent with known results in the literature, and some of them are new and give predictions.

In this paper we focused only on $\mathcal{I}_{(1,0,0)}$ and $\mathcal{I}_{(0,1,0)}$, the contributions of single-wrapping D3-branes around two cycles $X = 0$ and $Y = 0$, for simplicity. To improve our results we need to include higher-order contributions. The orders of the next-to-leading corrections are given in Eq. (48) for the superconformal index and in Eq. (49) for the Schur and Hall–Littlewood indices. For the latter, the $Z = 0$ cycle does not contribute and multiple-wrapping D3-branes with $n_x + n_y \geq 2$ give the higher-order corrections. As was studied in Refs. [17] and [18], in order to calculate multiple-wrapping contributions, we need to choose very carefully the integration contours in the holonomy integrals. In addition, we need to take account of the current algebra localized on the intersection of the D3-brane and the 7-brane. In the case of the superconformal index, the $Z = 0$ cycle also contributes. If n_z D3-branes are wrapped around the cycle, the theory realized on the worldvolume is $G[n_z]$, and perturbative treatment is not possible except for the D_4 case. Although direct calculation of such a contribution is difficult, it may be possible to extract some information about them from the error obtained in our analysis. For example, the tachyonic shifts $\delta_{H_n} = 3\Delta_G - 2$ in Eq. (41) should be somehow interpreted in $H_n[1]$ on the wrapped D3.

Recently, an expansion similar to Eq. (13) was proposed in Ref. [44] for Lagrangian gauge theories based on the direct analysis in gauge theories. Unlike the multiple expansion (13) the expansion in Ref. [44] is a simple expansion. It is interesting whether such a simple expansion exists for non-Lagrangian theories like those that we have studied in this paper.

Another important direction is to consider more complicated background geometries. Although only a limited class of Argyres–Douglas theories are realized by the D3-7-brane systems, more general Argyres–Douglas theories can be realized [45] as class S theories [46,47], and some supergravity solutions have been proposed [48–50]. It would be interesting to study to what extent the method can be applied in such, more complicated, backgrounds.

Acknowledgments

We would like to thank Reona Arai, Shota Fujiwara, Tatsuya Mori, and Daisuke Yokoyama for valuable discussions and comments. The work of Y.I. was partially supported by a Grant-in-Aid for Scientific Research (C) (No. 21K03569), Ministry of Education, Culture, Sports, Science and Technology, Japan.

Funding

Open Access funding: SCOAP^3.

Appendix A. Gravity multiplet

Kaluza–Klein modes of gravity multiplets in $AdS_5 \times S^5$ belong to the representations $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,n,0](0,0)}$ ($n = 1, 2, 3, \ldots$). We use the notation in Ref. [25] for superconformal representations. Each of them are constructed by acting $\mathcal{N} = 4$ supercharges on the lowest-energy states belonging to the $SU(4)_R$ representation $[0, n, 0]$. The $SU(2)_R \times SU(2)_F \times U(1)_{R_Z}$ decomposition of this representation is

$$[0, n, 0] \to \bigoplus_{m+k+l=n} \left(\tfrac{m}{2}, \tfrac{m}{2}\right)_{k-l} \tag{A1}$$

where the direct sum is taken over partitions of n into three non-negative integers m, k, and l. The $\mathcal{N} = 2$ representations are obtained by acting supercharges on the representations appearing in Eq. (A1). Although there may be other superconformal representations whose primaries do not belong to Eq. (A1), such representations do not contribute to the index and we can neglect them. Furthermore, only representations with $l = 0$ and $l = 1$ contribute to the index. The supersymmetry completions of the relevant representations are

$$\left(\tfrac{m}{2}, \tfrac{m}{2}\right)_k \to \mathcal{B}_{\frac{m}{2},k(0,0)} \otimes \left[\tfrac{m}{2}\right]_F, \quad \left(\tfrac{m}{2}, \tfrac{m}{2}\right)_{k-1} \to \mathcal{C}_{\frac{m}{2},k-1(0,0)} \otimes \left[\tfrac{m}{2}\right]_F. \tag{A2}$$

Both k and m run over non-negative integers. Representations with $k = 0$ and $k = 1$ have special structure and they are denoted in Ref. [25] as follows:

$$\mathcal{B}_{\frac{m}{2},0(0,0)} = \widehat{\mathcal{B}}_{\frac{m}{2}}, \quad \mathcal{B}_{\frac{m}{2},1(0,0)} = \widehat{\mathcal{D}}_{\frac{m}{2}(0,0)}, \quad \mathcal{C}_{\frac{m}{2},-1(0,0)} = \overline{\mathcal{D}}_{\frac{m}{2}(0,0)}, \quad \mathcal{C}_{\frac{m}{2},0(0,0)} = \widehat{\mathcal{C}}_{\frac{m}{2}(0,0)}. \tag{A3}$$

For the analysis of $G = H_n$ we need to extend the range of k [16] to fractional values. The unitarity requires $k = 0$ or $k \geq 1$. This means $r = 0$ or $r \geq 1$ for $\mathcal{B}_{\frac{m}{2},r(0,0)}$ and $r = -1$ or $r \geq 0$ for $\mathcal{C}_{\frac{m}{2},r(0,0)}$. For each value of r the contribution of the representations in Eq. (A2) with all allowed values of m to the index are as follows:

$$\sum_m i\left(\mathcal{B}_{m,0(0,0)}\right)\chi_m^F = \frac{1}{(\text{mom})}\left[\left\{\frac{1-qu_z}{(1-qu_x)(1-qu_y)} + qu_z\right\} - 1\right],$$

$$\sum_m i\left(\mathcal{B}_{m,1(0,0)}\right)\chi_m^F = \frac{1}{(\text{mom})}\left[(qu_z)\left(1 - q^{\frac{1}{2}}y^{\pm 1}u_z^{-1}\right)\left\{\frac{(1-qu_z)}{(1-qu_x)(1-qu_y)} + qu_z\right\} + q^3\right],$$

$$\sum_m i\left(\mathcal{B}_{m,r>1(0,0)}\right)\chi_m^F = (qu_z)^r \frac{\left(1 - q^{\frac{1}{2}}yu_z^{-1}\right)\left(1 - q^{\frac{1}{2}}y^{-1}u_z^{-1}\right)}{\text{mom}}\left\{\frac{1-qu_z}{(1-qu_x)(1-qu_y)} + qu_z\right\},$$

$$\sum_m i\left(\mathcal{C}_{m,-1(0,0)}\right)\chi_m^F = -\frac{1}{(\text{mom})}\frac{q^2u_xu_y(1-qu_z)}{(1-qu_x)(1-qu_y)},$$

$$\sum_m i\left(\mathcal{C}_{m,0(0,0)}\right)\chi_m^F = -(qu_z)\frac{\left(1 - q^{\frac{1}{2}}y^{\pm 1}u_z^{-1}\right)}{(\text{mom})}\frac{q^2u_xu_y(1-qu_z)}{(1-qu_x)(1-qu_y)},$$

$$\sum_m i\left(\mathcal{C}_{m,r>0(0,0)}\right)\chi_m^F = -(qu_z)^{r+1}\frac{\left(1 - q^{\frac{1}{2}}yu_z^{-1}\right)\left(1 - q^{\frac{1}{2}}y^{-1}u_z^{-1}\right)}{(\text{mom})}\frac{q^2u_xu_y(1-qu_z)}{(1-qu_x)(1-qu_y)}, \tag{A4}$$

where (mom) is the momentum factor

$$(\text{mom}) = (1 - q^{\frac{3}{2}}y)(1 - q^{\frac{3}{2}}y^{-1}). \tag{A5}$$

Appendix B. Fluctuation modes on D3

Let us consider a D3-brane wrapped on $X = 0$. The single-particle index for the fields living on the wrapped brane in S^5 without deficit angle was calculated in Ref. [2] by using variable changes from the index of the boundary $\mathcal{N} = 4$ vector multiplet. The explicit mode expansion is given in Ref. [18]. Let us first review the derivation and then we consider the effect of 7-branes.

The existence of a D3-brane wrapped on $X = 0$ respects only supercharges with quantum numbers $H = R_X$, and it breaks the superconformal algebra $usp(2, 2|4)$ to $psu(2|2) \times$

Table B1. Quantum numbers of the ground state and Q and $\overline{Q}$ used as raising operators.

	H	J	$\overline{J}$	R_X	R_Y	R_Z	A
Q	$+\frac{1}{2}$	$\pm\frac{1}{2}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	-1
$\overline{Q}$	$+\frac{1}{2}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+1$
R_-	0	0	0	0	$+1$	-1	0
$\|X^*\rangle$	$\ell-1$	0	0	-1	0	ℓ	0

Table B2. Fluctuation modes on a D3-brane wrapped on $X = 0$.

Rep.		H	J	$\overline{J}$	R_X	R_Y	R_Z	A	Range
$\mathcal{B}^{\mathrm{D3}(X=0)}_{m,r}$	$R_-^m\|X^*\rangle$	$m+r-1$	0	0	-1	m	r	0	
	$QR_-^m\|X^*\rangle$	$m+r-\frac{1}{2}$	$\pm\frac{1}{2}$	0	$-\frac{1}{2}$	$m+\frac{1}{2}$	$r-\frac{1}{2}$	-1	$r\geq 1$
	$Q^2R_-^m\|X^*\rangle$	$m+r$	0	0	0	$m+1$	$r-1$	-2	$r\geq 2$
$\mathcal{C}^{\mathrm{D3}(X=0)}_{m,r}$	$\overline{Q}R_-^m\|X^*\rangle$	$m+r+\frac{1}{2}$	0	$+\frac{1}{2}$	$-\frac{1}{2}$	$m-\frac{1}{2}$	$r+\frac{1}{2}$	$+1$	$m+r\geq 0$
	$Q\overline{Q}R_-^m\|X^*\rangle$	$m+r+1$	$\pm\frac{1}{2}$	$+\frac{1}{2}$	0	m	r	0	$r\geq 0$
	$Q^2\overline{Q}R_-^m\|X^*\rangle$	$m+r+\frac{3}{2}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$	$m+\frac{1}{2}$	$r-\frac{1}{2}$	-1	$r\geq 1$

psu(2|2). One of the *psu*(2|2) contains $SU(2)_J \times SU(2)_{\frac{R_Z-R_Y}{2}}$ and the other contains $SU(2)_{\overline{J}} \times SU(2)_{\frac{R_Z+R_Y}{2}}$ as the bosonic subgroups. There are eight supercharges with positive conformal dimension: Q and $\overline{Q}$ belonging to the bi-fundamental representations of $SU(2)_J \times SU(2)_{\frac{R_Z-R_Y}{2}}$ and $SU(2)_{\overline{J}} \times SU(2)_{\frac{R_Z+R_Y}{2}}$, respectively.

We want to construct short multiplets of excitations on the wrapped D3-brane. We apply Q and $\overline{Q}$ (and $SU(2)$ lowering operators) as raising operators on a ground state to form the whole multiplet. Because all components of Q and $\overline{Q}$ carry $R_X = +\frac{1}{2}$, the ground state must carry minimum R_X in the multiplet. We can use modes of X^* and denote the corresponding ground state by $|X^*\rangle$. There are different modes described by scalar S^3 harmonics on the wrapped D3. They are labeled by integer $\ell = 0, 1, 2, \ldots$, and belong to $[\frac{\ell}{2}, \frac{\ell}{2}]$ of $SU(2)_{\frac{R_Z-R_Y}{2}} \times SU(2)_{\frac{R_Z+R_Y}{2}}$. We start from the $SU(2)_{\frac{R_Z-R_Y}{2}} \times SU(2)_{\frac{R_Z+R_Y}{2}}$ highest-weight state that carries $\frac{R_Z-R_Y}{2} = \frac{R_Z+R_Y}{2} = \frac{\ell}{2}$, or, equivalently, $R_Z = \ell$, $R_Y = 0$. By applying raising operators on this state we obtain a *psu*(2|2) $\times$ *psu*(2|2) multiplet for each ℓ.

Due to the shortening conditions, we use only supercharges that do not increase either $\frac{R_Z-R_Y}{2}$ or $\frac{R_Z+R_Y}{2}$. Only two from Q and two from $\overline{Q}$ satisfy this condition. Furthermore, we are interested in BPS operators saturating the bound (7), and hence we use only supercharges carrying $\{\mathcal{Q}, \mathcal{Q}^\dagger\} = 0$. This condition excludes one of two components in $\overline{Q}$. For the same reason we do not use the $SU(2)_{\frac{R_Z+R_Y}{2}}$ lowering operator. As a result, we can use two components of Q, one component of $\overline{Q}$, and the $SU(2)_{\frac{R_Z-R_Y}{2}}$ lowering operator R_- in the construction of the representation. We show the quantum numbers of these operators and the ground state in Table B1.

A general state in the multiplet is schematically given as

$$Q^p\overline{Q}^qR_-^m|X^*\rangle, \quad p = 0, 1, 2, \quad q = 0, 1. \tag{B1}$$

The introduction of the 7-brane at $Z = 0$ breaks the supersymmetry $\overline{Q}$, and as a result the states in Eq. (A6) split into two types of representations: $\mathcal{B}^{\mathrm{D3}(X=0)}_{m,r}$ and $\mathcal{C}^{\mathrm{D3}(X=0)}_{m,r}$; see Table B2 for the states in each representation. The contributions of these representations to the index are as follows:

$$\sum_m i\left(\mathcal{B}^{\mathrm{D3}(X=0)}_{m,0}\right) = (qu_x)^{-1}\frac{1}{1-qu_y},$$

$$\sum_m i\left(\mathcal{B}^{\mathrm{D3}(X=0)}_{m,1}\right) = (qu_z)(qu_x)^{-1}\frac{1-q^{\frac{1}{2}}y^{\pm1}u_z^{-1}}{1-qu_y},$$

$$\sum_m i\left(\mathcal{B}^{\mathrm{D3}(X=0)}_{m,r>1}\right) = (qu_z)^r(qu_x)^{-1}\frac{\left(1-q^{\frac{1}{2}}yu_z^{-1}\right)\left(1-q^{\frac{1}{2}}y^{-1}u_z^{-1}\right)}{1-qu_y},$$

$$\sum_m i\left(\mathcal{C}^{\mathrm{D3}(X=0)}_{m,r=-1}\right) = -\frac{qu_y}{1-qu_y}+1,$$

$$\sum_m i\left(\mathcal{C}^{\mathrm{D3}(X=0)}_{m,r=0}\right) = -(qu_z)\frac{1-q^{\frac{1}{2}}y^{\pm1}u_z^{-1}}{1-qu_y},$$

$$\sum_m i\left(\mathcal{C}^{\mathrm{D3}(X=0)}_{m,r\geq0}\right) = -(qu_z)^{r+1}\frac{\left(1-q^{\frac{1}{2}}yu_z^{-1}\right)\left(1-q^{\frac{1}{2}}y^{-1}u_z^{-1}\right)}{1-qu_y}. \tag{B2}$$

Appendix C. Current algebra

We consider the current algebra of a simple Lie algebra G:

$$\left[J^a_m, J^b_n\right] = if_{abc}J^c_{m+n} + km\delta_{a,b}\delta_{m+n,0}, \tag{C1}$$

where f_{abc} are the structure constants of G. k is the level, and we are interested in the basic representation of the algebra, the Fock space constructed on a G-singlet ground state $|0\rangle$ with $k=1$.

The first few terms are

$$\chi^{\widehat{G}} = 1 + q\chi_\theta + q^2\left((\chi_\theta)^2_{\mathrm{sym}} - \chi_{2\theta} + \chi_\theta\right) + \cdots, \tag{C2}$$

where $(\chi_\theta)^2_{\mathrm{sym}}$ is the character of the symmetric product representation: $(\chi_\theta)^2_{\mathrm{sym}}(x) = (\chi_\theta(x)^2 + \chi_\theta(x^2))/2$. The first term and the second term correspond to the ground state $|0\rangle$ and the first excited states $J^a_{-1}|0\rangle$, respectively. The third term corresponds to two types of grade 2 states:

$$J^{\{a}_{-1}J^{b\}}_{-1}|0\rangle, \quad J^a_{-2}|0\rangle. \tag{C3}$$

The former belong to the symmetric product of two copies of the adjoint representation, which is decomposed into irreducible representations as

$$(R_\theta\otimes R_\theta)_{\mathrm{sym}} = R_{2\theta} + \cdots + R_0. \tag{C4}$$

We denote the representation with the highest weight w by R_w, and θ is the highest weight of the adjoint representation. With the commutation relation (C1) we can easily show that states in $R_{2\theta}$ are null states. By subtracting the null state contribution from the contribution of Eq. (C3) we obtain the q^2 term in Eq. (C2). Higher-order terms in $\chi^{\widehat{G}}$ can be effectively obtained by the free field realization:

$$\chi^{\widehat{G}}(q,x) = \frac{\sum_{p\in\Lambda_G} q^{\frac{p^2}{2}}x^p}{\prod_{n=1}^\infty(1-q^n)^r}, \quad r = \mathrm{rank}\, G. \tag{C5}$$

Λ_G is the root lattice of G. x collectively represents r fugacities, and $x^p = \prod_{i=1}^r x_i^{p_i}$. We show the first few terms of $\chi^{\widehat{G}}(q,1)$ for the seven types of G:

$$\chi^{\widehat{H}_0} = 1,$$
$$\chi^{\widehat{H}_1} = 1 + \chi^{H_1}_{\mathbf{3}} q + (1 + \chi^{H_1}_{\mathbf{3}})q^2 + (1 + 2\chi^{H_1}_{\mathbf{3}})q^3 + (2 + 2\chi^{H_1}_{\mathbf{3}} + \chi^{H_1}_{\mathbf{5}})q^4 + (2 + 4\chi^{H_1}_{\mathbf{3}} + \chi^{H_1}_{\mathbf{5}})q^5 + (4 + 5\chi^{H_1}_{\mathbf{3}} + 2\chi^{H_1}_{\mathbf{5}})q^6 + \cdots, \tag{C6a}$$

$$\chi^{\widehat{H}_2} = 1 + \chi^{H_2}_{\mathbf{8}} q + (1 + 2\chi^{H_2}_{\mathbf{8}})q^2 + (2 + 3\chi^{H_2}_{\mathbf{8}} + \chi^{H_2}_{\mathbf{10}} + \chi^{H_2}_{\overline{\mathbf{10}}})q^3 + (3 + 6\chi^{H_2}_{\mathbf{8}} + \chi^{H_2}_{\mathbf{10}} + \chi^{H_2}_{\overline{\mathbf{10}}} + \chi^{H_2}_{\mathbf{27}})q^4 + (4 + 10\chi^{H_2}_{\mathbf{8}} + 3\chi^{H_2}_{\mathbf{10}} + 3\chi^{H_2}_{\overline{\mathbf{10}}} + 2\chi^{H_2}_{\mathbf{27}})q^5 + (8 + 16\chi^{H_2}_{\mathbf{8}} + 5\chi^{H_2}_{\mathbf{10}} + 5\chi^{H_2}_{\overline{\mathbf{10}}} + 5\chi^{H_2}_{\mathbf{27}})q^6 + \cdots, \tag{C6b}$$

$$\chi^{\widehat{D}_4} = 1 + \chi^{D_4}_{\mathbf{28}} q + (1 + \chi^{D_4}_{\mathbf{28}} + \chi^{D_4}_{\mathbf{35}_V} + \chi^{D_4}_{\mathbf{35}_S} + \chi^{D_4}_{\mathbf{35}_C})q^2 + (1 + 4\chi^{D_4}_{\mathbf{28}} + \chi^{D_4}_{\mathbf{35}_V} + \chi^{D_4}_{\mathbf{35}_S} + \chi^{D_4}_{\mathbf{35}_C} + \chi^{D_4}_{\mathbf{350}})q^3 + (4 + 5\chi^{D_4}_{\mathbf{28}} + 3\chi^{D_4}_{\mathbf{35}_V} + 3\chi^{D_4}_{\mathbf{35}_S} + 3\chi^{D_4}_{\mathbf{35}_C} + \chi^{D_4}_{\mathbf{300}} + 3\chi^{D_4}_{\mathbf{350}})q^4 + (4 + 12\chi^{D_4}_{\mathbf{28}} + 5\chi^{D_4}_{\mathbf{35}_V} + 5\chi^{D_4}_{\mathbf{35}_S} + 5\chi^{D_4}_{\mathbf{35}_C} + \chi^{D_4}_{\mathbf{300}} + 7\chi^{D_4}_{\mathbf{350}} + \chi^{D_4}_{\mathbf{567}_V} + \chi^{D_4}_{\mathbf{567}_S} + \chi^{D_4}_{\mathbf{567}_C})q^5 + (9 + 18\chi^{D_4}_{\mathbf{28}} + 11\chi^{D_4}_{\mathbf{35}_V} + 11\chi^{D_4}_{\mathbf{35}_S} + 11\chi^{D_4}_{\mathbf{35}_C} + 5\chi^{D_4}_{\mathbf{300}} + 14\chi^{D_4}_{\mathbf{350}} + 2\chi^{D_4}_{\mathbf{567}_V} + 2\chi^{D_4}_{\mathbf{567}_S} + 2\chi^{D_4}_{\mathbf{567}_C} + \chi^{D_4}_{\mathbf{840}_V} + \chi^{D_4}_{\mathbf{840}_S} + \chi^{D_4}_{\mathbf{840}_C})q^6 + \cdots, \tag{C6c}$$

$$\chi^{\widehat{E}_6} = 1 + \chi^{E_6}_{\mathbf{78}} q + (1 + \chi^{E_6}_{\mathbf{78}} + \chi^{E_6}_{\mathbf{650}})q^2 + (1 + 2\chi^{E_6}_{\mathbf{78}} + 2\chi^{E_6}_{\mathbf{650}} + \chi^{E_6}_{\mathbf{2925}})q^3 + (2 + 4\chi^{E_6}_{\mathbf{78}} + 4\chi^{E_6}_{\mathbf{650}} + \chi^{E_6}_{\mathbf{2430}} + \chi^{E_6}_{\mathbf{2925}} + \chi^{E_6}_{\mathbf{5824}} + \chi^{E_6}_{\overline{\mathbf{5824}}})q^4 + (3 + 7\chi^{E_6}_{\mathbf{78}} + 7\chi^{E_6}_{\mathbf{650}} + \chi^{E_6}_{\mathbf{2430}} + 4\chi^{E_6}_{\mathbf{2925}} + 2\chi^{E_6}_{\mathbf{5824}} + 2\chi^{E_6}_{\overline{\mathbf{5824}}} + \chi^{E_6}_{\mathbf{34\,749}})q^5 + (6 + 11\chi^{E_6}_{\mathbf{78}} + 14\chi^{E_6}_{\mathbf{650}} + 3\chi^{E_6}_{\mathbf{2430}} + 7\chi^{E_6}_{\mathbf{2925}} + \chi^{E_6}_{\mathbf{3003}} + \chi^{E_6}_{\overline{\mathbf{3003}}} + 4\chi^{E_6}_{\mathbf{5824}} + 4\chi^{E_6}_{\overline{\mathbf{5824}}} + 3\chi^{E_6}_{\mathbf{34\,749}} + \chi^{E_6}_{\mathbf{70\,070}})q^6 + \cdots, \tag{C6d}$$

$$\chi^{\widehat{E}_7} = 1 + \chi^{E_7}_{\mathbf{133}} q + (1 + \chi^{E_7}_{\mathbf{133}} + \chi^{E_7}_{\mathbf{1539}})q^2 + (1 + 2\chi^{E_7}_{\mathbf{133}} + \chi^{E_7}_{\mathbf{1463}} + \chi^{E_7}_{\mathbf{1539}} + \chi^{E_7}_{\mathbf{8645}})q^3 + (2 + 3\chi^{E_7}_{\mathbf{133}} + \chi^{E_7}_{\mathbf{1463}} + 3\chi^{E_7}_{\mathbf{1539}} + \chi^{E_7}_{\mathbf{7371}} + \chi^{E_7}_{\mathbf{8645}} + \chi^{E_7}_{\mathbf{40\,755}})q^4 + (2 + 6\chi^{E_7}_{\mathbf{133}} + 3\chi^{E_7}_{\mathbf{1463}} + 4\chi^{E_7}_{\mathbf{1539}} + \chi^{E_7}_{\mathbf{7371}} + 3\chi^{E_7}_{\mathbf{8645}} + 2\chi^{E_7}_{\mathbf{40\,755}} + \chi^{E_7}_{\mathbf{152\,152}})q^5 + (5 + 8\chi^{E_7}_{\mathbf{133}} + 4\chi^{E_7}_{\mathbf{1463}} + 9\chi^{E_7}_{\mathbf{1539}} + 3\chi^{E_7}_{\mathbf{7371}} + 5\chi^{E_7}_{\mathbf{8645}} + 4\chi^{E_7}_{\mathbf{40\,755}} + \chi^{E_7}_{\mathbf{150\,822}} + 2\chi^{E_7}_{\mathbf{152\,152}} + \chi^{E_7}_{\mathbf{365\,750}})q^6 + \cdots, \tag{C6e}$$

$$\chi^{\widehat{E}_8} = 1 + \chi^{E_8}_{\mathbf{248}} q + q^2(1 + \chi^{E_8}_{\mathbf{248}} + \chi^{E_8}_{\mathbf{3875}})q + (1 + 2\chi^{E_8}_{\mathbf{248}} + \chi^{E_8}_{\mathbf{3875}} + \chi^{E_8}_{\mathbf{30\,380}})q^3 + (2 + 3\chi^{E_8}_{\mathbf{248}} + 2\chi^{E_8}_{\mathbf{3875}} + \chi^{E_8}_{\mathbf{27\,000}} + \chi^{E_8}_{\mathbf{30\,380}} + \chi^{E_8}_{\mathbf{147\,250}})q^4 + (2 + 5\chi^{E_8}_{\mathbf{248}} + 3\chi^{E_8}_{\mathbf{3875}} + \chi^{E_8}_{\mathbf{27\,000}} + 3\chi^{E_8}_{\mathbf{30\,380}} + \chi^{E_8}_{\mathbf{147\,250}} + \chi^{E_8}_{\mathbf{779\,247}})q^5 + (4 + 7\chi^{E_8}_{\mathbf{248}} + 6\chi^{E_8}_{\mathbf{3875}} + 3\chi^{E_8}_{\mathbf{27\,000}} + 4\chi^{E_8}_{\mathbf{30\,380}} + 2\chi^{E_8}_{\mathbf{147\,250}} + 2\chi^{E_8}_{\mathbf{779\,247}} + \chi^{E_8}_{\mathbf{2450\,240}})q^6 + \cdots. \tag{C6f}$$

Appendix D. Kaluza–Klein index

The bulk contributions calculated by Eq. (8) are shown below:

$$\mathcal{I}^{\rm KK}_{H_0} = 1 + u_z^{\frac{6}{5}} q^{\frac{6}{5}} - u_z^{\frac{1}{5}} \chi^J_1 q^{\frac{17}{10}} + u_z^{-1}\chi^F_2 q^2 + (u_z^{\frac{7}{10}}\chi^F_1 + u_z^{-\frac{4}{5}})q^{\frac{11}{5}} + 2u_z^{\frac{12}{5}} q^{\frac{12}{5}} + (u_z^{\frac{6}{5}} - u_z^{-\frac{3}{10}}\chi^F_1)\chi^J_1 q^{\frac{27}{10}} - 2u_z^{\frac{7}{5}}\chi^J_1 q^{\frac{29}{10}} + (-1 + u_z^{-\frac{3}{2}}\chi^F_3 - \chi^F_2)q^3 + \cdots, \tag{D1a}$$

$$\begin{aligned}
\mathcal{I}^{\mathrm{KK}}_{H_1} = {} & 1 + u_z^{\frac{4}{3}} q^{\frac{4}{3}} - u_z^{\frac{1}{3}} \chi_1^J q^{\frac{11}{6}} + \big(u_z^{-1}\chi_2^F + u_z^{-1}\chi_{\mathbf{3}}^{H_1}\big) q^2 + \big(u_z^{-\frac{2}{3}} + \chi_1^F u_z^{\frac{5}{6}}\big) q^{\frac{7}{3}} + 2u_z^{\frac{8}{3}} q^{\frac{8}{3}} \\
& + \big(- u_z^{-\frac{1}{6}}\chi_1^F + u_z^{\frac{4}{3}}\big)\chi_1^J q^{\frac{17}{6}} + \big(- 1 - \chi_2^F + u_z^{-\frac{3}{2}}\chi_3^F - \chi_{\mathbf{3}}^{H_1} + u_z^{-\frac{3}{2}}\chi_1^F\chi_{\mathbf{3}}^{H_1}\big) q^3 \\
& - 2u_z^{\frac{5}{3}}\chi_1^J q^{\frac{19}{6}} + \big(u_z^{-\frac{7}{6}}\chi_1^F - u_z^{\frac{1}{3}} + 2u_z^{\frac{1}{3}}\chi_2^F - uz^{\frac{11}{6}}\chi_1^F - u_z^{\frac{1}{3}}\chi_2^J + u_z^{\frac{1}{3}}\chi_{\mathbf{3}}^{H_1}\big) q^{\frac{10}{3}} \\
& + \big(u_z^{-1} + u_z^{-1}\chi_2^F + u_z^{-1}\chi_{\mathbf{3}}^{H_1}\big)\chi_1^J q^{\frac{7}{2}} + \big(3u_z^{\frac{2}{3}} + 2u_z^{\frac{13}{6}}\chi_1^F\big) q^{\frac{11}{3}} + \cdots ,
\end{aligned} \tag{D1b}$$

$$\begin{aligned}
\mathcal{I}^{\mathrm{KK}}_{H_2} = {} & 1 + u_z^{\frac{3}{2}} q^{\frac{3}{2}} + \big(u_z^{-1}\chi_2^F - u_z^{\frac{1}{2}}\chi_1^J + u_z^{-1}\chi_{\mathbf{8}}^{H_2}\big) q^2 + \big(u_z^{-\frac{1}{2}} + u_z\chi_1^F\big) q^{\frac{5}{2}} + \big(- 1 - \chi_2^F \\
& + u_z^{\frac{3}{2}}\chi_3^F + 2u_z^3 - \chi_1^F\chi_1^J + u_z^{\frac{3}{2}}\chi_1^J - \chi_{\mathbf{8}}^{H_2} + u_z^{-\frac{3}{2}}\chi_1^F\chi_{\mathbf{8}}^{H_2}\big) q^3 + \big(u_z^{-1}\chi_1^F - u_z^{\frac{1}{2}} + 2u_z^{\frac{1}{2}}\chi_2^F \\
& - u_z^2\chi_1^F + u_z^{-1}\chi_1^J + u_z^{-1}\chi_2^F\chi_1^J - 2u_z^2\chi_1^J - u_z^{\frac{1}{2}}\chi_2^J + u_z^{\frac{1}{2}}\chi_{\mathbf{8}}^{H_2} + u_z^{-1}\chi_{\mathbf{8}}^{H_2}\chi_1^J\big) q^{\frac{7}{2}} \\
& + \big(2u_z^{-2} + 2u_z^{-2}\chi_4^F - u_z^{-\frac{1}{2}}\chi_1^F - u_z^{-\frac{1}{2}}\chi_3^F + 4u_z + 2u_z^{\frac{5}{2}}\chi_1^F + u_z^{-\frac{1}{2}}\chi_1^J - 2u_z^{-\frac{1}{2}}\chi_2^F\chi_1^J \\
& + 2u_z\chi_1^F\chi_1^J + u_z^{-2}\chi_{\mathbf{8}}^{H_2} + 2u_z^{-2}\chi_2^F\chi_{\mathbf{8}}^{H_2} - u_z^{-\frac{1}{2}}\chi_1^F\chi_{\mathbf{8}}^{H_2} - u_z^{-\frac{1}{2}}\chi_1^J\chi_{\mathbf{8}}^{H_2} + u_z^2\chi_{\mathbf{27}}^{H_2}\big) q^4 \\
& + \big(- \chi_1^F + 3\chi_3^F + 2u_z^{-\frac{3}{2}}\chi_2^F - 2u_z^{\frac{3}{2}} - 2u_z^{\frac{3}{2}}\chi_2^F + 3u_z^{\frac{9}{2}} - 3\chi_1^J - \chi_2^F\chi_1^J + u_z^{-\frac{3}{2}}\chi_1^F\chi_1^J \\
& + u_z^{-\frac{3}{2}}\chi_3^F\chi_1^J - 3u_z^{\frac{3}{2}}\chi_1^F\chi_1^J + 2u_z^3\chi_1^J - \chi_1^F\chi_2^J + u_z^{\frac{3}{2}}\chi_2^J + 2\chi_1^F\chi_{\mathbf{8}}^{H_2} + u_z^{-\frac{3}{2}}\chi_{\mathbf{8}}^{H_2} \\
& - u_z^{\frac{3}{2}}\chi_{\mathbf{8}}^{H_2} - \chi_1^J\chi_{\mathbf{8}}^{H_2} + u_z^{-\frac{3}{2}}\chi_1^F\chi_1^J\chi_{\mathbf{8}}^{H_2}\big) q^{\frac{9}{2}} + \cdots ,
\end{aligned} \tag{D1c}$$

$$\begin{aligned}
\mathcal{I}^{\mathrm{KK}}_{D_4} = {} & 1 + \big(u_z^2 + \chi_2^F u_z^{-1} + u_z^{-1}\chi_{\mathbf{28}}^{D_4}\big) q^2 - u_z\chi_1^J q^{\frac{5}{2}} + \big(- \chi_2^F + \chi_1^F u_z^{\frac{3}{2}} + \chi_3^F u_z^{-\frac{3}{2}} - \chi_{\mathbf{28}}^{D_4} \\
& + \chi_1^F u_z^{-\frac{3}{2}}\chi_{\mathbf{28}}^{D_4}\big) q^3 + \big(u_z^2\chi_1^J - \chi_1^F u_z^{\frac{1}{2}}\chi_1^J + u_z^{-1}\chi_1^J + \chi_2^F u_z^{-1}\chi_1^J + u_z^{-1}\chi_1^J\chi_{\mathbf{28}}^{D_4}\big) q^{\frac{7}{2}} \\
& + \big(2u_z^4 - \chi_1^F u_z^{\frac{5}{2}} + 2\chi_2^F u_z - \chi_3^F u_z^{-\frac{1}{2}} + 2u_z^{-2} + 2\chi_4^F u_z^{-2} - u_z\chi_2^J + \big(u_z - \chi_1^F u_z^{-\frac{1}{2}} \\
& + 2\chi_2^F u_z^{-2}\big)\chi_{\mathbf{28}}^{D_4} + u_z^{-2}\chi_{\mathbf{300}}^{D_4} + u_z^{-2}\chi_{\mathbf{35}_v}^{D_4} + u_z^{-2}\chi_{\mathbf{35}_s}^{D_4} + u_z^{-2}\chi_{\mathbf{35}_c}^{D_4}\big) q^4 + \big(2u_z^{-\frac{5}{2}}\chi_1^F \\
& + u_z^{-\frac{5}{2}}\chi_3^F + 2u_z^{-\frac{5}{2}}\chi_5^F - u_z^{-1} - u_z^{-1}\chi_2^F - 2u_z^{-1}\chi_4^F + 3u_z^{\frac{1}{2}}\chi_3^F + u_z^2 - 2u_z^2\chi_2^F \\
& + 2u_z^{\frac{7}{2}}\chi_1^F + u_z^{-1}\chi_2^J + u_z^{-1}\chi_2^F\chi_2^J - u_z^{\frac{1}{2}}\chi_1^F\chi_2^J + u_z^2\chi_2^J + 2u_z^{-\frac{5}{2}}\chi_1^F\chi_{\mathbf{28}}^{D_4} \\
& + 3u_z^{-\frac{5}{2}}\chi_3^F\chi_{\mathbf{28}}^{D_4} - u_z^{-1}\chi_{\mathbf{28}}^{D_4} - 3u_z^{-1}\chi_2^F\chi_{\mathbf{28}}^{D_4} + 2u_z^{-\frac{1}{2}}\chi_1^F\chi_{\mathbf{28}}^{D_4} - u_z^2\chi_{\mathbf{28}}^{D_4} + u_z^{-1}\chi_2^J\chi_{\mathbf{28}}^{D_4} \\
& + u_z^{-\frac{5}{2}}\chi_1^F\big(\chi_{\mathbf{35}_v}^{D_4} + \chi_{\mathbf{35}_s}^{D_4} + \chi_{\mathbf{35}_c}^{D_4}\big) - u_z^{-1}\big(\chi_{\mathbf{35}_v}^{D_4} + \chi_{\mathbf{35}_s}^{D_4} + \chi_{\mathbf{35}_c}^{D_4}\big) + u_z^{-\frac{5}{2}}\chi_1^F\chi_{\mathbf{300}}^{D_4} \\
& - u_z^{-1}\chi_{\mathbf{300}}^{D_4} + u_z^{-\frac{5}{2}}\chi_1^F\chi_{\mathbf{350}}^{D_4} - u_z^{-1}\chi_{\mathbf{350}}^{D_4}\big) q^5 + \cdots ,
\end{aligned} \tag{D1d}$$

$$\begin{aligned}\mathcal{I}_{E_6}^{\mathrm{KK}} &= 1 + \left(u_z^{-1}\chi_2^F + u_z^{-1}\chi_{\mathbf{78}}^{E_6}\right)q^2 + \left(-1 - \chi_2^F + u_z^{-\frac{3}{2}}\chi_3^F + u_z^3 - \chi_{\mathbf{78}}^{E_6} + u_z^{-\frac{3}{2}}\chi_1^F\chi_{\mathbf{78}}^{E_6}\right)q^3 \\ &+ \left(u_z^{-1} + u_z^{-1}\chi_2^F - u_z^2 + u_z^{-1}\chi_{\mathbf{78}}^{E_6}\right)\chi_1^J q^{\frac{7}{2}} + \left(2u_z^{-2} + 2u_z^{-2}\chi_4^F - u_z^{-\frac{1}{2}}\chi_1^F - u_z^{-\frac{1}{2}}\chi_3^F\right. \\ &+ 2u_z + uz^{\frac{5}{2}}\chi_1^F + 2u_z^{-2}\chi_2^F\chi_{\mathbf{78}}^{E_6} - u_z^{-\frac{1}{2}}\chi_1^F\chi_{\mathbf{78}}^{E_6} + u_z^{-2}\chi_{\mathbf{650}}^{E_6} + u_z^{-2}\chi_{\mathbf{2430}}^{E_6}\Big)q^4 + \Big(-2 \\ &- \chi_2^F + u_z^{-\frac{3}{2}}\chi_1^F + u_z^{-\frac{3}{2}}\chi_3^F - uz^{\frac{3}{2}}\chi_1^F + u_z^3 - \chi_{78} + u_z^{-\frac{3}{2}}\chi_1^F\chi_{\mathbf{78}}^{E_6}\Big)\chi_1^J q^{\frac{9}{2}} + \Big(2u_z^{-\frac{5}{2}}\chi_1^F \\ &+ uz^{-\frac{5}{2}}\chi_3^F + 2u_z^{-\frac{5}{2}}\chi_5^F - u_z^{-1} - 3u_z^{-1}\chi_2^F - 2u_z^{-1}\chi_4^F + 2u_z^{\frac{1}{2}}\chi_1^F - u_z^2 + 2u_z^2\chi_2^F \\ &- u_z^{\frac{7}{2}}\chi_1^F + u_z^{-1}\chi_2^J + u_z^{-1}\chi_2^F\chi_2^J - u_z^2\chi_2^J + 2u_z^{-\frac{5}{2}}\chi_1^F\chi_{\mathbf{78}}^{E_6} + 3u_z^{-\frac{5}{2}}\chi_3^F\chi_{\mathbf{78}}^{E_6} - 2u_z\chi_{\mathbf{78}}^{E_6} \\ &- 3u_z\chi_2^F\chi_{\mathbf{78}}^{E_6} + u_z^2\chi_{\mathbf{78}}^{E_6} + u_z\chi_2^J\chi_{\mathbf{78}}^{E_6} + u_z^{-\frac{5}{2}}\chi_1^F\chi_{\mathbf{650}}^{E_6} - u_z^{-1}\chi_{\mathbf{650}}^{E_6} + u_z^{-\frac{5}{2}}\chi_1^F\chi_{\mathbf{2430}}^{E_6} \\ &- u_z^{-1}\chi_{\mathbf{2430}}^{E_6} + u_z^{-\frac{5}{2}}\chi_1^F\chi_{\mathbf{2925}}^{E_6} - u_z^{-1}\chi_{\mathbf{2925}}^{E_6}\Big)q^5 + \cdots, \qquad \text{(D1e)}\end{aligned}$$

$$\begin{aligned}\mathcal{I}_{E_7}^{\mathrm{KK}} &= 1 + \left(u_z^{-1}\chi_2^F + u_z^{-1}\chi_{\mathbf{133}}^{E_7}\right)q^2 + \left(-1 - \chi_2^F + u_z^{-\frac{3}{2}}\chi_3^F - \chi_{\mathbf{133}}^{E_7} + u_z^{-\frac{3}{2}}\chi_1^F\chi_{\mathbf{133}}^{E_7}\right)q^3 \\ &+ \left(u_z^{-1} + u_z^{-1}\chi_2^F + u_z^{-1}\chi_{\mathbf{133}}^{E_7}\right)\chi_1^J q^{\frac{7}{2}} + \Big(2u_z^{-2} + 2u_z^{-2}\chi_4^F - u_z^{-\frac{1}{2}}\chi_1^F - u_z^{-\frac{1}{2}}\chi_3^F \\ &+ u_z + u_z^4 + 2u_z^{-2}\chi_2^F\chi_{\mathbf{133}}^{E_7} - u_z^{-\frac{1}{2}}\chi_1^F\chi_{\mathbf{133}}^{E_7} + u_z^{-2}\chi_{\mathbf{1539}}^{E_7} + u_z^{-2}\chi_{\mathbf{7371}}^{E_7}\Big)q^4 + \Big(-2 \\ &- \chi_2^F + u_z^{-\frac{3}{2}}\chi_1^F + u_z^{-\frac{3}{2}}\chi_3^F - u_z^3 - \chi_{\mathbf{133}}^{E_7} + u_z^{-\frac{3}{2}}\chi_1^F\chi_{\mathbf{133}}^{E_7}\Big)\chi_1^J q^{\frac{9}{2}} + \Big(2u_z^{-\frac{5}{2}}\chi_1^F + u_z^{-\frac{5}{2}}\chi_3^F \\ &+ 2u_z^{-\frac{5}{2}}\chi_5^F - u_z^{-1} - 3u_z^{-1}\chi_2^F - 2u_z^{-1}\chi_4^F + u_z^{\frac{1}{2}}\chi_1^F + u_z^2 + u_z^{\frac{7}{2}}\chi_1^F + u_z^{-1}\chi_2^J \\ &+ u_z^{-1}\chi_2^F\chi_2^J + 2u_z^{-\frac{5}{2}}\chi_1^F\chi_{\mathbf{133}}^{E_7} + 3u_z^{-\frac{5}{2}}\chi_3^F\chi_{\mathbf{133}}^{E_7} - 2u_z^{-1}\chi_{\mathbf{133}}^{E_7} - 3u_z^{-1}\chi_2^F\chi_{\mathbf{133}}^{E_7} \\ &+ u_z^{-1}\chi_2^J\chi_{\mathbf{133}}^{E_7} + u_z^{-\frac{5}{2}}\chi_1^F\chi_{\mathbf{1539}}^{E_7} - u_z^{-1}\chi_{\mathbf{1539}}^{E_7} + u_z^{-\frac{5}{2}}\chi_1^F\chi_{\mathbf{7371}}^{E_7} - u_z^{-1}\chi_{\mathbf{7371}}^{E_7} \\ &+ u_z^{-\frac{5}{2}}\chi_1^F\chi_{\mathbf{8645}}^{E_7} - u_z^{-1}\chi_{\mathbf{8645}}^{E_7}\Big)q^5 + \cdots, \qquad \text{(D1f)}\end{aligned}$$

$$\begin{aligned}\mathcal{I}_{E_8}^{\mathrm{KK}} &= 1 + \left(u_z^{-1}\chi_2^F + u_z^{-1}\chi_{\mathbf{248}}^{E_8}\right)q^2 + \left(-1 - \chi_2^F + u_z^{-\frac{3}{2}}\chi_3^F - \chi_{\mathbf{248}}^{E_8} + u_z^{-\frac{3}{2}}\chi_1^F\chi_{\mathbf{248}}^{E_8}\right)q^3 \\ &+ \left(u_z^{-1} + u_z^{-1}\chi_2^F + u_z^{-1}\chi_{\mathbf{248}}^{E_8}\right)\chi_1^J q^{\frac{7}{2}} + \Big(2u_z^{-2} + 2u_z^{-2}\chi_4^F - u_z^{-\frac{1}{2}}\chi_1^F - u_z^{-\frac{1}{2}}\chi_3^F \\ &+ u_z + 2u_z^{-2}\chi_2^F\chi_{\mathbf{248}}^{E_8} - u_z^{-\frac{1}{2}}\chi_1^F\chi_{\mathbf{248}}^{E_8} + u_z^{-2}\chi_{\mathbf{3875}}^{E_8} + u_z^{-2}\chi_{\mathbf{27\,000}}^{E_8}\Big)q^4 + \Big(-2 - \chi_2^F \\ &+ u_z^{-\frac{3}{2}}\chi_1^F + u_z^{-\frac{3}{2}}\chi_3^F - \chi_{\mathbf{248}}^{E_8} + u_z^{-\frac{3}{2}}\chi_1^F\chi_{\mathbf{248}}^{E_8}\Big)\chi_1^J q^{\frac{9}{2}} + \Big(2u_z^{-\frac{5}{2}}\chi_1^F + u_z^{-\frac{5}{2}}\chi_3^F \\ &+ 2u_z^{-\frac{5}{2}}\chi_5^F - u_z^{-1} - 3u_z^{-1}\chi_2^F - 2u_z^{-1}\chi_4^F + u_z^{\frac{1}{2}}\chi_1^F + u_z^{-1}\chi_2^J + u_z^{-1}\chi_2^F\chi_2^J \\ &+ 2u_z^{-\frac{5}{2}}\chi_1^F\chi_{\mathbf{248}}^{E_8} + 3u_z^{-\frac{5}{2}}\chi_3^F\chi_{\mathbf{248}}^{E_8} - 2u_z^{-1}\chi_{\mathbf{248}}^{E_8} - 3u_z^{-1}\chi_2^F\chi_{\mathbf{248}}^{E_8} + u_z^{-1}\chi_{\mathbf{248}}^{E_8}\chi_2^J \\ &+ u_z^{-\frac{5}{2}}\chi_1^F\chi_{\mathbf{3875}}^{E_8} - u_z^{-1}\chi_{\mathbf{3875}}^{E_8} + u_z^{-\frac{5}{2}}\chi_1^F\chi_{\mathbf{27\,000}}^{E_8} - u_z^{-1}\chi_{\mathbf{27\,000}}^{E_8} + u_z^{-\frac{5}{2}}\chi_1^F\chi_{\mathbf{30\,380}}^{E_8} \\ &- u_z^{-1}\chi_{\mathbf{30\,380}}^{E_8}\Big)q^5 + \cdots. \qquad \text{(D1g)}\end{aligned}$$

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