universe

IMPACT FACTOR 2.5

CITESCORE 4.3

Review

Spinning Systems in Quantum Mechanics: An Overview and New Trends

E. Brito, Júlio E. Brandão and Márcio M. Cunha

MDPI

https://doi.org/10.3390/universe10100389 universe

MDPI

Review

Spinning Systems in Quantum Mechanics: An Overview and New Trends

E. Brito [1], Júlio E. Brandão [2] and Márcio M. Cunha [3,*]

[1] Centro de Ciências Exatas e das Tecnologias, Universidade Federal do Oeste da Bahia, Rua Bertioga 892, Barreiras 47810-059, BA, Brazil; eliasbaj@ufob.edu.br
[2] Centro de Ciência e Tecnologia em Energia e Sustentabilidade, Universidade Federal do Recôncavo da Bahia, Feira de Santana 44042-280, BA, Brazil; julio.brandao@ufrb.edu.br
[3] Unidade Educacional de Penedo, Campus Arapiraca, Universidade Federal de Alagoas, Av. Beira Rio, s/n—Centro Histórico, Penedo 57200-000, AL, Brazil
* Correspondence: marcio.cunha@penedo.ufal.br

Abstract: The study of spinning systems plays a question of interest in several research branches in physics. It allows the understanding of simple classical mechanical systems but also provides us with tools to investigate a wide range of phenomena, from condensed matter physics to gravitation and cosmology. In this contribution, we review some remarkable theoretical aspects involving the description of spinning quantum systems. We explore the nonrelativistic and relativistic domains and their respective applications in fields such as graphene physics and topological defects in gravitation.

Keywords: rotation; quantum mechanics; noninertial effects

Citation: Brito, E.; Brandão, J.E.; Cunha, M.M. Spinning Systems in Quantum Mechanics: An Overview and New Trends. *Universe* **2024**, *10*, 389. https://doi.org/10.3390/universe10100389

Academic Editor: Andreas Fring

Received: 12 August 2024
Revised: 1 October 2024
Accepted: 2 October 2024
Published: 4 October 2024

1. Introduction

Describing natural phenomena in physics demands intrinsic information about spatial coordinates and referential frames. Since the rise of classical mechanics, the connection between different referential frames is also a question of interest, related to the frame invariance. Indeed, the description of rotating frames and their implications for the properties of a physical system is an essential issue in general physics. While the physics of classical rotating systems is well established, there are quantum mechanical rotating systems of interest, such as low-dimensional and relativistic systems.

In a recent manuscript, Chernodub [1] argued that certain semiconductors should exhibit an asymmetric behavior in their mechanical and conducting properties concerning the clockwise/anticlockwise rotations. Chernodub has indicated that this feature could be employed to create a rotational diode. Wand et al. [2] designed simple nanoscale rotary machines enabled by electron tunneling. Tu et al. [3] reported the design process of a desalination device made of a rotating carbon nanotube. Stickler et al. [4] presented a review of experimental aspects of rotations at nanoscale devices and their applications. The influence of rotation on the properties of Bose-Einstein condensates has also been investigated. For instance, Arbunich et al. [5] examined the stability of Bose-Einstein condensates in rotating harmonic traps. They demonstrated that the energy functional is coercive and the ground states are stable when the angular velocity $|\Omega|$ is less than the smallest trapping frequency ω. Sakaguchi and Malomed [6] considered a model for a rotating attractive Bose-Einstein condensate in two dimensions. They have demonstrated the presence of quasisolitons in the lowest Landau level when the angular velocity is equal to the trapping frequency. Guo et al. [7] argued that the rotation velocity of a rotating Bose-Einstein condensate behaves as a critical parameter related to the existence of ground states. The quantum control of molecular rotations also has attracted attention, due to its wide range of applications. Koch et al. [8] discussed the main control phenomena in a review.

Spinning quantum systems have also been investigated in the context of high-energy physics and gravitation. For instance, Guvendi and Dogan [9] have considered the description of vector bosons in a rotating frame of a negative curvature wormhole. It was demonstrated that the interplay between the vector boson spin and the rotation can create real oscillation modes. Santos and Barros [10] have presented a study on the quantization of a massless scalar field in a rotating frame. It was shown that the noninertial effects limit the region where the particles can be observed. They also have demonstrated that the vacuum Casimir energy density is shifted due to noninertial effects. Recently, Švančara et al. [11] reported a scheme to create a simulator of a rotating curved spacetime by using superfluid helium. It was shown the possibility of investigating superradiance and black hole ringing by employing this analog system. Besides that, Švančara's work provides a method to investigate rotating spacetimes with tunable angular momentum.

Rotating effects also take place in quantum composite systems. In this context, Silverman [12] has studied a bounded two-particle system in a rotating frame. Silverman has shown that rotation can split the magnetic substates of an atomic system and has examined the equivalence principle for quantum systems. More recently, Guvendi and Hassanabadi [13] have reported a study of the dynamics of a fermion–antifermion pair in a rotating frame. It was shown that the rotation parameter affects the strength of the interaction between the particles. Also, rotation can influence the pair creation process. As argued by Maniharsingh [14], the pair creation will happen spontaneously in the presence of a strong gravitational field outside a rotating astrophysical object. In particular, Maniharsingh has studied the Robertson–Walker model and found a relationship between the number of created particles and the age of the universe. This proves that detectors at rest in a rotating frame find radiation in the vacuum.

Spinning systems also provide novel predictions regarding the quantum foundations. Vieira et al. [15] have considered the Aharonov–Bohm effect for light in a moving medium. The authors evaluated the quantum phase acquired by the photon when it passes around a rotating cylinder. Toroš et al. [16] have presented an experimental setup for studying a photon in path superposition, in which there are two rotating interferometers. They argued that it is possible to generate entanglement at low rotation frequencies.

Motivated by this wide range of phenomena that are influenced by the presence of rotation, in this manuscript we review the main aspects involving the quantum mechanical description of spinning systems. The manuscript is organized as follows. In Section 2, we discuss the description of classical rotating systems. Our goal is to obtain a generic form for the Hamiltonian, which will be useful in the next sections. Section 3 is dedicated to studying the effects of rotation on nonrelativistic quantum systems. We provide examples of studies dealing with rotating effects on the physical properties of a system. In Section 4, we consider the relativistic quantum mechanics in a rotating frame by introducing a line element containing information on the rotation and the corresponding Dirac equation. We also discuss some applications of this framework. In Section 5, we consider the interplay between topological defects and rotation in quantum systems. We make our conclusions and final remarks in Section 6.

2. Classical Rotating Systems

Following [17], let us revisit the equations describing a classical particle in a rotating frame. A free particle of mass m in a rotating frame experiences the inertial force given by

$$\vec{F}_{inertial} = 2m\vec{v} \times \vec{\Omega} - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}), \quad (1)$$

where $\vec{\Omega}$ is the angular velocity and $\vec{\Omega} \times \vec{r}$ is the relative velocity of rotation between two reference frames. The terms on the right-hand-side correspond to the Coriolis and centrifuge forces, respectively. Since in the present manuscript we are interested in the description of rotating effects, let us consider the Lagrangian and the corresponding Hamiltonian for a particle in a rotating frame. The Lagrangian is written as

$$\mathcal{L} = \frac{mv^2}{2} + m\vec{v}.\left(\vec{\Omega} \times \vec{r}\right) + \frac{m}{2}\left(\vec{\Omega} \times \vec{r}\right)^2. \tag{2}$$

The canonical momentum is given by

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + m\left(\vec{\Omega} \times \vec{r}\right). \tag{3}$$

Then, the Hamiltonian is

$$H = \vec{p}.\vec{v} - \mathcal{L}, \tag{4}$$

or, more explicitly,

$$H = \frac{1}{2m}(\vec{p} - 2m\vec{A}_{ine})^2 + mV_{ine}, \tag{5}$$

where $\vec{A}_{ine} = \frac{1}{2}(\vec{\Omega} \times \vec{r})$, is a vector inertial potential, and $V_{ine} = -\frac{1}{2}(\vec{\Omega} \times \vec{r})^2$, is a scalar inertial potential. We notice that the Hamiltonian for a free particle in a rotating frame is similar to that for a charged particle in the presence of a magnetic and an electric field. This similarity has been explored in the literature and will be discussed in the next sections. Another way to write the expression (5) is

$$H = \frac{p^2}{2m} - \vec{\Omega}.\vec{L}, \tag{6}$$

where L corresponds to the orbital angular momentum. The term $\vec{\Omega}.\vec{L}$ refers to a coupling between the rotation parameter and the angular momentum. In the following, we will dedicate our attention to the quantum counterpart of Equation (6) and its implications for the quantum motion of an electron in the non-relativistic domain. Before this, we will briefly discuss an interesting result dealing with rotation, which is the Sagnac effect.

The Sagnac Effect

The Sagnac effect is one of the most prominent phenomena involving rotating systems. It is a phenomenon that detects absolute rotations in relativistic systems. In 1913, Georges Sagnac demonstrated the feasibility of conducting an optical–mechanical experiment that characterized a measurement of rotation where the interferometer was at rest [18]. Although Sagnac and some critics of the theory of relativity used the phenomenon as proof of the existence of ether to the detriment of Einstein's theory, some time later, Langevin and Von Laue demonstrated the equations through relativistic arguments, proving that the Sagnac effect is a consequence of the theory of relativity [19,20].

Let us consider a source that emits, at the same time, two beams of light in opposite directions along circular paths, like in an optical fiber. If the detector is in a position diametrically opposite to the source, the beams will reach the detector in the same time interval, as they have traveled the same distance. On the other hand, now consider that the circle rotates with angular velocity ω in the clockwise direction after the beams are sent by the source. The beam that follows the path towards the approaching detector will arrive before the beam that moves in the same direction as the detector. The difference in the time interval between the arrivals of the beams is given by

$$\Delta t = \frac{4\pi\gamma^2 r^2 \omega}{c^2}, \tag{7}$$

where γ is the relativistic term, r is the radius of the disk, and c is the speed of light in a vacuum. The expression (7) indicates an interference pattern which causes a redshift given by

$$\Delta\phi = \frac{8\pi A\omega}{\lambda c}, \quad (8)$$

where $A = \pi R$ is the area of the disk and λ is the wavelength of the beam. According to [21], the speed in both directions will be

$$v = r\frac{d\theta}{dt} = -r\omega \pm c. \quad (9)$$

It allows us to measure the laboratory angular velocity optically, and determine the Earth's rotation velocity. The Sagnac effect continues to be studied and measured under different conditions, such as in [22], which describes tests of HeNe ring lasers in physical and geophysical systems. Matos et al. [23] reported a quantum electrodynamical analog of the Sagnac effect. Gautier et al. [24] reported an accurate test for the Sagnac effect for matter waves.

3. Nonrelativistic Quantum Mechanics in a Rotating Frame

3.1. The Schrödinger Equation in a Rotating Frame for the One-Dimensional Case

The general form of Schrödinger equation in the case of a rotating frame is given by

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\frac{\vec{\pi}^2}{2m_e} + V(r) - \vec{\Omega}\cdot(\vec{r}\times\vec{\pi})\right]\psi. \quad (10)$$

Here, $\vec{\pi} = \vec{p} - e\vec{A}/c$ is the momentum, $\vec{A}$ is the vector potential, m_e is the electron mass.

The simplest case of interest in studying the nonrelativistic quantum mechanics in the presence of rotation consists of considering an electron confined into a one-dimensional ring of radius r_0. In this case, the corresponding time-independent Schrödinger equation has the form

$$\frac{1}{2m_e}\left(-\frac{-\hbar^2}{r_0^2}\frac{\partial^2}{\partial\varphi^2}\right)\psi - \vec{\Omega}.(\vec{r}\times\vec{p})\psi = E\psi. \quad (11)$$

Let us consider that the system is rotating about the z-axis, such that $\vec{\Omega} = \Omega\hat{z}$. Then, we obtain

$$\frac{1}{2m_e}\left(-\frac{-\hbar^2}{r_0^2}\frac{\partial^2}{\partial\varphi^2}\right)\psi - \Omega r_0\left(-\frac{i\hbar}{r_0}\frac{\partial}{\partial\varphi}\right)\psi = E\psi. \quad (12)$$

We can use the ansatz $\psi = e^{im\varphi}$ because this equation does not depend on φ explicitly. Then, the corresponding energy levels are given by

$$E_m = \frac{1}{2m_e}\frac{\hbar^2}{r_0^2}m^2 - m\hbar\Omega, \quad (13)$$

with $m = 0, \pm1, \pm2, \ldots$

The first term of Equation (13) corresponds to the usual energy levels for an electron in a one-dimensional ring. The term $m\hbar\Omega$ depends on the rotation parameter, corresponding to momentum angular-rotation coupling. Thus, the system energy levels are shifted due to this coupling. Also, the degeneracy of energy levels concerning the allowed values for the quantum number m is broken.

3.2. Analogy with Aharonov–Bohm Effect

As commented above, Equation (13) exhibits an energy shift term due to rotation. It is similar to the Aharonov–Bohm (AB) effect [25] for an electron confined in a one-dimensional ring. Before introducing the equations describing the AB effect in this system, it is worth making some remarks. It is known that the AB effect provides us with an understanding of the electromagnetic potentials in quantum mechanics, showing that such potentials can affect the quantum dynamics of a system even in a region in which the corresponding fields are zero. The most famous version of the AB effect is related to a quantum phase in an interferometric experiment. However, we shall now discuss the AB effect for bound states. Following [26], let us consider an electron in a one-dimensional ring, which is subjected to a magnetic flux Φ, in such a way that the magnetic field is zero. In this case, the energy levels are

$$E_m = \frac{\hbar^2}{2m_e r_0^2}\left(m - \frac{q\Phi}{2\pi\hbar}\right)^2, \tag{14}$$

where q represents the particle charge. Thus, the AB effect for bound states manifests through the influence of the magnetic flux on the energy spectrum, even in the absence of a magnetic field. In a similar way to its version for interference experiments, it reflects the significance of the electromagnetic potentials in the quantum world. By comparing Equations (13) and (14), we notice that the rotation parameter Ω and the magnetic flux Φ play an analog role in the modification of the energy levels, by introducing an energy shift and lifting the degeneracy regarding the quantum number m. In the next sections, we will discuss in more detail some aspects involving the analogy between the effect of electromagnetic potentials and rotation.

Before we discuss more complicated problems, let us briefly comment on the case in which a one-dimensional ring is subjected to both rotation and the bounded AB effect. This issue was addressed in [27]. In this case, the energy levels are

$$E_m = \frac{\hbar^2}{2m_e r_0^2}(m - l')^2 - \hbar\Omega(m - l'). \tag{15}$$

Here, $l' = \Phi'/\Phi_0$, being Φ' the magnetic flux, $\Phi_0 = h/e$ and e is the electron charge. It is interesting to notice that Equation (15) contains a combined effect of rotation and magnetic flux, given by the quantity $\hbar\Omega l'$. The effects of rotation on persistent current and magnetization are also addressed in [27]. The expression for the current is

$$I_m = \frac{e\hbar}{2\pi m_e r_0^2}\left(m - l' - \frac{m_e\Omega r_0^2}{\hbar}\right). \tag{16}$$

Since the current is obtained from a derivative of E_m with respect to the magnetic flux, this time there is no combined effect of magnetic field and rotation. Despite that, rotation acts similarly to the magnetic flux, providing a linear contribution to the current. For the magnetization, the expression obtained is

$$M_m = \frac{e\hbar}{2m_e}\left(m - l' - \frac{m_e\Omega r_0^2}{\hbar}\right). \tag{17}$$

For magnetization, the most notable feature is the existence of a term depending on rotation, which has an analogy with the Barnett effect [28]. That effect is related to the arising of magnetization in a system due to rotation.

Now, we have conditions to explore the analogy with the AB effect in more detail. The seminal work of Aharonov and Bohm [25] has opened a large interest in the investigation of quantum foundations, referring to the role of electromagnetic potentials in the theory. Besides the magnetic version of the Aharonov–Bohm effect, there are analogs in other physical systems, including electric [29], optical [30], and gravitational [31] versions. As argued by Weder in [29], it is worth mentioning that the magnetic version is still the most known version, and it has been investigated from the experimental viewpoint by different setups. In addition to the analogues already mentioned, we can cite examples in other research topics. Davidowitz and Steinberg [32] proposed an experiment to a hydrodynamic analog of the AB effect in superfluid helium. Aharonov, in collaboration with Casher [33], also proposed an analog of the AB effect for neutral particles. It is now known as the Aharonov–Casher effect and it has been extensively investigated in the literature. Examples of works dealing with the Aharonov–Casher effect regarding quantum phases and bound states can be accessed in [34–36]. Hendricks and Nienhuise [37] discussed the analogy between the nonrelativistic limit of the Sagnac effect and the Aharonov–Bohm effect, arguing that the Sagnac effect is also an example of Berry's phas.

Aharonov and Carmi [38] have proposed an analogy with the Aharonov–Bohm effect for an inertial field. They suggested a thought experiment consisting of a ring spinning in the presence of electric and magnetic fields, in a way that Lorentz and inertial forces can cancel each other out [39]. Similarly to the Aharonov–Bohm effect, this experiment consists of studying an effect in a field-free region but considering the implications of a non-vanishing potential vector. Then, to guarantee this canceling, it is necessary to take the following field configuration:

$$\vec{B} = -(2m/q)\vec{\Omega}, \qquad \vec{E} = (m/q)\vec{\Omega} \times (\vec{\Omega} \times \vec{r}), \tag{18}$$

where $\vec{\Omega}$ is the rotation field. The Lorentz force corresponding to these fields cancels out the inertial force given by Equation (1). The Hamiltonian to the ring region is

$$H = \frac{(\vec{p} - 2m\vec{a})^2}{2m}, \tag{19}$$

where $\vec{a}$ is an inertial vector potential that satisfies

$$\vec{\Omega} = \vec{\nabla} \times \vec{a}. \tag{20}$$

On the other hand, the Hamiltonian for a charged particle subjected to a magnetic field is given by

$$H = \frac{(\vec{p} - q\vec{A})^2}{2m}, \tag{21}$$

where $\vec{A}$ is the potential vector that gives rise to the Aharonov–Bohm effect. Comparing Equations (19) and (21), notice that the Hamiltonians are analogous and the Coriolis force represented by the inertial potential vector $\vec{a}$ replaces the magnetic force denoted by the potential vector $\vec{A}$. For this reason, we expect an interference effect like Aharonov–Bohm and the arising of a quantum phase depending on the inertial potential vector, given by

$$\psi = \psi_0 \exp\left\{ i(2m/\hbar) \oint \vec{a} \cdot d\vec{l} \right\}. \tag{22}$$

Beyond this interference effect, the Aharonov–Carmi effect can be responsible for a shift in the spectrum of energy as shown by Shen and Zhuang, making the role of the Aharonov–Bohm effect for bound states [40].

3.3. The Schrödinger Equation for Electrons Confined in a Rotating Disk

Now, let us focus our attention on a two-dimensional problem. Johnson [41] considered a disk rotating about the z-axis containing a confined two-dimensional non-interacting electron gas. Johnson employed the Schrödinger equation describing the quantum dynamics for this rotating electron gas. Let us briefly revisit the main Johnson's results. The corresponding time-independent Schrödinger equation is

$$\left(-\frac{\hbar^2}{2m_e} \nabla^2 + i\hbar\Omega \frac{\partial}{\partial \varphi} \right) \psi = E\psi. \tag{23}$$

After the separation of variables $\psi = R(r)e^{-i\ell\varphi}$, we obtain the radial equation

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + \left(\lambda^2 r^2 - \ell^2\right)R = 0, \tag{24}$$

with $\lambda^2 = (2m/\hbar^2)(E - \ell\hbar\Omega)$. The solution for this differential equation is

$$R(r) = J_\ell(|\lambda| r), \tag{25}$$

where J_ℓ is the Bessel function. The energy levels are given by

$$E_\ell = \frac{\hbar^2\lambda^2}{2m_e} + \ell\hbar\Omega. \tag{26}$$

Again, rotation introduces an energy shift due to the coupling with the angular momentum. In this case, all energy levels are non-degenerated. A generalization of Johnson's work consists of considering the effect of rotation on the quantum mechanics of a non-interacting free electron gas in a rotating planar conductor with a perpendicular uniform magnetic field. It is carried out in [42]. More specifically, it deals with a perpendicular magnetic field, and the authors discuss the influence of rotation on the Landau levels and Hall conductivity. In this case, the Schrödinger equation is given by

$$\left(-\frac{\hbar^2}{2m_e}\nabla^2 + i\alpha\hbar\frac{\partial}{\partial\varphi} + \beta r^2\right)\psi = E\psi. \tag{27}$$

In this expression, $\alpha = (qB/2m_e + \Omega)$, and $\beta = q^2B^2/8m_e$. By employing again the *ansatz* $\psi = R(r)e^{-i\ell\varphi}$, results in the radial equation given by

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + \left(\sigma^2 r^4 + \lambda^2 r^2 - \ell^2\right)R = 0, \tag{28}$$

with $\sigma^2 = \frac{q^2B^2}{4\hbar^2}$, and $\lambda = \frac{2m}{\hbar}(\frac{E}{\hbar} - \frac{qB\ell}{2m_e} - \Omega\ell)$. By taking a solution of the form $R(\xi) = e^{\frac{\xi}{2}}\xi^{\frac{\ell}{2}}u(\xi)$, with $\xi = \sigma r^2$, the radial equation becomes a confluent hypergeometric equation

$$\xi\frac{d^2u}{d\xi^2} + [1 + |\ell| - \xi]\frac{du}{d\xi} + \left[\frac{\lambda}{4\sigma} - \frac{1}{2}(|\ell| + 1)\right]u = 0. \tag{29}$$

The energy spectrum is given by

$$E_{n,\ell} = \hbar\omega_c\left(n + \frac{\ell}{2} + \frac{|\ell|}{2} + \frac{1}{2}\right) + \ell\hbar\Omega. \tag{30}$$

The quantity $\omega_c = qB/m_e$ corresponds to the cyclotron frequency. By taking $\Omega = 0$ in Equation (30), we recover the usual expression for the Landau Levels. The existence of degenerated states depends on the relation between the magnetic field and the rotation, which is given like

$$\Omega = \frac{a}{2}\frac{qB}{m}, \tag{31}$$

where a is a real number. The expression (31) covers all possible combinations between B and Ω. The degenerated states occur only when a is an integer number. Two special cases are emphasized, $a = -2$ and $a = -1$. In the first one, the electronic structure is equivalent to the usual Landau levels, but with a reversed charge. In the second, the energy levels are similar to a harmonic oscillator.

3.4. Effects of Rotation in Two-Dimensional Quantum Rings

Studying two-dimensional quantum rings has been a widely researched line, due to their interesting physical properties. This type of system is a scenario for investigating phenomena such as the arising of persistent currents and magnetization, for example. There are several works in the literature investigating the effects of rotation on the physical properties of two-dimensional quantum rings. Before we summarize the main aspects of such works, it is important to notice that, in the case of two-dimensional rings, it is necessary to model the presence of the two physical boundaries. Often, these boundaries are described by potential barriers. Tan and Inkson [43] presented an exact model for describing a two-dimensional ring. They considered a confining potential given by

$$V(r) = \frac{a_1}{r^2} + a_2 r^2 - V_0, \tag{32}$$

with $V_0 = 2\sqrt{a_1 a_2}$. By employing this potential, the authors obtained the energy spectrum and the wavefunctions in the presence of a magnetic flux and a uniform magnetic field. The effect of rotation on a two-dimensional ring described by a confining potential like the expression (32) is investigated, for instance, by Pereira e Silva [44]. Recently, they have proposed a model to describe the effects of rotation on the energy spectrum and also on the edge states of two-dimensional rings. In their manuscript, the energy levels are given by

$$E_{nm} = \left(n + \frac{1}{2}\right)\hbar\omega_1 + V_{min}. \tag{33}$$

In this expression, $V_{min} = \frac{\hbar}{2}(|m|\omega_1 - m\omega_2)$. The parameters n and m are the quantum numbers related to the radial and angular degrees of freedom, respectively. The quantities ω_1 and ω_2 are defined by

$$\omega_1 = \left(\omega_c^2 + 4\omega_c\Omega\right)^{\frac{1}{2}}, \qquad \omega_2 = \omega_c + 2\Omega, \tag{34}$$

where $\omega_c = \frac{eB}{m_e}$ denotes the cyclotron frequency, and Ω corresponds to the angular velocity. Thus, the quantities ω_1 and ω_2 behave like effective frequencies for the system, incorporating effects from the magnetic field and the rotation. We notice an influence in the energy spectrum due solely to the field and also a contribution that comes uniquely from rotation, as expected. In addition, an interesting aspect in these expressions is the presence of a term describing a combined effect of rotation and the magnetic field, given by $4\omega_c\Omega$, which also occurs for the one-dimensional case. This combined effect in the two-dimensional case modifies the Landau levels. Another relevant aspect of this problem is that the rotation lifts the degeneracy of energy levels. In [44], it was discussed how the presence of rotation affects the system's magnetization, particularly altering the behavior of the Haas-Van Alphen oscillations.

The study of the influence of rotation on two-dimensional quantum rings is not restricted to investigating the electronic properties. Lima et al. [45] discussed how the rotating effects manifest in optical properties and demonstrated that the rotation parameter Ω can affect the behavior of optical absorption and refractive index. Another study dealing with a rotating two-dimensional ring was reported, investigating rotating effects on the photoionization process in the ring [46]. The thermodynamic properties of two-dimensional rings also have attracted attention. Ghanbari [47] reported results covering properties such as mean energy, specific heat, and free energy for a *GaAs* ring as a function of the rotation parameter. That study obtained the energy spectrum from the Schrödinger equation. The partition function was then derived, carrying out information about the rotation. It is shown that the presence of rotation brings significant changes in the physical properties analyzed.

3.5. The Pauli–Schrödinger Equation in a Rotating Frame

To accommodate the spin degree of freedom for an electron in the nonrelativistic domain, it is necessary to deal with the Pauli–Schrödinger equation. Since this equation is usually employed to take into account the Zeeman term, let us consider the Pauli–Schrödinger Hamiltonian in the presence of a magnetic field $\vec{B}$ and rotation. The corresponding Hamiltonian is given by

$$H = \frac{1}{2m_e}\left(\vec{p} - e\vec{A} - m_e\vec{\Omega} \times \vec{r}\right)^2 - \frac{1}{2}m_e(\Omega \times \vec{r})^2 - \frac{e\hbar}{2m_e}\vec{\sigma}.\vec{B} - \frac{\hbar}{2}\vec{\sigma}.\vec{\Omega}, \tag{35}$$

where $\vec{\sigma}$ denotes the Pauli matrices, the third term corresponds to the Zeeman interaction and the last term describes the spin-rotation coupling. By considering only the spin-dependent terms, we can write that

$$H_S = -\frac{\hbar}{2}\vec{\sigma}.\left(\frac{e}{m_e}\vec{B} + \vec{\Omega}\right). \tag{36}$$

In this context, the simple case involving both magnetic field and rotation consists of considering a uniform magnetic field $\vec{B} = B\hat{z}$ and that the system is rotating about the z-axis with $\vec{\Omega} = \Omega\hat{z}$. In this case, we obtain

$$H_S = \frac{\hbar}{2}\hat{\sigma}_z(\omega_c + \Omega), \tag{37}$$

where ω_c corresponds to the cyclotron frequency. Thus, regarding the spin degree of freedom, rotating effects manifest similarly to the Zeeman term, adding a shift in the usual Schrödinger Hamiltonian. Then, it is possible to lift the degeneracy of the energy spectrum of a system through rotation.

Another example of the application of Equation (35) is the case in which the electron is subjected to an Aharonov-Bohm flux as considered by Lima et al. in [48]. The inclusion of the Aharonov–Bohm flux and its corresponding magnetic field drastically changes the quantum mechanical description of the problem in comparison with the case of a uniform field. The definition of the magnetic field involves the inclusion of a delta function, introducing a singularity into the Hamiltonian. To deal with this problem, it is necessary to employ the method of the self-adjunct extensions that is often considered when the Hamiltonian operator is not self-adjunct anymore. More details about that method can be viewed in [49].

Including the spin degree of freedom in the Hamiltonian opens several possibilities for investigation, allowing the study of properties besides the energy spectrum. It is worth mentioning the distinguished work of Matsuo et al. [50], in which the Pauli–Schrödinger equation was obtained by taking the low energy limit of the Dirac equation in a non-inertial frame with electromagnetic interactions. The effect of rotation in the spin–orbit interaction and the spin-current components was explained. They also argued that there is a mechanism for transferring angular momentum involving rotations and spin currents.

3.6. Other Examples of Rotating Nonrelativistic Systems

Until this point, we have considered the quantum mechanical description of rotating systems like rings and disks and the inclusion of the spin degree of freedom in the nonrelativistic domain. Now, we will discuss other applications. It is known that, in the context of nonrelativistic quantum mechanics, a particle that has motion constrained to a curved surface is subjected to the action of a geometrical potential. It was introduced by Da Costa in [51,52]. This geometrical potential is given by

$$V_g = -\frac{\hbar^2}{2m_e}\left(M^2 - K\right) = -\frac{\hbar^2}{2m_e}(k_1 - k_2)^2. \tag{38}$$

Here, M denotes the mean curvature, while K is the Gaussian curvature. The quantities k_1 and k_2 correspond to the two principal curvatures of the surface. In this scenario, [53] has studied the problem of the Landau levels for a spherical surface. Based on this, Lima et al. [54] incorporated the effects of rotation on such a system, showing that the rotation can modulate how the magnetic field affects the electronic states. It was also shown that the rotation can suppress the appearance of the Landau levels. In this context of the geometrical potential approach, Atanasov and Dandoloff [55] have presented a scheme to obtain the effect of curvature that comes from a rotating two-dimensional disk. The authors have also discussed how this curvature can be measured as a function of the rotation parameter.

Another example of an investigation involving the interplay between magnetic fields and rotation on two-dimensional quantum systems refers to considering the dynamics of ballistic electrons in carbon nanotubes. Cunha et al. [56] have studied the problem of obtaining the energy spectrum, the charge currents, and the spin currents for a rotating tube in the presence of a uniform magnetic field. The authors demonstrated that, under specific cases, it is possible to vanish the charge current while the spin current is still present. In a recent contribution, Cheng et al. [57] have incorporated the spin degree of freedom in the Schrödinger equation for describing the quantum dynamics of a spin-$1/2$ particle that moves on a rotating curved surface. They also introduce time-varying fields and obtain the probability of flipping the spin particle on surfaces in the presence of rotation. It is shown that a rapid rotation has a significant influence on the spin-flip probability distribution. Another recent work dealing with the presence of electromagnetic fields and rotation can be accessed in [58], which addresses the problem of describing the effects of rotation and gravity on a Hall sample. It brings a different point of view by employing an algebraic approach to write down the corresponding Hamiltonian and its solutions. Rotating effects also have been considered for an atom with a magnetic quadrupole moment [59].

4. Relativistic Quantum Mechanics in a Rotating Frame

4.1. Equivalence Principle and Rotations

The equivalence principle of general relativity states that a body in an accelerating system behaves like a gravitational field. The same occurs in rotating systems, where non-inertial forces act similarly. To describe rotating systems, we use a metric that contains rotation terms and angular speed. All metrics with rotation are based on Minkowski space-time, and rotation creates non-inertial effects that change how a particle moves. These effects appear as extra terms in the metric that describes the surface the particle interacts with. For instance, Nelson [60] described the general Lorentz transformation for a rotating frame of reference in 1987, and there are several applications for rotating bodies and rotating reference frames. Brill and Cohen [61] looked at masses in slowly rotating bodies using the Schwarzschild metric for stars with a dense core. They found that as the core radius got closer to the shell radius, an induced rotation got closer to the rotation of its shell, which is the same thing that happens when a star's dust cloud grows and shrinks. Detweiler [62] examined the Klein–Gordon equation for a scalar field in the geometry of a rotating black hole and found that the Compton particle wavelength is often much larger than the black hole's size. This means that the scalar field is not stable. Tamburini et al. [63], using numerical results, found the new relativistic effects on the angular momentum of light surrounding black holes. In 2014, Altarirano et al. [64] investigated the thermodynamic properties of rotating black holes and black rings, as well as phase transitions of black hole Gibbs free energy. Pappas [65] has shown a nearly stationary and axisymmetric metric for rotating neutron stars that fits with the multipole moment data observations. Pesonen [66] came up with a contravariant metric tensor to describe the vibrating-rotation Eckart frame. The Coriolis coupling of the molecule structure, which was zero and best fit the theory of molecular vibrations and rotations, made this possible.

The connection between rotation effects and systems described by a metric allows us to explain different phenomena and calculate non-inertial relativistic effects. To illustrate how we describe these systems, let's consider a system in four-dimensional space-time. The metric for an observer in a circle with angular velocity Ω in Minkowski space is given by [67]

$$ds^2 = (1 - \Omega^2 r^2)dt^2 - 2\Omega r^2 d\phi dt - r^2 d\phi^2 - dr^2 - dz^2, \tag{39}$$

where the simplest case consists of considering cylindrical coordinates, and the rotation in a plane around the z-axis.

4.2. The Dirac Equation in a Rotating Frame

To work with relativistic quantum mechanics in rotating frames, we use the Dirac equation and modify it to accommodate rotation terms. Now, we can discuss two classic solutions for these systems and check which aspects differ from the standard. The Dirac equation in general coordinates is

$$\gamma^a \nabla_a \psi + i\mu\psi = 0, \tag{40}$$

where γ^a are the flat spacetime Dirac matrices. It satisfies the conditions

$$[\gamma^a, \gamma^b]_+ = 2\eta^{ab}, \tag{41}$$

and

$$\nabla_a \psi = e_a^\mu (\partial_\mu - \Gamma_\mu)\psi. \tag{42}$$

Here, e_a^μ is the tetrad field. The terms Γ_μ are the spinor affine connections given by

$$\Gamma_\mu = -\frac{1}{4}\gamma^a \gamma^b e_a^\nu e_{b\nu;\mu}. \tag{43}$$

In these cylindrical coordinates, the nonvanishing tetrad field components are

$$e_0^t = e_1^r = e_3^z = 1, \qquad e_0^\phi = -\Omega, \qquad e_2^\phi = \frac{1}{r}. \tag{44}$$

The resulting connections are

$$\Gamma_t = \Omega\Gamma_\phi, \qquad \Gamma_\phi = -\frac{1}{2}\gamma^1\gamma^2, \qquad \Gamma_r = \Gamma_z = 0. \tag{45}$$

The Dirac equation in rotating coordinates is given by

$$\left[\gamma^0(\partial_t - \Omega\partial_\phi) + \gamma^1\left(\partial_r + \frac{1}{2r}\right) + \frac{1}{r}\gamma^2\partial_\phi + \gamma^3\partial_z + i\mu\right]\psi = 0. \tag{46}$$

The first solution of Equation (46) is for neutrinos $\mu = 0$. The momentum of neutrino is anti-parallel to its spin and the additional condition satisfies

$$(1 + i\gamma_5)\psi = 0, \tag{47}$$

where

$$\gamma_5 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \tag{48}$$

From two component spinors, we have

$$\eta_1 = \eta_2. \tag{49}$$

The solutions will be

$$\psi(\omega, m, k; x) = \exp\{-i(\omega - m\Omega)t\} \times \exp\{i(m\phi + kz)\} \times (\eta(r), \eta(r))^T, \tag{50}$$

where $\eta(r)$ satisfies

$$\left[\sigma^1\left(\partial_r + \frac{1}{2r}\right) + \frac{im}{r}\sigma^2 + ik\sigma^3 - i\omega\right]\eta(r) = 0. \tag{51}$$

The resulting decoupled equation is

$$\left[r^2\frac{d^2}{dr^2} + r\frac{d}{dr} + Q^2r^2 - \left(m + \frac{1}{2}\right)^2\right]R(r) = 0, \tag{52}$$

where $Q = +(\omega^2 - k^2)^{1/2}$ and the equation is a Bessel equation of order $(m + \frac{1}{2})$. The resulting normal modes are

$$\psi'(\omega, m, k; x) = \exp\{-i(\tilde{\omega}t - m\phi - kz)\}(QJ_{m-\frac{1}{2}}, i(\omega - k)J_{m+\frac{1}{2}}, QJ_{m-\frac{1}{2}}, i(\omega - k)J_{m+\frac{1}{2}})^T. \tag{53}$$

Rotation terms such as the angular velocity are present in the solution, contributing to effects in the radial and angular direction. In Equation (53), we used

$$\tilde{\omega} = \omega - m\Omega, \qquad J_{m\pm\frac{1}{2}} \equiv J_{m\pm\frac{1}{2}}(Qr). \tag{54}$$

The second solution is the Dirac oscillator in a rotating frame which is another simple solution for relativistic quantum mechanics that has several applications like quantum optics [68], nuclear physics [69], general relativity [70] and quantum thermodynamics [71,72]. The effective vector potential is

$$\vec{P} \to \vec{P} + im\Omega\beta\vec{r}, \tag{55}$$

where $\vec{r}$ is the vector of the fermion from the origin and β is the Dirac matrix. According to [73], we restrict to look for solutions where $\Psi_2 = \Psi_3 = 0$. Therefore, the coupled differential equations for Ψ are

$$\begin{aligned} &i\hbar\frac{\partial\Psi_1}{\partial t} + \frac{\hbar\omega r}{c}\frac{\partial\Psi_4}{\partial t} + i\hbar c\frac{\partial\Psi_4}{\partial r} + \frac{\hbar c}{r}\frac{\partial\Psi_4}{\partial\phi} + \frac{i\hbar c}{2r}\Psi_4 - im\Omega cr\Psi_4 - \frac{\hbar\omega}{2}\Psi_1 = mc^2\Psi_1; \\ &i\hbar\frac{\partial\Psi_4}{\partial t} - \frac{\hbar\omega r}{c}\frac{\partial\Psi_4}{\partial t} + i\hbar c\frac{\partial\Psi_1}{\partial r} - \frac{\hbar c}{r}\frac{\partial\Psi_1}{\partial\phi} + \frac{i\hbar c}{2r}\Psi_1 + im\Omega cr\Psi_1 + \frac{\hbar\omega}{2}\Psi_4 = -mc^2\Psi_4. \end{aligned} \tag{56}$$

It is easier to find solutions by considering the time-dependent form as

$$\Psi_n(r, \phi, t) = \Psi_n(r, \phi)\exp\left\{\frac{-iWt}{\hbar}\right\}, \tag{57}$$

where W is the relativistic energy. We can make the separation and write the wave functions in the form

$$\Psi_1(r, \phi) = \psi_1(r)\Phi(\phi), \qquad \Psi_4(r, \phi) = \psi_4(r)\Phi(\phi). \tag{58}$$

In addition, we can make a separation of variables in Equation (56) by making

$$\Phi(\phi) = \exp\{i(\mu + 1/2)\phi\}. \tag{59}$$

For the wave function to be continuous, μ has to be half-integer. Thus, if we make the change of $\chi_1 = \psi_1$ and $i\chi_4 = \psi_4$, the radial part of the Dirac equation will be real and we can solve it separately.

$$
\begin{aligned}
&\left(W - mc + \frac{\hbar\omega}{2}\right)\chi_1(r) - \frac{\omega W r}{c}\chi_4(r) + \hbar c\frac{d\chi_4(r)}{dr} + \frac{\hbar c(\mu + 1/2)}{r}\chi_4(r) - m\Omega c r\chi_4(r) = 0; \\
&\left(W - mc + \frac{\hbar\omega}{2}\right)\chi_4(r) - \frac{\omega W r}{c}\chi_1(r) + \hbar c\frac{d\chi_1(r)}{dr} + \frac{\hbar c(\mu + 1/2)}{r}\chi_1(r) - m\Omega c r\chi_1(r) = 0.
\end{aligned} \tag{60}
$$

4.3. Applications in High Energy Physics and Gravitation

The Dirac equation in a rotating frame provides the theoretical tools for investigating diverse systems in gravitational and high-energy physics. In this framework, Hehl and Ni, in a remarkable contribution [74], have shown a method to describe the effects due to acceleration and rotation in the Dirac equation. Varjú and Ryder [75] have studied the problem of a Dirac particle in a rotating frame and under the influence of a gravitational field. They compared Kerr and Schwarzschild spacetimes, showing that the difference between these two should manifest at energies of the order of 10^{-19} eV. They argued that further experimental developments are required to employ general relativity in studying quantum systems. Dayi and Kilinçarslan [76] have obtained semiclassical transport equations for Dirac particles. The parallelism approach has also been employed to describe a Dirac particle in a rotating frame [77]. Another possible application of the Dirac equation is the study of rotating effects on spin currents. For example, Chowdhury and Basu [78] investigated how the spin Hall current and the momentum space Berry curvature are affected by acceleration effects. Papini [79] investigated how acceleration and rotation can create spin currents. Vilenkin [80] found the neutrino parity-violating current in rotating thermal radiation. Vilenkin argued that in rotating systems, the average velocity is antiparallel to angular velocity Ω, i.e., the direction where the neutrinos are partially polarized. It indicates the existence of this kind of current. In [10], it was shown that the spectrum of a massless scalar quantum field in a rotating frame depends on the system velocity.

4.4. Applications in Low-Dimensional Materials

Since the discovery of the graphene synthesis process, the physics of low-dimensional systems has attracted extensive interest. Gim and Novoselov, who received the Nobel Prize in physics in 2010, made the first communication about obtaining a graphene sample in a laboratory. It has opened the way to enormous literature due to the unique physical properties of graphene, which allow the development of novel materials. From a theoretical viewpoint, there are several possibilities, including studying models to describe graphene-based materials and remarkable aspects of the quantum theory itself. An important facet of graphene physics is the existence of a bridge with relativistic quantum mechanics. It is known that near the Fermi level, graphene presents a linear dispersion relation, given by

$$E_F = \pm\hbar v_F k, \tag{61}$$

where v_F is the Fermi velocity and k is the momentum. On the other hand, it is known from relativistic quantum mechanics that the Dirac equation admits solutions in the form

$$E = \pm\sqrt{p^2c^2 + m_0^2c^4} \tag{62}$$

for a free particle, where p is the momentum, c is the light speed and m_0 represents the rest mass. By taking $m_0 = 0$ in Equation (62), we obtain a linear behavior with respect to p. The comparison between Equations (61) and (62) when $m_0 = 0$ has motivated studying models to describe the physics of charge carriers in graphene from an effective Dirac equation. As argued by Novoselov et al. [81], these charge carriers behave like massless relativistic particles.

It has attracted attention in topics such as quantum dots [82], quantum Hall effect [83], effects of curvature [84], and analog gravity [85]. Regarding the effects of rotation on graphene-based materials, there are several examples in the literature dedicated to this issue. It is worth mentioning the existence of contributions that point to the experimental evidence that C_{60} molecules rotate [86,87]. Also, there are recent contributions regarding rotating fullerenes in the context of computational simulations [88,89]. It is possible to study the effects of rotation on the most diverse properties of C_{60}. For instance, Shen and He [90] investigate the arising of geometrical phases in C_{60} molecules due to rotating effects. In a recent contribution by Zhang et al. [91], it was shown that spinning C_{60} under thermal excitation produces large angular momentum. In [40], it was reported the existence of an energy shift in the valence electrons of C_{60} molecules due to the rotation. In [92], it was considered the problem of describing the effects of rotation on C_{60} by employing an effective Dirac equation. The obtained energy spectrum presents a shift due to the coupling between the orbital angular momentum and the rotation. Following a similar approach, Lima and Moraes [93] have incorporated the presence of electromagnetic fields and considered the combined effect of rotation and magnetic fields on the energy spectrum of C_{60}. Cavalcante et al. [94] adopted a theoretical description based on an analog of the Gödel metric for describing rotating fullerenes.

Rotating effects on carbon nanotubes have also been considered. From the experimental viewpoint, Král and Sadeghpour [95] have shown that carbon nanotubes can spin by employing polarized light. Cai et al. [96] used molecular dynamics simulations to look at how a double-walled carbon nanotube that has the inner tube rotating moves back and forth. Cunha et al. [97] employed the effective Dirac equation to obtain the energy spectrum for rotating nanotubes, demonstrating the presence of a spin splitting term due to the rotation.

5. Topological Defects and Rotating Effects in Quantum Systems

Topological defects occur in cosmology, statistical physics, and condensed matter physics. In a few words, topological defects represent a spacetime position in which some kind of symmetry is broken. In cosmology, the arising of topological defects is associated with phase transitions due to a cooling of the expanding universe [98]. Topological defects can also emerge in liquid crystals [99], liquid helium [100], and in type II superconductors [101]. In solid materials, topological defects are related to deviations of a lattice from its ideal form [102]. It is important to note that there are two possible approaches for describing these defects in solids. One can use either elasticity theory [103] or a geometrical approach [104–106].

Since our interest here is to discuss rotating quantum systems in the presence of topological defects, let us make some considerations on the geometrical approach. The most relevant aspect of this approach is that the topological defects are described by employing the theoretical framework of differential geometry. Thus, it is possible to characterize a topological defect using a metric tensor. Therefore, describing topological defects in solids in this way is similar to the case of a curved spacetime. It is possible due to the pioneer contribution reported in [107]. In this approach, the description of quantum systems in the presence of topological defects is made by incorporating the information about the corresponding metric into the Hamiltonian. As an example, let us consider the results reported by Furtado et al. [108], in which the Dirac equation for a conical geometry was considered. They have considered a graphitic cone, which is equivalent to a graphene with a topological defect known as disclination. In this case, the metric is given by

$$ds^2 = dt^2 - d\rho^2 - \alpha^2 \rho^2 d\phi^2. \tag{63}$$

In this expression, α is the parameter describing the topological defect. For $\alpha = 1$, we recover the line element corresponding to polar coordinates (ρ, ϕ). The corresponding Dirac equation for the metric (63) is

$$\left[\gamma^0 \partial_t + \gamma^\rho \left(\partial_\rho - \frac{1-\alpha}{\alpha\rho}\right) + \gamma^\phi \left(\frac{1}{\alpha\rho}\partial_\phi\right)\right]\psi = 0. \tag{64}$$

The geometric approach also allows the inclusion of rotational effects in these systems. For example, Dantas et al. [109] studied the quantum dynamics of a nonrelativistic system, which consists of a ring in a rotating frame in the presence of a topological defect known as dislocation. It was shown that the energy spectrum depends on the topological contribution as well as the rotation parameter. Brandão et al. [110] investigated the combined effect of rotation and magnetic field on a disk containing a topological defect. The corresponding Schrödinger equation is written as

$$\left[-\frac{\hbar^2}{2m_e}\nabla^2 + i\frac{\mu\hbar}{\alpha}\frac{\partial}{\partial\phi} + \beta r^2\right]\psi = E\psi. \tag{65}$$

Here, α corresponds to the parameter describing a conical defect. In addition,

$$\mu = \frac{qB}{2m_e\alpha} + \Omega, \qquad \beta = \frac{q^2B^2}{8m\alpha^2} + \frac{qB\Omega}{2}\left(\frac{1-\alpha}{\alpha}\right). \tag{66}$$

From these equations, we can notice a curious effect: the second term on the definition of β indicates the existence of a simultaneous coupling between rotation, magnetic field, and disclination. Rojas et al. [111] reported another contribution dealing with the presence of a topological defect and rotation. They studied a quasi-two-dimensional electron gas and described the influence of topological and noninertial effects on the energy levels and on the light interband absorption coefficient. Rotating systems with topological defects are also considered in the relativistic domain. For instance, Oliveira has investigated the problem of a Dirac particle in a rotating ring around a cosmic string and the effects on the Dirac oscillator in this background [112,113]. Additionally, Cuzinatto et al. [114] have examined the rotating effects on the Dirac oscillator in a cosmic string background, focusing on its non-commutative properties. Garcia et al. [115] have employed the geometric theory of defects to describe fullerenes in a rotating frame, obtaining a geometric quantum phase associated with the Aharonov–Carmi effect. Recently, Guvendi and Dogan [116] reported on a study of a relativistic spin-1 oscillator in a background induced by a spinning point source.

6. Conclusions

We have reviewed theoretical spinning systems in quantum mechanics. First, we examined the inertial forces acting on a massive particle in a rotating frame. From this, we derived the corresponding Lagrangian and Hamiltonian. Subsequently, we described non-relativistic systems by introducing the Schrödinger equation in a rotating frame and presented several examples. These examples allowed us to discuss the primary effects of rotation on properties such as the energy spectrum, current, and magnetization. We also discussed an analogy between rotating systems and the Aharonov–Bohm effect. It was shown that the presence of rotation affects the energy spectrum of bound systems, as well as the current and magnetization producing a shift that depends on the rotation parameter Ω.

Next, we included the spin degree of freedom in the description of non-relativistic systems by considering the Pauli–Schrödinger equation. In this case, the main effect of rotation consists of splitting the energy levels similarly to a Zeeman term. We also explored aspects of relativistic quantum mechanics in a rotating frame by employing the Dirac equation. Specifically, we introduced a line element describing a rotating frame into the Dirac equation and presented some examples of its application to spinning systems, particularly in the contexts of high-energy physics, gravitation, and low-dimensional materials. Finally, we discussed the quantum mechanical description of rotating systems in the presence of topological defects. We saw that rotating and topological effects can act in a combined way.

As demonstrated, the study of spinning systems in quantum mechanics encompasses a wide range of theoretical and experimental investigations, spanning various research fields. Despite this breadth, the central message of the present manuscript is that the presence of rotation in quantum systems alters physical properties of interest, leading to novel effects. Recently, the study of rotational effects has been explored in the development of quantum information protocols [117], the properties of quantum fluids [118], and photon bunching [119]. Another research trend refers to the investigation of quantum mechanical systems with position-dependent mass within metrics that include a rotation parameter [120,121].

Author Contributions: E.B.: writing, revision, conceptualization; J.E.B.: writing, revision, conceptualization, M.M.C.: writing,conceptualization, draft preparation. All authors have read and agreed to the published version of the manuscript.

Funding: This work was partially supported by the Brazilian agency Fundação de Amparo a Pesquisa do Estado da Bahia- Grant numbers APP0041/2023 and PPP0006/2024.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Chernodub, M.N. Rotational Diode: Clockwise/Counterclockwise Asymmetry in Conducting and Mechanical Properties of Rotating (semi)Conductors. *Symmetry* **2021**, *13*, 1569. [CrossRef]
2. Wang, B.; Vuković, L.; Král, P. Nanoscale Rotary Motors Driven by Electron Tunneling. *Phys. Rev. Lett.* **2008**, *101*, 186808. [CrossRef] [PubMed]
3. Tu, Q.; Yang, Q.; Wang, H.; Li, S. Rotating carbon nanotube membrane filter for water desalination. *Sci. Rep.* **2016**, *6*, 26183. [CrossRef] [PubMed]
4. Stickler, B.A.; Hornberger, K.; Kim, M. Quantum rotations of nanoparticles. *Nat. Rev. Phys.* **2021**, *3*, 589–597. [CrossRef]
5. Arbunich, J.; Nenciu, I.; Sparber, C. Stability and instability properties of rotating Bose–Einstein condensates. *Lett. Math. Phys.* **2019**, *109*, 1415–1432. [CrossRef]
6. Sakaguchi, H.; Malomed, B.A. Localized matter-wave patterns with attractive interaction in rotating potentials. *Phys. Rev. A* **2008**, *78*, 063606. [CrossRef]
7. Guo, Y.; Luo, Y.; Peng, S. Existence and Asymptotic Behavior of Ground States for Rotating Bose–Einstein Condensates. *SIAM J. Math. Anal.* **2023**, *55*, 773–804. [CrossRef]
8. Koch, C.P.; Lemeshko, M.; Sugny, D. Quantum control of molecular rotation. *Rev. Mod. Phys.* **2019**, *91*, 035005. [CrossRef]
9. Guvendi, A.; Dogan, S.G. Vector bosons in the rotating frame of negative curvature wormholes. *Gen. Relativ. Gravit.* **2024**, *56*, 32. [CrossRef]
10. Santos, L.C.N.; Barros, C.C. Rotational effects on the Casimir energy in the space–time with one extra compactified dimension. *Int. J. Mod. Phys. A* **2018**, *33*, 1850122. [CrossRef]
11. Švančara, P.; Smaniotto, P.; Solidoro, L.; MacDonald, J.F.; Patrick, S.; Gregory, R.; Barenghi, C.F.; Weinfurtner, S. Rotating curved spacetime signatures from a giant quantum vortex. *Nature* **2024**, *628*, 66–70. [CrossRef]
12. Silverman, M. Rotational degeneracy breaking of atomic substates: A composite quantum system in a noninertial reference frame. *Gen. Relativ. Gravit.* **1989**, *21*, 517–532. [CrossRef]
13. Guvendi, A.; Hassanabadi, H. Noninertial effects on a composite system. *Int. J. Mod. Phys. A* **2021**, *36*, 2150253. [CrossRef]
14. Maniharsingh, K. Particle creation in slowly-rotating Robertson-Walker universes. *Astrophys. Space Sci.* **1991**, *182*, 141–153. [CrossRef]
15. Vieira, M.; de M. Carvalho, A.M.; Furtado, C. Aharonov-Bohm effect for light in a moving medium. *Phys. Rev. A* **2014**, *90*, 012105. [CrossRef]
16. Toroš, M.; Cromb, M.; Paternostro, M.; Faccio, D. Generation of Entanglement from Mechanical Rotation. *Phys. Rev. Lett.* **2022**, *129*, 260401. [CrossRef]
17. Landau, L.D.; Lifshits, E.M. *Mechanics*, 3rd ed.; Course of Theoretical Physics; Pergamon Press: Oxford, UK, 1976; Volume 1, 169p.
18. Sagnac, G. Sur la preuve de la réalité de l'éther lumineux par l'expérience de l'interférographe tournant. *CR Acad. Sci.* **1913**, *157*, 1410–1413.
19. Laue, M. Zum versuch von f. harress. *Ann. Der Phys.* **1920**, *367*, 448–463. [CrossRef]
20. Pascoli, G. The Sagnac effect and its interpretation by Paul Langevin. *Comptes Rendus. Phys.* **2017**, *18*, 563–569. [CrossRef]
21. Grøn, Ø.; Hervik, S. *Einstein's General Theory of Relativity: With Modern Applications in Cosmology*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2007.
22. Stedman, G. Ring-laser tests of fundamental physics and geophysics. *Rep. Prog. Phys.* **1997**, *60*, 615. [CrossRef]
23. Matos, G.C.; Souza, R.d.M.e.; Neto, P.A.M.; Impens, F.m.c. Quantum Vacuum Sagnac Effect. *Phys. Rev. Lett.* **2021**, *127*, 270401. [CrossRef] [PubMed]

24. Gautier, R.; Guessoum, M.; Sidorenkov, L.A.; Bouton, Q.; Landragin, A.; Geiger, R. Accurate measurement of the Sagnac effect for matter waves. *Sci. Adv.* **2022**, *8*, eabn8009. [CrossRef] [PubMed]
25. Aharonov, Y.; Bohm, D. Significance of Electromagnetic Potentials in the Quantum Theory. *Phys. Rev.* **1959**, *115*, 485–491. [CrossRef]
26. Griffiths, D.J. *Introduction to Quantum Mechanics*, 2nd ed.; Addison-Wesley: San Francisco, CA, USA, 2005.
27. Pereira, L.F.C.; Cunha, M.M.; Silva, E.O. 1D Quantum ring: A Toy Model Describing Noninertial Effects on Electronic States, Persistent Current and Magnetization. *Few-Body Syst.* **2022**, *63*, 58. [CrossRef]
28. Barnett, S.J. Magnetization by Rotation. *Phys. Rev.* **1915**, *6*, 239–270. [CrossRef]
29. Weder, R. The electric Aharonov-Bohm effect. *J. Math. Phys.* **2011**, *52*, 052109. [CrossRef]
30. Dartora, C.; Nobrega, K.; Cabrera, G. Optical analogue of the Aharonov–Bohm effect using anisotropic media. *Phys. Lett. A* **2011**, *375*, 2254–2257. [CrossRef]
31. Bezerra, V.B. Gravitational analogs of the Aharonov–Bohm effect. *J. Math. Phys.* **1989**, *30*, 2895–2899. [CrossRef]
32. Davidowitz, H.; Steinberg, V. On an analog of the Aharonov-Bohm effect in superfluid helium. *Europhys. Lett. (EPL)* **1997**, *38*, 297–300. [CrossRef]
33. Aharonov, Y.; Casher, A. Topological Quantum Effects for Neutral Particles. *Phys. Rev. Lett.* **1984**, *53*, 319–321. [CrossRef]
34. König, M.; Tschetschetkin, A.; Hankiewicz, E.M.; Sinova, J.; Hock, V.; Daumer, V.; Schäfer, M.; Becker, C.R.; Buhmann, H.; Molenkamp, L.W. Direct Observation of the Aharonov-Casher Phase. *Phys. Rev. Lett.* **2006**, *96*, 076804. [CrossRef] [PubMed]
35. Bakke, K.; Furtado, C. The analogue of the Aharonov-Bohm effect for bound states for neutral particles. *Mod. Phys. Lett. A* **2011**, *26*, 1331–1341. [CrossRef]
36. Bell, M.T.; Zhang, W.; Ioffe, L.B.; Gershenson, M.E. Spectroscopic Evidence of the Aharonov-Casher Effect in a Cooper Pair Box. *Phys. Rev. Lett.* **2016**, *116*, 107002. [CrossRef] [PubMed]
37. Hendricks, B.H.W.; Nienhuis, G. Sagnac effect as viewed by a co-rotating observer. *Quantum Opt. J. Eur. Opt. Soc. Part B* **1990**, *2*, 13. [CrossRef]
38. Aharonov, Y.; Carmi, G. Quantum aspects of the equivalence principle. *Found. Phys.* **1973**, *3*, 493–498. [CrossRef]
39. Harris, J.H.; Semon, M.D. A review of the Aharonov-Carmi thought experiment concerning the inertial and electromagnetic vector potentials. *Found. Phys.* **1980**, *10*, 151–162. [CrossRef]
40. Shen, J.Q.; He, S.; Zhuang, F. Aharonov-Carmi effect and energy shift of valence electrons in rotating C60 molecules. *Eur. Phys. J. D-At. Mol. Opt. Plasma Phys.* **2005**, *33*, 35–38. [CrossRef]
41. Johnson, B.L. Inertial forces and the Hall effect. *Am. J. Phys.* **2000**, *68*, 649–653. [CrossRef]
42. Brandão, J.E.; Moraes, F.; Cunha, M.; Lima, J.R.; Filgueiras, C. Inertial-Hall effect: The influence of rotation on the Hall conductivity. *Results Phys.* **2015**, *5*, 55–59. [CrossRef]
43. Tan, W.C.; Inkson, J.C. Electron states in a two-dimensional ring - an exactly soluble model. *Semicond. Sci. Technol.* **1996**, *11*, 1635. [CrossRef]
44. Pereira, L.F.C.; Silva, E.O. Modification of Landau Levels in a 2D Ring Due to Rotation Effects and Edge States. *Ann. Der Phys.* **2023**, *535*, 2200371. [CrossRef]
45. Lima, D.F.; dos S. Azevedo, F.; Pereira, L.F.C.; Filgueiras, C.; Silva, E.O. Optical and electronic properties of a two-dimensional quantum ring under rotating effects. *Ann. Phys.* **2023**, *459*, 169547. [CrossRef]
46. Pereira, C.M.O.; Azevedo, F.d.S.; Pereira, L.F.C.; Silva, E.O. Rotating effects on the photoionization cross-section of a 2D quantum ring. *Commun. Theor. Phys.* **2024**, *76*, 105701. [CrossRef]
47. Ghanbari, A. Rotating effects on the thermophysical properties of a two-dimensional GaAs quantum ring. *Commun. Theor. Phys.* **2024**, *76*, 065504. [CrossRef]
48. Lima, D.F.; Cunha, M.M.; Pereira, L.F.C.; Silva, E.O. Bound States for the Spin-1/2 Aharonov-Bohm Problem in a Rotating Frame. *Universe* **2021**, *7*, 457. [CrossRef]
49. Reed, M.; Simon, B. *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness*; Academic Press: New York, NY, USA; London, UK, 1975.
50. Matsuo, M.; Ieda, J.; Saitoh, E.; Maekawa, S. Effects of Mechanical Rotation on Spin Currents. *Phys. Rev. Lett.* **2011**, *106*, 076601. [CrossRef]
51. da Costa, R.C.T. Quantum mechanics of a constrained particle. *Phys. Rev. A* **1981**, *23*, 1982–1987. [CrossRef]
52. da Costa, R.C.T. Constraints in quantum mechanics. *Phys. Rev. A* **1982**, *25*, 2893–2900. [CrossRef]
53. Aoki, H.; Suezawa, H. Landau quantization of electrons on a sphere. *Phys. Rev. A* **1992**, *46*, R1163–R1166. [CrossRef]
54. Lima, J.R.; de Pádua Santos, A.; Cunha, M.M.; Moraes, F. Effects of rotation on Landau states of electrons on a spherical shell. *Phys. Lett. A* **2018**, *382*, 2499–2505. [CrossRef]
55. Atanasov, V.; Dandoloff, R. The curvature of the rotating disk and its quantum manifestation. *Phys. Scr.* **2015**, *90*, 065001. [CrossRef]
56. Cunha, M.M.; Lima, J.R.F.; Moraes, F.; Fumeron, S.; Berche, B. Spin current generation and control in carbon nanotubes by combining rotation and magnetic field. *J. Phys. Condens. Matter* **2020**, *32*, 185301. [CrossRef] [PubMed]
57. Cheng, R.; Wang, L.; Zhao, H.; Wang, Y.L.; Wang, J. The spin-1/2 particle on a rotating curved surface in time-varying fields. *Results Phys.* **2022**, *42*, 105974. [CrossRef]

58. Landry, A.; Hammad, F.; Saadati, R. The Quantum Hall Effect under the Influence of Gravity and Inertia: A Unified Approach. *Universe* **2024**, *10*, 136. [CrossRef]
59. Fonseca, I.; Bakke, K. Rotating effects on an atom with a magnetic quadrupole moment confined to a quantum ring. *Eur. Phys. J. Plus* **2016**, *131*, 67. [CrossRef]
60. Nelson, R.A. Generalized Lorentz transformation for an accelerated, rotating frame of reference. *J. Math. Phys.* **1987**, *28*, 2379–2383. [CrossRef]
61. Brill, D.R.; Cohen, J.M. Rotating masses and their effect on inertial frames. *Phys. Rev.* **1966**, *143*, 1011. [CrossRef]
62. Detweiler, S. Klein-Gordon equation and rotating black holes. *Phys. Rev. D* **1980**, *22*, 2323. [CrossRef]
63. Tamburini, F.; Thidé, B.; Molina-Terriza, G.; Anzolin, G. Twisting of light around rotating black holes. *Nat. Phys.* **2011**, *7*, 195–197. [CrossRef]
64. Altamirano, N.; Kubizňák, D.; Mann, R.B.; Sherkatghanad, Z. Thermodynamics of rotating black holes and black rings: Phase transitions and thermodynamic volume. *Galaxies* **2014**, *2*, 89–159. [CrossRef]
65. Pappas, G. An accurate metric for the spacetime around rotating neutron stars. *Mon. Not. R. Astron. Soc.* **2017**, *466*, 4381–4394. [CrossRef]
66. Pesonen, J. Eckart frame vibration-rotation Hamiltonians: Contravariant metric tensor. *J. Chem. Phys.* **2014**, *140*, 074101. [CrossRef] [PubMed]
67. Iyer, B. Dirac field theory in rotating coordinates. *Phys. Rev. D* **1982**, *26*, 1900. [CrossRef]
68. Bermudez, A.; Martin-Delgado, M.; Solano, E. Exact mapping of the 2+ 1 Dirac oscillator onto the Jaynes-Cummings model: Ion-trap experimental proposal. *Phys. Rev. A—At. Mol. Opt. Phys.* **2007**, *76*, 041801. [CrossRef]
69. Yang, J.; Piekarewicz, J. Dirac oscillator: An alternative basis for nuclear structure calculations. *Phys. Rev. C* **2020**, *102*, 054308. [CrossRef]
70. Bakke, K. Rotating effects on the Dirac oscillator in the cosmic string spacetime. *Gen. Relativ. Gravit.* **2013**, *45*, 1847–1859. [CrossRef]
71. Pacheco, M.; Landim, R.; Almeida, C. One-dimensional Dirac oscillator in a thermal bath. *Phys. Lett. A* **2003**, *311*, 93–96. [CrossRef]
72. Myers, N.M.; Abah, O.; Deffner, S. Quantum Otto engines at relativistic energies. *New J. Phys.* **2021**, *23*, 105001. [CrossRef]
73. Strange, P.; Ryder, L.H. The Dirac oscillator in a rotating frame of reference. *Phys. Lett. A* **2016**, *380*, 3465–3468. [CrossRef]
74. Hehl, F.W.; Ni, W.T. Inertial effects of a Dirac particle. *Phys. Rev. D* **1990**, *42*, 2045–2048. [CrossRef]
75. Varjú, K.; Ryder, L.H. Comparing the effects of curved space and noninertial frames on spin 1/2 particles. *Phys. Rev. D* **2000**, *62*, 024016. [CrossRef]
76. Dayi, O.F.; Kilinçarslan, E. Semiclassical transport equations of Dirac particles in rotating frames. *Phys. Rev. D* **2020**, *102*, 045015. [CrossRef]
77. Zhang, C.M.; Beesham, A. Rotation intrinsic spin coupling: The parallelism description. *Mod. Phys. Lett. A* **2001**, *16*, 2319–2326. [CrossRef]
78. Chowdhury, D.; Basu, B. The effect of inertia on the Dirac electron, the spin Hall current and the momentum space Berry curvature. *Ann. Phys.* **2013**, *329*, 166–178. [CrossRef]
79. Papini, G. Spin currents in non-inertial frames. *Phys. Lett. A* **2013**, *377*, 960–963. [CrossRef]
80. Vilenkin, A. Quantum field theory at finite temperature in a rotating system. *Phys. Rev. D* **1980**, *21*, 2260. [CrossRef]
81. Novoselov, K.S.; Geim, A.K.; Morozov, S.V.; Jiang, D.; Katsnelson, M.I.; Grigorieva, I.V.; Dubonos, S.V.; Firsov, A.A. Two-dimensional gas of massless Dirac fermions in graphene. *Nature* **2005**, *438*, 197–200. [CrossRef]
82. Tang, C.; Yan, W.; Zheng, Y.; Li, G.; Li, L. Dirac equation description of the electronic states and magnetic properties of a square graphene quantum dot. *Nanotechnology* **2008**, *19*, 435401. [CrossRef]
83. Park, C.H.; Son, Y.W.; Yang, L.; Cohen, M.L.; Louie, S.G. Landau Levels and Quantum Hall Effect in Graphene Superlattices. *Phys. Rev. Lett.* **2009**, *103*, 046808. [CrossRef]
84. Castro-Villarreal, P.; Ruiz-Sánchez, R. Pseudomagnetic field in curved graphene. *Phys. Rev. B* **2017**, *95*, 125432. [CrossRef]
85. Gallerati, A. Graphene, Dirac equation and analogue gravity. *Phys. Scr.* **2022**, *97*, 064005. [CrossRef]
86. Johnson, R.D.; Yannoni, C.S.; Dorn, H.C.; Salem, J.R.; Bethune, D.S. C_{60} Rotation in the Solid State: Dynamics of a Faceted Spherical Top. *Science* **1992**, *255*, 1235–1238. [CrossRef] [PubMed]
87. Liang, Q.; Tsui, O.K.C.; Xu, Y.; Li, H.; Xiao, X. Effect of C_{60} Molecular Rotation on Nanotribology. *Phys. Rev. Lett.* **2003**, *90*, 146102. [CrossRef] [PubMed]
88. Bubenchikov, A.M.; Bubenchikov, M.A.; Mamontov, D.V.; Lun-Fu, A.V. Md-Simulation of Fullerene Rotations in Molecular Crystal Fullerite. *Crystals* **2019**, *9*, 496. [CrossRef]
89. Bubenchikov, A.M.; Bubenchikov, M.A.; Lun-Fu, A.V.; Ovchinnikov, V.A. Gyroscopic effects in fullerite crystal upon deformation. *Eur. Phys. J. Plus* **2021**, *136*, 388. [CrossRef]
90. Shen, J.Q.; He, S.L. Geometric phases of electrons due to spin-rotation coupling in rotating C_{60} molecules. *Phys. Rev. B* **2003**, *68*, 195421. [CrossRef]
91. Zhang, G.P.; Bai, Y.H.; George, T.F. Optically and thermally driven huge lattice orbital and spin angular momenta from spinning fullerenes. *Phys. Rev. B* **2021**, *104*, L100302. [CrossRef]

92. Lima, J.R.; Brandão, J.; Cunha, M.M.; Moraes, F. Effects of Rotation in the Energy Spectrum of C_{60}. *Eur. Phys. J. D* **2014**, *68*, 94. [CrossRef]
93. Lima, J.R.; Moraes, F. The combined effect of inertial and electromagnetic fields in a fullerene molecule. *Eur. Phys. J. B* **2015**, *88*, 63. [CrossRef]
94. Cavalcante, E.; Carvalho, J.; Furtado, C. Description for rotating C_{60} fullerenes via an analogue of Gödel-type metric. *Eur. Phys. J. Plus* **2016**, *131*, 288. [CrossRef]
95. Král, P.; Sadeghpour, H.R. Laser spinning of nanotubes: A path to fast-rotating microdevices. *Phys. Rev. B* **2002**, *65*, 161401. [CrossRef]
96. Cai, K.; Yin, H.; Qin, Q.H.; Li, Y. Self-excited oscillation of rotating double-walled carbon nanotubes. *Nano Lett.* **2014**, *14*, 2558–2562. [CrossRef] [PubMed]
97. Cunha, M.M.; Brandão, J.; Lima, J.R.; Moraes, F. Spin splitting at the Fermi level in carbon nanotubes in the absence of a magnetic field. *Eur. Phys. J. B* **2015**, *88*, 288. [CrossRef]
98. Durrer, R. Topological defects in cosmology. *New Astron. Rev.* **1999**, *43*, 111–156. [CrossRef]
99. Fumeron, S.; Berche, B. Introduction to topological defects: From liquid crystals to particle physics. *Eur. Phys. J. Spec. Top.* **2023**, *232*, 1813–1833. [CrossRef]
100. Kivotides, D. Superfluid helium-4 hydrodynamics with discrete topological defects. *Phys. Rev. Fluids* **2018**, *3*, 104701. [CrossRef]
101. Dodgson, M.J.W.; Moore, M.A. Topological defects in the Abrikosov lattice of vortices in type-II superconductors. *Phys. Rev. B* **1995**, *51*, 11887–11902. [CrossRef]
102. Lin, Z.K.; Wang, Q.; Liu, Y.; Xue, H.; Zhang, B.; Chong, Y.; Jiang, J.H. Topological phenomena at defects in acoustic, photonic and solid-state lattices. *Nat. Rev. Phys.* **2023**, *5*, 483–495. [CrossRef]
103. Braverman, L.; Scheibner, C.; VanSaders, B.; Vitelli, V. Topological Defects in Solids with Odd Elasticity. *Phys. Rev. Lett.* **2021**, *127*, 268001. [CrossRef]
104. Katanaev, M.O. Geometric theory of defects. *Physics-Uspekhi* **2005**, *48*, 675. [CrossRef]
105. Fumeron, S.; Berche, B.; Moraes, F. Geometric theory of topological defects: Methodological developments and new trends. *Liq. Cryst. Rev.* **2021**, *9*, 85–110. [CrossRef]
106. Fumeron, S.; Pereira, E.; Moraes, F. Modeling heat conduction in the presence of a dislocation. *Int. J. Therm. Sci.* **2013**, *67*, 64–71. [CrossRef]
107. Katanaev, M.O.; Volovich, I.V. Theory of defects in solids and three-dimensional gravity. *Ann. Phys.* **1992**, *216*, 1–28. [CrossRef]
108. Furtado, C.; Moraes, F.; de M. Carvalho, A. Geometric phases in graphitic cones. *Phys. Lett. A* **2008**, *372*, 5368–5371. [CrossRef]
109. Dantas, L.; Furtado, C.; Silva Netto, A. Quantum ring in a rotating frame in the presence of a topological defect. *Phys. Lett. A* **2015**, *379*, 11–15. [CrossRef]
110. Brandão, J.; Filgueiras, C.; Rojas, M.; Moraes, F. Inertial and topological effects on a 2D electron gas. *J. Phys. Commun.* **2017**, *1*, 035004. [CrossRef]
111. Rojas, M.; Filgueiras, C.; Brandão, J.; Moraes, F. Topological and non inertial effects on the interband light absorption. *Phys. Lett. A* **2018**, *382*, 432–439. [CrossRef]
112. Oliveira, R. Topological and noninertial effects in an Aharonov–Bohm ring. *Gen. Relativ. Gravit.* **2019**, *51*, 120. [CrossRef]
113. Oliveira, R. Noninertial and spin effects on the 2D Dirac oscillator in the magnetic cosmic string background. *Gen. Relativ. Gravit.* **2020**, *52*, 88. [CrossRef]
114. Cuzinatto, R.; De Montigny, M.; Pompeia, P. Non-commutativity and non-inertial effects on the Dirac oscillator in a cosmic string space–time. *Gen. Relativ. Gravit.* **2019**, *51*, 107. [CrossRef]
115. Garcia, G.Q.; Cavalcante, E.; de M. Carvalho, A.M.; Furtado, C. The geometric theory of defects description for c 60 fullerenes in a rotating frame. *Eur. Phys. J. Plus* **2017**, *132*, 183. [CrossRef]
116. Guvendi, A.; Dogan, S.G. Effect of internal magnetic flux on a relativistic spin-1 oscillator in the spinning point source-generated spacetime. *Mod. Phys. Lett. A* **2023**, *38*, 2350075. [CrossRef]
117. Yoon, J.; Pramanik, T.; Park, B.K.; Cho, Y.W.; Lee, S.Y.; Kim, S.; Han, S.W.; Moon, S.; Kim, Y.S. Experimental comparison of various quantum key distribution protocols under reference frame rotation and fluctuation. *Opt. Commun.* **2019**, *441*, 64–68. [CrossRef]
118. Mäkinen, J.; Autti, S.; Heikkinen, P.; Hosio, J.; Hänninen, R.; L'vov, V.; Walmsley, P.; Zavjalov, V.; Eltsov, V. Rotating quantum wave turbulence. *Nat. Phys.* **2023**, *19*, 898–903. [CrossRef]
119. Restuccia, S.; Toroš, M.; Gibson, G.M.; Ulbricht, H.; Faccio, D.; Padgett, M.J. Photon Bunching in a Rotating Reference Frame. *Phys. Rev. Lett.* **2019**, *123*, 110401. [CrossRef]
120. Mustafa, O. PDM Klein–Gordon particles in Gödel-type Som–Raychaudhuri cosmic string spacetime background. *Eur. Phys. J. Plus* **2023**, *138*, 21. [CrossRef]
121. Boumali, A.; Bouzenada, A.; Messai, N.; Mustafa, O. PDM-Coulombic effects of non-inertial cosmic string on a Klein-Gordon oscillator. *Rev. Mex. Física* **2024**, *70*, 050802. [CrossRef]