Effective Field Theories of Gravity: A Top-down Approach

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Abstract

From water boiling to the behaviour of magnets, the world around us is often described using the language of Effective Field Theory. This powerful framework is often employed to model a system in terms of its low energy degrees of freedom without much thought being given to the higher energy origins of the model. In this thesis we shall attempt to turn this idea on its head by constructing effective theories by beginning from the high energy theory and then arriving at the low energy description. Our main aim shall be to address this issue in the context of deriving the EFT of perturbations around a background solution of Type IIA Supergravity given by Crampton, Pope and Stelle that involves the non-compact hyperbolic space $\mathcal{H}^{2,2}$. Studying the subject of EFTs in the context of Type IIA Supergravity presents it's own challenges as this theory exists in ten spacetime dimensions, whilst our observed universe is four dimensional. We shall present how to address this issue of dimensional mismatch using Kaluza-Klein methods and discuss aspects of the resulting lower dimensional field with an eye towards achieving lower dimensional diffeomorphism invariance. This treatment shall lead us to the conclusion that a previously undiscovered Stueckelberg field must be accounted for and dealt with if one is to appropriately understand the lower dimensional theory. We also exhibit a generalisation of the solutions of Crampton, Pope and Stelle to include time dependence. Such solutions may have applications to cosmological models.

Declaration of originality

I hereby declare that the work contained in this thesis is my own, except where there are references to others works, or works done in collaboration with others. The work presented in this thesis was carried out between October 2015 and October 2019 at Imperial College London under the supervision of Professor K.S.Stelle. This thesis has not been submitted for a degree or diploma at any other university.

The work presented in Section 2 is based on unpublished work with K.S.Stelle. Sections 3-5 are based on unpublished work with C.W.Erickson, R.J.Y.Leung and K.S.Stelle. Section 6 is based on unpublished work with R.Gwyn, J.L.Lehners and K.S.Stelle. Appendix D is based on unpublished work with C.W.Erickson and K.S.Stelle.

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Copyright Declaration

Conventions and abbreviations

Greek letters $\mu, \nu, \ldots$ shall denote indices in a coordinate basis and Latin letters $a, b, \ldots$ shall denote abstract indices. Expressions involving abstract indices are valid in any basis.

We work, mostly, with Lorentzian metrics which we choose to have signature

$$(- + \ldots +).$$

The Christoffel symbol is defined by

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_{\nu} g_{\sigma\rho} + \partial_{\rho} g_{\nu\sigma} - \partial_{\sigma} g_{\nu\rho} \right) .$$

For covariant derivatives we choose

$$\nabla_{\mu} X^{\nu} = \partial_{\mu} X^{\nu} + \Gamma^{\nu}{}_{\rho\mu} X^{\rho} .$$

Our Laplacian is defined by

$$\nabla^2 = g^{ab} \nabla_a \nabla_b .$$

For curvature tensors we take the definition

$$R(X,Y)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

which, in a coordinate basis, reads

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho} \Gamma^{\mu}{}_{\nu\sigma} - \partial_{\sigma} \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\tau}{}_{\nu\sigma} \Gamma^{\mu}{}_{\tau\rho} - \Gamma^{\tau}{}_{\nu\rho} \Gamma^{\mu}{}_{\tau\sigma} .$$

For the associated Ricci tensor we adopt the convention

$$R_{ab} = R^{c}{}_{acb} .$$

When using differential forms we use the following conventions

$$\omega = \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} ,$$

$$d\omega = \frac{1}{p!} \partial_{\nu} \omega_{\mu_1 \ldots \mu_p} dx^{\nu} \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} ,$$

$$\omega \wedge \eta = \frac{1}{p!q!}\omega_{\mu_1\ldots\mu_p}\eta_{\nu_1\ldots\nu_q}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}\wedge dx^{\nu_1}\wedge\ldots\wedge dx^{\nu_q} \ ,$$

$$*\omega = \frac{1}{(n-p)!p!}\epsilon_{\mu_1\ldots\mu_{n-p}\nu_1\ldots\nu_p}\omega^{\nu_1\ldots\nu_p}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_{n-p}} \ .$$

Here and throughout this work ϵ denotes the Levi-Civita tensor with

$$\epsilon_{12\ldots n} = \sqrt{|g|} \ ,$$

while $\tilde{\epsilon}$ is the alternating symbol with

$$\tilde{\epsilon}_{12\ldots n} = \tilde{\epsilon}^{12\ldots n} = 1 \ ,$$

unless explicitly stated otherwise.

Symmetrisation and antisymmetrisation are done strength one so that

$$A_{(a_1\ldots a_p)} = \frac{1}{p!}\sum_{\sigma\in S_p} A_{a_{\sigma(1)}\ldots a_{\sigma(p)}} \ ,$$

$$A_{[a_1\ldots a_p]} = \frac{1}{p!}\sum_{\sigma\in S_p} \mathrm{sgn}(\sigma) A_{a_{\sigma(1)}\ldots a_{\sigma(p)}} \ .$$

$\mathbb{N}_0$ denotes the natural numbers including zero and $\mathbb{N} = \mathbb{N}_0 \setminus \{0\}$.

For Clifford algebras we use the convention

$$\{e_\mu, e_\nu\} = 2\eta_{\mu\nu} \ .$$

We also use the abbreviations B&E and CPS to refer to [4] and [17] respectively.

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1 Introduction

The twentieth century saw one of the most prolific periods of research in the area of theoretical physics, with the advent of the Theory of Relativity, in both its Special and General forms, and the discovery of Quantum Mechanics. These two theories have formed the pillars upon which approximately a hundred years of research has been based. Both have successfully passed numerous experimental tests, leading to these, once theoretical formalisms, being firmly accepted into the realm of real world physics. Aside from the experimental success of these theories they've also redefined how we understand the universe itself. All three of these theories fly in the face of the classical theory of Newtonian mechanics, with the theory of Special Relativity forcing us to abandon the notion of absolute time and replace the view of space and time as being distinct with the idea that they're a single entity, which leads us to the concept of spacetime. Quantum Mechanics compels us to take a non-deterministic viewpoint of the world at, and below, the atomic scale, while the lessons of General Relativity are no less drastic as we are led to conclude that gravity is not a force at all. Instead it should be considered as a manifestation of the underlying geometry of the spacetime on which we live. Aside from the shift in the paradigm of understanding of the physical world the introduction of these theories also altered the mathematical language used by physicists who, in light of this new world view, needed to incorporate the areas of abstract algebra and geometry into their repertoire.

However the physicists of the twentieth century weren't finished quite yet as they proceeded to combine the theory of Special Relativity with Quantum Mechanics into the formalism known as Quantum Field Theory (QFT). Armed with QFT the late twentieth century saw the construction of the Standard Model of Particle Physics (SM), which became complete in 2012 after the discovery of the Higgs boson. The SM, currently, represents the most thoroughly tested theory ever constructed and accounts for the Strong, Weak and Electromagnetic forces of nature in one consistent framework. Yet despite all of its successes the Standard Model is known to be an incomplete theory of nature, for one thing it has a very large gravity shaped hole in it!

We might hope that the union of General Relativity and Quantum Mechanics might overcome this issue. However, naively quantising General Relativity, as a field theory, leads to many issues. Principle amongst these is that the resulting theory is non-renormalisable, meaning that the introduction of a finite number of modifications to the theory, in the form of counter terms, isn't enough to render its predictions for physical quantities sensible. This seems to be a sticking point, as we'd expect that, at some scale, the effects of a quantum theory of gravity will take over[1] from the General Relativistic behaviour displayed at larger scales.

[1] For example, as we approach the Planck scale.

Many options for providing a harmonious union of these two subjects have been proposed, but in this thesis we shall follow the route provided by String Theory. String Theory is a fascinating subject where the point particles of classical field theory are replaced by one dimensional objects called strings. This one, seemingly innocuous, change sets in motion a collection of events that leads us to a theory that not only includes a quantum version of gravity, that in some limit reproduces General Relativity[2], but also provides mechanisms for introducing non-abelian gauge theories, of the type found in the standard model, into the game[3]. It may seem that String Theory is the answer to all of our problems. However, it comes along with its own issues. Phenomenologically the most pertinent amongst these are the requirement for us to deal with spacetimes with ten or eleven dimensions and the prediction of a large number of extra particles, both of which have not been observed experimentally.

We can mitigate the unobserved particles by taking a limit of the theory in which most of these additional particles become extremely heavy, thus becoming unobservable at the energies currently accessible to us. Taking such a limit is in fact how the connection between String Theory and Supergravity is best understood. While this limit may allow us to get around the plethora of unobserved particles it still leaves us with the issue of extra dimensions to deal with. This could been seen as a great weakness of the theory. However, if viewed in another light, it can be seen as one of the greatest sources of richness within the area. This richness relies upon the resurrection of ideas due to Kaluza [65] and Klein [66] whereby higher dimensional theories can be rewritten in such a way so as to appear as lower dimensional theories. In the usual setup the overall spacetime manifold M_d is assumed to have the topological structure $M_d = \mathcal{M}_n \times K_{d-n}$ where $\mathcal{M}_n$, usually, represents our universe and K_{d-n} is some compact space. The compact space comes with an associated characteristic length scale, l_c, and if this is made small enough then the space is undetectable by current experiments. As we'll see in this Section, a further aspect of the Kaluza-Klein setup is that higher dimensional fields have their dependence on the compact space expanded in the form of a generalised Fourier expansion. This expansion leads to the appearance of masses for the lower dimensional fields which are, in general, separated from each other by mass steps that are proportional to $\frac{1}{l_c}$. Hence another consequence of the compact space being assumed to be small is that these masses become extremely large, and so from an Effective Field Theory (EFT) point of view only the lightest of modes, which are often massless, are relevant to the lower dimensional physics[4]. The richness of Supergravity is now owed to the vast range of different choices of K_{d-n} one has available, meaning that from one higher dimensional theory we can receive

[2] Well, it's really an extension of General Relativity to include supersymmetry which results in a theory called Supergravity.

[3] Whilst this sounds very promising, we should emphasise that, to date, there has still not been an explicit construction of the Standard Model within String Theory.

[4] This very terse treatment wilfully disregards the issue of consistency of truncations in Kaluza-Klein approaches to theories. We'll have much more to say on this subtle issue later on in this Section.

a wide range of lower dimensional EFTs. This procedure of embedding lower dimensional theories into higher dimensional ones is itself desirable as the forms of these higher dimensional theories are highly constrained. As a result the Kaluza-Klein procedure leads to a small number of higher dimensional theories[5] from which a vast landscape of lower dimensional theories and EFTs can be obtained. One of the hopes of this embedding of theories into String Theory is that they, in some sense, inherit its seemingly very agreeable ultra-violet behaviour, which is yet to show any divergences.

We shall take a different approach in this thesis as, instead, we'll be interested in Kaluza-Klein theory in the case where the space K_{d-n} is non-compact. Given our previous discussion this may seem highly non-physical and appears to lead us into a whole lot of trouble. Firstly, we obviously only observe four large spacetime dimensions, and now we'd seemingly perceive additional ones. In addition to this we were able to develop an EFT description in the previous case, allowing us to ignore many of the fields in the expansion of higher dimensional fields, by making the compact space small enough, but now this clearly isn't an option. The first of these issues can, in some sense, be circumvented by considering the universe we occupy to be a four dimensional surface embedded in a higher dimensional spacetime. We then require a mechanism to localise the fields present in the lower dimensional theory, coming from the Kaluza-Klein expansion of the higher dimensional fields. Localisation in this context is required as the higher dimensional theory involves, at the level of the action, an integral over the higher dimensional spacetime manifold which includes the non-compact direction. Without a localisation mechanism this portion of the integral, which contributes to couplings in the lower dimensional EFT, will lead to results that are highly undesirable, for example the vanishing of the lower dimensional Newton's constant. Localisation is also required to ensure that the existence of large extra dimensions is not in conflict with experimental results, such as, but not limited to, the observed $\frac{1}{r^2}$ force law of Newtonian gravity. Assuming localisation can be achieved, we still have the issue that, in general, traditional EFT methods will fail because the spectrum of lower dimensional fields will, usually, be continuous, unlike in the case of a compact K_{d-n} where the characteristic scale l_c provides a natural separation of scales. This means there's, often, no sensible way to construct an EFT. One way this issue can be tamed is if the mass spectrum of the lower dimensional theory exhibits a mass gap as this provides the separation of scales required to render EFT methods admissible.

Models where our universe is taken to be a subsurface embedded in a higher dimensional spacetime with more than four non-compact directions often go under the name of braneworld models. This name derives from the original model of Randall and Sundrum [88] where localisation of gravity was achieved by using

[5] Which, from a String Theory point of view, are all related by duality symmetries.

a four dimensional brane embedded in a five dimensional spacetime. This model overcame all of the issues present with having an additional extended extra dimension. For example, the leading contribution to the four dimensional Newton's law potential was the expected $\frac{1}{r}$, in spite of the fact that the theory actually has a continuous spectrum of gravitons starting at zero mass, it should be noted that this continuum is separated from the lowest mass mode by a mass gap. So, whilst this model is extremely successful, it does require one to deal with the full collection of Kaluza-Klein modes of the graviton in order to show that these lead to subleading corrections. Interestingly the Randall-Sundrum braneworld model can be embedded in to ten dimensional Type IIB Supergravity [35].

The Randall-Sundrum model, whilst having many remarkable properties, is built using relatively simple and well understood components. These include branes, which are reasonably well understood in both String Theory and Supergravity, extra dimensions, on which there is a vast literature, and Kaluza-Klein theory. Over the years since its introduction into the literature the subject of braneworlds has seen a number of, increasingly sophisticated, models arise [2, 33, 70, 79]. However, whilst progress has been made, the subject remains difficult owing to the need for appropriate localisation mechanisms, which have proven hard to realise in a tractable manner.

As emphasised in the Randall-Sundrum case, a careful analysis of the lower dimensional graviton spectrum about a particular background solution is crucial in any attempt to develop a braneworld model. Particular care must be paid to the functional dependence of the graviton modes, interpreted as Kaluza-Klein modes of a higher dimensional graviton, on the directions transverse to the subspace which is to be interpreted as our universe. Fortunately, for a wide class of physically desirable backgrounds, a great deal of progress has been made in understanding the lower dimensional graviton spectrum [4, 19]. Particular focus has been devoted to understanding the allowed transverse dependence of the gravitons[6]. Interestingly detailed analysis shows that, under a certain collection of assumptions[7], the graviton spectrum can be determined entirely in terms of geometric quantities of the background without requiring knowledge concerning the specific matter content of the theory. However, despite strides being made in analysing the gravitational spectrum, the issue of requiring a localisation mechanism, which leads to sensible four dimensional physics, still has to be overcome.

In this thesis we shall predominately focus on the background solutions of Type IIA Supergravity provided by Crampton, Pope and Stelle (CPS) [17]. Building on work that considered Supergravity theories reduced on

[6] This is often called the transverse wavefunction of a graviton as the problem can often be rephrased as a Schrödinger like problem.

[7] Which shall be reviewed and considered further in Section 2.

the non-compact inhomogeneous hyperboloidal spaces $\mathcal{H}^{p,q}$ [20, 21] CPS were able to embed the Mink$_4$ $\times S^2$ solution of Salam and Sezgin [91] into Type IIA Supergravity. From this higher dimensional viewpoint the solution was identified as a particular instance of a brane resolved by transgression [22, 69]. This insight led to the discovery that the lifted Salam-Sezgin solution is in fact just one member of a family of solutions. The family was found to be parametrised by two parameters, one of which corresponded to the charge of an NS5-brane located within the geometry, which acts as a source for the system. The Salam-Sezgin solution was found to correspond to the case where the NS5-brane has zero charge and decouples from the system. The remarkable property of the CPS family of solutions is that the graviton spectrum has a single massless bound state, which one wishes to interpret as a graviton in the lower dimension, that is separated from a continuum of massive graviton modes by a mass gap. The height of the mass gap is specified by two parameters arising in the CPS solutions. Further to the existence of this mass gap, the bound state has a transverse wavefunction that ensures localisation occurs. It's the combination of these features that makes the CPS system appealing from a braneworld model building perspective.

The existence of a mass gap in the CPS system leads to a separation of the massless graviton from the Kaluza-Klein tower of massive gravitons which allows for a treatment of the system using the powerful tools of EFT methods[8]. As with any EFT only the low energy degrees of freedom, which have energies below the scale set by the mass gap, are to be included. Any high energy degrees of freedom will have been integrated out to arrive at the EFT. It is the construction of an EFT for the massless graviton that will be the central endeavour of this thesis. We shall take a top down approach to this EFT by beginning in ten dimensions and then performing various manipulations, detailed in the main body of this work, to attempt to arrive at a four dimensional EFT. Of particular interest shall be understanding how four dimensional diffeomorphism invariance manifests from its ten dimensional parent symmetry.

With all of the difficulties associated with having large extra dimensions the reader may ask why we should get involved in discussions of such configurations. From a classification point of view such solutions can be shown to lie within the space of solutions to Supergravity theories, and so if we're to understand this space we will eventually have to confront braneworld models. Another view point, which seems apt at the time of writing, is that such spaces avoid the powerful no-go theorems concerning the non-existence of de Sitter vacua in Supergravity theories, [50, 71] and more recently [90], as all assume compact internal spaces. Hence as braneworld models fall outside of the domain of applicability of these no-goes they appear to be an interesting place to go hunting for models where universes like our own may exist.

[8]For an introduction to this subject we recommend the reader consult the excellent review [10].

The outline of the rest of this thesis is as follows:

- In the remainder Section 1 we shall provide an introduction to Einstein gravity, in particular paying attention to its interpretation as a field theory of a spin-2 field as well as the presence of potential boundary actions. We shall then introduce the notion of supersymmetry and see how, when combined with Einstein gravity, this leads to theories of Supergravity. Various aspects of Supergravity theories will be presented with particular emphasis being given to branes and Kaluza-Klein theory.

- Section 2 will cover the problem of finding the graviton spectrum about a class of backgrounds contained within the class discussed in [4], with particular interest being paid to how the inclusion of a traceful perturbation affects the transverse wavefunction problem.

- In Section 3 we shall introduce, in detail, the CPS system and provide a first naive attempt at the construction of the EFT. We shall comment on how this doesn't lead to a system with an obvious lower dimensional diffeomorphism symmetry.

- The failure of lower dimensional diffeomorphism symmetry to emerge will lead us to reconsider the way in which perturbations to the CPS background are performed in Section 4. Insights gained from this will lead us to propose a modification to the perturbation problem, and hence the spectrum of the problem, that will cure the issues present with previous approaches allowing for the chance to realise four dimensional diffeomorphism symmetry.

- Then in Section 5 we shall apply the lessons learnt in previous Sections and provide a simplified example of the process we believe is required in order to appropriately construct a lower dimensional EFT that can display, in some manner, an analogue of higher dimensional diffeomorphism symmetry.

- We shall then change pace and look at a class of solutions, based on the CPS family, that may be of cosmological interest in Section 6. We shall demonstrate that the time dependent solutions obtained in this Section also localise gravity.

- Finally in Section 7 we shall provide a summary of the work undertaken, outline work that still needs to be completed, and consider possible future directions that should be explored.

1.1 Einstein gravity and the spin-2 field

General Relativity (GR) provides a beautiful example of the intimate relationship between geometry and physics[9]. From studying GR we learn that our universe should be considered as a spacetime, which

[9] Sadly this won't be the focus of this Section however excellent accounts of this can be found in [12, 101].

mathematically translates as a manifold[10] equipped with a Lorentzian metric. The mathematical domain of manifolds and metrics is that of differential geometry. If we follow this approach to GR we arrive at the picture that gravity is just a manifestation of the geometry, encoded in the choice of metric, of our spacetime. Owing to its construction using the language of differential geometry GR possess invariance under diffeomorphisms[11], often stated in the physics literature, in a more local form, as invariance under general coordinate transformations.

However, there exists another approach to GR in which it is modelled on a field theory. In this context it is considered about a fixed background spacetime and diffeomorphism invariance arises as a gauge symmetry. Whilst this field theoretic approach to GR obscures the geometric perspective it brings sharply into focus similarities with gauge theories, such as Yang-Mills theory[12], and also provides a way to address the question of quantising gravity using QFT methods.

Interestingly, it can be shown that the field theoretic approach to GR is in fact, if followed through to its conclusion, equivalent to the geometric approach [31]. It shall be our goal, in what follows, to give an account of how this occurs with an emphasis on the role of diffeomorphism invariance. However, before embarking on this task, we shall begin by reviewing the dynamics of Einstein's theory and it's formulation as a variational problem.

As previously stated, the geometric approach to GR involves taking a pair (M, g), with M a manifold and g a Lorentzian metric[13], along with a collection of matter fields Φ, where possible indices on the matter fields are suppressed. As opposed to pre-GR physics the metric is now a dynamical field and hence a priori isn't determined. Instead, the form it takes is now dynamically determined by the Einstein equation

$$R_{MN} - \frac{1}{2} g_{MN} R + \Lambda g_{MN} = \kappa^2 T_{MN}(\Phi, g) \ . \tag{1.1}$$

Here $N \in \{1, .., \dim M = d\}$, R_{MN} is the Ricci tensor of the metric, R is its associated Ricci scalar, $\Lambda \in \mathbb{R}$ is the cosmological constant, T_{MN} is the usual Hilbert energy momentum tensor of the system and $\kappa^2 = 8\pi G$ is a way of writing Newton's gravitational constant. This equation encompasses one of the key insights of GR, namely that matter and energy induce curvature of the spacetime manifold.

[10]Throughout, unless stated otherwise, all manifolds are assumed to be smooth manifolds.
[11]Recall if M and N are manifolds then a diffeomorphism is a map $f \in C^\infty(M, N)$ that has a smooth inverse. Here $C^\infty(M, N)$ is the set of smooth maps from M to N.
[12]This can be seen in the geometric picture if one takes a geometric approach to the subject of curvatures as detailed in Appendix B.
[13]Often when referring to GR authors take the, real, dimension of M to be four, however in this thesis when we refer to GR we shall keep this dimension arbitrary.

As with any equation of motion it is desirable that (1.1) be obtainable from an action principle, as this can aid us in formally attempting to quantise the theory via path integral methods. In order to make our life slightly easier we shall consider the case of (1.1) where $\Lambda = 0$ and where there are no matter fields present, meaning $T_{MN} = 0$, this is often referred to as the pure gravity system. Fortunately, the action principle required is the well known Einstein-Hilbert action

$$S_{EH}[g] = \frac{1}{2\kappa^2} \int_M R * 1 \ . \tag{1.2}$$

Where $*$ is the Hodge star associated to the metric g. If we take a local chart, (U, φ), on M, with local coordinates x^M, then in this patch (1.2) becomes

$$S_{EH}|_{\varphi(U)}[g] = \frac{1}{2\kappa^2} \int_{\varphi(U)} d^d x \sqrt{-g} R \ , \tag{1.3}$$

with $\varphi(U) \subset \mathbb{R}^d$ and $g = \det(g_{MN})$. We can now perform a variation of the metric in (1.3) and show that, with standard calculations, see for example [101], the variation of our action takes the form

$$\begin{aligned} \delta_g S_{EH} =& \frac{1}{2\kappa^2} \Big(\int_{\varphi(U)} d^d x \sqrt{-g} \ \delta g^{MN} (R_{MN} - \frac{1}{2} g_{MN} R) + \int_{\varphi(U)} d^d x \sqrt{-g} \ \nabla_P V^P \Big) \ , \\ V^P =& g^{MN} \delta \Gamma^P{}_{MN} - g^{PQ} \delta \Gamma^M{}_{QM} \ , \end{aligned} \tag{1.4}$$

where we have varied the inverse metric, as this is the simpler calculation, and ∇_M is the covariant derivative associated to the Levi-Civita connection of the metric. The first of the terms in this variation is precisely the specialised form of (1.1) we were seeking, however the second term isn't something we want. We can see this second term involves the variation of the Christoffel symbols, details of which are given in Appendix A. The presence of this term means that, even when the Einstein equations hold, $\delta S_{EH} \neq 0$. As such, we haven't yet achieved our goal of extremism the action. To begin with we note that the offending term is a total, covariant, derivative, and hence it's a boundary term. This might give us hope that imposing boundary conditions on our variation will eliminate it.

If we select the commonly used class of conditions $\delta g_{MN} = 0$ on our boundary, ∂M, then we can investigate this claim. In this case if we use

$$\delta \Gamma^P{}_{MN} = \frac{1}{2} g^{PQ} (\partial_M \delta g_{QN} + \partial_N \delta g_{MQ} - \partial_Q \delta g_{MN}) \ , \tag{1.5}$$

then the second term in (1.4) becomes

$$\int_{\varphi(U)} d^d x \sqrt{-g}\, \nabla_P V^P = \int_{\partial\varphi(U)} d^{d-1} y \sqrt{|\gamma|}\, n^P (g^{MN} (\partial_M \delta g_{PN} - \partial_P \delta g_{MN}))\ , \tag{1.6}$$

where we've assumed a portion of the boundary is included in our patch and then used the divergence theorem. In this expression y^m are coordinates on the boundary, $m \in \{1, .., d-1\}$, n^M is the unit normal to the boundary, $\gamma_{MN} = g_{MN} \pm n_M n_N$ is the transverse metric with the sign corresponding to if the boundary is timelike (-) or spacelike (+) and $\gamma = \det(\gamma_{mn})$ with γ_{mn} the induced metric on the boundary[14]. If we substitute $g^{MN} = \gamma^{MN} \mp n^M n^N$ and use that the the boundary term in (1.6) is antisymmetric on M and P then we obtain

$$\int_{\varphi(U)} d^d x \sqrt{-g}\, \nabla_P V^P = \int_{\partial\varphi(U)} d^{d-1} y \sqrt{|\gamma|}\, n^P \gamma^{MN} (\partial_M \delta g_{PN} - \partial_P \delta g_{MN})\ .$$

We now note that $\gamma^{MN} \partial_M \delta g_{PN}$ is the derivative of the variation parallel to the boundary. As such our boundary conditions ensure the contribution of this term is zero. However the second term involves a normal derivative and hence the boundary conditions can't cause it to be zero as the variation only vanishes on the boundary and this normal derivative peaks back into the bulk of the spacetime. So the variation (1.4) subject to the boundary conditions $\delta g_{MN}|_{\partial M} = 0$ is

$$\delta_g S_{EH} = \frac{1}{2\kappa^2} \Big(\int_{\varphi(U)} d^d x \sqrt{-g}\, \delta g^{MN} (R_{MN} - \frac{1}{2} g_{MN} R) - \int_{\partial\varphi(U)} d^{d-1} y \sqrt{|\gamma|}\, n^P \gamma^{MN} \partial_P \delta g_{MN} \Big)\ , \tag{1.7}$$

hence, even when the Einstein equations hold, the variation of the action is not zero, even subject to a choice of boundary conditions. This may not worry us, but a variational principle is only considered to be well posed if its variation vanishes on solutions [38].

We could just make peace with the fact that our variation is not giving an extremum of the action, which is highly defeatist, or we could look to modify the starting action, (1.3), in such a way that we ensure that the final term in (1.7) vanishes subject to our choice of boundary conditions. We know that the term we wish to add only has to affect the boundary value in the variational problem and so from a bulk point of view it won't affect the dynamics. This ensures we retain the Einstein equation as the Euler-Lagrange equation of our variational principle. There are many modifications one could possibly make, and so it might seem

[14] Recall if we have a boundary then we can locally specify it by a relationship $x^M(y^m)$ of bulk and boundary coordinates. Then the induced metric on the boundary is just the standard pullback of the spacetime metric which is given by $\gamma_{mn} = \frac{\partial x^M}{\partial y^m} \frac{\partial x^N}{\partial y^n} g_{MN}$. Since $\frac{\partial x^M}{\partial y^m}$ is transverse to the normal n^M of the boundary we can see, that on the boundary, $\gamma_{mn} = \frac{\partial x^M}{\partial y^m} \frac{\partial x^N}{\partial y^n} \gamma_{MN}$ thus showing us that the transverse metric is just an extension of the boundary metric into the bulk of the spacetime.

hard to find one that does the job we need. Fortunately, the work has already been done for us and the fix is the famous Gibbons, Hawking, York (GHY) term [51, 108]

$$S_{GHY}[g] = \frac{1}{\kappa^2}\int_{\partial M} K *_h 1 \ , \tag{1.8}$$

where $*_h$ is the Hodge star of the induced metric and $K = g^{MN}\nabla_M n_N = \gamma^{MN}\nabla_M n_N$ is the trace of the extrinsic curvature of the boundary. The second equality arises as n_M is the unit normal to the boundary so $n^N\nabla_M n_N = 0$. If we consider this term locally in our patch we find that the GHY boundary action takes the form

$$S_{GHY}[g]|_{\partial\varphi(U)} = \frac{1}{\kappa^2}\int_{\partial\varphi(U)} d^{d-1}y\sqrt{|\gamma|}K \ . \tag{1.9}$$

Now considering the variation of the GHY action under a variation of the spacetime metric we only have to calculate the variation of K, since $\delta(\sqrt{|\gamma|}) = 0$ due to our choice of boundary conditions for the variation δg_{MN}, which is

$$\delta K = -\gamma^{MN}\delta\Gamma^P{}_{MN} n_P = \frac{1}{2} n^P\gamma^{MN}\partial_P\delta g_{MN} \ ,$$

giving us

$$\delta_g S_{GHY}[g]|_{\partial\varphi(U)} = \frac{1}{2\kappa^2}\int_{\partial\varphi(U)} d^{d-1}y\sqrt{|\gamma|}n^P\gamma^{MN}\partial_P\delta g_{MN} \ , \tag{1.10}$$

which is exactly the same as the term that occurs in (1.7) but with the opposite sign. This suggests that we consider the new action

$$S[g] = S_{EH} + S_{GHY} \ , \tag{1.11}$$

whose variation is

$$\delta_g S|_{\varphi(U)} = \frac{1}{2\kappa^2}\int_{\varphi(U)} d^d x\sqrt{-g}\ \delta g^{MN}(R_{MN} - \frac{1}{2}g_{MN}R) \ , \tag{1.12}$$

which vanishes if the Einstein equation holds.

This process, whilst not directly used in this form in the rest of this thesis, teaches us a valuable lesson. We seem to have learnt that, once boundary conditions are selected, it can often be useful to modify our starting action, by boundary terms, in order to obtain a system with more desirable properties under variations. Further modifications may be made to our starting action if we desire other properties. One example of this is we may wish for action to be finite when evaluated on solutions of the Einstein equation.

Equipped with an understanding of boundary terms and the importance of boundary conditions for variations we can address the real goal of this Section and introduce the field theory description of gravity.

Often the subject is first encountered in the context of gravitational perturbation theory where we expand our metric around a flat background

$$g_{MN} = \eta_{MN} + h_{MN} \ , \tag{1.13}$$

with η_{MN} the Minkowski metric and h_{MN} a fluctuation of the geometry. One can then proceed to expand the various curvature tensors appearing in (1.1), where $\Lambda = T_{MN} = 0$, and develop equations of motion for fluctuation or to expand (1.3) to develop an effective action for the fluctuation. These two methods of performing gravitational perturbation theory are detailed in Appendix A. Both methods rely on the fact that the full non-linear theory of gravity is already known and hence we can just expand about a particular background and consider our fluctuation as a field on a fixed background of our choice. This allows us to develop a field theory of h_{MN} which is often referred to as the spin-2 field[15].

However, we might instead consider what would happen if we didn't know the full non-linear theory of GR, then we can't just cheat and expand this using (1.13), or an analogue about a curved space, in order to develop a field theory for our spin-2 field. So we might ask if there's any way to develop the field theory from scratch. We know that we want the theory to be built on the symmetric field h_{MN} and this field should represent the graviton field and hence it should be massless, but beyond this we have little to guide us. Fortunately, the demand of the field being massless perks us up as we know of another massless field, the Maxwell, or spin-1, field. In the case of a Maxwell field we know that the lack of appearance of a mass term in the theory isn't a neat coincidence but in fact points to the existence of a gauge symmetry, which forbids the occurrence of such a term, within the theory. This leads us to ask if the same could be true for our spin-2 theory? We could attempt to look for such a symmetry, but remember we have the answer already. So lets cheat once more, we know if we perform an infinitesimal diffeomorphism along the flow of a vector field X then (1.13) transforms, up to linear order in X, as

$$g_{MN} \rightarrow g'_{MN} = g_{MN} + 2\partial_{(M}X_{N)} + \mathcal{L}_X h_{MN} \ . \tag{1.14}$$

Here $\partial_{(M}X_{N)}$ denotes that we take the strength one symmetric part and $\mathcal{L}_X$ denotes the Lie derivative along X. Since diffeomorphisms are the gauge symmetry of GR two metrics related by a diffeomorphism lead to physically equivalent spacetimes. Hence, in light of (1.14), it seems sensible to conjecture that h_{MN} and

$$h'_{MN} = h_{MN} + 2\partial_{(M}X_{N)} \ , \tag{1.15}$$

[15] This name arises because, in four dimensions, representations of the proper Lorentz group, $SO(1,3)$, are characterised by the eigenvalues of two Casimir operators which correspond to spin and mass.

define equivalent theories of the spin-2 field, thus showing our theory should be a gauge theory, with (1.15) being its gauge invariance, at least at linear order in X and h_{MN}. Note we've dropped the $\mathcal{L}_X h_{MN}$ term. We shall see this term has consequences later on when we come to considering an interacting theory, but for now we can ignore it. It's the symmetry (1.15) that we hope will disallow the mass term in our theory.

The first authors to consider the relationship between massless spin-2 fields and the expansion of gravity were Fierz and Pauli [42]. We shall begin by reviewing their construction of the unique equations of motion of a spin-2 field without interactions, before turning to the subject of including self-interactions of the spin-2 field to the theory and seeing how this affects the system. Since the vast majority of this thesis shall be concerned with finding EFTs by constructing effective actions we shall focus on building the spin-2 theory by constructing an action principle for it. Since we are going to work about Minkowski spacetime we shall, for now, assume the more conventional indices $\mu \in \{0, .., d-1\}$.

Our initial goal is to construct and understand the quadratic theory of the spin-2 system. If we assume all terms are Lorentz scalars and the action is at most two derivative then the possible terms one can use to construct it are

$$
\begin{aligned}
&F_1 = h_{\mu\nu}h^{\mu\nu} \quad , \quad F_2 = hh \quad , \quad F_3 = \partial^\mu h_{\mu\nu}\partial^\nu h \quad , \quad F_4 = \partial^\mu h_{\mu\nu}\partial^\sigma h^\nu{}_\sigma \quad , \quad F_5 = \partial_\sigma h_{\mu\nu}\partial^\sigma h^{\mu\nu} \; , \\
&F_6 = \partial^\sigma h_{\mu\nu}\partial^\mu h_\sigma{}^\nu \quad , \quad F_7 = \partial^\sigma h \partial_\sigma h \quad , \quad F_8 = (\Box h)h \quad , \quad F_9 = (\Box h_{\mu\nu})h^{\mu\nu} \quad , \quad F_{10} = (\partial^\mu\partial^\nu h_{\mu\nu})h \; , \\
&F_{11} = (\partial^\mu\partial^\sigma h_{\mu\nu})h_\sigma{}^\nu \quad , \quad F_{12} = (\partial^\mu\partial^\nu h)h_{\mu\nu} \; ,
\end{aligned}
$$

for what seems like a total of twelve terms. Note $h = \eta^{\mu\nu}h_{\mu\nu}$ and in these expressions, and what follows, indices are raised and lowered with the Minkowski metric unless stated otherwise. However, the terms containing derivatives can be related as follows

$$
\begin{aligned}
F_3 &= \partial^\nu(\partial^\mu h_{\mu\nu}h) - F_{10} \; , \\
F_4 &= \partial^\sigma((\partial^\mu h_{\mu\nu})h^\nu{}_\sigma) - F_{11} \; , \\
F_5 &= \partial^\sigma(\partial_\sigma h_{\mu\nu}h^{\mu\nu}) - F_9 \; , \\
F_6 &= \partial^\mu((\partial^\sigma h_{\mu\nu})h^\nu{}_\sigma) - F_{11} \; , \\
F_7 &= \partial^\sigma((\partial_\sigma h)h) - F_8 \; , \\
F_{12} &= \partial^\mu(h_{\mu\nu}\partial^\nu h) - \partial^\nu(\partial^\mu h_{\mu\nu}h) + F_{10} \; ,
\end{aligned}
\tag{1.16}
$$

so we only require a total of six terms to write down our action. We choose these terms to be $F_1, F_2, F_8, F_9, F_{10}, F_{11}$.

This would leave us with potential boundary terms. However, if we select asymptotically flat boundary conditions, which ensures that $h_{\mu\nu} \to 0$ at spatial infinity, and conditions so the perturbation also falls to zero in the far future and far past, then all of the boundary terms for the spin-2 field drop out.

With the independent terms we've obtained we can now state the most general action, up to (1.16) and our boundary condition, for a spin-2 field as

$$S_{free\ spin-2}[h] = \int d^dx \left(a_1 h^{\mu\nu} h_{\mu\nu} + a_2 hh + a_3 h^{\mu\nu} \Box h_{\mu\nu} + a_4 h \Box h + a_5 h \partial^\mu \partial^\nu h_{\mu\nu} + a_6 h_\sigma{}^\nu \partial^\mu \partial^\sigma h_{\mu\nu} \right) , \quad (1.17)$$

with a_i arbitrary real constants and all terms appropriately symmetrised. In order to fix these constants we now demand that (1.17) be invariant under (1.15) at linear order in X. If we consider the transformation under (1.15) and choose our diffeomorphism to preserve the boundary conditions of the perturbation, meaning in the resulting transformation we can integrate by parts freely without incurring a boundary term, we obtain

$$\begin{aligned} \delta_X S_{free\ spin-2} = \int d^dx \Big(& 4a_1 h^{\mu\nu} \partial_\mu X_\nu + 4a_2 h \partial^\mu X_\mu + 2(2a_3 + a_6) h_{\mu\nu} \Box (\partial^\mu X^\nu) \\ & 2(2a_4 + a_5) h \Box (\partial^\mu X_\mu) + 2(a_5 + a_6) h_{\mu\nu} (\partial^\mu \partial^\nu \partial^\sigma X_\sigma) \Big) . \end{aligned}$$

Since we demand that this variation is zero we receive the following relationships

$$a_1 = a_2 = 0 \quad , \quad a_4 = -a_3 \quad , \quad a_5 = 2a_3 \quad , \quad a_6 = -2a_3 ,$$

the first two of these tell us that there are no zero derivative terms present and so our symmetry has done exactly what we wanted, since it has caused the potential mass terms in the problem to be excluded. The other three relationships tell us that only the constant a_3 is truly undetermined and we usually take it, by convention, to be equal to $\frac{1}{2}$. With this we have finally obtained the Fierz-Pauli action

$$S_{FP}[h] = \int d^dx \left(\frac{1}{2} h^{\mu\nu} \Box h_{\mu\nu} - \frac{1}{2} h \Box h + h \partial^\mu \partial^\nu h_{\mu\nu} - h_\sigma{}^\nu \partial^\mu \partial^\sigma h_{\mu\nu} \right) , \quad (1.18)$$

with all indices appropriately symmetrised.

With our action now obtained we can vary it to obtain the dynamics of the spin-2 field. If we perform the variation, subject to the conditions $\delta h_{\mu\nu} = 0$ on the boundary, and the previous boundary conditions on

$h_{\mu\nu}$, we find that the appropriate equations of motion are

$$\Box h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial^\sigma \partial_\mu h_{\sigma\nu} - \partial^\sigma \partial_\nu h_{\sigma\mu} + \eta_{\mu\nu}(\partial^\sigma \partial^\rho h_{\sigma\rho} - \Box h) = 0 \ , \tag{1.19}$$

which can be written more easily using the Fierz-Pauli operator

$$\begin{aligned} &\mathcal{O}_{FP}{}^{\sigma\rho}{}_{\mu\nu}(h_{\sigma\rho}) = 0 \ , \\ &\mathcal{O}_{FP}{}^{\sigma\rho}{}_{\mu\nu} = \delta^\sigma_\mu \delta^\rho_\nu \Box + \eta^{\sigma\rho}\partial_\mu\partial_\nu - \delta^\rho_\nu \partial^\sigma \partial_\mu - \delta^\rho_\mu \partial^\sigma \partial_\nu + \eta_{\mu\nu}\partial^\rho\partial^\sigma - \eta_{\mu\nu}\eta^{\sigma\rho}\Box \ , \end{aligned} \tag{1.20}$$

with δ^μ_ν the Kronecker delta.

It can be shown that

$$\partial^\mu(\mathcal{O}_{FP}{}^{\sigma\rho}{}_{\mu\nu}(h_{\sigma\rho})) = 0 \ , \tag{1.21}$$

identically, that is this relationship holds even off-shell. This relationship may at first seem a little mysterious but it can be shown to be a linearised version of the Bianchi identity for the Einstein tensor. Whilst, at the moment, this identity seems of little consequence, when we start considering interactions of the spin-2 field we shall see it has far reaching consequences.

Just before we move on to consider interactions we analyse (1.19) a little. Our treatment of this subject is heavily inspired by that of Reall in [89]. The main case of interest for us will be the case where d=4 since it corresponds to the spacetime dimension we observe. In this case it's useful to consider a change of variables and instead work with the trace reversed spin-2 field

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \qquad \mu \in \{0, .., 3\} \ . \tag{1.22}$$

We can then rewrite (1.19), using $\bar{h} = -h$, in terms of this new field which leads to the dynamics being given by

$$\Box \bar{h}_{\mu\nu} - \partial^\sigma \partial_\mu \bar{h}_{\sigma\nu} - \partial^\sigma \partial_\nu \bar{h}_{\sigma\mu} + \eta_{\mu\nu}\partial^\sigma\partial^\rho \bar{h}_{\sigma\rho} = 0 \ , \tag{1.23}$$

this equation now looks simpler than (1.19), but it still looks formidable to try to solve. However, we still have an ace up our sleeve. The equation (1.23) is still invariant under diffeomorphisms, which cause, as a consequence of (1.15) and (5.124), $\bar{h}$ to transform as

$$\bar{h}_{\mu\nu} \rightarrow \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + 2\partial_{(\mu} X_{\nu)} - \eta_{\mu\nu}\partial^\sigma X_\sigma \ . \tag{1.24}$$

We can fix this invariance in any, valid, way we want and then attempt to solve a gauge fixed version of (1.23) in our chosen gauge. A standard choice is the transverse gauge

$$\partial^{\mu}\bar{h}'_{\mu\nu} = 0 \ , \tag{1.25}$$

which, from (1.24), we can see requires the diffeomorphism parameter to obey the equation

$$\Box X_{\mu} = -\partial^{\nu}\bar{h}_{\nu\mu} \ , \tag{1.26}$$

with $\bar{h}_{\mu\nu}$ the original non-transverse field we started with. In this gauge (5.124) simplifies dramatically to

$$\Box\bar{h}'_{\mu\nu} = 0 \ . \tag{1.27}$$

We drop the prime from now on. This equation is now much more familiar and we propose the solution

$$\bar{h}_{\mu\nu} = H_{\mu\nu}e^{ik_{\sigma}x^{\sigma}} \ , \tag{1.28}$$

which constitutes a plane wave solution with wavevector k_{σ}, x^{σ} are our spacetime coordinates and $H_{\mu\nu}$ is a constant symmetric polarisation tensor. Note as we want real solutions we really take the real part of this expression, however for ease of notation we shall suppress this.

With (1.28) we can see that (1.27) implies that on shell

$$k_{\sigma}k^{\sigma} = 0 \ , \tag{1.29}$$

while (5.126) tells us that, even off shell,

$$k^{\nu}H_{\nu\mu} = 0 \ . \tag{1.30}$$

It turns out that (5.126) has not fully fixed our gauge invariance as transformations obeying

$$\Box X_{\mu} = 0 \ , \tag{1.31}$$

leave our spin-2 field in transverse gauge. Hence we have a residual gauge invariance which we'd like to fix. The usual choice that we want to implement is to try and make our perturbation traceless. Once again we can expand our gauge parameter in terms of plane waves and consider a single mode, since our system is linear we can then just superimpose modes and obtain a general transformation. This leads to us taking

$$X_\mu = \xi_\mu e^{ik_\sigma x^\sigma} \ , \tag{1.32}$$

with ξ_μ a constant, local, 1-form. If we substitute this transformation into (1.31) we learn that k^μ must be null, which is exactly the on-shell condition for the spin-2 field. So we can use our residual gauge transformations on the on-shell spin-2 field only, they won't be compatible with (5.126) except in this case. Once we accept this we can use (1.28) and (1.32) in (1.24) to show that under (1.28) $H_{\mu\nu}$ transforms as

$$H_{\mu\nu} \to H'_{\mu\nu} = H_{\mu\nu} + i(k_\mu \xi_\nu + k_\nu \xi_\mu - \eta_{\mu\nu} k^\sigma \xi_\sigma) \ . \tag{1.33}$$

This can be used to achieve the longitudinal gauge choice

$$H^{'}_{\mu 0} = 0 \ . \tag{1.34}$$

To do this we use (1.33) to set $H'_{i0} = 0$, $i \in \{1, 2, 3\}$, by the choice

$$\xi_i = \frac{i}{k_0}(H_{i0} + ik_i X_0) \ , \tag{1.35}$$

where we use that k_0 is non-zero, as if it was then (1.29) would mean that k^μ is the zero vector and we want to exclude this case as then we don't have a true plane wave[16]. With $H'_{i0} = 0$ and (1.30) we can conclude that $H'_{00} = 0$ and hence we've achieved the longitudinal gauge. However, ξ_0 is still unfixed and we can use this to set ${H'}^\mu{}_\mu = 0$. By tracing (1.33) we see that tracelessness will be achieved by choosing

$$\xi_0 = \frac{i}{2k_0}(H^\mu{}_\mu - 2ik_i \xi_i) \ , \tag{1.36}$$

with ξ_i as given in (1.35). So we now see that while a transverse traceless spin-2 field can be achieved it can only be done once we go on shell for the spin-2 field. We shall return to this issue in Section 2 where we will apply it in the context of determining the graviton spectrum around a wide class of backgrounds.

With the free spin-2 field now understood we'd like to move on to consider its possible self interactions, which will lead us to augment (1.19) by a source term. This means we have to ask what sort of sources, that only contain the spin-2 field, can we couple to the spin-2 field? Well, an obvious choice is to construct an

[16]It should be noted that due to this we are, wilfully, excluding the case where $k^\sigma = 0$ which corresponds to a constant $h_{\mu\nu}$ which is itself interesting. A similar analysis to the one we are performing can be carried out in the case of a constant $h_{\mu\nu}$ using a $3+1$ split of variables, see [101] for details on this.

energy momentum tensor for the spin-2 field, as we already know this is the sort of object gravity likes to couple to. Rather than dive straight in to the case of the gravitational field let's, following the work of [28], begin with a simpler one. As such we consider a scalar field, φ, with dynamics given by

$$S[\varphi] = \int d^d x \, (-\frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2) \; , \tag{1.37}$$

which leads to the equation of motion

$$\Box \varphi - m^2 \varphi = 0 \; . \tag{1.38}$$

The field theory given by (1.37) possesses global Poincaré invariance, and hence if we perform a translation we will, via the usual Noether procedure, obtain the Noether energy momentum tensor of the system

$$T^{\mu\nu}_{(0)} = \partial^\mu \varphi \partial^\nu \varphi - \frac{1}{2}\eta^{\mu\nu}(\partial^\sigma \varphi \partial_\sigma \varphi + m^2 \varphi^2) \; . \tag{1.39}$$

We note in this case that $T^{\mu\nu}_{(0)} = T^{\nu\mu}_{(0)}$, which is not guaranteed to be true for a general Noether stress energy tensor. As must be the case, when φ obeys (1.38), we find that

$$\partial_\mu T^{\mu\nu}_{(0)} = (\Box \varphi - m^2 \varphi)\partial^\nu \varphi = 0 \; . \tag{1.40}$$

We can now couple our two systems in the following manner

$$S[h,\varphi] = \int d^d x \, \left(\frac{1}{2} h^{\mu\nu}\Box h_{\mu\nu} - \frac{1}{2} h \Box h + h \partial^\mu \partial^\nu h_{\mu\nu} - h_\sigma{}^\nu \partial^\mu \partial^\sigma h_{\mu\nu} - \frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 - 2\kappa^2 h_{\mu\nu} T^{\mu\nu}_{(0)}\right) \; , \tag{1.41}$$

which leads to the equations of motion

$$\begin{aligned} \mathcal{O}_{FP}{}^{\sigma\rho}{}_{\mu\nu}(h_{\sigma\rho}) &= 2\kappa^2 T_{(0)\mu\nu} \; , \\ \Box\varphi - m^2\varphi &= -4\kappa^2 \partial^\mu (h_{\mu\nu}\partial^\nu \varphi) + 2\kappa^2 \partial^\sigma (h \partial_\sigma \varphi) - 2\kappa^2 m^2 h \varphi \; . \end{aligned} \tag{1.42}$$

However, now for φ obeying (1.42) we see that $\partial_\mu T^{\mu\nu}_{(0)} = 0$ only up to quadratic order in the fields. This would seem to be terrible, it means that (1.15) is no longer an invariance of our theory and this was what we wanted all along. However all is not lost, we could try to augment our theory, (1.41), and transformation, (1.15), by higher order terms in our spin-2 field. If we were just to consider terms involving at least one φ then we could derive a Noether stress energy tensor for the matter in (1.41). This can, and often does, lead to a stress energy tensor that isn't symmetric. If this is the case we subject the resulting Noether stress tensor to the usual modification process of Belinfante. This new stress tensor now contains not just φ but also $h_{\mu\nu}$ and, so, while it contains (1.39), it also contains a higher order piece that includes one copy of $h_{\mu\nu}$ and two copies of φ. As such, we often denote this tensor as $T^{\mu\nu} = T^{\mu\nu}_{(0)} + h_{\sigma\rho}T^{\mu\nu\sigma\rho}_{(1)}$. We now know that this stress energy tensor is conserved subject to φ obeying its e.o.m in (1.42).

Now if repeat our procedure and, instead of $T^{\mu\nu}_{(0)}$, couple our new stress energy tensor to the system we can try and see how the situation now plays out. Again this new coupling will alter the equations of motion of our system, but it will ensure that our stress energy tensor is conserved at third order in the fundamental fields. However, it still isn't conserved at all orders since we've now modified the scalar field equation by terms cubic in fundamental fields. However, we again have a new theory and so can derive a new stress tensor that is conserved under our new equations and now contains terms with two spin-2 fields in it. We can now, hopefully, see where the story is going. We're forced to keep modifying our theory by higher order couplings of the spin two field in order to ensure that our stress tensor is conserved at higher and higher orders in the spin-2 field. This all seems fine and it turns out that if we keep on going we'll reconstruct the usual GR minimal coupling of a massive scalar field. However along the way we've completely ruined our spin-2 gauge invariance, and this was what ensured that our spin-2 field remained massless. So it seems that introducing interactions to another field has completely washed away our precious symmetry. However, much like the stress energy tensor's conservation law, we can modify (1.15) to include higher order terms, in X_μ and linear in $h_{\mu\nu}$, and then use these to reinstate gauge invariance at each order. This isn't so out of the question as we saw previously that the transformation (1.14) contained higher order terms, and so we may have hope that this could work. Miraculously it does.

We've just sketched how coupling the spin-2 field to a scalar field sets in motion a chain of events that ends with us being forced to modify both our theory and transformation laws at infinite order. However, when we first started we were really interested in self-couplings of the spin-2 field, rather than its coupling to matter, so can we do something similar to what we did for the scalar field in this case? Well, the previous story started with us finding a Noether stress energy tensor and then coupling it to the system, but our free spin-2 field theory is also Poincaré invariant and so we can find an associated stress tensor by the usual Noether procedure and then symmetrise it with the Belinfante procedure. This stress-energy tensor will then be conserved if the spin-2 field obeys (1.19). As can be checked in various sources, for example [101], recalling our conventions differ by a factor of 2 so $\mathcal{O}^{\rho\sigma}_{FP\mu\nu}(h_{\rho\sigma}) = 2\kappa^2 t_{\mu\nu}$, we find that the stress tensor is given by

$$
\begin{aligned}
t_{\mu\nu} = -\frac{1}{\kappa^2}\Big(&\frac{1}{2}h^{\sigma\rho}\partial_\mu\partial_\nu h_{\sigma\rho} - h^{\sigma\rho}\partial_\rho\partial_{(\mu}h_{\nu)\sigma} + \frac{1}{4}(\partial_\mu h_{\rho\sigma})\partial_\nu h^{\sigma\rho} \\
&+ \partial^\sigma h^\rho{}_\nu \partial_{[\sigma}h_{\rho]\mu} + \frac{1}{2}\partial_\sigma(h^{\sigma\rho}\partial_\rho h_{\mu\nu}) - (\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h)\partial_{(\mu}h_{\nu)\rho} \\
&- \frac{1}{2}\eta_{\mu\nu}[\frac{1}{2}h^{\sigma\rho}\Box h_{\sigma\rho} - h^{\sigma\rho}\partial_\sigma\partial^\tau h_{\tau\rho} + \frac{1}{4}(\partial_\tau h_{\sigma\rho})\partial^\tau h^{\sigma\rho} \\
&\partial^\sigma h^{\rho\tau}\partial_{[\sigma}h_{\rho]\tau} + \frac{1}{2}\partial_\sigma(h^{\sigma\rho}\partial_\rho h) - \frac{1}{4}\partial^\sigma h \partial_\sigma h - (\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h)\partial^\tau h_{\tau\rho}]\Big) \; .
\end{aligned}
\tag{1.43}
$$

If we now couple in our stress energy tensor we will have terms involving three spin-2 fields, at the level of the action, which fulfils our desire to have an interacting theory[17]. We however know that this will augment (1.19) with terms quadratic in $h_{\mu\nu}$, and so our conservation law will only hold at linear order and we will have to modify the stress tensor to be appropriate for our new theory if we want conservation to persist at higher orders in the spin-2 field. Following the scalar field example we should be happy that we can do this and then we just continue with the process. However, it is well know that (1.43) is not invariant under (1.15) and hence again we see we're forced to consider modifications to our transformation. At this point, taking a lesson from (1.14), we find that

$$
h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + 2\partial_{(\mu}X_{\nu)} + \mathcal{L}_X h_{\mu\nu} \; , \tag{1.44}
$$

is the correction required at this order to allow the transformation of our third order term in the action, which involves the coupling of the energy momentum tensor, to be invariant under the transformation (1.15) as the transformation of (1.18) under $\mathcal{L}_X h_{\mu\nu}$ cancels with it. However, now our third order term isn't invariant under this higher order piece of the transformation, but it exactly cancels higher order terms arising from coupling the various modified versions of the stress energy tensor to the theory.

The story is now becoming clear. While our spin-2 theory is happy to exist as a free theory, if we add in even a single interaction, at the lowest order, we'll be confronted with a deluge of higher order interactions which allow the stress tensor to be conserved at higher and higher orders in the spin-2 field. Further to this we'll be forced to modify the spin-2 field's transformation behaviour, although it will always be at most linear in the spin-2 field, to ensure our theory is invariant under some transformation of the spin-2 field. At this point we'd love to be able to just say that as we go further into the expansion all we're doing is

[17] We have to augment the original action (1.18) by a term third order in the spin-2 field, say $S^{(3)}[h]$, whose variation is $\delta S^{(3)} = \int d^dx - \delta h^{\mu\nu}(2\kappa^2 t_{\mu\nu})$ and hence the coupling is more involved than in the case of a scalar field. For some details on this see [81], note this article raises some issue with the spin-2 treatment of gravity which are addressed in [30].

reconstructing the full non-linear theory of GR order by order and our spin-2 fields transformation is just destined to become the full non-linear diffeomorphism symmetry of GR. However can we be sure that this is the case? There could be numerous ways to complete the interactions of the spin-2 field and to extend the transformations of the field so that we could end up at many non-linear completions of the theory. However an argument, due to Deser, ensures this isn't the case, so long as we demand our theory remains a two derivative theory[18]. At this stage it may seem perilous to confront the infinite expansions we've been led to, and even more daunting to show they spit out the results known in GR. Luckily though the argument of Deser [31] evades all of this by making use of the Palatini formalism of gravity, see Appendix B for more details, and a novel choice of variables for the problem. In this form the discussion becomes finite with the full interacting theory being obtained from the free theory by coupling the stress energy tensor obtained from the free theory. When this is done the full non-linear Palatini theory is obtained and since this is, classically, equivalent to the Einstein theory of gravity the argument shows that, under mild assumptions, the only non-linear completion of the spin-2 field theory is GR.

So we've now learnt that the interacting spin-2 field theory and GR are just two sides of the same coin, and from each we can obtain the other. Up until now we have mainly focused on the spin-2 field theory side, however we now know we can obtain this, up to an overall constant, order by order by expanding (1.3) using (1.13), this procedure is explained in Appendix A. This will lead us to find, for now ignoring boundary terms as we hope to be able to find boundary actions that cancel them, that our action takes the form

$$S_{EH}[\eta + h] = \frac{1}{2\kappa^2} S_{free-spin-2}[h] + \frac{1}{2\kappa^2} S^{(3)}[h] + \frac{1}{2\kappa^2} S^{(4)}[h] + \dots \;, \tag{1.45}$$

where $S^{(3)}$ and $S^{(4)}$ contain three and four copies of the spin-2 field respectively and dots represent higher order terms that we don't write explicitly. Since the original Einstein-Hilbert action is a two derivative theory all of the terms at each order in the expansion (1.45) contain two derivatives. It's often convenient to rescale our field so that we work with the new field

$$\tilde{h}_{\mu\nu} = \frac{1}{\kappa} h_{\mu\nu} \;, \tag{1.46}$$

using this field to re-write (1.45), absorbing factors of two into redefined actions, we obtain

$$S_{EH}[\eta + \kappa \tilde{h}] = \tilde{S}^{(2)}[\tilde{h}] + \kappa \tilde{S}^{(3)}[\tilde{h}] + \kappa^2 \tilde{S}^{(4)}[\tilde{h}] + \dots \;, \tag{1.47}$$

[18] We also have to pick a prescription for choosing how we derive the stress-energy tensor for the system, but if we pick the one used by covariantising our action with respect to an auxiliary metric and then using the Hilbert prescription everything works out as one expects.

which we see converts the expansion in to one involving higher and higher powers of the gravitational coupling κ. The redefinition means that (1.44) becomes

$$\tilde{h}_{\mu\nu} \rightarrow \tilde{h}'_{\mu\nu} = \tilde{h}_{\mu\nu} + \frac{2}{\kappa}\partial_{(\mu}X_{\nu)} + \mathcal{L}_X \tilde{h}_{\mu\nu} \ . \tag{1.48}$$

Since we know that lower orders in (1.47) are required to ensure invariance, under (1.48), of the term directly above them, in (1.47), we now see it's crucial that the expansion works out in powers of κ, as if it didn't invariance couldn't be achieved. As such it's no coincidence that κ appears in both (1.47) and (1.48). This seemingly simple fact will be one of the guiding principles for us going forward in the rest of this thesis.

This completes our survey of some of the salient features of GR that we shall require. Throughout the rest of this thesis we shall use these lessons to inform the decisions we make and provide direction in the construction of our desired EFT.

1.2 Supergravity- an abridged introduction

We've now seen how to write gravity as the field theory of a spin-2 field. However, we have yet to address any of the issues regarding it's non-renormalisability. If we can't eventually make sense of GR as a quantum theory then is there really any point in putting all of this effort into a field theory formulation of it? Luckily for us there are ways to embed GR into other theories where the issue of renormalisability is, seemingly, resolved. Within this thesis our bias is towards a String Theory treatment of this issue where the divergences usually encountered in quantising GR are avoided altogether. While we won't deal with String Theory good and proper, except possibly in passing, we shall use its low energy limit. This low energy limit is the theory of Supergravity and it's our aim to now give a whirlwind tour of some aspects of this fascinating theory. Supergravity constitutes the union of GR and Supersymmetry (SUSY). It was hoped that the inclusion of SUSY into GR would render the theory renormalisable, and whilst it does improve the behaviour, it turns out to not be enough [9]. Fortunately, since Supergravity arises, as we'll see, as the low energy limit of String theory, these divergences aren't something to loose sleep over. In order to introduce Supergravity we shall begin by reviewing rigid SUSY and then proceed to gauge this symmetry which will naturally lead us into the realm of Supergravity. We will only have time to introduce the very basics of this vast subject and would recommend the reader that wishes to learn more about the subject to consult any of [44, 76, 99, 100].

1.2.1 Supersymmetry-the cliff notes

When first introduced to quantum mechanics we're taught that particles, or waves- who really knows, come in two distinct forms[19], namely Bosons and Fermions. These two types of particle exhibit wildly different properties and so one may reasonably assume there's no link between them. However, in some theories this couldn't be further from the truth. The types of theories we're referring to are, of course, those that possess the remarkable property of having supersymmetry. This symmetry acts by transforming bosons into fermions and vice versa, and as such in these theories the boundaries between these types of particles becomes a lot less clear. So far tests to see if our universe possesses such a symmetry have not boded well for a low energy, around the TeV scale and energies accessible at the LHC, realisation of such a symmetry.

This may dampen our spirits somewhat as, in physics, we're meant to try and describe the world we actually inhabit and if our world doesn't possess SUSY then that's just too bad. We have to break up with it and move on with our lives. However, far from being left in the theory wasteland, SUSY is an active research area. Why is this the state of affairs? Well to begin with we've only, seemingly, ruled SUSY out at low energies[20], our observations don't tell us anything about energies much higher than this and, as such, SUSY may still be a symmetry of nature at these higher energies. However this could be just wishful thinking, so surely we must have other reasons for sticking by SUSY? It's common knowledge that QFT is hard and calculations within it often have to be performed in a perturbative regime. However, if our theory possesses SUSY, we can do better and in a lot of cases much better. The presence of SUSY in a theory often allows us to calculate results exactly, which is practically unheard of in QFT[21]. So SUSY helps make QFT tractable. This alone would be enough to study it, but SUSY isn't done giving yet! Aside from just aiding with physics SUSY has led to rich results in mathematics as well. For example the work of Seiberg and Witten [93] which was shown to relate to invariants of four-manifolds [105], the classification of simple finite dimensional complex Lie superalgebras [64], or work on mirror symmetry of Calabi-Yau manifolds [59] to name but a few.

So SUSY has a wide range of applicability, but how did the idea of such a symmetry arise? The origins of SUSY begin with the classic no-go theorem of Coleman and Mandula [14]. Roughly this theorem states that if a QFT, with a mass gap[22], has a continuous global symmetry group $\tilde{G}$ which contains, as a subgroup, the Poincaré group $IO(1, d-1) = \mathbb{R}^{1,d-1} \rtimes O(1, d-1)$, then if we desire the S-matrix to be analytic in scattering

[19] This isn't strictly true, for example in condensed matter physics we often deal with anyons.
[20] Here this means energies of the order of those reached at the LHC- so not really low energy by our everyday standards!
[21] Another example where this is possible is the case where our theory possesses conformal invariance.
[22] If we relax this assumption and have a massless spectrum then we can extend the Poincaré group to the conformal group- but we won't deal with this technical detail here.

angles and non-trivial, locally, $\tilde{G} = IO(1,d-1) \times G$ with G an internal symmetry group[23].

The theorem can be more easily understood if we consider it at the level of algebras. Assuming $\tilde{G}$ is a Lie group, where we have generators P_M, $M_{MN} = -M_{NM}$ and T_A for the Lie algebra, $\tilde{\mathfrak{g}}$, of $\tilde{G}$. Then the Lie algebra $\tilde{\mathfrak{g}}$ is specified by the following Lie bracket relations

$$\begin{aligned}
[P_M, P_N] &= 0 \ , \\
[P_M, M_{NQ}] &= i(\eta_{MQ} P_N - \eta_{MN} P_Q) \ , \\
[M_{MN}, M_{QP}] &= i(\eta_{MQ} M_{NP} - \eta_{MP} M_{NQ} - \eta_{NQ} M_{MP} + \eta_{NP} M_{MQ}) \ , \\
[T_A, P_M] &= 0 \ , \\
[T_A, M_{MN}] &= 0 \ , \\
[T_A, T_B] &= f_{AB}{}^C T_C \ ,
\end{aligned} \tag{1.49}$$

the first three of these tell us P_M is the translation operator and M_{MN} are the generators of the Lorentz group, so $N \in \{1, ..., d\}$, and thus, together, they form the generators of $\mathfrak{io}(1, d-1)$. We also see that T_A, $A \in \{1, ..., \dim G\}$, form the generators of $\mathfrak{g}$, the Lie algebra of G, which has structure constants $f_{AB}{}^C$ with respect to this basis. However, the crucial set of Lie brackets are the pair which mix T_A with the Poincaré generators. These tell us our algebra decomposes as a direct sum of algebras. As such $\tilde{\mathfrak{g}} = \mathfrak{io}(1, d-1) \oplus \mathfrak{g}$, which is the algebra analogue of the Coleman Mandula theorem. Since each continuous symmetry, via Noether's theorem, corresponds to a conserved charge of the theory we now see that the theorem means we can't have charges that are simultaneously in non-trivial representations of both the Poincaré and internal symmetry groups.

This all leads us to the conclusion that there are very stringent restrictions on the form of the symmetry algebras we can consider. However, as everyone always tells us, a no-go theorem is all in the assumptions, if we can just evade one of these then we can get around the no-go. In the case of Coleman and Mandula the whole argument is built on groups and if we were to use something else then we could circumvent the no-go completely. This led to the introduction of the concept of supergroups and superalgebras [52, 104] which in turn led to the introduction of SUSY into the literature. At this point we'd be forgiven for feeling free as we've just unshackled ourselves from the Coleman Mandula theorem. However, it turns out that even by running to the land of supergroups we can't escape the spectre of the theorem as a result of Haag,

[23] The original theorem of Coleman and Mandula was set in four dimensions but it was later extended to higher dimensions [83].

Lopuszanski and Sohnius [57] extends the Coleman Mandula theorem to the setting of supergroups.

Rather than jumping head first into the subject of supergroups we'll consider the slightly easier topic of Lie superalgebras as this will be enough to allow us develop the extension of (1.49) to the supersymmetry algebra. Our approach here is in no way novel and relies heavily upon the excellent exposition of the subject provided by Freund [45].

In light of the fact that SUSY deals with particles of two different types we need to develop a language that encompasses this. The obvious setting, in analogy with Lie algebras, is that of vector spaces. However our space somehow needs to be aware that our theory contains two types of object. The correct structure to use here is that of a $\mathbb{Z}_2$-graded vector space

Definition 1. *Let V, V^0 and V^1 be $\mathbb{F}$ vector spaces such that $V = V^0 \oplus V^1$ and V is equipped with an operation $|\cdot| : V \to \{0,1\}$ such that $|(u, 0_1)| = 0\ \forall u \in V^0$ and $|(0_0, w)| = 1\ \forall w \in V^1$, with 0_0 and 0_1 the zero vectors of V^0 and V^1 respectively, and $|\cdot|$ is extended to other elements by linearity, with mod 2 addition being performed. In this case we call V a $\mathbb{Z}_2$-graded vector space with parity operation $|\cdot|$. Often V^0 is referred to as the even or Bose part of V while V^1 is referred to as the odd or Fermi part of V.*

With the groundwork laid it is now a simple matter to generalise the notion of a Lie algebra to this new setting

Definition 2. *Let V be a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$ with a binary operation $[\cdot,\cdot] : V \times V \to V$ such that*
(i) $[\cdot,\cdot]$ is bilinear
(ii) $\forall\, a, b \in V \quad [a,b] = (-1)^{1+|a||b|}[b,a]$
(iii) $\forall\, a,b,c \in V \quad (-1)^{|a||c|}[a,[b,c]] + (-1)^{|b||a|}[b,[c,a]] + (-1)^{|c||b|}[c,[a,b]] = 0$
with $|\cdot|$ the parity operation of V. Such a pair $(V, [\cdot,\cdot])$ is called a Lie superalgebra over $\mathbb{C}$.

A classification of the simple[24] finite dimensional Lie superalgebras over $\mathbb{C}$ was given by Kac [64].

Aside from the introduction of this new algebraic structure, designed to allow us to circumvent Coleman-Mandula, we also need to develop the notion of spinor representations. Such representations allow for the inclusion of fields that have half integer spin, defined via the Lorentz group. Thus, due to the spin-statistics theorem, such representations play the role of fermions in our theory. As we'll see later in this introduction supersymmetric theories arise in various spacetime dimensions, and as such we now provide a survey of the forms of spinor representations available in various dimensions.

[24] Recall an algebra is simple if it has no non-trivial ideals.

To make progress we consider a d-dimensional vector space V over a field $\mathbb{F}$, which is either $\mathbb{R}$ or $\mathbb{C}$, and consider a quadratic form[25] $Q : V \to \mathbb{F}$. The existence of such a quadratic form implies the existence of a symmetric bilinear form on V which is defined by $(u, w) = \frac{1}{2}(Q(u + w) - Q(u) - Q(w))$ for $v, w \in V$. The existence of such a bilinear form leads to a privileged set of bases of V, namely those whose basis vectors are orthogonal with respect to the associated bilinear form. Such an orthogonal basis, $\{e_a | a \in \{1, .., d\}\}$, obeys

$$(e_a, e_b) = 0 \quad a \neq b \ , \ (e_a, e_a) = Q(e_a) \ . \tag{1.50}$$

We can now consider the formal relationship

$$e_a e_b + e_b e_a = 2\delta_{ab} Q(e_a) 1 \ , \tag{1.51}$$

where 1 is a formal symbol and the product and addition on the left hand side of this expression are formal operations. However we could push this further and decide to consider the unital associative algebra generated by the basis $\{e_a\}$, over the field $\mathbb{F}$, subject to the relationship (1.51). In this context it becomes obvious that the formal symbol 1 is to be the unit element of our algebra. This algebra is called the Clifford algebra of V and Q and is often denoted by $\mathcal{C}\ell(V, Q)$[26]. We can see from (1.51) that if $a \neq b$ then e_a and e_b anticommute and hence one can easily show that a basis of $\mathcal{C}\ell(V, Q)$ is given by[27]

$$1 \ , \ e_{a_1} \ , \ e_{a_1} e_{a_2} \ , \ ... \ , \ e_{a_1} e_{a_2} .. e_{a_d} \qquad a_1 < a_2 < ... < a_d$$

from which one easily concludes that

$$dim_{\mathbb{F}} \mathcal{C}\ell(V, Q) = \sum_{n=0}^{d} \binom{d}{n} = 2^d \tag{1.52}$$

Using both of these it can be seen that a general element $\alpha \in \mathcal{C}\ell(V, Q)$ can be expanded as

$$\alpha = \alpha_0 1 + \alpha_1^{a_1} e_{a_1} + \alpha_2^{a_1 a_2} e_{a_1} e_{a_2} + .. + \alpha_d^{a_1 a_2 .. a_d} e_{a_1} e_{a_2} .. e_{a_d} \tag{1.53}$$

with $\alpha_i^{a_1 .. a_i}$ totally antisymmetric, for $i \in \{1, .., d\}$, and all coefficients being $\mathbb{F}$ valued.

This is all well and good but it does little to help us understand the structure of these algebras, or their

[25]Recall a quadratic form on a vector space V is a map $Q : V \to \mathbb{F}$ such that $\forall \lambda \in \mathbb{F}, \forall v \in V : \ Q(\lambda v) = \lambda^2 Q(v)$.

[26]Note a less basis dependent description of Clifford algebras can be developed but for our current purposes the basis one shall suffice.

[27]Since $e_a^2 \propto 1$ we don't have to worry about including elements where $a = b$.

relations to spinors. Luckily a theorem due to Chevalley states that $\mathcal{C}\ell(V,Q)$ admits an irreducible matrix representation which is unique up to equivalence. Hence this allows us to characterise Clifford algebras as matrix algebras. To make further progress we specialise to $\mathbb{F}=\mathbb{R}$ as in this case any, non-degenerate, quadratic form is equivalent to the quadratic form

$$Q(e_a) = \begin{cases} \text{-1} & \text{a} \in \{1,..,q\} \\ 1 & \text{a} \in \{q+1,..,q+p=d\} \end{cases} , \tag{1.54}$$

where $q,p \in \mathbb{N}_0$ and if $q=0$ the first indexing set is empty, while if $p=0$ the second indexing set is empty. We shall denote[28] Clifford algebras with this quadratic form as $\mathcal{C}\ell(q,p)$. It turns out that these algebras can be understood easily for low p and q values

$$\mathcal{C}\ell(0,0)=\mathbb{R}\ ,\ \mathcal{C}\ell(0,1)=\mathbb{R}\oplus\mathbb{R}\ ,\ \mathcal{C}\ell(1,0)=\mathbb{C}\ ,\ \mathcal{C}\ell(0,2)=\mathbb{R}(2)\ ,\ \mathcal{C}\ell(1,1)=\mathbb{R}(2)\ ,\ \mathcal{C}\ell(2,0)=\mathbb{H}\ , \tag{1.55}$$

with $\mathbb{C}$ and $\mathbb{H}$ being treated as two and four dimensional algebras over the reals respectively[29], and $\mathbb{K}(n)$ denotes the algebra of $n\times n$ matrices over the appropriate $\mathbb{K}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$.

In order to obtain higher values of p and q we can make use of the isomorphisms

$$\begin{aligned} \mathcal{C}\ell(p,q+2) &\cong \mathcal{C}\ell(q,p)\otimes\mathcal{C}\ell(0,2)\ , \\ \mathcal{C}\ell(p+1,q+1) &\cong \mathcal{C}\ell(q,p)\otimes\mathcal{C}\ell(1,1)\ , \\ \mathcal{C}\ell(p+2,q) &\cong \mathcal{C}\ell(q,p)\otimes\mathcal{C}\ell(2,0)\ , \end{aligned} \tag{1.56}$$

from these results, and several well known isomorphisms, we find that the form of $\mathcal{C}\ell(q,p)$ is dependent on the signature of the quadratic form, mod eight, leading to the classification given in Table 1.

We're also interested in Clifford algebras where $\mathbb{F}=\mathbb{C}$, which we denote by $\mathcal{C}\ell_{\mathbb{C}}(d)$. In this case signature doesn't matter, as we can use i to remove signs, meaning all non-degenerate quadratic forms are equivalent to the Kronecker delta. In this case we can just complexify our real algebra, with all real algebras of the same $d=p+q$ but varying signatures, leading to the same complex algebra. This leads us to the relationship

$$\mathcal{C}\ell_{\mathbb{C}}(d)=\mathcal{C}\ell(d-p,p)\otimes\mathbb{C}\ , \tag{1.57}$$

[28]Our notation here differs from that in the literature where $\mathcal{C}\ell(p,q)$ is used.
[29]Here we use $\mathbb{H}$ to denote the quaternions.

$p-q$ mod 8	$\mathcal{C}\ell(q,p)$
0	$\mathbb{R}(2^l)$
1	$\mathbb{R}(2^l)\oplus\mathbb{R}(2^l)$
2	$\mathbb{R}(2^l)$
3	$\mathbb{C}(2^l)$
4	$\mathbb{H}(2^{l-1})$
5	$\mathbb{H}(2^{l-1})\oplus\mathbb{H}(2^{l-1})$
6	$\mathbb{H}(2^{l-1})$
7	$\mathbb{C}(2^l)$

Table 1: List of real Clifford algebras of varying signature. $l=[\frac{d}{2}]$ with $[\cdot]$ being the integer part operator with downwards rounding always implemented.

d mod 2	$\mathcal{C}\ell_{\mathbb{C}}(d)$
0	$\mathbb{C}(2^l)$
1	$\mathbb{C}(2^l)\oplus\mathbb{C}(2^l)$

Table 2: List of complex Clifford algebras. $l=[\frac{d}{2}]$ with $[\cdot]$ being the integer part operator with downwards rounding always implemented

which are the complex Clifford algebras we were seeking. Since the notion of signature is no longer in the game its no surprise that the classification becomes much simpler. This is shown in Table 2.

We've made progress in understanding Clifford algebras but how does this really help us in understanding spinors? Spinors are usually introduced as representations of the $Spin(q,p|\mathbb{R})$ groups, which are double covers of the $SO(q,p)$ groups, however here we'll take a slightly different approach. The crucial piece of insight is that we've classified Clifford algebras as matrix algebras and these matrices can then naturally act on vector spaces and it's these representation spaces which our spinors are elements of. It is in this way that Dirac spinors first enter into our lives in courses on QFT. To see this recall (1.51) is precisely the gamma matrix algebra if we set p=3 and q=1, so it shouldn't come as a surprise that the story generalises. In our new picture Dirac spinors are elements of the representation space of $\mathcal{C}\ell_{\mathbb{C}}(d)$. In even dimensions, d=2n, we see from Table 2 that this gives us a representation of complex dimension 2^n. In odd dimensions, d=2n+1, $\mathcal{C}\ell_{\mathbb{C}}(d)$ is built from two matrix algebras each of which have a representation of complex dimension 2^n. However, from the point of view of $Spin(q,p)$, these representations are isomorphic so our Dirac spinor still has complex dimension 2^n. If we wish to explicitly construct such a representation it is generated by the commutator of our basis. For example the elements $\frac{1}{4}(e_ae_b-e_be_a)$ form a representation of the appropriate $\mathfrak{spin}(q,p)$ algebra.

The Dirac spinor has the nice property that it is defined in any dimension. However, often we can impose conditions upon this representation to further reduce it. The simplest reduction occurs in even dimensions, where $d=2n$. We note that our Clifford algebra decomposes into an even and odd part. The even part

$\mathcal{C}\ell_{\mathbb{C}}(d)^0$ is given by taking elements of the form (1.53) with vanishing coefficients with odd numbers of indices, so $\alpha_1^{a_1} = \alpha_3^{a_1 a_2 a_3} = .. = 0$, and is a 2^{n-1} complex dimensional subalgebra[30]. However this subalgebra can be further decomposed as the top element of our algebra

$$\epsilon = e_1 e_2 \ldots e_d \ , \tag{1.58}$$

can be used to construct the projection operators

$$P_{\pm} = \frac{1}{2}(1 \pm \epsilon) \ . \tag{1.59}$$

These objects are projection operators[31] if $\epsilon^2 = 1$. However, as we're in the complex numbers and since, by (1.51), we have $\epsilon^2 = \pm 1$, we can see if $\epsilon^2 = -1$ then replacing ϵ with $i\epsilon$ in (1.59) will again lead to properly constructed projection operators. In either case this allows a decomposition of $\mathcal{C}\ell_{\mathbb{C}}(d)^0$ in the d even case, as $\epsilon \in \mathcal{C}\ell_{\mathbb{C}}(d)^0$, into

$$\mathcal{C}\ell_{\mathbb{C}}(d)^0 = P_+(\mathcal{C}\ell_{\mathbb{C}}(d)^0) \oplus P_-(\mathcal{C}\ell_{\mathbb{C}}(d)^0) \ . \tag{1.60}$$

We can now use our representation of $\mathcal{C}\ell_{\mathbb{C}}(d)$ to form a representation of $\mathcal{C}\ell_{\mathbb{C}}(d)^0$ on the same representation space. We can also see that for any ψ in this space we have $\psi = P_+(\psi) + P_-(\psi)$ hence the representation space decomposes under our projection operators as $V = P_+(V) \oplus P_-(V)$. Further to this, since $e_a\epsilon + \epsilon e_a = 0$ for any generator of our algebra, we have for any $A \in \mathcal{C}\ell_{\mathbb{C}}(d)^0$ that $P_{\pm}A = AP_{\pm}$. Using this one can show that if $\psi \in P_{\pm}(V)$ then $A(\psi) \in P_{\pm}(V)$ and hence our representation is reducible. This now allows us to decompose our Dirac spinor into one of two inequivalent Weyl spinors, of complex dimension 2^{n-1}, by applying one of the projection operators, given in (1.59), to our original Dirac spinor. So in even dimensions we can always decompose our complex Dirac spinor into a pair of complex Weyl spinors.

The dynamics of our Dirac spinor is determined by the Dirac equation which relates the components of our spinor. If there's a real representation of the Clifford algebra, for a given $d = p + q$ and signature $\sigma = p - q$, then the Dirac equation just shuffles real components amongst themselves and the same is true for imaginary components. As such it becomes compatible with the dynamics to impose a reality condition on our Dirac spinor. Imposing such a reality condition leads to Majorana spinors which have real dimension $2^{[\frac{d}{2}]}$. Since the existence of a Majorana spinor is dependent upon $\mathcal{C}\ell(q,p)$ having a real representation[32] we

[30]Note the odd part of $\mathcal{C}\ell_{\mathbb{C}}(d)$ doesn't give rise to an algebra as the product of two odd elements can lead to even pieces.
[31]Meaning $P_{\pm}^2 = P_{\pm}$ and are also orthogonal so $P_+P_- = P_-P_+ = 0$.
[32]If $\mathcal{C}\ell(q,p)$ has a purely imaginary representation then if we only have one field and it is massless we can have a Pseudo Majorana spinor. This representation is of real dimension $2^{[\frac{d}{2}]}$ and is available in the cases where $p - q \in \{0, 6, 7 \mod 8\}$, since if $\mathcal{C}\ell(q,p)$ has a real representation then $\mathcal{C}\ell(p,q)$ has an imaginary representation.

q \ d	1	2	3	4	5	6	7	8	9	10	11
0	M	M,W		W		PM,W	PM	M,PM,W,MW	M	M,W	
1	PM	M,PM,W,MW	M	M,W		W		PM,W	PM	M,PM,W,MW	M

Table 3: List of possible conditions on spinors in various dimensions for zero and one timelike direction. In this list M= Majorana, PM=Pseudo Majorana, W=Weyl and MW=Majorana-Weyl.

see from Table 1 that Majorana spinors are possible in cases where $p - q \in \{0, 1, 2 \mod 8\}$.

The final case of interest to us is the possibility of imposing both Majorana and Weyl conditions simultaneously. In this case we're dealing with the real Clifford algebras $\mathcal{C}\ell(q,p)$ and we know a Majorana condition is possible if $p-q \in \{0, 1, 2 \mod 8\}$ and that Weyl conditions are possible in even dimensions $d = p+q = 2n$. However, we now have to be more careful as we're in the reals. To define Weyl spinors we used (1.59) which were projection operators for any even dimension in the complex numbers, since ϵ could be replaced by $i\epsilon$, however this redefinition is no longer possible when we work over the reals and so we can only impose a Weyl condition to perform the split

$$\mathcal{C}\ell(q,p)^0 = P_+(\mathcal{C}\ell(q,p)^0) \oplus P_-(\mathcal{C}\ell(q,p)^0) \ , \tag{1.61}$$

when ϵ^2=1. Using (1.58) and (1.51) we can see this occurs when $p - q \in \{0 \mod 4\}$. This means that Majorana and Weyl conditions can simultaneously be imposed if $p - q \in \{0 \mod 8\}$, which leads to the concept of a Majorana-Weyl spinor. Such spinors have real dimension $2^{[\frac{d}{2}]-1}$.

Our treatment so far has been extremely general and has left the signature of spacetime unfixed. However, for us, the cases of interest are those where $q \in \{0, 1\}$. In these settings the possible conditions available for spinors are summarised in Table 3.

With everything assembled we can now consider how to construct the supersymmetry algebra. Rather than attempt a general construction we shall firmly place our feet in the case of $d = 4$ with a signature $\sigma = 2$ and shall only provide the schematics of the construction so as to illustrate the general structures appearing in SUSY algebras, for a general treatment see [74]. Once more our treatment closely parallels that of Freund [45].

It is well known that the Poincaré algebra, $\mathfrak{io}(1,3)$, can be obtained from the (Anti) de Sitter algebra $(\mathfrak{o}(2,3))\mathfrak{o}(1,4)$ by Wigner-Inönü contraction [62]. This procedure allows non-isomorphic algebras to be related without a reduction in the number of generators and is fortunately available to us even in the case of Lie superalgebras. In order to obtain a superalgebra version of the Poincaré algebra we now just look for a superalgebra that contains the (Anti) de Sitter algebra in its even part. Since the even part of a Lie superalgebra is just a Lie algebra the Coleman-Mandula theorem rears its head once more and tells us we require the superalgebra to have an even piece that is the direct sum of $(\mathfrak{o}(2,3))\mathfrak{o}(1,4)$ and an internal algebra $\mathfrak{g}$. This restriction can be seen because once the contraction is performed the (Anti) de Sitter algebra becomes the Poincaré algebra dropping us right back into the path of Coleman and Mandula[33].

It turns out that we can find a Lie super algebra that will work, its one of the orthosymplectic superalgebras[34] $\mathfrak{osp}(N|4)$ whose even part is $\mathfrak{osp}(N|4)^0 = \mathfrak{o}(N) \oplus \mathfrak{sp}(4)$ where we make use of the accidental isomorphism $\mathfrak{sp}(4) \cong \mathfrak{o}(2,3)$ to ensure our Anti de Sitter algebra is in place. We can now decompose the generators of this algebra into $M_{\mu\nu} = -M_{\nu\mu}$ and P_μ for $\mu \in \{0,..,3\}$ which are the generators of $\mathfrak{o}(2,3)$, B_a for $a \in \{1,..,\frac{N(N-1)}{2}\}$ which are the generators of $\mathfrak{o}(N)$ and Q^i_α with $i \in \{1,..,N\}$ and $\alpha \in \{1,..,4\}$ being the generators of the odd part of the superalgebra, which has (complex) dimension $4N$. Performing a Wigner-Inönü contraction requires rescaling the generators of this algebra as

$$\bar{M}_{\mu\nu} = M_{\mu\nu} \quad , \quad \bar{P}_\mu = \lambda P_\mu \quad , \quad \bar{B}_a = \lambda^b B_a \quad , \quad \bar{Q}^i_\alpha = \lambda^q Q^i_\alpha \; , \tag{1.62}$$

for λ,b,$q \in \mathbb{R}$. The superalgebra, in terms of these redefined variables, very schematically, takes the form

$$\begin{aligned} &[\bar{M},\bar{M}] \propto \bar{M} \quad , \quad [\bar{B},\bar{M}] = 0 \quad , \quad [\bar{M},\bar{Q}] \propto \bar{Q} \quad , \quad [\bar{M},\bar{P}] \propto \bar{P} \quad , \quad [\bar{B},\bar{P}] = 0 \quad , \quad [\bar{P},\bar{Q}] \propto \lambda\bar{Q} \; , \\ &[\bar{P},\bar{P}] \propto \lambda^2\bar{M} \quad , \quad [\bar{B},\bar{B}] \propto \lambda^b\bar{B} \qquad [\bar{B},\bar{Q}] \propto \lambda^b\bar{Q} \quad , \quad [\bar{Q},\bar{Q}] \propto \lambda^{2q}\bar{M} + \lambda^{2q-1}\bar{P} + \lambda^{2q-b}\bar{B} \; , \end{aligned} \tag{1.63}$$

where terms on the right hand side indicate the generators that appear rather than the exact way in which they appear. The heart of the contraction is now in taking the limit $\lambda \to 0$. However, we can't just do this freely as we require that no divergences occur in the algebra. This forces the conditions $b \geqslant 0$, $2q-1 \geqslant 0$ and $2q-b \geqslant 0$ on us. This turns out to lead to two interesting cases.

The first case is $b = 0$ and $q = \frac{1}{2}$ where (1.63) becomes

$$\begin{aligned} &[\bar{M},\bar{M}] \propto \bar{M} \quad , \quad [\bar{B},\bar{M}] = 0 \quad , \quad [\bar{M},\bar{Q}] \propto \bar{Q} \quad , \quad [\bar{M},\bar{P}] \propto \bar{P} \quad , \quad [\bar{B},\bar{P}] = 0 \qquad [\bar{P},\bar{Q}] = 0 \; , \\ &[\bar{P},\bar{P}] = 0 \quad , \quad [\bar{B},\bar{B}] \propto \bar{B} \quad , \quad [\bar{B},\bar{Q}] \propto \bar{Q} \qquad [\bar{Q},\bar{Q}] \propto \bar{P} \; . \end{aligned} \tag{1.64}$$

[33] It is possible to envision a situation where the even sector of the Lie superalgebra doesn't begin as a direct sum but under the contraction becomes one, however if we restrict ourselves to simple Lie superalgebras this does not occur.

[34] All that follows is done at the level of complex algebras and so at the end an appropriate real form shall have to be taken but we won't concern ourselves with this technical detail.

In this case we have the N extended Poincaré superalgebra, or supersymmetry algebra, which is generated by $\bar{M}_{\mu\nu}, \bar{P}_\mu, \bar{Q}^i_\alpha$ and the internal algebra is seen to only behave non-trivially in the odd sector of the algebra.

The second case is $b = 1$ and $q = \frac{1}{2}$ where (1.63) becomes

$$[\bar{M}, \bar{M}] \propto \bar{M} \quad , \quad [\bar{B}, \bar{M}] = 0 \quad , \quad [\bar{M}, \bar{Q}] \propto \bar{Q} \quad , \quad [\bar{M}, \bar{P}] \propto \bar{P} \quad , \quad [\bar{B}, \bar{P}] = 0 \quad , \quad [\bar{P}, \bar{Q}] = 0 \; ,$$
$$[\bar{P}, \bar{P}] = 0 \quad , \quad [\bar{B}, \bar{B}] = 0 \quad , \quad [\bar{B}, \bar{Q}] = 0 \quad , \quad [\bar{Q}, \bar{Q}] \propto \bar{P} + \bar{B} \; . \tag{1.65}$$

In this case we see the $\bar{B}_a$ commutes with all elements of the algebra and hence they are central elements, which we often call central charges. We see the only place these generators appear is on the right hand side of the Fermi Fermi bracket. Thus in this case we've obtained a central extension of the N-fold extended Poincaré superalgebra which is also called the centrally extended supersymmetry algebra.

By selecting a consistent collection of b values, some being zero and some being one, we can obtain a combination of a central extension while also retaining a non abelian internal symmetry algebra. It can further be shown that the N-extended Poincaré algebra, with a possible central extension, is the largest possible superalgebra compatible with the Haag, Lopuszanski and Sohnius theorem.

If we now analyse the new commutators in the central extension of the N-fold extended Poincaré algebra the $[\bar{M}, \bar{Q}]$ bracket just tells us that $\bar{Q}$ transforms as a collection of N Majorana spinors. While the crucial content of the algebra is contained in the $[\bar{Q}, \bar{Q}]$ bracket which takes the form

$$[\bar{Q}^i_\alpha, \bar{Q}^j_\beta] = 2(\gamma^\mu C)_{\alpha\beta}\delta^{ij}P_\mu + C_{\alpha\beta}Z^{[ij]} + (\gamma_5 C)_{\alpha\beta}Y^{[ij]} \; , \tag{1.66}$$

with γ^μ and γ_5 the usual four dimensional Dirac gamma matrices and $Z^{[ij]}$ and $Y^{[ij]}$ being central charges, the origins of which we shall discuss further later in this Section. In addition $C_{\alpha\beta}$ is the charge conjugation matrix which obeys $(\gamma^\mu C)^T = \gamma^\mu C$. Analogues of this algebra can be obtained in other dimensions where very similar structures arise. With the algebra in hand it's possible to look for representations of it. This is done in the usual way, with a subset of the $\bar{Q}^i_\alpha$ acting as raising operators that increase the, Lorentz, spin of states by $\frac{1}{2}$ hence ensuring that these representations contain both bosonic and fermionic states. Crucially we can find finite dimensional representations of this algebra which we call supermultiplets and tend to label by the highest spin present in the multiplet. A standard calculation can be used to show that supermultiplets contain the same number of bosonic and fermionic states[35].

[35]The one way to get out of this is for the multiplet to have $p_0 = 0$ so the multiplet has zero energy. In the case of a supersymmetric theory this means that we are dealing with the ground state of the system.

When we started out with our desire to understand supersymmetry we mostly described the situation in terms of particles, however we know that we're really interested in fields, from which particles are derived entities obtained via the procedure of quantisation. As such we have to seek representations of SUSY algebras that act on fields. Interestingly such representations fall into two classes. The first is the case where the algebra is represented in such a way that it only closes when the fields satisfy their classical equations of motion. Representations of this type are referred to as on-shell representations of the SUSY algebra. The other class of representation, unsurprisingly, is one where the algebra closes on the fields without any requirements being put on the fields, in particular they are not obliged to obey any equations of motion. We referred to such representations as off-shell representations of the SUSY algebra. It can be possible, by the inclusion of auxiliary fields, to turn an on-shell representation into an off-shell representation. Both forms of representation arise frequently in supersymmetric theories and each have their own benefits and challenges.

Sadly in this brief introduction we have been very specific in our choice of material and have certainly underserved the wonderful world of supersymmetry. For those interested there are many beautiful treatments of the subject with [48, 94] providing but two of these.

1.2.2 Gauging supersymmetry- the basics of Supergravity

So far we've introduced supersymmetry mostly in the abstract and have been more concerned with the structure of this new symmetry and the consequences of it rather than directly finding models exhibiting the symmetry. That's all about to change. Like any symmetry it is possible to find theories which exhibit a global invariance and those which exhibit a local invariance under the proposed symmetry. In usual treatments one is introduced to the symmetry in a rigid (global) form first before moving on to gauge the symmetry to display examples of theories where the symmetry is realised locally. However in the interest of expediency we shall skip this first step and launch straight into the case of local supersymmetry. Our treatment here relies heavily on the excellent exposition of the subject given by Nastase in [76].

One of the interesting features of supersymmetric field theories is that the symmetry transformations of the theory relate bosonic and fermionic fields. This leads to infinitesimal symmetry transformations schematically of the form

$$\delta Boson = \epsilon Fermion \quad , \quad \delta Fermion = \epsilon \partial Boson \; , \tag{1.67}$$

with ϵ being a fermionic parameter, hence it's a spinor, with the derivative included in the fermionic transformation on dimensional grounds. In a local theory of supersymmetry the fermionic parameter ϵ is dependent upon the point in spacetime at which we perform the transformation. An interesting observation is that any theory of local supersymmetry must also be invariant under diffeomorphisms[36] and hence is a theory of gravity. Owing to this observation theories with local supersymmetry are called Supergravity theories.

Since we know we're going to be dealing with fermions in a theory of gravity, and hence will require the formalism of spinors in curved space, we know that we shall have to deal with gravity in its vielbein formulation. An introduction to the vielbein formulation of gravitational theories is provided in Appendix B. Aside from the, inverse, vielbein $E^a{}_\mu$ we require, as with all gauge theories, a gauge field. We know that gauge fields have the same spin-statistics as the parameter of the transformation they're associated to. Hence, from (1.67), we can see that we're looking for a gauge theory with a fermionic gauge field! This is highly contrary to the cases one encounters in the Standard Model where all of the gauge fields are spin-1 bosons. So aside from being fermionic what else can we say about this gauge field? Well owing, once more to (1.67), we expect that $E^a{}_\mu$ and our gauge field will transform into each other, hence telling us they should be in the same, massless[37], supermultiplet. This gives us the vital insight we need as we now recall that within a supermultiplet we need the same number of fermionic and bosonic states, and so if we can construct a supermultiplet that includes a graviton then we will find our gauge field.

Our insight provides the perfect opportunity to delve into the subject of on-shell and off-shell degrees of freedom counting for fields. Since we've seen earlier in this introduction that on-shell a four dimensional graviton has 2 real degrees of freedom we know if we can find a fermionic field with the same number of on-shell degrees of freedom then we can try to make an informed guess about the appropriate gauge field. Rather than just provide the counting for fermionic fields we shall present it for all of the types of fields we shall encounter throughout this thesis. For this purpose we shall work in an arbitrary spacetime dimension d. For a quick reference showing these results please consult Table 4.

Off-shell any real scalar has one degree of freedom, as it is a function, while once we go on-shell it will obey a Klein-Gordon like equation. This doesn't alter the number of degrees of freedom hence leading to one off-shell degree of freedom. The exception is if the scalar is an auxiliary field and hence has an algebraic equation of motion. In this case the field has zero on-shell degrees of freedom.

Spin-$\frac{1}{2}$ fields in any dimension have the same number of off-shell degrees of freedom as the appropriate

[36]Interestingly if we have a supersymmetric theory and it contains gravity then the supersymmetry must be realised as a local symmetry thus, seemingly, indicating an intimate relationship between local supersymmetry and diffeomorphism invariance.

[37]Since, as we've seen, a sensible graviton is massless.

spinor used to describe them as described in the last Subsection. We denote this number by m, which at this point could correspond to real or complex degrees of freedom. On-shell an equation like the Dirac equation will apply which acts to relate $\frac{1}{2}$ of the components to the other half meaning that we're left with $\frac{m}{2}$ components. Usually we shall use the minimal spinor, which is the one with the fewest real degrees of freedom, in a given dimension and as such this is the number m will usually represent. For example in four dimensions the minimal spinor is of either Majorana or Weyl type and has $m = 4$ off-shell real components leading to 2 on-shell real degrees of freedom.

A gauge field is specified by a one-form $A_{[1]}$ which obviously has d components, however we have a gauge transformation $d\lambda_{[0]}$ which allows us to fix one component of the gauge field thus leading to $d-1$ real degrees of freedom off-shell. Once we go on-shell the field will obey a Maxwell like equation which can be shown to impose one relationship between the components of the on-shell field thus reducing the number of real on-shell degrees of freedom to $d-2$.

For a gravitino ψ_μ^α which has a spinor index $\alpha \in \{1, ..., m\}$, where m is as in the spin $\frac{1}{2}$ case, and a coordinate index μ we might expect that this field has md off-shell degrees of freedom, where now we take m to denote the number of real degrees of freedom of the minimal spinor. However this isn't the case because this field possesses a gauge invariance, which in the free field case takes the form $\psi_\mu^\alpha \to \psi_\mu^\alpha + \partial_\mu \epsilon^\alpha$, and allows us to remove a full spinors worth of components giving us $m(d-1)$ real degrees of freedom. We can probably see where our search for a gauge field is pointing us. On-shell we might expect that we have $\frac{m}{2}(d-2)$ degrees of freedom, however a careful treatment[38] shows that there is an additional constraint that has to be taken into account. This careful treatment leads to us discovering that there are $\frac{m}{2}(d-3)$ on-shell real degrees of freedom.

For a graviton we've seen previously that there are $\frac{d(d-1)}{2}$ off-shell degrees of freedom. This can been seen since $E^a{}_\mu$ has d^2 real components but it also has both local Lorentz and diffeomorphism gauge transformations which remove $\frac{d(d-1)}{2} + d$ components. On-shell we loose a further d degrees of freedom due to constraints which leads to a total of $\frac{(d-1)(d-2)}{2} - 1$ on-shell real degrees of freedom.

It is also possible to consider higher order forms, such as p-forms $A_{[p]}$ $p \geqslant 2$, which possess slightly more involved gauge invariances than one forms. This can be seen as follows: p-form theories possess invariance under transformations of the form $d\lambda_{[p-1]}$, however $\lambda_{[p-1]} + d\omega_{[p-2]}$ defines the same gauge transformation as $\lambda_{[p-1]}$, so the gauge parameter has its own gauge invariance. This pattern continues until we get to a 0-form

[38]For example see chapter six of [44].

	auxiliary scalar	scalar	spin $\frac{1}{2}$	1-form	gravitino	graviton	p-form
off-shell d.o.f	1	1	m	$d-1$	$m(d-1)$	$\frac{d(d-1)}{2}$	$\binom{d-1}{p}$
on-shell d.o.f	0	1	$\frac{m}{2}$	$d-2$	$\frac{m}{2}(d-3)$	$\frac{(d-1)(d-2)}{2}-1$	$\binom{d-2}{p}$

Table 4: List of number of off-shell and on-shell real degrees of freedom for various fields encountered in Supergravity theories. Here m denotes the number of real degrees of freedom of the minimal spinor representation in the appropriate dimension.

potential. If the counting is performed accurately we find that such a field has $\binom{d-1}{p}$ real off-shell degrees of freedom. Much like the case of a 1-form it can be shown that constraints reduce the on-shell degrees of freedom to only transverse ones. This gives a total of $\binom{d-2}{p}$ real on-shell degrees of freedom.

At this point it's obvious that the gauge field for supersymmetry is going to be the gravitino, so we might ask what theories we can write down. We'll focus on theories in low dimensions and try to consider the simplest theory possible so as to illustrate the features of supergravity theories without the clutter that can occur in higher dimensions. We shall begin by considering on-shell representations of the supersymmetry algebra as this will allow us to consider how to convert on-shell representations into off-shell ones with the use of auxiliary fields. In $d=3$ we see that $E^a{}_\mu$ and ψ^α_μ both have zero on-shell degrees of freedom[39]. This gives us an on-shell representation of supersymmetry, but it's pretty trivial from a dynamics standpoint. In $d=4$ we find that both $E^a{}_\mu$ and ψ^α_μ have 2 on-shell degrees of freedom. Thus, again, they from a supermultiplet on their own. This is often called the $N=1$ supergravity multiplet and it is this multiplet that we shall focus on. The first authors to consider this theory were [43]. Interestingly invariance under supersymmetry of the theory described was one of the first calculations in theoretical physics that could only be completed by computer based computational methods.

When writing an action for the vielbein we know there are two approaches we can take, for a full recap please refer to Appendix B. Here we shall begin with the second order formulation of gravity, in this case the dynamics of $E^a{}_\mu$ are encoded by the action

$$S_{EH}[E^a{}_\mu] = \frac{1}{2\kappa^2}\int d^4x\ \det(E) e_a{}^\mu e_b{}^\nu R^{ab}{}_{\mu\nu}(\omega[E]). \tag{1.68}$$

Here $e_a{}^\mu$ is the inverse of $E^a{}_\mu$, $R^{ab}{}_{\mu\nu}(\omega[E])$ denotes the components of the curvature 2-form which are given in terms of the spin connection $\omega_\mu{}^a{}_b$. Formulae for both of these objects are given in Appendix B.

We then need to couple in the gravitino which requires nothing but a curved space generalisation of the

[39]Here $E^a{}_\mu$ is defined by $E^a = E^a{}_\mu dx^\mu$ and diagonalises the metric.

Rarita-Schwinger coupling for spin-$\frac{3}{2}$ fields. This is given by

$$S_{\frac{3}{2}}[\psi_\mu, E^a{}_\mu] = -\frac{1}{2\kappa^2}\int_M d^4x\ \det(E)\bar{\psi}_\mu \gamma^{\mu\nu\sigma} D_\nu \psi_\sigma \ , \tag{1.69}$$

In this expression γ^μ are the curved space Dirac gamma matrices, $\gamma^{\mu\nu\sigma} = \gamma^{[\mu}\gamma^\nu\gamma^{\sigma]}$, and $\bar{\psi}_\mu = \psi_\mu{}^T C$ is the Majorana conjugate of the gravitino, which is a Majorana spinor. The covariant derivative of the gravitino is given by

$$D_\mu \psi_\nu = \partial_\mu \psi_\nu + \frac{1}{2}\omega_{\mu ab} T^{ab}\psi_\nu \ , \tag{1.70}$$

where $T^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$ with γ^a the flat space Dirac gamma matrices[40]. It may have been expected that a term involving the Christoffel connection $\Gamma^\sigma{}_{\mu\nu}$ would be present, however owing to the presence of the $\gamma^{\mu\nu\sigma}$ such a contribution would evaluate to zero.

The on-shell $N = 1, d = 4$ Supergravity action is then given by

$$S^{N=1,d=4}_{Sugra}[E^a{}_\mu, \psi_\mu] = S_{EH}[E^a{}_\mu] + S_{\frac{3}{2}}[\psi_\mu, E^a{}_\mu] \ , \tag{1.71}$$

which is invariant under the following supersymmetry transformations

$$\delta_\epsilon E^a{}_\mu = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu \quad , \quad \delta_\epsilon \psi_\mu = D_\mu \epsilon \ , \tag{1.72}$$

in this expression ϵ is a local Majorana spinor. We note that the gravitino transforms like the covariant derivative of the gauge parameter, which is exactly how we expect a gauge field to transform. Showing invariance of this action is not a task for the faint of heart and we shall discuss ways to make this task simpler later in this Section.

Aside from local supersymmetry (1.71) also possesses local Lorentz invariance and diffeomorphism invariance. If we now calculate the algebra of two supersymmetry transformations, with Majorana spinor parameters ϵ_1 and ϵ_2, we find that

$$[\delta_{Susy}(\epsilon_1), \delta_{Susy}(\epsilon_2)] = \delta_{dfm}(\xi^\mu) + \delta_{Lorentz}(\xi^\mu \omega_\mu{}^a{}_b) + \delta_{Susy}(-\xi^\mu\psi_\mu) + (e.o.m) \quad , \quad \xi^\mu = \frac{1}{2}\bar{\epsilon}_2\gamma^\mu\epsilon_1 \ , \tag{1.73}$$

where δ_{dfm} represents a diffeomorphism transformation, $\delta_{Lorentz}$ represents a local Lorentz transformation and δ_{Susy} is a supersymmetry transformation all with the appropriate parameter given in the brackets. The

[40] Please note the difference of conventions between this and Appendix B, this has been done here to make it simpler for the reader to compare the results presented in this Section to those of the original material [43].

first term is exactly what we'd expect for a local theory of supersymmetry given (1.66). The additional terms are possibly unexpected but as they're associated to gauge invariances their presence isn't harmful. Crucially the appearance of terms involving the equations of motion means that this is an on-shell representation of supersymmetry.

This gives us an impression of the structures at play in Supergravity theories but everything has been at the on-shell level. A natural question one might have is if it's possible to give an off-shell formulation of the theory? We know that we'll need the number of off-shell bosonic and fermionic degrees of freedom to match so the first thing to do is to start counting again. As Table 4 shows off-shell $E^a{}_\mu$ has 6 real degrees of freedom while ψ_μ has 12. This mismatch means we need to add in 6 bosonic degrees of freedom. We also note that we need these new off-shell degrees of freedom to not contribute to the on-shell degrees of freedom counting, hence the fields we add must be auxiliary fields. There are several choices, but one is to take two auxiliary real scalars P, S and a single auxiliary vector A_μ which contributes 4 real off-shell degrees of freedom, since such a vector doesn't possess a gauge invariance[41].

The dynamics of our theory are now given by the off-shell $N = 1, d = 4$ Supergravity action

$$S^{N=1,d=4}_{Off-shell\ sugra}[E^a{}_\mu, \psi_\mu, P, S, A_\mu] = S^{N=1,d=4}_{Sugra} - \frac{1}{3\kappa^2}\int d^4x\ \det(E)(S^2 + P^2 - A_\mu A^\mu)\ . \tag{1.74}$$

We can see from this action that, under their own equations of motion, all of the new auxiliary fields are set to zero on-shell. Hence once we go on-shell for these fields we return to our original on-shell description of the $N = 1, d = 4$ theory. Aside from augmenting the action principle these fields augment the supersymmetry transformations, (1.72), of our fields. It's this change in the supersymmetry transformations that allows us to go from an on-shell representation to an off-shell one. The new transformations are given by

$$\begin{gathered}\delta_\epsilon E^a{}_\mu = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu \quad , \quad \delta_\epsilon\psi_\mu = (D_\mu + \frac{i}{2}A_\mu\gamma_5)\epsilon - \frac{1}{2}\gamma_\mu\eta\epsilon \quad , \quad \eta = -\frac{1}{3}(S - i\gamma_5 P - iA_\sigma\gamma^\sigma\gamma_5)\ , \\ \delta_\epsilon S = \frac{1}{4}\bar{\epsilon}\gamma^\mu R^{cov}_\mu \quad , \quad \delta_\epsilon P = -\frac{i}{4}\bar{\epsilon}\gamma_5\gamma^\mu R^{cov}_\mu \quad , \quad \delta_\epsilon A_\mu = \frac{3i}{4}\bar{\epsilon}\gamma_5(R^{cov}_\mu - \frac{1}{3}\gamma_\mu\gamma^\sigma R^{cov}_\sigma)\ , \\ R^{\mu,cov} = \tilde{\epsilon}^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\Big(D_\rho\psi_\sigma - \frac{i}{2\kappa}A_\rho\gamma_5\psi_\sigma + \frac{1}{2\kappa}\gamma_\rho\eta\psi_\sigma\Big).\end{gathered} \tag{1.75}$$

Once more ϵ is a Majorana spinor parameter, with $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ being the Levi-Civita symbol and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ is the usual product of four dimensional Dirac matrices.

As in the on-shell case (1.74) is invariant under (1.75). However, the big change, which is precisely what

[41] P is really a pseudoscalar as can be observed by it's behaviour under a parity transformation.

we were aiming for, is that the supersymmetry algebra now closes off-shell. In this off-shell case the algebra of two supersymmetry transformations, again with parameters ϵ_1 and ϵ_2, is

$$[\delta_{Susy}(\epsilon_1), \delta_{Susy}(\epsilon_2)] = \delta_{dfm}(\xi^\mu) + \delta_{Lorentz}(\xi^\mu \omega_\mu{}^a{}_b - \frac{i}{3\kappa}\xi^\mu \tilde{\epsilon}_\mu{}^{ac}{}_b A_c + \frac{1}{3\kappa}\bar{\epsilon}_2 \sigma^a{}_b (S - i\gamma_5 P)\epsilon_1) + \delta_{Susy}(-\xi^\mu \psi_\mu) \ ,$$
$$\xi^\mu = \frac{1}{2}\bar{\epsilon}_2 \gamma^\mu \epsilon_1 \ , \tag{1.76}$$

with $\sigma^a{}_b$ the standard antisymmetric product of Van der Waerden matrices. The crucial difference between this and (1.73) is that now there are no terms proportional to the equations of motion appearing, thus ensuring that, as promised, these fields form an off-shell representation of local supersymmetry.

So the game seems pretty clear, the gauge field of supersymmetry is the gravitino and under a SUSY transformation it transforms into $E^a{}_\mu$. Hence to get models with more supersymmetry we'll need to include more gravitinos. These can't come from supergravity multiplets, as the graviton should be unique, and so they must come from other matter supermultiplets. As such if we want more supersymmetry we'll need to include more fields into our theory. The other lesson we should take is that given an on-shell representation of supersymmetry we can attempt to turn it into an off-shell representation by including auxiliary fields. Then we need to augment the action and supersymmetry transformations of the theory in order to ensure closure of the off-shell supersymmetry algebra. This process is easy to describe but is extremely difficult to implement, and as a result of this there are many theories for which an off-shell formulation isn't known.

Despite having waxed lyrically about our actions having local supersymmetry we've been very laxed in showing it, and that won't change really. The reason for this is that it's extremely difficult to show. We will however pause to consider how one might go about it. For this we shall focus on the on-shell supergravity action, as this task is simpler and less cluttered. When we come to trying to show the invariance of the second order action, (1.71), it's useful to note that its variation under any change of the fundamental fields, up to boundary terms, takes the form

$$\delta S_{on-shell} = \int d^4x \ \det(E)\Big(\frac{\delta S_{on-shell}}{\delta E^a{}_\mu}\delta E^a{}_\mu + \frac{\delta S_{on-shell}}{\delta \psi_\mu}\delta\psi_\mu + \frac{\delta S_{on-shell}}{\delta \omega_\mu{}^a{}_b}\frac{\delta \omega_\mu{}^a{}_b}{\delta E^c{}_\sigma}\delta E^c{}_\sigma\Big) \ , \tag{1.77}$$

where final term arises because the spin connection depends upon $E^a{}_\mu$. It can be shown that in the case where the gravitino vanishes this term is identically zero. This occurs, as explained in Appendix B, because $\frac{\delta S_{on-shell}}{\delta \omega_\mu{}^a{}_b}$ is just Cartan's first equation of structure and the solution to this, when the torsion tensor is zero, is just the spin connection. However the inclusion of the gravitino complicates things as, like all propagating fermions, it leads to a torsion term in the first equation of structure. However, we can then see that if we replace the spin connection with the unique connection with torsion that satisfies the torsionful version of Cartan's equation of structure then we shall again find that this term vanishes identically. So if we replace ${\omega_\mu}^a{}_b[E]$ with the new connection ${\hat{\omega}_\mu}{}^a{}_b[E,\psi]$ in (1.71) then we have the new action

$$S_{On-shell\ 1.5} = \frac{1}{2\kappa^2}\int d^4x\ \det(E)\Big({e_a}^\mu {e_b}^\nu {R^{ab}}_{\mu\nu}(\hat{\omega}[E,\psi]) - \bar{\psi}_\mu\gamma^{\mu\nu\sigma}\hat{D}_\nu\psi_\sigma\Big)\ , \tag{1.78}$$

with ${R^{ab}}_{\mu\nu}(\hat{\omega}[E,\psi])$ being the curvature 2-form associated to our now torsionful connection and $\hat{D}_\nu\psi_\sigma$ being the analogue of (1.70) with our new connection in place of the spin connection. Note we might think this covariant derivative could include an additional term relating to the connection for diffeomorphisms also becoming torsionful however it turns out such a term isn't permitted by local supersymmetry. Since the torsion tensor for our connection is quadratic in fermions our action now contains the famous quartic fermion terms often associated with supergravity theories. In addition to augmenting our action we also have to modify our supersymmetry transformations to use this new connection. Hence (1.72) becomes

$$\delta_\epsilon {E^a}_\mu = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu \quad , \quad \delta_\epsilon\psi_\mu = \hat{D}_\mu\epsilon\ . \tag{1.79}$$

We now note that a variation of (1.78) is given by

$$\begin{aligned}\delta S_{on-shell\ 1.5} = \int d^4x\ \det(E)\Big(&\frac{\delta S_{on-shell\ 1.5}}{\delta {E^a}_\mu}\delta {E^a}_\mu + \frac{\delta S_{on-shell\ 1.5}}{\delta\psi_\mu}\delta\psi_\mu \\ &+ \frac{\delta S_{on-shell\ 1.5}}{\delta {\hat{\omega}_\mu}{}^a{}_b}\Big[\frac{\delta {\hat{\omega}_\mu}{}^a{}_b}{\delta {E^c}_\sigma}\delta {E^c}_\sigma + \frac{\delta {\hat{\omega}_\mu}{}^a{}_b}{\delta\psi_\sigma}\delta\psi_\sigma\Big]\Big)\ ,\end{aligned} \tag{1.80}$$

and now because of our choice of connection the final term vanishes and the job of checking invariance under (1.79) comes down to checking just the first two terms, which while more manageable is still extremely difficult. This approach to Supergravity is called the 1.5 order formalism because it combines aspects of the first and second order formalisms of gravity[42]. It is in this form that supergravity theories are most commonly encountered.

This completes our brief survey of the properties of Supergravity theories, most of the material covered, and much more, can be found in [44, 76, 100] which we highly recommend to the interested reader.

[42] It should be noted that our action is still a second order action.

1.3 d=11 Supergravity

So far our approach to Supergravity has had a distinctly lower dimensional feel to it, however we'll mostly be interested in theories residing in higher dimensions. Eleven dimensions holds a special place amongst dimensions as it is the maximum dimension in which a Supergravity theory can exist[43]. This comes about because in eleven dimensions supersymmetry is generated by a 32-component Majorana spinor. If this spinor were to be any larger, as it would have to be in higher dimensions, then upon a dimensional reduction of our theory on the seven-torus[44] we'd obtain massless fields with spins greater than 2 in four dimensions. We may wonder what the issue is? Well, as always, there's a collection of theorems that make this situation, seemingly, perilous. In this case they're due to Weinberg [102] and Weinberg and Witten [103]. The combination of these amounts to forbidding interactions for fields of spin higher than 2 in flat space[45]. As such, if we want to have Minkowski vacua for our theory, we're forced to conclude eleven dimensions is the final dimension in which we can have a supersymmetric field theory of massless fields, and we know how important it is that we keep the graviton massless.

With our dimensional accountancy now in order we can move on to discussing the eleven dimensional theory. As we previously stated the minimal spinor in eleven dimensions is a 32-component Majorana spinor, Q_α for $\alpha \in \{1, \ldots, 32\}$, which generates the following Fermi-Fermi bracket for the eleven dimensional centrally extended supersymmetry algebra

$$\{Q_\alpha, Q_\beta\} = (\Gamma^M C)_{\alpha\beta} P_M + \frac{1}{2!}(\Gamma^{MN} C)_{\alpha\beta} Z_{MN} + \frac{1}{5!}(\Gamma^{MNPQR} C)_{\alpha\beta} Z_{MNPQR} \,, \tag{1.81}$$

with $M \in \{1, \ldots, 11\}$, Γ^M eleven dimensional curved space Dirac gamma matrices and $\Gamma^{M \ldots N}$ are antisymmetric products of these gamma matrices. The totally antisymmetric objects Z_{MN} and Z_{MNPQR} are central charges of the algebra whose origin shall be discussed later in this introduction.

Having the supersymmetry algebra, however, is only one part of the story, we still need a theory that realises it. Such a feat was achieved by Cremmer, Julia and Scherk [18]. The theory is an on-shell realisation of local supersymmetry and as such contains a metric g_{MN}, with 44 on-shell degrees of freedom[46], and a gravitino ψ_M, which has 128 on-shell degrees of freedom. So we've got a mismatch of bosonic and fermionic

[43] Here and throughout, unless otherwise stated, we work with metrics with one timelike direction, hence Lorentzian metrics, and it is for these that eleven dimensions is special. If we add in more time directions we can go to higher dimensions, but at the expense of closed timelike curves being present.

[44] To be introduced a little later on in this introduction.

[45] Again this constitutes a no-go theorem, but as we know we can often wriggle out of these and this case is no exception. The proofs, and statements, of these theorems rely on flat space, so if we go to a curved space we're outside of their range of applicability and so we might have more luck. If we do this then, happily, interactions abound.

[46] Which is related to $E^A{}_M$ by $g_{MN} = E^A{}_M E^B{}_N \eta_{AB}$.

degrees of freedom, and as such we need to add in more bosonic fields. It turns out that the additional degrees of freedom we need are supplied by a 3-form gauge potential $A_{[3]}$ which contributes the required 84 on-shell bosonic degrees of freedom. Hence the on-shell supermultiplet we're dealing with is $(g, A_{[3]}, \psi_M)$.

To further add to the interesting properties of our eleven dimensional theory, its dynamics are uniquely determined by its gauge invariances[47], and are given by

$$\begin{aligned} S_{11d\ Sugra} &= S_{Bosonic} + S_{Fermionic} \\ &= \frac{1}{2\kappa_{11}^2}\int_{M_{11}} \Big(R *_{11} 1 - \frac{1}{2} *_{11} F_{[4]} \wedge F_{[4]} - \frac{1}{6} F_{[4]} \wedge F_{[4]} \wedge A_{[3]} \Big) \\ &- \frac{1}{2\kappa_{11}^2}\int_{M_{11}} \Big(\bar{\psi}_M \Gamma^{MNP} D_N(\frac{\hat{\omega}+\tilde{\omega}}{2})\psi_P \\ &+ \frac{1}{192}(\bar{\psi}_M \Gamma^{MNPQRS}\psi_S + 12\bar{\psi}^N\Gamma^{PQ}\psi^R)(F_{NPQR} + \tilde{F}_{NPQR})\Big) *_{11} 1 \ , \end{aligned} \tag{1.82}$$

here $F_{[4]} = dA_{[3]}$ is the field strength associated to our gauge potential, $*_{11}$ is the Hodge star associated to g and R is the associated Ricci scalar. Further to these $\bar{\psi}_M = i\psi^\dagger \Gamma^0_{flat}$, with Γ^A_{flat}, $A \in \{0 \dots 10\}$, a flat space gamma matrix[48]. We have that the connections are defined as follows

$$\hat{\omega}_M{}^A{}_B = \omega_M{}^A{}_B[E] - \frac{1}{4}\left(\bar{\psi}_M \Gamma_B \psi^A + \bar{\psi}_B \Gamma^A \psi_M - \bar{\psi}^A \Gamma_M \psi_B\right) \ , \tag{1.83}$$

which is the usual connection used in the 1.5 order formalism. As expected the torsion terms are fermion bilinears. While

$$\tilde{\omega}_M{}^A{}_B = \hat{\omega}_M{}^A{}_B - \frac{1}{4}\bar{\psi}^Q \Gamma_M{}^A{}_{BQR}\psi^R \ , \tag{1.84}$$

is a supercovariant tensor. This means its supersymmetry transformation contains no derivatives of the supersymmetry parameter. The same is true of

$$\tilde{F}_{MNPQ} = F_{MNPQ} - 6\bar{\psi}_{[M}\Gamma_{NP}\psi_{Q]} \ . \tag{1.85}$$

From these equations we can see that (1.82) contains both quadratic and quartic terms in the gravitino. Throughout this thesis the quartic terms will, mostly, be ignored as we shall be interested in setups where only the bosonic fields are non-zero. As such we shall ignore the precise forms of terms that arise from these in the expressions to follow.

[47]Under the assumption that no terms higher than two derivatives occur.

[48]If we use unevaluated indices we shall drop the specification flat from flat space gamma matrices.

The action (1.82) possesses $N=1$ supersymmetry in eleven dimensions with each field transforming as

$$\begin{aligned} \delta_\epsilon g_{MN} &= 2\bar{\epsilon}\Gamma_{(M}\psi_{N)} \ , \\ \delta_\epsilon A_{MNP} &= -3\bar{\epsilon}\Gamma_{[MN}\psi_{P]} \ , \\ \delta_\epsilon \psi_M &= 2\tilde{D}_M\epsilon \ , \end{aligned} \tag{1.86}$$

with ϵ a Majorana spinor and

$$\begin{aligned} \tilde{D}_M\epsilon &= \Big(D_M(\tilde{\omega}) + \frac{1}{288}(\Gamma_M{}^{NPQR} - 8\delta_M^N\Gamma^{PQR})\tilde{F}_{NPQR}\Big)\epsilon \\ &= \Big(D_M(\omega) + \frac{1}{288}(\Gamma_M{}^{NPQR} - 8\delta_M^N\Gamma^{PQR})F_{NPQR}\Big)\epsilon + (\psi)^2\epsilon \ , \end{aligned} \tag{1.87}$$

where the final term represents terms containing two copies of the gravitino. We shall return to this equation in later Sections when we consider solutions to eleven dimensional supergravity that preserve some amount of supersymmetry. However to even ask this question we need to be able to find solutions to this theory, which obviously requires knowledge of the equations of motion of the theory. If one varies (1.82) in the usual manner the equations of motion are found to be

$$R_{MN} - \frac{1}{2}Rg_{MN} = \frac{1}{12}F_{MP_1P_2P_3}F_N{}^{P_1P_2P_3} - \frac{1}{96}g_{MN}F_{P_1P_2P_3P_4}F^{P_1P_2P_3P_4} \ , \tag{1.88}$$

$$d*_{11}F_{[4]} + \frac{1}{2}F_{[4]}\wedge F_{[4]} = 0 \ , \tag{1.89}$$

$$\Gamma^{MNP}\tilde{D}_N\psi_P = 0 \ , \tag{1.90}$$

where in the metric and 3-form equations we have ignored all contributions due to the gravitino as we won't require these.

As a final note, in passing, we highlight that this eleven dimensional supergravity theory has been the subject of much interest due to its status as the low energy description of the elusive M-theory, which is a theory that is meant to unify all five of the known perturbative string theories [106]. Often in the literature both 11 dimensional supergravity and M-theory are referred to as M-theory. However, throughout this thesis we shall endeavour to make a distinction between them.

1.4 Branes two ways

We've now sped through some of the features of Supergravity theories in general and focused in on the unique theory in eleven dimensions. However, as we know from GR, we can often learn a lot about a theory by studying solutions to it. In this Section we shall do just this and look at some of the more exotic objects that occur in Supergravity theories[49]. We shall discover that Supergravity allows for solutions that correspond to objects with extended spatial dimensions which generalise the point particle of classical GR. Such objects are called branes and their appearance in Supergravity theories owes to the fact that these theories contain p-form gauge fields, such as the 3-form in eleven dimensions. We might now have our interest peaked as we know objects that are supported by gauge fields often carry some sort of charge and we're still looking for the origin of the central charges in the supersymmetry algebra. It will turn out that it is precisely the presence of branes that leads to the central charges arising in the supersymmetry algebra. This picture may paint branes as interesting solutions but it may also give the impression that this is all they are. This couldn't be further from the truth and, as we'll see, branes actually get a new lease of life when one moves into the land of string theory[50].

We shall begin our survey of branes in a very innocuous manner and ask how do central charges arise within a supersymmetry algebra? We know that our supercharges and Poincaré generators arise as Noether charges associated to global symmetries of our theory. Once these are obtained the algebra they obey can be constructed directly. This provides a nice method to gain insight into how central charges occur as we can consider explicit models that realise supersymmetry and see which ones lead to central extensions of the algebra. This procedure was followed by Olive and Witten [107]. The crucial insight of their work was that the calculation of the two Fermi bracket, corresponding to different supercharges, included a surface term that traditionally had been set to zero. However, in the presence of certain backgrounds this term is in fact non-zero and hence leads to a non-zero contribution to the algebra. The class of backgrounds within which they found this term to be non-zero were shown to correspond to solitonic states of the system. In these cases the surface term is related to the soliton charge. However, Olive and Witten went further and considered $d = 4$, $N = 2$ Super Yang-Mills theory where they showed that the charges arising in (1.66) are surface integrals of the Yang-Mills field strength, corresponding to long range 'electric' and 'magnetic' fields, in a background where the scalar, or pseudo-scalar, of the $N = 2$ vector multiplet has a non-trivial vacuum expectation value. This starts to paint a picture that the central charges we're seeking are carried by objects that give rise to long range forces hence leading to non-trivial surface integrals at infinity. Since we have form

[49]Well they were exotic when they were first discovered, but now they're just seen as nothing more than the mundane!

[50]We still haven't seen how Supergravity and String Theory are related, but be patient, we'll get there.

fields arising in Supergravity theories these provide obvious candidates for objects to be integrated at infinity to give non-zero charges that could arise in the supersymmetry algebra. The validity of this assumption was shown in [26]. However, we still need to find out what sort of objects carry such charges.

So our job now becomes to try and explicitly find the types of objects that might carry these charges. To do this we shall try to understand some of their properties, with our approach closely following [98]. We could just plough in and try and guess a solution or we could be a little smarter and use the SUSY algebra to help us. Since our main focus is in eleven dimensions we shall focus on the algebra (1.81) and, owing to hindsight, lets focus on the 2-form charge and thus set the 5-form charge to zero. This helps to simplify the supersymmetry algebra somewhat, but we can go further and assume that only one, independent, component of the 2-form is non-zero and the object carrying this is massive. This allows us to go to its rest frame where $P^A = (P^0, 0, \ldots, 0)$ if we further assume the non-zero component is $Z_{12} = Q$ and also assume that we use the Majorana rep of the Dirac gamma matrices, so that $C = \Gamma^0$, then the SUSY algebra becomes

$$\{Q_\alpha, Q_\beta\} = (P^0 \mathbb{1} + Q\Gamma^{012}_{flat})_{\alpha\beta} \ , \tag{1.91}$$

with $\Gamma^{012}_{flat} = \Gamma^0_{flat}\Gamma^1_{flat}\Gamma^2_{flat}$. If we now consider a quantum theory[51] that realises this symmetry then we can derive the bound

$$P^0 \geqslant |Q| \ , \tag{1.92}$$

which owes to the fact that the left hand side of (1.91) is positive definite. Of course, the interesting case is the one where such a bound is saturated. If we consider such a state in the theory, what sort of properties would it have? Originally the theory possessed 32 real supercharges, all of which annihilate the vacuum state, and we know that if a state preserves some of our symmetries then it will be annihilated by some linear combination of the symmetry generators. This will be seen by the existence of zero-eigenspinors of $\{Q_\alpha, Q_\beta\}$. From (1.91) such spinors, ϵ, will obey

$$\Gamma^{012}_{flat}\epsilon = \pm\epsilon \ . \tag{1.93}$$

Since the gamma matrices we're using are flat space ones we know that $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$ and hence it can be shown[52] that $(\Gamma^{012}_{flat})^2 = \mathbb{1}$ and $\mathrm{tr}(\Gamma^{012}_{flat}) = 0$. This ensures that the dimension of the nullity of $\{Q_\alpha, Q_\beta\}$ is sixteen, hence a state saturating (1.92) preserves 16 real supercharges.

[51] While we are considering Supergravity, which is a classical theory, we can use its quantum completion to try and inform us about the types of objects that may appear even at the classical level.

[52] For those who haven't done gamma matrix manipulations in a while we can see this by noting that $(\Gamma^3)^2 = \mathbb{1}$ and then considering $\mathrm{tr}((\Gamma^3)^2\Gamma^{012})$.

Whilst this may all seem a little far away from our goal of constructing solutions to our theory we've gained valuable insight. We can see that the easiest place to look for a solution carrying our desired charge is to consider the case where we preserve some amount of supersymmetry. Further to this insight we should ask what sort of object can carry charge under the 3-form gauge potential of eleven dimensional Supergravity? It turns out that to answer this we should look at how particles couple to the 1-form Maxwell potential. We know that particles trace out a worldline in spacetime and the natural coupling[53] of a Maxwell field to this worldline is via a pullback of this 1-form. This leads to a term $\int_{W_1} \varphi^*(A_{[1]})$ being added to the usual worldline action of the particle, where W_1 is the manifold describing the particle worldline, $\varphi : W_1 \to M$ provides the embedding functions of the worldline into spacetime and φ^* is the pullback associated to this function. So if we want to mimic this coupling we should be looking for an object which is described by a three-dimensional manifold. Such an object is called a membrane. Given these one begins to look for a solution representing a membrane that preserves 16 supersymmetries. This was done by Duff and Stelle [37] who proposed the following ansatz for a solution of eleven dimensional Supergravity

$$\begin{aligned} ds^2_{11} &= e^{2A(r)} ds^2(\mathbb{E}^{1,2}) + e^{2B(r)} ds^2(\mathbb{E}^8) \ , \\ A_{[3]} &= \pm e^{C(r)} dx^0 \wedge dx^1 \wedge dx^2, \ \ \psi_M = 0 \ , \end{aligned} \tag{1.94}$$

where x^μ, with $\mu \in \{0, 1, 2\}$, are coordinates on the worldvolume of the brane, y^m, for $m \in \{1, \dots, 8\}$, are coordinates on the space transverse to worldvolume and A, B, C are three undetermined functions with $r^2 = \delta_{mn} y^m y^n$. We can see that this solution clearly has an object with a three dimensional worldvolume in it and the ansatz preserves the $ISO(1, 2) \times SO(8)$ subgroup of the full $ISO(1, 10)$ of the vacuum. The ansatz also has a non-trivial 3-form and so can lead to a non-trivial Maxwell like charge. This ansatz can also be seen to be purely bosonic, as the gravitino is set to zero.

However, how can we determine if the solution is in some way supersymmetric? Well the solution could be invariant under the supersymmetry transformations (1.86) for some choices of spinors ϵ. We can see that, since the ansatz is purely bosonic, the only non-trivial transformation is that of the gravitino and for our ansatz to preserve some degree of supersymmetry we need this variation to vanish. This consideration leads to the equation

$$\delta_\epsilon \psi_M = 0 \implies \Big(D_M(\omega) + \frac{1}{288} (\Gamma_M{}^{NPQR} - 8 \delta^N_M \Gamma^{PQR}) F_{NPQR} \Big) \epsilon = 0 \ . \tag{1.95}$$

[53]We use the word natural here loosely, maybe a better term to use is the electric style coupling.

Equations of this sort are called Killing spinor equations and spinors, ϵ, solving this equation are called Killing spinors. Given the symmetry of the ansatz any spinor must decompose as

$$\epsilon = \tilde{\epsilon} \otimes \omega(r) \ , \tag{1.96}$$

with $\tilde{\epsilon}$ a constant Majorana spinor of $Spin(1,2)$ and $\omega(r)$ a Majorana spinor of $Spin(8)$, hence it is not the minimal spinor as this is a Majorana-Weyl spinor so we could choose to further decompose it using $P_{\pm} = \frac{1}{2}(1 \pm \Sigma_9)$ with Σ_9 the ninth gamma matrix in eight dimensions. If we calculate the form of the Killing spinor equation for the membrane ansatz we find that in order for there to be preserved supersymmetry the following conditions must hold

$$(\mathbb{1} \pm \Sigma_9)\omega = 0, \quad \omega = e^{-\frac{C}{6}}\omega_0, \quad A = \frac{1}{3}C, \quad B = -\frac{1}{6}C + \tilde{d} \ . \tag{1.97}$$

Here ω_0 is a constant spinor of $Spin(8)$, the sign in the first equation corresponds to the sign of the form ansatz in (1.94) and $\tilde{d} \in \mathbb{R}$. Since we can see the spinor of $Spin(8)$ is forced to be Majorana Weyl it has real dimension 8 and hence with the 2 dimensional spinor $\tilde{\epsilon}$ we can see we have a 16 component Killing spinor. As a result our ansatz preserves half of the supersymmetry of the full theory. So our time spent looking at the eleven dimensional supersymmetry algebra doesn't look like such a waste of time now. Even better we've managed to reduce the number of undetermined functions to just one. However, we still haven't appealed to the equations of motion of the theory. If we do this then it can be shown that they reduce to

$$\delta^{mn}\partial_m\partial_n e^{-C} = 0 \ , \tag{1.98}$$

which is a Laplace equation in the transverse dimensions. Selecting boundary conditions so that the metric is asymptotically flat we find that

$$H(r) := e^{-C} = 1 + \frac{k}{r^6} \ , \tag{1.99}$$

for k an arbitrary constant. With this and (1.97), where $\tilde{d} = 0$ as it can be absorbed by a coordinate rescaling, (1.94) becomes

$$\begin{aligned} ds^2_{11} &= H^{-\frac{2}{3}}(\mathbb{E}^{1,2}) + H^{\frac{1}{3}}ds^2(\mathbb{E}^8) \ , \\ A_{[3]} &= \pm H^{-1}dx^0 \wedge dx^1 \wedge dx^2, \quad \psi_M = 0 \ . \end{aligned} \tag{1.100}$$

This is the famous M2-brane solution of eleven dimensional supergravity. This solution appears to be singular at $r = 0$ but this is just a coordinate singularity and is in fact a horizon of the solution. Near this horizon the metric takes the form

$$ds^2_{11} \sim k^{-\frac{2}{3}} r^4 ds^2(\mathbb{E}^{1,2}) + \frac{k^{\frac{1}{3}}}{r^2} dr^2 + k^{\frac{1}{3}} d\Omega^2_7, r \to 0 \ , \tag{1.101}$$

with $d\Omega^2_7$ the unit round metric on S^7. If we now set $R_{AdS} = \frac{k^{\frac{1}{6}}}{2}$ and let $u = \frac{1}{4}\frac{r^2}{R_{AdS}}$ then we find that the metric takes the form

$$ds^2_{11} \sim (R_{AdS})^2 \Big(\frac{du^2}{u^2} + \big(\frac{u}{R^2_{AdS}}\big)^2 ds^2(\mathbb{E}^{1,2}) \Big) + k^{\frac{1}{3}} d\Omega^2_7, r \to 0 \ , \tag{1.102}$$

which is an $AdS_4 \times S^7$ metric[54], showing that the M2 brane solution interpolates between two vacuum solutions, 11-dimensional Minkowski and $AdS_4 \times S^7$, of eleven dimensional Supergravity. The geometry as $r \to 0$ is often referred to as the near horizon geometry of the M2 brane.

The M2 brane also has another very nice property since it enjoys a no-force condition, much like the Reissner-Nordström black hole or various solitons. This property is very powerful and arises due to the preserved supersymmetry of the solution [37]. The key upshot of it is that the equation governing the solution is (1.98) which is a linear equation. As such we are allowed to superimpose solutions to the equation which means we can replace the harmonic function in (1.99) with the more general solution

$$e^{-C} = 1 + \sum_{i=1}^{n} \frac{k_i}{|\mathbf{r} - \mathbf{r}_i|} \ , \tag{1.103}$$

which represents n parallel[55] M2 branes at positions $\mathbf{r}_i$ in the transverse space each with its own associated integration constant k_i. The M2 brane solution can also be generalised in other ways. For example we might expect to be able to allow ripples along the surface of the brane thus obtaining a wave travelling on our brane much like waves travel on the skin of a drum. A solution of this type can be obtained by a Garfinkle-Vachaspati transformation [47] which is explained in Appendix C.

The M2 brane marks our first encounter with a brane, but it's certainly not our last. In fact branes abound in Supergravity theories. We're now going to look at a class of branes, with features very similar to the M2 brane, which go under the name of p-branes. Our approach to this will be heavily influenced by

[54] Where we use Bertotti-Robinson coordinates for the AdS_4 space.

[55] This means their worldvolumes are in the same directions, we'll have a little to say about setups where this isn't the case later on.

[95]. In order to see how branes can arise in different contexts lets consider a toy model system[56]

$$S[g, A_{[n-1]}, \phi] = \int R * 1 - \frac{1}{2} d\phi \wedge * d\phi - \frac{e^{a\phi}}{2} F_{[n]} \wedge * F_{[n]} \ , \tag{1.104}$$

with $F_{[n]} = dA_{[n-1]}$, ϕ a scalar, $a \in \mathbb{R}$ and $*$ and R the Hodge star and Ricci scalar of the metric g. The equations of motion of this system can then be found to be

$$\begin{aligned} R_{MN} &= \frac{1}{2} \nabla_M \phi \nabla_N \phi + \frac{e^{a\phi}}{2(n-1)!} \Big(F_{MQ_1 \dots Q_{n-1}} F_N{}^{Q_1 \dots Q_{n-1}} - \frac{n-1}{n(d-2)} F_{Q_1 \dots Q_n} F^{Q_1 \dots Q_n} g_{MN} \Big) \ , \\ \Box \phi &= \frac{a e^{a\phi}}{2(n!)} F_{Q_1 \dots Q_n} F^{Q_1 \dots Q_n} \ , \qquad \nabla_M (e^{a\phi} F^{M N_1 \dots N_{n-1}}) = 0 \ , \end{aligned} \tag{1.105}$$

where ∇_M is the covariant derivative associated to the Levi-Civita connection of the metric and $M \in \{1, \dots, d\}$. Following the M2 brane ansatz we propose the following solution to these equations

$$\begin{aligned} ds^2 &= e^{2A(r)} ds^2(\mathbb{E}^{1,n-2}) + e^{2B(r)} ds^2(\mathbb{E}^{d-n-1}) \ , \\ \phi &= \phi(r) \ , \ \ A_{[n-1]} = e^{C(r)} dx^0 \wedge \dots \wedge dx^{n-2} \ , \end{aligned} \tag{1.106}$$

where we split the full spacetime coordinates as x^μ, with $\mu \in \{0, \dots, n-2\}$, which parametrise the worldvolume of our proposed brane, and y^m, where $m \in \{1, \dots, d-n+1\}$, denoting the coordinates on the transverse space with $r^2 = \delta_{mn} y^m y^n$. Once more this ansatz preserves $ISO(1, n-2) \times SO(d-n+1)$ symmetry. If we now calculate the form of (1.105) for (1.106) we find the following system of equations

$$\begin{aligned} & \Delta A + A'((n-1)A' + (d-n-1)B') = \frac{d-n-1}{2(d-2)} S^2 \ , \\ & \Delta B + B'((n-1)A' + (d-n-1)B') + \frac{1}{r}((n-1)A' + (d-n-1)B') = -\frac{n-1}{2(d-2)} S^2 \ , \\ & (d-n-1)B'' + (n-1)A'' - 2(n-1)A'B' + (n-1)(A')^2 - (d-n-1)(B')^2 \\ & - \frac{1}{r}((n-1)A' + (d-n-1)B') + \frac{1}{2}(\phi')^2 = \frac{1}{2} S^2 \ , \\ & \Delta \phi + \phi'((n-1)A' + (d-n-1)B') = -\frac{1}{2} a S^2 \ , \\ & \Delta C + C'(C' + (d-n-1)B' - (n-1)A' + a\phi') = 0 \ , \\ & S = \Big(e^{\frac{a}{2}\phi - (n-1)A + C} \Big) C' \ . \end{aligned} \tag{1.107}$$

Here Δ is the Laplacian of the metric $ds^2(\mathbb{E}^{d-n-1})$. Following what we learnt from the M2 brane it's useful to relate the arbitrary functions in our ansatz to each other, underlying these choices is often the desire for

[56] In fact such a system arises as a subset of many Supergravity theories.

the solution to preserve some degree of supersymmetry. One can see an obvious choice, given (1.107), is

$$(n-1)A' + (d-n-1)B' = 0 \ , \tag{1.108}$$

which reduces our system of equations to

$$\begin{aligned}
&\Delta A = \frac{d-n-1}{2(d-2)} S^2 \ , \\
&(n-1)(d-2)(A')^2 + \frac{d-n-1}{2}(\phi')^2 = \frac{d-n-1}{2} S^2 \ , \\
&\Delta \phi = -\frac{1}{2} a S^2 \ , \\
&\Delta C + C'(C' + (d-n-1)B' - (n-1)A' + a\phi') = 0 \ .
\end{aligned} \tag{1.109}$$

We can now see that selecting

$$\phi' = -\frac{a(d-2)}{d-n-1} A' \ , \tag{1.110}$$

will further simplify our equations to

$$\begin{aligned}
&S^2 = \frac{1}{a^2}\Big(a^2 + \frac{2(n-1)(d-n-1)}{d-2}\Big)(\phi')^2 \ , \\
&\Delta \phi = -\frac{1}{2} a S^2 \ , \\
&\Delta C + C'(C' + (d-n-1)B' - (n-1)A' + a\phi') = 0 \ .
\end{aligned} \tag{1.111}$$

The first two of these can be consolidated in to

$$\Delta(e^{\frac{\alpha}{2a}\phi}) = 0 \ , \qquad \alpha = \Big(a^2 + \frac{2(n-1)(d-n-1)}{d-2}\Big) \ , \tag{1.112}$$

which, if we decide upon boundary conditions so $\phi \to 0$ as $r \to \infty$, is solved by

$$H(r) = e^{\frac{\alpha}{2a}\phi} = 1 + \frac{k}{r^{d-n-1}} \ , \tag{1.113}$$

with $k \in \mathbb{R}$. This can now be used to solve for A and B with the integration constants from this being determined by the condition that the metric is asymptotically the Minkowski metric. We can now solve for C by using

$$\frac{d}{dr} e^C = \pm \frac{\sqrt{\alpha}}{a} e^{-\frac{a}{2}\phi + (n-1)A} \phi' \ , \tag{1.114}$$

whose solution is compatible with the equation $\Delta C + C'(C' + (d-n-1)B' - (n-1)A' + a\phi') = 0$. Once we assemble all of the functions we find the final solution takes the form

$$\begin{aligned}
&ds^2 = H^{-\frac{4(d-n-1)}{\alpha(d-2)}} ds^2(\mathbb{E}^{1,n-2}) + H^{\frac{4(n-1)}{\alpha(d-2)}} ds^2(\mathbb{E}^{d-n-1}) \ , \\
&e^{\phi} = H^{\frac{2a}{\alpha}} \ , \qquad A_{[n-1]} = \frac{2}{\sqrt{\alpha}} H^{-1} dx^0 \wedge \dots dx^{n-2} \ , \\
&H = 1 + \frac{k}{r^{d-n-1}} \ .
\end{aligned} \tag{1.115}$$

For such a solution the dimension of the worldvolume of the brane is often denoted by $p+1 = n-1$ and as such we call it a p-brane solution. Interestingly, again, the whole solution is determined by the form of a harmonic function.

So now we have a whole host of branes and we might be content with these. However our initial analogy with point particles in Maxwell theory can be pushed further. So far we've been concerned with form fields that couple to the worldvolume of the brane in an electric type manner, but we know Maxwell theory can also allow for magnetic couplings. This leads us to the natural question of if there are branes that can carry magnetic like charges? Rather unsurprisingly it turns out there are. The next question that comes to mind is how would such a magnetic object couple to our form field? Well again we just look at the story in the case of Maxwell theory, here magnetically charged objects couple to the dual of the vector potential which is obtained by a procedure that involves, often only as part of it, taking the Hodge dual of the field strength and then deriving a potential from this object. So lets assume we're in a d-dimensional space, if we have a gauge field $A_{[n-1]}$ its field strength is an n-form $F_{[n]}$ whose dual is $\tilde{F}_{[d-n]} = *F_{[n]} + \dots$ which has an underlying potential $\tilde{A}_{[d-n-1]}$. Then following the electric case we now expect this dual field to couple to a $d-n-2$ brane. We then propose the following ansatz for a solution to our toy model system (1.104)

$$\begin{aligned}
&ds^2 = e^{2A(r)} ds^2(\mathbb{E}^{1,d-n-2}) + e^{2B(r)} ds^2(\mathbb{E}^{n+1}) \ , \\
&\phi = \phi(r) \ , \qquad F_{[n]} = \frac{\lambda}{n!} \tilde{\epsilon}_{m_1 \dots m_n p} \frac{y^p}{r^{n+1}} dy^{m_1} \wedge \dots dy^{m_n} \ ,
\end{aligned} \tag{1.116}$$

where $\tilde{\epsilon}_{m_1 \dots m_n p}$ is the Levi-Civita symbol, y^m are coordinates on the transverse space and $\lambda \in \mathbb{R}$. The power of $r = \sqrt{\delta_{mn} y^m y^n}$ in the field strength is fixed by the requirement that it solve the Bianchi identity $dF_{[n]} = 0$. If we now follow a similar procedure to that for the electric ansatz, but with a few signs changed[57], we find

[57]See [95] for details.

that the magnetic brane solution is

$$
\begin{aligned}
ds^2 &= H^{-\frac{4(n-1)}{\alpha(d-2)}} ds^2(\mathbb{E}^{1,d-n-2}) + H^{\frac{4(d-n-1)}{\alpha(d-2)}} ds^2(\mathbb{E}^{n+1}) \ , \\
\phi &= H^{-\frac{2a}{\alpha}} \ , \qquad F_{[n]} = -\frac{2}{\sqrt{\alpha} n!} \tilde{\epsilon}_{m_1 \ldots m_n m_{n+1}} \partial_{m_{n+1}} H dy^{m_1} \wedge \ldots \wedge dy^{m_n} \ , \qquad (1.117) \\
H(r) &= 1 + \frac{k}{r^{n-1}} \ , \qquad k = \frac{\sqrt{\alpha}}{2(n-1)} \lambda \ . \qquad (1.118)
\end{aligned}
$$

In order to obtain this solution we have assumed asymptotically Minkowski boundary conditions for the metric.

The obvious magnetic brane we want to find is dual of the M2 brane. To get to this we have to be a little careful as eleven dimensional Supergravity doesn't contain a dilaton, so in (1.105) both ϕ and a have to be set to zero, which is the easy part. However the potentially more complicated part is that the form field equation (1.88) contains a term proportional to $F_{[4]} \wedge F_{[4]}$ which is not present in the equations of our model system. Fortunately explicit calculation of this term shows that in the case of the magnetic ansatz (1.117) it evaluates to zero. This means that the analysis leading to (1.117) goes through in eleven dimensions and we obtain the famous M5 brane solution [56]

$$
\begin{aligned}
ds_{11}^2 &= H^{-\frac{1}{3}} ds^2(\mathbb{E}^{1,5}) + H^{\frac{2}{3}} ds^2(\mathbb{E}^5) \ , \\
F_{[4]} &= 3k \tilde{\epsilon}_{m_1 \ldots m_n p} \frac{y^p}{r^5 4!} dy^{m_1} \wedge \ldots dy^{m_n} \ , \qquad (1.119) \\
H(r) &= 1 + \frac{k}{r^3} \ .
\end{aligned}
$$

Once more as $r \to \infty$ the solution tends to eleven dimensional Minkowski space. As in the M2 brane case this solution looks singular at $r = 0$ but again this is just a coordinate singularity and is a horizon of the solution. If we look at the near horizon geometry in this case we find that it takes the form $AdS_7 \times S^4$ and so the M5 brane also interpolates between two vacuum solutions of eleven dimensional Supergravity. If we were to study the Killing spinor equation in the M5 background we'd find that this background possesses 16 Killing spinors. It is also no surprise that we can replace the harmonic function in our solution with a multicentre solution which represents a series of parallel M5 branes at various points in the transverse space.

We've now seen electric and magnetic branes so it's logical to ask if there are branes that carry both magnetic and electric charge? Such branes are referred to as dyonic branes and are beyond the scope of this introduction, for details refer to [82].

So far we seem to be doing quite well, we've managed to find solutions corresponding to both electric and magnetic branes that we've claimed are applicable to a wide range of Supergravity theories. These solutions are controlled by harmonic functions, which are given in the M2 and M5 brane cases by (1.99) and (1.119), and are supposed to solve Laplace equations. It's here that we run in to a constitutional issue of honesty. While the harmonic forms do solve a Laplace equation almost everywhere, at the centres of the branes they solve a Poisson equation. This, again, is very much like a point charge in Maxwell theory where we know that the resolution is to find a source action to couple in to the system. It's this desire to include sources that leads to the famous $\int J^\mu A_\mu$ coupling of a conserved current in Maxwell theory. So we're at an impasse, do we tell the truth and search for a source for our system or do we just accept the solutions we've presented aren't solutions everywhere[58]? Here we'll strike a compromise and tell the truth, but only in the case of the M2 brane. The story works out similarly for other cases but a general presentation is beyond our current needs.

So we need to find an action that we can couple to (1.82) to act as a source for the M2 brane. Obviously we're looking for an action describing a membrane and so the obvious choice is the supermembrane of Bergshoff, Sezgin and Townsend [8]. To understand the supermembrane in all of its glory requires the use of curved superspace methods which generalise the structures of geometry to the setting of supermanifolds. Fortunately the M2 brane is a bosonic solution, hence we can avoid a trip in to the realm of super geometry by merely concerning ourselves with the bosonic pieces of the supermembrane action

$$S_{memb} = -T_2 \int_{W_3} *_3 1 + T_2 \int_{W_3} \varphi^*(A_{[3]}) \,, \tag{1.120}$$

with W_3 the worldvolume manifold of the supermembrane, T_2 its tension and $\varphi : W_3 \to M$ the embedding functions into the spacetime manifold, which is replaced by a supermanifold in the case of the full supermembrane action. The component functions of φ are denoted by $X^M(\xi)$ with ξ^i being coordinates on the world volume, $i \in \{0, 1, 2\}$, and $*_3$ is the Hodge star of the pullback of the spacetime metric, $\partial_i X^M \partial_j X^N g_{MN}$, and $\varphi^*(A_{[3]})$ is the pullback of the spacetime 3-form. This is the Nambu-Goto form of the action, there is also the Polyakov/Howe-Tucker form of this action given by

$$S[X,\gamma] = T_2 \int d^3\xi \Big(-\frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i X^M \partial_j X^N g_{MN} + \frac{1}{2}\sqrt{-\gamma} + \frac{1}{3!}\tilde{\epsilon}^{ijk}\partial_i X^M \partial_j X^N \partial_k X^p A_{MNP} \Big) \,, \tag{1.121}$$

where γ_{ij} is an independent worldvolume metric. The equation of motion of γ_{ij} sets it equal to the pullback

[58] Such issues to do with telling the truth about sourcing of systems are sometimes referred to as George Washington problems.

of the spacetime metric. Hence on-shell we have

$$\gamma_{ij} = \partial_i X^M \partial_j X^N g_{MN} \ , \tag{1.122}$$

which can be used along with (1.121) to show that the Polyakov action is classically equivalent to (1.120), which is a statement at the level of the equations of motion of both systems.

With the correct action now obtained we can consider coupling it to the action of eleven dimensional Supergravity, where we only care about bosonic fields. This gives us the action

$$S = \frac{1}{2\kappa_{11}^2} \int_M R *_{11} 1 - \frac{1}{2} F_{[4]} \wedge *_{11} F_{[4]} - \frac{1}{6} F_{[4]} \wedge F_{[4]} \wedge A_{[3]} - T_2 \int_{W_3} *_3 1 + T_2 \int_{\varphi(W_3)} A_{[3]} \ , \tag{1.123}$$

where $\varphi(W_3) \subset M$ is the image of the worldvolume under the embedding function φ. If we begin by focusing on the form field sector then we can attempt to find a form $J_{[3]}$ such that

$$T_2 \int_{\varphi(W_3)} A_{[3]} = T_2 \int_M *_{11} J_{[3]} \wedge A_{[3]} \ . \tag{1.124}$$

For now we shall assume the existence of such a form whose sole purpose is to turn our integral over a submanifold into one defined on the entire spacetime manifold, later we'll consider how to construct such a form. With this the form portion of our action becomes

$$S_{A_{[3]}} = \frac{1}{2\kappa_{11}^2} \int_M \Big(- \frac{1}{2} F_{[4]} \wedge *_{11} F_{[4]} - \frac{1}{6} F_{[4]} \wedge F_{[4]} \wedge A_{[3]} + 2\kappa_{11}^2 T_2 *_{11} J_{[3]} \wedge A_{[3]} \Big) \ , \tag{1.125}$$

whose equation of motion is

$$d *_{11} F_{[4]} + \frac{1}{2} F_{[4]} \wedge F_{[4]} = 2\kappa_{11}^2 T_2 *_{11} J_{[3]} \ , \tag{1.126}$$

from which we can see that

$$d *_{11} J_{[3]} = 0 \ , \tag{1.127}$$

meaning that $*_{11} J_{[3]}$ acts as a conserved source for the form field equation. This confirms that, if we can find a $J_{[3]}$ such that (1.124) holds, we have successfully sourced our system.

In order to see how such a source may be constructed we resort to a coordinate based version of the action

$$
\begin{aligned}
S = \frac{1}{2\kappa_{11}^2}&\int d^{11}x\sqrt{-g}\,\Big(R - \frac{1}{48}F_{MNPQ}F^{MNPQ}\Big)\\
-\frac{1}{2\kappa_{11}^2}&\int d^{11}x\;\frac{1}{(4!)^2(3!)^2}\tilde{\epsilon}^{M_1\dots M_4M_5\dots M_8M_9\dots M_{11}}F_{M_1\dots M_4}F_{M_5\dots M_8}A_{M_9\dots M_{11}}\\
+\frac{T_2}{2}&\int d^3\xi\sqrt{-\gamma}\Big(-\gamma^{ij}\partial_iX^M\partial_jX^Ng_{MN}+1\Big)+\frac{T_2}{3!}\int d^3\xi\tilde{\epsilon}^{ijk}\partial_iX^M\partial_jX^N\partial_kX^PA_{MNP}\ ,
\end{aligned}
\tag{1.128}
$$

where we have used the Polyakov form of the bosonic supermembrane action. We can then look at the variation under a change of the spacetime metric. This leads to

$$
\begin{aligned}
\delta_gS = \frac{1}{2\kappa_{11}^2}&\int d^{11}x\sqrt{-g}\;\delta g_{MN}\Big(-R^{MN}+\frac{1}{2}g^{MN}R-\frac{1}{96}g^{MN}F_{PQRS}F^{PQRS}+\frac{1}{12}F^{MPQR}F^{N}{}_{PQR}\Big)\\
-\frac{T_2}{2}&\int d^{11}x\sqrt{-g}\;\frac{\delta^{11}(x-X)}{\sqrt{-g}}\int d^3\xi\sqrt{-\gamma}\gamma^{ij}\partial_iX^M\partial_jX^N\delta g_{MN}\ ,
\end{aligned}
\tag{1.129}
$$

which gives a modified version of the Einstein equation given in (1.88) which now includes a source term

$$
\begin{aligned}
R^{MN}-\frac{1}{2}g^{MN}R =&\frac{1}{12}F^{MPQR}F^{N}{}_{PQR}-\frac{1}{96}g^{MN}F_{PQRS}F^{PQRS}\\
&-\kappa_{11}^2T_2\int d^3\xi\sqrt{-\gamma}\gamma^{ij}\partial_iX^M\partial_jX^N\frac{\delta^{11}(x-X)}{\sqrt{-g}}\ .
\end{aligned}
\tag{1.130}
$$

So we seem to be seeing that it's delta distributions that will allow us to localise our sources. How about in the case of the form field equation? If we follow the same steps and employ the field redefinition $A_{[3]}\to -A_{[3]}$ then we obtain

$$
\begin{aligned}
\partial_M(\sqrt{-g}F^{MNPQ})+&\frac{1}{2(4!)^2}\tilde{\epsilon}^{NPQM_1\dots M_4M_5\dots M_8}F_{M_1\dots M_4}F_{M_5\dots M_8}\\
&=2\kappa_{11}^2T_2\int d^3\xi\;\tilde{\epsilon}^{ijk}\partial_iX^N\partial_jX^P\partial_kX^Q\delta^{11}(x-X)\ ,
\end{aligned}
\tag{1.131}
$$

which is a sourced version of the original 3-form equation. We see that the source corresponds to a 3-form, hence it is the promised J_3 that localises our integral, as given in (1.124). There are also equations of motion for the auxiliary metric, which is given by (1.122), and the embedding coordinates X^M

$$
\nabla_i\nabla^iX^M+\Gamma(g)^M{}_{NP}\nabla^iX^N\nabla_iX^P=\frac{1}{3!}\epsilon^{ijk}\partial_iX^N\partial_jX^P\partial_kX^QF^M{}_{NPQ}\ ,
\tag{1.132}
$$

with ∇_i the covariant derivative associated to the Levi-Civita connection of γ, $\Gamma(g)^M{}_{NP}$ the Christoffel symbol of the metric g and the sign arises due to our field redefinition of $A_{[3]}$. If we now use the fact that we have world volume diffeomorphism invariance then it's possible to set the static gauge choice

$$X^i = \xi^i, \quad i \in \{0, 1, 2\} , \tag{1.133}$$

and then propose the constant solution

$$X^{2+m} = Y^m, \quad m \in \{1, \dots, 8\} , \tag{1.134}$$

then provided

$$k = \frac{\kappa_{11}^2 T_2}{3\Omega_7} , \tag{1.135}$$

(1.100) solves the sourced equations everywhere in the spacetime and all other equations are solved for their appropriate variables. This all means we have successfully found a solution of the sourced eleven dimensional system.

The preceding work thus shows us that in order for p-branes to be solutions everywhere we need to introduce source actions. These source actions represent theories on the worldvolume of a brane and can be seen to couple to appropriate spacetime fields. Whilst the subject of coupling branes to supergravity systems is very interesting, we'll leave our treatment of it here and instead sleep easy in the knowledge that, if we desired, we could find the appropriate sources.

Our digression into the territory of sourcing the Supergravity equations actually allows us to answer how central charges arise in the supersymmetry algebra. We can see that we've introduced a conserved current in to the system, (1.127), and thus an appropriate integral of $*_{11} J_{[3]}$ will yield a conserved charge

$$Q[\Sigma_8] = \int_{\Sigma_8} *_{11} J_{[3]} , \tag{1.136}$$

with the value of this integral being dependent on if the eight dimensional manifold Σ_8 encircles the brane source. Hence the charge is in fact aware of the spatial orientation of the brane, which in the case of the M2 brane requires a 2-form to specify. As such the charge becomes two form valued as expected in the supersymmetry algebra. An easier way to see this two form nature was found in [26]. Here the authors noted that the Wess-Zumino term of the supermembrane action led to a topologically conserved current

$$J^{iM_1M_2} = \tilde{\epsilon}^{ijk} \partial_j X^{M_1} \partial_k X^{M_2}, \qquad \partial_i J^{iM_1M_2} , \tag{1.137}$$

which leads to the conserved charges

$$Z^{M_1M_2} = T_2 \int d\xi_2 d\xi_1 J^{0M_1M_2} \; , \tag{1.138}$$

which provides a 2-form of charges that are only non-zero along the spatial directions of a brane. The equivalence of these two descriptions can be established, but we'll leave it here. Unsurprisingly the 5-form in (1.81) is related to the charge of the M5 brane. The interested reader can learn more about this issue, as well as charges arising in supersymmetry algebras in general, by referring to [60].

So far our approach to branes has had a very Supergravity feel to it, however they also arise within the context of string theory. We shall very briefly recap this in the simplest setting, which is the bosonic string, however the reader interested in superstrings should consult [63, 85]. For ease consider the Polyakov action for a bosonic string propagating in a flat target space

$$S_{Poly}[X,\gamma] = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} \; , \tag{1.139}$$

with $\alpha \in \{0,1\}$, $\mu \in \{0,\ldots,d-1\}$, σ^α being coordinates on the string worldsheet and α' a constant that's related to the string tension, T, by $\alpha' = (2\pi T)^{-1}$. Considering the variation of this action under a variation of the embedding coordinates, X^μ, gives

$$\delta_X S_{Poly} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\Big(\partial_\alpha(\sqrt{-\gamma}\gamma^{\alpha\beta}\delta X^\mu\partial_\beta X^\nu\eta_{\mu\nu}) - \big(\partial_\alpha(\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\beta X^\nu\eta_{\mu\nu})\big)\delta X^\mu\Big) \; , \tag{1.140}$$

the second term is the equations of motion of the system while the first is a boundary term. Lets consider an open string with $\sigma^\alpha = (\tau,\sigma)$ where $\tau \in (-\infty,\infty)$ and $\sigma \in (0,\pi)$. If we assume at the τ boundary all fields have zero variation then the only boundary piece left is

$$\frac{1}{2\pi\alpha'}\int d\tau\sqrt{-\gamma}\gamma^{1\beta}\partial_\beta X^\nu\eta_{\mu\nu}\delta X^\mu\Big|_0^\pi \; , \tag{1.141}$$

where for our variational problem to give an extremum we now need this term to vanish, and hence we can have either

$$\delta X^\mu\Big|_{\sigma_0} = 0 \; , \tag{1.142}$$

which are called Dirichlet boundary conditions or

$$\partial^1 X^\mu\Big|_{\sigma_0} = 0 \; , \tag{1.143}$$

which are referred to as Neumann boundary conditions. Note in these expressions $\sigma_0 \in \{0, \pi\}$. We can select a combination of these conditions[59], and we can see that selecting Neumann boundary conditions doesn't fix the spacetime position of the endpoint of the string whilst choosing Dirichlet boundary conditions does. If we consider a system with the following boundary conditions

$$\partial^1 X^a = 0 \;\; a \in \{0, \dots, p\} \quad , \quad X^I = C^I \;\; I \in \{p+1, \dots, d-1\} \; , \tag{1.144}$$

where C^I are constants and we assume these boundary conditions hold at one of the string endpoints, the conditions at the other endpoint may be different. So the string is free to move in $p+1$ directions and is fixed in the rest of them. This may seem a little odd as the string endpoint is now firmly rooted in some directions but free in the others. This may feel like a very strange setup, and indeed this was the feeling for a long time. This led to such boundary conditions being mostly overlooked. However it was realised that such conditions led to objects that could carry charge under the Ramond-Ramond forms occurring in superstring theories [84]. This led to a flurry of interest in the area and it was realised that the directions the string end could move in were in fact objects in their right which became known as D-branes. By considering the beta functions for a string with such boundary conditions it is possible to arrive at an effective action for a Dp-brane which, in string frame variables, is given by

$$S_{Dp} = -T_p \int d^{p+1}\xi \; e^{-\phi} \sqrt{-det(g_{ab} + \mathcal{F}_{ab})} + T_p \int_{W_{p+1}} \sum_n e^{\mathcal{F}_{[2]}} \wedge \varphi^*(A_{[n]}) \; . \tag{1.145}$$

with

$$g_{ab} = \partial_a X^M \partial_b X^N g_{MN}, \;\; \mathcal{F}_{ab} = 2\pi\alpha' F_{ab} + B_{ab}, \;\; B_{ab} = \partial_a X^M \partial_b X^N B_{MN} \; , \tag{1.146}$$

where T_p is the brane tension, g_{MN} is the spacetime metric, B_{MN} is the spacetime NS-NS 2-form and ϕ is the spacetime dilaton with g_{ab} and B_{ab} being the pullbacks of the first two of these fields to the worldvolume by the embedding function $\varphi : W_{p+1} \to M$ which has coordinate functions X^M and ξ^a, for $a \in \{0, \dots, p\}$, are coordinates on the brane[60]. $F_{[2]} = dA_{[1]}$ is a worldvolume gauge field, under which the string endpoint can carry charge. The $A_{[n]}$ are then forms that appear in the superstring theory and the sum only receives contributions from terms that give $p+1$-forms. This action shows us that within the context of string theory branes develop their own dynamics, a fact we've seen in Supergravity due to the sourcing of the M2 brane by the supermembrane. This leads us to the conclusion that branes can support their own worldvolume theories and so actually have a life far beyond just being solutions of Supergravity theories.

[59] We assume the timelike coordinate X^0 obeys Neumann boundary conditions. If we relax this assumption then we include the option of having a D-instanton [54].

[60] Here M is a supermanifold as we're really dealing with superstrings at this point, but we omit all fermionic contributions.

The fact that an open string, which is really a two dimensional brane, can end on a D-brane actually alludes to another phenomena in Supergravity and String Theory. That is the fact that branes can end on other branes. However we can't just have any brane ending on any other brane, there are rules for what can happen. These go under the name of brane surgery rules, who says physicists don't have a sense of humour, and can be seen to be related to the appearance of Chern-Simons terms in Supergravity theories [97]. The possibility for more complicated brane configurations is also captured in so called intersecting brane solutions which are reviewed in [49]. This brings our introduction to branes to a close, sadly we've only managed to touch on the very basics of this fascinating subject, the interested reader is recommend to look at any of the references given in this Section if they want more details or other presentations on the subject.

1.5 Kaluza-Klein theory

We've already discussed the basic idea of Kaluza-Klein theory as a mechanism for obtaining lower dimensional theories from higher dimensional ones. So rather than talking in the hypothetical lets jump straight into an example. Consider a scalar field, Φ, propagating in $\mathbb{R}^{1,d}$ and assume its dynamics are given by

$$S_{d+1}[\Phi] = \int_{M_{d+1}} d^{d+1}X\Big(-\frac{1}{2}\partial^M\Phi\partial^N\Phi\eta_{MN} + \frac{g}{3!}\Phi^3\Big) \ , \tag{1.147}$$

where $M \in \{0, \ldots, d\}$ and we've decided that our field is a massless interacting scalar with equation of motion

$$\Box_{d+1}\Phi + \frac{g}{2}\Phi^2 = 0 \ , \tag{1.148}$$

with $\Box_{d+1} = \eta_{MN}\partial^M\partial^N$.

If we now consider the case where we take

$$M_{d+1} = \mathbb{R}^{1,d-1} \times I, \quad ds^2_{d+1} = \eta_{\mu\nu}dx^\mu dx^\nu + dz^2 \ , \tag{1.149}$$

where $X^M = (x^\mu, z)$ for x^μ, $\mu \in \{0, \ldots, d-1\}$, coordinates on $\mathbb{R}^{1,d-1}$ and z a coordinate on I. Lets now compactify the z coordinate by imposing the periodicity condition

$$z \sim z + 2\pi R \ , \tag{1.150}$$

so our manifold becomes $M_{d+1} = \mathbb{R}^{1,d-1} \times S^1$ with the S^1 having, finite, radius R. The structure of the space suggests that we propose our scalar field obey the periodicity condition

$$\Phi(x,z) = \Phi(x, z + 2\pi R) \ . \tag{1.151}$$

At this stage we'd usually just jump into expressing Φ as a Fourier series, but lets stop and run through the analysis in full as we'll see it many more times throughout this thesis. In performing any expansion of our fields we're looking for the transverse dependence they're permitted to have. This dependence usually results from considering eigenfunctions of a particular differential operator in the problem subject to a set of boundary conditions. We have the boundary condition but what is the operator? Well in this simple example the only operator we have is $\Box_{d+1} = \eta_{\mu\nu}\partial^\mu\partial^\nu + \partial_z^2$. So the obvious choice is to take ∂_z^2 and consider its eigenfunctions

$$\frac{d^2}{dz^2}\psi(z,\lambda) = \lambda\psi(z,\lambda), \ \psi(z,\lambda) = \psi(z + 2\pi R, \lambda) \ , \tag{1.152}$$

where the 'boundary' condition of the problem is inherited from (1.148). If we just aim to solve the equation, rather than imposing this condition for now, we find the following eigenfunctions

$$\begin{aligned} &\psi(z,0) = az + b \ , && \\ &\psi(z,\lambda) = ae^{\sqrt{\lambda}z} + be^{\sqrt{\lambda}z} && \lambda > 0 \ , \\ &\psi(z,\lambda) = ae^{i\sqrt{|\lambda|}z} + be^{i\sqrt{|\lambda|}z} && \lambda < 0 \ , \end{aligned} \tag{1.153}$$

where $a, b \in \mathbb{R}$. So it appears that we have 3 types of behaviour, however we now have to impose our 'boundary' condition. Doing this we find that

$$\begin{aligned} &az + b = a(z + 2\pi R) + b \implies a = 0 \ , \\ &ae^{\sqrt{\lambda}z} + be^{\sqrt{\lambda}z} = ae^{\sqrt{\lambda}z}e^{2\pi\sqrt{\lambda}R} + be^{\sqrt{\lambda}z}e^{-2\pi\sqrt{\lambda}R} \implies a = b = 0 \ \lambda > 0 \ , \\ &ae^{i\sqrt{|\lambda|}z} + be^{i\sqrt{|\lambda|}z} = ae^{i\sqrt{|\lambda|}z}e^{2\pi i\sqrt{|\lambda|}R} + be^{i\sqrt{|\lambda|}z}e^{-2\pi i\sqrt{|\lambda|}R} \\ &\implies e^{2\pi i\sqrt{|\lambda|}R} = 1 \implies \sqrt{|\lambda|} = \frac{n}{R} \ n \in \mathbb{N}_0 \setminus \{0\} \ \lambda > 0 \ . \end{aligned} \tag{1.154}$$

We can see that the imposition of the periodicity condition removes certain transverse dependence, for example the zero mode is forced to have constant transverse dependence, the eigenfunctions that behave as exponentials are completely removed and the oscillatory solutions have a quantisation condition imposed on the allowed eigenvalues. We note that the condition on the eigenvalues of our oscillatory modes is precisely the condition expected for a Fourier series. We can now expand[61] our field Φ using the admissible eigenfunctions

$$\Phi(x,z) = \phi(x,0) + \sum_{n\in\mathbb{N}\setminus\{0\}} \left(\phi(x,n)e^{\frac{in}{R}z} + \tilde{\phi}(x,n)e^{-\frac{in}{R}z}\right) = \sum_{n\in\mathbb{Z}} \phi(x,n)e^{\frac{in}{R}z} , \tag{1.155}$$

with $\phi(x,-n) = \tilde{\phi}(x,n)$ for $n > 0$. We often call the $\phi(x,n)$ the Kaluza-Klein modes of the expansion. So we've managed to arrive at a Fourier series for our field, which now seems to have been expressed in terms of an infinite set of d-dimensional scalar fields $\phi(x,n)$. We can also see that had we imposed different boundary conditions then different eigenfunctions from (1.153) may have been admissible and our expansion would have looked rather different.

Having now expanded our field Φ we can attempt to determine the dynamics of our theory by using (1.148) and (1.155) to obtain

$$\sum_{\tilde{n}\in\mathbb{Z}} e^{i\frac{\tilde{n}}{R}z}\left(\Box_d\phi(x,\tilde{n}) - \left(\frac{\tilde{n}}{R}\right)^2\phi(x,\tilde{n})\right) + \frac{g}{2}\sum_{\tilde{n},m\in\mathbb{Z}} e^{i\frac{\tilde{n}+m}{R}z}\phi(x,\tilde{n})\phi(x,m) = 0 , \tag{1.156}$$

with $\Box_d = \eta_{\mu\nu}\partial^\mu\partial^\nu$. In order to obtain equations of motion for our fields $\phi(x,n)$ we need to project this equation on to a single mode. Within the context of the circle the basis of functions we use for expansions are $\{e^{i\frac{n}{R}z}|n\in\mathbb{Z}\}$ and to go further in constructing a lower dimensional theory we'll need to evaluate integrals of these functions. In the current context the integrals we require are

$$\begin{aligned}\int_0^{2\pi R} dz\ e^{i\frac{n}{R}z}e^{i\frac{m}{R}z} &= 2\pi R\delta_{n+m,0} , \\ \int_0^{2\pi R} dz\ e^{i\frac{n}{R}z}e^{i\frac{m}{R}z}e^{i\frac{p}{R}z} &= 2\pi R\delta_{n+m+p,0} .\end{aligned} \tag{1.157}$$

The first of these integrals tells us our eigenfunctions are orthogonal and normalisable and is required to understand the linear part of the lower dimensional theory. On the other hand the second is a triple integral and, as we'll see, it determines the types of interactions present in our lower dimensional theory. Crucially the structure of these overlap integrals of the transverse eigenfunctions determines the structure of the lower dimensional theory and the interactions it contains. To see this we now apply the operator $\int_0^{2\pi R} dze^{-i\frac{n}{R}z}$ to

[61] Since Φ is a real scalar we must have $\bar{\phi}(x,n) = \tilde{\phi}(x,n)$ where $\bar{\phi}$ denotes complex conjugation of ϕ.

(1.156) which leads to the equations[62]

$$\left(\Box_d - \left(\frac{n}{R}\right)^2\right)\phi(x,n) + \frac{g}{2}\sum_{m\in\mathbb{Z}}\phi(x,n-m)\phi(x,m) = 0\ ,\ n\in\mathbb{Z}\ . \tag{1.158}$$

We see that this leads to massive lower dimensional scalars, except in the case of the zero mode, with various interactions between these fields. So we've managed to rephrase our d+1 dimensional theory in terms of d dimensional fields and it is in this sense that we have a lower dimensional theory.

However from a solution point of view we've just traded a PDE in d+1 dimensions in for a discrete infinity of PDEs in d dimensions. This doesn't look any easier to solve, however it would be if we could set some of the $\phi(x,n)$ to zero. Whilst this wouldn't lead to the most general solution of (1.148) it would lead to a solution and this solution would be easier to obtain. However, we can see that our interactions lead to sources for fields. So if we try to set a field to zero, initially, but it has a non-zero source then the equation of motion of that field means it can't be consistently set to zero. One choice is to consider a system where

$$\phi(x,0)\neq 0,\ \phi(x,n) = 0,\ n\in\mathbb{Z} \tag{1.159}$$

but is this consistent? If we look at the equation of a massive mode, $\phi(x,n)$, then only sources involving the massless mode, $\phi(x,0)$, are dangerous. Our equations become

$$\left(\Box_d - \left(\frac{n}{R}\right)^2\right)\phi(x,n) + g\phi(x,n)\phi(x,0) = 0\ ,\ n\in\mathbb{Z}\setminus\{0\}\ , \tag{1.160}$$

from which we can see the potentially dangerous source term isn't an issue. This is because every time there is a zero mode it's accompanied by a massive mode and so the source term is zero. So if we perform the truncation of modes given in (1.159) then the only non-trivial equation arising from (1.158) is

$$\Box_d\phi(x,0) + \frac{g}{2}\phi(x,0)^2 = 0\ . \tag{1.161}$$

If we solve this then we have a solution of the equation (1.148) where we take Φ to be defined by (1.155) and use our truncated solution to the lower dimensional equation. Such a truncation of the Kaluza-Klein modes that leads to a sensible system of lower dimensional equations, where a solution of this truncated system leads to a solution of the full system, is called a consistent truncation[63]. Consistent truncations are

[62] In reality we are using the fact that we can write (1.156) as a new Fourier series $\sum_{n\in\mathbb{Z}} f(x,n)e^{i\frac{n}{R}z}$ and then we're extracting the components $f(x,n)$.

[63] Consistent truncations don't have to be between a higher and lower dimensional theory but can be between two theories in the same dimension where one has some of the fields present in the other set to specific values.

mathematically very appealing as they make life easier. However consider the following proposed truncation for our system

$$\phi(x,2n)\neq 0,\ \phi(x,2n+1)=0,\ n\in\mathbb{Z}\ , \tag{1.162}$$

then the equations of motion of the odd modes read

$$\begin{aligned}\Big(\Box_d-\Big(\frac{2n+1}{R}\Big)^2\Big)\phi(x,2n+1)+\frac{g}{2}\sum_{m\in\mathbb{Z}}\phi(x,2n+1-(2m+1))\phi(x,2m+1)\\ +\frac{g}{2}\sum_{m\in\mathbb{Z}}\phi(x,2n+1-2m)\phi(x,2m)=0,\ n\in\mathbb{Z}\ .\end{aligned} \tag{1.163}$$

We can see that the source terms always contain an odd term and hence will evaluate to zero. So this truncation is mathematically consistent. However on physical grounds we're exciting modes that are much heavier than say $\phi(x,1)$ and this isn't sensible so the truncation is physically inconsistent. We can see that our previous truncation (1.159) is physically consistent so long as we are at energies $E << R^{-2}$ otherwise we risk exciting higher Kaluza-Klein modes and the system would then no longer be physically consistent.

Interestingly if we perform the truncation (1.159) then the equation (1.161) can be obtained from the action

$$S_d[\phi(x,0)]=\int d^dx\Big(-\frac{1}{2}\eta_{\mu\nu}\partial^\mu\phi(x,0)\partial^\nu\phi(x,0)+\frac{g}{3!}\phi(x,0)^3\Big)\ . \tag{1.164}$$

We notice that if we were to use (1.155) in the action (1.147) and perform the truncation (1.159) we'd receive the action

$$S[\phi(x,0)]=2\pi R\int d^dx\Big(-\frac{1}{2}\eta_{\mu\nu}\partial^\mu\phi(x,0)\partial^\nu\phi(x,0)+\frac{g}{3!}\phi(x,0)^3\Big)\ , \tag{1.165}$$

where we have performed the z integral. This action leads to the same equations of motion as we obtained by direct substitution in to (1.148). This may cause use to conjecture that it doesn't matter if we substitute a consistent Kaluza-Klein ansatz into the higher dimensional equations of motion or the action as in the end we'll arrive at the same system of equations. This was long though to be the case, however it was shown that it isn't actually true [68] and so when we consider if a truncation is consistent it is a statement that must be made at the level of the equations of motion.

There is a big industry for finding consistent truncations within theories, however, also of interest is the question of why are some truncations consistent but others aren't? This is an extremely hard question to answer, however in some cases it is possible to understand why a truncation is consistent. The simplest example is that of a circle reduction. We can see from the sources in (1.158) that if we were to add the mode numbers of the two fields in the source for the $\phi(x,n)$ field then we'd always have $n-m+m=n$, so the source has to have the same overall mode number as the field it sources. This looks curious, is it just a coincidence or is there more to it? Lets add a term $\frac{\lambda}{4!}\Phi^4$ to (1.147) which would lead to a $\frac{\lambda}{3!}\Phi^3$ term in (1.148). If we then make use of the overlap integral

$$\int_0^{2\pi R} dz\; e^{i\frac{n}{R}z}e^{i\frac{m}{R}z}e^{i\frac{p}{R}z}e^{i\frac{q}{R}z} = 2\pi R\delta_{n+m+p+q,0}\ , \tag{1.166}$$

we see that (1.158) gets augmented by a term

$$+\frac{\lambda}{3!}\sum_{m,p\in\mathbb{Z}}\phi(x,n-m-p)\phi(x,m)\phi(x,p)\ , \tag{1.167}$$

from which we can see that sum of mode numbers is $n-m-p+m+p=n$. This can't just be a coincidence, and it isn't. We seem to be seeing some sort of 'charge' conservation law. Under this only sources carrying the same charge as the field they source can occur. We might now ask what is the origin of this charge? At this point we recall that S^1 can be given a group structure under which it becomes the Lie group $U(1)$. We then recognise our eigenfunctions $\{e^{i\frac{n}{R}z}|n\in\mathbb{Z}\}$ as representations of this group. The zero mode is the trivial representation, whilst $e^{i\frac{n}{R}z}$ and $e^{-i\frac{n}{R}z}$ are dual representations of the group, in the sense that their product is in the trivial rep. So when we perform our truncation, (1.159), we're restricting to the trivial representation of the underlying group. Then no matter how many trivial representations we tensor product together we can't produce a non-trivial representation. This manifests in the fact that any source for a heavy mode that contains a massless field must contain some number of heavy fields to balance the associated charges and, overall, give the charge of the non-trivial representation it's sourcing. As such, in the case where we restrict to the trivial representations of a group, we're guaranteed a consistent truncation! Interestingly this piece of group theory also explains why we received the same lower dimensional system if we put our truncation into the action or the field equations. This very simple example shows the crucial role group theory plays in our understanding of the consistency of truncations. We'll return to this issue, in a more complicated setting, a little later on.

Whilst consistent truncations are nice they're also rare. As such we have to develop methods to deal with cases where no consistent truncations are available to us. In order to understand this lets concoct an example. We'll take a case where we have a single massless scalar, $\phi(x,0)$, and a tower of massive scalars,

$\phi(x,n)$ $n \in \mathbb{N}_0 \setminus \{0\}$. Lets assume they obey the equations

$$\Box_d \phi(x,0) + \frac{g}{2}\phi(x,0)^2 = \sum_{n \in \mathbb{N}_0 \setminus \{0\}} \lambda(n)\phi(x,n)^2 \ ,$$
$$(\Box_d - m^2(n))\phi(x,n) = g(n)\phi(x,0)^2, \ n \in \mathbb{N}_0 \setminus \{0\} \ , \tag{1.168}$$

with $m^2(n)$ the mass of the heavy fields, with $m^2(n) < m^2(\tilde{n})$ if $n < \tilde{n}$. We can clearly see that we can't perform a truncation like (1.161) in this case due to the source for the heavy modes. If we were to set the heavy fields to zero, but not set the massless field to zero, then we'd run into inconsistencies in the equations. As such we call such a truncation an inconsistent truncation. If we were to deal with this theory in a setting in which the masses of the massive fields were much greater than the energies we wished to probe could we do anything? Ideally we'd like to be able to deal with just the mode $\phi(x,0)$, but we know we can't just set the massive modes to zero. So is there a way to write our system just in terms of $\phi(x,0)$, but taking in to account the effects of the $\phi(x,n)$? The procedure we employ is called, classically, integrating out the massive fields. It involves us solving the heavy field equations, for the heavy fields, and then substituting these solutions wherever the heavy field occurs in other equations. The procedure is more easily understood by seeing it in action. The equation we want to solve is

$$(\Box_d - m^2(n))\phi(x,n) = g(n)\phi(x,0)^2, \ n \in \mathbb{N}_0 \setminus \{0\} \ ,$$

if we could find a Green's function for $\Box_d - m^2(n)$ then we could invert the operator and would have solved these equations for the massive fields. Lets denote this Green's function by $(\Box_d - m^2(n))^{-1}$ then

$$\phi(x,n) = g(n)(\Box_d - m^2(n))^{-1}(\phi(x,0)^2), \ n \in \mathbb{N}_0 \setminus \{0\} \ , \tag{1.169}$$

this is now the solution for our massive field, given the specific massless field source. We can now use this in the massless field equation to obtain

$$\Box_d \phi(x,0) + \frac{g}{2}\phi(x,0)^2 - \sum_{n \in \mathbb{N}_0 \setminus \{0\}} \lambda(n)\Big(g(n)(\Box_d - m^2(n))^{-1}(\phi(x,0)^2)\Big)^2 = 0 \ . \tag{1.170}$$

So the integrating out procedure has produced a whole host of new interactions, which are non-local due to the nature of Green's functions. We can now solve this equation, or seek an action principle from which it can be derived. This gives us a description of the system, in terms of just the massless field, but with all of the effects of the heavy fields appropriately accounted for. This ensures our system isn't going to lead to inconsistencies and has allowed us to develop an EFT for the massless mode.

As interesting as scalars are we need to go beyond them. Our main aim will be to look at Supergravity theories and how Kaluza-Klein methods can be applied to them. However, to begin with, let's just look at a gravitational theory. We've discussed such theories in some detail and hence we know we want to use the Einstein-Hilbert action

$$S_{EH}[\hat{g}] = \frac{1}{2\kappa_{d+1}^2} \int_{M_{d+1}} \hat{R} \hat{*}_{d+1} 1 \ ,$$

with a hat put on all d+1 dimensional quantities. We now wish to propose that, topologically, $M_{d+1} = M_d \times S^1$ and truncate to the zero mode sector of the S^1 Fourier expansion, which we know to be a consistent truncation based on the group theoretic arguments given earlier. Truncating to the zero Fourier mode means we loose any dependence on the circle coordinate, but we still need to workout how to parametrise the reduction ansatz. This involves working out what fields we will have in the lower dimensional theory. This can be done by group theoretic means by looking at how representations of the appropriate little group branch upon the reduction and then using character methods. However an easier method is to split the index $M = (\mu, z)$ with $M \in \{0, \ldots, d\}$, $\mu \in \{0, \ldots, d-1\}$ and z the circle direction. If we now consider $\hat{g}_{MN}$ and split it based upon this index splitting then we obtain $g_{\mu\nu}, g_{\mu z}, g_{zz}$ which seems to tell us our lower dimensional theory should contain a d-dimensional metric, $g_{\mu\nu}$, a vector, from $g_{\mu z}$, and a scalar, from g_{zz}. This, in fact, is the correct content, as can be confirmed by the more formal group theoretic methods, however if we were just to propose the following reduction ansatz for the metric

$$d\hat{s}_{d+1}^2 = g_{\mu\nu}(x) dx^\mu dx^\nu + 2A_\mu(x) dx^\mu dz + \Phi dz^2 \ , \tag{1.171}$$

we'd end up with a very odd looking lower dimensional theory whose dynamics we'd find very hard to interpret. We could try to do field redefinitions to put this system in a more recognisable form, but wouldn't it be nice if we didn't have to? What if we could set up a dimensional reduction ansatz in a way that the lower dimensional theory is one we already recognise? This, along with consistency considerations, is the real art of Kaluza-Klein methods.

In light of the previous comment we might ask can we do better? While we could we've been beaten to the punchline by Kaluza. It turns out that a more preferable reduction ansatz is

$$d\hat{s}_{d+1}^2 = e^{2\alpha\phi(x)} g_{\mu\nu}(x) dx^\mu dx^\nu + e^{2\beta\phi(x)} (dz + \mathcal{A}_\mu(x) dx^\mu)^2 \ , \tag{1.172}$$

with $\alpha, \beta \in \mathbb{R} \setminus \{0\}$. These constants allow us to tune the ansatz, somewhat, as we go along and will become parameters of the lower dimensional theory that we can use to establish certain structures within it. We note that this means that

$$\hat{g}_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} \mathcal{A}_\mu \mathcal{A}_\nu, \;\; \hat{g}_{\mu z} = e^{2\beta\phi} \mathcal{A}_\mu, \;\; \hat{g}_{zz} = e^{2\beta\phi} \; , \tag{1.173}$$

which can be compared to (1.171) and we can see the difference is that one is linear in the fields while the other is non-linear. It may seem odd that we choose to work with something non-linear over something linear, but as we'll see below the non-linear ansatz leads to a very simple system.

In order to proceed we need to calculate the higher dimensional curvature tensors $\hat{R}_{MN}$ and $\hat{R}$ for the ansatz (1.172). This is most simply performed by employing vielbein methods, which are discussed in Appendix B. Since we're working with a circle case we know that we can work at the level of the action or equations of motion and these will produce the same results. Since the Einstein equation involves both $\hat{R}_{MN}$ and $\hat{R}$ but the action only involves $\hat{R}$ we choose to work at the level of the action. If we do this then we find that we obtain, up to a total derivative, the action

$$S_{EH} = \frac{1}{2\kappa_{d+1}^2} \int_{M_d \times S^1} \Big(e^{(\beta+(d-2)\alpha)\phi} \big(R *_d 1 - \alpha^2 (d-1)(d-2) d\phi \wedge *_d d\phi \big) - \frac{1}{2} e^{(\beta - d\alpha)\phi} \mathcal{F}_{[2]} \wedge *_d \mathcal{F}_{[2]} \Big) \wedge dz \; , \tag{1.174}$$

here R and $*_d$ are associated to the d-dimensional metric $g_{\mu\nu}$, d is the d-dimensional exterior derivative and $\mathcal{F}_{[2]} = d\mathcal{A}_{[1]}$. We can now see how useful α and β are. If we set them to the values

$$\beta = -(d-2)\alpha, \quad \alpha^2 = \frac{1}{2(d-1)(d-2)} \; , \tag{1.175}$$

then this puts our action into Einstein frame variables and canonically normalises the scalar. If we also perform the circle integral we arrive at the d-dimensional action

$$S_d[g, \mathcal{A}, \phi] = \frac{2\pi\tilde{R}}{2\kappa_{d+1}^2} \int_{M_d} \Big(R *_d 1 - \frac{1}{2} d\phi \wedge *_d d\phi - \frac{1}{2} e^{-2(d-1)\alpha\phi} \mathcal{F}_{[2]} \wedge *_d \mathcal{F}_{[2]} \Big) \; , \tag{1.176}$$

which can be seen to be an Einstein-Maxwell-scalar system with Newton's constant $\kappa_d^2 = \frac{\kappa_{d+1}^2}{2\pi\tilde{R}}$, where $\tilde{R}$ is the radius of the S^1. This result is incredible, we've managed to derive a lower dimensional theory that includes both electromagnetism and gravity from a theory of just pure gravity by a choice of geometry for our higher dimensional manifold. We'd love to be able to set the scalar to zero, as then we'd have a true Einstein-Maxwell theory. However this is only consistent, at the level of the e.o.m, if we also set $\mathcal{F}_{[2]} = 0$, which defeats the point of the truncation. It's hard to emphasise how interesting this result is, we've managed to unify gravity and electromagnetism as a purely geometric theory in one higher dimension and we got all of this by just curling up one dimension[64]!

So far we've taken for granted that $g_{\mu\nu}$ is a metric, $\mathcal{A}_\mu$ is a vector and ϕ is a scalar. However, we know these names are really derived from the transformation properties of the fields, so we need to confirm that our lower dimensional fields transform correctly. Our original d+1-dimensional theory has $Diff(M_{d+1})$ invariance so the symmetry group of our lower dimensional theory must embed into this in some way. So let's try to understand how this can happen. To begin with we note an interesting invariance of the dimensional reduction ansatz

$$\phi \to \phi + c\ ,\ \mathcal{A}_{[1]} \to e^{-\beta c}\mathcal{A}_{[1]}\ ,\ g_{\mu\nu} \to e^{-2\alpha c} g_{\mu\nu}\ ,\ z \to e^{-\beta c} z \tag{1.177}$$

where $c \in \mathbb{R}$. This in fact derives from a symmetry of the higher dimensional Einstein equation under which $\hat{g}_{MN} \to k^2 \hat{g}_{MN}$ for a constant non zero k. This is not an invariance at the level of the action, however, it only leads to an overall scaling of the action functional giving the famed trombone symmetry of GR[65]. So overall we have d+1 dimensional diffeomorphism invariance and a trombone symmetry. All of our lower dimensional symmetries have to descend from these. In order to make life tractable we shall work at the infinitesimal level. In this case we find that a general d+1 dimensional transformation is given by

$$\delta \hat{g}_{MN} = \hat{\xi}^P \partial_P \hat{g}_{MN} + \hat{g}_{PN} \partial_M \xi^P + \hat{g}_{MP} \partial_N \hat{\xi}^P + 2a \hat{g}_{MN}\ , \tag{1.178}$$

with $\hat{\xi}^M$ parametrising the infinitesimal diffeomorphism and a the infinitesimal trombone symmetry parameter. Any d+1 dimensional diffeomorphism that we're permitted to use must preserve the form of the ansatz (1.172) and, as such, we can use (1.173) to obtain

$$\begin{aligned}
\delta \hat{g}_{\mu\nu} &= e^{2\alpha\phi}(2\alpha\delta\phi + \delta g_{\mu\nu}) + e^{2\beta\phi}(2\beta\delta\phi \mathcal{A}_\mu \mathcal{A}_\nu + \delta\mathcal{A}_\mu \mathcal{A}_\nu + \mathcal{A}_\mu \delta\mathcal{A}_\nu)\ , \\
\delta \hat{g}_{\mu z} &= e^{2\beta\phi}(2\beta\delta\phi \mathcal{A}_\mu + \delta\mathcal{A}_\mu)\ , \\
\delta \hat{g}_{zz} &= 2\beta e^{2\beta\phi}\delta\phi\ .
\end{aligned} \tag{1.179}$$

We also know we want $g_{\mu\nu}$ to behave as a d-dimensional metric, $\mathcal{A}_\mu$ to be a d-dimensional vector and ϕ to be a scalar. Hence under a combination of an infinitesimal d dimensional diffeomorphism, a shift and $U(1)$

[64] So imagine what we could do if we used more exotic spaces.
[65] This symmetry also extends to eleven dimensional Supergravity.

gauge transformation with parameters $\xi^\mu(x)$, c and $\lambda(x)$, respectively, our fields transform as

$$\begin{aligned}
\delta g_{\mu\nu} &= \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu \xi^\rho + g_{\mu\rho}\partial_\nu \xi^\rho - 2\alpha c g_{\mu\nu} \ , \\
\delta \mathcal{A}_\mu &= \xi^\nu \partial_\nu \mathcal{A}_\mu + \mathcal{A}_\nu \partial_\mu \xi^\nu - \beta c \mathcal{A}_\mu + \partial_\mu \lambda \ , \\
\delta\phi &= \xi^\mu \partial_\mu \phi + c \ .
\end{aligned} \tag{1.180}$$

If we now compare (1.180) and (1.179) to (1.178) then we find, since these relationships hold for any value of the fields, that

$$\hat{\xi}^\mu = \xi^\mu(x) \quad , \quad \hat{\xi}^z = c\beta z + \lambda(x) \ , \tag{1.181}$$

which shows us, at the infinitesimal level, how to embed our lower dimensional symmetries into our higher dimensional ones. It also ensures that our interpretation of the lower dimensional fields is correct.

We're making progress as we can now deal with scalars and metrics but, as we know, Supergravity theories also contain other types of fields. Principle among these are p-form fields and fermionic fields, but since we won't require fermionic fields in this thesis we shall omit them in our treatment. However, this still leaves form fields to deal with, so let's plough on with them. Let's look at the case of a form field $\hat{A}_{[n-1]}$ whose dynamics are encoded by the d+1-dimensional action

$$S_{form}[\hat{g}, \hat{A}] = \int_{M_{d+1}} -\frac{1}{2}\hat{F}_{[n]} \wedge \hat{*}_{d+1}\hat{F}_{[n]} \ , \tag{1.182}$$

with $\hat{F}_{[n]} = d\hat{A}_{[n-1]}$. The obvious ansatz for such a form is

$$\hat{A}_{[n-1]} = A_{[n-1]} + A_{[n-2]} \wedge dz \ , \tag{1.183}$$

where $A_{[n-1]}$ and $A[n-2]$ are forms on M_d. From this we find that

$$\hat{F}_{[n]} = dA_{[n-1]} + dA_{[n-2]} \wedge dz \ ,$$

from which we may want to conclude that we get the lower dimensional field strengths

$$F_{[n]} = dA_{[n-1]}, \qquad F_{[n-1]} = dA_{[n-2]} \ .$$

This would seem sensible, but if one proceeds with the calculation we would find it leads us in to difficulties. This can be understood in the following way. If we look at the metric ansatz, (1.172), we can see dz always appears in the combination $dz + \mathcal{A}_{[1]}$ and so it would suggest we modify our form field strengths[66] to

$$\hat{F}_{[n]} = dA_{[n-1]} - dA_{[n-2]} \wedge \mathcal{A}_{[1]} + dA_{[n-2]} \wedge (dz + \mathcal{A}_{[1]}) \ , \qquad (1.184)$$

which gives rise to the lower dimensional field strengths

$$F_{[n]} = dA_{[n-1]} - dA_{[n-2]} \wedge \mathcal{A}_{[1]}, \qquad F_{[n-1]} = dA_{[n-2]} \ , \qquad (1.185)$$

the second field strength is unchanged, but the first one now has a term proportional to the Kaluza-Klein vector $\mathcal{A}_{[1]}$. This modification is called a transgression term and it has some important consequences for our lower dimensional field strengths. To see this lets recall that $\hat{F}_{[n]}$ obeys the Bianchi identity $d\hat{F}_{[n]} = 0$. We can see that our ansatz (1.184) ensures this holds. However now given (1.185) we can see that the lower dimensional field strengths also obey Bianchi identities

$$dF_{[n]} = (-1)^n F_{[n-1]} \wedge \mathcal{F}_{[2]}, \qquad dF_{[n-1]} = 0 \ , \qquad (1.186)$$

hence we see that the transgression term leads to a modification of the usual Bianchi identity obeyed by an exact form. Having established the ansatz for both the form and metric we can perform the reduction of (1.182) and find that

$$S_{form} = \int_{M_d \times S^1} \Big(-\frac{1}{2} e^{-2(n-1)\alpha\phi} F_{[n]} \wedge *_d F_{[n]} - \frac{1}{2} e^{2(d-n)\alpha\phi} F_{[n-1]} \wedge *_d F_{[n-1]} \Big) \wedge dz \ ,$$

which leads to

$$S[A_{[n-1]}, A_{[n-2]}, \phi, g] = 2\pi \tilde{R} \int_{M_d} -\frac{1}{2} e^{-2(n-1)\alpha\phi} F_{[n]} \wedge *_d F_{[n]} - \frac{1}{2} e^{2(d-n)\alpha\phi} F_{[n-1]} \wedge *_d F_{[n-1]} \ . \qquad (1.187)$$

We are now in a position to apply the results of dimensional reduction to Supergravity. We shall do this by applying the results to eleven dimensional Supergravity. We shall focus on the bosonic sector, but the

[66] The reason for the modification is that a simple vielbein basis for the metric involves a 1-form proportional to $dz + \mathcal{A}_{[1]}$, so it becomes natural to use this combination throughout the ansatz.

fermionic sector can also be reduced and will lead us to the same conclusion. The reduction ansatz we use is

$$
\begin{aligned}
&d\hat{s}_{11}^2 = e^{-\frac{1}{6}\phi} ds_{10}^2 + e^{\frac{4}{3}\phi}(dz + \mathcal{A}_{[1]}) \ , \\
&\hat{A}_{[3]} = A_{[3]} + B_{[2]} \wedge dz, \ \hat{F}_{[4]} = F_{[4]} + H_{[3]} \wedge (dz + \mathcal{A}_{[1]}) \ , \\
&F_{[4]} = dA_{[3]} - H_{[3]} \wedge \mathcal{A}_{[1]}, \ H_{[3]} = dB_{[2]} \ .
\end{aligned} \tag{1.188}
$$

Using this in the bosonic part of (1.82), along with the previously established results (1.176) and (1.187), we obtain the action

$$
\begin{aligned}
S_{IIA}[g, \mathcal{A}_{[1]}, B_{[2]}, A_{[3]}] = \frac{1}{2\kappa_{10}^2} \int_{M_d} & R *_d 1 - \frac{1}{2} d\phi \wedge *_d d\phi - \frac{1}{2} e^{\frac{3}{2}\phi} \mathcal{F}_{[2]} \wedge *_d \mathcal{F}_{[2]} \\
& - \frac{1}{2} e^{\frac{1}{2}\phi} F_{[4]} \wedge *_d F_{[4]} - \frac{1}{2} e^{-\phi} H_{[3]} \wedge *_d H_{[3]} - \frac{1}{2} dA_{[3]} \wedge dA_{[3]} \wedge B_{[2]} \ ,
\end{aligned} \tag{1.189}
$$

which is nothing but the bosonic action of Type IIA Supergravity in ten dimensions. Had we also carried out the fermionic reduction we'd have obtained the fermionic sector of Type IIA Supergravity. This is one of the two maximal, 32 real supercharge, Supergravity theories in ten dimensions, the other being Type IIB Supergravity. The difference between these is, aside from the forms fields they contain, that IIA is a non-chiral theory while IIB is a chiral theory, where chirality refers to the eigenvalue of spinors under Γ_{11} as the minimal spinor in ten dimensions is of Majorana-Weyl type. The fact that this theory has maximal supersymmetry demonstrates the well known fact that compactification on a circle doesn't break supersymmetry. We also note we previously coupled eleven dimensional Supergravity to the supermembrane, so it's natural that we'd ask what happens if we compactify the supermembrane action. This procedure involves wrapping the supermembrane on a circle so the worldvolume takes the topological form $W_3 = W_2 \times S^1$. If this procedure is performed then it is found that the Type IIA superstring action is obtained [34].

The IIA action contains many interesting features but one of the main ones is it contains pieces that look like our model system (1.104). This hints at the fact that our system, if we can consistently truncate some fields, which we can, will contain brane solutions. So can we obtain some of these? We could just find the equations of motion and then use our p-brane ansatz to find these solutions. However we know we've availed ourselves of a consistent truncation of eleven dimensional Supergravity, hence any solution of Type IIA Supergravity will lift to a solution of eleven dimensional Supergravity by using the reduction ansatz (1.188). This procedure of lifting solutions is called dimensional oxidation. However we already have a solution, namely the M2 brane solution, is there any way we can go the other way and see if it arises from a lower dimensional solution? Well we begin by noticing that the solution (1.100) has, spatial, isometry directions along the x^1 and x^2 directions. These are obvious candidates for the circle directions so let's just choose the x^2 direction and relabel it as z. We can now rewrite the M2 brane solution in the very suggestive manner

$$
\begin{aligned}
&d\hat{s}^2_{11} = H^{\frac{1}{12}}\left(H^{-\frac{3}{4}}ds^2(\mathbb{E}^{1,1}) + H^{\frac{1}{4}}ds^2(\mathbb{E}^8)\right) + H^{-\frac{2}{3}}dz^2 \ , \\
&\hat{A}_{[3]} = (\pm H^{-1}dx^0 \wedge dx^1) \wedge dz \ , H(r) = 1 + \frac{k}{r^6} \ ,
\end{aligned}
$$

which can be compared to (1.188) and leads to the identification of

$$
\begin{aligned}
&ds^2_{10} = H^{-\frac{3}{4}}ds^2(\mathbb{E}^{1,1}) + H^{\frac{1}{4}}ds^2(\mathbb{E}^8) \ , \\
&B_{[2]} = \pm H^{-1}dx^0 \wedge dx^1 \ , \ e^{\phi} = H^{-\frac{1}{2}} \ , \ H(r) = 1 + \frac{k}{r^6} \ ,
\end{aligned}
\tag{1.190}
$$

as the only non-zero fields. This is the famous fundamental string solution of Dabholkar et al [24], unsurprisingly the source for it comes from the wrapped Supermembrane. This procedure of dimensionally reducing a theory with a brane action, or a brane solution, along a common spacetime and worldvolume direction is referred to as diagonal dimensional reduction. A similar procedure can be applied to the M5 brane solution to obtain the D4-brane solution of Type IIA Supergravity.

From looking at the forms present in (1.189) we also expect there to be a 2-brane solution. So the question now becomes can we obtain this from the M2 brane solution? Well in order to get a 2-brane we would need to reduce the dimension of spacetime by one yet keep the brane worldvolume the same dimension. This suggests that the circle coordinate should be one of the directions transverse to the brane worldvolume. However, now we run into an issue as our dimensional reduction ansatz has an isometry direction along the circle but our brane solution depends on all of the transverse coordinates through the harmonic function and its dependence on $r = \sqrt{\delta_{mn}y^m y^n}$. We seem to be in a pickle. However, recall, we were able to place multiple M2 branes in a space and still obtain a solution. So how about we select a transverse coordinate, say y^8, and place branes along it. This breaks isotropy in the y^8 direction but it is retained in all other directions of the transverse space. If this is done then our harmonic function becomes

$$
H(y^m) = 1 + \sum_{\alpha} \frac{k_\alpha}{|\mathbf{y} - \mathbf{y}_\alpha|^6} \ . \tag{1.191}
$$

Now taking all of the branes to have the same charge k and taking the limit of infinitesimally separated branes we obtain the following harmonic function

$$\tilde{H}(y^i) = 1 + \int_{-\infty}^{\infty} dy^8 \, \frac{k}{|\mathbf{y} - \mathbf{y}_\alpha|^6} = 1 + \frac{3k\pi}{8\tilde{r}^5} = 1 + \frac{\tilde{k}}{\tilde{r}^5} \ , \tag{1.192}$$

with $\tilde{r}^2 = \sum_{i=1}^{7} y^i y^i$. This system now has an isometry along the y^8 direction owing to the stacking of the branes. If we reduce the solution along this direction in a manner similar to before then we obtain the following

$$\begin{aligned} ds_{10}^2 &= \tilde{H}^{-\frac{15}{24}} ds^2(\mathbb{E}^{1,2}) + \tilde{H}^{\frac{9}{24}} ds^2(\mathbb{E}^7) \ , \\ A_{[3]} &= \pm \tilde{H}^{-1} dx^0 \wedge dx^1 \wedge dx^2 \ , \ e^{\phi} = \tilde{H}^{\frac{1}{4}} \ , \ \tilde{H} = 1 + \frac{\tilde{k}}{\tilde{r}^5} \ , \end{aligned} \tag{1.193}$$

which is the, Einstein frame, D2-brane solution of Type IIA Supergravity, which we have obtained by a process called vertical dimensional reduction [67]. If the same procedure is applied to the M5 brane then we obtain the NS5 brane solution of Type IIA Supergravity.

So far we've just looked at the case of a circle and we've already seen how rich it is. We might now wonder would happen if we reduced on a larger space? The obvious first case is to try T^n, as this is just a sequence of n S^1 reductions. If we were to pursue this direction we'd obtain the maximal Supergravity theories in every decreasing dimension, passing through the notable case of $N = 8$, $d = 4$ Supergravity, which was once hoped to provide a renormalisable generalisation of GR. However we've only just scratched the surface, the circle is just one of an infinite number of possible manifolds we could have picked. What if we'd picked a different splitting of our manifold, say $M_d = M_n \times \mathcal{M}_{d-n}$, with M_n meant to represent our universe and $\mathcal{M}_{d-n}$ the space on which we reduce. This space could be compact or non-compact the choice really is all up to us. The reduction procedure is, hopefully, now fairly obvious. We begin with fields defined on M_d and then we require a method to split them into fields on M_n that are in some way expanded in terms of objects on $\mathcal{M}_{d-n}$. In order to do this we isolate an operator on $\mathcal{M}_{d-n}$ whose eigenfunctions we need to understand, once we've understood the eigenspectrum of this operator, along with which eigenfunctions are admissible for the problem, we can expand our fields in this eigenbasis and then go forth. Schematically this would look like

$$\Phi(x, y) = \sum_{\tilde{n}} \phi(x, \tilde{n}) \xi(y, \tilde{n}) \ , \tag{1.194}$$

where our sum is only formal and tells us to expand over the eigenbasis $\xi(y, \tilde{n})$ in the space $\mathcal{M}_{d-n}$ which has

[67] The terms diagonal and vertical dimensional reduction arise from the way in which these procedures move one in the brane scan of p-branes in Supergravity theories [95]. The rough idea is a diagonal reduction takes a p-brane in d dimensions to a (p-1)-brane in d-1 dimensions, whilst a vertical reduction takes a p-brane in d dimensions to a p-brane in d-1 dimensions.

coordinates y^i. In this case the $\phi(x,\tilde{n})$ are to be interpreted as the dynamical fields on M_n, we've suppressed any index structure such a field may have for ease of presentation. We can then derive appropriate equations for their dynamics by taking the equations of motion obeyed by Φ, inserting the generalised Fourier expansion (1.194) and using an appropriate integral to extract the component we want. This process forces us to tackle the issue of calculating overlap integrals of the transverse eigenfunctions, but for now lets assume this can be achieved. We're now in possession of a system of equations for our $\phi(x,\tilde{n})$, but we've done precisely nothing[68]. The theories involving Φ and $\phi(x,\tilde{n})$ are equivalent and equally difficult to solve, but to get the $\phi(x,\tilde{n})$ theory we had to do a lot more work! The key to the circle case was our ability to perform a consistent truncation of the $\phi(x,\tilde{n})$ fields to a subset of the full complement of fields. Now the problem involving the $\phi(x,\tilde{n})$ has become more tractable and if we can solve it then we know we're guaranteed a solution to the Φ system of equations. Sure, again it is a specific solution, not the general one, but how many general solutions to non-linear PDEs do you know? It's here that we run into an issue as finding consistent truncations of a general system is perilously hard. The key to the S^1 case was to appealing to group theory, however in a general case there won't be any group theory to help us. As such it becomes practically very challenging to deal with a general treatment of reducing theories.

Given the fairly bleak picture painted by the preceding paragraph we'd be forgiven for giving up and going for a coffee. However, as always, there are exceptions which turn out to very interesting. Given the power of group theory we'd be silly to ignore it. So the first class of systems that might be considered are those where $\mathcal{M}_{d-n}$ is a Lie group. Such cases were considered in work by DeWitt [32]. Given a Lie group G on which we reduce, so that $M_d = M_n \times G$, we will expand our fields using eigenfunctions on the Lie group G. Unsurprisingly we can now use group theory arguments to find a consistent truncation of the generalised Fourier expansion. As in the circle case the key is to keep fields on M_n that have transverse dependence in the trivial representations of the G_L, or G_R, action of G and set all other fields on M_n to zero. This may seem unremarkable at first but every good trick involves a slight of hand. In our case we've been focused on the behaviour on the transverse space, but the real magic is what is happening on M_n. When we turn to look at the spectrum on this space we're staring straight at a gauge theory with gauge group G. This is a truly profound result, we've managed to obtain Yang-Mills theory from geometry and, what's more, get it to reside in the same theory as gravity. This is one of the most elegant results in all of theoretical physics and is why Supergravity and String Theory are so attractive to many people. However, before we get ahead of ourselves lets recall if we hope to reproduce the Standard Model we need to obtain a gauge group $SU(3) \times SU(2) \times U(1)$. It's now that our hubris catches up to us, since $\dim(SU(3) \times SU(2) \times U(1)) = 12$

[68] If we assume that all transverse dependence occuring within our equations is in the span of the functions $\{\xi(y,\tilde{n})\}$.

and the maximal dimension we can work with is eleven we seem to have -1 dimensions left over for ourselves to live in, which would be awfully cramped. So whilst reductions on group manifolds are calculationally attractive, they can't yield the universe we live in.

If we could just make the dimensions involved smaller, but somehow keep our non-abelian gauge bosons, we'd be in with a chance of obtaining something like our universe but from a theory in greater than four dimensions. This led to the introduction of what have become known as Pauli reductions[69] into the literature, with the name deriving from the fact that they were first considered by Pauli in unpublished correspondences [96]. These sorts of reductions are performed on spaces $M_d = M_n \times \mathcal{M}_{d-n}$ where $\mathcal{M}_{d-n}$ has isometry group G, often used are the examples of the spheres $S^n = SO(n+1)/SO(n)$. They allow us to have the gauge bosons of the G isometry group of $\mathcal{M}_{d-n}$ and overcome the dimensionality problems we had with group manifold reductions. However, now there isn't, in general, any obvious group theory we can use to help guide us. This in particular leads to issues in trying to find consistent truncations in order to make our lower dimensional theories more tractable [36]. Often we can perform a truncation but we will have to sacrifice some of our precious gauge bosons. As such very few consistent Pauli reductions [70] of theories are known. However, there are several very important examples. These are eleven dimensional Supergravity on S^7 [29] and S^4 [77, 78] and Type IIB Supergravity on S^5 [5]. The original example due to Pauli was of $M_6 = M_4 \times S^2$. It is informative, even though this compactification of gravity is inconsistent, to look at the the form of the metric ansatz which is

$$d\hat{s}_6^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + g_{mn}(y)(dy^m + K^{mI}A^I_{[1]})(dy^n + K^{mJ}A^J_{[1]}) \;, \tag{1.195}$$

with x^μ coordinates on the 4 dimensional space and y^m are coordinates on S^2 which is equipped with the usual round metric g_{mn}. The $A^I_{[1]} = A^I_\mu(x)dx^\mu$ are gauge bosons of $SO(3)$, as can be shown by their transformation properties under a subset of diffeomorphisms, and the K^I are the Killing vectors of S^2. This can be compared to the S^1 case where $y \to z$. We see, aside from the scalars, the ansatz are very similar. With $g_{zz} = 1$ being replaced by g_{mn} and the S^1 Killing vector, which is $\frac{d}{dz}$ thus having $K^z = 1$, being replaced by the components of the non-abelian Killing vectors K^I. The crucial feature that we should note is that the non-abelian gauge bosons come in alongside the Killing vectors of the reduction space. Recently some interesting work in to the consistency of Pauli reductions has been undertaken. A particularly elegant approach to this is a case where S^2 reductions were viewed as arising from S^3 reductions via the Hopf fibration. This makes these cases admissible to group theoretic methods to understand the issue of consistency [3].

[69]Which often provide examples of reductions on coset spaces. These are nicely reviewed in [92].
[70]These are defined to be the cases where we can keep all of the gauge bosons of G.

In this last Section we have been very schematic in our details of reductions on more involved spaces. This has partly been due to the difficulty undertaking such calculations would lead to and also because our aim was to provide a brief glimpse of how rich the subject is. For those that desire more technical details please consult [86].

1.6 Supergravity from String Theory

We have continually said that there is an intimate relationship between Supergravity and String Theory, however, up until now, we have done nothing to substantiate this. As such it's about time that we resolve this. Recall that String Theory is usually described as a curved space two dimensional field theory whose fundamental fields are a set of scalar, X^M, and fermionic, ψ^M, fields. If we ignore the fermions then the scalars X^M act as embedding functions of the worldsheet into a specific manifold, often called the target space. It is possible to quantise such a string if we take a Minkowski target space. In this case we find the massless bosonic spectrum of the theory contains a symmetric tensor g_{MN}, an antisymmetric tensor B_{MN} and a scalar ϕ. These fields can be coupled to the string world sheet to obtain an action for a string propagating in a target space M_d with coordinates $X^M(\sigma)$, a metric g_{MN}, a background 2-form B_{MN} and a scalar ϕ, called the dilaton. This action is given by

$$\begin{aligned}S[\gamma, X] = \frac{1}{4\pi\alpha'}\int d^2\sigma\Big(&\sqrt{-\gamma}\,\left(g_{MN}(X)\gamma^{ij}\partial_i X^M\partial_j X^N + \alpha'\phi(X)R(\gamma)\right)\\ &+ B_{MN}(X)\tilde{\epsilon}^{ij}\partial_i X^M\partial_j X^N\Big)\ ,\end{aligned} \tag{1.196}$$

with γ_{ij} an independent worldvolume metric, with $i \in \{0,1\}$ and α' being related to the string tension, T_2, by $\alpha' = \frac{1}{2\pi T_2}$. From this point of view the fields on M_d appear as couplings on the worldvolume. Crucial to our ability to understand String Theory is the conformal symmetry of the theory. Whilst classically this symmetry is intact; it becomes anomalous at the quantum level and in such a case the theory becomes inconsistent. However the symmetry can be restored at the quantum level by forcing the beta functions of the worldvolume theory to vanish. In order to make progress these beta functions are calculated order by order in α'. The lowest order contributions to these, from the bosonic action (1.196), are given by [11]

$$\begin{aligned}\beta(g_{MN}) &= \alpha'(R_{MN} + 2\nabla_M\nabla_N\phi - \frac{1}{4}H_{MPQ}H_N{}^{PQ}) + \mathcal{O}(\alpha'^2)\ ,\\ \beta(B_{MN}) &= \alpha'(-\frac{1}{2}\nabla^P H_{MNP} + H_{MNP}\nabla^P\phi) + \mathcal{O}(\alpha'^2)\ ,\\ \beta(\phi) &= \frac{d-10}{6} + \alpha'(-\frac{1}{2}\nabla^2\phi + \nabla_M\phi\nabla^M\phi - \frac{1}{24}H_{MNP}H^{MNP}) + \mathcal{O}(\alpha'^2)\ ,\end{aligned} \tag{1.197}$$

with the $\mathcal{O}(\alpha'^0)$ term setting $d = 10$, thus providing a calculation of the string critical dimension. The vanishing of the beta functions at $\mathcal{O}(\alpha')$ places non-trivial equations on the target space fields g_{MN}, B_{MN} and ϕ. It can then be noted that these are precisely the fields arising in the NSNS sector of the Type IIA Supergravity action and that the vanishing of these beta functions is equivalent[71] to a consistent truncation of the string frame equations of motion arising from (1.189). Hence at length scales $l \gg \sqrt{\alpha'}$ String Theory preferentially selects out backgrounds obeying the Supergravity equations to propagate on. This finally shows the relationship between these two theories. As we can see we've ignored terms of $\mathcal{O}(\alpha'^2)$ which would provide corrections to the Supergravity equations. These corrections are of higher derivative type and have been explored in the literature, but we won't dedicate any time to them here.

[71] Terms involving R have been removed from the equations arising directly from the variation by combining the Einstein and dilaton equations.

2 The spin-2 spectrum: a tale of the trace

Having discussed the importance of the spin-2 mode in the construction of a consistent field theoretic formulation of General Relativity, it should come as no shock that the problem of fluctuations around a background geometry has drawn much attention. Of course the problem of perturbing a solution to a set of equations can be performed for any system. However, for us, it is in the context of Supergravity that we shall explore this problem. Of particular interest are backgrounds which take on a warped product structure, since many of the interesting solutions to Supergravity are of this form, for example brane solutions. By studying fluctuations of the metric around a background we are probing the allowed forms of gravitational waves on that background, which has become more popular in recent years. Within the context of a warped product manifold it's often the case that one portion of the spacetime represents our universe and the other portion represents some form of transverse space. This is highly reminiscent of the setup seen in Kaluza-Klein theory. As such, it's obvious to ask what is the allowed behaviour of a graviton on this transverse space. In answering this question we will gain insight in to the allowed spectrum of gravitons around our chosen background.

We may think that this problem requires a different treatment for each of the many Supergravity theories available to us. Obvious differences could arise due to their varying matter content. However it was shown by Bachas and Estes, B&E, building on work in [19], that this is in fact not the case [4]. In their work B&E showed that, under certain assumptions, the graviton spectrum around a class of warped product backgrounds could be computed without specific knowledge of the matter content of the theory. This allowed for the construction of a universal equation governing the allowed transverse dependence of gravitons in a particular background. From this the entire graviton mass spectrum could be determined. Crucial to the argument was the choice that admissible gravitons obeyed a transverse traceless condition. However, as we've seen, at least in Minkowski spacetime, tracelessness is only an on-shell gauge condition, and so it's a little presumptive to use it in deriving the equations of motion. We could just add it to the list of assumptions required for us to obtain a universal equation for the transverse behaviour of gravitons, or we could reinstate the trace and see if it affects the derivation. Here we shall take the latter approach. We shall begin with a review of the construction of the universal transverse equation given by B&E . We then address the role of imposing the tracelessness in the argument by initially assuming the trace to be non-zero and then exploring the consequences of this for the linearised form of the equations of motion. Our presentation may at times seem to be drawn out and calculations may seem to be overly laboured however, as we'll see later in this thesis, we shall be forced to return to the perturbation problem in a more general context. It is hoped that the insight gained by a step by step presentation, in this simpler, context will help orient readers when the more general problem is tackled.

Over the course of this Section we'll explore the consequences of the linearised equations of motion of Supergravity systems for the trace of our proposed perturbations. It will be shown, for a special class of perturbations, that while the Einstein equations may allow for a non-zero trace the linearised matter equations force the trace to be zero.

2.1 Review of the universal equation governing the spin-2 spectrum

We shall begin by reviewing the work of B&E. They considered solutions to the equations of motion arising from a Supergravity theory, in Einstein frame, where the metric, on a manifold M_d, took the form

$$d\hat{s}^2 = e^{2A(y)}\bar{g}_{\mu\nu}(x)dx^\mu dx^\nu + \hat{g}_{ab}(y)dy^a dy^b \ . \tag{2.1}$$

Here $\mu \in \{0, \ldots, 3\}$, $\bar{g}_{\mu\nu}$ is the metric for one of the maximally symmetric spaces AdS_4, dS_4 or Mink$_4$ and y^a, for $a \in \{4, .., d-1\}$, are coordinates on the transverse space.

B&E then considered the, linearly, perturbed metric given by

$$d\hat{s}^2_{perturbed} = e^{2A(y)}(\bar{g}_{\mu\nu}(x) + H_{\mu\nu}(x,y))dx^\mu dx^\nu + \hat{g}_{ab}(y)dy^a dy^b \ . \tag{2.2}$$

This choice of perturbation amounts to the choice that the fluctuations $H_{\mu a}$ and H_{ab} are set to zero and, further to this, that all perturbations of the matter fields are set to zero. The linearised Einstein equations for (2.2) were obtained and a factorisable solution was proposed

$$H_{\mu\nu}(x,y) = h_{\mu\nu}(x,\lambda)\psi(y,\lambda) \ . \tag{2.3}$$

In this context λ arose as a separation constant between the internal and external parts of the problem. $h_{\mu\nu}$ was assumed to obey the world volume equation

$$\bar{\Box}_x h_{\mu\nu} = \lambda h_{\mu\nu} \ , \tag{2.4}$$

where $\bar{\Box}_x$ is the Laplace-Beltrami operator on the unwarped world volume with respect to the Levi-Civita connection of the metric $\bar{g}$. This was imposed as it was the equation that was predicted to arise on the world volume from the perspective of a 4 dimensional observer.

Imposed on $h_{\mu\nu}$ were, what seemed to be, a pair of four dimensional "gauge" choices:

$$\bar{\nabla}^{\mu} h_{\mu\nu} := \bar{g}^{\mu\rho} \bar{\nabla}_{\rho} h_{\mu\nu} = 0 \ , \tag{2.5}$$

$$\bar{g}^{\mu\nu} h_{\mu\nu} = 0 \ . \tag{2.6}$$

With (2.3)-(2.6) the linearised Einstein equations become a second order differential equation for $\psi(y, \lambda)$. B&E showed that (2.2), (2.5) and (2.6) implied that

$$\hat{\nabla}^{M} \delta\hat{g}_{MN} = 0 \ , \tag{2.7}$$

$$\hat{g}^{MN} \delta\hat{g}_{MN} = 0 \ , \tag{2.8}$$

with $\hat{\nabla}$ the Levi-Civita connection associated to the metric $\hat{g}$ given in (2.1) and $\delta\hat{g}_{\mu\nu} = e^{2A(y)} H_{\mu\nu}$, as given in (2.2). We will have to reconsider the forms of (2.4)-(2.8) when we come to consider tracefull perturbations.

The variations of the Riemann and Ricci tensors were then calculated, and (2.7) and (2.8) were applied to simplify the resulting expressions. The Weyl transformation

$$\hat{g}_{MN} = e^{2A(y)} \bar{g}_{MN} \ , \tag{2.9}$$

was performed as in the metric $\bar{g}$ the manifold is a product manifold. As a result all quantities of interest, such as the connection, curvature tensors, ..., decompose in to objects on one of the two sub manifolds of the full geometry. This leads to objects like $\bar{R}_{\mu\nu}$ being mathematically well defined. Then due to maximal symmetry of the external space, in the $\bar{g}$ metric, one has

$$\bar{R}_{\mu\nu\rho\sigma} = k(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\rho\nu}) \ , \tag{2.10}$$

with $k \in \{-1, 0, 1\}$ depending on which maximally symmetric space was present in the background solution (2.1).

The variation of the stress-energy tensor, T_{MN}, was then addressed and it was argued, that at linear order, the variation took the form

$$\kappa^2 \delta\hat{T}_{\mu\nu} = \frac{1}{4} \delta\hat{g}_{\mu\nu} \kappa^2 \hat{T}^{\rho}_{\rho} \ , \tag{2.11}$$

with all the other components vanishing and $\hat{T}^{\rho}_{\rho} = \hat{g}^{\mu\nu}\hat{T}_{\mu\nu}$. Since $\hat{T}$ is the background stress-energy tensor the Einstein equation for the background relates it to expressions involving the curvature of the background. This allowed the analysis to be carried out independent of the matter content and allowed a universal equation for $\psi(y, \lambda)$ to be obtained

$$-[\bar{\Box}_y + (d-2)\bar{g}^{ab}(\partial_a A)\partial_b]\psi = m^2\psi \ . \tag{2.12}$$

A useful rewriting of (2.12), using $\hat{g}$, was also given

$$-\frac{e^{-2A}}{\sqrt{[\hat{g}]}}(\partial_a\sqrt{[\hat{g}]}\hat{g}^{ab}e^{4A}\partial_b)\psi = m^2\psi \ , \tag{2.13}$$

where $[\hat{g}]$ is the determinant of $\hat{g}_{ab}$. Another rewriting of the problem was presented

$$(-\bar{\Box}_y + V(y))\Psi = m^2\Psi \ , \ \Psi := e^{\frac{1}{2}(d-2)A}\psi \ , \ V(y) := e^{-\frac{1}{2}(d-2)A}\bar{\Box}_y e^{\frac{1}{2}(d-2)A} \ , \tag{2.14}$$

which revealed a Schrödinger type interpretation of the transverse wavefunction problem.

The eigenmode equations were supplemented by the selection of the usual norm, from curved space QFT, for a scalar field

$$||\delta\Phi||^2 := i\int_{\Sigma} dS^M(\delta\Phi_-\partial_M\delta\Phi_+ - \delta\Phi_+\partial_M\delta\Phi_-) \ . \tag{2.15}$$

Here Σ is a spacelike hypersurface of codimension 1, dS^M is the volume element on it and $\delta\Phi_\pm$ denote the positive and negative frequency parts of the solution. If one has has a stationary spacetime, so there exists a time like Killing vector, then the form of the volume element can be given. B&E showed that the norm factorised, for separable solutions $\Phi = \phi(x)\psi(y)$, as

$$||\delta\Phi||^2 = ||\phi||_4^2||\psi||^2 \quad , \quad ||\psi||^2 = \int d^{d-4}y\sqrt{[\hat{g}]}e^{2A}|\psi|^2 \ , \tag{2.16}$$

with $||\phi||_4^2$ the norm of $\phi(x)$ in $\bar{M}_4$, which is the four dimensional piece of M_d equipped with the appropriate, unit radius, metric $\bar{g}_{\mu\nu}$. B& E showed the same form of norm can be used for the spin-2 perturbations. Then, by integrating (2.13) over the transverse space, and assuming no boundary terms occur when integration by parts is performed[72], they found that

$$m^2||\phi||^2 = -\int d^{d-4}y\ \psi\partial_a(\sqrt{[\hat{g}]}\hat{g}^{ab}e^{4A}\partial_b)\psi = \int d^{d-4}y\sqrt{[\hat{g}]}e^{4A}|\partial\psi|^2 \tag{2.17}$$

thus showing that $m^2 \geq 0$. They also concluded that for saturation of this bound $\psi(y) = $ constant, since the integrand of (2.17) is positive definite and we're on a Riemannian sub manifold. If this is to give a normalisable mode then (2.16) requires

$$\int d^{d-4}y\sqrt{[\hat{g}]}e^{2A} < \infty\ . \tag{2.18}$$

It will be our aim to replicate the procedure leading to the universal transverse equation but without assuming (2.6).

2.2 The traceful spin-2 spectrum

We now embark on our journey to understand the spin-2 spectrum in the presence of a non-zero trace. In a minor generalisation of the work of B&E we shall work with a worldvolume of dimension n. As such our background metric takes the form given in (2.1) except that now $\mu \in \{0, .., n-1\}$ and $a \in \{n+1, .., d-1\}$. This generalisation is performed to allow for branes of dimension higher than 4 to be involved in the discussion. Our perturbed metric is of the form (2.2) but with μ and a running over these new values, hence we set the possible perturbation fields $H_{\mu a}$ and H_{ab} to zero. The form of the perturbed metric then implies that, to linear order in perturbations, the inverse metric takes on the form

$$\hat{g}^{MN}_{perturbed} = \begin{pmatrix} e^{-2A(y)}(\bar{g}^{\mu\nu} - H^{\mu\nu}) & 0 \\ 0 & \hat{g}^{ab}(y) \end{pmatrix}\ , \tag{2.19}$$

with $H^{\mu\nu} = \bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}H_{\rho\sigma}$. Like B&E we assume that (2.3) holds, and so our perturbation is separable. However we will not assume a world volume equation for $h_{\mu\nu}$ and shall only impose the n dimensional version of the condition (2.5), on the assumption that it will be an admissible off-shell gauge condition[73]. We would now like to find the analogue of (2.7) for our case.

Claim 1.

$$\bar{\nabla}^M H_{MN} = 0\ . \tag{2.20}$$

[72] This provides an obvious chink in the armour of any conclusions derived from this line of thought. This was exploited in [17], a point we'll return to later in this thesis.

[73] We shall have a lot more to say about this in Section 4, but for now keep this, somewhat foreboding, comment in your back pocket.

Proof.

$$\bar{\nabla}^M H_{MN} = \bar{g}^{MP}\bar{\nabla}_P H_{MN} = \bar{g}^{MP}(\partial_P H_{MN} - \bar{\Gamma}^R{}_{MP}H_{RN} - \bar{\Gamma}^R{}_{NP}H_{MR})$$
$$= \bar{g}^{\mu\nu}(\partial_\nu H_{\mu N} - \bar{\Gamma}^\rho{}_{\mu\nu}H_{\rho N} - \bar{\Gamma}^\rho{}_{N\nu}H_{\mu\rho}) \ .$$

Hence

$$\bar{\nabla}^M H_{Ma} = \bar{g}^{\mu\nu}(\partial_\nu H_{\mu a} - \bar{\Gamma}^\rho{}_{\mu\nu}H_{\rho a} - \bar{\Gamma}^\rho{}_{a\nu}H_{\mu\rho}) = 0 \ .$$

Where the fact that in the metric $\bar{g}$ the manifold possesses a product structure and the form of the perturbation, as given in (2.2), have been used.

$$\bar{\nabla}^M H_{M\sigma} = \bar{g}^{\mu\nu}(\partial_\nu H_{\mu\sigma} - \bar{\Gamma}^\rho{}_{\mu\nu}H_{\rho\sigma} - \bar{\Gamma}^\rho{}_{\sigma\nu}H_{\mu\rho})$$
$$= \bar{g}^{\mu\nu}\psi(y;\lambda)(\partial_\nu h_{\mu\sigma} - \bar{\Gamma}^\rho{}_{\mu\nu}h_{\rho\sigma} - \bar{\Gamma}^\rho{}_{\sigma\nu}h_{\mu\rho}) \ ,$$

however, by (2.5), we have that

$$\bar{\nabla}^\mu h_{\mu\nu} = \bar{g}^{\mu\nu}\big(\partial_\nu h_{\mu\sigma} - \bar{\Gamma}^\rho{}_{\mu\nu}h_{\rho\sigma} - \bar{\Gamma}^\rho{}_{\sigma\nu}h_{\mu\rho}\big) = 0 \ ,$$

which leads to

$$\bar{\nabla}^M H_{M\sigma} = \bar{g}^{\mu\nu}\psi(y;\lambda)(\partial_\nu h_{\mu\sigma} - \bar{\Gamma}^\rho{}_{\mu\nu}h_{\rho\sigma} - \bar{\Gamma}^\rho{}_{\sigma\nu}h_{\mu\rho}) = 0 \ .$$

□

We now wish to see what the consequences for (2.7) & (2.8) are

Claim 2.

$$H := \bar{g}^{MN}H_{MN} = \hat{g}^{MN}\delta\hat{g}_{MN} =: \delta\hat{g} \ . \tag{2.21}$$

Proof.

$$\delta\hat{g} = \hat{g}^{MN}\delta\hat{g}_{MN} = (e^{-2A}\bar{g}^{MN})(e^{2A}H_{MN}) = H \ .$$

□

Claim 3.

$$\hat{\nabla}^M \delta\hat{g}_{MN} = -(\hat{\nabla}_N A)\delta\hat{g} \ . \tag{2.22}$$

Proof. One can show that under a Weyl transformation of the form (2.9) the Levi-Civita connection coefficients of the metrics $\bar{g}$ and $\hat{g}$ are related as follows:

$$\hat{\Gamma}^M{}_{NP} = \bar{\Gamma}^M{}_{NP} + \delta^M{}_N \bar{\nabla}_P A + \delta^M{}_P \bar{\nabla}_N A - \bar{g}_{NP} \bar{\nabla}^M A \ . \tag{2.23}$$

Hence

$$\begin{aligned}
\hat{\nabla}_Q \delta\hat{g}_{MN} &= \bar{\nabla}_Q \delta\hat{g}_{MN} - (\delta^R_M \bar{\nabla}_Q A + \delta^R_Q \bar{\nabla}_M A - \bar{g}_{MQ} \bar{\nabla}^R A)\delta\hat{g}_{RN} \\
&\quad - (\delta^R_N \bar{\nabla}_Q A + \delta^R_Q \bar{\nabla}_N A - \bar{g}_{NQ} \bar{\nabla}^R A)\delta\hat{g}_{MR} \\
&= \bar{\nabla}_Q \delta\hat{g}_{MN} - (\bar{\nabla}_Q A)\delta\hat{g}_{MN} - (\bar{\nabla}_M A)\delta\hat{g}_{QN} - (\bar{\nabla}_Q A)\delta\hat{g}_{MN} - (\bar{\nabla}_N A)\delta\hat{g}_{MQ} \\
&= \bar{\nabla}_Q \delta\hat{g}_{MN} - 2(\bar{\nabla}_Q A)\delta\hat{g}_{MN} - (\bar{\nabla}_M A)\delta\hat{g}_{QN} - (\bar{\nabla}_N A)\delta\hat{g}_{MQ} \ .
\end{aligned}$$

Where (2.2) has been used to remove terms of the form $(\bar{\nabla}^R A)\delta\hat{g}_{RM}$. If we expand $\delta\hat{g}_{MN}$, using (2.2), and employ the definition of a covariant derivative then we find that

$$\hat{\nabla}_Q \delta\hat{g}_{MN} = e^{2A(y)}(\bar{\nabla}_Q H_{MN} - (\bar{\nabla}_M A)H_{QN} - (\bar{\nabla}_N A)H_{MQ}) \ . \tag{2.24}$$

Hence applying $\hat{g}^{MQ}$ to this yields

$$\begin{aligned}
\hat{\nabla}^M \delta\hat{g}_{MN} &= \bar{\nabla}^M H_{MN} - (\bar{\nabla}_N A)H \\
&= -(\hat{\nabla}_N A)\delta\hat{g} \ ,
\end{aligned}$$

where (2.2), (2.20) and (2.21) have all been used. □

With the preceding results, we are in a position to calculate the linear variations of the required curvature tensors. Under a perturbation of the form (2.2) the change in the components of the Christoffel symbols can be shown to be:

$$\delta\hat{\Gamma}^M{}_{NP} = \frac{1}{2}\hat{g}^{MQ}(\hat{\nabla}_P \delta\hat{g}_{NQ} + \hat{\nabla}_N \delta\hat{g}_{QP} - \hat{\nabla}_Q \delta\hat{g}_{NP}) \ . \tag{2.25}$$

From which it can be shown that the linear variation of the Riemann tensor is

$$\delta\hat{R}^M{}_{NPQ} = \hat{\nabla}_P(\delta\hat{\Gamma}^M{}_{NQ}) - \hat{\nabla}_Q(\delta\hat{\Gamma}^M{}_{NP}) \ . \tag{2.26}$$

In our conventions the Ricci tensor is defined by $R_{MN} = R^P{}_{MPN}$, hence its linear variation, given (2.26), is

$$\begin{aligned}\delta\hat{R}_{MN} =& \hat{\nabla}_P(\delta\hat{\Gamma}^P{}_{MN}) - \hat{\nabla}_N(\delta\hat{\Gamma}^P{}_{MP}) \\ =& \frac{1}{2}(\hat{\nabla}_P\hat{\nabla}_M\delta\hat{g}^P{}_N + \hat{\nabla}_P\hat{\nabla}_N\delta\hat{g}^P{}_M - \hat{\nabla}_P\hat{\nabla}^P\delta\hat{g}_{MN} - \hat{\nabla}_N\hat{\nabla}_M\delta\hat{g}) \ .\end{aligned} \tag{2.27}$$

If we now use (2.22) this becomes

$$\begin{aligned}\delta\hat{R}_{MN} =& \frac{1}{2}([\hat{\nabla}_P,\hat{\nabla}_M]\delta\hat{g}^P{}_N + [\hat{\nabla}_P,\hat{\nabla}_N]\delta\hat{g}^P{}_M - \hat{\nabla}_M((\hat{\nabla}_N A)\delta\hat{g}) \\ & - \hat{\nabla}_N((\hat{\nabla}_M A)\delta\hat{g}) - \hat{\nabla}_P\hat{\nabla}^P\delta\hat{g}_{MN} - \hat{\nabla}_N\hat{\nabla}_M\delta\hat{g}) \ ,\end{aligned} \tag{2.28}$$

which, using the standard result for curvatures in gauge theories

$$[\hat{\nabla}_Q,\hat{\nabla}_M]\delta\hat{g}^Q{}_N = \hat{R}_{QM}\delta\hat{g}^Q{}_N - \hat{R}^Q{}_{NPM}\delta\hat{g}^P{}_Q \ , \tag{2.29}$$

leads to the expression

$$\begin{aligned}\delta\hat{R}_{MN} = \frac{1}{2}\Big(&\hat{R}_{QM}\delta\hat{g}^Q{}_N - \hat{R}^Q{}_{NPM}\delta\hat{g}^P{}_Q + \hat{R}_{QN}\delta\hat{g}^Q{}_M - \hat{R}^Q{}_{MPN}\delta\hat{g}^P{}_Q \\ & - \hat{\nabla}_M((\hat{\nabla}_N A)\delta\hat{g}) - \hat{\nabla}_N((\hat{\nabla}_M A)\delta\hat{g}) - \hat{\nabla}_P\hat{\nabla}^P\delta\hat{g}_{MN} - \hat{\nabla}_N\hat{\nabla}_M\delta\hat{g}\Big) \ .\end{aligned} \tag{2.30}$$

The final linearised curvature we need is the Ricci scalar, $\hat{R} = \hat{g}^{MN}\hat{R}_{MN}$, which is found to be

$$\delta\hat{R} = -\hat{g}^{MP}\hat{g}^{NQ}\delta\hat{g}_{PQ}\hat{R}_{MN} + \hat{g}^{MN}\delta\hat{R}_{MN} \ . \tag{2.31}$$

Later on in this Section we shall explicitly calculate the expressions obtained above for our particular situation.

From a computational perspective it is often easier to deal with a manifold that is the Cartesian product of two lower dimensional manifolds. This ease derives form the fact that the tangent bundle separates into a direct sum bundle, and as such components of curvature tensors with mixed indices vanish. In our present case this will make our calculations easier to manage and so we shall employ the various results relating conformally equivalent metrics to aid us. Two metrics, g and $\tilde{g}$, are related by a Weyl transformation if there exists a function ϕ such that

$$\tilde{g} = e^{2\phi} g \ . \tag{2.32}$$

Under such a transformation one can show that the Riemann tensors, associated to the Levi-Civita connections of the respective metrics, are related as follows

$$\begin{aligned}\tilde{R}^M{}_{NPQ} =& R^M{}_{NPQ} + 2\delta^M{}_{[Q}\nabla_{P]}\nabla_N\phi - 2g_{N[Q}\nabla_{P]}\nabla^M\phi + 2\delta^M{}_{[P}(\nabla_{Q]}\phi)(\nabla_N\phi) \\ & -2\delta^M{}_{[P}g_{Q]N}(\nabla\phi)^2 - 2g_{N[P}(\nabla_{Q]}\phi)(\nabla^M\phi) \ ,\end{aligned} \tag{2.33}$$

with ∇ the covariant derivative associated the Levi-Civita connection of g. If we contract this to find the Ricci tensor, which can be done without requiring a metric, we obtain

$$\begin{aligned}\tilde{R}_{MN} =& R_{MN} - (d-2)\nabla_M\nabla_N\phi - g_{MN}\nabla^2\phi \\ & +(d-2)(\nabla_M\phi)(\nabla_N\phi) - (d-2)g_{MN}(\nabla\phi)^2 \ ,\end{aligned} \tag{2.34}$$

with d being the dimension of the manifold on which our metrics are defined. Finally contracting, using the appropriate metric, $\tilde{g}$, we find that the Ricci scalar transforms as

$$\tilde{R} = e^{-2\phi}(R - 2(d-1)\nabla^2\phi - (d-2)(d-1)(\nabla\phi)^2) \ . \tag{2.35}$$

With (2.32)-(2.35) we can use (2.9) to rewrite (2.30) and (2.31) in terms of quantities with respect to the product metric, $\bar{g}$, on M_d. We begin by calculating each of the individual terms in (2.30) and then assemble the full results later on.

Claim 4.

$$\hat{\nabla}_M((\hat{\nabla}_N A)\delta\hat{g}) = (\bar{\nabla}_M\bar{\nabla}_N A)H - 2(\partial_M A)(\partial_N A)H + \bar{g}_{NM}(\bar{\nabla}_P A)(\bar{\nabla}^P A)H + (\partial_N A)(\partial_M H) \ . \tag{2.36}$$

Proof.

$$\begin{aligned}\hat{\nabla}_M((\hat{\nabla}_N A)\delta\hat{g}) &= (\hat{\nabla}_M\hat{\nabla}_N A)\delta\hat{g} + (\hat{\nabla}_N A)(\hat{\nabla}_M\delta\hat{g}) \\ &= (\partial_M\partial_N A - \hat{\Gamma}^P_{NM}\partial_P A)\delta\hat{g} + (\hat{\nabla}_N A)(\hat{\nabla}_M\delta\hat{g}) \ .\end{aligned}$$

If we now use (2.21) and (2.23) then we obtain

$$
\begin{aligned}
\hat{\nabla}_M((\hat{\nabla}_N A)\delta\hat{g}) &= \Big(\partial_M\partial_N A - (\bar{\Gamma}^P{}_{NM} + \delta^P{}_N\partial_M A + \delta^P{}_M\partial_N A - \bar{g}_{NM}\partial^P A)\partial_P A\Big)H + (\partial_N A)(\partial_M H) \\
&= (\bar{\nabla}_M\bar{\nabla}_N A)H - 2(\partial_M A)(\partial_N A)H + \bar{g}_{NM}(\bar{\nabla}_P A)(\bar{\nabla}^P A)H + (\partial_N A)(\partial_M H)\ .
\end{aligned}
$$

□

Following a similar proof it is possible to show that

Claim 5.

$$
\hat{\nabla}_M\hat{\nabla}_N\delta\hat{g} = \bar{\nabla}_M\bar{\nabla}_N H - (\partial_M A)(\partial_N H) - (\partial_N A)(\partial_M H) + \bar{g}_{NM}(\bar{\nabla}^P A)(\bar{\nabla}_P H)\ . \tag{2.37}
$$

To proceed further we need to calculate the Weyl transformation of the covariant Laplacian on our perturbation. This can be shown to be given by the following

Claim 6.

$$
\hat{\nabla}^Q\hat{\nabla}_Q\delta\hat{g}_{MN} = (d-2)(\bar{\nabla}^Q A)(\bar{\nabla}_Q H_{MN}) + \bar{\Box}H_{MN} - 2(\bar{\nabla}^Q A)(\bar{\nabla}_Q A)H_{MN} + 2(\partial_N A)(\partial_M A)H\ . \tag{2.38}
$$

Proof. In order to prove this it is useful to show

$$
\begin{aligned}
\hat{\nabla}_P\hat{\nabla}_Q\delta\hat{g}_{MN} = e^{2A}\Big(&2(\partial_P A)\big(\bar{\nabla}_Q H_{MN} - (\partial_M A)H_{QN} - (\partial_N A)H_{MQ}\big) + \bar{\nabla}_P\bar{\nabla}_Q H_{MN} \\
&-(\bar{\nabla}_P\bar{\nabla}_M A)H_{QN} - (\partial_M A)\bar{\nabla}_P H_{QN} - (\bar{\nabla}_P\bar{\nabla}_N A)H_{MQ} - (\partial_N A)\bar{\nabla}_P H_{MQ} \\
&-3(\partial_P A)\bar{\nabla}_Q H_{MN} + 4(\partial_M A)(\partial_P A)H_{QN} + 4(\partial_P A)(\partial_N A)H_{MQ} \\
&+2(\partial_M A)(\partial_N A)H_{QP} - (\partial_Q A)\bar{\nabla}_P H_{MN} + (\partial_Q A)(\partial_M A)H_{PN} \\
&+(\partial_Q A)(\partial_N A)H_{MP} + \bar{g}_{QP}(\bar{\nabla}^R A)\bar{\nabla}_R H_{MN} - (\partial_M A)\bar{\nabla}_Q H_{PN} \\
&+\bar{g}_{MP}(\bar{\nabla}^R A)\bar{\nabla}_Q H_{RN} - \bar{g}_{MP}(\bar{\nabla}^R A)(\bar{\nabla}_R A)H_{QN} - (\partial_N A)\bar{\nabla}_Q H_{MP} \\
&+\bar{g}_{NP}(\bar{\nabla}^R A)\bar{\nabla}_Q H_{MR} - \bar{g}_{NP}(\bar{\nabla}^R A)(\bar{\nabla}_R A)H_{MQ}\Big)\ .
\end{aligned} \tag{2.39}
$$

If one now contracts (2.39) with $\hat{g}^{PQ}$ and uses

$$
(\bar{\nabla}^Q\bar{\nabla}_N A)H_{MQ} = 0\ , \tag{2.40}
$$

$$
(\bar{\nabla}^R A)(\bar{\nabla}_M H_{RN}) = 0\ , \tag{2.41}
$$

which hold because of the form of A and the perturbation H_{MN}, as given in (2.2), then the required result is obtained. □

Using (2.39) and the appropriate contraction, (2.37) is obtained, thus confirming our previous result. We are now in a position to calculate the form of the terms involving curvature tensors appearing in (2.30)

Claim 7.

$$\hat{R}_{QN}\delta\hat{g}^{Q}{}_{M} = \bar{R}_{QN}H^{Q}{}_{M} - H_{MN}\Big(\bar{\nabla}^2 A + (d-2)(\bar{\nabla}^Q A)(\bar{\nabla}_Q A)\Big) \ . \tag{2.42}$$

Proof. If we begin by using (2.34) then we obtain

$$\hat{R}_{QN}\delta\hat{g}^{Q}{}_{M} = \Big(\bar{R}_{QN} - (d-2)\bar{\nabla}_Q\bar{\nabla}_N A - g_{QN}\bar{\nabla}^2 A \\ +(d-2)(\bar{\nabla}_Q A)(\bar{\nabla}_N A) - (d-2)g_{QN}(\bar{\nabla}^P A)(\bar{\nabla}_Q A\Big)\delta\hat{g}^{Q}{}_{M} \ .$$

We can convert the hat perturbation to a bar perturbation by employing the following relationship

$$\delta\hat{g}^{Q}{}_{M} = \hat{g}^{QP}\delta\hat{g}_{MP} = e^{-2A}\bar{g}^{QP}e^{2A}H_{MP} = \bar{g}^{QP}H_{MP} = H^{Q}{}_{M} \ .$$

If we now expand our previous expression, making use of (2.40) and the form of our perturbation, then the required result follows. □

If we perform similar manipulations we find that

$$\hat{R}^{Q}{}_{NPM}\delta\hat{g}^{P}{}_{Q} = \bar{R}^{Q}{}_{NPM}H^{P}{}_{Q} - (\bar{\nabla}_M\bar{\nabla}_N A)H + (\bar{\nabla}_M A)(\bar{\nabla}_N A)H \\ -\bar{g}_{MN}(\bar{\nabla}^P A)(\bar{\nabla}_P A)H + (\bar{\nabla}^P A)(\bar{\nabla}_P A)H_{MN} \ , \tag{2.43}$$

with $H = \bar{g}^{MN}H_{MN}$

With (2.36)-(2.38), (2.42) and (2.43) we can calculate an expression for the perturbation of $\hat{R}_{MN}$, (2.30), in the Weyl related variables. When this is done we obtain

$$
\begin{aligned}
\delta\hat{R}_{MN} = \frac{1}{2}\Big(&\bar{R}_{QM}H^{Q}{}_{N} - H_{NM}\big(\bar{\nabla}^2 A + (d-2)(\bar{\nabla}^P A)(\bar{\nabla}_P A)\big)\\
&-\big(\bar{R}^{Q}{}_{NPM}H^{P}{}_{Q} - (\bar{\nabla}_M\bar{\nabla}_N A)H + (\bar{\nabla}_M A)(\bar{\nabla}_N A)H - \bar{g}_{MN}(\bar{\nabla}^P A)(\bar{\nabla}_P A)H + (\bar{\nabla}^P A)(\bar{\nabla}_P A)H_{MN}\big)\\
&+\bar{R}_{QN}H^{Q}_{M} - H_{MN}(\bar{\nabla}^2 A + (d-2)(\bar{\nabla}^P A)(\bar{\nabla}_P A))\\
&-\big(\bar{R}^{Q}{}_{MPN}H^{P}{}_{Q} - (\bar{\nabla}_N\bar{\nabla}_M A)H + (\bar{\nabla}_N A)(\bar{\nabla}_M A)H - \bar{g}_{NM}(\bar{\nabla}^P A)(\bar{\nabla}_P A)H + (\bar{\nabla}^P A)(\bar{\nabla}_P A)H_{NM}\big)\\
&-\big((\bar{\nabla}_M\bar{\nabla}_N A)H - 2(\partial_M A)(\partial_N A)H + \bar{g}_{NM}(\nabla_P A)(\nabla^P A)H + (\partial_N A)(\partial_M H)\big)\\
&-\big((\bar{\nabla}_N\bar{\nabla}_M A)H - 2(\partial_N A)(\partial_M A)H + \bar{g}_{MN}(\nabla_P A)(\nabla^P A)H + (\partial_M A)(\partial_N H)\big)\\
&-((d-2)\big(\bar{\nabla}^Q A)(\bar{\nabla}_Q H_{MN}) + \bar{\Box}H_{MN} - 2(\bar{\nabla}^Q A)(\bar{\nabla}_Q A)H_{MN} + 2(\partial_N A)(\partial_M A)H\big)\\
&-\big(\bar{\nabla}_M\bar{\nabla}_N H - (\partial_M A)(\partial_N H) - (\partial_N A)(\partial_M H) + \bar{g}_{NM}(\bar{\nabla}^P A)(\bar{\nabla}_P H)\big)\Big)\ .
\end{aligned}
$$

Which if one collects terms appropriately gives

$$
\begin{aligned}
\delta\hat{R}_{MN} = \frac{1}{2}\Big(&\bar{R}_{QM}H^{Q}{}_{N} + \bar{R}_{QN}H^{Q}{}_{M} - 2H_{MN}\big(\bar{\nabla}^2 A + (d-2)(\bar{\nabla}^P A)(\bar{\nabla}_P A)\big) - 2\bar{R}^{Q}{}_{MPN}H^{P}{}_{Q}\\
&-(d-2)(\bar{\nabla}^Q A)(\bar{\nabla}_Q H_{MN}) - \bar{\Box}H_{MN} - \bar{\nabla}_M\bar{\nabla}_N H - \bar{g}_{NM}(\bar{\nabla}^P A)(\bar{\nabla}_P H)\Big)\ ,
\end{aligned}
\tag{2.44}
$$

where the fact that $\bar{R}^{Q}{}_{NPM}H^{P}{}_{Q} = \bar{R}^{Q}{}_{MPN}H^{P}{}_{Q}$ has been used. If one contracts this with $\hat{g}^{MN}$ we obtain

$$
\hat{g}^{MN}\delta\hat{R}_{MN} = -e^{-2A}\Big(\bar{\Box}H + H\big(\bar{\nabla}^2 A + (d-2)(\bar{\nabla}^P A)(\bar{\nabla}_P A)\big) + (d-1)(\bar{\nabla}^Q A)(\bar{\nabla}_Q H)\Big)\ . \tag{2.45}
$$

Using this in (2.31) the variation of the Ricci scalar can be found to be

$$
\delta\hat{R} = -e^{-2A}\Big(\bar{g}^{MP}\bar{g}^{NQ}H_{PQ}\bar{R}_{MN} + \bar{\Box}H + (d-1)(\bar{\nabla}^Q A)(\bar{\nabla}_Q H)\Big)\ . \tag{2.46}
$$

Having obtained these results we are now in a position to calculate the linearised Einstein tensor. The Einstein tensor is defined as

$$
\hat{G}_{MN} = \hat{R}_{MN} - \frac{1}{2}\hat{g}_{MN}\hat{R}\ , \tag{2.47}
$$

thus its linear variation is given by

$$
\delta\hat{G}_{MN} = \delta\hat{R}_{MN} - \frac{1}{2}\delta\hat{g}_{MN}\hat{R} - \frac{1}{2}\hat{g}_{MN}\delta\hat{R}\ . \tag{2.48}
$$

We will now proceed to evaluate (2.48) for each of the 3 possible, independent, combinations of world volume and transverse index structures. If we start with the worldvolume transverse combination then only the first term in (2.48) contributes

Claim 8.

$$\delta\hat{G}_{\mu a} = -\frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_a H = -\frac{1}{2}\partial_\mu\partial_a H \ . \tag{2.49}$$

Proof.

$$\begin{aligned}\delta\hat{R}_{\mu a} = \frac{1}{2}\Big(&\bar{R}_{Q\mu}H^Q{}_a + \bar{R}^Q_{Qa}H_\mu - 2H_{\mu a}\big(\bar{\nabla}^2 A + (d-2)(\bar{\nabla}^P A)(\bar{\nabla}_P A)\big) - 2\bar{R}^Q{}_{\mu P a}H^P{}_Q \\ &-(d-2)(\bar{\nabla}^Q A)(\bar{\nabla}_Q H_{\mu a}) - \bar{\Box}H_{\mu a} - \bar{\nabla}_\mu\bar{\nabla}_a H - \bar{g}_{a\mu}(\bar{\nabla}^P A)(\bar{\nabla}_P H)\Big) \\ = \frac{1}{2}\Big(&0+0-0-0-(d-2)(\bar{\nabla}^Q A)(\bar{\nabla}_Q H_{\mu a}) - \bar{\Box}H_{\mu a} - \bar{\nabla}_\mu\bar{\nabla}_a H - 0\Big) \ .\end{aligned}$$

Since

$$\bar{\nabla}_M H_{NP} = \partial_M H_{NP} - \bar{\Gamma}^R{}_{NM}H_{RP} - \bar{\Gamma}^R{}_{PM}H_{NR} \ ,$$

we find that

$$\bar{\nabla}_M H_{\mu a} = \partial_M H_{\mu a} - \bar{\Gamma}^R{}_{\mu M}H_{Ra} - \bar{\Gamma}^R{}_{aM}H_{\mu R} = 0 \ .$$

Using this result, one can show that $\bar{\Box}H_{\mu a} = 0$, and hence

$$\delta\hat{G}_{\mu a} = -\frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_a H \ .$$

The final step just requires one to note that for a product manifold $\bar{\Gamma}^R{}_{\mu a} = 0$. □

Using similar manipulations it can be shown that

$$\delta\hat{G}_{ab} = \frac{1}{2}\Big(- \bar{\nabla}_a\bar{\nabla}_b H + (d-2)\bar{g}_{ab}(\bar{\nabla}^P A)(\bar{\nabla}_P H) + \bar{g}_{ab}(\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}H_{\rho\sigma}\bar{R}_{\mu\nu} + \bar{\Box}H)\Big) \ , \tag{2.50}$$

which, if (2.10) is used, leads to

$$\delta\hat{G}_{ab} = \frac{1}{2}\Big(- \bar{\nabla}_a\bar{\nabla}_b H + (d-2)\bar{g}_{ab}(\bar{\nabla}^P A)(\bar{\nabla}_P H) + \bar{g}_{ab}((n-1)kH + \bar{\Box}H)\Big) \ . \tag{2.51}$$

We can then calculate the final piece of the linearised Einstein tensor to be

$$\begin{aligned}\delta\hat{G}_{\mu\nu} = \frac{1}{2}\Big(&\bar{R}_{\rho\mu}H^{\rho}{}_{\nu} + \bar{R}_{\rho\nu}H^{\rho}{}_{\mu} - 2H_{\mu\nu}\big(\bar{\nabla}^2 A + (d-2)(\bar{\nabla}^P A)(\bar{\nabla}_P A)\big) - 2\bar{R}^{\sigma}{}_{\mu\rho\nu}H^{\rho}{}_{\sigma} \\ &-(d-2)(\bar{\nabla}^Q A)(\bar{\nabla}_Q H_{\mu\nu}) - \bar{\Box}H_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu H + (d-2)\bar{g}_{\mu\nu}(\bar{\nabla}^Q A)(\bar{\nabla}_Q H) \\ &-\delta\hat{g}_{\mu\nu}\hat{R} + \bar{g}_{\mu\nu}(\bar{g}^{\tau\rho}\bar{g}^{\lambda\sigma}H_{\rho\sigma}\bar{R}_{\tau\lambda} + \bar{\Box}H)\Big)\ ,\end{aligned} \tag{2.52}$$

which, if we use (2.10), becomes

$$\begin{aligned}\delta\hat{G}_{\mu\nu} = \frac{1}{2}\Big(&2(n-1)kH_{\mu\nu} - 2H_{\mu\nu}\big(\bar{\nabla}^2 A + (d-2)(\bar{\nabla}^P A)(\bar{\nabla}_P A)\big) - 2k\bar{g}_{\mu\nu}H + 2kH_{\mu\nu} \\ &-(d-2)(\bar{\nabla}^Q A)(\bar{\nabla}_Q H_{\mu\nu}) - \bar{\Box}H_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu H + (d-2)\bar{g}_{\mu\nu}(\bar{\nabla}^Q A)(\bar{\nabla}_Q H) \\ &-\delta\hat{g}_{\mu\nu}\hat{R} + \bar{g}_{\mu\nu}\big(k(n-1)H + \bar{\Box}H\big)\Big)\ .\end{aligned} \tag{2.53}$$

In order to make further progress we will need to calculate the linearised energy momentum tensor of the system. Our main focus throughout this thesis will be on the massless modes that arise from the NSNS sector of Type II string theories, which give rise to the NSNS sectors of 10 dimensional Supergravity theories. As such we shall now undertake the task of calculating the linearised variation of the energy momentum tensor for such a system. The dynamics of the massless bosonic field content is governed by the Einstein frame action

$$S[\hat{g}, B_{[2]}, \phi] = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-det\hat{g}}\Big(\hat{R} - \frac{1}{12}e^{-\phi}H_{MNP}H^{MNP} - \frac{1}{2}\hat{\nabla}_M\phi\hat{\nabla}^M\phi\Big)\ . \tag{2.54}$$

Here all indices are raised and lowered using the metric $\hat{g}$ and $H_{[3]} = dB_{[2]}$. By varying the matter parts of (2.54) with respect to the metric, we obtain the, Einstein frame, stress energy tensor of this system

$$\kappa^2\hat{T}_{MN} = \frac{1}{2}(\hat{\nabla}_M\phi)(\hat{\nabla}_N\phi) - \frac{1}{4}\hat{g}_{MN}(\hat{\nabla}_P\phi)(\hat{\nabla}^P\phi) + \frac{e^{-\phi}}{4}H_{MPQ}H_N{}^{PQ} - \frac{e^{-\phi}}{24}\hat{g}_{MN}H_{PQR}H^{PQR}\ . \tag{2.55}$$

In order to calculate the linearised variation of (2.55) we use the definition

$$\delta\hat{T}_{MN} := \hat{T}_{MN}(\hat{g} + \delta\hat{g}, \phi + \delta\phi, B_{[2]} + \delta B_{[2]})|^{lin} - \hat{T}_{MN}(\hat{g}, \phi, B_{[2]})\ . \tag{2.56}$$

We can then use this to calculate the variation of (2.55). In the case where matter perturbations have been set to zero we find

$$\kappa^2 \delta\hat{T}_{MN} = -\frac{1}{4}\delta\hat{g}_{MN}(\hat{\nabla}^Q\phi)(\hat{\nabla}_Q\phi) + \frac{1}{4}\hat{g}_{MN}\delta\hat{g}^{PQ}(\hat{\nabla}_P\phi)(\hat{\nabla}_Q\phi) - \frac{e^{-\phi}}{2}\delta\hat{g}^{QS}H_{MQP}H_{NS}{}^P$$
$$-\frac{e^{-\phi}}{24}\delta\hat{g}_{MN}H_{PQR}H^{PQR} + \frac{e^{-\phi}}{8}\hat{g}_{MN}\delta\hat{g}^{RK}H_{RPQ}H_K{}^{PQ} \quad (2.57)$$

where $\delta\hat{g}^{MN} := \hat{g}^{MP}\hat{g}^{NQ}\delta\hat{g}_{PQ}$. By using the form of our perturbation we find that

$$\kappa^2 \delta\hat{T}_{\mu\nu} = \delta\hat{g}_{\mu\nu}\Big(-\frac{1}{4}\hat{\nabla}^P\phi\hat{\nabla}_P\phi - \frac{e^{-\phi}}{24}H_{PQR}H^{PQR}\Big) = \frac{\kappa^2}{n}\delta\hat{g}_{\mu\nu}\hat{T}^\rho{}_\rho \ , \quad (2.58)$$

with $\hat{T}^\mu{}_\mu = \hat{g}^{\mu\nu}\hat{T}_{\mu\nu}$ and all other components vanish. This agrees with the result of B&E where n=4.

The advantage of using a background of the form (2.1) and a perturbation of the form (2.2) is that this leads to a variation of the stress energy tensor of the form (2.58). This can be arrived at without a detailed understanding of the matter fields present in the theory. As such the results derived are more far reaching than may have originally been anticipated. The Einstein equation of the background metric allows us to express (2.58) in terms of background curvatures as follows

$$\kappa^2 \hat{T}^\mu{}_\mu = \hat{g}^{\mu\nu}\hat{R}_{\mu\nu} - \frac{1}{2}n\hat{R} \ , \quad (2.59)$$

where $\hat{R}$ is the Ricci scalar for the full geometry. If we now use (2.34), recalling that the metrics $\hat{g}$ and $\bar{g}$ are related by (2.9), then (2.59) gives

$$\kappa^2 \delta\hat{T}_{\mu\nu} = \frac{1}{n}\delta\hat{g}_{\mu\nu}\Big(e^{-2A}\big(\bar{g}^{\rho\sigma}\bar{R}_{\rho\sigma} - n\bar{\Box}_y A - n(d-2)(\bar{\nabla}^P A)(\bar{\nabla}_P A)\big) - \frac{1}{2}n\hat{R}\Big) \ . \quad (2.60)$$

If we now use the maximal symmetry of M_n, thus invoking (2.10), we obtain

$$\kappa^2 \delta\hat{T}_{\mu\nu} = \delta\hat{g}_{\mu\nu}\Big(e^{-2A}\big((n-1)k - \bar{\Box}_y A - (d-2)(\bar{\nabla}^P A)(\bar{\nabla}_P A)\big) - \frac{1}{2}\hat{R}\Big) \ . \quad (2.61)$$

We are now in a position to explore the consequences of the linearised Einstein equations. We begin with the worldvolume transverse equation, (2.49). Recalling our assumption, (2.3), we write $H(x,y) = h(x)\psi(y)$ with $h(x) = \bar{g}^{\mu\nu}h_{\mu\nu}$ which leads to

$$\partial_\mu h \partial_a \psi = 0 \implies h \in \mathbb{R} \quad \vee \quad \psi \in \mathbb{R} \ . \quad (2.62)$$

Considering the transverse transverse equation, (2.51), we obtain 4 options. If we decide to take $h \in \mathbb{R}$ then we find:

$$h = 0 \qquad \vee \qquad -\bar{\nabla}_a \bar{\nabla}_b \psi + \bar{g}_{ab}\big((n-1)k\psi + (d-2)(\bar{\nabla}^c A)(\bar{\nabla}_c \psi) + \bar{\Box}_y \psi\big) = 0 \; . \tag{2.63}$$

The first of these options means that our perturbation is traceless, hence we are in the case considered by B&E, while the second option appears to be a new route one could investigate, see later in this Section. The other option is to take $\psi \in \mathbb{R}$ in which case we find two further cases

$$\psi = 0 \qquad \vee \qquad \bar{\Box}_x h + (n-1)kh = 0 \tag{2.64}$$

The $\psi = 0$ case means the full perturbation is zero and hence we shall not peruse it further. However, the second option provides a scalar wave equation for the trace of the perturbation which deserves further investigation. Note in this case, as $\psi \in \mathbb{R}$, we are reliant on an analogue of (2.18) holding for the perturbation to be normalisable.

Finally we look at the worldvolume worldvolume equation. Firstly consider the case where the option $h \in \mathbb{R}$ is taken in order to solve (2.62), if we then select $h = 0$ to solve (2.63) then, as expected, the results of Bachas and Estes are recovered. In the cases where $h \neq 0$ we obtain the equation:

$$\begin{aligned} &2kh_{\mu\nu}\psi - 2k\bar{g}_{\mu\nu}h\psi - (d-2)h_{\mu\nu}\bar{g}^{ab}\partial_a A \partial_b \psi - \psi \bar{\Box}_x h_{\mu\nu} \\ &-h_{\mu\nu}\bar{\Box}_y \psi + (d-2)\bar{g}_{\mu\nu} h \bar{g}^{ab}\partial_a A \partial_b \psi + \bar{g}_{\mu\nu}\big(k(n-1)h\psi + h\bar{\Box}_y\psi\big) = 0 \; . \end{aligned} \tag{2.65}$$

However it can be seen that (2.65) is, in general, incompatible with separability of the perturbation (2.3). This can more easily be seen by considering a component of this tensor equation. For example, by setting $\mu, \nu = 0$ in (2.65), we obtain

$$\begin{aligned} &2kh_{00}\psi - 2k\bar{g}_{00}h\psi - (d-2)h_{00}\bar{g}^{ab}\partial_a A \partial_b \psi - \psi \bar{\Box}_x h_{00} \\ &-h_{00}\bar{\Box}_y \psi + (d-2)\bar{g}_{00} h \bar{g}^{ab}\partial_a A \partial_b \psi + \bar{g}_{00}(k(d_{WV}-1)h\psi + h\bar{\Box}_y\psi) = 0 \; , \end{aligned} \tag{2.66}$$

which is clearly not separable. Since we're discussing the case where $h \in \mathbb{R}$ an obvious way to achieve separability is to take $h_{\mu\nu}(x) = \epsilon \bar{g}_{\mu\nu}(x)$, with ϵ input to carry the order of the perturbation. This choice leads to trivial dynamics of $h_{\mu\nu}$ as all derivatives acting on it are covariant derivatives of $\bar{g}$. So whilst this option is available to us mathematically it is physically not very desirable. However, ignoring this issue, we find that (2.65) leads to

$$\bar{\Box}_y\psi + (d-2)\bar{g}^{ab}\partial_a A\partial_b\psi + k(n-2)\psi = 0\ . \tag{2.67}$$

This has the potential to be in conflict with the traced version of (2.63) which gives

$$\Big((d-n-1)\bar{\Box}_y + (d-n)(n-1)k + (d-n)(d-2)\bar{\nabla}^a A\bar{\nabla}_a\Big)\psi = 0\ . \tag{2.68}$$

If we use (2.67) to replace the $(d-2)\partial^a A\partial_a\psi$ term in (2.68) then we obtain the equation

$$\bar{\Box}_y\psi - (d-n)k\psi = 0\ . \tag{2.69}$$

This then allows for $\bar{\Box}_y\psi$ to be solved for and then used in (2.67), to check the self consistency of this condition, giving

$$\bar{g}^{ab}\partial_a A\partial_b\psi = -k\psi \tag{2.70}$$

Assuming a non-trivial solution exists to these equations one can now attack the problem of (2.63). We substitute (2.69) and (2.70) in for the derivative terms to obtain

$$-\bar{\nabla}_a\bar{\nabla}_b\psi + \bar{g}_{ab}k\psi = 0\ . \tag{2.71}$$

Hence, if a non-trivial solution of (2.69)-(2.71) can be found then there is the possibility of finding a traceful perturbation with non trivial dependence on the internal space. However the form of $h_{\mu\nu} = \epsilon\bar{g}_{\mu\nu}$ renders this option undesirable.

This leaves us with the case where $\psi \in \mathbb{R}\setminus\{0\}$ is used to solve (2.62) and so the second option is used to solve (2.64). Under these assumptions the worldvolume worldvolume equation becomes

$$2kh_{\mu\nu} + k(n-3)\bar{g}_{\mu\nu}h - \bar{\Box}_x h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h + \bar{g}_{\mu\nu}\bar{\Box}_x h = 0\ . \tag{2.72}$$

If we trace this equation then we obtain

$$(n-2)\bar{\Box}_x h + (n^2-3n+2)kh = (n-2)(\bar{\Box}_x h + (n-1)kh) = 0 \tag{2.73}$$

This shows that the worldvolume worldvolume equations and the transverse transverse equations are consistent and hence it seems there is a way to allow for perturbations with a non-zero trace. However, it has forced us in to the case of a constant transverse dependence.

However, before we declare victory, we need to recall that, so far, we have only considered perturbations to the metric. The real power of the argument of B&E is that it is meant to apply to all Supergravity theories which possess a solution of the form (2.1). The result is meant to hold even in the presence of matter perturbations. Stated in other terms the metric perturbations and the matter perturbations, at the linear level [74] are decoupled from each other, hence there are no interactions between these perturbations [75]. This means that one can solved the metric perturbation equation without worrying about the perturbations of the other fields. In effect we have a consistent truncation of the matter perturbations. We'd now like to see how this occurs.

To keep the discussion concrete we shall work with the, portion of the, Type IIA Supergravity theory given by (2.54). The metric obeys the usual Einstein equation, with the energy momentum tensor given by (2.55). In order to proceed we shall need the dilaton and 3-form equations of motion. These are

$$\frac{\delta S}{\delta\phi} = 0 \implies \nabla^M \nabla_M \phi + \frac{e^{-\phi}}{12} H_{MNP} H^{MNP} = 0 \,, \tag{2.74}$$

$$\frac{\delta S}{\delta B_{MN}} = 0 \implies \nabla_P (e^{-\phi} H^{PMN}) = 0 \,. \tag{2.75}$$

We can now consider the following perturbation of a solution of these equations

$$(\hat{g}_{MN}, \phi, H_{MNP}) \to (\hat{g}_{MN} + \delta\hat{g}_{MN}, \phi + \delta\phi, H_{MNP} + \partial_M \delta B_{NP}) \,, \tag{2.76}$$

where we take δg_{MN} as in (2.2). The equations obeyed by the perturbations of all three fields are obtained by linearising (2.74) and (2.75) along with including matter perturbations in (2.56). Linearising the dilaton equation leads to

$$\begin{aligned} &-\delta\hat{g}^{MN} \hat{\nabla}_M \hat{\nabla}_N \phi - \hat{g}^{MN} \delta\hat{\Gamma}^R{}_{MN} \hat{\nabla}_R \phi + \hat{\nabla}^2 \delta\phi \\ &-\frac{e^{-\phi}}{12} \delta\phi H_{MNP} \hat{H}^{MNP} - \frac{e^{-\phi}}{4} \delta\hat{g}^{MI} H_{MNP} \hat{H}_I{}^{NP} + \frac{e^{-\phi}}{6} \partial_M \delta B_{NP} \hat{H}^{MNP} = 0 \,, \end{aligned} \tag{2.77}$$

where $\delta\Gamma^R{}_{MN}$ is given in (2.25), $\hat{H}$ refers to 3 form components raise by the metric $\hat{g}$ and $\delta\hat{g}^{MN} := \hat{g}^{MP}\hat{g}^{NQ}\delta\hat{g}_{PQ}$.

[74] Or at the quadratic level in the effective action.
[75] Note that at higher orders this no longer holds.

The linearised 2-form equation is

$$
\begin{aligned}
&-\delta\hat{g}^{PI}\hat{\nabla}_P(e^{-\phi}\hat{H}_I{}^{MN}) - \delta\hat{g}^{MJ}\hat{\nabla}_P(e^{-\phi}H^P{}_J{}^N) - \delta\hat{g}^{NK}\hat{\nabla}_P(e^{-\phi}H^{PM}{}_K) - \hat{\nabla}_P(e^{-\phi}\delta\phi\hat{H}^{PMN}) \\
&+\hat{g}^{PI}\hat{g}^{MJ}\hat{g}^{NK}\hat{\nabla}_P(e^{-\phi}\partial_I\delta B_{JK}) + \delta\hat{\Gamma}^P{}_{RP}e^{-\phi}\hat{H}^{RMN} + \delta\hat{\Gamma}^M{}_{RP}e^{-\phi}\hat{H}^{PRN} + \delta\hat{\Gamma}^N{}_{RP}e^{-\phi}\hat{H}^{PMR} = 0 \ , \quad (2.78)
\end{aligned}
$$

where all symbols have the same definition as in the previous paragraph.

The variation of the energy momentum tensor, including matter perturbations, for a metric perturbation of the from (2.2), is given by

$$
\begin{aligned}
\kappa^2\delta\hat{T}_{MN} = &-\frac{1}{4}\delta\hat{g}_{MN}(\hat{\nabla}^Q\phi)(\hat{\nabla}_Q\phi) - \frac{e^{-\phi}}{24}\delta\hat{g}_{MN}H_{PQR}\hat{H}^{PQR} + \frac{1}{2}(\hat{\nabla}_M\delta\phi\hat{\nabla}_N\phi + \hat{\nabla}_M\phi\hat{\nabla}_N\delta\phi) \\
&-\frac{1}{2}\hat{g}_{MN}(\hat{\nabla}_P\phi)(\hat{\nabla}^P\delta\phi) + \frac{e^{-\phi}}{4}(\partial_M\delta B_{PQ}\hat{H}_N{}^{PQ} + \partial_N\delta B_{PQ}\hat{H}_M{}^{PQ}) \qquad (2.79) \\
&-\frac{e^{-\phi}}{4}\delta\phi H_{MPQ}\hat{H}_N{}^{PQ} + \frac{e^{-\phi}}{24}g_{MN}\delta\phi H_{PQR}\hat{H}^{PQR} - \frac{e^{-\phi}}{12}\hat{g}_{MN}\partial_P\delta B_{QR}\hat{H}^{PQR} \ .
\end{aligned}
$$

We are now in a position to see how matter perturbations affect the linearised Einstein equation. Let's begin with the worldvolume transverse equation which, using (2.49) and recalling the above perturbations are in Einstein frame, gives

$$
-\bar{\nabla}_\mu\bar{\nabla}_a H = \bar{\nabla}_\mu\delta\phi\bar{\nabla}_a\phi + \frac{e^{-\phi}}{2}\partial_\mu\delta B_{PQ}\hat{H}_a{}^{PQ} \ . \qquad (2.80)
$$

The transverse transverse equation now reads

$$
\begin{aligned}
&-\bar{\nabla}_a\bar{\nabla}_b H + (d-2)\bar{g}_{ab}\big(\bar{\nabla}^P A)(\bar{\nabla}_P H) + \bar{g}_{ab}((n-1)kH + \bar{\Box}H\big) \\
&= \bar{\nabla}_a\delta\phi\bar{\nabla}_b\phi + \bar{\nabla}_a\phi\bar{\nabla}_b\delta\phi - \hat{g}_{ab}(\hat{\nabla}^Q\phi)(\hat{\nabla}_Q\delta\phi) + \frac{e^{-\phi}}{2}\big(\partial_a\delta B_{PQ}\hat{H}_b{}^{PQ} + \partial_b\delta B_{PQ}\hat{H}_a{}^{PQ}\big) \qquad (2.81) \\
&-\frac{e^{-\phi}}{2}\delta\phi H_{aPQ}\hat{H}_b{}^{PQ} + \frac{e^{-\phi}}{12}\hat{g}_{ab}\delta\phi H_{IJK}\hat{H}^{IJK} - \frac{e^{-\phi}}{6}\hat{g}_{ab}\partial_P\delta B_{QR}\hat{H}^{PQR} \ .
\end{aligned}
$$

Finally the WVWV equation, making use of (2.3), becomes

$$2kh_{\mu\nu}\psi - 2k\bar{g}_{\mu\nu}h\psi - (d-2)h_{\mu\nu}\bar{\nabla}^P A\bar{\nabla}_P\psi - \psi\bar{\Box}_x h_{\mu\nu} - h_{\mu\nu}\bar{\Box}_y\psi$$

$$+(d-2)\bar{g}_{\mu\nu}h\bar{\nabla}^P A\bar{\nabla}_P\psi + \bar{g}_{\mu\nu}\big(k(n-1)h\psi + h\bar{\Box}_y\psi\big) = \hat{g}_{\mu\nu}(\hat{\nabla}^Q\phi)(\hat{\nabla}_Q\delta\phi) \tag{2.82}$$

$$+\frac{e^{-\phi}}{12}\hat{g}_{\mu\nu}\delta\phi H_{PQR}\hat{H}^{PQR} - \frac{e^{-\phi}}{6}\hat{g}_{\mu\nu}\partial_P\delta B_{QR}\hat{H}^{PQR} \; . \tag{2.83}$$

We'd like to be able to truncate to a sector in which only the metric perturbations are considered. This means we want to be able to, consistently, set the matter perturbations to zero. If this is done then (2.80)-(2.82) assume the forms given previously. The perturbed matter equations, reinstating hats to show that the quantities are associated to the metric $\hat{g}$, become

$$-\delta\hat{g}^{MN}\hat{\nabla}_M\hat{\nabla}_N\phi - \hat{g}^{MN}\delta\hat{\Gamma}^R{}_{MN}\hat{\nabla}_R\phi = 0 \; , \tag{2.84}$$

$$-\delta\hat{g}^{PI}\hat{\nabla}_P(e^{-\phi}\hat{H}_I{}^{MN}) - \delta\hat{g}^{MJ}\hat{\nabla}_P(e^{-\phi}\hat{H}^P{}_J{}^N) - \delta\hat{g}^{NK}\hat{\nabla}_P(e^{-\phi}\hat{H}^{PM}{}_K) + \delta\hat{\Gamma}^P{}_{RP}e^{-\phi}\hat{H}^{RMN} = 0 \; . \tag{2.85}$$

Beginning with the perturbed dilaton equation we find that

$$h(\hat{g}^{ab}\partial_a\psi\partial_b\phi) = 0 \; , \tag{2.86}$$

while the 2 form equation gives us

$$h\frac{e^{-\phi}}{2}(\hat{H}^{aMN}(\partial_a\psi - 2\psi\partial_a A)) = 0 \; . \tag{2.87}$$

As we've seen the only really feasible path to a traceful perturbation was to take $\psi \in \mathbb{R}\setminus\{0\}$. In this case (2.86) is trivially solved while (2.87) would impose

$$\frac{h}{2}\hat{H}^{aMN}\partial_a A = 0 \; .$$

If we want our conclusions to hold for any solution of the Supergravity equations that has a metric of the form (2.1) then we are forced to set the trace equal to zero. We could try and solve this equation with the condition $\hat{H}^{aMN}\partial_a A = 0$. However it would seem this would greatly restrict the types of backgrounds one could discuss. Hence, if one wants to decouple perturbations on a general background of the form (2.1), then it would seem that traceless perturbations are all one can consider.

So it seems that the only sensible route to follow, at least in the case of a Supergravity theory of the form (2.54), is to consider traceless perturbations. We might, with good reason, ask what have we really gained. Well, to begin with, we have taken the first steps towards understanding the problem of spin-2 fluctuations. As we've mentioned already, understanding this problem will prove critical in the process of constructing an EFT for the spin-2 field about a, specific, background of the form (2.1). Further to this, while we have just confirmed the results of B&E we have now dealt with any niggling doubts about the validity of the trace condition. Whilst it may not be a valid gauge condition off-shell, if transversality has already been implemented, it now follows dynamically from the consistency of a truncation of the perturbation problem. These conclusions have been shown in the context of a specific theory, however we believe that they could be extended to other theories with relative ease.

3 Effective Field Theories: a DIY kit

Having covered all of the prerequisite material we now turn to attempting to construct an Effective Field Theory (EFT) for gravity. We've referred to the idea of constructing an EFT for gravity a fair bit, but have done little to explain what we actually mean, in this Section we hope to remedy this. Roughly speaking, when we refer to an EFT of gravity what we mean is a, usually four dimensional, theory in which the fundamental fields originate as perturbations of a background solution to a system of field equations. In the context of an EFT of gravity, obviously, some of these perturbations will arise from fluctuations of a background metric. In this way the spin-2 field gives rise to an EFT of gravity around a Minkowski background. In our case we shall be interested in looking at the EFT of gravity that arises from fluctuations around a background solution of Type IIA Supergravity due to Crampton, Pope and Stelle[76] [17]. In this case the background solution involves a ten dimensional metric which has a four dimensional Minkowski worldvolume and a non-compact hyperbolic space making up, a portion of, the transverse space. Since the transverse space is non-compact an appropriate localising mechanism is required in order to render the final theory, effectively, four dimensional. The mismatch of dimensions also means we need to decide how our perturbation fields, which naturally are ten dimensional, should be expanded into four dimensional fields. This requires us to determine the dependence of our perturbations on the transverse space, which, unsurprisingly, leads us in to the territory of a Bachas and Estes style transverse wavefunction problem for the graviton spectrum.

We begin this Section with an overview of the various aspects involved in the development of a general EFT of gravity. This includes the use of perturbative methods to develop an expansion of our action by deforming a background solution of our theory, with these deformations going on to play the role of the fundamental fields of our EFT. In the cases we consider our actions will lead to a theory in more than four dimensions, and as a result dimensional reduction techniques, specifically the use of generalised Fourier expansions, shall be required in order to render the final theory, effectively, four dimensional. It is expected that the resulting lower dimensional theory will display the usual diffeomorphism symmetry expected to be present in gravitational theories. After this general introduction we shall discuss both the background solution of CPS and the graviton spectrum, which is at the heart of the localisation mechanism, around this background. After reviewing this work we shall apply the general EFT methods discussed to the CPS case. We shall show, by direct calculation, that these methods lead to a lower dimensional field theory that doesn't possess the well known diffeomorphism invariance of gravity. This will lead us, in future Sections, to reconsider and modify the methods used to construct EFTs of gravity.

[76]Referred to as CPS from now on.

3.1 How to build an Effective Field Theory of gravity: a general treatment

The field of EFTs is vast, so here we'll try to provide a brief introduction to the process of constructing what is usually called a top down EFT of gravity. A top down approach to EFT involves beginning with a theory containing degrees of freedom that separate out, based on energy, into high energy and low energy fields. This leads to us having some fields, ϕ_h, with energies $E_h > \Lambda$ and other fields, ϕ_l, with energies $E_l < \Lambda$, for some energy scale Λ. When discussing physics at energies below Λ all phenomena can be described in terms of just the ϕ_l fields. However, we can't just set the ϕ_h fields to zero, we have to correctly integrate them out so their effects are accounted for. The process of integrating out fields, as we've seen, leads to a whole host of new interactions in the effective theory of the ϕ_l fields that, in general, weren't present in the original theory. These new terms appear in the effective theory suppressed by an energy scale, M, which is large compared to Λ. This leads to the contributions of these effects being subleading. This top down approach is the opposite of the usual construction of an EFT, which are normally considered in the context of a bottom up approach. In this case it is known that the physics representing a situation can adequately be described by the degrees of freedom φ as long as we work at scales below a given cutoff scale $\tilde{\Lambda}$. Once we want to probe energies comparable to $\tilde{\Lambda}$ our EFT breaks down as new degrees of freedom become relevant to the problem. However, if we stay safely below $\tilde{\Lambda}$, then to develop an EFT description of our theory we just write down an action involving all possible terms, with undetermined coefficients, involving φ and derivatives of it that are compatible with the symmetries of the system. This approach leads to terms of ever increasing inverse powers of $\tilde{\Lambda}$ and as such these terms are suppressed in the EFT and give rise to subleading effects. The difference between these two approaches is that in the top down approach we begin with the full high energy theory and then write it in a low energy form, while in the bottom up approach we only ever deal with the low energy description. The bottom up approach has the advantage of being much simpler to write down. However the top down approach does allow for the determination of the coefficients of terms in the EFT action in terms of the parameters of the high energy theory and we already have knowledge of the degrees of freedom present at higher energies. The classic example of a top down EFT construction is the Fermi four point theory of the weak interaction which arises from the full $SU(2)$ gauge theory by integrating out the massive gauge bosons, although this was certainly not the way these theories, initially, arose in the literature.

We're now going to try and understand the steps required to construct a top-down EFT of gravity. Our treatment here will be mostly schematic, with detailed examples being given later in this thesis. To begin with lets assume we have a theory described by an action principle of the form

$$S[g,\Phi] = \int_{M_d} d^d X \sqrt{-g}\mathcal{L}(g,\Phi) \ , \tag{3.1}$$

with g a, Lorentzian, metric field and Φ representing any matter fields in the problem[77]. This action can be varied to obtain

$$\delta S = \int_{M_d} d^d X \sqrt{-g}\Big(\frac{\delta S}{\delta g_{MN}}\delta g_{MN} + \frac{\delta S}{\delta \Phi}\delta\Phi\Big) + \int_{\partial M_d} d\Sigma_M V^M \ , \tag{3.2}$$

where in this expression $d\Sigma_M$ is the surface element on the boundary of M_d. The first term represents the equations of motion of the system and the second accounts for any possible boundary terms incurred in the variation procedure, for example terms like those we saw arising from δR_{MN} in the variation of the Einstein Hilbert action in Section 1. We can use the second term to set boundary conditions for the variation or, if we desire to have a specific set of boundary conditions, modify it by the addition of a boundary term to the original action. Once we have the equations of motion we can find solutions to them. If we assume we've found a solution, say $(\mathring{g},\mathring{\Phi})$, then we can consider a perturbation of this solution

$$g_{MN} = \mathring{g}_{MN} + H_{MN}, \qquad \Phi = \mathring{\Phi} + \varphi \ . \tag{3.3}$$

Often one wishes to only consider metric perturbations, hence we set $\varphi = 0$[78]. We can then use (3.3) in (3.1) and perform an expansion of the resulting expression to obtain

$$S[g,\phi] = S[\mathring{g},\mathring{\Phi}] + \int_{\partial M_d} d\Sigma_M V^M\Big|_{\delta\Phi=0} + S_d^{(2)}[H;\mathring{g},\mathring{\Phi}] + S_d^{(3)}[H;\mathring{g},\mathring{\Phi}] + \dots \ , \tag{3.4}$$

Here no, bulk, terms linear in H_{MN} occur because they would arise from the variation (3.2) and hence would evaluate to zero since $\mathring{g}$ solves the equations of motion of this action. The contributions $S_d^{(n)}[H;\mathring{g},\mathring{\Phi}]$ denote terms in the expansion with n copies of the perturbation H in them and omitted terms are those associated to $n \geqslant 4$. This expanded action now starts to look very similar, in a highly schematic fashion, to that of the spin-2 field we encountered earlier. There $\mathring{g} = \eta$ and $\mathring{\Phi} = 0$ and our interpretation was clear, the theory described a symmetric field $h_{\mu\nu}$ propagating on Minkowski spacetime. This suggests that the correct interpretation of (3.4) is as a theory of a field H_{MN} propagating on the background provided by $\mathring{g}$ and $\mathring{\Phi}$, so in this sense we are describing gravitational waves on a curved background.

[77]We, willingly, ignore any terms not involving a metric. Such terms are often referred to as topological terms with a classic example being the $F_{[4]} \wedge F_{[4]} \wedge A_{[3]}$ Chern-Simons term of eleven dimensional Supergravity.

[78]Some care needs to be taken here, as in Section 2 we saw that the consistent truncation of matter perturbations can lead to conditions on our metric perturbation. However, we'll ignore this subtle issue for now.

Since our theory comes from one involving gravity we expect it to possess diffeomorphism invariance. In the current context this symmetry has the infinitesimal form

$$H_{MN} \to H'_{MN} = H_{MN} + \mathring{\nabla}_M X_N + \mathring{\nabla}_N X_M + \mathcal{L}_X H_{MN} \ , \tag{3.5}$$

where X is a vector field, $\mathring{\nabla}$ is the covariant derivative associated to the Levi-Civita connection of $\mathring{g}$ and indices are raised and lowered by $\mathring{g}$ and its inverse. We note that this is just the curved space version of (1.44). However, we now recall, that in the flat space case, the second order action is happy to live all by itself. So it is probably no surprise that the term $S_d^{(2)}$, in (3.4), also enjoys this property. It turns out that this term is just a curved space analogue of the Fierz-Pauli action. So we might suspect that the theory described by $S_d^{(2)}$ enjoys it's own gauge symmetry

$$H_{MN} \to H'_{MN} = H_{MN} + \mathring{\nabla}_M X_N + \mathring{\nabla}_N X_M \ , \tag{3.6}$$

which it does. So we see that the $S_d^{(2)}$ term acts as the free part of the theory and then all higher order terms, in n, arise from the self-coupling of our field H_{MN} and so represent interactions of our field. This story now appears to be exactly analogues to what we've seen in the Minkowski space case. Now recall that gravitational actions come with a copy of the Newton constant so, up to a numerical constant that we absorb in to the definition of $S_d^{(n)}$, (3.4) would really take the form

$$\begin{aligned} S[g,\phi] =& \frac{1}{\kappa_d^2} S[\mathring{g},\mathring{\Phi}] + \frac{1}{\kappa_d^2}\int_{\partial M_d} d\Sigma_M V^M\Big|_{\delta\Phi=0} + \frac{1}{\kappa_d^2} S_d^{(2)}[H;\mathring{g},\mathring{\Phi}] + \frac{1}{\kappa_d^2} S_d^{(3)}[H;\mathring{g},\mathring{\Phi}] \\ &+ \frac{1}{\kappa_d^2} S_d^{(4)}[H;\mathring{g},\mathring{\Phi}] + \dots \ . \end{aligned} \tag{3.7}$$

As before we can redefine our field as

$$\tilde{H}_{MN} = \frac{1}{\kappa_d} H_{MN} \ , \tag{3.8}$$

which leads to (3.7) assuming the form

$$\begin{aligned} S[g,\phi] =& \frac{1}{\kappa_d^2} S[\mathring{g},\mathring{\Phi}] + \frac{1}{\kappa_d}\int_{\partial M_d} d\Sigma_M V^M\Big|_{\delta\Phi=0} + S_d^{(2)}[\tilde{H};\mathring{g},\mathring{\Phi}] + \kappa_d S_d^{(3)}[\tilde{H};\mathring{g},\mathring{\Phi}] \\ &+ \kappa_d^2 S_d^{(4)}[\tilde{H};\mathring{g},\mathring{\Phi}] + \dots \ , \end{aligned} \tag{3.9}$$

and (3.3) becomes

$$g_{MN} = \mathring{g}_{MN} + \kappa_d \tilde{H}_{MN} \ . \tag{3.10}$$

This redefinition means that (3.6) becomes

$$\tilde{H}_{MN} \to \tilde{H}'_{MN} = \tilde{H}_{MN} + \frac{1}{\kappa_d}\big(\mathring{\nabla}_M X_N + \mathring{\nabla}_N X_M\big) + \mathcal{L}_X \tilde{H}_{MN} \ . \tag{3.11}$$

So, as in the Minkowski space case, the invariance of (3.9) comes down to a knitting together of orders. This is carefully orchestrated by the occurrence of the same parameter, κ_d, in the transformation law and the EFT action.

So far we've only addressed the top part of the theory, but the crucial piece for us is really the down piece of the process. In order to perform this we need the background metric $\mathring{g}$ to have a warped product structure

$$d\mathring{s}_d^2 = e^{2\mathring{A}(y)}\mathring{g}_{\mu\nu}(x)dx^\mu dx^\nu + \mathring{g}_{ab}(y)dy^a dy^b \ . \tag{3.12}$$

Such a metric naturally splits up in to a worldvolume, which is parametrised by the coordinates x^μ with $\mu \in \{1,\ldots,p\}$, and a transverse space with coordinates y^a for $a \in \{1,\ldots,d-p\}$. It's then the usual practice to specialise our perturbation to be in the worldvolume directions only. As such our perturbed metric takes the form

$$ds_d^2 = e^{2\mathring{A}(y)}\big(\mathring{g}_{\mu\nu}(x) + \kappa_d \tilde{H}_{\mu\nu}(x,y)\big)dx^\mu dx^\nu + \mathring{g}_{ab}(y)dy^a dy^b \ . \tag{3.13}$$

If the background matter $\mathring{\Phi}$ also respects this split then if we could just split up the dependence of $\tilde{H}_{\mu\nu}(x,y)$ in to worldvolume pieces and transverse pieces then each of the terms in (3.9), including those that comprise each of the $S_d^{(n)}$, will split into a worldvolume integral and a transverse integral. If we then performed these transverse integrals, assuming they lead to sensible results[79], then we will have removed the transverse space from the problem and shall be left with an action principle only over the worldvolume. This process acts to tame the issue of any mismatching of dimensions of the original theory and the final theory one wishes to obtain, for example allowing one to go from say ten dimensions to four.

So how are we meant to split up $\tilde{H}_{\mu\nu}$? Well, by now we might have already guessed, if we choose $\mathring{g}_{\mu\nu}$ to be a maximally symmetric space then we can use the work of Bachas and Estes to determine the graviton spectrum on the background $(\mathring{g}, \mathring{\Phi})$. Lets assume the transverse dependence is given by the set of eigenfunctions $\{\xi(y;\alpha)\}$ with α parametrising the eigenvalues of the transverse wavefunction problem. Then

[79]Conditions such as the integrals being finite are crucial here.

our perturbation can be expanded as

$$\tilde{H}_{\mu\nu} = \sum_{\alpha} h_{\mu\nu}(x;\alpha)\xi(y;\alpha) \ , \tag{3.14}$$

where the sum is merely symbolic of the statement that we take a combination of eigenfunctions over all possible eigenvalues. As such our expansion can contain a genuine discrete sum alongside an integral if our spectrum has both discrete and continuous parts. We can now substitute this expansion into (3.9) and then perform the transverse integrals to arrive at the p dimensional action

$$S[g,\phi] = \tilde{S}_p^{(2)}[h_{\mu\nu}(x;\alpha);\mathring{g},\mathring{\Phi}] + \kappa_d \tilde{S}_p^{(3)}[h_{\mu\nu}(x;\alpha);\mathring{g},\mathring{\Phi}] + \kappa_d^2 \tilde{S}_p^{(4)}[h_{\mu\nu}(x;\alpha);\mathring{g},\mathring{\Phi}] + \dots \ ,$$

here the first two terms in (3.9) have been omitted for ease of presentation. The new terms $\tilde{S}_p^{(n)}$ arise from performing the transverse integrals present in the corresponding $S_d^{(n)}$, which involves performing integrals of ever increasing numbers of products of the $\xi(y;\alpha)$ basis elements along side various quantities associated to the transverse dependence of the background solution $(\mathring{g},\mathring{\Phi})$. Since each term within each $S_d^{(n)}$ comes with its own transverse integral this can lead to the coefficients present in the corresponding $\tilde{S}_p^{(n)}$ being vastly different. As such, while on a tensorial level many terms appear in the same form as they did in $S_d^{(n)}$, the coefficient accompanying these and, more importantly, the ratio of coefficients within $\tilde{S}_p^{(n)}$ will in general change after performing the transverse integrals[80].

We might now specialise a little further and assume that our transverse wavefunction spectrum contains a zero-eigenvalue mode, say $\xi(y;0)$. Such a mode is expected to correspond to a massless graviton, $h_{\mu\nu}(x;0)$. Now if further to this the next eigenvalue of the transverse problem is separated from this zero eigenvalue then this provides a mass gap for the problem. In this setting, if we only consider energies below this mass gap then we shall only require a theory containing $h_{\mu\nu}(x;0)$ and so, using our EFT philosophy, we can integrate out the higher mass modes and just use the resulting effective theory in order to describe phenomena of interest. In general our heavy modes $h_{\mu\nu}(x;\alpha)$, with $\alpha \neq 0$, will couple to the massless mode and so the integrating out procedure will leave us with a theory described by

$$S[g,\phi] = \bar{S}_p^{(2)}[h_{\mu\nu}(x;0);\mathring{g},\mathring{\Phi}] + \kappa_d \bar{S}_p^{(3)}[h_{\mu\nu}(x;0);\mathring{g},\mathring{\Phi}] + \kappa_d^2 \bar{S}_p^{(4)}[h_{\mu\nu}(x;0);\mathring{g},\mathring{\Phi}] + \dots \ , \tag{3.15}$$

where the integrating out procedure means that the terms in the action are further changed in form, now being represented by $\bar{S}_p^{(n)}$. Given that we now have a p-dimensional theory of a massless graviton we may

[80]There are also some new types of terms that arise. These will be discussed later on in this Section.

ask if it possesses a symmetry that is the p-dimensional analogue of (3.11). We already know that for our original action (3.9) to possess this symmetry there needs to be a carefully laid out conspiracy between terms at differing orders to cancel with each other. This in turn requires a careful alignment of coefficients at each order in the expansion. Hence, after we perform the transverse integrals and then integrate out the heavy fields the action needs to assume the form

$$S[g,\phi] = S_p^{(2)}[h_{\mu\nu}(x;0);\mathring{g},\mathring{\Phi}] + \kappa_p S_p^{(3)}[h_{\mu\nu}(x;0);\mathring{g},\mathring{\Phi}] + \kappa_p^2 S_p^{(4)}[h_{\mu\nu}(x;0);\mathring{g},\mathring{\Phi}] + \dots \ , \tag{3.16}$$

with $S_p^{(n)}$ having the exact same functional form as $S_d^{(n)}$ and κ_p being a new gravitational constant in p-dimensions, that must be identified during the procedure. Not only does the action have to assume this form but the transformation law for $h_{\mu\nu}(x;0)$, which we usually think descends from (3.11), must also assume the form

$$h_{\mu\nu}(x;0) \to h'_{\mu\nu}(x;0) = h_{\mu\nu}(x;0) + \frac{1}{\kappa_p}\left(\mathring{\nabla}_\mu\chi_\nu + \mathring{\nabla}_\nu\chi_\mu\right) + \mathcal{L}_\chi h_{\mu\nu}(x,0) \ , \tag{3.17}$$

where $\chi_\mu = \chi_\mu(x)$ and the same κ_p arises as in (3.16). It is only through such an alignment that the delicate symphony of cancellations required can occur for our p-dimensional theory[81]. So far we have spoken mainly in hypotheticals. The goal of much of the remainder of this thesis is to understand how this procedure is realised concretely.

3.2 Review of braneworld localisation in hyperbolic spacetime

In order to develop an EFT of gravity we can see that we'll require a background around which to perturb. In our case this background is provided by the family of solutions of Crampton et al. We shall now give an introduction to the work of CPS and also discuss aspects of the graviton spectrum around one member of this family of solutions.

It was shown by Cvetic, Gibbons and Pope [20] that the six dimensional Salam-Sezgin Supergravity [91] could be consistently embedded in to Type I Supergravity. This embedding was achieved by a dimensional reduction with transverse space $\mathcal{H}^{(2,2)} \times S^1$. The reduction involved the non-compact, three dimensional, hyperbolic space $\mathcal{H}^{(2,2)}$ and avoided the usual issues of incorrect signs for kinetic terms which can lead to ghosts[82]. There has been discussion in the literature of circumstances under which such ghostly fields can be avoided within the context of non-compact dimensional reductions [21]. In the current context this is

[81] Well at least in the case where the worldvolume is p-dimensional Minkowski space as here the argument of Deser holds.

[82] Recall these are the sort of ghosts that lead to a violation of unitarity. They are not to be confused with Faddev-Poppov ghosts, which are present in the theory of the quantisation of gauge theories.

avoided by defining $\mathcal{H}^{(2,2)}$ by the embedding condition

$$\mu_1^2 + \mu_2^2 - \mu_3^2 - \mu_4^2 = 1 \ , \tag{3.18}$$

with $(\mu_1, \mu_2, \mu_3, \mu_4) \in \mathbb{R}^4$. The metric on $\mathbb{R}^4$ is taken to be

$$ds_4^2 = d\mu_1^2 + d\mu_2^2 + d\mu_3^2 + d\mu_4^2 \ . \tag{3.19}$$

The embedding condition (3.18) possesses invariance under $SO(2,2)$ while the metric (3.19) is invariant under $SO(4)$. This leads to a metric on $\mathcal{H}^{(2,2)}$ that has isometry group $SO(2,2) \cap SO(4) = SO(2) \times SO(2)$ given by

$$ds_3^2 = \cosh(2\rho) d\rho^2 + \cosh(2\rho)^2 d\alpha^2 + \sinh(2\rho)^2 d\beta^2 \ , \tag{3.20}$$

$\rho \in [0, \infty)$, $\alpha \in [0, 2\pi)$ and $\beta \in [0, 2\pi)$. It can clearly be seen that this is an inhomogeneous metric of cohomogeneity one[83]. Building on this work CPS showed that the Salam-Sezgin model could be consistently embedded into Type IIA Supergravity and found the consistent reduction ansatz.

The Salam-Sezgin model is a six dimensional $(1,0)$ Supergravity theory with bosonic field content $(g, B_{[2]}, A_{[1]}, \sigma)$ with σ a scalar field. As such this theory is a chiral model, which is very attractive for those trying to understand how the chirality of the Standard Model fermions could arise in Supergravity. This model involves coupling $(1,0)$ 6d Supergravity to a 6d 2-form[84] multiplet and a 6d Maxwell multiplet. This additional matter is added as the Supergravity multiplet in 6 dimensions contains a real self dual 2-form[85]. As is known from Type IIB Supergravity such self-dual forms are difficult to encompass into an action principle. The inclusion of the 2-form multiplet adds an anti-self dual 2-form to the theory allowing these issues to be circumvented. The Maxwell multiplet is added to provide a vector field in the field content of the theory. This model has an interesting solution [91] given by

$$\begin{aligned} &ds_6^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{4g^2}\big(d\theta^2 + \sin(\theta)^2 d\varphi^2\big) \ , \quad A_{[1]} = -\Big(\frac{1}{\sqrt{2}g}\Big)\big(\cos(\theta) \mp 1\big) d\varphi \ , \\ &\sigma = 0 \in \mathbb{R} \ , \quad B_{[2]} = 0 \ , \end{aligned} \tag{3.21}$$

from which we see the spacetime adopts the form $\text{Mink}_4 \times S^2$ while the Maxwell field takes on the form of a $U(1)$ monopole with $g \in \mathbb{R} \setminus \{0\}$. Using the embedding of the Salam-Sezgin model into ten dimensions the

[83]This is a fancy way of saying it depends on the coordinates of the manifold but actually only depends on one of them.
[84]Also referred to as a tensor multiplet.
[85]Self duality here refers to that of the field strength derived from this 2-form.

non-zero fields were found to be [20]

$$\begin{aligned} ds^2_{10\ str} &= \eta_{\mu\nu}dx^\mu dx^\nu + dy^2 + \frac{1}{4g^2}\big(d\psi + \mathrm{sech}(2\rho)(d\chi + \cos(\theta)d\varphi)\big)^2 + \frac{1}{g^2}\,\mathrm{sech}(2\rho)ds^2_{EH}\ , \\ ds^2_{EH} &= \cosh(2\rho)d\rho^2 + \frac{\sinh(2\rho)^2}{4\cosh(2\rho)}(d\chi + \cos(\theta)d\varphi)^2 + \frac{\cosh(2\rho)}{4}(d\theta^2 + \sin(\theta)^2 d\varphi^2)\ , \\ e^{2\phi} &= \mathrm{sech}(2\rho)\ , \quad A_{[2]} = \frac{1}{4g^2}(d\chi + \mathrm{sech}(2\rho)d\psi) \wedge (d\chi + \cos(\theta)d\varphi)\ , \end{aligned} \tag{3.22}$$

where $\chi = \alpha - \beta \in [0, 2\pi)$, $\psi = \alpha + \beta \in [0, 4\pi)$ and g is a parameter related to the size of the waist of the $\mathcal{H}^{2,2}$ space. In this expression the metric is given in string frame[86] and we notice, if we use the change of coordinates $\cosh(2\rho) = r^2$, that ds^2_{EH} is the Eguchi-Hanson metric [40]. This background was found to possess eight Killing spinors, thus it is a supersymmetric background. It was also shown that in the large ρ limit (3.22) approaches the form

$$\begin{aligned} ds^2_{10\ str} &\sim \eta_{\mu\nu}dx^{\mu\nu} + \frac{1}{4g^2} + \frac{1}{g^2}d\rho^2 + \frac{1}{4g^2}(d\chi + \cos(\theta)d\varphi)^2 + \frac{1}{4g^2}(d\theta^2 + \sin(\theta)^2 d\varphi^2)\ , \\ e^{2\phi} &\sim 2e^{-2\rho}\ , \quad A_{[2]} \sim \frac{1}{4g^2}\cos(\theta)d\chi \wedge d\varphi\ . \end{aligned} \tag{3.23}$$

This was noted to be the same as the near-horizon geometry of the NS5-brane solution of Type IIA Supergravity.

The perturbation problem, for perturbations of the form (3.13), around this background was studied using the work of [4]. The final result of this[87] was, with $H_{\mu\nu} = h_{\mu\nu}(x, \lambda)\xi(\rho, \lambda)$, the system of equations[88]

$$\Box_4 h_{\mu\nu} = m^2 h_{\mu\nu}\ , \quad \Big(\frac{d^2}{d\rho^2} + \frac{2}{\tanh(2\rho)}\frac{d}{d\rho}\Big)\xi = -\lambda\xi\ , \quad m^2 = g^2\lambda\ . \tag{3.24}$$

In order to have a massless graviton we require a solution with $\lambda = 0$. In this case the ξ equation has the solution

$$\xi(\rho, 0) = c_1 + c_2\log(\tanh(\rho))\ , \tag{3.25}$$

with c_1 and c_2 real numbers. The argument of Bachas and Estes, (2.17), now required that $c_2 = 0$ and hence the massless graviton was found to have constant dependence on the transverse space. However this was found to lead to a non-normalisable massless graviton since the transverse ρ integral took the form $\int_0^\infty d\rho \sinh(2\rho)$ which gives a non-finite result. This non-normalisability was detected in the four dimensional

[86] Which is related to the Einstein frame metric, $ds^2_{10\ Ein}$, by $ds^2_{10\ Ein} = e^{-\frac{1}{2}\phi}ds^2_{10\ str} = (\cosh(2\rho))^{\frac{1}{4}}ds^2_{10\ str}$.

[87] Where all transverse dependence has been made trivial except for that of the non-compact coordinate. The reasoning behind this is that it represents the trivial mode of the S^2 and so is an S-wave truncation of the system.

[88] This eigenvalue problem is carefully explained in Appendix D where its solutions are also discussed.

theory by the vanishing of the four dimensional Newton's constant.

However, it was noted by CPS that the second option presented by (3.25), namely the $c_1 = 0$, case does allow for a normalisable massless graviton in the four dimensional theory. The setup also avoided the argument of Bachas and Estes since the integration by parts used in deriving (2.17) was found to give rise to a non-vanishing boundary term. Analysis of the eigenvalue equation showed it could be converted to a Schrödinger problem if the substitution

$$\Psi(\rho,\lambda) = \sqrt{\sinh(2\rho)}\xi(\rho,\lambda) \ , \tag{3.26}$$

was used. In these variables the ρ part of (3.24) was shown to become

$$\Big(- \frac{d^2}{d\rho^2} + 2 - \frac{1}{\tanh(2\rho)^2}\Big)\Psi = \lambda\Psi \ . \tag{3.27}$$

It turns out this eigenfunction problem is in fact a Legendre equation, see Appendix D for full details on this. CPS noted that as $\rho \to 0$ this problem asymptotes to a $-\frac{1}{4\rho^2}$ potential. This interesting potential has been studied in the context of quantum mechanics extensively and is excellently reviewed in [41]. It was found to possess a single normalisable boundstate separated from a continuum of scattering states by a mass gap[89]. As such it was observed that the graviton spectrum around the background (3.22) naturally incorporated a localisation mechanism, namely $\xi(\rho,0)$ being normalisable, and a mass gap. The combination of these features made it admissible to EFT methods. However, whilst the spectrum of the problem can be elucidated from its asymptotic form the eigenvalue of the unique boundstate is undetermined and in fact can be assumed to be any value we desire. The desired choice was for it to be zero, however putting this in by hand would have been artificial. Instead CPS noted that the presence of the Eguchi-Hanson space in (3.22), along with the asymptotic form (3.23), made the solution a possible candidate for having arisen from a brane resolved by transgression procedure [22]. This was found to be the case and (3.22) was found to be the fully resolved, singularity free, member of a family of solutions. The one parameter family of solutions found by CPS corresponds to the unresolved brane solutions and as such they include a singularity

$$\begin{aligned} ds^2_{10\ str} &= \eta_{\mu\nu}dx^\mu dx^\nu + dy^2 + \frac{1}{4g^2}\big(d\psi + \mathrm{sech}(2\rho)(d\chi + \cos(\theta)d\varphi)\big)^2 + \frac{H(\rho)}{g^2}\,\mathrm{sech}(2\rho)ds^2_{EH} \ , \\ e^{2\phi} &= H(\rho) \ , \ A_{[2]} = \frac{1}{4g^2}\big((1+k)d\chi + \mathrm{sech}(2\rho)d\psi\big)\wedge(d\chi + \cos(\theta)d\varphi) \ , \\ H(\rho) &= \mathrm{sech}(2\rho) - k\log(\tanh(\rho)) \ , \end{aligned} \tag{3.28}$$

[89] Again for a much fuller review please refer to Appendix D.

where $k \in \mathbb{R}$ and represents the charge of an NS5 brane. This family was also shown to still possess eight Killing spinors. Since (3.28) has a singularity as $\rho \to 0$ it was deduced that it was in fact a solution to Type IIA Supergravity coupled to an NS5-brane source. The inclusion of the NS5-brane source, which was smeared in the S^2 directions, was performed by adding the Einstein frame Nambu-Goto style bosonic source action

$$S^{Ein}_{NG}[x,g,\phi] = -\frac{T}{\Omega_2}\int_{S^2} d^2\Omega \int_{W_6} d^6\zeta \Big(-\det(\partial_i x^M \partial_j x^N g^{Ein}_{MN}(x(\zeta))\Big)^{\frac{1}{2}} e^{\frac{-\phi}{2}} \ , \tag{3.29}$$

with T the tension of the brane, ζ^i, for $i \in \{0, ..., 5\}$, coordinates on the NS5 brane worldvolume and x^M embedding coordinates from the worldvolume into spacetime. The smearing[90] of the brane over the S^2 directions is performed by the $d^2\Omega = \sin(\theta)d\theta d\varphi$ integral and is averaged by division by the volume, Ω_2, of the unit round S^2. It was found to be more natural to work in string frame where

$$g^{Ein}_{MN} = e^{\frac{-\phi}{2}} g^{str}_{MN} \ , \tag{3.30}$$

and to switch to the, classically equivalent, Polyakov form of the action

$$S^{Str}_{Poly}[\gamma, x, g, \phi] = -\frac{T}{2\Omega_2}\int d^2\Omega \int_{W_6} d^6\zeta \sqrt{-\det\gamma}\Big(\gamma^{ij}\partial_i x^M \partial_j x^N g^{str}_{MN}(x(\zeta)) - 4\Big)e^{-2\phi} \ , \tag{3.31}$$

where γ_{ij} is an independent worldvolume metric obeying the equation of motion

$$\gamma_{ij} = \partial_i x^M \partial_j x^N g^{str}_{MN} \ . \tag{3.32}$$

Hence, on shell, it's the pullback of the spacetime metric to the worldvolume.

CPS then considered the system resulting from coupling (3.31) to Type IIA Supergravity, written in string frame. The inclusion of the NS5 brane action meant that the equations of motion of Type IIA Supergravity now included source terms. Of principle concern were the source terms arising in the Einstein equation.

Varying (3.31) with respect to the spacetime metric led to

$$\delta_g S^{str}_{Poly} = -\frac{T}{2\Omega_2}\int_{S^2} d^2\Omega \int_{W_6} d^6\zeta \sqrt{-\det\gamma}\gamma^{ij}\partial_i x^M \partial_j x^N \delta g^{str}_{MN} e^{-2\phi} \ . \tag{3.33}$$

In order for this to enter into the equations of motion of Type IIA Supergravity this variation was written

[90] Which is assumed to be valid on the basis of preserved supersymmetry, however no explicit proof of this was given. It would be nice to have a proof of this.

as an integral on the spacetime manifold. This was achieved by the use of a spacetime delta distribution

$$\delta_g S^{str}_{Poly} = -\frac{T}{2\Omega_2}\int_{M_{10}} d^2\Omega d^8\tilde{x}\sqrt{-\tilde{g_8}}\int_{W_6} d^6\zeta \frac{\delta^{(8)}(x(\zeta)-\tilde{x})}{\sqrt{-\tilde{g_8}}}\sqrt{-\det\gamma}\gamma^{ij}\partial_i x^M \partial_j x^N \delta g^{str}_{MN} e^{-2\phi} \ , \tag{3.34}$$

where $\tilde{x} = (x^\mu, y, \psi, \rho, \chi)$ and $\tilde{g_8}$ is the determinant of the matrix obtained by setting θ and φ equal to a constant in (3.28).

To obtain the source terms' contribution to the Type IIA Supergravity equations a term of the form $\int_{M_{10}} d^{10}x\sqrt{-g_{10}}$ was required in (3.34). Explicit calculation showed that

$$\sin(\theta)\sqrt{-\tilde{g_8}} = \frac{4g^2\sqrt{-g_{10}}}{H(\rho)\cosh(2\rho)} \ , \tag{3.35}$$

which gave the source term for the Type IIA Supergravity, string frame, Einstein equation as

$$\begin{aligned}&\frac{e^{-2\phi}}{2\kappa^2_{10}}\big(R_{MN} - \frac{1}{2}g_{MN}R - 2g_{MN}\nabla^2\phi + 2\nabla_M\nabla_N\phi + 2g_{MN}(\nabla_P\phi)(\nabla^P\phi) - \frac{1}{4}H_{MPQ}H_N{}^{PQ} + \frac{1}{24}g_{MN}H_{PQR}H^{PQR}\big)\\ &\quad + \frac{T}{2\Omega_2}\int_{W_6} d^6\zeta \frac{4g^2\delta^{(8)}(x(\zeta)-\tilde{x})}{H(\rho)cosh(2\rho)\sqrt{-\tilde{g_8}}}\sqrt{-\det\gamma}\gamma^{ij}g_{MP}g_{NQ}(\partial_i x^P\partial_j x^Q)e^{-2\phi} = 0 \ .\end{aligned} \tag{3.36}$$

In this expression all quantities are in string frame and the explicit *str* label has been dropped for clarity. The worldvolume worldvolume portion of this equation was considered and was found to lead to

$$\begin{aligned}&\frac{e^{-2\phi}}{2\kappa^2_{10}}\Big(-\frac{1}{2}\eta_{\mu\nu}R - 2\eta_{\mu\nu}g^{PQ}\nabla_P\nabla_Q\phi + 2\eta_{\mu\nu}g^{PQ}(\nabla_P\phi)(\nabla_Q\phi) + \frac{1}{24}\eta_{\mu\nu}H_{PQR}H^{PQR})\Big)\\ &\quad = -\frac{T}{2\Omega_2}\int_{W_6} d^6\zeta \frac{4g^2\delta^{(8)}(x(\zeta)-\tilde{x})}{H(\rho)cosh(2\rho)\sqrt{-\tilde{g_8}}}\sqrt{-\det\gamma}\gamma^{ij}\eta_{\mu\lambda}\eta_{\nu\tau}(\partial_i x^\lambda\partial_j x^\tau)e^{-2\phi} \ .\end{aligned} \tag{3.37}$$

The static gauge choice

$$\zeta^\mu = x^\mu, \zeta^4 = z, \zeta^5 = \psi \ , \tag{3.38}$$

was then used, while the other embedding coordinates were set to constant values. This meant that (3.32) evaluated to

$$ds^2_{WV} = \eta_{\mu\nu}dx^\mu dx^\nu + dz^2 + \frac{1}{4g^2}d\psi^2 \ , \tag{3.39}$$

$$\det\gamma = \frac{-1}{4g^2} \ , \tag{3.40}$$

from which the source term[91] was found to be

$$\frac{-4Tg^4e^{-2\phi}}{\Omega_2 H^3\cosh(2\rho)\sinh(2\rho)}\eta_{\mu\nu}\delta^{(2)}(\alpha) \ , \tag{3.41}$$

with $\alpha = (\rho, \chi)$. Hence the source was found to be localised in the ρ-χ plane.

Calculation of the other terms in (3.37) established the relationship

$$\eta_{\mu\nu}\frac{\Delta_{EH}H}{2\kappa_{10}^2} = \frac{-2Tg^2}{\pi\cosh(2\rho)\sinh(2\rho)}\eta_{\mu\nu}\delta^2(\alpha) \ , \tag{3.42}$$

with

$$\Delta_{EH} = \mathrm{sech}(2\rho)\Big(\frac{d^2}{d\rho^2} + 2\coth(2\rho)\frac{d}{d\rho}\Big) \ . \tag{3.43}$$

From which it was established that

$$k = 2\kappa_{10}^2 Tg^2 \ , \tag{3.44}$$

showing that as the brane charge was sent to zero the brane smoothly decoupled from the system. Following a similar method a source term was found for the ξ portion of the eigenvalue problem (3.24). From this a ring regularisation of the delta distribution $\delta^2(\alpha)$ was used and it was found that the eigenvalue of the bound state was selected to be zero. Thus a massless graviton was selected by the inclusion of the NS5 brane source and the zero mode was found to be the same as in the unsourced case. The smooth decoupling of the brane as the charge was sent to zero was then used to establish the choice of the existence of a massless graviton in the original unsourced problem. Thus the NS5 brane source was found to provide a mechanism for selecting the appropriate eigenvalue of the boundstate of the $\frac{1}{\rho^2}$ potential.

3.3 The effective field theory around the CPS background: a first attempt

Having considered both the general process of constructing an EFT and the class of backgrounds of interest, we are now well positioned to tackle the construction of our first EFT. As our background we take the $k = 0$ member of the family of solutions (3.28) which is a solution of the equations of motion following from the action

$$S = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-\det g}\; e^{-2\phi}(R - \frac{1}{12}H_{MNP}H^{MNP} + 4\nabla_M\phi\nabla^M\phi), \tag{3.45}$$

[91] Which corrects (5.2) in [17] which has not had the θ integral performed.

which can be noticed to be the NSNS sector action for Type IIA Supergravity written in string frame variables. A detailed analysis of the graviton spectrum around this background can be performed and is detailed extensively in Appendix D. The crucial features are that the mass spectrum of the gravitons is given by $\mathcal{S} = \{0\} \cup \{\lambda|\lambda > g^2 =: \lambda_{edge}\}$ and the mode associated to the zero eigenvalue has transverse wavefunction

$$\xi(\rho; 0) = \frac{2\sqrt{3}}{\pi} \log(\tanh(\rho)) \ . \tag{3.46}$$

By calculation of the invariant measure $d^{10}x\sqrt{-g}\, e^{-2\phi}$, for the background (3.28), the appropriate transverse inner product, for S-wave modes, can be found to be given by

$$(\omega, \zeta) = \frac{4\pi^2}{g^5} \int_0^\infty d\rho\ \sinh(2\rho)\bar{\omega}(\rho)\zeta(\rho) \ . \tag{3.47}$$

The transverse wavefunctions of the massive gravitons can be shown to be given by Legendre functions and integrals of multiple copies of these transverse wavefunctions can be performed as shown in Appendix D. If we denote the transverse wavefunctions by $\{\xi(\rho;\alpha)|\alpha \in \mathcal{S}\}$ then our ten dimensional metric perturbation can be expressed as

$$H_{\mu\nu}(x, \rho) = h_{\mu\nu}(x; 0)\xi(\rho; 0) + \int_{\lambda_{edge}}^\infty d\lambda\ h_{\mu\nu}(x; \lambda)\xi(\rho; \lambda) \ . \tag{3.48}$$

With this form of perturbation we could just charge forward with our calculation of the exact form of the EFT action for our particular case. However, it should come as no surprise that this is a long endeavour. So rather than rushing straight in, we might ask if it's possible to get some intuition for how the story proceeds by working in an area somewhere between the abstract case and our specific example? Fortunately it turns out this is very illuminating. When we begin expanding our action in ten dimensions we end up with an action, given here up to fourth order, of the form

$$\begin{aligned}
S_{10} = &\int_{M_{10}} d^{10}x\sqrt{-\det \mathring{g}}\ e^{-2\mathring{\phi}}\mathcal{L}_{10}^{(2)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \tilde{H}_{PQ}, \tilde{H}_{RS}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi}) \\
+\kappa_{10} &\int_{M_{10}} d^{10}x\sqrt{-\det \mathring{g}}\ e^{-2\mathring{\phi}}\mathcal{L}_{10}^{(3)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \tilde{H}_{PQ}, \tilde{H}_{RS}, \tilde{H}_{TU}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi}) \\
+\kappa_{10}^2 &\int_{M_{10}} d^{10}x\sqrt{-\det \mathring{g}}\ e^{-2\mathring{\phi}}\mathcal{L}_{10}^{(4)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \tilde{H}_{PQ}, \tilde{H}_{RS}, \tilde{H}_{TU}, \tilde{H}_{VW}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi}) \ .
\end{aligned} \tag{3.49}$$

This action is schematic and is meant to illustrate the fact that every term in the expansion has two derivatives and also is meant to define the order by order Lagrangian densities $\mathcal{L}_{10}^{(n)}$. We should note that $\mathring{B}_{[2]}$ can only occur in this action in the gauge invariant form $\mathring{H}_{[3]} = d\mathring{B}_{[2]}$, where we are referring to the gauge invariance $\mathring{B}_{[2]} \to \mathring{B}_{[2]} + d\Lambda_{[1]}$. Under a diffeomorphism that only has components in the worldvolume directions only the metric $g = \mathring{g} + H$ transforms. This occurs due to the maximal symmetry of the worldvolume of the background metric, which forbids background matter fields to depend upon the worldvolume coordinates or have components in those directions. We already know, that at linear order in the diffeomorphism parameter, the transformation is implemented by the Lie derivative and so

$$\delta\tilde{H} = \frac{1}{\kappa_{10}}\big(\mathring{\nabla}_M X_N + \mathring{\nabla}_N X_M\big) + \mathcal{L}_X \tilde{H}_{MN} =: \frac{1}{\kappa_{10}}\mathcal{L}_X \mathring{g}_{MN} + \mathcal{L}_X \tilde{H}_{MN}\ . \tag{3.50}$$

If we use this in (3.49) then invariance of the action demands that

$$\begin{aligned}
\delta S_{10} = \frac{1}{\kappa_{10}} \int_{M_{10}} & d^{10}x\sqrt{-\det\mathring{g}}\; e^{-2\mathring{\phi}}\mathcal{L}^{(2)}_{10}(\mathring{\nabla}_M,\mathring{\nabla}_N,\mathcal{L}_X\mathring{g}_{PQ},\tilde{H}_{RS};\mathring{g},\mathring{B}_{[2]},\mathring{\phi}) \\
+\frac{1}{\kappa_{10}} \int_{M_{10}} & d^{10}x\sqrt{-\det\mathring{g}}\; e^{-2\mathring{\phi}}\mathcal{L}^{(2)}_{10}(\mathring{\nabla}_M,\mathring{\nabla}_N,\tilde{H}_{PQ},\mathcal{L}_X\mathring{g}_{RS};\mathring{g},\mathring{B}_{[2]},\mathring{\phi}) \qquad (3.51)\\
+ \int_{M_{10}} & d^{10}x\sqrt{-\det\mathring{g}}\; e^{-2\mathring{\phi}}\mathcal{L}^{(2)}_{10}(\mathring{\nabla}_M,\mathring{\nabla}_N,\mathcal{L}_X\tilde{H}_{PQ},\tilde{H}_{RS};\mathring{g},\mathring{B}_{[2]},\mathring{\phi}) \\
+ \int_{M_{10}} & d^{10}x\sqrt{-\det\mathring{g}}\; e^{-2\mathring{\phi}}\mathcal{L}^{(2)}_{10}(\mathring{\nabla}_M,\mathring{\nabla}_N,\tilde{H}_{PQ},\mathcal{L}_X\tilde{H}_{RS};\mathring{g},\mathring{B}_{[2]},\mathring{\phi}) \qquad (3.52)\\
+ \int_{M_{10}} & d^{10}x\sqrt{-\det\mathring{g}}\; e^{-2\mathring{\phi}}\mathcal{L}^{(3)}_{10}(\mathring{\nabla}_M,\mathring{\nabla}_N,\mathcal{L}_X\mathring{g}_{PQ},\tilde{H}_{RS},\tilde{H}_{TU};\mathring{g},\mathring{B}_{[2]},\mathring{\phi}) \\
+ \int_{M_{10}} & d^{10}x\sqrt{-\det\mathring{g}}\; e^{-2\mathring{\phi}}\mathcal{L}^{(3)}_{10}(\mathring{\nabla}_M,\mathring{\nabla}_N,\tilde{H}_{PQ},\mathcal{L}_X\mathring{g}_{RS},\tilde{H}_{TU};\mathring{g},\mathring{B}_{[2]},\mathring{\phi}) \qquad (3.53)\\
+ \int_{M_{10}} & d^{10}x\sqrt{-\det\mathring{g}}\; e^{-2\mathring{\phi}}\mathcal{L}^{(3)}_{10}(\mathring{\nabla}_M,\mathring{\nabla}_N,\tilde{H}_{PQ},\tilde{H}_{RS},\mathcal{L}_X\mathring{g}_{TU};\mathring{g},\mathring{B}_{[2]},\mathring{\phi}) \\
+\kappa_{10} \int_{M_{10}} & d^{10}x\sqrt{-\det\mathring{g}}\; e^{-2\mathring{\phi}}\mathcal{L}^{(3)}_{10}(\mathring{\nabla}_M,\mathring{\nabla}_N,\mathcal{L}_X\tilde{H}_{PQ},\tilde{H}_{RS},\tilde{H}_{TU};\mathring{g},\mathring{B}_{[2]},\mathring{\phi}) \\
+\kappa_{10} \int_{M_{10}} & d^{10}x\sqrt{-\det\mathring{g}}\; e^{-2\mathring{\phi}}\mathcal{L}^{(3)}_{10}(\mathring{\nabla}_M,\mathring{\nabla}_N,\tilde{H}_{PQ},\mathcal{L}_X\tilde{H}_{RS},\tilde{H}_{TU};\mathring{g},\mathring{B}_{[2]},\mathring{\phi}) \qquad (3.54)\\
+\kappa_{10} \int_{M_{10}} & d^{10}x\sqrt{-\det\mathring{g}}\; e^{-2\mathring{\phi}}\mathcal{L}^{(3)}_{10}(\mathring{\nabla}_M,\mathring{\nabla}_N,\tilde{H}_{PQ},\tilde{H}_{RS},\mathcal{L}_X\tilde{H}_{TU};\mathring{g},\mathring{B}_{[2]},\mathring{\phi}) \\
+\kappa_{10} \int_{M_{10}} & d^{10}x\sqrt{-\det\mathring{g}}\; e^{-2\mathring{\phi}}\mathcal{L}^{(4)}_{10}(\mathring{\nabla}_M,\mathring{\nabla}_N,\mathcal{L}_X\mathring{g}_{PQ},\tilde{H}_{RS},\tilde{H}_{TU},\tilde{H}_{VW};\mathring{g},\mathring{B}_{[2]},\mathring{\phi}) \\
+\kappa_{10} \int_{M_{10}} & d^{10}x\sqrt{-\det\mathring{g}}\; e^{-2\mathring{\phi}}\mathcal{L}^{(4)}_{10}(\mathring{\nabla}_M,\mathring{\nabla}_N,\tilde{H}_{PQ},\mathcal{L}_X\mathring{g}_{RS},\tilde{H}_{TU},\tilde{H}_{VW};\mathring{g},\mathring{B}_{[2]},\mathring{\phi}) \\
+\kappa_{10} \int_{M_{10}} & d^{10}x\sqrt{-\det\mathring{g}}\; e^{-2\mathring{\phi}}\mathcal{L}^{(4)}_{10}(\mathring{\nabla}_M,\mathring{\nabla}_N,\tilde{H}_{PQ},\tilde{H}_{RS},\mathcal{L}_X\mathring{g}_{TU},\tilde{H}_{VW};\mathring{g},\mathring{B}_{[2]},\mathring{\phi}) \\
+\kappa_{10} \int_{M_{10}} & d^{10}x\sqrt{-\det\mathring{g}}\; e^{-2\mathring{\phi}}\mathcal{L}^{(4)}_{10}(\mathring{\nabla}_M,\mathring{\nabla}_N,\tilde{H}_{PQ},\tilde{H}_{RS},\tilde{H}_{TU},\mathcal{L}_X\mathring{g}_{VW};\mathring{g},\mathring{B}_{[2]},\mathring{\phi}) \qquad (3.55)
\end{aligned}$$

$$+\kappa_{10}^2 \int_{M_{10}} d^{10}x \sqrt{-\det \mathring{g}}\; e^{-2\mathring{\phi}} \mathcal{L}_{10}^{(4)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \mathcal{L}_X \tilde{H}_{PQ}, \tilde{H}_{RS}, \tilde{H}_{TU}, \tilde{H}_{VW}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi})$$
$$+\kappa_{10}^2 \int_{M_{10}} d^{10}x \sqrt{-\det \mathring{g}}\; e^{-2\mathring{\phi}} \mathcal{L}_{10}^{(4)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \tilde{H}_{PQ}, \mathcal{L}_X \tilde{H}_{RS}, \tilde{H}_{TU}, \tilde{H}_{VW}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi})$$
$$+\kappa_{10}^2 \int_{M_{10}} d^{10}x \sqrt{-\det \mathring{g}}\; e^{-2\mathring{\phi}} \mathcal{L}_{10}^{(4)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \tilde{H}_{PQ}, \tilde{H}_{RS}, \mathcal{L}_X \tilde{H}_{TU}, \tilde{H}_{VW}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi})$$
$$+\kappa_{10}^2 \int_{M_{10}} d^{10}x \sqrt{-\det \mathring{g}}\; e^{-2\mathring{\phi}} \mathcal{L}_{10}^{(4)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \tilde{H}_{PQ}, \tilde{H}_{RS}, \tilde{H}_{TU}, \mathcal{L}_X \tilde{H}_{VW}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi})\;, \tag{3.56}$$

be zero[92]. The first two terms are known to be zero on there own, which is the statement that the curved space Fierz-Pauli is gauge invariant. Then we can see that the terms with the appropriate κ_{10} factors must cancel amongst themselves, as we've previously stated, which leads to the relationships[93]

$$\begin{aligned}
&\mathcal{L}_{10}^{(2)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \mathcal{L}_X \tilde{H}_{PQ}, \tilde{H}_{RS}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi}) + \mathcal{L}_{10}^{(2)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \tilde{H}_{PQ}, \mathcal{L}_X \tilde{H}_{RS}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi}) = \\
&-\mathcal{L}_{10}^{(3)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \mathcal{L}_X \mathring{g}_{PQ}, \tilde{H}_{RS}, \tilde{H}_{TU}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi}) - \mathcal{L}_{10}^{(3)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \tilde{H}_{PQ}, \mathcal{L}_X \mathring{g}_{RS}, \tilde{H}_{TU}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi}) \\
&-\mathcal{L}_{10}^{(3)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \tilde{H}_{PQ}, \tilde{H}_{RS}, \mathcal{L}_X \mathring{g}_{TU}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi})\;,
\end{aligned} \tag{3.57}$$

$$\begin{aligned}
&\mathcal{L}_{10}^{(3)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \mathcal{L}_X \tilde{H}_{PQ}, \tilde{H}_{RS}, \tilde{H}_{TU}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi}) + \mathcal{L}_{10}^{(3)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \tilde{H}_{PQ}, \mathcal{L}_X \tilde{H}_{RS}, \tilde{H}_{TU}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi}) \\
&+\mathcal{L}_{10}^{(3)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \tilde{H}_{PQ}, \tilde{H}_{RS}, \mathcal{L}_X \tilde{H}_{TU}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi}) = -\mathcal{L}_{10}^{(4)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \mathcal{L}_X \mathring{g}_{PQ}, \tilde{H}_{RS}, \tilde{H}_{TU}, \tilde{H}_{VW}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi}) \\
&-\mathcal{L}_{10}^{(4)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \tilde{H}_{PQ}, \mathcal{L}_X \mathring{g}_{RS}, \tilde{H}_{TU}, \tilde{H}_{VW}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi}) - \mathcal{L}_{10}^{(4)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \tilde{H}_{PQ}, \tilde{H}_{RS}, \mathcal{L}_X \mathring{g}_{TU}, \tilde{H}_{VW}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi}) \\
&-\mathcal{L}_{10}^{(4)}(\mathring{\nabla}_M, \mathring{\nabla}_N, \tilde{H}_{PQ}, \mathcal{L}_X \mathring{g}_{RS}, \tilde{H}_{TU}, \tilde{H}_{VW}; \mathring{g}, \mathring{B}_{[2]}, \mathring{\phi})\;,
\end{aligned} \tag{3.58}$$

the order κ_{10}^2 terms cancel higher order contributions that haven't been written. We now need to consider what types of structures can be present when one integrates over the transverse space. In order to illustrate this lets consider a few model terms

$$\mathring{\nabla}_M \tilde{H}_{PQ} \mathring{\nabla}^M \tilde{H}^{PQ}\;,\; \tilde{H}_{PQ}\tilde{H}_{RS}F^{RPT}F^{QS}{}_T\;,\; (\mathring{\nabla}_M \tilde{H}_{PQ})\tilde{H}^{PQ}\mathring{\nabla}^M\phi\;, \tag{3.59}$$
$$(\mathring{\nabla}_M \tilde{H}_{PQ})(\mathring{\nabla}_N \tilde{H}^{PM})\tilde{H}^{QN}\;,\; \tilde{H}_{PQ}\tilde{H}^{PQ}\mathring{\nabla}^M\mathring{\nabla}_M\phi\;,\; (\mathring{\nabla}_M \tilde{H}_{PQ})\tilde{H}^{MP}\mathring{\nabla}^Q\phi\;. \tag{3.60}$$

These terms have been chosen as they exemplify, respectively, the following structures, a term involving only perturbations where the derivatives are contracted, perturbations contracted into background fields where both derivatives are on the background fields, perturbations along with background fields with one derivative on a perturbation and the other on a background field and the derivatives contracted, a term with

[92] This rather long expression is included here mostly to illustrate the size of the calculations required when undertaking an EFT description of gravity. Hopefully its inclusion motivates why we're not adding many explicit details at this point.

[93] Which hold up to total derivative terms, but we'll omit this technical detail for now.

only perturbations where the derivatives are contracted into the perturbations, a term with both background fields and perturbations with both derivatives on the background contracted into each other and finally a term with both perturbations and background fields in it with both having a derivative which are not contracted into each other. We can now, schematically, expand these terms, using the form of our perturbation (3.48), which leads to the terms

$$\begin{aligned}
\mathring{\nabla}_M \tilde{H}_{PQ} \mathring{\nabla}^M \tilde{H}^{PQ} =& \partial_\mu h_{\nu\lambda} \partial^\mu h^{\nu\lambda} \xi^2 + h_{\mu\nu} h^{\mu\nu} \partial_\sigma \xi \partial^\sigma \xi \ , \\
\tilde{H}_{PQ} \tilde{H}_{RS} F^{RPT} F^{QS}{}_T =& h_{\mu\nu} h_{\sigma\tau} F^{\sigma\mu a} F^{\nu\tau}{}_a = 0 \ , \\
(\mathring{\nabla}_M \tilde{H}_{PQ}) \tilde{H}^{PQ} \mathring{\nabla}^M \phi =& h_{\mu\nu} h^{\mu\nu} \xi \partial_\rho \xi \partial^\rho \phi \ , \\
(\mathring{\nabla}_M \tilde{H}_{PQ})(\mathring{\nabla}_N \tilde{H}^{PM}) \tilde{H}^{QN} =& (\partial_\mu h_{\nu\sigma})(\partial_\tau h^{\nu\mu}) h^{\sigma\tau} \xi^3 \ , \\
\tilde{H}_{PQ} \tilde{H}^{PQ} \mathring{\nabla}^M \mathring{\nabla}_M \phi =& h_{\mu\nu} h^{\mu\nu} \xi^2 \mathring{\nabla}^a \mathring{\nabla}_a \phi \ , \\
(\mathring{\nabla}_M \tilde{H}_{PQ}) \tilde{H}^{MP} \mathring{\nabla}^Q \phi =& (\partial_\mu h_{\nu\sigma}) h^{\mu\nu} \xi^2 \partial^\sigma \phi + (\partial_\mu h_{\nu a}) h^{\mu\sigma} \xi^2 \partial^a \phi = 0 \ ,
\end{aligned} \tag{3.61}$$

where we have used the maximal symmetry of the background worldvolume and the fact that our background metric is a product. We can clearly see that when expanded into four dimensional fields our terms either have two or zero worldvolume derivatives. As such the 4 dimensional action can be organised into two derivative and zero derivative pieces at each order. Since our infinitesimal diffeomorphism transformation, (3.50), is a one derivative transformation these terms can't contribute to the lower dimensional diffeomorphism invariance of each other so they can be dealt with separately. We shall focus on the two derivative terms in four dimensions which must all come from expanding the Ricci scalar in (3.45). It can be shown, that at any given order in our perturbative expansion, the transverse integral accompanying these two derivative terms is given by

$$I(\alpha_1, \dots, \alpha_n) := \int_{M_6} d^6y \sqrt{g_6} e^{-2\phi} \xi(\rho; \alpha_1) \dots \xi(\rho; \alpha_n) \ . \tag{3.62}$$

Within this the integral involving n-copies of the zero mode, $\xi(\rho; 0)$, is of particular interest

$$I_n := I(0, \dots, 0) = \int_{M_6} d^6y \sqrt{g_6} e^{-2\phi} \xi(\rho; 0)^n = \frac{4\pi^2}{g^5} \Big(\frac{-2\sqrt{3}}{\pi} \Big)^n \frac{n!}{2^{n-1}} \zeta(n) \ , \tag{3.63}$$

with $\zeta(n)$ the Riemann zeta function. Since this zero mode is expected to be our massless graviton it's two derivative action is the one we really want to focus on. When we focus on this sector we find that (3.49) takes on the form

$$
\begin{aligned}
S_4\Big|_{zero-mode\ only} &= I_2 \int_{M_4} d^4x \mathcal{L}_4^{(2)}(\partial_\mu, \partial_\nu, h_{\sigma\tau}(x;0), h_{\lambda\theta}(x;0); \eta, 0, 0) \\
&+\kappa_{10} I_3 \int_{M_4} d^4x \mathcal{L}_4^{(3)}(\partial_\mu, \partial_\nu, h_{\sigma\tau}(x;0), h_{\lambda\theta}(x;0), h_{\alpha\beta}(x;0); \eta, 0, 0) \\
&+\kappa_{10}^2 I_4 \int_{M_4} d^4x \mathcal{L}_4^{(4)}(\partial_\mu, \partial_\nu, h_{\sigma\tau}(x;0), h_{\lambda\theta}(x;0), h_{\alpha\beta}(x;0), h_{\gamma\delta}(x;0); \eta, 0, 0) \ .
\end{aligned}
\tag{3.64}
$$

In this expression the Lagrangian densities $\mathcal{L}_4^{(n)}$ have the same structure as the corresponding term in ten dimensions. We now rescale our four dimensional field $h_{\mu\nu}(x;0)$ as

$$
\tilde{h}_{\mu\nu}(x;0) = \sqrt{I_2} h_{\mu\nu}(x;0) \ , \tag{3.65}
$$

meaning that (3.64) takes on the form

$$
\begin{aligned}
S_4\Big|_{zero-mode\ only} &= \int_{M_4} d^4x \ \mathcal{L}_4^{(2)}(\partial_\mu, \partial_\nu, \tilde{h}_{\sigma\tau}(x;0), \tilde{h}_{\lambda\theta}(x;0); \eta, 0, 0) \\
&+ \frac{\kappa_{10} I_3}{I_2^{\frac{3}{2}}} \int_{M_4} d^4x \ \mathcal{L}_4^{(3)}(\partial_\mu, \partial_\nu, \tilde{h}_{\sigma\tau}(x;0), \tilde{h}_{\lambda\theta}(x;0), \tilde{h}_{\alpha\beta}(x;0); \eta, 0, 0) \\
&+ \frac{\kappa_{10}^2 I_4}{I_2^2} \int_{M_4} d^4x \ \mathcal{L}_4^{(4)}(\partial_\mu, \partial_\nu, \tilde{h}_{\sigma\tau}(x;0), \tilde{h}_{\lambda\theta}(x;0), \tilde{h}_{\alpha\beta}(x;0), \tilde{h}_{\gamma\delta}(x;0); \eta, 0, 0) \ .
\end{aligned}
\tag{3.66}
$$

To proceed further we need the transformation property of our four dimensional field. In ten dimensions we know that

$$
\tilde{H}'_{\mu\nu} = \tilde{H}_{\mu\nu} + \frac{1}{\kappa_{10}} \Big(\partial_\mu X_\nu + \partial_\nu X_\mu \Big) + \mathcal{L}_X \tilde{H}_{\mu\nu} \ , \tag{3.67}
$$

which given (3.48) and using $X_\mu = \chi(x)_\mu \xi(x;0)$ leads to

$$
\tilde{h}'_{\mu\nu}(x;\alpha) = \tilde{h}_{\mu\nu}(x;\alpha) + \delta_{0\alpha} \frac{I_2^{\frac{1}{2}}}{\kappa_{10}} \Big(\partial_\mu \chi_\nu + \partial_\nu \chi_\mu \Big) + I(0,0,\alpha) \mathcal{L}_\chi h_{\mu\nu}(x;0) + \int_{\lambda_{edge}}^{\infty} d\lambda \ I(0,\lambda,\alpha) \mathcal{L}_\chi h_{\mu\nu}(x;\lambda) \ , \tag{3.68}
$$

which for the zero mode reads

$$
\tilde{h}'_{\mu\nu}(x;0) = \tilde{h}_{\mu\nu}(x;0) + \frac{I_2^{\frac{1}{2}}}{\kappa_{10}} \Big(\partial_\mu \chi_\nu + \partial_\nu \chi_\mu \Big) + I_3 \mathcal{L}_\chi h_{\mu\nu}(x;0) + \int_{\lambda_{edge}}^{\infty} d\lambda \ I(0,\lambda,0) \mathcal{L}_\chi h_{\mu\nu}(x;\lambda) \ . \tag{3.69}
$$

This is a little bothering because it's clearly not the transformation we're used to, unless $I(0,\lambda,0) = 0$- which it isn't[94]. However, ignoring this for a moment, if things were to work out as we expect them to, then

[94]See Appendix D.

the arguments leading to (3.58) could be repeated and we'd find that it would lead to the requirement that

$$\frac{\kappa_{10} I_3}{I_2^{\frac{3}{2}}} I_3 - \frac{\kappa_{10}^2 I_4}{I_2^2} \frac{I_2^{\frac{1}{2}}}{\kappa_{10}} = 0 \implies I_3^2 = I_4 \; . \tag{3.70}$$

However the form of (3.63) shows this doesn't hold! The story of diffeomorphism invariance in four dimensions seems to not be working as we'd expect. At this stage one might think that the issue is we still have the heavy gravitons, $\tilde{h}_{\mu\nu}(x;\lambda)$, present. From (3.68) they clearly transform under our diffeomorphism into terms involving $\tilde{h}_{\mu\nu}(x;0)$ and so they can clearly be seen to be part of the story, no matter how it eventually plays out. So it may be expected that integrating them out will fix all of our issues. However if this is done it can be shown to not be enough to overcome all of our problems. So what's going on? We seem to have in fact written down a theory that isn't behaving under diffeomorphisms as we expect in four dimensions. Integrating out fields[95] doesn't help us, and as such we can conclude that the sickness runs deeper than we'd like. As we'll see in the next Section, our EFT starter pack, like all good DIY kits, is missing a few parts!

To recap, we began by outlining a, supposedly, general method for building top-down EFTs of gravity. The method involved the use of generalised Fourier series, using the operator identified by Bachas and Estes [4] to determine the expansion basis. It was expected that the overlap integrals of these basis functions would provide coupling coefficients in the lower dimensional theory that would ensure the presence of lower dimensional diffeomorphism invariance. However a direct calculation, in the context of the EFT about the background solution of Crampton, Pope and Stelle [17], showed that this was not the case, with lower dimensional diffeomorphism symmetry being shown to not be present in the EFT.

[95] Or attempting field redefinitions.

4 The perturbation problem revisited

In the previous Section we saw that the proposed EFT construction procedure, when applied to the $k = 0$ CPS background solution, produced a four dimensional theory that lacked diffeomorphism symmetry. This is a terminal flaw in the process as we know that our universe displays such a symmetry. In this Section we shall attempt to diagnose the cause of the sickness in our procedure and then cure it. There are a few places that we could look, but the obvious first choice is the perturbation problem discussed in Section 2 since we initially wanted to set four dimensional 'gauge' conditions in order to obtain the results given. Since we now seem to lack this symmetry it seems rather suspect to enforce these conditions at the start of the problem[96].

We shall begin this Section by providing details of the perturbation problem of B&E but without assuming any form of gauge fixing condition or truncating any of the fields. It will then be shown that once the truncation performed by B&E is implemented the resulting system of equations doesn't possess lower dimensional diffeomorphism invariance. As such, we find that underlying the loss of diffeomorphism symmetry, as seen in the previous Section, is an inconsistent truncation of the field content in the EFT. This inconsistency will be shown to arise precisely because of the non-trivial transverse dependence of the lower dimensional graviton. The resolution to this loss of diffeomorphism symmetry can then be understood to be the restoration of the fields truncated in the procedure of deriving the B&E wavefunction equation. In particular it is found that off diagonal components of the metric perturbation, which are found to behave as Stueckelberg fields, are crucial in establishing that the lower dimensional system possesses diffeomorphism invariance. We shall then highlight how the usual Kaluza-Klein circle reduction avoids any of these issues, which owes to the trivial dependence of the lower dimensional graviton on the transverse space and the structure of the overlap integrals in this setting.

4.1 How to lose diffeomorphisms: the sting in the tail of inconsistent truncations

Having isolated that there is an issue with lower dimensional diffeomorphism symmetry the results derived in Section 2 are on a shaky foundation, if we wish to achieve them with lower dimensional gauge conditions rather than higher dimensional ones. As such let's back out of our gauge choice and see what we can learn. In this Section we shall provide analogues of the results given in Section 2 however we won't impose any gauge conditions, be they higher or lower dimensional in nature. We shall, however, later on consider perturbations

[96]We can, and did, still use higher dimensional gauge conditions which means our results still hold.

of only the worldvolume portion of the metric, as in (2.2), setting the rest of the perturbations, including matter ones, to zero. This truncation shall be referred to as the worldvolume sector truncation and it shall be indicated, explicitly, when this truncation has been implemented.

Since we're no longer allowing ourself to make use of gauge choices we can't make use of the condition

$$\bar{\nabla}^M H_{MN} = 0 \ , \tag{4.1}$$

which leads to the occurrence of various new terms in the expressions previously given in Section 2. The first of these changes is that (2.30) now reads

$$\begin{aligned}\delta\hat{R}_{MN} = \frac{1}{2}\Big(&\hat{R}_{QM}\delta\hat{g}^Q{}_N - \hat{R}^Q{}_{NPM}\delta\hat{g}^P{}_Q + \hat{R}_{QN}\delta\hat{g}^Q{}_M - \hat{R}^Q{}_{MPN}\delta\hat{g}^P{}_Q \\ &+ \hat{\nabla}_N\hat{\nabla}_Q\delta\hat{g}_M{}^Q + \hat{\nabla}_M\hat{\nabla}_Q\delta\hat{g}_N{}^Q - \hat{\nabla}_P\hat{\nabla}^P\delta\hat{g}_{MN} - \hat{\nabla}_N\hat{\nabla}_M\delta\hat{g}\Big) \ ,\end{aligned} \tag{4.2}$$

while (2.31) remains unchanged. In order to proceed we need to calculate the constituent terms of (4.2) in terms of the conformally related metric

$$\bar{g}_{MN} = e^{-2A}\hat{g}_{MN} \ , \tag{4.3}$$

and the perturbation

$$H_{MN} = e^{-2A}\delta\hat{g}_{MN} \ . \tag{4.4}$$

We find that $\hat{\nabla}_M\hat{\nabla}_N\delta\hat{g}$ is still given by (2.37), whilst the terms $\hat{\nabla}_N\hat{\nabla}_Q\delta\hat{g}_M{}^Q$, $\hat{\nabla}_M\hat{\nabla}_Q\delta\hat{g}_N{}^Q$ and $\hat{\nabla}_P\hat{\nabla}^P\delta\hat{g}_{MN}$ all arise from appropriate contractions of (2.39). This leads to the results

$$\begin{aligned}\hat{\nabla}_N\hat{\nabla}_Q\delta\hat{g}_M{}^Q =& dH_{MQ}\bar{\nabla}_N\bar{\nabla}^Q A - dH_{MQ}\bar{\nabla}_N A\bar{\nabla}^Q A - H\bar{\nabla}_N\bar{\nabla}_M A + 2H\bar{\nabla}_N A\bar{\nabla}_M A \\ &-\bar{g}_{NM}H\bar{\nabla}^Q A\bar{\nabla}_Q A - dH_{NQ}\bar{\nabla}_M A\bar{\nabla}^Q A + d\bar{g}_{NM}H_{RS}\bar{\nabla}^R A\bar{\nabla}^S A - \bar{\nabla}^Q H_{MQ}\bar{\nabla}_N A \\ &+d\bar{\nabla}_N H_{MR}\bar{\nabla}^R A - \bar{\nabla}_N H\bar{\nabla}_M A + \bar{\nabla}_N\bar{\nabla}^Q H_{MQ} - \bar{\nabla}^Q H_{NQ}\bar{\nabla}_M A + \bar{g}_{NM}\bar{\nabla}^Q H_{RQ}\bar{\nabla}^R A \ ,\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\hat{\nabla}^Q\hat{\nabla}_Q\delta\hat{g}_{MN} = & -2(\bar{\nabla}^Q A)(\bar{\nabla}_Q A)H_{MN} - dH_{MP}\bar{\nabla}^P A\bar{\nabla}_N A - dH_{NP}\bar{\nabla}^P A\bar{\nabla}_M A + 2H\bar{\nabla}_M A\bar{\nabla}_N A \\ &+2\bar{g}_{MN}H_{RS}\bar{\nabla}^R A\bar{\nabla}^S A + (d-2)\bar{\nabla}_P H_{MN}\bar{\nabla}^P A - 2\bar{\nabla}^P H_{MP}\bar{\nabla}_N A - 2\bar{\nabla}^P H_{NP}\bar{\nabla}_M A \\ &+2\bar{\nabla}_M H_{RN}\bar{\nabla}^R A + 2\bar{\nabla}_N H_{RM}\bar{\nabla}^R A + \bar{\nabla}^P\bar{\nabla}_P H_{MN} \ ,\end{aligned} \tag{4.6}$$

where d is the dimension of the full spacetime manifold.

Using standard results on conformal transformations we find that (2.42) now takes the form

$$\hat{R}_{QN}\delta\hat{g}^{Q}{}_{M} = \bar{R}_{QN}H^{Q}{}_{M} - H_{MN}\Big(\bar{\nabla}^2 A + (d-2)(\bar{\nabla}^{Q}A)(\bar{\nabla}_{Q}A)\Big) \\ -(d-2)H^{Q}{}_{M}\bar{\nabla}_{N}\bar{\nabla}_{Q}A + (d-2)H^{Q}{}_{M}\bar{\nabla}_{N}A\bar{\nabla}_{Q}A \; , \tag{4.7}$$

and (2.43) becomes

$$\hat{R}^{Q}{}_{NPM}\delta\hat{g}^{P}{}_{Q} = \bar{R}^{Q}{}_{NPM}H^{P}{}_{Q} + H^{P}{}_{M}\bar{\nabla}_{P}\bar{\nabla}_{N}A - H\bar{\nabla}_{M}\bar{\nabla}_{N}A - \bar{g}_{NM}H^{P}{}_{Q}\bar{\nabla}_{P}\bar{\nabla}^{Q}A + H_{QN}\bar{\nabla}_{M}\bar{\nabla}^{Q}A \\ +H\bar{\nabla}_{M}A\bar{\nabla}_{N}A - H^{P}{}_{M}\bar{\nabla}_{P}A\bar{\nabla}_{N}A - \bar{g}_{MN}(\bar{\nabla}^{P}A)(\bar{\nabla}_{P}A)H + H_{MN}(\bar{\nabla}^{P}A)(\bar{\nabla}_{P}A) \\ -H_{QN}\bar{\nabla}_{M}A\bar{\nabla}^{Q}A + \bar{g}_{NM}H_{PQ}\bar{\nabla}^{P}A\bar{\nabla}^{Q}A \; . \tag{4.8}$$

We can now use these results in (4.2) to obtain

$$\delta\hat{R}_{MN} = \frac{1}{2}\Big(\bar{R}_{QM}H^{Q}{}_{N} + \bar{R}_{QN}H^{Q}{}_{M} - 2H_{MN}\big(\bar{\nabla}^2 A + (d-2)(\bar{\nabla}^{P}A)(\bar{\nabla}_{P}A)\big) - 2\bar{R}^{Q}{}_{MPN}H^{P}{}_{Q} \\ +2\bar{g}_{NM}H_{PQ}\bar{\nabla}^{P}\bar{\nabla}^{Q}A + 2(d-2)\bar{g}_{MN}H_{RS}\bar{\nabla}^{R}A\bar{\nabla}^{S}A + (d-2)\bar{\nabla}_{N}H_{MR}\bar{\nabla}^{R}A \\ +(d-2)\bar{\nabla}_{M}H_{NR}\bar{\nabla}^{R}A + \bar{\nabla}_{N}\bar{\nabla}^{Q}H_{MQ} + \bar{\nabla}_{M}\bar{\nabla}^{Q}H_{NQ} + 2\bar{g}_{NM}\bar{\nabla}^{Q}H_{RQ}\bar{\nabla}^{R}A \\ -(d-2)(\bar{\nabla}^{Q}A)(\bar{\nabla}_{Q}H_{MN}) - \bar{\Box}H_{MN} - \bar{\nabla}_{M}\bar{\nabla}_{N}H - \bar{g}_{NM}(\bar{\nabla}^{P}A)(\bar{\nabla}_{P}H)\Big) \; , \tag{4.9}$$

which along with (2.31) means that

$$\delta\hat{R} = e^{-2A}\Big(2(d-1)H_{MN}\bar{\nabla}^{M}\bar{\nabla}^{N}A + (d-1)(d-2)H_{MN}\bar{\nabla}^{M}A\bar{\nabla}^{N}A + 2(d-1)\bar{\nabla}^{N}H_{MN}\bar{\nabla}^{M}A \\ -H^{MN}\bar{R}_{MN} - \bar{\Box}H - (d-1)(\bar{\nabla}^{M}A)(\bar{\nabla}_{M}H) + \bar{\nabla}^{M}\bar{\nabla}^{N}H_{MN}\Big) \; . \tag{4.10}$$

Utilising (4.9) and (4.10) we find that the perturbed Einstein tensor takes the form

$$\begin{aligned}\delta\hat{G}_{MN} = \frac{1}{2}\Big(&\bar{R}_{QM}H^{Q}{}_{N} + \bar{R}_{QN}H^{Q}{}_{M} - 2H_{MN}\big(\bar{\Box}A + (d-2)(\bar{\nabla}^{P}A)(\bar{\nabla}_{P}A)\big) - 2\bar{R}^{Q}{}_{MPN}H^{P}{}_{Q}\\ &-2(d-2)\bar{g}_{NM}H_{PQ}\bar{\nabla}^{P}\bar{\nabla}^{Q}A - (d-2)(d-3)\bar{g}_{MN}H_{RS}\bar{\nabla}^{R}A\bar{\nabla}^{S}A + (d-2)\bar{\nabla}_{N}H_{MR}\bar{\nabla}^{R}A\\ &+(d-2)\bar{\nabla}_{M}H_{NR}\bar{\nabla}^{R}A + \bar{\nabla}_{N}\bar{\nabla}^{Q}H_{MQ} + \bar{\nabla}_{M}\bar{\nabla}^{Q}H_{NQ} + 2(2-d)\bar{g}_{NM}\bar{\nabla}^{Q}H_{RQ}\bar{\nabla}^{R}A\\ &-(d-2)(\bar{\nabla}^{Q}A)(\bar{\nabla}_{Q}H_{MN}) - \bar{\Box}H_{MN} - \bar{\nabla}_{M}\bar{\nabla}_{N}H + (d-2)\bar{g}_{NM}(\bar{\nabla}^{P}A)(\bar{\nabla}_{P}H)\\ &+\bar{g}_{MN}(\bar{\nabla}^{2}H + H^{PQ}\bar{R}_{PQ} - \bar{\nabla}^{P}\bar{\nabla}^{Q}H_{PQ})\Big) - \frac{1}{2}\delta\hat{g}_{MN}\hat{R}\,.\end{aligned} \tag{4.11}$$

The mixed index components of this lead to

$$\begin{aligned}\delta\hat{G}_{\mu a} = \frac{1}{2}\Big(&\bar{R}_{\nu\mu}H^{\nu}{}_{a} + \bar{R}_{ba}H^{b}{}_{\mu} - 2H_{\mu a}\big(\bar{\Box}A + (d-2)(\bar{\nabla}^{P}A)(\bar{\nabla}_{P}A)\big) + (d-2)\bar{\nabla}_{a}H_{\mu b}\bar{\nabla}^{b}A\\ &+(d-2)\bar{\nabla}_{\mu}H_{ab}\bar{\nabla}^{b}A + \bar{\nabla}_{a}\bar{\nabla}^{\nu}H_{\mu\nu} + \bar{\nabla}_{a}\bar{\nabla}^{b}H_{\mu b} + \bar{\nabla}_{\mu}\bar{\nabla}^{\nu}H_{a\nu} + \bar{\nabla}_{\mu}\bar{\nabla}^{b}H_{ab}\\ &-(d-2)(\bar{\nabla}^{b}A)(\bar{\nabla}_{b}H_{\mu a}) - \bar{\Box}H_{\mu a} - \bar{\nabla}_{\mu}\bar{\nabla}_{a}H\Big) - \frac{1}{2}\delta\hat{g}_{\mu a}\hat{R}\,,\end{aligned} \tag{4.12}$$

where the fact that $\bar{g}$ is a product metric has been used to drop various terms. The transverse components lead to the expression

$$\begin{aligned}\delta\hat{G}_{ab} = \frac{1}{2}\Big(&\bar{R}_{ca}H^{c}{}_{b} + \bar{R}_{cb}H^{c}{}_{a} - 2H_{ab}\big(\bar{\Box}A + (d-2)(\bar{\nabla}^{P}A)(\bar{\nabla}_{P}A)\big) - 2\bar{R}^{c}{}_{adb}H^{d}{}_{c}\\ &-2(d-2)\bar{g}_{ba}H_{cd}\bar{\nabla}^{c}\bar{\nabla}^{d}A - (d-2)(d-3)\bar{g}_{ab}H_{cd}\bar{\nabla}^{c}A\bar{\nabla}^{d}A + (d-2)\bar{\nabla}_{b}H_{ac}\bar{\nabla}^{c}A\\ &+(d-2)\bar{\nabla}_{a}H_{bc}\bar{\nabla}^{c}A + \bar{\nabla}_{b}\bar{\nabla}^{\mu}H_{a\mu} + \bar{\nabla}_{b}\bar{\nabla}^{c}H_{ac} + \bar{\nabla}_{a}\bar{\nabla}^{\mu}H_{b\mu} + \bar{\nabla}_{a}\bar{\nabla}^{c}H_{bc}\\ &+2(2-d)\bar{g}_{ba}\bar{\nabla}^{\mu}H_{c\mu}\bar{\nabla}^{c}A + 2(2-d)\bar{g}_{ba}\bar{\nabla}^{d}H_{cd}\bar{\nabla}^{c}A - (d-2)(\bar{\nabla}^{c}A)(\bar{\nabla}_{c}H_{ab})\\ &-\bar{\Box}H_{ab} - \bar{\nabla}_{a}\bar{\nabla}_{b}H + (d-2)\bar{g}_{ba}(\bar{\nabla}^{c}A)(\bar{\nabla}_{c}H) + \bar{g}_{ab}(\bar{\Box}H + H^{\mu\nu}\bar{R}_{\mu\nu} + H^{cd}\bar{R}_{cd} - \bar{\nabla}^{P}\bar{\nabla}^{Q}H_{PQ})\Big)\\ &-\frac{1}{2}\delta\hat{g}_{ab}\hat{R}\,,\end{aligned} \tag{4.13}$$

and the worldvolume components give

$$
\begin{aligned}
\delta\hat{G}_{\mu\nu} = \frac{1}{2}\Big(&\bar{R}_{\sigma\mu}H^{\sigma}{}_{\nu} + \bar{R}_{\sigma\nu}H^{\sigma}{}_{\mu} - 2H_{\mu\nu}\big(\bar{\Box}A + (d-2)(\bar{\nabla}^a A)(\bar{\nabla}_a A)\big) - 2\bar{R}^{\sigma}{}_{\mu\tau\nu}H^{\tau}{}_{\sigma} \\
&-2(d-2)\bar{g}_{\nu\mu}H_{ab}\bar{\nabla}^a\bar{\nabla}^b A - (d-2)(d-3)\bar{g}_{\mu\nu}H_{ab}\bar{\nabla}^a A\bar{\nabla}^b A + (d-2)\bar{\nabla}_{\nu}H_{\mu a}\bar{\nabla}^a A \\
&+(d-2)\bar{\nabla}_{\mu}H_{\nu a}\bar{\nabla}^a A + \bar{\nabla}_{\nu}\bar{\nabla}^{\sigma}H_{\mu\sigma} + \bar{\nabla}_{\nu}\bar{\nabla}^a H_{\mu a} + \bar{\nabla}_{\mu}\bar{\nabla}^{\sigma}H_{\nu\sigma} + \bar{\nabla}_{\mu}\bar{\nabla}^a H_{\nu a} \\
&+2(2-d)\bar{g}_{\nu\mu}\bar{\nabla}^{\sigma}H_{a\sigma}\bar{\nabla}^a A + 2(2-d)\bar{g}_{\nu\mu}\bar{\nabla}^b H_{ab}\bar{\nabla}^a A - (d-2)(\bar{\nabla}^a A)(\bar{\nabla}_a H_{\mu\nu}) - \bar{\Box}H_{\mu\nu} \\
&-\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}H + (d-2)\bar{g}_{\mu\nu}(\bar{\nabla}^a A)(\bar{\nabla}_a H) + \bar{g}_{\mu\nu}(\bar{\Box}H + H^{\sigma\tau}\bar{R}_{\sigma\tau} + H^{ab}\bar{R}_{ab} - \bar{\nabla}^P\bar{\nabla}^Q H_{PQ})\Big) \\
&-\frac{1}{2}\delta\hat{g}_{\mu\nu}\hat{R} \, .
\end{aligned} \tag{4.14}
$$

So far these results hold for a completely generic metric of the form (2.1) and an arbitrary perturbation H_{MN}. As a result we must surely have diffeomorphism symmetry at this stage.

However, rather than just taking it for granted let's consider the linearised Einstein equations for a pure gravity setup. In this case we consider a perturbation of the form

$$
g_{MN} = \mathring{g}_{MN} + H_{MN} \, , \tag{4.15}
$$

with $M \in \{1,\ldots,d\}$ and $\mathring{g}_{MN}$ a solution of the vacuum Einstein equations. Using the results given in Appendix A we can show that the linearised Einstein equations take the form

$$
\mathring{\Box}H_{MN} - \mathring{g}_{MN}\mathring{\Box}H + \mathring{g}_{MN}\mathring{\nabla}^P\mathring{\nabla}^Q H_{PQ} + \mathring{\nabla}_N\mathring{\nabla}_M H - \mathring{\nabla}^P\mathring{\nabla}_M H_{PN} - \mathring{\nabla}^P\mathring{\nabla}_N H_{PM} = 0 \, , \tag{4.16}
$$

where $\mathring{\nabla}$ is the covariant derivative associated to the Levi-Civita connection of $\mathring{g}$ and $H = \mathring{g}^{MN}H_{MN}$. By now we know the form of an infinitesimal diffeomorphism by heart, so let's check if (4.16) is invariant under

$$
H_{MN} \to H'_{MN} = H_{MN} + \mathring{\nabla}_M\mathcal{X}_N + \mathring{\nabla}_N\mathcal{X}_M \, . \tag{4.17}
$$

Under (4.17) the change in (4.16) is

$$
\begin{aligned}
&\mathring{\Box}\mathring{\nabla}_M\mathcal{X}_N + \mathring{\Box}\mathring{\nabla}_N\mathcal{X}_M - 2\mathring{g}_{MN}\mathring{\Box}\mathring{\nabla}^P\mathcal{X}_P + \mathring{g}_{MN}\mathring{\nabla}^P\mathring{\nabla}^Q\mathring{\nabla}_P\mathcal{X}_Q + \mathring{g}_{MN}\mathring{\nabla}^P\mathring{\nabla}^Q\mathring{\nabla}_Q\mathcal{X}_P \\
&+2\mathring{\nabla}_N\mathring{\nabla}_M\mathring{\nabla}^P\mathcal{X}_P - \mathring{\nabla}^P\mathring{\nabla}_M\mathring{\nabla}_P\mathcal{X}_N - \mathring{\nabla}^P\mathring{\nabla}_N\mathring{\nabla}_P\mathcal{X}_M - \mathring{\nabla}^P\mathring{\nabla}_M\mathring{\nabla}_N\mathcal{X}_P - \mathring{\nabla}^P\mathring{\nabla}_N\mathring{\nabla}_M\mathcal{X}_P \, .
\end{aligned} \tag{4.18}
$$

If we use

$$\mathring{\nabla}_P\mathring{\nabla}_Q\mathcal{X}^Q = \mathring{\nabla}_Q\mathring{\nabla}_P\mathcal{X}^Q - \mathring{R}^Q{}_{MQP}\mathcal{X}^M \ , \tag{4.19}$$

and its generalisations to higher rank tensors to commute covariant derivatives then (4.18), upon recalling $\mathring{R}_{MN} = 0$, (4.18) becomes

$$\mathring{\nabla}^P\mathcal{X}^Q\Big(\mathring{R}_{MPNQ} - \mathring{R}_{MQNP} - \mathring{R}_{PQMN}\Big) + \mathcal{X}^Q\mathring{\nabla}^P\Big(\mathring{R}_{NQMP} - \mathring{R}_{MQNP} - \mathring{R}_{PQMN}\Big) \ . \tag{4.20}$$

If we now note that our background curvature tensor obeys the first Bianchi identity

$$\mathring{R}_{MNPQ} + \mathring{R}_{MPQN} + \mathring{R}_{MQNP} = 0 \ , \tag{4.21}$$

then we can see that (4.20) vanishes thus confirming we have linearised diffeomorphism symmetry for a perturbed theory of gravity at this order. So, with this to back us up, it seems sensible to presume our system starts with diffeomorphism symmetry. So where do we loose it?

The next step we performed was the worldvolume sector truncation so let's go ahead with that. In order to have a concrete setting to discuss let's again consider our theory to be that given by the NSNS sector of Type IIA Supergravity, written in Einstein frame variables. If we perform the worldvolume sector truncation then the components of the linearised Einstein tensor (4.12)-(4.14) become

$$\delta\hat{G}_{\mu a} = \frac{1}{2}\Big(\bar{\nabla}_a\bar{\nabla}^\nu H_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_a H\Big) \ , \tag{4.22}$$

$$\delta\hat{G}_{ab} = \frac{1}{2}\Big(- \bar{\nabla}_a\bar{\nabla}_b H + (d-2)\bar{g}_{ab}\bar{\nabla}^c A\bar{\nabla}_c H + \bar{g}_{ab}\big(\bar{\Box}H + H^{\mu\nu}\bar{R}_{\mu\nu} - \bar{\nabla}^\mu\bar{\nabla}^\nu H_{\mu\nu}\big)\Big) \ , \tag{4.23}$$

$$\begin{aligned}\delta\hat{G}_{\mu\nu} = \frac{1}{2}\Big(&\bar{R}_{\sigma\mu}H^\sigma{}_\nu + \bar{R}_{\sigma\nu}H^\sigma{}_\mu - 2H_{\mu\nu}\big(\bar{\Box}A + (d-2)(\bar{\nabla}^a A)(\bar{\nabla}_a A)\big) - 2\bar{R}^\sigma{}_{\mu\tau\nu}H^\tau{}_\sigma + \bar{\nabla}_\nu\bar{\nabla}^\sigma H_{\mu\sigma}\\ &+\bar{\nabla}_\mu\bar{\nabla}^\sigma H_{\nu\sigma} - (d-2)(\bar{\nabla}^a A)(\bar{\nabla}_a H_{\mu\nu}) - \bar{\Box}H_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu H + (d-2)\bar{g}_{\mu\nu}(\bar{\nabla}^a A)(\bar{\nabla}_a H)\\ &+\bar{g}_{\mu\nu}(\bar{\Box}H + H^{\sigma\tau}\bar{R}_{\sigma\tau} - \bar{\nabla}^\sigma\bar{\nabla}^\tau H_{\sigma\tau})\Big) - \frac{1}{2}\delta\hat{g}_{\mu\nu}\hat{R} \ .\end{aligned} \tag{4.24}$$

Since the worldvolume sector truncation sets the matter perturbations to zero, the linearised perturbation of the energy momentum tensor is still given by (2.60). If we now collect all of the linearised equations of motion of this system, using that the worldvolume is a maximally symmetric space, so (2.10) holds, we find that our perturbation must obey

$$\bar{\nabla}_a\bar{\nabla}^\nu H_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_a H = 0 \ , \tag{4.25}$$

$$-\bar{\nabla}_a\bar{\nabla}_b H + (d-2)\bar{g}_{ab}\bar{\nabla}^c A\bar{\nabla}_c H + \bar{g}_{ab}\big(\bar{\Box} H + k(n-1)H - \bar{\nabla}^\mu\bar{\nabla}^\nu H_{\mu\nu}\big) = 0\ , \tag{4.26}$$

$$\begin{aligned}&k(n-3)\bar{g}_{\mu\nu}H + 2kH_{\mu\nu} + \bar{\nabla}_\mu\bar{\nabla}^\sigma H_{\nu\sigma} + \bar{\nabla}_\nu\bar{\nabla}^\sigma H_{\mu\sigma} - (d-2)\bar{\nabla}_a H_{\mu\nu}\bar{\nabla}^a A - \bar{\Box} H_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu H\\ &+(d-2)\bar{g}_{\mu\nu}\bar{\nabla}^a A\bar{\nabla}_a H - \bar{g}_{\mu\nu}\bar{\nabla}^\sigma\bar{\nabla}^\tau H_{\sigma\tau} + \bar{g}_{\mu\nu}\bar{\Box} H = 0\ .\end{aligned} \tag{4.27}$$

$$\bar{\nabla}^a H\bar{\nabla}_a\phi = 0\ , \tag{4.28}$$

$$H^{aMN}\big(\bar{\nabla}_a H - 2H\bar{\nabla}_a A\big) = 0\ , \tag{4.29}$$

where H^{MNP} is the background 3-form, ϕ is the background dilaton and n is the dimension of the world-volume. Upon assuming the separable form, (2.3), for $H_{\mu\nu}$, these equations become

$$\bar{\nabla}_a\psi\big(\bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}_\mu h\big) = 0\ , \tag{4.30}$$

$$-h\bar{\nabla}_a\bar{\nabla}_b\psi + \bar{g}_{ab}\big((d-2)h\bar{\nabla}^c A\bar{\nabla}_c\psi + \psi\bar{\Box}_x h + h\bar{\Box}_y\psi + k(n-1)h\psi - \psi\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu}\big) = 0\ , \tag{4.31}$$

$$\begin{aligned}&k(n-3)\bar{g}_{\mu\nu}h\psi + 2kh_{\mu\nu}\psi + \psi\bar{\nabla}_\mu\bar{\nabla}^\sigma h_{\nu\sigma} + \psi\bar{\nabla}_\nu\bar{\nabla}^\sigma h_{\mu\sigma} - (d-2)h_{\mu\nu}\bar{\nabla}_a\psi\bar{\nabla}^a A - h_{\mu\nu}\bar{\Box}_y\psi\\ &-\psi\bar{\Box}_x h_{\mu\nu} - \psi\bar{\nabla}_\mu\bar{\nabla}_\nu h + (d-2)\bar{g}_{\mu\nu}h\bar{\nabla}^a A\bar{\nabla}_a\psi - \bar{g}_{\mu\nu}\psi\bar{\nabla}^\sigma\bar{\nabla}^\tau h_{\sigma\tau} + \bar{g}_{\mu\nu}h\bar{\Box}_y\psi + \bar{g}_{\mu\nu}\psi\bar{\Box}_x h = 0\ ,\end{aligned} \tag{4.32}$$

$$h\bar{\nabla}^a\psi\bar{\nabla}_a\phi = 0\ , \tag{4.33}$$

$$H^{aMN}h\big(\bar{\nabla}_a\psi - 2\psi\bar{\nabla}_a A\big) = 0\ . \tag{4.34}$$

It is expected that these equations are invariant under the transformation

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \bar{\nabla}_\mu\chi_\nu + \bar{\nabla}_\nu\chi_\mu\ , \quad \chi_\mu = \chi_\mu(x)\ . \tag{4.35}$$

However if we take (4.33) then we can see that under (4.35) this equation transforms as

$$\bar{\nabla}^\mu\chi_\mu\bar{\nabla}^a\psi\bar{\nabla}_a\phi\ , \tag{4.36}$$

while (4.30) transforms into

$$\bar{\nabla}_a\psi\big(\bar{\nabla}^\nu\bar{\nabla}_\mu\chi_\nu + \bar{\Box}_x\chi_\mu - 2\bar{\nabla}_\mu\bar{\nabla}^\nu\chi_\nu\big)\ . \tag{4.37}$$

Since both of these must vanish for any ψ we can see this leads to constraints on our diffeomorphism parameter. For example (4.36) leads to

$$\bar{\nabla}^{\mu}\chi_{\mu} = 0 \; , \tag{4.38}$$

while, in combination with (4.38), (4.37) leads to the constraint

$$\bar{\Box}_x\chi_{\mu} + k(n-1)\chi_{\mu} = 0 \; . \tag{4.39}$$

So even to make two of our equations invariant we've had to constrain the parameter χ_{μ}. However, we've previously seen, (4.20), that the parameters of a diffeomorphism are unconstrained by demanding invariance of the linearised equations. As such, our current system can clearly be seen to have, at most, a reduced diffeomorphism symmetry. This explains why our EFT in the previous Section failed to show a four dimensional diffeomorphism symmetry, it wasn't present in ten dimensions! We can also see that setting the conditions

$$\bar{\nabla}^{\mu}h_{\mu\nu} = 0 \; , \quad h = 0 \; , \tag{4.40}$$

makes (4.30),(4.31),(4.33) and (4.34) trivial and, hence, we'll never notice this issue, as we appear to have 'gauge' fixed our equations and obtained exactly the expected result on the worldvolume. We might then think that we can just remove this lower dimensional 'gauge' fixing and write down the gauge unfixed equations. However we now see that (4.40) are not gauge conditions, as we didn't initially possess the gauge symmetry required to set them, and it's just a nice coincidence that setting them drops us straight in to the lap of a system we think we know!

4.2 Finding diffeomorphisms: going beyond the worldvolume sector truncation

It should now be obvious that the issue is with the worldvolume sector truncation, since before applying it our symmetry was intact. The lack of a worldvolume gauge symmetry suggests that there are fields, that have been set to zero, which are required to transform to non-zero values in order for the lower dimensional system to display the desired symmetry. As such we need to identify what these fields may be.

In order to find our missing fields, let's recap what we've done. We began with a worldvolume perturbation that had dependence upon the transverse space. So in our higher dimensional theory we had a perturbation $H_{\mu\nu} = H_{\mu\nu}(x, y)$. In order to get to a lower dimensional theory we then expanded this perturbation in terms of some transverse wavefunctions[97] as in (3.14). We then assumed that within this expansion there was a

[97] These were the eigenfunctions of the transverse wavefunction operator of B&E. Since the choice of expansion of the higher dimensional fields is entirely up to us we can still use these functions even though it turns out the perturbation problem arising

mode that corresponded to a massless graviton in the lower dimension. We took this to be the mode

$$h_{\mu\nu}(x;0)\xi(y;0)\ , \tag{4.41}$$

and we wanted our lower dimensional field, $h_{\mu\nu}(x;0)$, to have a sensible lower dimensional diffeomorphism symmetry. It's at this point that we remember that all of our lower dimensional symmetries must embed into the higher dimensional symmetries. As such if we wish to perform diffeomorphisms that we can interpret in our lower dimensional theory we must expand them in the same manner as our fields. As worldvolume diffeomorphisms are the ones we're, currently, interested in, this means we should expand our parameter X_μ as

$$X_\mu(x,y) = \sum_\alpha \chi_\mu(x;\alpha)\xi(y;\alpha)\ , \tag{4.42}$$

with the sum again symbolic of taking one mode per eigenvalue and hence could correspond to a combination of a discrete and continuous label. As we saw in (3.68), to get the correct leading order transformation, meaning that $h_{\mu\nu}(x;0)$ transforms as in (4.35) at order zero in fields, we need to perform the diffeomorphism

$$X^{(0)}_\mu(x,y) := \chi_\mu(x;0)\xi(y;0)\ . \tag{4.43}$$

We can now look at the affect of this specific higher dimensional diffeomorphism on the fields present in the theory.

We already know that, at linear order in $X^{(0)}_\mu$, H_{MN} will transform as

$$H_{MN} \to H'_{MN} = H_{MN} + \bar\nabla_M X^{(0)}_N + \bar\nabla_N X^{(0)}_M + \mathcal{L}_{X^{(0)}} H_{MN}\ . \tag{4.44}$$

However we're more interested in how this transformation behaves once we split indices. Once this is done we find that

$$H_{\mu\nu} \to H'_{\mu\nu} = H_{\mu\nu} + \xi(y;0)\big(\bar\nabla_\mu\chi_\nu + \bar\nabla_\nu\chi_\mu\big) + \xi(y;0)\mathcal{L}_\chi H_{\mu\nu}\ , \tag{4.45}$$

$$H_{\mu a} \to H'_{\mu a} = H_{\mu a} + \chi_\mu\bar\nabla_a\xi(y;0) + \xi(y;0)\mathcal{L}_\chi H_{\mu a} + H_{\mu\nu}\chi^\nu\bar\nabla_a\xi(y;0)\ , \tag{4.46}$$

$$H_{ab} \to H'_{ab} = H_{ab} + \xi(y;0)\mathcal{L}_\chi H_{ab} + H_{\mu b}\chi^\mu\bar\nabla_a\xi(y;0) + H_{a\mu}\chi^\mu\bar\nabla_b\xi(y;0)\ . \tag{4.47}$$

The reason for our loss of lower dimensional diffeomorphism symmetry now becomes painfully obvious. When we performed the worldvolume sector truncation we set $H_{\mu a}$ and H_{ab} to zero. However when we

in a worldvolume sector truncation doesn't lead to a gauge system.

perform our desired diffeomorphism these fields are transformed to non-zero values! So the worldvolume sector truncation is incompatible with lower dimensional diffeomorphism symmetry. As a result any EFT built using it, clearly, shouldn't be expected to display lower dimensional diffeomorphism symmetry.

Fortunately our discovery of the cause of our loss of lower dimensional diffeomorphism symmetry also tells us how we might be able to restore it. We just have to keep around all of the fields present in an arbitrary higher dimensional perturbation H_{MN}. Then, since we haven't truncated anything, our theory definitely possesses higher dimensional diffeomorphism symmetry and so this could[98] give rise to such a symmetry in our EFT. In order to be able to investigate lower dimensional diffeomorphism symmetry we need to interpret (4.45)-(4.47) in terms of lower dimensional fields. This involves a choice of expansion, in terms of their transverse dependence, for $H_{\mu a}$ and H_{ab} rather than detail this here we shall defer doing this until the next Section where we shall discuss it in detail. Instead here let's just look at the worldvolume portion of the perturbation. Since the wavefunctions $\{\xi(y;\alpha)\}$, by assumption, form an orthonormal basis[99] our perturbations $H_{\mu\nu}$ and $H'_{\mu\nu}$ can both be expanded in terms of this set of functions. From this the transformation of the lower dimensional modes is found to be[100]

$$h'_{\mu\nu}(x;\alpha) = h_{\mu\nu}(x;\alpha) + \delta_{0\alpha}\big(\bar{\nabla}_\mu \chi_\nu + \bar{\nabla}_\nu \chi_\mu\big) + \sum_\beta \Big(\int d^{d-n}y \; w(y)\xi(y;0)\xi(y;\beta)\xi(y;\alpha)\Big)\mathcal{L}_\chi h_{\mu\nu}(x;\beta) \ , \quad (4.48)$$

where $w(y)$ is the measure factor appropriate for the inner product used in the procedure of orthonormalising the basis. For the zero mode this leads to

$$h'_{\mu\nu}(x;0) = h_{\mu\nu}(x;0)+\bar{\nabla}_\mu\chi_\nu+\bar{\nabla}_\nu\chi_\mu+I_3\mathcal{L}_\chi h_{\mu\nu}(x;0)+\sum_i \Big(\int d^{d-n}y \; w(y)\xi(y;0)^2\xi(y;i)\Big)\mathcal{L}_\chi h_{\mu\nu}(x;i) \ , \quad (4.49)$$

while for the non-zero modes we have

$$h'_{\mu\nu}(x;i) = h_{\mu\nu}(x;i) + \sum_\beta \Big(\int d^{d-n}y \; w(y)\xi(y;0)\xi(y;\beta)\xi(y;i)\Big)\mathcal{L}_\chi h_{\mu\nu}(x;\beta) \ . \quad (4.50)$$

Here i runs over the same values as α except it excludes zero and

$$I_3 = \int d^{d-n}y \; w(y)\xi(y;0)^3 \ . \quad (4.51)$$

[98]We're being deliberately careful here since the main lesson of this and the last Section is that we shouldn't take things for granted!

[99]With respect to some inner product $(\xi(\alpha),\xi(\beta)) = \int d^{d-n}y \; w(y)\xi(y;\alpha)\xi(y;\beta)$.

[100]Here we assume that products of the basis elements and derivatives are also in the span of the basis. In later Sections we'll see we need to revise this assumption, but for ease let's assume, for now, that our system possesses this property.

These transformations should be a little troubling. We know that a lower dimensional graviton $\mathfrak{h}_{\mu\nu}$ should transform as

$$\mathfrak{h}_{\mu\nu} \to \mathfrak{h}'_{\mu\nu} = \mathfrak{h}_{\mu\nu} + \bar{\nabla}_\mu \chi_\nu + \bar{\nabla}_\nu \chi_\mu + I_3 \mathcal{L}_\chi \mathfrak{h}_{\mu\nu} \ , \tag{4.52}$$

however, while our proposed graviton, $h_{\mu\nu}(x;0)$, starts out transforming correctly it picks up contributions from all of the heavy modes $h_{\mu\nu}(x;i)$. This highlights a problem that we'll have to address in more detail in the next Section. Namely the issue is if we can perform a field redefinition to obtain a new field that transforms as in (4.52), but we're jumping the gun by discussing that here. The other feature we note is that the heavy fields can't just be set to zero as (4.56) shows their transformation, in general, picks up a piece proportional to $h_{\mu\nu}(x;0)$, hence to correctly deal with these fields in our EFT we expect to have to integrate them out[101].

With all of these issues to deal with we may feel a little overwhelmed and confused. How come we don't see all of these issues arising in the normal systems we know and love? Well the answer to this is fairly simple, the systems we know and love are in fact just the Kaluza-Klein circle reduction. This is a case where all of the required explicit calculations can be performed, so how come we're not made aware of the issues highlighted above in this case? The crucial ingredient is the fact that the transverse wavefunction of the massless mode in this case is the function $e^0 = 1$. As such in (4.43) we have no dependence on the coordinate z of the circle. This then filters through to the transformations (4.45)-(4.47) which now become

$$\begin{aligned} H_{\mu\nu} \to H'_{\mu\nu} =& H_{\mu\nu} + \left(\bar{\nabla}_\mu \chi_\nu + \bar{\nabla}_\nu \chi_\mu\right) + \mathcal{L}_\chi H_{\mu\nu} \ , \\ H_{\mu z} \to H'_{\mu z} =& H_{\mu z} + \mathcal{L}_\chi H_{\mu z} \ , \\ H_{zz} \to H'_{zz} =& H_{zz} + \mathcal{L}_\chi H_{zz} \ , \end{aligned} \tag{4.53}$$

these are already a significant simplification as we can see only the worldvolume fields receive a transformation that includes no fields[102]. Beyond this at the order field level everything is transforming tensorially and we don't have any mixing of fields. So in the higher dimensions life is already significantly simpler. However life is even easier, as in terms of the lower dimensional fields, (4.54) becomes

$$h'_{\mu\nu}(x;0) = h_{\mu\nu}(x;0) + \bar{\nabla}_\mu \chi_\nu + \bar{\nabla}_\nu \chi_\mu + I_3 \mathcal{L}_\chi h_{\mu\nu}(x;0) \ , \tag{4.54}$$

[101] This is dependent upon our previous assumptions holding, as we'll see if they don't then the story differs.
[102] Often also called the inhomogeneous piece of the transformation.

owing to the fact that

$$\int dz\ \xi(z;0)\xi(z;i) = \int dz\ \xi(z;i) = 0\ , \tag{4.55}$$

since $\xi(z;0) = 1$. So our $h_{\mu\nu}(x;0)$ transforms exactly like a graviton! Even better (4.56)

$$h'_{\mu\nu}(x;i) = h_{\mu\nu}(x;i) + \mathcal{L}_\chi h_{\mu\nu}(x;i)\ , \tag{4.56}$$

since

$$\int dz\ \xi(z;0)\xi(z;i)\xi(z;j) = \int dz\ \xi(z;i)\xi(z;j) = \delta_{ij}\ , \tag{4.57}$$

with the final equality arising because our basis is orthonormalised. So not only does $h_{\mu\nu}(x;0)$ transform as we want but the heavy fields also transform as we expect heavy gravitons too! So all of our previous issues have disappeared. This teaches us a critical lesson, our issues with having lost diffeomorphism symmetry in the previous Section are related to allowing for non-trivial dependence of the zero mode wavefunction on the transverse directions. Hence whilst we've encountered this problem in one particular setting it will occur in generic reductions where the reduction allows for non-trivial dependence on the transverse space. It's this point that we shall pursue in the following Section.

Before moving on let's take stock of what we've learnt in this Section. We've seen, by studying the higher dimensional equations of motion, that the loss of lower dimensional diffeomorphism symmetry observed in Section 3 arises due to the truncation of the metric perturbation to only components along the worldvolume. This leads us to include all possible metric perturbations into the problem. In particular we've seen that off diagonal components of the metric perturbation, that might have been expected to behave as vector fields, become Stueckelberg fields for lower dimensional diffeomorphism symmetry. We showed that this arose because of the non-trivial dependence of the lower dimensional graviton on the transverse space. As a result such behaviour is concluded to occur in all theories where the massless lower dimensional graviton has non-trivial dependence on the transverse space. The obvious counter example to this being the case of the Kaluza-Klein S^1 reduction, which was checked, and shown to workout precisely because of the trivial dependence of the lower dimensional graviton on the transverse space.

5 Effective field theories of gravity: a new approach

In the previous Section we saw that the procedure outlined in Section 3 isn't, in general, robust enough to lead to a lower dimensional theory that possesses diffeomorphism symmetry. The major barrier to this is that during the procedure a truncation of the graviton modes to only include worldvolume perturbations is performed. Whilst we can get away with this in the case where the transverse dependence of our gravitons is trivial, the instant we allow for non-trivial transverse dependence, as is forced on us if the transverse space is non-compact[103], this truncation runs us head first into trouble. The main issue we find is that when we attempt to perform our chosen diffeomorphism, which from a higher dimensional perspective must have non-trivial transverse dependence, we find, from a lower dimensional point of view, fields that our truncation sets to zero are turned on. The fact that these fields become non-zero is a tell tale sign that they're involved in the story of lower dimensional diffeomorphism symmetry.

While we may have learnt about this issue in the context of a specific example, it obviously applies to a far wider class of systems. It's this general problem that we'd really like to address. The problem based around the $k = 0$ CPS solution is complicated not only by the issue of having non-trivial transverse dependence but also by the fact that there are numerous other fields, both perturbative and background in nature, that muddy the waters. As such, rather than dealing with all of these difficulties in one go, we shall consider a simplified model. To this end we shall consider a 5d to 4d reduction in the context of pure gravity. The exact model involves perturbations about $\mathbb{R}^{1,3} \times [0, 1]$ equipped with the usual Minkowski metric. To mirror the general case we're trying to study, boundary conditions for the B&E problem are chosen so that the graviton spectrum only allows for non-trivial transverse dependence. This setup closely mirrors that of the usual Kaluza-Klein circle system, aside from the obvious fact that in the Kaluza-Klein context the zero mode has trivial transverse dependence, and as such it shall prove valuable to consider this system in detail.

Our end goal is to be able to write down a lower dimensional action that possesses a lower dimensional form of diffeomorphism symmetry[104]. In order to do this we shall have to correctly identify a quadratic action that exhibits such a symmetry, it's this task that we shall turn to in the beginning of this Section. Once we have understood how the quadratic action should be constructed we shall turn to consider aspects involved in the construction of the theory at higher order in fields[105]. In this case it will prove invaluable to

[103] Well it is forced on us if we wish to have a non-zero value for the lower dimensional Newton constant.

[104] At this stage we're not able to say if this will take the same form as that usually expected for gravity so we'll remain agnostic on this point for now.

[105] At the time of writing it is believed that a resolution to all issues has been found and an action displaying diffeomorphism invariance up to fourth order has been obtained. Unfortunately, due to time constraints, this hasn't had time to be verified but it is expect to be made available in an upcoming publication, the reference for which shall be added here as soon as it can be.

consider the case of a circle Kaluza-Klein reduction, as in this setting the full non-linear lower dimensional theory and ansatz, at least at the level of the zero mode sector, is known.

Through the course of this Section we shall see that the Stueckelberg field, that we previously identified for it's role in allowing lower dimensional diffeomorphism invariance to appropriately be embedded into the parent higher dimensional symmetry, can be excised from the quadratic action by the inclusion of boundary terms in the higher dimensional action. This will lead us to a lower dimensional field theory whose physical spectrum includes a massless graviton and scalar along with a tower of massive graviton states. We shall then provide the full quadratic action for the system. This can be shown to display manifest lower dimensional diffeomorphism invariance. After this we proceed to restore the gauge degrees of freedom and give the full un-gauge fixed lower dimensional quadratic action. We then investigate the system at beyond quadratic order, with a focus on how the transformations of the lower dimensional fields are modified. It's shown that attempts to understand the interacting system using the basis of functions discovered at the quadratic order are insufficient. This is shown to owe to the appearance of transverse functions that are not in the span of the functions used at quadratic order. We then propose a method for embedding the desired transformations of our lower dimensional fields into the known higher dimensional symmetry. This procedure will be seen to lead to a redefinition of not only the perturbation, but also of the diffeomorphism parameter. This method is applied to the Kaluza-Klein S^1 reduction and the usual non-linear reduction ansatz is obtained at second order in fields.

So in summary in this section we shall investigate the construction an EFT of gravity by looking at pure gravity with the background taking the form of $\mathbb{R}^{1,3} \times [0,1]$ equipped with the Minkowski metric. By choosing Dirichlet-Robin boundary conditions for the perturbation problem we shall select a graviton spectrum with a massless mode that has non-trivial dependence on the transverse space. As expected this leads to the off diagonal metric perturbations assuming the role of a Stueckelberg field for lower dimensional diffeomorphism symmetry. We shall obtain a quadratic action for the EFT and show that it possesses lower dimensional diffeomorphism symmetry once a boundary term is added to the original five dimensional theory [106]. This higher dimensional boundary term is found to excise the kinetic term of the Stueckelberg in the lower dimensional theory. Going beyond the quadratic theory we find that the functions identified in the Sturm-Liouville problem at quadratic order do not form a basis for functions arising at beyond quadratic order. This leads to difficulty in determining the form of the non-linear transformations of the lower dimensional fields. To mitigate this we shall introduce a method for embedding the desired lower dimensional transformations

[106] It is our belief that this term is required for higher dimensional diffeomorphism invariance to be in tact in the theory, however this is yet to be verified.

into the higher dimensional transformation. It will be shown that this method leads to a modification of the perturbation and the diffeomorphism parameter determined at quadratic order, and by direct calculation we shall show this method leads to the usual non-linear Kaluza-Klein reduction ansatz.

Throughout this Section the symbol $'$ is used in two different contexts. On four and five dimensional fields it denotes a transformed version of a field we were previously working with. Whilst on fields that only have dependence on the coordinate z, to be introduced in the main body of this Section, it denotes the result of applying $\frac{d}{dz}$ to the field. We apologise for this overload of notation but hope this comment and context will be sufficient that no confusion is caused to the reader.

5.1 Obtaining a diffeomorphism invariant quadratic action

As we have stated we shall be interested in considering the case of five dimensional pure gravity which is described by the action

$$S_5[g] = \frac{1}{2\kappa_5^2}\int_{M_5} d^5X\sqrt{-g}R\ , \tag{5.1}$$

with X^M, for $M \in \{0, \ldots, 4\}$, coordinates on a patch of the spacetime manifold M_5. For our current purposes we shall take

$$M_5 = \mathbb{R}^4 \times [0, 1]\ , \tag{5.2}$$

where $\mathbb{R}^4$ represents the worldvolume and $[0, 1]$ is the transverse space. We now equip this manifold with the metric

$$ds_5^2 = \eta_{\mu\nu}dx^\mu dx^\nu + dz^2\ , \tag{5.3}$$

where x^μ are coordinates on $\mathbb{R}^4$, for $\mu \in \{0, \ldots, 3\}$, and z is the coordinate on $I = [0, 1]$. We can then consider perturbations around this background solution

$$g_{MN} = \eta_{MN} + \mathcal{H}_{MN}(x, z)\ . \tag{5.4}$$

If we use this in (5.1), along with the methods described in Appendix A, then we find that at second order the action, ignoring $2\kappa_5^2$ factors, for now, takes the form

$$\begin{aligned}\int_{M_5} d^5X\Big(&\frac{3}{2}\partial_P\mathcal{H}_{MN}\partial^P\mathcal{H}^{MN} + 2\mathcal{H}^{MN}\partial_P\partial^P\mathcal{H}_{MN} - \frac{1}{2}\partial_P\mathcal{H}^M{}_M\partial^P\mathcal{H}^N{}_N - \mathcal{H}^M{}_M\partial_P\partial^P\mathcal{H}^N{}_N \\ &+2\partial^N\mathcal{H}^M{}_M\partial_P\mathcal{H}_N{}^P + 2\mathcal{H}^{MN}\partial_N\partial_M\mathcal{H}^P{}_P + \mathcal{H}^M{}_M\partial_P\partial_N\mathcal{H}^{NP} - 2\partial_M\mathcal{H}^{MN}\partial_P H_N{}^P \\ &-\partial_N\mathcal{H}_{MP}\partial^P\mathcal{H}^{MN} - 2\mathcal{H}^{MN}\partial_N\partial_P\mathcal{H}_M{}^P - 2\mathcal{H}^{MN}\partial_P\partial_N\mathcal{H}_M{}^P\Big)\ .\end{aligned} \tag{5.5}$$

To this we add the total derivative term

$$\int_{M_5} d^5X\partial^M\Big(-\frac{3}{2}\mathcal{H}_{NP}\partial_M\mathcal{H}^{NP}+\frac{1}{2}\mathcal{H}^P{}_P\partial_M\mathcal{H}^N{}_N-\frac{3}{2}(\partial^N\mathcal{H}^P{}_P)\mathcal{H}_{MN}-\frac{1}{2}(\partial^N\mathcal{H}_{MN})\mathcal{H}^P{}_P \\ +2\mathcal{H}_{MN}\partial_P\mathcal{H}^{PN}+\mathcal{H}^{PN}\partial_P\mathcal{H}_{MN}\Big)\ , \tag{5.6}$$

which allows us to write the quadratic five dimensional action as

$$\mathcal{S}^{(2)}_{5d}[\mathcal{H}]=\int d^5X\Big(\frac{1}{2}\mathcal{H}_{MN}(\eta^{MP}\eta^{NQ}-\eta^{MN}\eta^{PQ})\Box_5\mathcal{H}_{PQ}+\frac{1}{2}\mathcal{H}_{MN}\eta^{MN}\partial^P\partial^Q\mathcal{H}_{PQ} \\ +\frac{1}{2}\mathcal{H}_{PQ}\eta^{MN}\partial^P\partial^Q\mathcal{H}_{MN}-\mathcal{H}_{MN}\eta^{NQ}\partial^P\partial^M\mathcal{H}_{PQ}\Big)\ , \tag{5.7}$$

which is the form of the Fierz-Pauli action we have been using throughout this thesis. Employing a split between world volume and transverse indices, this may be written as

$$\mathcal{S}^{(2)}_{5d}[\mathcal{H}_{\mu\nu},\mathcal{A}_\mu,\Phi]=\int d^5X\Big(\frac{1}{2}\mathcal{H}_{\mu\nu}(\Box_4+\partial_z^2)\mathcal{H}^{\mu\nu}+\mathcal{A}_\mu(\Box_4+\partial_z^2)\mathcal{A}^\mu+\frac{1}{2}\Phi(\Box_4+\partial_z^2)\Phi \\ -\frac{1}{2}\mathcal{H}^\mu{}_\mu(\Box_4+\partial_z^2)\mathcal{H}^\nu{}_\nu-\frac{1}{2}\mathcal{H}^\mu{}_\mu(\Box_4+\partial_z^2)\Phi-\frac{1}{2}\Phi(\Box_4+\partial_z^2)\mathcal{H}^\mu{}_\mu \\ -\frac{1}{2}\Phi(\Box_4+\partial_z^2)\Phi+\frac{1}{2}\mathcal{H}^\mu{}_\mu\partial^\nu\partial^\sigma\mathcal{H}_{\nu\sigma}+\mathcal{H}^\mu{}_\mu\partial^\nu\partial^z\mathcal{A}_\nu+\frac{1}{2}\mathcal{H}^\mu{}_\mu\partial_z^2\Phi \\ +\frac{1}{2}\Phi\partial^\nu\partial^\sigma\mathcal{H}_{\nu\sigma}+\Phi\partial^\nu\partial^z\mathcal{A}_\nu+\frac{1}{2}\Phi\partial_z^2\Phi+\frac{1}{2}\mathcal{H}_{\mu\nu}\partial^\mu\partial^\nu\mathcal{H}^\sigma{}_\sigma \\ +\frac{1}{2}\mathcal{H}_{\mu\nu}\partial^\mu\partial^\nu\Phi+\mathcal{A}_\mu\partial^\mu\partial^z\mathcal{H}^\nu{}_\nu+\mathcal{A}_\mu\partial^\mu\partial^z\Phi+\frac{1}{2}\Phi\partial_z^2\mathcal{H}^\mu{}_\mu+\frac{1}{2}\Phi\partial_z^2\Phi \\ -\mathcal{A}_\mu\partial^\nu\partial^\mu\mathcal{A}_\nu-\mathcal{A}_\mu\partial^z\partial^\mu\Phi-\mathcal{A}^\nu\partial^\mu\partial^z\mathcal{H}_{\mu\nu}-\mathcal{A}^\mu\partial_z^2\mathcal{A}_\mu-\Phi\partial^\mu\partial^z\mathcal{A}_\mu \\ -\Phi\partial_z^2\Phi-\mathcal{H}_\mu{}^\nu\partial^\sigma\partial^\mu\mathcal{H}_{\sigma\nu}-\mathcal{H}_\mu{}^\nu\partial^z\partial^\mu\mathcal{A}_\nu\Big)\ , \tag{5.8}$$

where $\mathcal{A}_\mu=\mathcal{H}_{\mu z}$ and $\Phi=\mathcal{H}_{zz}$. This may be simplified to obtain

$$\int_{M_5} d^5X\Big(\frac{1}{2}\mathcal{H}_{\mu\nu}(\Box_4+\partial_z^2)\mathcal{H}^{\mu\nu}-\frac{1}{2}\mathcal{H}^\mu{}_\mu(\Box_4+\partial_z^2)\mathcal{H}^\nu{}_\nu+\frac{1}{2}\mathcal{H}^\mu{}_\mu\partial^\nu\partial^\sigma\mathcal{H}_{\nu\sigma}+\frac{1}{2}\mathcal{H}_{\mu\nu}\partial^\mu\partial^\nu\mathcal{H}^\sigma{}_\sigma-\mathcal{H}_\mu{}^\nu\partial^\sigma\partial^\mu\mathcal{H}_{\sigma\nu} \\ +\mathcal{A}_\mu(\Box_4\eta^{\mu\nu}-\partial^\mu\partial^\nu)\mathcal{A}_\nu+\mathcal{H}^\mu{}_\mu\partial^\nu\partial^z\mathcal{A}_\nu-\mathcal{H}_\mu{}^\nu\partial^z\partial^\mu\mathcal{A}_\nu+\mathcal{A}_\mu\partial^\mu\partial^z\mathcal{H}^\nu{}_\nu-\mathcal{A}^\nu\partial^\mu\partial^z\mathcal{H}_{\mu\nu} \\ -\frac{1}{2}\mathcal{H}^\mu{}_\mu(\Box_4)\Phi-\frac{1}{2}\Phi(\Box_4)\mathcal{H}^\mu{}_\mu+\frac{1}{2}\Phi\partial^\nu\partial^\sigma\mathcal{H}_{\nu\sigma}+\frac{1}{2}\mathcal{H}_{\mu\nu}\partial^\mu\partial^\nu\Phi\Big)\ . \tag{5.9}$$

In order to proceed we need to identify a method for calculating the allowed transverse dependence of our graviton spectrum. For this, of course, we just employ the procedure provided by B&E.

5.1.1 The transverse graviton spectrum and the role of boundary conditions

So we can properly appreciate the role of choosing boundary conditions on the transverse graviton spectrum we'll look at how different choices lead to transverse wavefunctions with vastly differing properties. To this end we shall compare the conditions that lead to the Kaluza-Klein circle transverse spectrum and a set of conditions leading to a spectrum with a zero mode with non-trivial transverse dependence. It is the second system that more closely mirrors the setup of the CPS problem, and hence it's the one that will be of principle interest to us in our attempts to obtain a quadratic lower dimensional action.

In order to proceed we need to isolate the B&E transverse operator for our problem, this is easily found to be

$$\frac{d^2}{dz^2} \, , \tag{5.10}$$

as a result we are seeking the eigenfunctions and eigenvalues of the following problem

$$\frac{d^2}{dz^2}\psi(z;m) = \lambda(m)\psi(z;m) \, . \tag{5.11}$$

Associated to this problem is a natural inner product, which arises from the calculation of the invariant measure in (5.1) for the background solution (5.3), in this case it's given by

$$(f,g) = \int_0^1 dz \; \bar{f}(z)g(z) \, , \tag{5.12}$$

with $\bar{f}$ denoting the complex conjugate of f, which won't be necessary for us as we shall only deal with real functions.

In order to complete our specification of the problem we are required to provide boundary conditions. We shall begin with the choice

$$\psi(0;m) = \psi(1;m) \, , \quad \int_I dz \; \psi(z;m)^2 = 1 \, , \quad \int_I dz \; \psi(z;m)\psi(z;n)|_{m\neq n} = 0 \, . \tag{5.13}$$

The resulting system of solutions to (5.11) shall be referred to, interchangeably, as the periodic or Kaluza-Klein wavefunctions. We can see that the first of these conditions ensures that the eigenfunctions take the same value at both endpoints of the interval, while the second condition just states that we normalise these eigenfunctions with respect to (5.12). The final condition is the statement that the eigenfunctions are orthogonal to each other and is the statement that our operator (5.10) is self adjoint. All of this is telling us that we've crossed in to the territory of Sturm-Liouville theory.

The first case we can consider is when $\lambda(m) = 0$. Here we find that the corresponding eigenfunction obeying (5.13) is

$$\psi(z;0) = 1\ . \tag{5.14}$$

In the case where $\lambda(m) > 0$ the solution to (5.11) is given by

$$\psi(z;m) = \alpha(m)\cosh\left(\lambda(m)^{\frac{1}{2}}z\right) + \beta(m)\sinh\left(\lambda(m)^{\frac{1}{2}}z\right)\ , \tag{5.15}$$

while for $\lambda(m) < 0$ we find

$$\psi(z;m) = \tilde{\alpha}(m)\cos\left(|\lambda(m)|^{\frac{1}{2}}z\right) + \tilde{\beta}(m)\sin\left(|\lambda(m)|^{\frac{1}{2}}z\right)\ . \tag{5.16}$$

Whilst these options solve (5.11) we still need to enforce the boundary conditions (5.13).

To begin with, to ensure orthogonality to (5.14) for (5.15), we find that

$$\begin{aligned}\int_I dz\ \psi(z;0)\psi(z;m) =& \frac{1}{\lambda(m)^{\frac{1}{2}}}\Big(\beta(m)\big(1-\cosh(\lambda(m)^{\frac{1}{2}})\big) + \alpha(m)\sinh(\lambda(m)^{\frac{1}{2}})\Big) = 0\\ \implies& \beta(m)\big(1-\cosh(\lambda(m)^{\frac{1}{2}})\big) + \alpha(m)\sinh(\lambda(m)^{\frac{1}{2}}) = 0\ ,\ \lambda(m) > 0.\end{aligned} \tag{5.17}$$

Demanding $\psi(0;m) = \psi(1;m)$ leads to the equation

$$\alpha(m)\big(1-\cosh(\lambda(m)^{\frac{1}{2}})\big) - \beta(m)\sinh(\lambda(m)^{\frac{1}{2}}) = 0\ . \tag{5.18}$$

If we assume $\alpha(m), \beta(m) \neq 0$ then we can multiply (5.17) by $\alpha(m)$ and (5.18) by $\beta(m)$ and then subtract the results to obtain

$$\big(\alpha(m)^2 + \beta(m)^2\big)\sinh(\lambda(m)^{\frac{1}{2}}) = 0 \quad \implies \sinh(\lambda(m)^{\frac{1}{2}}) = 0\ , \tag{5.19}$$

with the last equality due to our assumption that $\alpha(m), \beta(m) \neq 0$. However, this condition would require $\lambda(m) = 0$, which contradicts $\lambda(m) > 0$ hence we must discount this possible collection of solutions. If instead $\alpha(m) = 0$ then (5.18) leads to $\beta(m) = 0$ and if $\beta(m) = 0$ then (5.17) implies $\alpha(m) = 0$. So the only solution for hyperbolic functions is the trivial one[107]!

For the eigenfunctions given in (5.16) orthogonality to the zero mode, (5.14), requires

$$\int_I dz\ \psi(z;0)\psi(z;m) = \frac{1}{|\lambda(m)|^{\frac{1}{2}}}\big(\tilde{\alpha}(m)\sin(|\lambda(m)|^{\frac{1}{2}}) - \tilde{\beta}(m)\cos(|\lambda(m)|^{\frac{1}{2}}) + \tilde{\beta}(m)\big) = 0\ ,$$
$$\implies \tilde{\alpha}(m)\sin(|\lambda(m)|^{\frac{1}{2}}) - \tilde{\beta}(m)\cos(|\lambda(m)|^{\frac{1}{2}}) + \tilde{\beta}(m) = 0\ . \tag{5.20}$$

If we then demand that our solutions also obey $\psi(0;m) = \psi(1;m)$ we obtain the condition

$$\tilde{\alpha}(m)\big(1 - \cos(|\lambda(m)|^{\frac{1}{2}})\big) - \tilde{\beta}(m)\sin(|\lambda(m)|^{\frac{1}{2}}) = 0\ . \tag{5.21}$$

Assuming $\tilde{\alpha}(m), \tilde{\beta}(m) \neq 0$ we can multiply (5.20) by $\tilde{\alpha}(m)$ and (5.21) by $\tilde{\beta}(m)$ and subtract the resulting equations to obtain

$$\big(\tilde{\alpha}(m)^2 + \tilde{\beta}(m)^2\big)\sin(|\lambda(m)|^{\frac{1}{2}}) = 0 \quad \implies \sin(|\lambda(m)|^{\frac{1}{2}}) = 0\ , \tag{5.22}$$

Thus sinusoidal solutions to this problem are compatible with the boundary conditions if $\sin(|\lambda_m|^{\frac{1}{2}}) = 0$ and $\lambda_m < 0$. Hence we have quantised eigenvalues for the problem. If $\tilde{\alpha}(m) = 0$ or $\tilde{\beta}(m) = 0$ then exactly the same condition is found. As such $\tilde{\alpha}(m)$ and $\tilde{\beta}(m)$ are unconstrained by (5.21) and (5.20). The only possible exception is that both can't simultaneously be zero.

We can now check if our admissible eigenfunctions are orthogonal to each other with respect to (5.12). This requires that

$$\int_I dz\ \psi(z;m)\psi(z;n) = \frac{1}{\lambda(n) - \lambda(m)}\Big(\psi(z;m)\psi'(z;n) - \psi(z;n)\psi'(z;m)\Big)\Big|_{\partial I} = 0\ , \tag{5.23}$$

[107] By trivial we mean the zero function- which isn't really a valid solution as it has $\lambda(m) = 0$ which we've already discounted.

where $m \neq n$. Evaluating this for our admissible modes gives

$$\int_I dz\ \psi_m(z)\psi_n(z) = \frac{1}{\lambda(n)-\lambda(m)}\Big(|\lambda(n)|^{\frac{1}{2}}\big((\tilde{\alpha}(m)\cos(|\lambda(m)|^{\frac{1}{2}}z)+\tilde{\beta}(m)\sin(|\lambda(m)|^{\frac{1}{2}}z)\big)$$
$$\times(-\tilde{\alpha}(n)\sin(|\lambda(n)|^{\frac{1}{2}}z)+\tilde{\beta}(n)\cos(|\lambda(n)|^{\frac{1}{2}}z))\big)$$
$$-|\lambda(m)|^{\frac{1}{2}}\big((\tilde{\alpha}(n)\cos(|\lambda(n)|^{\frac{1}{2}}z)+\tilde{\beta}(n)\sin(|\lambda(n)|^{\frac{1}{2}}z)\big) \tag{5.24}$$
$$\times(-\tilde{\alpha}(m)\sin(|\lambda(m)|^{\frac{1}{2}}z)+\tilde{\beta}(m)\cos(|\lambda(m)|^{\frac{1}{2}}z))\big)\Big)\Big|_{\partial I} = 0\ ,$$

with the final equality arising from the fact that $\sin(|\lambda(p)|^{\frac{1}{2}}) = 0 \implies |\lambda(p)|^{\frac{1}{2}} = 2p\pi$ for $p \in \mathbb{N}_0 \setminus \{0\}$.

Interestingly calculating (5.23) using the $m = 0$ mode, (5.14), leads to the condition

$$\psi'(0;n) = \psi'(1;n)\ , \tag{5.25}$$

then by (5.11) we can see that

$$\psi''(0;n) = \psi''(1;n)\ , \tag{5.26}$$

and this keeps going for all of the derivatives that exist. We can also see that despite our specification of the boundary data (5.13) we haven't determined the coefficients $\tilde{\alpha}(m)$ or $\tilde{\beta}(m)$. This isn't how ODEs with boundary conditions usually work, we're meant to obtain unique solutions. So what's going on? Well in reality we're actually dealing with functions $\psi(z;m)$ such that $\psi(z+1;m) = \psi(z;m)$, which means our functions are periodic with period length the same as the length of our interval. So while it looked like we specified boundary data we really didn't. Instead we specified our space has no boundary, so it is a circle, and that our functions are continuous on this domain. As such it's no surprise that our supposed boundary conditions haven't lead to a unique solution to the ODE (5.11). Instead to fix our constants we shall split our solutions in to the case $\tilde{\alpha}(m) = 0, \tilde{\beta}(m) \neq 0$ and $\tilde{\alpha}(m) \neq 0, \tilde{\beta}(m) = 0$ and normalise these to unity. This gives as our admissible eigenfunctions for the problem

$$\begin{aligned} \psi(z;0) &= 1\ , \\ \psi_e(z;m) &= \sqrt{2}\cos(|\lambda(m)|^{\frac{1}{2}}z)\ , \\ \psi_o(z;m) &= \sqrt{2}\sin(|\lambda(m)|^{\frac{1}{2}}z)\ , \\ |\lambda(m)|^{\frac{1}{2}} &= 2\pi m\ ,\ m \in \mathbb{N}_0 \setminus \{0\}\ , \end{aligned} \tag{5.27}$$

which is precisely the set of eigenfunctions one expects to see in a standard S^1 Kaluza-Klein reduction.

However, here we have derived them based upon imposing (5.13), along with splitting into odd and even parts[108], on the solutions of (5.11). We should note that for each non-zero eigenvalue there is a 2-fold degeneracy of eigenfunctions with the corresponding eigenvalue.

Lets now consider the same eigenvalue equation, (5.11), but instead we shall impose the boundary data

$$\xi(0;m) = 0 \ , \quad \int_I dz\ \xi(z;m)^2 = 1, \quad \int_I dz\ \xi(z;m)\xi(z;n)\Big|_{m\neq n} = 0 \ , \tag{5.28}$$

where we have decided to denote this new set of eigenfunctions by $\xi(z;m)$.

In this case we find there is a single eigenfunction, obeying (5.28), associated to a zero eigenvalue of the Laplacian, which is given by

$$\xi_0(z) = \sqrt{3}z \ . \tag{5.29}$$

There are also solutions of the eigenvalue equation, with $\lambda(i) < 0$, which obey $\xi(0;i) = 0$. These are found to take the form

$$\xi(z;i) \ = \ n(i)\sin(\sqrt{|\lambda(i)|}z) \ , \quad \lambda(i) < 0 \ , \quad n(i) = \Big(\frac{4\sqrt{|\lambda(i)|}}{2\sqrt{|\lambda(i)|} - \sin(2\sqrt{|\lambda(i)|})}\Big)^{\frac{1}{2}} \ . \tag{5.30}$$

In addition to which there are also eigenfunctions with $\lambda(i) > 0$ which obey $\xi(0;i) = 0$. We find these take the form

$$\xi(z;i) = \tilde{n}(i)\sinh\sqrt{\lambda(i)}z) \ , \quad \lambda(i) > 0 \ , \quad n(i) = \Big(\frac{4\sqrt{\lambda(i)}}{\sinh(2\sqrt{\lambda(i)}) - 2\sqrt{\lambda(i)}}\Big)^{\frac{1}{2}} \ . \tag{5.31}$$

The orthogonality condition, to the zero mode, may now be shown to be equivalent to the more standard Robin boundary information

$$\int_I dz\ \xi(z;0)\xi(z;i) = \frac{1}{\lambda(i)}(\xi(z;0)\xi'(z;i) - \xi'(z;0)\xi(z;i))\Big|_{\partial I} = 0 \implies \xi'(1;i) = \xi(1;i) \ . \tag{5.32}$$

In the case of solutions of the type given in (5.31) this condition becomes

$$\begin{aligned}\int_I dz\ \xi(z;0)\xi(z;i) =& \frac{\sqrt{3}\tilde{n}(i)}{\lambda(i)}\Big(\sqrt{\lambda(i)}\cosh(\sqrt{\lambda(i)}) - \sinh(\sqrt{\lambda(i)})\Big) = 0 \\ &\implies \tanh(\sqrt{\lambda(i)}) = \sqrt{\lambda(i)} \ , \quad \lambda(i) > 0 \ ,\end{aligned} \tag{5.33}$$

which has as its only solution $\sqrt{\lambda(i)} = 0$. Hence solutions of the type given in (5.31) are excluded by the

[108] With this concept referring to symmetry about $z = \frac{1}{2}$.

demand that $\xi(z;0)$ be in the self-adjoint domain of $\frac{d^2}{dz^2}$.

Considering, instead, solutions of the type give in (5.30) we find that

$$\begin{aligned}\int_I dz\ \xi(z;0)\xi(z;i) =& \frac{\sqrt{3}n(i)}{\lambda(i)}(\sqrt{|\lambda(i)|}\cos(\sqrt{|\lambda(i)|}) - \sin(\sqrt{|\lambda(i)|})) \\ &\implies \tan(\sqrt{|\lambda(i)|}) = \sqrt{|\lambda(i)|}\ , \quad \lambda(i) < 0\ . \end{aligned} \tag{5.34}$$

Hence this theory has a discrete spectrum of sinusoidal solutions with eigenvalues given by the solutions of $\tan(x) = x$ for $x > 0$. This equation also shows that the system possesses a mass gap, making it admissible to EFT methods. One can then show that

$$\begin{aligned}\int_I dz\ \xi(z;i)\xi(z;j) = \frac{n(j)n(i)}{\lambda(j)-\lambda(i)}\Big(&\sqrt{|\lambda(j)|}\sin(\sqrt{|\lambda(i)|})\cos(\sqrt{|\lambda(j)|}) \\ &-\sqrt{|\lambda(i)|}\sin(\sqrt{|\lambda(j)|})\cos(\sqrt{|\lambda(i)|})\Big) = 0\ , \quad \lambda(i), \lambda(j) < 0\ , \end{aligned} \tag{5.35}$$

where $i \neq j$ and the final equality holds because of the quantisation condition (5.34).

As a result of this, the admissible eigenfunctions for the transverse wavefunction problem, subject to (5.28), are

$$\begin{aligned}\xi_0 &= \sqrt{3}z\ , \\ \xi(z;i) &= n(i)\sin(\sqrt{|\lambda(i)|}z)\ , \end{aligned} \tag{5.36}$$

with the constants and eigenvalues taking the values

$$n(i) = \Big(\frac{4\sqrt{\lambda(i)}}{2\sqrt{|\lambda(i)|} - \sin(2\sqrt{|\lambda(i)|})}\Big)^{\frac{1}{2}}\ , \qquad \sqrt{|\lambda(i)|} = \tan(\sqrt{|\lambda(i)|})\ , \quad \lambda(i) < 0\ . \tag{5.37}$$

It is this set of eigenfunctions that we shall use in later subsections to develop expansions of our five dimensional fields in terms of four dimensional fields. We can see in this case that the zero mode has non-trivial dependence on the transverse space and that for each non-zero eigenvalue there is a unique eigenfunction.

As we can see from (5.27) and (5.36) the specification of boundary conditions has a dramatic effect on the form of the transverse graviton spectrum. It should, as such, come as no surprise that this difference is also present in the form of the lower dimensional theory that we obtain from performing a dimensional reduction using both types of spectrum. Whilst our main concern is to look at the system obtained using (5.36) it will prove valuable to have considered the usual Kaluza-Klein system in this slightly odd manner. In particular insights gained from it, and the usual non-linear Kaluza-Klein, case will prove invaluable when, later in this Section, we turn to going beyond the quadratic level.

5.1.2 The second order gauge fixed action

Now that we have obtained a transverse graviton spectrum which exhibits a zero mode with non-trivial dependence, (5.36), we can begin to try to write (5.9) as a four dimensional field theory. Starting at (5.7) our aim is to deal with a theory involving $\mathcal{H}_{\mu\nu}$ that is the Fierz-Pauli action. However, currently there are interaction terms between this variable and the other fields $\mathcal{A}_\mu$ and Φ. However, as mentioned before, the system (5.7) enjoys linearised diffeomorphism invariance in the bulk. That is $S^{(2)}_{5d}[\mathcal{H}] = S^{(2)}_{5d}[\mathcal{H}']$, up to a boundary term, for transformations of the form

$$\mathcal{H}_{MN} \to \mathcal{H}'_{MN} = \mathcal{H}_{MN} + \partial_M \mathfrak{X}_N + \partial_N \mathfrak{X}_M \ , \tag{5.38}$$

where, $\mathfrak{X}^M$ is an arbitrary vector field parametrising the transformation. Under any variation $\delta\mathcal{H}_{MN}$, the variation of the action $S^{(2)}_{5d}[\mathcal{H}]$ is

$$\begin{aligned}\delta S^{(2)}_{5d}[\mathcal{H}] = &\int d^5X \ \delta\mathcal{H}_{MN}\Big(\Box_5\mathcal{H}^{MN} - \eta^{MN}\Box_5\mathcal{H} + \partial^M\partial^N\mathcal{H} + \eta^{MN}\partial^P\partial^Q\mathcal{H}_{PQ} - \partial_P\partial^M\mathcal{H}^{PN} - \partial_P\partial^N\mathcal{H}^{PM}\Big)\\ &+\frac{1}{2}\int d^5X\partial_P J^P \ ,\end{aligned} \tag{5.39}$$

where $\mathcal{H} = \eta^{MN}\mathcal{H}_{MN}$, and

$$\begin{aligned}J^P =&\mathcal{H}^{MN}\partial^P\delta\mathcal{H}_{MN} - \delta\mathcal{H}_{MN}\partial^P\mathcal{H}^{MN} - \eta^{MN}\mathcal{H}\partial^P\delta\mathcal{H}_{MN} + \eta^{MN}\delta\mathcal{H}_{MN}\partial^P\mathcal{H}\\ &+\mathcal{H}\partial_Q\delta\mathcal{H}^{PQ} - \delta\mathcal{H}^{PQ}\partial_Q\mathcal{H} + \eta^{MN}\mathcal{H}^{PQ}\partial_Q\delta\mathcal{H}_{MN} - \eta^{MN}\delta\mathcal{H}_{MN}\partial_Q\mathcal{H}^{PQ}\\ &-2\mathcal{H}_{MN}\partial^M\delta\mathcal{H}^{NP} + 2\delta\mathcal{H}_{MN}\partial^M\mathcal{H}^{NP} \ .\end{aligned} \tag{5.40}$$

The first line of (5.39) gives the equation of motion for H_{MN} while the second line is a total derivative term and hence is a boundary term. The system obeys the following, off-shell, identity

$$\partial_M\Big(\Box_5\mathcal{H}^{MN} - \eta^{MN}\Box_5\mathcal{H} + \partial^M\partial^N\mathcal{H} + \eta^{MN}\partial^P\partial^Q\mathcal{H}_{PQ} - \partial_P\partial^M\mathcal{H}^{PN} - \partial_P\partial^N\mathcal{H}^{PM}\Big) = 0 \ , \tag{5.41}$$

which we recognise as the linearised version of the Bianchi identity. If we let $\delta\mathcal{H}_{MN} = 2\partial_{(M}\mathfrak{X}_{N)}$, as is the case for the (5.38), we find that the variation of the action is a pure boundary term which is given by

$$\delta S^{(2)}_{5d}[\mathcal{H}] = \int d^5X \ \partial_M V^M \ , \tag{5.42}$$

where

$$V^M = 2\mathfrak{X}_N\Big(\Box_5\mathcal{H}^{MN} - \eta^{MN}\Box_5\mathcal{H} + \partial^M\partial^N\mathcal{H} + \eta^{MN}\partial^P\partial^Q\mathcal{H}_{PQ} - \partial_P\partial^M\mathcal{H}^{PN} - \partial_P\partial^N\mathcal{H}^{PM}\Big) + \frac{1}{2}J^M \tag{5.43}$$

It's our goal, after choosing appropriate boundary conditions, to be able to write down a boundary action $S_B[\mathcal{H}]$ that under a gauge variation transforms as

$$\delta S_B[\mathcal{H}] = -\int d^5X \ \partial_M V^M \ . \tag{5.44}$$

If this can be achieved then the total action $S[\mathcal{H}] = S^{(2)}_{5d}[\mathcal{H}] + S_B[\mathcal{H}]$ will be invariant under gauge variations. In what follows our main concern shall be with the boundary of the interval. As such we shall be rather careless about the Minkowski boundary since this is already a well known story.

To determine the appropriate boundary terms and conditions, we note that the transformation in (5.38) can be interpreted in terms of the $4+1$ split variables as

$$\mathcal{H}_{\mu\nu} \to \mathcal{H}'_{\mu\nu} = \mathcal{H}_{\mu\nu} + \partial_\mu\mathfrak{X}_\nu + \partial_\nu\mathfrak{X}_\mu \ , \quad \mathcal{A}_\mu \to \mathcal{A}'_\mu = \mathcal{A}_\mu + \partial_\mu\mathfrak{X}_z + \partial_z\mathfrak{X}_\mu \ , \quad \Phi \to \Phi' = \Phi + 2\partial_z\mathfrak{X}_z \ . \tag{5.45}$$

Looking at the types of interactions occurring between $\mathcal{H}_{\mu\nu}$, $\mathcal{A}_\mu$ and Φ, we notice that Φ decouples from $\mathcal{H}_{\mu\nu}$ if $\Phi' = 0$. We now split $\Phi = \Phi_t + \Phi_{nt}$. This split originates from the lower dimensional expansion that will take place in later parts of this Subsection. The reason to account for it now is that once the theory is expanded in a basis of functions on the transverse space some five dimensional gauge transformations have no sensible four dimensional interpretation. Hence such transformations are no longer allowed once we decide to interpret our five dimensional theory in a truly four dimensional sense. We shall show what happens explicitly later on, but for now we just take as a given that under (5.45) we have

$$\Phi'_{nt} = \Phi_{nt} \ , \quad \Phi'_t = \Phi_t + 2\partial_z\mathfrak{X}_z \ . \tag{5.46}$$

Taking account of (5.46) the condition $\Phi_t = 0$ is set by the transformation:

$$\Phi_t + 2\,\partial_z \mathfrak{X}_z = 0 \implies \partial_z \mathfrak{X}_z = -\frac{1}{2}\Phi_t \implies \mathfrak{X}_z(x,z) = -\frac{1}{2}\int^z d\tilde{z}\;\Phi_t(x,\tilde{z}) =: -\frac{1}{2}\partial_z^{-1}\Phi_t\;. \tag{5.47}$$

Note the constant from this integral is set to zero. Under this transformation we can see, using (5.45), that

$$\mathcal{H}'_{\mu\nu} = \mathcal{H}_{\mu\nu}\;,\quad \mathcal{A}'_\mu = \mathcal{A}_\mu - \frac{1}{2}\partial_\mu \partial_z^{-1}\Phi_t\;. \tag{5.48}$$

With this (5.9) becomes

$$\begin{aligned}\int d^5X\Big(&\frac{1}{2}\mathcal{H}_{\mu\nu}(\Box_4+\partial_z^2)\mathcal{H}^{\mu\nu} - \frac{1}{2}\mathcal{H}^\mu{}_\mu(\Box_4+\partial_z^2)\mathcal{H}^\nu{}_\nu + \frac{1}{2}\mathcal{H}^\mu{}_\mu\partial^\nu\partial^\sigma\mathcal{H}_{\nu\sigma} + \frac{1}{2}\mathcal{H}_{\mu\nu}\partial^\mu\partial^\nu\mathcal{H}^\sigma{}_\sigma - H_\mu{}^\nu\partial^\sigma\partial^\mu\mathcal{H}_{\sigma\nu}\\ &+ \mathcal{A}'_\mu(\Box_4\eta^{\mu\nu} - \partial^\mu\partial^\nu)\mathcal{A}'_\nu + \mathcal{H}^\mu{}_\mu\partial^\nu\partial^z\mathcal{A}'_\nu - \mathcal{H}_\mu{}^\nu\partial^z\partial^\mu\mathcal{A}'_\nu + \mathcal{A}'_\mu\partial^\mu\partial^z\mathcal{H}^\nu{}_\nu - \mathcal{A}'^\nu\partial^\mu\partial^z\mathcal{H}_{\mu\nu}\\ &- \frac{1}{2}\mathcal{H}^\mu{}_\mu(\Box_4)\Phi_{nt} - \frac{1}{2}\Phi_{nt}(\Box_4)\mathcal{H}^\mu{}_\mu + \frac{1}{2}\Phi_{nt}\partial^\nu\partial^\sigma\mathcal{H}_{\nu\sigma} + \frac{1}{2}\mathcal{H}_{\mu\nu}\partial^\mu\partial^\nu\Phi_{nt}\Big)\;,\end{aligned} \tag{5.49}$$

the scalar Φ_{nt} appearing in this is very strange as it lacks a kinetic term. However, this situation is well known in Kaluza-Klein theory where, given our $4+1$ split of $\mathcal{H}_{MN}$, there is no kinetic term for the lower dimensional scalar field. In this case there is a field redefinition that allows Φ to have a kinetic term. Being inspired by this we propose the following field redefinition

$$\mathcal{H}_{\mu\nu} = H_{\mu\nu} + 2a\widetilde{\Phi}_{nt}\eta_{\mu\nu}\;,\quad \Phi_{nt} = 2b\widetilde{\Phi}_{nt}\;,\quad \Phi_t = 2b\tilde{\Phi}_t\;. \tag{5.50}$$

Then, by expanding (5.9) in terms of these new variables, we find that there exists a $\widetilde{\Phi}_{nt}\Box_4\widetilde{\Phi}_{nt}$ term with coefficient

$$-12a(a+b)\;. \tag{5.51}$$

Choosing $b = -2a$, we find that $\widetilde{\Phi}_{nt}$ will have a normalised four dimensional kinetic term if

$$a^2 = \frac{1}{12}\;. \tag{5.52}$$

With these choices we find that (5.9) becomes

$$\begin{aligned}\int d^5X\Big(&\frac{1}{2}H_{\mu\nu}(\Box_4+\partial_z^2)H^{\mu\nu} - \frac{1}{2}H^\mu{}_\mu(\Box_4+\partial_z^2)H^\nu{}_\nu + \frac{1}{2}H^\mu{}_\mu\partial^\nu\partial^\sigma H_{\nu\sigma} + \frac{1}{2}H_{\mu\nu}\partial^\mu\partial^\nu H^\sigma{}_\sigma - H_\mu{}^\nu\partial^\sigma\partial^\mu H_{\sigma\nu}\\ &+ \mathcal{A}'_\mu(\Box_4\eta^{\mu\nu} - \partial^\mu\partial^\nu)\mathcal{A}'_\nu + H^\mu{}_\mu\partial^\nu\partial^z\mathcal{A}'_\nu - H_\mu{}^\nu\partial^z\partial^\mu\mathcal{A}'_\nu + \mathcal{A}'_\mu\partial^\mu\partial^z H^\nu{}_\nu - \mathcal{A}'^\nu\partial^\mu\partial^z H_{\mu\nu}\\ &+ \tilde{\Phi}_{nt}(\Box_4 - 2\partial_z^2)\tilde{\Phi}_{nt} - 3aH^\mu{}_\mu\partial_z^2\tilde{\Phi}_{nt} - 3a\tilde{\Phi}_{nt}\partial_z^2 H^\mu{}_\mu + 6a\tilde{\Phi}_{nt}\partial^\mu\partial^z\mathcal{A}'_\mu + 6a\mathcal{A}'_\mu\partial^\mu\partial^z\tilde{\Phi}_{nt}\Big)\;.\end{aligned} \tag{5.53}$$

If we consider a further transformation with parameter $\tilde{X}^M$. Our fields will transform as

$$\begin{aligned} H_{\mu\nu} \to H'_{\mu\nu} &= H_{\mu\nu} + \partial_\mu \tilde{X}_\nu + \partial_\nu \tilde{X}_\mu \ , \quad \mathcal{A}'_\mu \to \mathcal{A}''_\mu = \mathcal{A}_\mu - b\partial_\mu \partial_z^{-1}\tilde{\Phi}_t + \partial_\mu \tilde{X}_z + \partial_z \tilde{X}_\mu \ , \\ \tilde{\Phi}_t \to \tilde{\Phi}'_t &= \frac{1}{b}\partial_z \tilde{X}_z \ , \quad \tilde{\Phi}_{nt} \to \tilde{\Phi}'_{nt} = \tilde{\Phi}_{nt} \ . \end{aligned} \tag{5.54}$$

In order to decouple $H'_{\mu\nu}$ and $\mathcal{A}'_\mu$ we note that all of the interaction terms involve a ∂_z term. As such, this suggests that we set $\partial_z \mathcal{A}''_\mu = 0$ via another gauge transformation. However we have to be careful not to be overzealous and undo our first gauge transformation, which would result in $\tilde{\Phi}'_t \neq 0$. Looking at (5.54) we can see to avoid this issue we can just set $\partial_z \tilde{X}_z = 0$. We then observe that

$$\partial_z \mathcal{A}''_\mu = \partial_z \mathcal{A}_\mu - b\partial_\mu \tilde{\Phi}_t + \partial_\mu \partial_z \tilde{X}_z + \partial_z^2 \tilde{X}_\mu \ , \tag{5.55}$$

with the third term being zero by our previous comment. So if we wish to set $\partial_z \mathcal{A}''_\mu = 0$ we require that

$$\partial_z^2 \tilde{X}_\mu = -\partial_z \mathcal{A}_\mu + b\partial_\mu \tilde{\Phi}_t \ . \tag{5.56}$$

With this choice of gauge transformations and variables, (5.53) becomes

$$\begin{aligned} \mathcal{S}^{(2)}_{5d}[H', \mathcal{A}'', \tilde{\Phi}'] = \int d^5X \Big(&\frac{1}{2} H'_{\mu\nu}(\Box_4 + \partial_z^2) H'^{\mu\nu} - \frac{1}{2} H'^\mu{}_\mu (\Box_4 + \partial_z^2) H'^\nu{}_\nu + \frac{1}{2} H'^\mu{}_\mu \partial^\nu \partial^\sigma H'_{\nu\sigma} \\ &+ \frac{1}{2} H'_{\mu\nu} \partial^\mu \partial^\nu H'^\sigma{}_\sigma - H'^\nu_\mu \partial^\sigma \partial^\mu H'_{\sigma\nu} + \mathcal{A}''_\mu (\Box_4 \eta^{\mu\nu} - \partial^\mu \partial^\nu) \mathcal{A}''_\nu \\ &+ \tilde{\Phi}'_{nt}(\Box_4 - 2\partial_z^2)\tilde{\Phi}'_{nt} - 3aH'^\mu{}_\mu \partial_z^2 \tilde{\Phi}'_{nt} - 3a\tilde{\Phi}'_{nt}\partial_z^2 H'^\mu{}_\mu \\ &+ \partial^z \left(\mathcal{A}''_\mu \partial^\mu H'^\nu{}_\nu - \mathcal{A}''^\nu \partial^\mu H'_{\mu\nu} + 6a\mathcal{A}''_\mu \partial^\mu \tilde{\Phi}'_{nt} \right) \Big) \ , \end{aligned} \tag{5.57}$$

which is a Fierz-Pauli term for $H'_{\mu\nu}$, a Maxwell term for $\mathcal{A}''_\mu$, a kinetic term for $\tilde{\Phi}'_{nt}$ and several interaction terms between $\tilde{\Phi}'_{nt}$ and $H'_{\mu\nu}$. We can see that the gauge transformation of the Maxwell field comes from a, linearised, diffeomorphism $\mathcal{X}^M = (0, \mathcal{X}^z(x))$. Also, since this is a Fierz-Pauli system, $H'_{\mu\nu}$ enjoys the gauge symmetry

$$H'_{\mu\nu} \to H''_{\mu\nu} = H'_{\mu\nu} + \partial_\mu \mathfrak{X}_\nu + \partial_\nu \mathfrak{X}_\mu \ , \tag{5.58}$$

for $\mathfrak{X}_\mu$ such that $\partial_z^2 \mathfrak{X}_\mu = 0$, which neatly looks like something in the kernel of (5.56). Note, due to the term $-3aH'^\mu{}_\mu \partial_z^2 \tilde{\Phi}'_{nt}$, this transformation only holds up to z boundary terms.

Our action is starting to look more like something we recognise. However, the terms in the last line of (5.57) are a little odd. They appear to be boundary terms but they're not full covariant five dimensional boundary terms as they involve only total z derivatives. One might wonder what effect such terms might have. To understand this, consider Maxwell theory on five dimensional Minkowski space. This theory has its dynamics governed by the action

$$S_5[A] = -\frac{1}{4}\int d^5X\ F_{MN}F^{MN}\ , \tag{5.59}$$

where $F_{MN} = 2\partial_{[M}A_{N]}$. Calculating the variation of this one finds

$$\delta S[A] = \int d^5X\ \delta A_N \partial_M F^{MN} - \int d^5X\ \partial_M\big(F^{MN}\delta A_N\big)\ . \tag{5.60}$$

For the boundary term to vanish, a natural condition to impose is for δA_N to vanish at the boundary. While in the bulk we find the expected Euler-Lagrange equation $\partial_M F^{MN} = 0$. In order to understand what's going on in (5.57), let's consider a $4+1$ split of A_M and consider a slightly modified action

$$S'[A_\mu, A_z] = \int_{M_5} d^5X\ \Big(-\frac{1}{4}F_{MN}F^{MN} + \partial^z\big(A_z\partial^\mu A_\mu\big)\Big)\ . \tag{5.61}$$

If we vary this action we find that

$$\begin{aligned}\delta S'[A_\mu, A_z] = &\int d^5X\Big(\delta A_\mu \partial_M F^{M\mu} + \delta A_z\partial_\mu F^{\mu z}\Big) - \int_0^1 dz \int d^3y\sqrt{-h}n_\mu\Big(F^{\mu\nu}\delta A_\nu + F^{\mu z}\delta A_z\Big) \\ &+ \int d^4x\Big(F^{\mu z}\delta A_\mu + \delta A_z\partial^\mu A_\mu + A_z\partial^\mu\delta A_\mu\Big)\Big|_{z=0}^{z=1}\ ,\end{aligned} \tag{5.62}$$

where y^i are coordinates on the boundary of $\mathbb{R}^{1,3}$, h is the determinant of the induced metric on this boundary and n_μ is the normal vector to the boundary. From this, it can be seen that imposing $\delta A_\mu = 0$ and $\delta A_z = 0$ on both the $\mathbb{R}^{1,3}$ and z boundaries does not lead to all of the boundary terms present vanishing. In particular, the term $A_z\partial^\mu\delta A_\mu$, which comes from the z derivative term that we added in (5.61), does not vanish. However, although the conventional boundary conditions do not remove the boundary terms in our problem, it is clear that there exist more exotic boundary conditions such that this can be done. For example, setting $\delta A_\mu = 0$ and $\delta A_z = 0$ on both boundaries and $A_z = 0$ on the z boundary would work. With such a choice of boundary conditions, the variation of the action becomes

$$\delta S'[A_\mu, A_z] = \int d^5X\Big(\delta A_\mu\partial_M F^{M\mu} + \delta A_z\partial_\mu F^{\mu z}\Big)\ , \tag{5.63}$$

which gives the same equations of motion as standard Maxwell theory, albeit it arrived at in a non-covariant manner.

Having now understood the effect of the terms in the last line of (5.57) we see that their role is purely to inform us about boundary conditions on $H_{\mu\nu}, \mathcal{A}_\mu$, and their respective variations. Given this insight we can modify our original action (5.7) by adding to it the terms

$$\int d^5X\Big(\partial^z\big(\mathcal{A}^\nu\partial^\mu H_{\mu\nu}\big) - \partial^z\big(\mathcal{A}_\mu\partial^\mu H^\nu{}_\nu\big)\Big) \ . \tag{5.64}$$

If this is done then the action (5.9) can be written as

$$\begin{aligned}\int d^5X\Big(&\frac{1}{2}\mathcal{H}_{\mu\nu}(\Box_4+\partial_z^2)\mathcal{H}^{\mu\nu} - \frac{1}{2}\mathcal{H}^\mu{}_\mu(\Box_4+\partial_z^2)\mathcal{H}^\nu{}_\nu + \frac{1}{2}\mathcal{H}^\mu{}_\mu\partial^\nu\partial^\sigma\mathcal{H}_{\nu\sigma} + \frac{1}{2}\mathcal{H}_{\mu\nu}\partial^\mu\partial^\nu\mathcal{H}^\sigma{}_\sigma - \mathcal{H}_\mu{}^\nu\partial^\sigma\partial^\mu\mathcal{H}_{\sigma\nu}\\ &+\mathcal{A}_\mu(\Box_4\eta^{\mu\nu}-\partial^\mu\partial^\nu)\mathcal{A}_\nu + \mathcal{H}^\mu{}_\mu\partial^\nu\partial^z\mathcal{A}_\nu - \mathcal{H}_\mu{}^\nu\partial^z\partial^\mu\mathcal{A}_\nu - (\partial^z\mathcal{A}_\mu)\partial^\mu\mathcal{H}^\nu{}_\nu + (\partial^z\mathcal{A}^\nu)\partial^\mu\mathcal{H}_{\mu\nu}\\ &-\frac{1}{2}\mathcal{H}^\mu{}_\mu(\Box_4)\Phi - \frac{1}{2}\Phi(\Box_4)\mathcal{H}^\mu{}_\mu + \frac{1}{2}\Phi\partial^\nu\partial^\sigma\mathcal{H}_{\nu\sigma} + \frac{1}{2}\mathcal{H}_{\mu\nu}\partial^\mu\partial^\nu\Phi\Big) \ ,\end{aligned} \tag{5.65}$$

where we have performed integration by parts in the z direction on the final two terms of the second line of (5.9) and then the boundary terms cancel against those added in (5.64).

The analysis leading to (5.57) still holds but now the action we arrive at takes the form

$$\begin{aligned}\mathcal{S}^{(2)}_{5d}[H'_{\mu\nu},\mathcal{A}''_\mu,\tilde{\Phi}'_{nt}] = \int d^5X\Big(&\frac{1}{2}H'_{\mu\nu}(\Box_4+\partial_z^2)H'^{\mu\nu} - \frac{1}{2}H'^\mu{}_\mu(\Box_4+\partial_z^2)H'^\nu{}_\nu + \frac{1}{2}H'^\mu{}_\mu\partial^\nu\partial^\sigma H'_{\nu\sigma}\\ &+\frac{1}{2}H'_{\mu\nu}\partial^\mu\partial^\nu H'^\sigma{}_\sigma - H'^\nu_\mu\partial^\sigma\partial^\mu H'_{\sigma\nu} + \mathcal{A}''_\mu(\Box_4\eta^{\mu\nu}-\partial^\mu\partial^\nu)\mathcal{A}''_\nu\\ &+\tilde{\Phi}'_{nt}(\Box_4-2\partial_z^2)\tilde{\Phi}'_{nt} - 3aH'^\mu{}_\mu\partial_z^2\tilde{\Phi}'_{nt} - 3a\tilde{\Phi}'_{nt}\partial_z^2H'^\mu{}_\mu\Big) \ ,\end{aligned} \tag{5.66}$$

which really is a Fierz-Pauli system along with a Maxwell system and a scalar.

We'd now like to understand how we descend to 4d. In order to do this we have to decide how to write our 5d fields in terms of 4d ones. We begin by ignoring the terms involving $\tilde{\Phi}'_{nt}$ as by considering the other fields first we shall understand how to deal with it. The field $\mathcal{A}''_\mu$ is independent of z and so an obvious[109] expansion is

$$\mathcal{A}''_\mu = A_\mu(x) \ , \tag{5.67}$$

while our expansion for $H'_{\mu\nu}$ is chosen to be

$$H'_{\mu\nu}(x,z) = \sum_\alpha h_{\mu\nu}(x;\alpha)\xi(z;\alpha) \ , \tag{5.68}$$

[109] We'll have to re-examine this later on.

where $\alpha \in \{0\} \cup \{x|\tan(\sqrt{|x|}) = \sqrt{|x|}, x < 0\}$ and $\xi(z;\alpha)$ is as given in (5.36). This choice of expansion means that

$$H'_{\mu\nu}(x,0) = 0\ . \tag{5.69}$$

If we input these expansions into (5.66) then we obtain the four dimensional action

$$\begin{aligned}\mathcal{S}^{(2)}_{4d}[h_{\mu\nu}(\alpha), A_\mu] + \mathcal{S}[H', \tilde{\Phi}'_{nt}] = \int d^4x \Big(\sum_\alpha \big(\frac{1}{2} h_{\mu\nu}(\alpha)(\Box_4 + \lambda(\alpha)) h^{\mu\nu}(\alpha) - \frac{1}{2} h^\mu{}_\mu(\alpha)(\Box_4 + \lambda(\alpha)) h^\nu{}_\nu(\alpha) \\ + \frac{1}{2} h^\mu{}_\mu(\alpha) \partial^\nu \partial^\sigma h_{\nu\sigma}(\alpha) + \frac{1}{2} h_{\mu\nu}(\alpha) \partial^\mu \partial^\nu h^\sigma{}_\sigma(\alpha) - h_\mu{}^\nu(\alpha) \partial^\sigma \partial^\mu h_{\sigma\nu}(\alpha)\big) \\ + A_\mu(\eta^{\mu\nu}\Box_4 - \partial^\mu\partial^\nu) A_\nu \Big) + \mathcal{S}[H', \tilde{\Phi}'_{nt}]\ ,\end{aligned} \tag{5.70}$$

where in this expression we have only included eigenvalue arguments on our fields for notational simplicity. We have also included a term $\mathcal{S}[H', \tilde{\Phi}'_{nt}]$ to remind us that there are terms involving $\tilde{\Phi}'_{nt}$ that we are currently ignoring. This action describes a massless spin-2 field, $\alpha = 0$, along with a collection of massive spin-2 fields, $\alpha \neq 0$, and what seems to be a Maxwell field.

In order for the field $h_{\mu\nu}(0)$ to truly be a Fierz-Pauli field it must enjoy the gauge symmetry

$$h_{\mu\nu}(x;0) \to h'_{\mu\nu}(x;0) = h_{\mu\nu}(x,0) + \partial_\mu \chi_\nu + \partial_\nu \chi_\mu\ , \tag{5.71}$$

with $\chi_\mu = \chi_\mu(x)$. The transformation noted in (5.58) would appear to be an obvious choice. However, the $\mathfrak{X}_\mu$ there has constant dependence on the z direction and hence can't be interpreted in the $\xi(\alpha)$ basis we are using for our perturbation since all elements of the basis are zero at $z = 0$. However if we parametrise a general worldvolume diffeomorphism parameter as

$$\mathfrak{X}_\mu(x,z) = \sum_\alpha \chi_\mu(x;\alpha)\xi(z;\alpha)\ , \tag{5.72}$$

then

$$\delta H'_{\mu\nu} = \sum_\alpha \Big(\partial_\mu \chi_\nu(x;\alpha) + \partial_\nu \chi_\mu(x;\alpha)\Big)\xi(z;\alpha)\ , \quad \delta \mathcal{A}''_\mu = \sum_\alpha \chi_\mu(x;\alpha) \frac{d}{dz}\xi(z;\alpha)\ . \tag{5.73}$$

Recalling that we must preserve the gauge choice $\partial_z \mathcal{A}''_\mu = 0$. This leads to

$$\partial_z \delta \mathcal{A}''_\mu = \sum_\alpha \chi_\mu(x;\alpha) \frac{d^2}{dz^2}\xi(z;\alpha) = \sum_{\lambda(i)} \chi_\mu(x;i)\lambda(i)\xi(z;i) = 0\ ,$$

with $\lambda(i) \in \{x|\tan(\sqrt{|x|}) = \sqrt{|x|}, x < 0\}$. Thus, by linear independence of the basis, all modes $\chi_\mu(x;i) = 0$. This leaves only the mode $\chi_\mu(x;0)$, which is associated to the zero mode $\xi(z;0)$, for us to use. So under such a diffeomorphism

$$\mathfrak{X}_\mu(x,z) = \chi_\mu(x)\xi(z;0) \ , \tag{5.74}$$

we find, in terms of 4d fields, this corresponds to

$$\begin{aligned} h_{\mu\nu}(0) \to h'_{\mu\nu}(0) = h_{\mu\nu}(0) + \partial_\mu\chi_\nu + \partial_\nu\chi_\mu \ , \quad h_{\mu\nu}(i) \to h'_{\mu\nu}(i) = h_{\mu\nu}(i) \ , \\ A_\mu \to A'_\mu = A_\mu + \sqrt{3}\chi_\mu \ . \end{aligned} \tag{5.75}$$

We now demand that (5.70) is invariant under this transformation. It's here that we run in to an issue. While the Fierz-Pauli term for $h_{\mu\nu}(0)$ behaves as expected, the Maxwell term doesn't play ball. Let's see how things work in detail. The terms in (5.70) that transform are

$$\begin{aligned} \mathcal{S}^{(2)}_{4d}[h_{\mu\nu}(\alpha), A_\mu]\big|_{transforming\ part} = \int d^4x\Big(h_{\mu\nu}(0)\Big(\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma})\Box_4 \\ +\frac{1}{2}\eta^{\mu\nu}\partial^\rho\partial^\sigma + \frac{1}{2}\eta^{\rho\sigma}\partial^\mu\partial^\nu - \eta^{\nu\sigma}\partial^\rho\partial^\mu\Big)h_{\rho\sigma}(0) \\ +A_\mu(\eta^{\mu\nu}\Box_4 - \partial^\mu\partial^\nu)A_\nu\Big) \end{aligned} \tag{5.76}$$

whose transformation under (5.75) is given by

$$\begin{aligned} \delta S^{(2)}_{4d} = \int d^4x\Big(2\sqrt{3}A_\mu(\eta^{\mu\nu}\Box_4 - \partial^\mu\partial^\nu)\chi_\nu + \partial_\mu\big(2\chi_\nu\mathcal{O}^{\mu\nu\rho\sigma}_{FP}h_{\rho\sigma}(0)\big) \\ +\partial_\mu\big(\sqrt{3}\chi^\nu\partial^\mu A_\nu\big) - \partial_\mu\big(\sqrt{3}(\partial^\mu\chi^\nu)A_\nu\big) - \partial_\mu\big(\sqrt{3}\chi^\mu\partial^\nu A_\nu\big) + \partial_\nu\big(\sqrt{3}(\partial_\mu\chi^\mu)A^\nu\big)\Big) \ , \end{aligned} \tag{5.77}$$

where

$$\mathcal{O}^{\mu\nu\rho\sigma}_{FP} = \frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma})\Box_4 + \frac{1}{2}\eta^{\mu\nu}\partial^\rho\partial^\sigma + \frac{1}{2}\eta^{\rho\sigma}\partial^\mu\partial^\nu - \eta^{\nu\sigma}\partial^\rho\partial^\mu \ . \tag{5.78}$$

All but the first term are total derivatives on the worldvolume. Hence they evaluate to zero, given our choice of boundary conditions on $\partial\mathbb{R}^{1,3}$. The first term, however, is troubling as it is zero only if

$$\Box_4\chi^\mu - \partial^\mu\partial^\nu\chi_\nu = 0 \ ,$$

which would impose a non-trivial condition on our diffeomorphism parameter. We can't have this if we want full four dimensional diffeomorphism invariance of the theory. So what's the issue? Well in the case of a standard Maxwell field there would be no inhomogeneous transformation for the field, so (5.75) would read

$$A'_\mu = A_\mu \ .$$

Hence we see this field, even though it has a Maxwell like kinetic term, is behaving more like a Stueckelberg field than a true Maxwell field. So what can we do about it? Let's re-examine how our field is defined and transforms in five dimensions

$$\mathcal{A}''_\mu = A_\mu(x), \quad \delta\mathcal{A}''_\mu = \chi_\mu(x)\partial_z\xi_0 = \sqrt{3}\chi_\mu \ .$$

So we have a bit of a mismatch between our expansion and transformation. Sure they both have constant transverse dependence, but the way it arises differs. Since the transformation is forced on us by our previous choices, and we want all of them to remain valid, it seems sensible to let the transformation inform our decision on the expansion front. As such we trade (5.67) in for the less obvious expansion

$$\mathcal{A}''_\mu = A_\mu(x)\partial_z\xi(z,0) = \sqrt{3}A_\mu(x) \ . \tag{5.79}$$

If we now use this in the Maxwell term of (5.65), we obtain the term

$$S^{(2)}_{5d}\big|_{Maxwell} = \int d^5X\mathcal{A}''_\mu(\eta^{\mu\nu}\Box_4 - \partial^\mu\partial^\nu)\mathcal{A}''_\nu = \int d^4x A_\mu(\eta^{\mu\nu}\Box_4 - \partial^\mu\partial^\nu)A_\nu \int dz\ \xi'(0)\xi'(0) \ , \tag{5.80}$$

where $\xi' = \frac{d}{dz}\xi$. We can then note that

$$\int dz\ \xi'(0)\xi'(0) = \int dz\ \frac{d}{dz}\big(\xi(0)\xi'(0)\big) \ . \tag{5.81}$$

Using this we can write (5.80) as

$$\begin{aligned} S^{(2)}_{5d}\big|_{Maxwell} &= \int d^4x dz\ \partial_z\Big(A_\mu\xi(0)(\eta^{\mu\nu}\Box_4 - \partial^\mu\partial^\nu)A_\nu\partial_z\xi(0)\Big) \\ &= \int d^5X\ \partial_z\Big((\partial_z^{-1}\mathcal{A}''_\mu)(\eta^{\mu\nu}\Box_4 - \partial^\mu\partial^\nu)\mathcal{A}''_\nu\Big) \ . \end{aligned} \tag{5.82}$$

Where the constant from ∂_z^{-1} is set to zero. We now see that the supposed Maxwell term is actually just a boundary term, so we can deal with it in precisely the same way as we have with other boundary terms, noting that this will alter the boundary conditions of our problem but happily it'll give us manifest invariance under (5.75). This leads us to modify (5.66) by adding the term

$$-\int d^5X\,\partial_z\Big((\partial_z^{-1}\mathcal{A}''_\mu)(\eta^{\mu\nu}\Box_4-\partial^\mu\partial^\nu)\mathcal{A}''_\nu\Big)\ . \tag{5.83}$$

So our final four dimensional action is

$$S^{(2)}_{4d}[h_{\mu\nu}(\alpha)]=\int d^4x\Big(\sum_\alpha\big(h_{\mu\nu}(\alpha)\mathcal{O}^{\mu\nu\rho\sigma}_{FP}h_{\rho\sigma}(\alpha)\big)+\frac{1}{2}\sum_i\lambda(i)\big(h_{\mu\nu}(i)h^{\mu\nu}(i)-h^\mu{}_\mu(i)h^\nu{}_\nu(i)\big)\Big)\ , \tag{5.84}$$

which is a massless Fierz-Pauli field along with a tower of massive Fierz-Pauli fields. This action clearly has the manifest gauge symmetry

$$\begin{aligned}h_{\mu\nu}(0)\to h'_{\mu\nu}(0)=h_{\mu\nu}(0)+\partial_\mu\chi_\nu+\partial_\nu\chi_\mu\ ,\quad h_{\mu\nu}(i)\to h'_{\mu\nu}(i)=h_{\mu\nu}(i)\ ,\\ A_\mu\to A'_\mu=A_\mu+\chi_\mu\ .\end{aligned} \tag{5.85}$$

here the A_μ transformation differs from that given in (5.75) due to the re-expression of the expansion (5.79).

This is all well and good but we have neglected $\mathcal{S}[H',\tilde{\Phi}'_{nt}]$ throughout, so while (5.100) looks very promising it still isn't quite the full story. In order to proceed we need to workout how to expand our field $\tilde{\Phi}$. Given (5.68) it seems sensible to take

$$\tilde{\Phi}'(x,z)=\sum_\alpha\phi(x;\alpha)\xi(z;\alpha)\ , \tag{5.86}$$

however, can we do more to justify this choice? Lets assume we return to our original variable choices then we'd have the expansion

$$\mathcal{H}_{\mu\nu}(x,z)=\sum_\alpha\tilde{h}_{\mu\nu}(x;\alpha)\xi(z;\alpha)\ . \tag{5.87}$$

Which transforms under a diffeomorphism $\mathfrak{X}^M$ as

$$\mathcal{H}_{\mu\nu}\to\mathcal{H}'_{\mu\nu}=\mathcal{H}_{\mu\nu}+\partial_\mu\mathcal{X}_\nu+\partial_\nu\mathcal{X}_\mu\ . \tag{5.88}$$

In order to be able to interpret this diffeomorphism in terms of the fields $\tilde{h}_{\mu\nu}(x;\alpha)$ it makes sense to expand $\mathcal{X}_\mu$ as in (5.72). However under such a world volume diffeomorphism we know that $\mathcal{A}_\mu$ transforms as

$$\mathcal{A}_\mu\to\mathcal{A}'_\mu=\mathcal{A}_\mu+\partial_z\mathcal{X}_\mu\ , \tag{5.89}$$

hence the expansion becomes

$$\partial_z \mathcal{X}_\mu = \sum_\alpha \chi(x;\alpha) \frac{d}{dz}\xi(z;\alpha) \ . \tag{5.90}$$

We then note that $\{\frac{d}{dz}\xi(z;\alpha)\} = \{\sqrt{3}, n_i\sqrt{|\lambda_i|}\cos\left(\sqrt{|\lambda_i|}z\right)\}$ which clearly do not tend to zero as $z \to 0$. As such this is clearly not equivalent to the basis described by the $\{\xi(z;\alpha)\}$. Given this and our previous insight in to the expansion of this field we suggest that the full expansion, as in for the un-gauge fixed field, should be

$$\mathcal{A}_\mu = \sum_\alpha A_\mu(x;\alpha) \frac{d}{dz}\xi(z;\alpha) \ , \tag{5.91}$$

note that this is not an orthonormalised basis. We still have the z directed diffeomorphisms to deal with and we know that if we use the parameter $\mathfrak{X}^M = (0, \mathfrak{X}^z(x,z))$ then

$$\mathcal{A}_\mu \to \mathcal{A}'_\mu = \mathcal{A}_\mu + \partial_\mu \mathfrak{X}_z \ , \tag{5.92}$$

Since this should be able to be interpreted in terms of the basis given by $\{\frac{d}{dz}\xi(z;\alpha)\}$ this suggests writing

$$\mathfrak{X}_z = \sum_\alpha \chi_z(x;\alpha) \frac{d}{dz}\xi(z;\alpha) \ . \tag{5.93}$$

We then note that under such a transformation

$$\Phi(x,z) \to \Phi'(x,z) = \Phi + 2\partial_z \mathfrak{X}_z \ , \tag{5.94}$$

from which we see that (5.93) gives us

$$\partial_z \mathfrak{X}_z = \sum_\alpha \chi_z(x;\alpha) \frac{d^2}{dz^2}\xi(z;\alpha) = \sum_i \lambda(i)\chi_z(x;i)\xi(z;i) \ . \tag{5.95}$$

This suggests that the appropriate expansion for Φ should be

$$\Phi(x,z) = \sum_\alpha \phi(x;\alpha)\xi(z;\alpha) \ . \tag{5.96}$$

Given (5.96) and (5.93) we can see how Φ splits into Φ_t and Φ_{nt} since

$$\sum_\alpha \phi'(x;\alpha)\xi(z;\alpha) = \sum_\alpha \phi(x;\alpha)\xi(z;\alpha) + \sum_i \lambda(i)\chi_z(x;i)\xi(z;i) \ , \tag{5.97}$$

thus giving

$$\Phi_{nt} = \phi(x;0)\xi(z;0)\ , \quad \Phi_t = \sum_i \phi(x;i)\xi(z;i)\ . \tag{5.98}$$

With this we can finally expand the term $\mathcal{S}[H, \tilde{\Phi}_{nt}]$ in (5.70)

$$\begin{aligned}\mathcal{S}_{4d}[\phi] &= \int d^5X \Big(\tilde{\Phi}'_{nt}(\Box_4 - 2\partial_z^2)\tilde{\Phi}'_{nt} - 3a\tilde{H}^{\mu}{}_{\mu}\partial_z^2\tilde{\Phi}'_{nt} - 3a\tilde{\Phi}'_{nt}\partial_z^2\tilde{H}^{\mu}{}_{\mu}\Big)\\ &= \int d^5X \Big(\xi(0)^2\phi(0)(\Box_4)\phi(0) + a\sum_i \phi(0)\lambda(i)h^{\mu}{}_{\mu}(i)\xi(0)\xi(i)\Big)\\ &= \int d^4x\ \phi(x;0)\Box_4\phi(x;0)\ ,\end{aligned} \tag{5.99}$$

which leads to the final full action

$$\begin{aligned}S^{(2)}_{4d}[h_{\mu\nu}(\alpha), \phi(0)] = \int d^4x \Big(&\sum_\alpha \big(h_{\mu\nu}(\alpha)\mathcal{O}^{\mu\nu\rho\sigma}_{FP}h_{\rho\sigma}(\alpha)\big) + \frac{1}{2}\sum_i \lambda(i)\big(h_{\mu\nu}(i)h^{\mu\nu}(i) - h^{\mu}{}_{\mu}(i)h^{\nu}{}_{\nu}(i)\big)\\ &+\phi(0)\Box_4\phi(0)\Big)\ ,\end{aligned} \tag{5.100}$$

which describes a massless graviton, a tower of massive gravitons and a massless scalar field. As such we finally have the physical spectrum of our lower dimensional theory and have constructed a lower dimensional action that has explicit lower dimensional diffeomorphism invariance. Interestingly the zero mode spectrum contains a massless graviton and scalar but no Maxwell field, which should be contrasted to the usual Kaluza-Klein case.

5.1.3 The second order un-gauge fixed action

In the previous Subsection we finished by obtaining the lower dimensional action (5.100) in terms of the physical lower dimensional fields, up to having the unphysical gauge degrees of freedom associated to the four dimensional linearised gauge symmetry of $h_{\mu\nu}(x;0)$ present. However, most of the gauge theories we're use to are presented in a gauge unfixed manner. For example consider the case of Maxwell theory where no gauge is selected at the level of the action. This has the advantage of not committing to a particular choice of gauge fixing condition and hence any admissible gauge condition can then be implemented at a later stage, rather than always being fixed in the gauge that was chosen when writing down the original system. As such we shall now look to back out of our gauge choices $\tilde{\Phi}_t = 0$ and $\partial_z\mathcal{A}_\mu = 0$ and derive the analogue of (5.100). However, all of this new material will be pure gauge degrees of freedom and once our previous gauge conditions are implemented it must return to the from given in (5.100).

The full gauge symmetry we are expecting to find in five dimensions is summarised in (5.45). In the end these have to be interpreted in terms of lower dimensional fields. To do this we use (5.68), (5.91) and (5.96) along with (5.72) and (5.93) to obtain[110]

$$\begin{aligned}
\mathcal{H}'_{\mu\nu} &= \sum_{\alpha} \tilde{h}'_{\mu\nu}(x;\alpha)\xi(z;\alpha) = \sum_{\alpha} \big(\tilde{h}_{\mu\nu}(x;\alpha) + \partial_\mu \chi_\nu(x;\alpha) + \partial_\nu \chi_\mu(x;\alpha)\big)\xi(z;\alpha) \ , \\
\mathcal{H}'_{\mu z} &= \sum_{\alpha} A'_\mu(x;\alpha)\frac{d}{dz}\xi(z;\alpha) = \sum_{\alpha} \big(A_\mu(x;\alpha) + \partial_\mu \chi_z(x;\alpha) + \chi_\mu(x;\alpha)\big)\frac{d}{dz}\xi(z;\alpha) \ , \\
\mathcal{H}'_{zz} &= \sum_{\alpha} \varphi'(x;\alpha)\xi(z;\alpha) = \sum_{\alpha} \big(\varphi(x;\alpha) + 2\lambda(\alpha)\chi_z(x;\alpha)\big)\xi(z;\alpha) \ .
\end{aligned} \tag{5.101}$$

From this, linear independence of the $\xi(z;\alpha)$ allows us to peel off transformations for our four dimensional fields in a unique manner. However, up until now we've been very relaxed about the properties of the $\{\xi'(z;\alpha)\}$. Ideally we'd be able to show these are linearly independent and then we could again uniquely peel off the transformations of our four dimensional fields. To this end, let's take a finite set of these functions $\{\frac{d}{dz}\xi(z;\alpha)|\alpha \in \{1,\ldots,n\}\}$ and assume that they're linearly dependent so

$$\frac{d}{dz}\xi(z;n) = \sum_{\alpha=1}^{n-1} c(\alpha)\frac{d}{dz}\xi(z;\alpha) \ , \tag{5.102}$$

with $c(\alpha) \in \mathbb{R}$. We can now integrate this expression and use (5.28) to discard the constants of integration, from which we obtain that

$$\xi(z;n) = \sum_{\alpha=1}^{n-1} c(\alpha)\xi(z;\alpha) \ , \tag{5.103}$$

which would imply that $\{\xi(z;\alpha)|\alpha \in \{1,\ldots,n\}\}$ are linearly dependent. However this is false and so we have a contradiction thus meaning our original assumption (5.102) is incorrect. This logic demonstrates linear independence of a finite set of these functions. So assuming we can commute the sum and integral in the infinite case we can also show linear independence in this way. With this we can determine the transformation properties of our fields. We shall split the transformations into two types depending on the form of the diffeomorphism we perform. We shall call transformations of the form $\mathfrak{X}^M = (\chi^\mu(x)\xi(z;\beta), 0)$ spin-2 like transformations and $\mathfrak{X}^M = (0, \chi^z(x)\frac{d}{dz}\xi(z;\beta))$ Maxwell like transformations. Under a spin-2

[110] Here we are expanding the variables $\mathcal{H}_{\mu\nu}$, $\mathcal{A}_\mu$ and Φ.

transformation with transverse dependence $\xi(z;\beta)$, we have the following transformation properties

$$\begin{aligned}\tilde{h}'_{\mu\nu}(x;\alpha) =& \tilde{h}_{\mu\nu}(x;\alpha) + \delta_{\alpha\beta}\big(\partial_\mu\chi_\nu(x) + \partial_\nu\chi_\mu(x)\big) \ , \\ A'_\mu(x;\alpha) =& A_\mu(x;\alpha) + \delta_{\alpha\beta}\chi_\mu(x) \ , \\ \varphi'(x;\alpha) =& \varphi(x;\alpha) \ .\end{aligned} \tag{5.104}$$

While under a Maxwell transformation with transverse dependence $\frac{d}{dz}\xi(z;\beta)$, we find that

$$\begin{aligned}\tilde{h}'_{\mu\nu}(x;\alpha) =& \tilde{h}_{\mu\nu}(x;\alpha) \ , \\ A'_\mu(x;\alpha) =& A_\mu(x;\alpha) + \delta_{\alpha\beta}\partial_\mu\chi_z(x) \ , \\ \varphi'(x;\alpha) =& \varphi(x;\alpha) + 2\lambda(\beta)\delta_{\alpha\beta}\chi_z(x) \ .\end{aligned} \tag{5.105}$$

We can see that in both cases one field transforms exactly as we expect, one field is invariant and another undergoes a Stueckelberg-like transformation, which is characterised by a shift in the field by the parameter of the transformation.

Our goal is to now find a four dimensional action that is invariant, including boundary terms, under (5.104) and (5.105) for any transverse dependence, and which upon the implementation of our previous gauge conditions leads to the action (5.100). For this we begin with (5.65) and then perform the variable redefinition

$$\mathcal{H}_{\mu\nu} = H_{\mu\nu} + 2a\eta_{\mu\nu}\tilde{\Phi}_{nt} \ , \quad \Phi = 2b\big(\tilde{\Phi}_t + \tilde{\Phi}_{nt}\big) \ , \quad b = -2a \ , \quad a^2 = \frac{1}{12} \ . \tag{5.106}$$

This redefinition leads to the action assuming the form

$$\begin{aligned}S_{5d}[H,\mathcal{A},\tilde{\Phi}_{nt},\tilde{\Phi}_t] = \int d^5X\Big(&\frac{1}{2}H_{\mu\nu}(\Box_4+\partial_z^2)H^{\mu\nu} - \frac{1}{2}H^\mu{}_\mu(\Box_4+\partial_z^2)H^\nu{}_\nu + \frac{1}{2}H^\mu{}_\mu\partial^\nu\partial^\sigma H_{\nu\sigma} + \frac{1}{2}H_{\mu\nu}\partial^\mu\partial^\nu H^\sigma{}_\sigma \\ &-H_\mu{}^\nu\partial^\sigma\partial^\mu H_{\sigma\nu} + \mathcal{A}_\mu(\Box_4\eta^{\mu\nu} - \partial^\mu\partial^\nu)\mathcal{A}_\nu + H^\mu{}_\mu\partial^\nu\partial^z\mathcal{A}_\nu - H_\mu{}^\nu\partial^z\partial^\mu\mathcal{A}_\nu - (\partial^z\mathcal{A}_\mu)\partial^\mu H^\nu{}_\nu \\ &+(\partial^z\mathcal{A}^\nu)\partial^\mu H_{\mu\nu} + 6a\tilde{\Phi}_{nt}\partial^\mu\partial^z\mathcal{A}_\mu - 6a(\partial^z\mathcal{A}_\mu)\partial^\mu\tilde{\Phi}_{nt} + 2aH^\mu{}_\mu(\Box_4)\tilde{\Phi}_t + 2a\tilde{\Phi}_t(\Box_4)H^\mu{}_\mu \\ &-2a\tilde{\Phi}_t\partial^\nu\partial^\sigma H_{\nu\sigma} - 2aH_{\mu\nu}\partial^\mu\partial^\nu\tilde{\Phi}_t + \tilde{\Phi}_{nt}\Box_4\tilde{\Phi}_t + \tilde{\Phi}_t\Box_4\tilde{\Phi}_{nt} + \tilde{\Phi}_{nt}(\Box_4 - 2\partial_z^2)\tilde{\Phi}_{nt} \\ &-3aH^\mu{}_\mu\partial_z^2\tilde{\Phi}_{nt} - 3a\tilde{\Phi}_{nt}\partial_z^2H^\mu{}_\mu\Big) \ ,\end{aligned} \tag{5.107}$$

which if we now use the expansions

$$H_{\mu\nu} = \sum_\alpha h_{\mu\nu}(x;\alpha)\xi(z;\alpha)\ , \quad \mathcal{A}_\mu = \sum_\alpha A_\mu(x;\alpha)\frac{d}{dz}\xi(z;\alpha)\ ,$$
$$\tilde{\Phi}_{nt} = \phi(x;0)\xi(z;0)\ , \quad \tilde{\Phi}_t = \sum_i \phi(x;i)\xi(z;i)\ , \tag{5.108}$$

becomes

$$\begin{aligned}
S^{(2)}_{4d}[h_{\mu\nu}(\alpha),\phi(0),\phi(i),A_\mu(i)] = \int d^4x\Big(&\sum_\alpha \big(h_{\mu\nu}(\alpha)\mathcal{O}_{FP}^{\mu\nu\rho\sigma}h_{\rho\sigma}(\alpha)\big) + \frac{1}{2}\sum_i \lambda(i)\big(h_{\mu\nu}(i)h^{\mu\nu}(i) - h^\mu{}_\mu(i)h^\nu{}_\nu(i)\big) \\
&+\phi(0)\Box_4\phi(0)\Big) \\
+\int d^4x\Big(&\sum_{\alpha,\beta} A_\mu(\alpha)\big(\Box_4\eta^{\mu\nu} - \partial^\mu\partial^\nu\big)A_\nu(\beta)\int dz\ \xi'(\alpha)\xi'(\beta) \\
&+\sum_{\alpha,\beta}\big(h^\mu{}_\mu(\alpha)\partial^\nu A_\nu(\beta)\int dz\ \xi(\alpha)\xi''(\beta) - h_\mu{}^\nu(\alpha)\partial^\mu A_\nu(\beta)\int dz\ \xi(\alpha)\xi''(\beta)\big) \\
&-\sum_{\alpha,\beta}\big(\partial^\mu h^\nu{}_\nu(\alpha)A_\mu(\beta)\int dz\ \xi(\alpha)\xi''(\beta) - \partial^\mu h_{\mu\nu}(\alpha)A^\nu(\beta)\int dz\ \xi(\alpha)\xi''(\beta)\big) \\
&+6a\sum_\alpha\big(\phi(0)\partial^\mu A_\mu(\alpha)\int dz\ \xi(0)\xi''(\alpha) - A_\mu(\alpha)\partial^\mu\phi(0)\int dz\ \xi(\alpha)\xi''(0)\big) \\
&+2a\sum_{\alpha,i} h^\mu{}_\mu(\alpha)\Box_4\phi(i)\int dz\ \xi(\alpha)\xi(i) + \sum_{i,\alpha}\phi(i)\Box_4 h^\mu{}_\mu(\alpha)\int dz\ \xi(i)\xi(\alpha) \\
&-2a\sum_{i,\alpha}\big(\phi(i)\partial^\mu\partial^\nu h_{\nu\mu}(\alpha)\int dz\ \xi(i)\xi(\alpha) + h_{\mu\nu}(\alpha)\partial^\mu\partial^\nu\phi(i)\int dz\ \xi(\alpha)\xi(i)\big) \\
&+\sum_i\big(\phi(0)\Box_4\phi(i)\int dz\ \xi(0)\xi(i) + \phi(i)\Box_4\phi(0)\int dz\ \xi(i)\xi(0)\big)\Big)\ ,
\end{aligned}$$

which is just our previous gauge fixed action plus a collection of terms involving pure gauge degrees of freedom- precisely as expected.

If we now add a boundary term analogous to (5.83), use the transverse eigenfunction problem and (5.28), this becomes

$$S^{(2)}_{4d}[h_{\mu\nu}(\alpha),\phi(0),\phi(i),A_\mu(i)] = \int d^4x\Big(\sum_\alpha\big(h_{\mu\nu}(\alpha)\mathcal{O}^{\mu\nu\rho\sigma}_{FP}h_{\rho\sigma}(\alpha)\big) + \phi(0)\Box_4\phi(0) \tag{5.109}$$
$$+\sum_i\Big(\lambda(i)\big(\frac{1}{2}(h_{\mu\nu}(i)h^{\mu\nu}(i) - h^\mu{}_\mu(i)h^\nu{}_\nu(i)) - A_\mu(i)\big(\Box_4\eta^{\mu\nu} - \partial^\mu\partial^\nu\big)A_\nu(i)$$
$$+h^\mu{}_\mu(i)\partial^\nu A_\nu(i) - h_\mu{}^\nu(i)\partial^\mu A_\nu(i) - \partial^\mu h^\nu{}_\nu(i)A_\mu(i) + \partial^\mu h_{\mu\nu}(i)A^\nu(i)\big)$$
$$+2ah^\mu{}_\mu(i)\Box_4\phi(i) + 2a\phi(i)\Box_4 h^\mu{}_\mu(i) - 2a\phi(i)\partial^\mu\partial^\nu h_{\nu\mu}(i)$$
$$-2ah_{\mu\nu}(i)\partial^\mu\partial^\nu\phi(i)\Big)\Big)\ .$$

We now need to interpret our gauge transformations (5.104) and (5.105) in terms of our fields (5.108). Using all of the previous definitions we find that under a Maxwell like transformation, given by $\chi_z(x)\frac{d}{dz}\xi(z;\beta)$, we have the following transformations

$$h'_{\mu\nu}(x;\alpha) = h_{\mu\nu}(x;\alpha)\ ,\quad A'_\mu(x;\alpha) = A_\mu(x;\alpha) + \delta_{\alpha\beta}\partial_\mu\chi_z(x)\ ,$$
$$\phi'(x;0) = \phi(x;0)\ ,\quad \phi'(x;i) = \phi(x;i) - \frac{1}{2a}\lambda(\beta)\delta_{i\beta}\chi_z(x)\ , \tag{5.110}$$

while under a spin-2 transformation, $\chi_\mu(x)\xi(z;\beta)$, we have

$$h'_{\mu\nu}(x;\alpha) = h_{\mu\nu}(x;\alpha) + \delta_{\alpha\beta}\big(\partial_\mu\chi_\nu(x) + \partial_\nu\chi_\mu(x)\big)\ ,\quad A'_\mu(x;\alpha) = A_\mu(x;\alpha) + \delta_{\alpha\beta}\chi_\mu(x)\ ,$$
$$\phi'(x;0) = \phi(x;\alpha)\ ,\quad \phi'(x;i) = \phi(x;i)\ . \tag{5.111}$$

Under (5.110) we find that (5.109) transforms, if we use the parameter $\chi_\mu(x)\xi(z;j)$, as

$$\delta S^{(2)}_{4d} = \int d^4x\Big(\lambda(j)\big(\frac{1}{2}(2(\partial_\mu\chi_\nu + \partial_\nu\chi_\mu)h^{\mu\nu}(j) - 4h^\mu{}_\mu(j)\partial^\nu\chi_\nu) - \chi_\mu(\Box_4\eta^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu(j)$$
$$-A_\mu(j)(\Box_4\eta^{\mu\nu} - \partial^\mu\partial^\nu)\chi_\nu + 2\partial^\mu\chi_\mu\partial^\nu A_\nu(j) + h^\mu{}_\mu(j)\partial^\nu\chi_\nu$$
$$+(\partial_\mu\chi_\nu + \partial_\nu\chi_\mu)\partial^\mu A^\nu(j) - h_{\mu\nu}(j)\partial^\mu\chi^\nu - 2(\partial^\mu\partial^\nu\chi_\nu)A_\mu(j)$$
$$-\partial^\mu h^\nu{}_\nu(j)\chi_\mu + \partial^\mu(\partial_\mu\chi_\nu + \partial_\nu\chi_\mu)A^\nu(j) + \partial^\mu h_{\mu\nu}(j)\chi^\nu\big) + 4a\partial^\mu\chi_\mu\Box_4\phi(j)$$
$$+4a\phi(j)\Box_4\partial^\mu\chi_\mu - 2a\phi(j)\partial^\mu\partial^\nu(\partial_\mu\chi_\nu + \partial_\nu\chi_\mu) - 2a(\partial_\mu\chi_\nu + \partial_\nu\chi_\mu)\partial^\mu\partial^\nu\phi(j)\Big)\ .$$

We can see that these terms cancel, up to worldvolume boundary terms which we shall not worry about in the current setting. The crucial insight we gain here is how gauge invariance is achieved. Under a spin-2 transformation, we see firstly that the terms coupling $h_{\mu\nu}(i)$ and $\phi(i)$ are gauge invariant on their own, which signals this coupling is precisely as expected from the massless case. The most important lesson we learn is that the mass term for our massive gravitons, which on its own, is infamously, not gauge invariant, combines with the couplings between $h_{\mu\nu}(i)$ and $A_\mu(i)$, along with the kinetic term for $A_\mu(i)$ to give us gauge invariance. So we see that it's crucial that the Maxwell like fields $A_\mu(i)$ behave as Stueckelberg fields, under world volume directed diffeomorphisms, as they allow for a gauge invariant action to be written. Even neater, this approach leads us to understand that we can, after a few worldvolume integration by parts, write the action (5.109) in the form

$$
\begin{aligned}
S^{(2)}_{4d}[h_{\mu\nu}(\alpha),\phi(0),\phi(i),A_\mu(i)] = \int d^4x \Big(& \sum_\alpha \big(h_{\mu\nu}(\alpha)\mathcal{O}^{\mu\nu\rho\sigma}_{FP}h_{\rho\sigma}(\alpha)\big) + \phi(0)\Box_4\phi(0) \\
& + \sum_i \Big(\frac{\lambda(i)}{2}\big((h_{\mu\nu}(i) - 2\partial_{(\mu}A_{\nu)}(i))(h^{\mu\nu}(i) - 2\partial^{(\mu}A^{\nu)}(i)) \\
& - (h^\mu{}_\mu(i) - 2\partial^\mu A_\mu(i))(h^\nu{}_\nu(i) - 2\partial^\nu A_\nu(i))\big) \\
& + 4a\big(\Box_4 h^\mu{}_\mu(i) - \partial^\mu\partial^\nu h_{\mu\nu}(i)\big)\phi(i)\Big)\Big) \ ,
\end{aligned}
\tag{5.112}
$$

which is manifestly gauge invariant under our spin-2 like transformations.

How about under the Maxwell like transformations? If we perform the transformation $\chi_z \frac{d}{dz}\xi(z;j)$ then we find the variation of the action (5.109) is

$$
\begin{aligned}
\delta S^{(2)}_{4d} = \int d^4x \ \Big(\lambda(j)\big(& -\partial_\mu\chi_z(\Box_4\eta^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu(j) - A_\mu(j)(\Box_4\eta^{\mu\nu} - \partial^\mu\partial^\nu)\partial_\nu\chi_z \\
& + h^\mu{}_\mu(j)\Box_4\chi_z - h_{\mu\nu}(j)\partial^\mu\partial^\nu\chi_z - \partial^\mu h^\nu{}_\nu(j)\partial_\mu\chi_z + \partial^\mu h_{\mu\nu}(j)\partial^\nu\chi_z\big) \\
& - \lambda(j)h^\mu{}_\mu(j)\Box_4\chi_z - \lambda(j)\chi_z\Box_4 h^\mu{}_\mu(j) + \lambda(j)\chi_z\partial^\mu\partial^{nu}h_{\mu\nu}(j) + \lambda(j)h_{\mu\nu}(j)\partial^\mu\partial^\nu\chi_z\Big) \ .
\end{aligned}
$$

From this we can see that the interaction terms between $h_{\mu\nu}(i)$ and $A_\mu(i)$, which are of one derivative type, are made gauge invariant by the interactions of $h_{\mu\nu}(i)$ with $\phi(i)$, which are of two derivative type. Again this suggests a rewriting of (5.109) to exhibit the Maxwell like gauge invariance more readily

$$
\begin{aligned}
S^{(2)}_{4d}[h_{\mu\nu}(\alpha),\phi(0),\phi(i),A_\mu(i)] = \int d^4x \Big(& \sum_\alpha \big(h_{\mu\nu}(\alpha)\mathcal{O}^{\mu\nu\rho\sigma}_{FP}h_{\rho\sigma}(\alpha)\big) + \phi(0)\Box_4\phi(0) \\
& + \sum_i \Big(\lambda(i)\big(\frac{1}{2}(h_{\mu\nu}(i)h^{\mu\nu}(i) - h^\mu{}_\mu(i)h^\nu{}_\nu(i)) - A_\mu(i)\big(\Box_4\eta^{\mu\nu} - \partial^\mu\partial^\nu\big)A_\nu(i) \\
& + 2h^\mu{}_\mu(i)\big(\partial^\nu A_\nu(i) + \frac{2a}{\lambda(i)}\Box_4\phi(i)\big) - 2h_{\mu\nu}(i)\big(\partial^\mu A^\nu(i) + \frac{2a}{\lambda(i)}\partial^\mu\partial^\nu\phi(i)\big)\big)\Big)\Big) \ ,
\end{aligned}
\tag{5.113}
$$

which shows manifest Maxwell like invariance but the spin-2 invariance is more obscured.

At this point we have achieved our original goal of obtaining a quadratic second order action that displays the full gauge invariance we expected. We can see that in order to display the spin-2 invariance or the Maxwell-like invariance it is beneficial to write the quadratic action in two quite different ways, which emphasise the Stueckelberg nature of different fields. It would be nice to be able to write our action in terms of objects that are gauge inert under both the spin-2 and Maxwell like transformations. One object that has this property is

$$h_{\mu\nu}(i) - 2\partial_{(\mu}A_{\nu)}(i) - \frac{4a}{\lambda(i)}\partial_\mu\partial_\nu\phi(i) \,, \tag{5.114}$$

so we might naturally ask if there is a way to write our mass term using this object. By direct calculation, involving several integration by parts on the worldvolume, we find that (5.115) can be written, using (5.114), as

$$\begin{aligned} S^{(2)}_{4d}[h_{\mu\nu}(\alpha), \phi(0), \phi(i), A_\mu(i)] = \int d^4x \Big(\sum_\alpha \big(h_{\mu\nu}(\alpha)\mathcal{O}^{\mu\nu\rho\sigma}_{FP} h_{\rho\sigma}(\alpha)\big) + \phi(0)\Box_4\phi(0) \\ + \sum_i \frac{\lambda(i)}{2}\Big((h_{\mu\nu}(i) - 2\partial_{(\mu}A_{\nu)}(i) - \frac{4a}{\lambda(i)}\partial_\mu\partial_\nu\phi(i))(h^{\mu\nu}(i) - 2\partial^{(\mu}A^{\nu)}(i) - \frac{4a}{\lambda(i)}\partial^\mu\partial^\nu\phi(i)) \\ -(h^\mu{}_\mu(i) - 2\partial^\mu A_\mu(i) - \frac{4a}{\lambda(i)}\Box_4\phi(i))(h^\nu{}_\nu(i) - 2\partial^\nu A_\nu(i) - \frac{4a}{\lambda(i)}\Box_4\phi(i))\Big)\Big) \,. \end{aligned} \tag{5.115}$$

This action is manifestly invariant, up to the usual boundary term associated to the Fierz-Pauli term, under both the spin-2 and Maxwell-like transformations. However, note, in order to achieve this invariance, we've had to, seemingly, include higher derivative terms in the action. Fortunately these higher derivative terms cancel out when we expand all of the terms and perform integration by parts on the worldvolume.

5.2 Going beyond quadratic order

So far we've mostly been interested in treating our system at quadratic order. However, we know that much of the interesting physics of gravitational systems occurs at the non-linear order. As a result we're going to have to go beyond our free theory and start including interactions. This leads us to consider higher orders in our perturbative expansion and so to introduce terms cubic and higher in fields into our effective action. Doing this not only increases the complexity of our effective action, but also alters the transformational properties of our fields. In upcoming work we shall present the full solution, up to quartic order in fundamental fields, to obtain an effective four dimensional action that displays the expected diffeomorphism invariance of gravity. Here, however, our goal is much more humble, as we shall attempt to understand how the expected lower dimensional field transformations emerge once we go beyond linear order. This seemingly simple task is crucial in the construction of the lower dimensional theory as it's the combination of the action and transformations assuming a particular form that signals that lower dimensional diffeomorphism symmetry has emerged in the expected manner.

We know that given a general five dimensional diffeomorphism, $(X^\mu(x,z), X^z(x,z))$, our five dimensional fields, written in terms of $4+1$ split variables, transform as

$$\begin{aligned}
\mathcal{H}'_{\mu\nu} =& \mathcal{H}_{\mu\nu} + \partial_\mu X_\nu + \partial_\nu X_\mu + X^\sigma\partial_\sigma\mathcal{H}_{\mu\nu} + \mathcal{H}_{\sigma\nu}\partial_\mu X^\sigma + \mathcal{H}_{\mu\sigma}\partial_\nu X^\sigma + X^z\partial_z\mathcal{H}_{\mu\nu} + \mathcal{A}_\nu\partial_\mu X^z + \mathcal{A}_\mu\partial_\nu X^z \ , \\
\mathcal{A}'_\mu =& \mathcal{A}_\mu + \partial_z X_\mu + X^\sigma\partial_\sigma\mathcal{A}_\mu + \mathcal{A}_\sigma\partial_\mu X^\sigma + \mathcal{H}_{\mu\sigma}\partial_z X^\sigma + \partial_\mu X_z + X^z\partial_z\mathcal{A}_\mu + \Phi\partial_\mu X^z + \mathcal{A}_\mu\partial_z X^z \ , \quad (5.116) \\
\Phi' =& \Phi + X^\sigma\partial_\sigma\Phi + 2\mathcal{A}_\mu\partial_z X^\mu + 2\partial_z X_z + X^z\partial_z\Phi + 2\Phi\partial_z X^z \ ,
\end{aligned}$$

from which we'd like to interpret the non-linear transformations of our lower dimensional fields. Since our lower dimensional fields arise from the expansions of our higher dimensional fields, in terms of our chosen basis of functions for the transverse space, we might expect to be able to use the expansions

$$\mathcal{H}_{\mu\nu} = \sum_\alpha h_{\mu\nu}(x;\alpha)\xi(z;\alpha) \ , \quad \mathcal{A}_\mu = \sum_\alpha A_\mu(x;\alpha)\xi'(z;\alpha) \ , \quad \Phi = \sum_\alpha \phi(x;\alpha)\xi(z;\alpha) \ , \qquad (5.117)$$

which were proposed at quadratic order[111], along with the following, also learnt at quadratic order, expansions

$$X_\mu = \sum_\alpha \chi_\mu(x;\alpha)\xi(z;\alpha) \ , \quad X_z = \sum_\alpha \chi_z(x;\alpha)\xi'(z;\alpha) \ , \qquad (5.118)$$

for our diffeomorphism parameter. Making use of these in (5.116) we might expect to find that

$$\begin{aligned}
\sum_\alpha h'_{\mu\nu}(x;\alpha)\xi(z;\alpha) =& \sum_\alpha \big(h_{\mu\nu}(x;\alpha) + \partial_\mu\chi(x;\alpha) + \partial_\nu\chi_\mu(x;\alpha)\big)\xi(z;\alpha) \qquad (5.119) \\
&+ \sum_{\beta,\gamma} \big(\chi^\sigma(x;\gamma)\partial_\sigma h_{\mu\nu}(x;\beta) + h_{\sigma\nu}(x;\beta)\partial_\mu\chi^\sigma(x;\gamma) + h_{\mu\sigma}(x;\beta)\partial_\nu\chi^\sigma(x;\gamma)\big)\xi(z;\beta)\xi(z;\gamma) \\
&+ \sum_{\beta,\gamma} \big(\chi^z(x;\gamma)h_{\mu\nu}(x;\beta) + A_\nu(x;\beta)\partial_\mu\chi^z(x;\gamma) + A_\mu(x;\beta)\partial_\nu\chi^z(x;\gamma)\big)\xi'(z;\beta)\xi'(z;\gamma) \ ,
\end{aligned}$$

[111]Note here we don't include the term involving $\eta_{\mu\nu}\phi(0)$ into $\mathcal{H}_{\mu\nu}$ just for convenience. It could and will, later on in this Section, be added to the discussion, but as it won't affect the chain of thought we're currently in the middle of we omit it for now.

$$\sum_{\alpha} A'_\mu(x;\alpha)\xi'(z;\alpha) = \sum_{\alpha} \big(A_\mu(x;\alpha) + \chi_\mu(x;\alpha) + \partial_\mu \chi_z(x;\alpha)\big)\xi'(z;\alpha)$$
$$+ \sum_{\beta,\gamma} \big(\chi^\sigma(x;\gamma)\partial_\sigma A_\mu(x;\beta) + A_\sigma(x;\beta)\partial_\mu \chi^\sigma(x;\gamma) + h_{\mu\sigma}(x;\gamma)\chi^\sigma(x;\beta) \tag{5.120}$$
$$+\chi^z(x;\beta)\lambda(\gamma)A_\mu(x;\gamma) + \phi(x;\gamma)\partial_\mu\chi^z(x;\beta) + A_\mu(x;\beta)\lambda(\gamma)\chi^z(x;\gamma)\big)\xi(z;\gamma)\xi'(z;\beta) \ ,$$

$$\sum_{\alpha} \phi'(x;\alpha)\xi(z;\alpha) = \sum_{\alpha} \big(\phi(x;\alpha) + 2\lambda(\alpha)\chi^z(x;\alpha)\big)\xi(z;\alpha)$$
$$+ \sum_{\beta,\gamma} \big(\chi^\sigma(x;\gamma)\partial_\sigma\phi(x;\beta) + 2\phi(x;\beta)\lambda(\gamma)\chi^z(x;\gamma)\big)\xi(z;\beta)\xi(z;\gamma) \tag{5.121}$$
$$+ \sum_{\beta,\gamma} \big(2A_\mu(x;\beta)\chi^\mu(x;\gamma) + \chi^z(x;\gamma)\phi(x;\beta)\big)\xi'(z;\beta)\xi'(z;\gamma) \ ,$$

where the left hand sides represent a new expansion of the form given in (5.117) with transformed lower dimensional fields. From these we might hope to be able to peel off lower dimensional non-linear transformations by applying operators such as $\int dz\ \xi(z;\tilde{\alpha})$ and then using appropriate overlap integrals[112]. This is a procedure that is commonly advocated. However, there is an implicit assumption in these equations. The assumption is that all objects arising on the right hand side of these expressions reside within the span of our chosen transverse wavefunctions. It's here that issues can arise. Recall that the basis of functions $\{\xi(z;\alpha)\}$ obeys the condition

$$\xi(0;\alpha) = 0 \ , \tag{5.122}$$

which ensures any function, $f(z)$, expanded in terms of them should also obey the same condition, namely $f(0) = 0$. It's now that our problems become apparent. Looking at the terms in last line of both (5.119) and (5.121) we see that they don't conform to the boundary conditions we've selected for our basis, since $\xi'(0;\alpha)$ is a non-zero constant for all α. This insight tells us that the expressions given in (5.119) and (5.121) are mathematical nonsense, as we've tried to expand a function using a basis for a space it doesn't reside in!

So we're in the position where we need to change our approach, but what can we do? We can start by considering a problem that occurs in four dimensional linearised gravity. Here we know that our graviton, $h_{\mu\nu}$, enjoys, at quadratic order, the gauge symmetry

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu X_\nu + \partial_\nu X_\mu \ . \tag{5.123}$$

[112] Recall the $\xi'(z;\alpha)$ are not orthogonal in our case and so dealing with them is more subtle, but again we can ignore this complication in the current discussion.

We often choose, since it simplifies the linearised equations of motion, to deal with the trace reversed field

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\sigma}{}_{\sigma} \ , \tag{5.124}$$

which inherits the gauge symmetry

$$\tilde{h}'_{\mu\nu} = \tilde{h}_{\mu\nu} + \partial_\mu X_\nu + \partial_\nu X_\mu - \eta_{\mu\nu}\partial^\sigma X_\sigma \ , \tag{5.125}$$

this is often used to set the transverse gauge condition

$$\partial^\mu \tilde{h}'_{\mu\nu} = 0 \ . \tag{5.126}$$

For a general trace reversed perturbation, $\tilde{h}_{\mu\nu}$, we can see, from (5.125), that it's possible to obtain a perturbation obeying (5.126) if our diffeomorphism parameter is chosen so it solves[113]

$$\Box_4 X_\nu = -\partial^\mu \tilde{h}_{\mu\nu} \ . \tag{5.127}$$

At linear order this is the full story. However, at some point we want to go beyond our original free action and include interactions. Once this is done our simple notion of gauge invariance, (5.123), is modified to

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu X_\nu + \partial_\nu X_\mu + \mathcal{L}_X h_{\mu\nu} \ , \tag{5.128}$$

which means the trace reversed transformation, (5.125), becomes

$$\begin{aligned}\tilde{h}'_{\mu\nu} =& \tilde{h}_{\mu\nu} + \partial_\mu X_\nu + \partial_\nu X_\mu - \eta_{\mu\nu}\partial^\sigma X_\sigma + \mathcal{L}_X \tilde{h}_{\mu\nu} - \frac{1}{2}\tilde{h}^{\rho}{}_{\rho}\big(\partial_\mu X_\nu + \partial_\nu X_\mu\big) \\ & - \eta_{\mu\nu}\tilde{h}_{\rho\sigma}\partial^\rho X^\sigma + \frac{1}{2}\eta_{\mu\nu}\tilde{h}^{\sigma}{}_{\sigma}\partial_\rho X^\rho \ . \end{aligned} \tag{5.129}$$

If we now try to implement (5.126) using the parameter given in (5.127) then we will find that, while at first order in fields the condition is as we'd like, once we include the second order terms our condition no longer holds. This seems to be an issue as the non-linear terms in our transformation are leading to effects we weren't aware of at the linear order. However as with most problems there is a solution. In this case our analysis is telling us that our previous derivation isn't enough to give us the full picture once we allow for

[113] Yes this expression does have issues regarding to inverting a d'Alembertian operator, but again we're concerned with the rough argument rather than the precise details.

interactions. The fix is to understand that our previous result held at linear order in fields, and that's it. If we want to go to higher orders in fields in one part of our analysis we'll have to include corrections from higher orders to everything we derived at linear order. This suggests that we expand our diffeomorphism parameter as

$$X_\mu = \sum_{n=1}^{N} X_\mu^{(n)} \ , \tag{5.130}$$

where $X_\mu^{(i)}$ is i-th order in fields and $X_\mu^{(1)}$ is given by (5.127). If we use this newly defined diffeomorphism parameter in (5.129) then we find that, for $i > 1$, the relationship

$$\Box_4 X_\nu^{(i+1)} = -\Big(\partial^\mu\big(\mathcal{L}_{X^{(i)}}\tilde{h}_{\mu\nu}\big) - \frac{1}{2}\partial^\mu\big(\tilde{h}^\rho{}_\rho\big(\partial_\mu X_\nu^{(i)} + \partial_\nu X_\mu^{(i)}\big)\big) - \partial_\nu\big(\tilde{h}_{\rho\sigma}\partial^\rho X^\sigma_{(i)}\big) + \frac{1}{2}\partial_\nu\big(\tilde{h}^\sigma{}_\sigma\partial_\rho X^\rho_{(i)}\big)\Big) \ , \tag{5.131}$$

must hold in order for (5.126) to hold, at a given order. So we can see that we require a compensating diffeomorphism in order to remain in our desired gauge after we go beyond linear order. This teaches us that once we go beyond the safety of lowest order in fields we have to amend all of our results to include corrections from terms that are higher order in the number of fields. So this seems to provide an insight that could allow us to resolve the issues plaguing (5.119) and (5.121).

We can now see from our example that we can try and allow our diffeomorphism parameter to be expanded in powers of fields, which we keep track of by the use of a parameter ϵ. Then it's our hope that these higher order terms in the diffeomorphism parameter can be used to cancel off terms that don't conform to our specified boundary conditions. Since we're attempting to perform a dimensional reduction we're required to specify a transverse dependence at each level of our power series. As such, we shall take as our diffeomorphism parameter

$$X^M = \Big(\sum_{n=0}^{N}\sum_{a_n}\chi^\mu_{(n)}(x;a_n)f(z;a_n,n), \sum_{n=0}^{N}\sum_{\tilde{a}_n}\chi^z_{(n)}(x;\tilde{a}_n)g(z;\tilde{a}_n,n)\Big) \ , \tag{5.132}$$

where terms labelled by n are $\mathcal{O}(\epsilon^n)$ and $f(z;a_n,n)$ and $g(z;\tilde{a}_n,n)$ are functions to be determined, as we'll see below. The label a_n is included to act as a counter for modes and it's exact form will be explained later in this Section, crucially we shall see we need the form of this counter to depend on n. When $n = 0$ the transverse dependence is as given in (5.118) and N denotes the highest order in fields we wish to go to[114]. With this we can consider if it's possible to find a compensating transformation that allows us to make sense

[114]We should note that if we want a solution at all orders in perturbation theory we have to go to the limit $N \to \infty$. For us we shall mostly be concerned with fourth order terms in the action. As a result shall have to deal with $N = 3$.

of expressions like (5.119) and (5.121).

We shall begin by considering a z-directed diffeomorphism, by which we mean we take $\chi^{\mu}_{(0)}=0$ in (5.132). Considering such a diffeomorphism in (5.116), while making use of (5.117) and suppressing sums over α and a_n, for clarity of presentation[115], we find that the equations we wish to be able to make sense of are

$$\begin{aligned}h'_{\mu\nu}(x;\alpha)\xi(z;\alpha) =&h_{\mu\nu}(x;\alpha)\xi(z;\alpha)+\sum_{n=1}^{N}\big(\partial_\mu\chi_\nu(x;a_n)+\partial_\nu\chi_\mu(x;a_n)\big)f(z;a_n)\\&+\sum_{n=1}^{N}\big(\chi^\sigma(x;a_n)\partial_\sigma h_{\mu\nu}(x;\alpha)+h_{\sigma\nu}(x;\alpha)\partial_\mu\chi^\sigma(x;a_n)+h_{\mu\sigma}(x;\alpha)\partial_\nu\chi^\sigma(x;a_n)\big)\xi(z;\alpha)f(z;a_n)\\&+\chi^z(x;\alpha)h_{\mu\nu}(x;\beta)\xi'(z;\alpha)\xi'(z;\beta)+\sum_{n=1}^{N}\chi^z(x;a_n)h_{\mu\nu}(x;\alpha)g(z;a_n)\xi'(z;\alpha)\\&+\big(A_\nu(x;\alpha)\partial_\mu\chi^z(x;\beta)+A_\mu(x;\alpha)\partial_\nu\chi^z(x;\beta)\big)\xi'(z;\alpha)\xi'(z;\beta)\\&+\sum_{n=1}^{N}\big(A_\nu(x;\alpha)\partial_\mu\chi^z(x;a_n)+A_\mu(x;\alpha)\partial_\nu\chi^z(x;a_n)\big)\xi'(z;\alpha)g(z;a_n)\ ,\end{aligned}\tag{5.133}$$

$$\begin{aligned}\phi'(x;\alpha)\xi(z;\alpha) =&\phi(x;\alpha)\xi(z;\alpha)+\sum_{n=1}^{N}\chi^\sigma(x;a_n)\partial_\sigma\phi(x;\alpha)f(z;a_n)\xi(z;\alpha)+\sum_{n=1}^{N}2A_\mu(x;\alpha)\chi^\mu(x;a_n)\xi'(z;\alpha)f'(z;a_n)\\&+2\lambda(\alpha)\chi_z(x;\alpha)\xi(z;\alpha)+\sum_{n=1}^{N}2\chi_z(x;a_n)g'(z;a_n)+\chi_z(x;\alpha)\phi(x;\beta)\xi'(z;\alpha)\xi'(z;\beta)\\&+\sum_{n=1}^{N}\chi_z(x;a_n)\phi(x;\alpha)g(z;a_n)\xi'(z;\alpha)+2\lambda(\beta)\phi(x;\alpha)\chi_z(x;\beta)\xi(z;\alpha)\xi(z;\beta)\\&+\sum_{n=1}^{N}2\phi(x;\alpha)\chi_z(x;a_n)\xi(z;\alpha)g'(z;a_n)\ .\end{aligned}\tag{5.134}$$

We have omitted the transformation of $\mathcal{A}_\mu$ here as it is the boundary conditions of the other fields that are of principle concern to us at this stage. Of principal concern to us is if it's possible to find compensating diffeomorphisms such that all terms that arise that don't obey our chosen boundary conditions can be removed.

In order to appropriately account for such a compensating diffeomorphism, if it exists, we need to understand the sorts of terms that can be expanded in our selected basis. To understand this we have to recall that our basis $\{\xi(z;\alpha)\}$ arises as the eigenfunctions of the differential operator

$$\mathcal{L}=\frac{d^2}{dz^2}\ ,\tag{5.135}$$

[115] Thus instating a summation convention over such arguments when they're repeated.

which was made to be self-adjoint[116] by the choice of the boundary conditions

$$\xi(0;\alpha) = 0\ ,\quad \xi'(1;\alpha) = \xi(1;\alpha)\ . \tag{5.136}$$

As mentioned previously this puts us in the realm of Sturm-Liouville theory. In this context it can be shown that the eigenbasis of a Sturm-Liouville operator forms a complete basis for functions that satisfy the boundary conditions we have selected. This provides us with simple way to determine which of the terms in (5.133) and (5.134) can be expanded in terms of our basis.

Let's begin with the term $\xi'(z;\alpha)\xi'(z;\beta)$. We know that $\xi'(0;\alpha) \neq 0$ and hence this function doesn't obey our boundary conditions and so can't be expanded in terms of the functions $\xi(z;\alpha)$. We already knew this would be the case as it was this issue that led us to initially consider the idea of compensating gauge transformations. The other term to check is $\xi(z;\alpha)\xi(z;\beta)$. This obeys the boundary condition at $z=0$. While we have $\frac{d}{dz}\Big(\xi(z;\alpha)\xi(z;\beta)\Big) = \xi'(z;\alpha)\xi(z;\beta) + \xi(z;\alpha)\xi'(z;\beta)$ which at $z=1$ gives $2\xi(1;\alpha)\xi(1;\beta) \neq \xi(1;\alpha)\xi(1;\beta)$ and so this term also fails to obey our specified boundary conditions. This is critical as it means all of the terms at first order in the fields in our transformations have to be excised by our compensating transformation. This means that our fields don't obtain higher order transformations[117]. This is not how gravity usually works and so it seems we're missing something.

One might wonder why we don't see this issue is Kaluza-Klein theory? In the Kaluza-Klein case the basis of functions we use is $\{1, \sin(2\pi m), \cos(2\pi m) | m \in \mathbb{N}\}$. For such a basis the derivatives of the basis elements are $\{0, 2\pi m\cos(2\pi m), -2\pi m \sin(2\pi m) | m \in \mathbb{N}\}$ which can obviously be expanded in the original basis, whilst products of basis elements, via standard trigonometric identities, can also be expanded in the basis. As a result Kaluza-Klein circle reductions again prove to be very special and actually obscure the more general picture.

At this stage we may wonder what we should do. We need the transformations of our fields to receive corrections when we go beyond the linear order but our choice of boundary conditions appear to be obstructing this. Since we'd like to obtain a four dimensional theory that includes gravity we need our lower dimensional field transformations to be correct, and so we're forced to abandon the boundary conditions we were so attached to at quadratic order. In doing this we can quite clearly alter the form of our field $H_{\mu\nu} = \tilde{H}_{\mu\nu} + 2a\eta_{\mu\nu}\tilde{\Phi}_{nt}$ when we go to second order, and beyond, in fields. It's this that we now investigate.

[116]With respect to the inner product $\big(f,g\big) = \int_0^1 dz\ \bar{f}(z)g(z)$ with $^-$denoting complex conjugation.

[117]Unless we somehow input them using the freedom in our transformation (5.132). We will not explore such an approach here.

Recalling we're perturbing around a flat background, our metric takes the form

$$g_{MN} = \eta_{MN} + \mathcal{H}_{MN} \ , \quad M \in \{0, \dots, 4\} \ , \tag{5.137}$$

where we now take our perturbation to have the following form

$$\mathcal{H}_{MN} = \sum_{n=1}^{N} \epsilon^n \mathcal{H}_{MN}(x, z; n) \ , \tag{5.138}$$

with $\mathcal{H}_{MN}(x, z; n)$ containing n copies of lower dimensional fields. We found previously that

$$\begin{aligned} \mathcal{H}_{\mu\nu}(x, z; 1) &= \big(h_{\mu\nu}(x; 0) + 2a\eta_{\mu\nu}\phi(x; 0)\big)\xi(z; 0) + \sum_i h_{\mu\nu}(x; i)\xi(z; i) \ , \\ \mathcal{H}_{\mu z}(x, z; 1) &= \sum_\alpha A_\mu(x; \alpha)\xi'(z; \alpha) \ , \\ \mathcal{H}_{zz}(x, z; 1) &= \sum_\alpha \phi(x; \alpha)\xi(z; \alpha) \ , \end{aligned} \tag{5.139}$$

leads to an appealing writing of the action at second order.

We now wish to determine the higher order terms, $\mathcal{H}_{MN}(x, z; n)$ for $n \geqslant 2$, in $\mathcal{H}_{MN}$. All such terms are written in terms of our lower dimensional fields, which for our interval system are $h_{\mu\nu}(x; \alpha)$, $A_\mu(x; \alpha)$ and $\phi(x; \alpha)$. In order to account for this we shall re-express (5.138) as[118]

$$\mathcal{H}_{MN} = \sum_{n=1}^{N} \epsilon^n \mathcal{H}_{MN}(h_{\mu\nu}, A_\mu, \phi, z; n) \ . \tag{5.140}$$

In doing this we've managed to remedy the issue we had with boundary conditions. The resolution of this problem can be understood by recalling that boundary conditions initially entered our lives while we were dealing with the perturbation problem at quadratic order in the action and were directly applied to the perturbation itself. However, in reality, the object on which we should be providing boundary conditions is the full metric, g_{MN}, given in (5.137). Whilst the conditions we picked originally were sufficient if we only ever wanted a solution to linearised gravity, if we wish to go beyond it then these conditions must be altered, and one might hope, in our perturbative treatment, that it would be possible to understand the boundary conditions one must apply to g_{MN} order by order. In this way we see that the emergence of terms in the

[118]We should note that in a case similar to the one at hand, but where the transverse dependence of the functions arising in this sum can be expanded in terms of the functions that were used at linear order, then the new perturbation (5.140) can also be interpreted in terms of a field redefinition of the lower dimensional fields. We will see, later in this Section, that if the procedure outlined here is applied to the usual case of Kaluza-Klein theory on a circle then the known non-linear ansatz is derived. In addition to this it can be shown that the diffeomorphism parameter we use receives no corrections.

transformations (5.119) and (5.121) that don't admit an expansion in terms of the functions $\{\xi(z;\alpha)\}$ isn't so surprising. So we've seen to our boundary condition issue, but we haven't really addressed the problem of how we obtain lower dimensional field transformations at the non-linear order. So far we've been seeking to intuit the transformations of the lower dimensional fields from the known transformation of the higher dimensional field, which is

$$\mathcal{H}'_{MN} = \mathcal{H}_{MN} + \gamma(\partial_M X_N + \partial_N X_M) + \gamma \mathcal{L}_X \mathcal{H}_{MN} \ , \tag{5.141}$$

with γ acting as a parameter to keep track of the size of our diffeomorphism. However, we already have a good idea about the form of the lower dimensional field transformations we want. As a result we might wonder if there's any way we can just input the transformations we want for the lower dimensional fields and then embed them appropriately into the transformation of the higher dimensional theory. This may sound a little like the tail wagging the dog, but it's exactly the same procedure that we used in Section 1 when we were considering the S^1 reduction of pure gravity.

Let's see if we can formalise this rough idea. We know we have a collection of lower dimensional fields, which we shall denote collectively as Ψ, that will have the transformation properties

$$\Psi' = \Psi + \frac{\gamma}{\epsilon}\mathfrak{i}(\Psi) + \gamma \mathfrak{h}(\Psi) \ , \tag{5.142}$$

where $\mathfrak{i}$ denotes the inhomogeneous operator which implements the inhomogeneous[119] piece of the fields transformation, and $\mathfrak{h}$ denotes an operator that performs the homogeneous[120] piece of the fields transformation. For ease, let's start by considering the case where our diffeomorphism parameter is field independent. If we now perform a diffeomorphism, X^M, under which our lower dimensional fields transform as in (5.142), then using our new perspective (5.141) reads

$$\begin{aligned}\mathcal{H}'_{MN} = \sum_{n=1}^{N} \epsilon^n \mathcal{H}_{MN}(h'_{\mu\nu}, A'_\mu, \phi', z; n) = &\sum_{n=1}^{N} \epsilon^n \mathcal{H}_{MN}(h_{\mu\nu}, A_\mu, \phi, z; n) + \gamma(\partial_M X_N(x,z) + \partial_N X_M(x,z)) \\ &+ \sum_{n=1}^{N} \gamma \epsilon^n \mathcal{L}_X \mathcal{H}_{MN}(h_{\mu\nu}, A_\mu, \phi, z; n) \ ,\end{aligned} \tag{5.143}$$

with $h'_{\mu\nu}$, A'_μ and ϕ' being the transformed lower dimensional fields. We can then use the transformations of the lower dimensional fields, which we now specify as input information to the process, to expand

[119] This term is usually used to refer to the piece of the transformation that only involves the diffeomorphism parameter and no fields. However since we're allowing for diffeomorphisms that include fields in them we're being a little sloppy in our use of the term. As an example, for a normal graviton transformation the inhomogeneous piece is the $2\partial_{(\mu}X_{\nu)}$ part of the transformation.

[120] Which are the modifications to the transformation that arise once we go past linear order.

$\mathcal{H}_{MN}(h'_{\mu\nu}, A'_\mu, \phi', z; n)$ as

$$\mathcal{H}_{MN}(h'_{\mu\nu}, A'_\mu, \phi', z; n) = \mathcal{H}_{MN}(h_{\mu\nu}, A_\mu, \phi, z; n) + \gamma\epsilon^{-1}\mathfrak{i}\big(\mathcal{H}_{MN}(h_{\mu\nu}, A_\mu, \phi, z; n)\big) + \gamma\mathfrak{h}\big(\mathcal{H}_{MN}(h_{\mu\nu}, A_\mu, \phi, z; n)\big) \ , \tag{5.144}$$

where we assume $\mathfrak{i}$ and $\mathfrak{h}$ are linear, obey a Leibnitz property and we only work to linear order[121] in γ. If we use this in (5.143), then we find that

$$\sum_{n=0}^{N-1} \epsilon^n \mathfrak{i}\big(\mathcal{H}_{MN}(h_{\mu\nu}, A_\mu, \phi, z; n)\big) + \epsilon^{n+1}\mathfrak{h}\big(\mathcal{H}_{MN}(h_{\mu\nu}, A_\mu, \phi, z; n+1)\big) \tag{5.145}$$

$$= \partial_M X_N + \partial_N X_M + \sum_{n=0}^{N-1} \epsilon^{n+1} \mathcal{L}_X \mathcal{H}_{MN}(h_{\mu\nu}, A_\mu, \phi, z; n+1) \ , \tag{5.146}$$

from which we can peel off the following transformation properties

$$\mathfrak{i}\big(\mathcal{H}_{MN}(h_{\mu\nu}, A_\mu, \phi, z; 1)\big) = \partial_M X_N + \partial_N X_M \ , \tag{5.147}$$

$$\mathfrak{i}\big(\mathcal{H}_{MN}(h_{\mu\nu}, A_\mu, \phi, z; n) = \mathcal{L}_X \mathcal{H}_{MN}(h_{\mu\nu}, A_\mu, \phi, z; n-1) - \mathfrak{h}\big(\mathcal{H}_{MN}(h_{\mu\nu}, A_\mu, \phi, z; n-1)\big) \ , \quad n \geqslant 2 \ . \tag{5.148}$$

These relations are interesting for a number of reasons. Firstly, if we can solve them then we've managed to embed our desired lower dimensional symmetry transformations into the higher dimensional symmetry we started with. This is excellent as it means we've got a chance to obtain the transformations we need in order to display the lower dimensional invariance we're seeking. Further to this we see that we can use these relations to solve for the various terms in (5.138), as these are now specified in terms of the lower order terms in the series. We also notice that the first term is defined by the inhomogeneous transformation properties of the fields which we can learn from a treatment of the quadratic action of the system. Interestingly this whole procedure is, somewhat, insensitive[122] to terms that vanish under the application of the inhomogeneous operator. As such, when we attempt to use (5.147) and (5.148) to determine terms in (5.138), we shall have the ability to freely input such terms by hand.

So far we've only considered diffeomorphisms that are field independent, but we've seen already that these are often not sufficient. As such we now consider a diffeomorphsim parameter of the form (5.130)

[121] As is customary in theories of gravity.

[122] We have to be careful here as such terms will, in general, have homogeneous transformations which will affect the higher order terms in the expansion.

which we shall, for continuity of notation, denote as

$$\gamma X^M(x,z) = \sum_{n=0}^{N} \gamma \epsilon^n X^M(x,z;n) \ . \tag{5.149}$$

Rerunning the arguments leading to (5.147) and (5.143) leads to the new result

$$\begin{aligned}
&\mathfrak{i}_{X(x,z;0)}\big(\mathcal{H}_{MN}(\Psi,z;1)\big) + \sum_{m=0}^{N}\sum_{n=1}^{N-1} \epsilon^{m+n}\Big(\mathfrak{i}_{X(x,z;m)}\big(\mathcal{H}_{MN}(\Psi,z;n+1)\big) + \mathfrak{h}_{X(x,z;m)}\big(\mathcal{H}_{MN}(\Psi,z;n)\big)\Big) \\
&+ \sum_{m=0}^{N} \epsilon^{m+N}\mathfrak{h}_{X(x,z;m)}\big(\mathcal{H}_{MN}(\Psi,z;N)\big) = \partial_M X_N(x,z;0) + \partial_N X_M(x,z;0) \\
&+ \sum_{n=1}^{N} \epsilon^N\big(\partial_M X_N(x,z;n) + \partial_N X_M(x,z;n)\big) + \sum_{m=0}^{N}\sum_{n=1}^{N-1} \epsilon^{m+n}\mathcal{L}_{X(x,z;m)}\big(\mathcal{H}_{MN}(\Psi,z;n)\big) \\
&+ \sum_{m=0}^{N} \epsilon^{m+N}\mathcal{L}_{X(x,z;m)}\big(\mathcal{H}_{MN}(\Psi,z;N)\big) \ ,
\end{aligned} \tag{5.150}$$

which leads to the new relations

$$\mathfrak{i}_{X_M(x,z;0)}\big(\mathcal{H}_{MN}(\Psi,z;1)\big) = \partial_M X_N(x,z;0) + \partial_N X_M(x,z;0) \ , \tag{5.151}$$

$$\begin{aligned}
&\sum_{\substack{m=0,n=1\\ m+n=q}} \Big(\mathfrak{i}_{X(x,z;m)}\big(\mathcal{H}_{MN}(\Psi,z;n+1) + \mathfrak{h}_{X(x,z;m)}\big(\mathcal{H}_{MN}(\Psi,z;n)\big)\Big) = \partial_M X_N(x,z;q) + \partial_N X_M(x,z;q) \\
&+ \sum_{\substack{m=0,n=1\\ m+n=q}} \Big(\mathcal{L}_{X(x,z;m)}\mathcal{H}_{MN}(\Psi,z;n)\Big) \ , \quad q < N \ .
\end{aligned} \tag{5.152}$$

Whilst powerful this relationship is unilluminating in this form. As a result let's look at what the $q = 1$ case yields

$$\begin{aligned}
\mathfrak{i}_{X(x,z;0)}\big(\mathcal{H}_{MN}(\Psi,z;2)\big) - \partial_M X_N(x,z;1) - \partial_N X_M(x,z;1) =& \mathcal{L}_{X(x,z;0)}\big(\mathcal{H}_{MN}(\Psi,z;1)\big) \\
&- \mathfrak{h}_{X(x,z;0)}\big(\mathcal{H}_{MN}(x,z;1)\big) \ ,
\end{aligned} \tag{5.153}$$

from which we can see the modification to the previous result is to include an analogue of the higher dimensional inhomogeneous term for the field dependent piece of the diffeomorphism parameter.

In order to get a feel for how this process works let's look at the case encountered in the Kaluza-Klein expansion on S^1. This proves to be an extremely useful testing ground as the all orders, in n, ansatz is known[123]. Within the context of Kaluza-Klein theory the four dimensional quadratic action is obtained by expanding the five dimensional fields in terms of the eigenfunctions of (5.135) subject to (5.13). This leads to the transverse functions[124]

$$1 \ , \quad \xi(z;m) = \sqrt{2}\sin(2\pi m z) \ , \quad \zeta(z;m) = \sqrt{2}\cos(2\pi m z) \ , \quad m \in \mathbb{N}_0 \setminus \{0\} \ , \tag{5.154}$$

showing there is a single function associated to a zero eigenvalue solution and then 2 degenerate solutions associated to all other eigenvalues. Our goal is to study the effect of the diffeomorphism with trivial transverse dependence

$$X_M(x,z) = (\chi_\mu(x), \chi_z(x)) \ , \tag{5.155}$$

where we shall ignore any possible field dependent terms in this transformation parameter[125]. With this choice we find that (5.147) leads to

$$\mathfrak{i}\big(\mathcal{H}_{\mu\nu}(h,A,\phi,z;1)\big) = \partial_\mu \chi_\nu(x;0) + \partial_\nu \chi_\mu(x;0) \ , \tag{5.156}$$

$$\mathfrak{i}\big(\mathcal{H}_{\mu z}(h,A,\phi,z;1)\big) = \partial_\mu \chi_z(x;0) \ , \tag{5.157}$$

$$\mathfrak{i}\big(\mathcal{H}_{zz}(h,A,\phi,z;1)\big) = 0 \ , \tag{5.158}$$

where our lower dimensional fields are the massless fields $h_{\mu\nu}(x)$, $A_\mu(x)$ and $\phi(x)$[126]. We propose the following solution to these equations

$$\mathcal{H}_{\mu\nu}(h,A,\phi,z;1) = h_{\mu\nu}(x) + C_{\mu\nu}(x,z) \ , \tag{5.159}$$

$$\mathcal{H}_{\mu z}(h,A,\phi,z;1) = A_\mu(x) + C_\mu(x,z) \ , \tag{5.160}$$

$$\mathcal{H}_{zz}(h,A,\phi,z;1) = \phi(x) + C(x,z) \ , \tag{5.161}$$

[123] At least for the zero mode sector.
[124] Which we relabel from earlier on to be more in keeping with the notation we've been using.
[125] This can be justified by attempting to solve (5.152) and showing all higher order pieces in (5.149) can consistently be set to zero. However, for the sake of expediency of presentation, we shall not undertake such a calculation.
[126] We omit terms due to the massive states as we know already group theory will guarantee us a consistent truncation. However, if one desires they can be restored and the whole procedure carried out with them included.

where $\mathfrak{i}\big(C_{\mu\nu}\big) = 0$, $\mathfrak{i}\big(C_{\mu}\big) = 0$ and $\mathfrak{i}\big(C\big) = 0$ and with

$$\mathfrak{i}\big(h_{\mu\nu}(x)\big) = \partial_\mu \chi_\nu(x) + \partial_\nu \chi_\mu(x) \ , \quad \mathfrak{i}\big(A_\mu(x)\big) = \partial_\mu \chi_z(x;0) \ , \quad \mathfrak{i}\big(\phi(x)\big) = 0 \ . \tag{5.162}$$

This leaves us with $C_{\mu\nu}$, C_μ and C to fix. We propose, since other choices lead to undesirable terms in the second order action, that

$$C_{\mu\nu} = 2a\eta_{\mu\nu}\phi(x;0) \ , \quad C_\mu = 0 \ , \quad C = \big(2b-1\big)\phi(x;0) \ , \tag{5.163}$$

which gives us the following choice for $\mathcal{H}_{MN}(x,z;1)$

$$\mathcal{H}_{\mu\nu}(h,A,\phi,z;1) = h_{\mu\nu}(x) + 2a\eta_{\mu\nu}\phi(x) \ , \ \mathcal{H}_{\mu z}(h,A,\phi,z;1) = \quad A_\mu(x) \ , \ \mathcal{H}_{zz}(h,A,\phi,z;1) = 2b\phi(x) \ . \tag{5.164}$$

In order to proceed further we need to propose homogeneous transformations for our lower dimensional fields. Given our prejudice for the interpretation of our fields, these are chosen to be

$$\begin{aligned}
(\mathfrak{i}+\mathfrak{h})\big(h_{\mu\nu}(x)\big) =& \partial_\mu \chi_\nu(0) + \partial_\nu \chi_\mu(0) + \mathcal{L}_\chi h_{\mu\nu} \ , \\
(\mathfrak{i}+\mathfrak{h})_0\big(A_\mu(x)\big) =& \partial_\mu \lambda(0) + \mathcal{L}_{\chi(0)} A_\mu(0) \ , \\
(\mathfrak{i}+\mathfrak{h})\big(\phi(x)\big) =& \mathcal{L}_\chi \phi(0) \ ,
\end{aligned} \tag{5.165}$$

if we use these and (5.164) in (5.143) then we obtain the following equations

$$\begin{aligned}
\mathfrak{i}\big(\mathcal{H}_{\mu\nu}(\Psi,z;2)\big) =& 4a\phi(0)\partial_{(\mu}\chi_{\nu)}(0) + 2A(0)_{(\mu}\partial_{\nu)}\lambda(0) \ , \\
\mathfrak{i}\big(\mathcal{H}_{\mu z}(\Psi,z;2)\big) =& 2b\phi(0)\partial_\mu\lambda(0) \ , \\
\mathfrak{i}\big(\mathcal{H}_{zz}(\Psi,z;2)\big) =& 0
\end{aligned}$$

With Ψ denoting all of our lower dimensional fields. We can solve (5.166) which leads to the following result

$$\begin{aligned}
\mathcal{H}_{\mu\nu}(\Psi,z;2) =& A_\mu A_\nu + 2a\phi h_{\mu\nu} + C^{(1)}_{\mu\nu} \ , \\
\mathcal{H}_{\mu z}(\Psi,z;2) =& 2b\phi A_\mu + C^{(1)}_\mu \ , \\
\mathcal{H}_{zz}(\Psi,z;2) =& C^{(1)} \ ,
\end{aligned} \tag{5.166}$$

where $\mathfrak{i}\big(C^{(1)}_{\mu\nu}\big) = \mathfrak{i}\big(C^{(1)}_{\mu}\big) = \mathfrak{i}\big(C^{(1)}\big) = 0$. if we choose

$$C^{(1)}_{\mu\nu} = 2a^2\phi^2\eta_{\mu\nu} \ , \quad C^{(1)}_{\mu} = 0 \ , \quad 2b^2\phi^2 \ , \tag{5.167}$$

then we find that

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} + 2a\phi\eta_{\mu\nu} + 2a\phi h_{\mu\nu} + A_\mu A_\nu + 2a^2\phi^2 \ , \\ g_{\mu z} &= A_\mu + 2b\phi A_\mu \ , \quad g_{zz} = 1 + 2b\phi + 2b^2\phi^2 \ , \end{aligned} \tag{5.168}$$

which is precisely the perturbative expansion of the non-linear Kaluza-Klein ansatz given in (1.173). Yes we've cheated a little and picked the terms in the kernel of $\mathfrak{i}$ at each order to ensure this occurred, but the highly non-obvious $A_\mu A_\nu$ term in $g_{\mu\nu}$ has emerged as a result of solving our recursive relation. This demonstrates one of the powerful applications of the recursive formula we've derived. Whilst it initially was conceived of to allow us to embed our desired lower dimensional symmetries into the higher dimensional one, we can now see the fact that it links orders can be used as a method to derive non-linear reduction ansätze. This shouldn't come as a huge surprise since we're allowing the lower dimensional theory to influence the higher dimensional one and non-linear ansätze are used precisely to ensure the lower dimensional theory assumes a form that we can recognise. This process can be continued to higher orders where it can be shown that the correct non-linear ansatz, up to inhomogeneously invariant terms which we have to put in by hand, is reproduced order by order.

Having now understood the implications of the recursive formula we've derived, we can begin the process of solving (5.152) for the interval system. Since our goal is to work up to quartic order in the action we're required to calculate the terms up to and including $N = 3$ in (5.138), and also terms up to $N = 2$ in (5.149). The full details of this shall be given in upcoming publications. Crucially it can be shown that a solution does exist that allows our lower dimensional fields, and in particular $h_{\mu\nu}(x;0)$, to transform as we expect them to. Here we shall present the results for $\mathcal{H}_{MN}(\Psi, z; 2)$ and leave the results for $\mathcal{H}_{MN}(\Psi, z; 3)$ for an upcoming publication[127].

In order to begin the process of solving the recursive formula we must specify the homogeneous piece of the lower dimensional transformations of our fields. For $h_{\mu\nu}(0)$ we wish it to transform as a lower dimensional graviton, $\phi(0)$ is expected to transform as a lower dimensional scalar and $h_{\mu\nu}(i)$ is predicted to transform as matter. However, this leaves the Stueckelberg field, $A_\mu(0)$, to deal with. There isn't a general theory of

[127]This is done as we're yet to decide on the precise form of inhomogeneously invariant terms to add to $\mathcal{H}_{MN}(\psi, z; 3)$.

how such a field should transform, and as such we take the approach of beginning with the most general transformation involving at most one copy of $A_\mu(0)$ and the four dimensional diffeomorphism parameter $\chi_\mu(x)$. Overall this leads to us taking the transformations of our lower dimensional fields to be[128]

$$
\begin{aligned}
(\mathfrak{i}+\mathfrak{h})(h_{\mu\nu}(x;0)) =&\partial_\mu\chi_\nu + \partial_\mu\chi_\nu + \kappa_4\mathcal{L}_\chi h_{\mu\nu}(x;0)\ ,\ (\mathfrak{i}+\mathfrak{h})(h_{\mu\nu}(x;i)) = \mathcal{L}_\chi h_{\mu\nu}(x;i)\ ,\ (\mathfrak{i}+\mathfrak{h})(\phi(x;0)) = \mathcal{L}_\chi\phi(x;0)\ ,\\
(\mathfrak{i}+\mathfrak{h})(A_\mu(x;0)) =&\chi_\mu + a_1\chi^\sigma\partial_\sigma A_\mu(x;0) + a_2A^\sigma(x;0)\partial_\sigma\chi_\mu + a_3\chi_\mu\partial_\sigma A^\sigma(x;0) + a_4A_\mu(x;0)\partial_\sigma\chi^\sigma\\
&+ a_5\chi^\sigma\partial_\mu A_\sigma(x;0) + a_6A^\sigma(x;0)\partial_\mu\chi_\sigma\ ,
\end{aligned}
\tag{5.169}
$$

where $\kappa_4 = \kappa_5 I_3$, with κ_5 being the five dimensional gravitational coupling, $I_3 = \int dz\ \xi(z;0)^3$ and $a_i \in \mathbb{R}$ for $i \in \{1,\ldots,6\}$.

Using these transformations and the following results

$$
\begin{aligned}
\mathcal{H}_{\mu\nu}(\Psi,z;1) =&(h_{\mu\nu}(x;0) + 2a\eta_{\mu\nu}\phi(x;0))\xi(z;0) + \sum_i h_{\mu\nu}(x;i)\xi(z;i)\ ,\\
\mathcal{H}_{\mu z}(\Psi,z;1) =&A_\mu(x;0)\xi'(z;0)\ ,\ \mathcal{H}_{zz} = 2b\phi(x;0)\xi(z;0)\ ,\\
X^\mu(x,z;0) =&\chi^\mu(x)\xi(z;0)\ ,\ X^z(x,z;0) = 0\ ,
\end{aligned}
\tag{5.170}
$$

which were obtained from our quadratic order treatment of the action, we can solve (5.153) to obtain

$$
\begin{aligned}
\mathcal{H}_{\mu\nu}(\Psi,z;2) =&\big(\mathcal{L}_A h_{\mu\nu}(x;0) - \partial_\mu A^\sigma\partial_\nu A_\sigma\big)\xi(z;0)(\xi(z;0)-\kappa_4) + 2a\eta_{\mu\nu}\mathcal{L}_A\phi(x;0)\xi(z;0)(\xi(z;0)-1)\\
&+ 2a\phi(x;0)h_{\mu\nu}(x;0)\xi(z;0)^2 + \sum_i \mathcal{L}_A h_{\mu\nu}(x;i)\xi(z;i)(\xi(z;0)-1)\ ,\\
\mathcal{H}_{\mu z}(\Psi,z:2) =&h_{\mu\sigma}(x;0)A^\sigma\xi'(z;0)\xi(z;0) + \sum_i h_{\mu\sigma}(x;i)A^\sigma\xi'(z;0)\xi(z;i) + A^\sigma\partial_\sigma A_\mu\,\xi'(z;0)(\xi(z;0)-\kappa_4)\ ,\\
\mathcal{H}_{zz}(\Psi,z;2) =&2b\mathcal{L}_A\phi(x;0)\,\xi(z;0)(\xi(z;0)-1) + A_\mu A^\mu(\xi'(z;0))^2\ ,\\
X^\mu(x,z;1) =&A^\sigma\partial_\sigma\chi_\mu\,\xi(z;0)(\xi(z;0)-\kappa_4)\ ,\ X^z(x,z;1) = 0\ ,
\end{aligned}
\tag{5.171}
$$

where $\mathcal{L}_A$ denotes the Lie derivative with respect to $A^\mu(x;0)$, which we write as A^μ for ease. We should note that (5.171) is not the unique solution to (5.153). However, any two solutions are related by a solution of

$$
\mathfrak{i}_{X(x,z;0)}\big(\mathcal{H}_{MN}(\Psi,z;2)\big) - \partial_M X_N(x,z;1) - \partial_N X_M(x,z;1) = 0\ ,
\tag{5.172}
$$

[128]We omit the fields $\phi(i)$ and $A_\mu(i)$ by the assumption that there exists a gauge in which they can be set to zero and that our current transformation doesn't cause them to become non-zero. This assumption should be checked at a later date.

which is precisely the freedom we used to introduce inhomogeneously invariant terms in the Kaluza-Klein case. In solving (5.153) for (5.171) it is found that certain relationships between the a_i in the transformation for $A_\mu(x;0)$ are required. These are found to be

$$-a_1 + a_2 + \kappa_4 = 0\ ,\ a_3 = a_4\ ,\ a_5 = -a_6\ , \tag{5.173}$$

which we solve by choosing $a_2 = a_3 = a_4 = a_5 = a_6 = 0$ and $a_1 = \kappa_4$. This leads to the transformation of $A_\mu(x;0)$ assuming the form

$$(\mathfrak{i} + \mathfrak{h})(A_\mu(x;0)) = \chi_\mu(x) + \kappa_4 \chi^\sigma \partial_\sigma A_\mu(x;0)\ . \tag{5.174}$$

These choices are made as they aid in the construction[129] of a $\mathcal{H}_{MN}(\Psi, z; 3)$.

With the form of $H_{MN}(\Psi, z; 2)$ and $X^M(x, z; 1)$ found, the job of ensuring the fields $h_{\mu\nu}(z;\alpha)$ and $\phi(x;0)$ transform as we desire up to first order in fields is completed. As a by product of the procedure we've also discovered the beginning of the higher dimensional diffeomorphism we'd have to perform in order to achieve the lower dimensional diffeomorphism we want. This provides a nice constructive method for finding how to embed a lower dimensional symmetry into a higher dimensional one.

Having now established a manner in which we can achieve the desired lower dimensional transformations we now, briefly, consider the consequences of what we've done for the effective action. We can see that our process has caused us to consider perturbations that generalise those we had at quadratic order. This procedure introduces additional terms into the effective action, which will alter the form of the coefficient multiplying the quartic order term. Direct calculation shows this is the case and that for the $h_{\mu\nu}(x;0)$ only sector of the theory the coefficient is now exactly as expected. However it turns out that the action also contains terms involving the inhomogeneously gauge inert field

$$\tilde{h}_{\mu\nu}(x;0) = h_{\mu\nu}(x;0) - 2\partial_{(\mu} A_{\nu)}(x;0)\ . \tag{5.175}$$

The effective action can then be shown to assume the form

$$S^{(2)}_{grav}[h_{\mu\nu}(0)] + \kappa_4 S^{(3)}_{grav}[h_{\mu\nu}(0)] + (\kappa_4)^2 S^{(4)}_{grav}[h_{\mu\nu}(0)] + w\tilde{S}^{(4)}[\tilde{h}_{\mu\nu}(0)] + \dots\ , \tag{5.176}$$

where $S^{(i)}_{grav}[h_{\mu\nu}(0)]$ are the usual actions expected for gravity, w is a non-zero constant and $\tilde{S}^{(4)}[\tilde{h}_{\mu\nu}(0)]$

[129]Again see upcoming publications.

is an action involving only terms of quartic order in the field $\tilde{h}_{\mu\nu}(x;0)$. It can be shown that the term involving $\tilde{h}_{\mu\nu}$ can't be removed using the ambiguity in the recurrence relation. However, by the inclusion of five dimensional boundary terms, which are written in terms of the four dimensional fields, it can be shown that this term can be removed. This leaves behind an effective action for $h_{\mu\nu}(x;0)$ that is precisely of the form expected for four dimensional gravity. As such it can be shown that it's possible to obtain a lower dimensional field theory that does display the expected diffeomorphism symmetry in four dimensions. Here we've been very schematic about how this occurs and have offered none of the details of the construction. These details shall be fully fleshed out in an upcoming publication.

To summarise we've managed, in the setting of pure gravity about a flat $\mathbb{R}^{1,3} \times [0,1]$ background, to obtain an EFT of gravity where the massless graviton has non-trivial dependence on the transverse space. It was shown that the resulting Stueckelberg field is excised from the quadratic action by the inclusion of a boundary term in the higher dimensional theory. This leads to a lower dimensional, quadratic, system that possesses the usual diffeomorphism symmetry of gravity. In going beyond the quadratic order we found that the eigenfunctions of the Sturm-Liouville operator didn't form a basis for functions appearing at beyond quadratic order. This led to us introducing the recursion relation as a method for embedding the desired lower dimensional transformations of fields into the higher dimensional transformation. It was demonstrated that this resulted in both the perturbation and diffeomorphism parameters being redefined. Applying this method to the setting of the Kaluza-Klein reduction was shown to result in the well known non-linear ansatz.

6 Time dependent localisation

In Section 3 we discussed, in detail, the solutions of Type IIA Supergravity found by Crampton, Pope and Stelle. These solutions are interesting because, despite the presence of a non-compact transverse direction, there exists the chance to develop a lower dimensional EFT description of fluctuations around such a background. Crucial to this was the localisation mechanism offered by the transverse wavefunction problem. The important feature was that the transverse wavefunction of the lowest lying mode in the graviton spectrum, which turned out to be massless, had the following transverse dependence

$$\xi(\rho; 0) \propto \log(\tanh(\rho)) \, , \tag{6.1}$$

with ρ the coordinate of the non-compact direction of the background geometry.

Whilst the CPS family of solutions possess many interesting features there is one obvious feature that is absent. It is well known that on cosmological scales the universe is expanding and hence exhibits time dependence. The solution family of CPS has an operative worldvolume that is a warped version of Minkowski space. Given our previous comments it would be desirable, if our aim is to study physics on cosmological scales, to search for similar types of solutions but where the worldvolume develops time dependence. If such solutions exist then we could consider these as backgrounds about which we can attempt to develop an EFT description of fluctuations.

It might be asked if the existence of time dependent solutions is expected or if it is just a punt in the dark? Fortunately the Salam-Sezgin model, on which the solution of CPS is based, has a rich history of cosmological solutions. One particularly pertinent example is given by the family of cosmological solutions of Halliwell [58][130] which shares many features, such as including an S^2 and a monopole configuration for the Maxwell field, with the original $\text{Mink}_4 \times S^2$ solution of Salam and Sezgin [91].

Within the remainder of this Section we shall present a family of time dependent solutions to Supergravity, modelled on the CPS solution, and then study the allowed transverse spectrum for fluctuations about these solutions.

[130]It should be noted this reference was brought to the attention of the author after the work carried out in this Section was completed. As such it would seem that the work presented here could be further generalised to include more general time dependence. This shall be carried out and documented in future work.

6.1 Time dependent ansatz and solutions

Our goal is to now deform (3.28) in to a time dependent solution to the equations arising from the string frame action (3.45). A standard assumption is that the universe is both homogeneous and isotropic, which holds on cosmological scales after a suitable statistical averaging has been performed. As such, to focus our search, we look for solutions that are of FLRW-type. This leads to the following ansatz[131]

$$
\begin{aligned}
d\bar{s}_{10}^2 =& - dt^2 + \alpha_1(t)^2 dx_i dx^i + \alpha_3(t)^2 dz^2 + \frac{1}{4g^2}\big(d\psi + \mathrm{sech}(2\rho)(d\chi + \cos\theta d\varphi)\big)^2 + \frac{H(\rho)}{g^2} ds_{EH}^2 \ , \\
e^{2\phi} =& \alpha_2(t) H \ , \quad A_{(2)} = \frac{1}{4g^2}\big((1+k)d\chi + \mathrm{sech}\, 2\rho d\psi\big) \wedge \big(d\chi + \cos\theta d\varphi\big) \ , \quad H = \mathrm{sech}\, 2\rho - k\log(\tanh\rho) \ ,
\end{aligned}
\tag{6.2}
$$

where the metric is given in string frame with $i \in \{1, 2, 3\}$ and ds_{EH}^2 the Eguchi-Hanson metric, as given in (3.22). This ansatz must then solve the following system of equations

$$
\begin{aligned}
&\bar{R}_{MN} - \frac{1}{4} H_{MPQ} H_N{}^{PQ} + 2\bar{\nabla}_M \bar{\nabla}_N \phi = 0 \ , \quad \bar{\nabla}^2(e^{-2\phi}) - \frac{1}{6} e^{-2\phi} H_{MNP} H^{MNP} = 0 \ , \\
&\bar{\nabla}_M (e^{-\phi} H^{MNP}) = 0 \ ,
\end{aligned}
\tag{6.3}
$$

where $\bar{\nabla}$ is the covariant derivative associated to the Levi-Civita connection of the string frame metric $\bar{g}_{MN}$ and $\bar{R}_{MN}$ is the associated Ricci tensor.

If we select a vielbein basis, see Appendix B, for (6.2) of the form

$$
e^0 = dt \ , \quad e^i = \alpha_1(t) dx^i \ , \quad e^4 = \alpha_3(t) dz \ , \quad e^5 = \frac{1}{2g}\big(d\psi + \mathrm{sech}\, 2\rho(d\chi + \cos\theta d\varphi)\big) \ , \quad e^\alpha = \frac{H^{\frac{1}{2}}}{g} \hat{e}^\alpha \ , \tag{6.4}
$$

with $\alpha \in \{6, \ldots, 9\}$ and $\hat{e}^\alpha$ being vielbeins for the Eguchi-Hanson space that are chosen to be

$$
\begin{aligned}
&\hat{e}^6 = \frac{\sinh 2\rho}{2(\cosh 2\rho)^{\frac{1}{2}}}\big(d\chi + \cos\theta d\varphi\big) \ , \quad \hat{e}^7 = (\cosh 2\rho)^{\frac{1}{2}} d\rho \ , \quad \hat{e}^8 = \frac{1}{2}(\cosh 2\rho)^{\frac{1}{2}} d\theta \ , \\
&\hat{e}^9 = \frac{1}{2}(\cosh 2\rho)^{\frac{1}{2}} \sin\theta d\varphi \ .
\end{aligned}
\tag{6.5}
$$

The resulting non-zero components of the connection 1-forms and the curvature 2-form are presented in Appendix B.

[131]As stated previously the work of Halliwell suggests that this ansatz could be further generalised to include non-flat three space slices and additional time dependence.

The effective equations of motion for the functions $\alpha_i(t)$ are then found to be

$$3\frac{\ddot{\alpha}_1}{\alpha_1} = -\frac{\ddot{\alpha}_3}{\alpha_3} + \frac{\ddot{\alpha}_2}{\alpha_2} - \left(\frac{\dot{\alpha}_2}{\alpha_2}\right)^2 , \tag{6.6}$$

$$\frac{\ddot{\alpha}_1}{\alpha_1} = -\frac{\dot{\alpha}_1\dot{\alpha}_3}{\alpha_1\alpha_3} + \frac{\dot{\alpha}_1\dot{\alpha}_2}{\alpha_1\alpha_2} - 2\left(\frac{\dot{\alpha}_1}{\alpha_1}\right)^2 , \tag{6.7}$$

$$\frac{\ddot{\alpha}_2}{\alpha_2} = 2\left(\frac{\dot{\alpha}_2}{\alpha_2}\right)^2 - 3\frac{\dot{\alpha}_1\dot{\alpha}_2}{\alpha_1\alpha_2} - \frac{\dot{\alpha}_2\dot{\alpha}_3}{\alpha_2\alpha_3} , \tag{6.8}$$

$$\frac{\ddot{\alpha}_3}{\alpha_3} = -3\frac{\dot{\alpha}_1\dot{\alpha}_3}{\alpha_1\alpha_3} + \frac{\dot{\alpha}_2\dot{\alpha}_3}{\alpha_2\alpha_3} . \tag{6.9}$$

Our aim is to now analyse these equations in an attempt to find solutions that may be of interest.

To begin with, assume α_2 and α_3 are non-zero constants, then (6.6) implies that $\alpha_1 = at + b$ for a,$b \in \mathbb{R}$. However (6.7) implies $a = 0$ and hence α_1 is also a constant. A similar conclusion can be reached for α_2 if α_1 and α_3 are set equal to, non-zero, constants. However, in the case of α_1 and α_2 being non-zero constants we find the non-trivial solution

$$\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\} , \quad \alpha_3 = at + b , \quad a, b \in \mathbb{R} , \tag{6.10}$$

where a and b can't simultaneously be zero.

The next case to address is when one of the alphas becomes constant. This leads to a general solution in two cases

$$\alpha_1 \in \mathbb{R} \setminus \{0\} , \quad \alpha_2 = \frac{C_1}{(t + C_2)^2} , \quad \alpha_3 = \frac{-C_3}{t + C_2} , \quad C_1, C_3 \in \mathbb{R} \setminus \{0\} , \quad C_2 \in \mathbb{R} , \tag{6.11}$$

$$\alpha_2 \in \mathbb{R} \setminus \{0\} , \quad \alpha_1 = C_1(2t - C_2)^{\frac{1}{2}} , \quad \alpha_3 = \frac{C_3}{(-2t + C_2)^{\frac{1}{2}}} , \quad C_1, C_3 \in \mathbb{R} \setminus \{0\} , \quad C_2 \in \mathbb{R} . \tag{6.12}$$

Solutions for $\alpha_3 \in \mathbb{R} \setminus \{0\}$ could not be found. However, it can be shown that they can't be of power law or exponential form, meaning $\alpha_a = t^{n_a} \;\; i \in \{1, 2\}$ for $n_a \in \mathbb{R}$, or the exponential alternative, $\alpha_a = e^{n_a t} \;\; i \in \{1, 2\}$ for $n_a \in \mathbb{R}$, type solutions don't exist. It should be noted that the occurrence of the square root of the function $-2t + C_2$ and the square root of its negative in (6.12) leads to a solution which will have multiple timelike directions and hence is not physically relevant for us in the current setting.

The final case, where all α_i are non-zero, is the most difficult to tackle, and while it may be possible to find a general solution we won't attempt this. Instead we shall focus on a family of power law solutions that are of particular physical interest

$$\alpha_i = t^{n_i} \ , \quad i \in \{1, 2, 3\} \ . \tag{6.13}$$

Note this ansatz could clearly include multiplicative constants without changing any of the conclusions we shall draw since only ratios appear in (6.6)-(6.9). If any of the $n_i = 0$ we return to one of the cases previously discussed. If we input (6.13) into (6.6)-(6.9) we find that

$$n_1 \in \mathbb{R} \ , \quad n_2 = 3n_1 - 1 \pm (1 - 3(n_1)^2)^{\frac{1}{2}} \ , \quad n_3 = \pm(1 - 3(n_1)^2)^{\frac{1}{2}} \ , \tag{6.14}$$

which will give real solutions if $n_1 \in [-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]$. Some explicit examples of solutions are $(n_1, n_2, n_3) \in \{(\frac{1}{2}, 0, -\frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2})\}$.

6.2 Gravitational fluctuations about time dependent backgrounds

Having obtained a collection of time dependent solutions we can now consider using these as backgrounds about which we can consider fluctuations. In order to begin to understand the graviton spectrum around such a background it's useful to mimic the analysis of Bachas and Estes, [4], see Section 2, but now allowing for the worldvolume to have time dependence, rather than being maximally symmetric. We have learnt, through out this thesis, that the conditions set on perturbations and truncations of fields used in [4] need to be reconsidered when one is attempting to construct an EFT of gravity. However, the treatment of [4] is sufficient to isolate a crucial transverse operator that aids in understanding the graviton spectrum around the chosen background. Since our goal is to see if the class of backgrounds (6.2) preserves the localising mechanism of the original time independent background, which is intimately related to the transverse operator identified in [4], it is sufficient for us to consider a procedure similar to that of Bachas and Estes[132].

To this end we consider gravitational perturbations about the time dependent backgrounds given by (6.2). We shall consider perturbations in string frame, as in this case the metric of (6.2) separates into a product of two 5 dimensional manifolds with coordinates $X^{\hat{\mu}} = (t, x^i, z)$ and $y^a = (\psi, \chi, \rho, \theta, \varphi)$. Due to the

[132] With the understanding that, while it provides calculational simplicity, if we wish to develop a full EFT about such backgrounds we shall have to remove any conditions and truncations we impose. Thus this approach provides a quick and dirty method to ascertain if there is any chance of localisation in the current context.

loss of maximal symmetry we shall now consider a perturbation of the form

$$ds^2_{10perturbed} = -dt^2 + \big(\alpha_1(t)^2\delta_{ij} + H_{ij}(X,y)\big)dx^i dx^j + \alpha_3(t)^2 dz^2 \\ + \frac{1}{4g^2}\big(d\psi + sech(2\rho)(d\chi + cos\theta d\varphi)\big)^2 + \frac{1}{g^2 cosh2\rho} ds^2_{EH} \ , \tag{6.15}$$

where we've truncated to a spatial three perturbation $H_{ij}(X,y)$.

In order to reduce the number of terms appearing in subsequent expressions it is useful to select a condition on this perturbation. As such we shall choose to impose the gauge condition $\nabla^M H_{MN} = 0$ where ∇ is the covariant derivative associated to the, string frame, metric given in (6.2).

With such a choice, the variation of the, string frame, Ricci tensor is given by (2.30) as

$$\delta R_{MN} = \frac{1}{2}\big(R_{QM}H_N{}^Q - R^Q{}_{NPM}H_Q{}^P + R_{QN}H_M{}^Q - R^Q{}_{MPN}H^P_Q - \nabla_P\nabla^P H_{MN} - \nabla_N\nabla_M H\big) \ , \tag{6.16}$$

where all of the curvature tensors are associated to the string frame metric and $A = 0$. We can calculate the equations of motion obeyed by our perturbation by considering the linearised version of the equation

$$R_{MN} - \frac{1}{4}H_{MPQ}H_N{}^{PQ} + 2\nabla_M\nabla_N\phi = 0 \ , \tag{6.17}$$

which arises from the Einstein equation of (3.45) with the Ricci scalar eliminated using the dilaton equation. Since we have the variation of the Ricci tensor, the next task to undertake is the calculation of the linearised version of the stress energy tensor

$$8\pi t_{MN} = -2\nabla_M\nabla_N\phi + \frac{1}{4}H_{MPQ}H_N{}^{PQ} \ , \tag{6.18}$$

which yields

$$8\pi\delta t_{MN} = -g^{RS}g^{QT}H_{ST}\big(\partial_M g_{QN} + \partial_N g_{MQ} - \partial_Q g_{MN}\big)\nabla_R\phi + g^{RQ}\big(\partial_M H_{QN} + \partial_N H_{MQ} - \partial_Q H_{MN}\big)\nabla_R\phi \\ -\frac{1}{2}H_{MPQ}H_{NR}{}^Q H^{PR} \ . \tag{6.19}$$

For a perturbation of the form (6.15) and a background of the form (6.2) all but the sixth term of (6.19) vanishes.

The linearised version of (6.17) now leads to the off-diagonal equation

$$\nabla_{\hat{\mu}}\nabla_a H = \partial_a\psi(y)\partial_{\hat{\mu}}h(X) = 0 \ , \tag{6.20}$$

where $h = g^{ij}h_{ij}$ and we've assumed that $H_{ij} = h_{ij}(X)\psi(y)$. This equation forces us to set the transverse wavefunction to a be constant or to set the $X^{\hat{\mu}}$ dependence of the trace to be a constant. Since our goal is to seek normalisable modes we shall select the latter condition. Next if we investigate the transverse transverse part of the linearised equation we find that

$$\nabla_a\nabla_b H = h\nabla_a\nabla_b\psi = 0 \ , \tag{6.21}$$

thus if we wish to leave ψ unconstrained then we must set $h = 0$ telling us that the perturbation must be traceless[133].

If one now looks at the $M, N = 0$ equation then it can be shown that it assumes the form

$$2R^k{}_{0l0}H_k^{\ l} + \nabla^2 H_{00} - \nabla_0\nabla_0 H = 0 \ . \tag{6.22}$$

This equation is trivially satisfied due to our selection of $H = 0$ and the choice of gauge condition. Considering the $M = 0, N = i$ equation we arrive at

$$\frac{\dot{\alpha}_1}{\alpha_1}(\partial^l H_{li} - \nabla^l H_{li}) = 0 \ , \tag{6.23}$$

however a simple calculation shows that the bracket in this expression is identically zero for a background of the form (6.2) with a perturbation as given by (6.15). The $M = 0, N = 4$ and $M = i, N = 4$ equations turn out to be trivially zero for the situation at hand. The $M, N = 4$ equation yields

$$2\frac{\dot{\alpha}_3}{\alpha_3}\frac{\dot{\alpha}_1}{\alpha_1}H = 0 \ , \tag{6.24}$$

which is zero since we've specialised to the case of traceless perturbations.

If we now consider the $M = i, N = j$ equation, which is where we expect the equation governing the

[133]However recall we are working in a setup that we know isn't the complete story and so this conclusion almost certainly must be revised when the full problem is considered.

evolution of our perturbation to arise from, we find that

$$R_{ki}H^k{}_j + R_{kj}H^k{}_i - 2R^k{}_{jli}H^l_k - \Box_{10}H_{ij} = -2\big(\nabla^Q\phi\big)\partial_Q H_{ij} \ . \tag{6.25}$$

Using (6.2), this leads to

$$R^k{}_{jli} = (\dot\alpha_1)^2(\delta_{ji}\delta^k_l - \delta_{jl}\delta^k_i) \ , \quad R_{ki} = \Big(2\Big(\frac{\dot\alpha_1}{\alpha_1}\Big)^2 + \frac{\dot\alpha_1\dot\alpha_3}{\alpha_1\alpha_3} + \frac{\ddot\alpha_1}{\alpha_1}\Big)g_{ki} \ . \tag{6.26}$$

With this and (6.25) we obtain the equation obeyed by the fluctuation

$$\Box_{10}H_{ij} - 2\Big(\frac{\dot\alpha_1}{\alpha_1}\Big)^2 H_{ij} - 2\Big(2\Big(\frac{\dot\alpha_1}{\alpha_1}\Big)^2 + \frac{\dot\alpha_1\dot\alpha_3}{\alpha_1\alpha_3} + \frac{\ddot\alpha_1}{\alpha_1}\Big)H_{ij} - 2\nabla^Q\phi\partial_Q H_{ij} = 0 \ , \tag{6.27}$$

however this can be simplified by noting that (6.26), (6.17) and (6.2) mean that $\partial_0\phi = \frac{\dot\alpha_2}{2\alpha_2}$ and thus

$$2\Big(\frac{\dot\alpha_1}{\alpha_1}\Big)^2 + \frac{\dot\alpha_1\dot\alpha_3}{\alpha_1\alpha_3} + \frac{\ddot\alpha_1}{\alpha_1} = \frac{\dot\alpha_1\dot\alpha_2}{\alpha_1\alpha_2} \ , \tag{6.28}$$

which leads to

$$\Box_{10}H_{ij} - 2\Big(\Big(\frac{\dot\alpha_1}{\alpha_1}\Big)^2 + \frac{\dot\alpha_1\dot\alpha_2}{\alpha_1\alpha_2}\Big)H_{ij} - 2\nabla^Q\phi\partial_Q H_{ij} = 0 \ . \tag{6.29}$$

Recalling that we have supposed our perturbation is separable and given the from of ϕ, as in (6.2), we can show that

$$\psi(y)\Big(\Box_X h_{ij}(X) + \frac{\dot\alpha_2}{\alpha_2}\partial_0 h_{ij}(X) - 2\Big(\Big(\frac{\dot\alpha_1}{\alpha_1}\Big)^2 + \frac{\dot\alpha_1\dot\alpha_2}{\alpha_1\alpha_2}\Big)h_{ij}(X)\Big) + \Big(\Box_y\psi(y) - 2g^{\rho\rho}\partial_\rho\phi\partial_\rho\psi(y)\Big)h_{ij}(X) \ , \tag{6.30}$$

where $\Box_X$ is the laplacian on the 5d space parametrised by coordinates $X^{\hat\mu} = (t, x^i, z)$ and $\Box_y$ is the laplacian on the remaining 5 dimensions. Clearly (6.30) is separable[134], hence validating our earlier assumption of separability. This leads us to seek solutions to the eigenvalue problems

$$\begin{aligned} &\Box_X h_{ij}(X) + \frac{\dot\alpha_2}{\alpha_2}\partial_0 h_{ij}(X) - 2\Big(\Big(\frac{\dot\alpha_1}{\alpha_1}\Big)^2 + \frac{\dot\alpha_1\dot\alpha_2}{\alpha_1\alpha_2}\Big)h_{ij}(X) = m^2 h_{ij}(X) \ , \\ &\Box_y\psi(y) - 2g^{\rho\rho}\partial_\rho\phi\partial_\rho\psi(y) = -\frac{m^2}{g^2}\psi(y) \ . \end{aligned} \tag{6.31}$$

[134]The possible troubling term is the one involving $\phi = \phi(t,\rho)$. However, since $\phi = \frac{1}{2}\log(\alpha_2) + \frac{1}{2}\log(H(\rho))$, $\partial_\rho\phi$ is a function of ρ only, ensuring separability.

Finally, to ensure we have a true linear solution, we must check that the linearised version of the matter equations arising from (3.45) are also solved. These can be calculated and shown to hold since we are considering perturbations where the trace is zero.

6.3 Does massless gravity persist?

We now will seek to solve (6.31) and see if the system retains the property of localising gravity. To do this we shall follow the example of [17] and consider only S-wave perturbations where the only dependence on the transverse coordinates is through the non-compact ρ coordinate[135]. This leads to an expansion of the perturbation in terms of the eigenfunctions of the transverse operator

$$H_{ij}(X,y) = \sum_k h_{ij}(x,t;\lambda_k)\xi(\rho;\lambda_k) + \int_{\Lambda_{edge}}^{\infty} d\lambda \, h_{ij}(x,t;\lambda)\xi(\rho;\lambda) \ , \tag{6.32}$$

so the spectrum has been assumed to retain the form of a discrete part followed by the usual continuous spectrum associated to non-compact reductions.

We shall study the case where the NS5 brane charge, k, in (6.2) is zero. In this case, assuming the S-wave form of the perturbation, the transverse part of (6.31) can be obtained by noting that

$$\Box_y f(\rho) = g^2\Big(\frac{d^2}{d\rho^2}f + 4\,\text{cosech}(4\rho)\frac{d}{d\rho}f\Big) \ , \quad -2g^{\rho\rho}\partial_\rho\phi = 2g^2\tanh(2\rho) \ . \tag{6.33}$$

Combining these, we obtain the equation obeyed by the S-wave transverse parts of the perturbation

$$g^2\Big(\frac{d^2}{d\rho^2} + \frac{2}{\tanh(2\rho)}\frac{d}{d\rho}\Big)\xi(\rho) = -m^2\xi(\rho) \implies \Big(\frac{d^2}{d\rho^2} + \frac{2}{\tanh(2\rho)}\frac{d}{d\rho}\Big)\xi(\lambda)(\rho) = -\lambda\xi(\rho;\lambda) \ , \quad m^2 = g^2\lambda \ , \tag{6.34}$$

which is exactly the same as for the time independent case [17]. As such one can still solve this when $\lambda = 0$ to yield

$$\xi_0(\rho) = c_1 + c_2\log(\tanh\rho) \ , \quad c_1, c_2 \in \mathbb{R} \ . \tag{6.35}$$

Since the transverse eigenfunction equation, (6.34), is exactly the same as was found in [17]. This ensures that the graviton spectrum is the same in the current case as was found in [17] and expanded in Appendix D.

[135] This is consistent for the types of perturbations under consideration here. However, once all of the other components of the perturbation are turned on, there are sectors in which non-trivial dependence on other transverse coordinates, in particular those of the S^2, are required in order to avoid divergent integrals in the effective action.

The next question to address is if the zero mode remains normalisable. To do this we need to calculate the form of the invariant measure for this geometry. The kinetic term of the perturbation will arise from (3.45) via the Ricci scalar and as such will be multiplied by

$$\sqrt{-\det g}\, e^{-2\phi} = \frac{(\alpha_1)^3\alpha_3 \sin(\theta)\sinh(2\rho)}{16g^5\alpha_2}\ , \tag{6.36}$$

which means that the natural inner product between transverse S-wave modes is given by

$$\int_0^\infty d\rho \sinh 2\rho\ \xi(\rho;\lambda)\xi(\rho;\tilde{\lambda})\ . \tag{6.37}$$

If we set $\lambda = \tilde{\lambda} = 0$ then demanding that the zero mode be normalisable forces c_1 in (6.35) to be zero. Further demanding the norm be one sets the constant c_2 which yields

$$\xi(\rho;0) = \frac{2\sqrt{3}}{\pi}\log(\tanh\rho)\ . \tag{6.38}$$

Hence in the case where the NS5-brane charge is set to zero massless normalisable gravity is retained[136].

We now see that, not only can the hyperbolic time independent solution be embedded into a larger family of time dependent solutions, but these retain the crucial localising mechanism and mass gap of the original family of solutions. Were we to turn to attempting to find an EFT of gravity built from one of these time dependent backgrounds, we know we'd have to include modes associated to all of the other components of a generic perturbation and also consider transverse dependence beyond that of the S-wave modes considered here.

[136]One might wonder about the inclusion of an NS5 brane source in the cases where we set $k \neq 0$. Previously this was facilitated by the smearing of an NS5 brane source action over the S^2 directions with the smearing being motivated, but not proven, by arguments to do with preserved supersymmetry. In the current time dependent case it would appear this argument will no longer hold as the time dependence is expected to break the supersymmetry of the CPS family of solutions. As such we shall not attempt to include a source in this thesis and shall defer discussion of the $k \neq 0$ case to future work.

7 Conclusions

7.1 Summary of results

Within the main body of this thesis we have presented many technical and detailed results, and as a consequence it may be a little hard to see the overarching picture we've tried to present. In an attempt to remedy this we now take an opportunity to step back from the details and reflect on the main findings of this work.

We began in Section 2 by considering the transverse wavefunction problem described by Bachas and Estes [4]. This problem describes the graviton spectrum around a class of backgrounds that are warped products of a maximally symmetric worldvolume and a general transverse space. This background, which also includes matter fields in addition to the metric, was assumed to solve a system of field equations arising from a Supergravity theory. In their original treatment B&E studied perturbations of the world volume metric with non-trivial dependence on the transverse space. In addition to this they set all other possible perturbations to zero and assumed the non-zero perturbation was both transverse and traceless, both of which were claimed to be the result of a gauge condition on the worldvolume. Since it's known that tracelessness of perturbations arises as a result of the linearised equations of motion it seems a little presumptive to impose it in the derivation of them. As a result we undertook the calculation performed by B&E, in the context of the NSNS sector of Type IIA Supergravity, but without demanding tracelessness. It was found that if the linearised matter equations of motion were to be satisfied then the perturbation was indeed forced to be traceless.

We then moved on in Section 3 to consider the construction of top down EFTs of gravity. Here we outlined a, supposedly, general method for constructing EFTs. This procedure was based on deforming a chosen background solution of a theory with the deformations assuming the role of the fundamental fields in the EFT. Dimensional reduction techniques, based on generalised Fourier expansions, were then used to render the theory effectively four dimensional. It was then the belief that the resulting lower dimensional theory would inherit diffeomorphism symmetry from the higher dimensional theory. We then proceeded to show, in the context of the EFT around a background provided by the solutions of Crampton, Pope and Stelle [17], that this is not the case. This result was shown by direct calculation of the terms arising in this EFT and in particular required the calculation of the coefficients multiplying the cubic and quartic expansion terms of the effective action. From this it was possible to show that these coefficients didn't obey the relationships required for diffeomorphism symmetry to hold. This was then used as evidence to signify the loss of lower dimensional diffeomorphism invariance in its usual form.

Having observed the loss of diffeomorphism symmetry, in Section 4 we returned to the original perturbation problem that we discussed in Section 2. We repeated the calculation of the linearised Einstein and matter equations for the NSNS sector of Type IIA Supergravity around a background of the type discussed by Bachas and Estes. However, crucially, we didn't apply any gauge fixing conditions or truncations to the field content of the perturbation problem. Once the complete system of equations was obtained we performed a truncation of the perturbation fields to the case where only the perturbation of the world volume metric was taken to be non-zero. It was then demonstrated that the equations arising from this truncation did not possess linearised gauge symmetry and, as a result, it was surmised that it was this inconsistent truncation that was leading to the loss of lower dimensional gauge invariance. This demonstrated that the loss of diffeomorphism symmetry is a generic feature of theories in which this truncation is applied and is not just special to the case of the EFT around the CPS background. Within the process of fixing this issue the role of a perturbation field that transformed as a Stueckelberg field under diffeomorphisms was identified. It was found that the transformation properties of this field arose as a direct consequence of the graviton having a non-trivial transverse wavefuntion.

In order to fix the issues with the general construction of an EFT of gravity we considered the toy model of a five dimensional theory of gravity reduced on an interval, subject to Dirichlet-Robin boundary conditions, in Section 5. We proceeded to construct both a four dimensional gauge fixed and gauge unfixed action at quadratic order that displayed manifest diffeomorphism symmetry. Interestingly it was shown that the Stueckelberg field, that was previously identified for it's role in diffeomorphism symmetry, could be excised from the action, at this order, by the introduction of five dimensional boundary terms, which we believe are related to five dimensional diffeomorphism invariance of the theory. We then moved on to consider how the system behaved beyond lowest order, with our efforts mainly focused on determining the transformation properties of our lower dimensional fields. We encountered the issue that the eigenfunctions of the Sturm-Liouville operator, which were used in generalised Fourier expansions at linear order, no longer provided a basis for various functions arising in the higher dimensional transformations at beyond this order. To work around this we decided to redefine the perturbation of the metric and the diffeomorphism parameter to include terms that were quadratic and higher in lower dimensional fields. This allowed for the derivation of a recurrence relation that enabled the lower dimensional field transformations we desired to be embedded into the prescribed higher dimensional transformations. The use of this recurrence relation allowed us to determine the form of the non-linear perturbation and the results of this, up to quadratic order in fields, were presented.

Finally in Section 6 we considered the addition of time dependence to the backgrounds of CPS. We presented a class of FLRW like solutions to Type IIA Supergravity and showed that the perturbation problem around these backgrounds lead to the same transverse eigenvalue problem as in the CPS case. This crucially meant that the time dependent backgrounds presented also localised gravity in a sensible manner.

7.2 Work to be completed and future directions

The we've work presented is in no way exhaustive, and whilst we have provided answers to some questions there are many that we haven't treated. As such we shall take a moment to consider the work that is still to be completed and the further questions that the research presented in this thesis leads onto.

The obvious first piece of work that should be completed and documented is the method required to restore four dimensional diffeomorphism invariance in the case of the interval problem presented in Section 5. Crucial to this is the solution of the recurrence relation at third order in fields. Once this has been settled upon the desired lower dimensional transformation of the fields have been obtained at the order required to be able to investigate the gauge invariance of the lower dimensional system up to quartic order. As we've already mentioned this can be shown to be sufficient to solve the problem of the fourth order coefficient, the solution to which needs to be worked out in detail. However, it also leaves over a remaining residual fourth order term involving a gauge inert combination of the graviton and the Stueckelberg field. We've stated that this term can be dealt with by the introduction of a boundary term in the five dimensional theory. The calculation of the precise form of this boundary term should be completed and then the complete quartic order five dimensional theory used, before the reduction is performed, along with the four dimensional action it reduces to should be worked out.

The completion of this work naturally leads on to several open questions. The first and most pressing of which is the origin of the boundary terms added to the five dimensional theory, as it seems very unsatisfying to just add these terms in by hand. While one could claim they're present on the basis of requiring diffeomorphism invariance of the lower dimensional theory it would be nice to develop a method where these terms are present in the initial five dimensional action one begins with. This means that we'd like to see if it's possible to augment the five dimensional Einstein-Hilbert action with a boundary action that when expanded precisely reproduces the terms added in by hand. Since our five dimensional problem is posed on a manifold with a boundary it would seem natural to conjecture that, on the basis of well posedness of the variational problem, that a term like the standard Gibbons, Hawking, York term [51, 108] should be included into the total action of the five dimensional theory. The expansion of this boundary action should then be computed up to quartic order and it should be compared to the terms calculated on the basis of lower dimensional diffeomorphism invariance. It may be hoped that these two expressions will be the same. However, if this isn't the case then it could lead to an interesting new direction involving admissible boundary actions that one could explore.

So far many of our comments have been about the graviton only sector of the effective action of the interval problem. However, it's also of vital importance that we understand the interactions between the various other fields in the problem. This requires the development of the effective action for all of the physical lower dimensional fields, again at, say, up to quartic order. Whilst this follows in precisely the same manner as the graviton only sector the task of computing it accurately is still required to be undertaken. It seems obvious that this process should be computerised for efficiency of computation. Given that the results can be obtained, we'd expect the graviton to couple to matter in the way prescribed by the usual minimal coupling prescription. The expected form of the couplings between the other matter fields in the problem is less clear and would have to be deduced by direct calculation.

Another question that warrants further investigation is the use of a gauge condition to set the unphysical fields in the interval problem to zero at beyond the quadratic order in the action. Clearly, given our acknowledgement of the subtlety of gauge conditions in this thesis, this choice must be checked carefully. Assuming that this choice of gauge is admissible it would be interesting to see if one could back out of it and develop an un-gauge fixed version of the action. If this could be achieved it would mark an interesting step in attempting to prove the full equivalence of the five dimensional and four dimensional theories at the levels of the action.

The previous paragraphs raise an old and highly pertinent question. Namely that we've worked at the level of the action for the vast majority of the procedure we've outlined. We've willingly been putting dimensional reduction ansätze into the higher dimensional action and then using this to derive the lower dimensional action. From the lower dimensional action we're able to derive field equations for the fields arising within it. However, it's known that the insertion of a dimensional reduction ansatz into the higher dimensional action versus the insertion into the higher dimensional field equations can, even in the case of a consistent truncation, lead to different lower dimensional equations of motion [68]. As a result we're not guaranteed that the procedure we've outlined will lead to lower dimensional field equations whose solutions can be lifted to solutions of the higher dimensional equations. In a preliminary check to compare the results of these two methods it would seem that we need to insert our perturbative reduction ansatz into the higher dimensional field equations and see if these are equivalent to those derived from the lower dimensional effective action. A moments thought tells us that once we input our dimensional reduction ansatz into the higher dimensional field equations we will expect terms involving the basis of functions discovered from a treatment of the field equations at linear order along with products, derivatives etc of these. This lands us directly back in the path of the span issue that caused us to introduce the recurrence relation in Section 5. In this context addressing the span issue is critical. This owes to the fact that the usual method for obtaining equations of motion for lower dimensional fields from higher dimensional ones is to make use of linear independence and an appropriate projection operator onto a particular Kaluza-Klein mode of the higher dimensional theory. This issue seems to be intimately tied up with the introduction of the higher order terms into the perturbative ansatz, as mandated by the recurrence relation, and resolutions of both this problem and the issue of equivalence of field equations are crucial if the method outline here is to be of practical use.

As a final comment on the interval system our approach has focused on going up to quartic order in the expansion. This owes to the fact that this was where the issue of loss of diffeomorphism invariance was first observed. However, this clearly isn't the end of the story. While we don't foresee any issues in going beyond quartic order it would be interesting to see if the system could be understood at quintic order. This would hopefully help provide insight into how one might realise the procedure at the full non-linear level. However, this extension seems to still be out of our grasp at the current moment. This owes mostly to the computational complexity in extending the procedure to higher orders rather than a lack of understanding of the steps required.

So far our outlook has mainly focused on the toy model system of the five dimensional interval. However, when we started out our goal was to understand the EFT around the CPS background. Once the interval problem has been completely understood, with particular interest being paid to the resolution of the issues of how the boundary terms arise and the problem of inserting the reduction ansatz into the action versus inserting it into the higher dimensional field equations, then the case of deriving the EFT around the CPS background should be undertaken. We believe many of the problems arising in the CPS case will be resolved in the same manner as the interval case. For example we predict that a redefinition of the perturbation using the recurrence relation and also the inclusion of ten dimensional boundary terms shall be required. Should this prove to be correct we will once more have to interpret the boundary terms arising in the higher dimensional theory. It is our belief that the most fruitful place to start looking in this case is the NS5 brane source action. We would also have to address the issue of working at the level of the action, which we hope would have a similar resolution, assuming it has one, to the interval case.

The way that we have presented it above would seem to suggest that solving the CPS problem is completely analogous to the solution of the toy model system. However, this isn't the case. The CPS system brings along further complications owing to the more diverse background field content of the problem and the curved space nature of the metric. In addition to the increase in dimensionality of the higher dimensional field theory, the occurrence of matter fields in the theory means that we have more perturbation fields to deal with. This increase in fields leads to an increase in the computational complexity of the problem as well as to new features that weren't observed in the interval problem. The most striking amongst these is that several of the perturbations are forced to have non-trivial dependence on the transverse coordinates of the problem that are not the radial direction. As such we are forced to go beyond the S-wave truncation originally proposed by Crampton, Pope and Stelle. This requirement can be seen by noticing that the effective action about the CPS background develops divergent integrals, in particular in the S^2 directions, if all perturbations are forced to have trivial dependence on all transverse directions except the radial coordinate. This issue seems to be arising due to the existence of non-trivial Killing vectors of the CPS background. We believe the occurrence of such an issue isn't of great concern as the Killing vectors of the geometry are well known to arise in the dimensional reduction ansatz for the metric in the case of sphere reductions.

Assuming the issues described above can be overcome in the CPS case it then becomes natural to ask if the EFT around the time dependent backgrounds discussed in Section 6 could be constructed. Further to this, in light of the work of Halliwell [58], it seems as if we should ask if the family of time dependent solutions we've found can be embedded into a larger family. If it's possible to extend this time dependent family of solutions then one should check to see if the extended family retains the gravitational localisation mechanism of the original CPS solution.

We hope that this Section has provided the reader with an account of the future work that should be undertaken beyond the work outline in this thesis. It's our hope that this gives the reader the impression that this research area still contains many interesting unanswered questions whose resolution could have application outside of the initial example in which they were first conceived.

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A Gravitational perturbation theory around a curved background

In this appendix we aim to provide an account of the general formulae used to perform gravitational perturbation theory about a curved background geometry. After providing all of the results required in generality, an example of expanding a gravitational action up to second order around a background is presented.

A.1 General gravitational perturbation theory

We shall focus on the example of pure gravity, which is governed by the Einstein-Hilbert action

$$S[g] = \frac{1}{2\kappa^2} \int d^d x \sqrt{-g} R \ , \tag{A.1}$$

where g_{MN} is the metric on spacetime, R is its associated Ricci scalar, $g = \det(g_{MN})$, κ is the gravitational coupling and $M, N \in \{1, ..., d\}$. The field equations arising from this action are the Einstein equations

$$R_{MN} - \frac{1}{2} g_{MN} R = 0 \implies R_{MN} = 0 \ , \tag{A.2}$$

with R_{MN} the Ricci tensor of the spacetime metric. Any metric satisfying (A.2) which leads to a non-vanishing Riemann tensor represents a curved spacetime geometry.

We can consider a geometry, g, that is a deformation of any solution, $\mathring{g}$, of these field equations given by

$$g_{MN} = \mathring{g}_{MN} + \epsilon h_{MN} \ , \tag{A.3}$$

where we have added the real parameter ϵ that can be used to keep track of the size of our deformation[137], h_{MN}. Within a perturbative approach to gravity the condition $\epsilon \ll 1$ holds. While $\mathring{g}$ will solve its own form of (A.2) g, for an arbitrary h, will not. However by calculating the quantities present in (A.2) for g we can derive the equations that h would have to solve in order for g to be a solution of the field equations (A.2).

Once we have decided to consider a perturbative deformation of the geometry we can justify working order by order in the parameter ϵ. In such a setup it is usual to view h, in (A.3), as a field that exists on the geometry defined by $\mathring{g}$ and then interpret the equations on h as field equations pertaining to the dynamics of h in the fixed background given by $\mathring{g}$. With an eye to this interpretation it becomes desirable to be able

[137]It should be noted deformations are only inequivalent up to diffeomorphisms- see Section 1 for more details on this subtle issue.

to derive the equations governing the dynamics of h from an action principle. Conveniently this can be achieved by considering (A.1) for a g of the form given in (A.3) and then expanding the quantities appearing in (A.1) as a power series in ϵ. As such we need to be able to find expansions for $\sqrt{-g}$ and R.

Any variation of the metric will always lead to a variation of the inverse metric, and given a variation of the form (A.3) we can parametrise the inverse metric by

$$g^{MN} = \mathring{g}^{MN} + \tilde{h}^{MN} \ , \tag{A.4}$$

with $\mathring{g}^{MN}$ being the inverse of $\mathring{g}_{MN}$ and $\tilde{h}^{MN}$ being present to ensure that

$$g^{MP} g_{PN} = \delta^{M}{}_{N} \ , \tag{A.5}$$

holds at all orders in ϵ. This requirement leads us to propose that

$$\tilde{h}^{MN} = \sum_{i=1}^{n} \epsilon^{i} \tilde{h}^{MN}_{(i)} \ , \tag{A.6}$$

with n a currently undetermined positive integer. Once (A.6) is used in conjunction with (A.4) and (A.5) we find that

$$\epsilon(\mathring{g}^{MP} h_{PN} + \tilde{h}^{MP}_{(1)} \mathring{g}_{PN}) + \epsilon^2(\tilde{h}^{MP}_{(1)} h_{PN} + \tilde{h}^{MP}_{(2)} \mathring{g}_{PN}) + \mathcal{O}(\epsilon^3) = 0 \ , \tag{A.7}$$

where we have made use of the fact that $\mathring{g}^{MP} \mathring{g}_{PN} = \delta^{M}{}_{N}$.

We can now solve (A.7) order by order in ϵ, which leads to the following recursive relationship

$$\tilde{h}^{MN}_{(m)} = -\tilde{h}^{MP}_{(m-1)} \mathring{g}^{NQ} h_{PQ} \ , \tag{A.8}$$

for $m \in \{1, ..., n\}$ and $\tilde{h}^{MN}_{(0)} := \mathring{g}^{MN}$.

It now becomes apparent that when performing perturbation theory in general relativity series of the form (A.6) will naturally be infinite series. This can be seen by considering (A.7) in the case where we terminate our series at order n. There will then be a term at $\mathcal{O}(\epsilon^{n+1})$ in (A.7) of the from $\tilde{h}^{MP}_{(n)} h_{PN}$ which must now be equal to zero. Since h is arbitrary this means that $\tilde{h}^{MP}_{(n)}$ will be set to zero and this will cause a cascade effect via (A.8) that causes all terms in (A.6) to become zero and hence leads to $h_{MN} = 0$ which we know to be untrue. The only way to avoid this is for the series (A.6) to be infinite. While this patches up one issue it does introduce its own complications, namely the issue of convergence. We shall not concern ourselves with such subtleties since we shall only work to finite order in perturbation theory, however our sins will catch up to us as our proposed geometries shall only be approximate solutions of (A.2), being correct up to a given order in our perturbation parameter.

With the inverse metric in hand we can move on to calculate the effect of our perturbation on $\sqrt{-g}$. In order to make progress we note that

$$g_{MN} = \mathring{g}_{MP}(\delta^P{}_N + \epsilon \mathring{g}^{PQ} h_{QN}) \ , \tag{A.9}$$

which then using standard results concerning determinants and the identity $\det(A) = \exp(\mathrm{tr}(\log(A))$ leads us to find

$$\sqrt{-g} = \sqrt{-\mathring{g}} \exp(\frac{1}{2} \mathrm{tr}(\log(\mathbb{1} + \epsilon \mathring{g}^{-1} h)) \ , \tag{A.10}$$

Now expanding the logarithm in this expression along with the exponential will yield the desired perturbation of the square root of the metric determinant at any order. For example up to order ϵ^2 we find

$$\sqrt{-g} = \sqrt{-\mathring{g}}(1 + \epsilon \frac{1}{2}(\mathring{g}^{-1}h)^M{}_M + \epsilon^2(\frac{1}{8}((\mathring{g}^{-1}h)^M{}_M)^2 - \frac{1}{4}((\mathring{g}^{-1}h)^2)^M{}_M)) \ . \tag{A.11}$$

Finally we require an expression for the Ricci tensor and scalar. In General Relativity these curvatures are taken to be those of the Levi-Civita connection which locally, in a coordinate basis, can be described by the Christoffel symbols

$$\Gamma^M{}_{NP} = \frac{1}{2} g^{MQ}(\partial_N g_{QP} + \partial_P g_{NQ} - \partial_Q g_{NP}) \ . \tag{A.12}$$

If we now use (A.3) and (A.6) and work in a Riemann normal coordinate system associated to the metric $\mathring{g}$ then, recalling the difference of connections is tensorial, we find

$$\delta\Gamma^M{}_{NP} = \frac{1}{2}(\mathring{g} + \tilde{h})^{MQ}(\mathring{\nabla}_N h_{QP} + \mathring{\nabla}_P h_{NQ} - \mathring{\nabla}_Q h_{NP}) \ , \tag{A.13}$$

where $\mathring{\nabla}$ is the Levi-Civita covariant derivative associated to $\mathring{g}$. If (A.6) and (A.8) are used then the perturbation of the Christoffel symbols at any order can be shown to be

$$\delta\Gamma^M_{(m)\ NP} = \frac{1}{2}\tilde{h}^{MQ}_{(m-1)}(\mathring{\nabla}_N h_{QP} + \mathring{\nabla}_P h_{NQ} - \mathring{\nabla}_Q h_{NP}) \ , \tag{A.14}$$

with

$$\delta\Gamma^{M}{}_{NP} = \sum_{i=1}^{\infty} \epsilon^{i} \delta\Gamma^{M}_{(m)\ NP} \ . \tag{A.15}$$

Using our conventions we can now write the all orders perturbed Riemann tensor as

$$R^{M}{}_{NPQ} = \mathring{R}^{M}{}_{NPQ} + \mathring{\nabla}_{P}\delta\Gamma^{M}{}_{NQ} - \mathring{\nabla}_{Q}\delta\Gamma^{M}{}_{NP} + \delta\Gamma^{R}{}_{NQ}\delta\Gamma^{M}{}_{RP} - \delta\Gamma^{R}{}_{NP}\delta\Gamma^{M}{}_{RQ} \ , \tag{A.16}$$

which can be used to find the all orders Ricci tensor

$$R_{MN} = \mathring{R}_{MN} + \mathring{\nabla}_{P}\delta\Gamma^{P}{}_{MN} - \mathring{\nabla}_{N}\delta\Gamma^{P}{}_{MP} + \delta\Gamma^{R}{}_{MN}\delta\Gamma^{P}{}_{RP} - \delta\Gamma^{R}{}_{MP}\delta\Gamma^{P}{}_{RN} \ , \tag{A.17}$$

contracting this with (A.4) yields the all orders Ricci scalar

$$R = g^{MN}(\mathring{R}_{MN} + \mathring{\nabla}_{P}\delta\Gamma^{P}{}_{MN} - \mathring{\nabla}_{N}\delta\Gamma^{P}{}_{MP} + \delta\Gamma^{R}{}_{MN}\delta\Gamma^{P}{}_{RP} - \delta\Gamma^{R}{}_{MP}\delta\Gamma^{P}{}_{RN}) \ . \tag{A.18}$$

We can now expand (A.17) and (A.18) to any order we want and then, using (A.1) or (A.2), obtain field equations for h at any order in ϵ.

A.2 Expansion of gravitational actions

Using the results of the previous Subsection we can now easily expand a gravitational action of the form (A.1) to any order we desire. As an example of this we now consider an expansion up to second order with the hope that it provides a template for the more complicated expansions performed in the main body of the text.

The required expression for the square root of the metric determinant has already been given in (A.11) so only the inverse metric and Ricci tensor are required. Using (A.8) we find that

$$g^{MN} = \mathring{g}^{MN} - \epsilon h^{MN} + \epsilon^{2} h^{MQ} h^{N}{}_{Q} \ , \tag{A.19}$$

where indices have been raised using $\mathring{g}$, so for example $h^{MN} = \mathring{g}^{MP}\mathring{g}^{NQ}h_{PQ}$. While using (A.17) we find

$$\begin{aligned}R_{MN} =&\mathring{R}_{MN} + \epsilon\frac{1}{2}(\mathring{\nabla}_Q\mathring{\nabla}_M h^Q{}_N + \mathring{\nabla}_Q\mathring{\nabla}_N h^Q{}_M - \mathring{\nabla}^2 h_{MN} - \mathring{\nabla}_N\mathring{\nabla}_M h)\\ &+\epsilon^2\Big(- \frac{1}{2}\mathring{\nabla}_Q h^{QP}(\mathring{\nabla}_M h_{PN} + \mathring{\nabla}_N h_{MP} - \mathring{\nabla}_P h_{MN}) - \frac{1}{2}h^{QP}(\mathring{\nabla}_Q\mathring{\nabla}_M h_{PN}\\ &+\mathring{\nabla}_Q\mathring{\nabla}_N h_{MP} - \mathring{\nabla}_Q\mathring{\nabla}_P h_{MN}) + \frac{1}{4}\mathring{\nabla}_N h^{QP}\mathring{\nabla}_M h_{QP} + \frac{1}{2}h^{PQ}\mathring{\nabla}_N\mathring{\nabla}_M h_{QP}\\ &+\frac{1}{4}\mathring{\nabla}_Q h(\mathring{\nabla}_M h^Q{}_N + \mathring{\nabla}_N h^Q{}_M - \mathring{\nabla}_Q h_{MN}) - \frac{1}{2}\mathring{\nabla}_Q h^P{}_M\mathring{\nabla}_P h^Q{}_N + \frac{1}{2}\mathring{\nabla}_Q h_{MP}\mathring{\nabla}^Q h^P{}_N\Big) ,\end{aligned} \tag{A.20}$$

with $h = \mathring{g}^{MN}h_{MN}$.

If we now use (A.11), (A.19) and (A.20) in (A.1) we find that the second order expanded action takes the form

$$\begin{aligned}2\kappa^2 S[g] =&2\kappa^2 S[\mathring{g}] + S^{(1)}[h,\mathring{g}] + S^{(2)}[h,\mathring{g}]\\ =&\int d^dx\sqrt{-\mathring{g}}\mathring{R}\\ &+\epsilon\int d^dx\sqrt{-\mathring{g}}\Big(- h^{MN}\mathring{R}_{MN} + \frac{h}{2}\mathring{R} + (\mathring{\nabla}_M\mathring{\nabla}_N h^{MN} - \mathring{\nabla}^2 h)\Big)\\ &+\epsilon^2\int d^dx\sqrt{-\mathring{g}}\Big(- \frac{1}{2}hh^{MN}\mathring{R}_{MN} + \frac{1}{8}h^2\mathring{R} - \frac{1}{4}h^{MN}h_{MN}\mathring{R}\\ &+ h^{MQ}h^N{}_Q\mathring{R}_{MN} - \mathring{\nabla}_Q h^{QP}\mathring{\nabla}_M h^M{}_P + \mathring{\nabla}_Q h^{QP}\mathring{\nabla}_P h\\ &- h^{QP}\mathring{\nabla}_Q\mathring{\nabla}^M h_{PM} + h^{PQ}\mathring{\nabla}_P\mathring{\nabla}_Q h + \frac{3}{4}\mathring{\nabla}^M h^{PQ}\mathring{\nabla}_M h_{PQ}\\ &+ h^{PQ}\mathring{\nabla}^2 h_{PQ} - \frac{1}{4}\mathring{\nabla}^M h\mathring{\nabla}_M h - \frac{1}{2}\mathring{\nabla}_P h^{MN}\mathring{\nabla}_N h^P{}_M\\ &- h^{MN}\mathring{\nabla}_P\mathring{\nabla}_M h^P{}_N + \frac{1}{2}h\mathring{\nabla}_P\mathring{\nabla}_Q h^{QP} - \frac{1}{2}h\mathring{\nabla}^2 h\Big) ,\end{aligned} \tag{A.21}$$

from which we can see that $S^{(0)}$ and $S^{(1)}$ vanish, up to a boundary term, since $\mathring{g}$ solves (A.2). This also causes the first 4 terms to be zero in $S^{(2)}$ and then we can interpret this term as an action describing the dynamics of the field h in the background given by $\mathring{g}$.

If we use the Leibniz property of $\mathring{\nabla}$ then up to total derivative terms, which we ignore here, we find that

$$S^{(2)}[h] = \int d^dx\sqrt{-\mathring{g}}\Big(\frac{1}{2}h\mathring{\nabla}_M\mathring{\nabla}_N h^{MN} + \frac{1}{4}h^{MN}\mathring{\nabla}^2 h_{MN} - \frac{1}{4}h\mathring{\nabla}^2 h - \frac{1}{2}h^{MN}\mathring{\nabla}^P\mathring{\nabla}_N h_{MP}\Big) , \tag{A.22}$$

with appropriate symmetrisation implicit. Taking a variation of this yielding the field equation

$$\mathring{\nabla}^2 h_{MN} + \mathring{\nabla}_M \mathring{\nabla}_N h - \mathring{\nabla}^P \mathring{\nabla}_M h_{PN} - \mathring{\nabla}^P \mathring{\nabla}_N h_{PM} + \mathring{g}_{MN} (\mathring{\nabla}_P \mathring{\nabla}_Q h^{PQ} - \mathring{\nabla}^2 h) = 0 \;, \tag{A.23}$$

which is equivalent to the linearised version of (A.2).

B The Cartan approach to gravity- an introduction to vielbeins and General Relativity

In this appendix we shall review the approach of Cartan to General Relativity (GR). An obvious first question is why, given the success of GR as it is, we would want another formulation of the same theory? To see this we need to understand what goes in to the mathematical formulation of GR and what restrictions this puts on the applicability of the theory. Once we've understood this we shall delve into the methods developed by Cartan for dealing with connections and curvatures in differential geometry. This will provide a natural route to a discussion of orthonormal frames and vielbeins which will allow us to express gravity in a manner that allows the inclusion of elements missing from GR, principle amongst these is the notion of torsion.

It may be thought that these modifications, while mathematically interesting, don't aid our endeavour to do physics in any way. However this couldn't be further from the truth. As we'll see these ideas are precisely what is needed to allow the introduction of fermions into GR, and after all they make up just a smidgen of the matter in the universe. Finally at the end of this appendix the non-zero connection one-forms and curvature two-forms for the time dependent geometries discussed in Section 6 are given.

The language of bundles is used extensively throughout this appendix, the reader not familiar with this subject is advised to consult [39], [75] or [15]. Note in this Section **Einstein summation convention** is used unless explicitly stated otherwise and the term manifold is used as a shorthand for differentiable manifold.

B.1 Connections, curvatures and General Relativity

In order to understand the choices made and approach usually taken to GR we shall begin by reviewing the formal mathematical aspects that constitute its foundation. We shall then be in a position to precisely state the specialisations made in writing GR in the form that it is usually encountered. From this we can seek other formulations that relax some of these restrictions.

The natural mathematical arena for GR is that of differential geometry. Within this framework our universe is described by a pair (M, g), with M a manifold and g a Lorentzian metric. M represents the space we live on while g determines the geometry our spacetime assumes[138]. However this isn't enough for us to be able to do physics just yet. In addition to the current structure we need the notions of a derivative. In differential geometry this is supplied by a **connection**. Formally this is defined on a bundle and here we shall look at a vector bundle E over M. In this case a connection is defined as a linear map $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$ such that $\nabla(fv) = df \otimes v + f\nabla v$ for all $f \in C^\infty(M)$ and $v \in \Gamma(E)$ [139], often denoted by ∇, and fields are sections of a given vector bundle E. A connection gives us a notion of a **covariant derivative** which, once more in the language of vector bundles, means that if $v \in \Gamma(E)$ we define the covariant derivative of this section by $\nabla_X v = (\nabla v)(X)$ with $\nabla_X v \in \Gamma(E)$ and $X \in \Gamma(TM)$. This gives us a notion of the covariant derivative along any vector field, $X \in \Gamma(TM)$, which we denote by ∇_X. The presence of this structure enables us to differentiate the fields defined on our manifold.

Once we have specified a connection, ∇, on a vector bundle, E, over our manifold we can define the **curvature tensor**[140], F, associated to the connection by

$$F(X,Y)(v) = \nabla_X \nabla_Y v - \nabla_Y \nabla_X v - \nabla_{[X,Y]} v \ , \tag{B.1}$$

with $X, Y \in \Gamma(TM)$, $v \in \Gamma(E)$ and $[\cdot,\cdot]$ being the Lie bracket of vector fields. Note that, by definition, for any $X, Y \in \Gamma(TM)$, $F(Y,X) = -F(X,Y)$. We say a connection is a **flat connection** if its associated curvature is zero for any $X, Y \in \Gamma(TM)$.

Of principle interest are connections on TM, which can naturally be extended to connections on T^*M and tensor product bundles of these, thus encompassing the full complement of tensors, usually, required in GR. Such a connection is often called an **affine connection**. Given an affine connection , ∇, we can define its **torsion**[141] by

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \qquad X, Y \in \Gamma(TM) \ . \tag{B.2}$$

If the torsion tensor of an affine connection vanishes for all $X, Y \in \Gamma(TM)$ then we say the connection is a **symmetric/torsion free connection**. Note that, for any $X, Y \in \Gamma(TM)$, $\tau(Y,X) = -\tau(X,Y)$.

Thus far we have taken a global approach to connections, curvatures and torsion, however these objects

[138] Recall a circle is just an amorphous closed loop before a metric is specified on it, only once we imbue it with a very specific metric does it assume the shape we're use to.

[139] Where $\Gamma(E)$ denotes the space of smooth sections of E over M, similarly for $\Gamma(E \otimes T^*M)$, and $C^\infty(M)$ is the space of smooth functions on the manifold M.

[140] In our current language a curvature tensor is, given $X, Y \in \Gamma(TM)$, the map $F(X,Y) : \Gamma(E) \to \Gamma(E)$. It can also be formulated as a section of the bundle $E \otimes \bigwedge^2 T^*M$.

[141] The notion of torsion can be defined on bundles other than TM however it requires a choice of solder form for the bundle, whereas on TM there is a canonical choice. For full details see [72].

are most commonly encountered and dealt with in a local patch $U \subset M$. If we assume on the patch we have a **local frame**[142], $\{e_i\}$ $i \in \{1, ..., r = rank(E)\}$, for our bundle E then any section $v \in \Gamma(E)$ can locally be expanded as

$$v = v^i e_i \qquad v^i \in C^\infty(M|_U, \mathbb{R}) \ . \tag{B.3}$$

Now given a connection, ∇, we define its action on our local frame as

$$\nabla e_i = \omega^j{}_i \otimes e_j \ , \tag{B.4}$$

where we have introduced the **connection one-forms** $\omega^j{}_i \in \Gamma(T^*M|_U)$ with i, j being matrix indices. Since a connection is linear, specifying it on our local frame allows us to locally understand its action on any section v

$$\nabla v = (dv^j + \omega^j{}_i v^i) \otimes e_j \ . \tag{B.5}$$

Since we're working in a local patch (U, ϕ) on the base manifold we have a set of coordinates $\{x^\mu\}$ $\mu \in \{1, ..., n = dimM\}$. These coordinates provide a natural local frame for TM in the patch[143] $\{\partial_\mu\}$ which leads to the local dual frame $\{dx^\mu\}$ for T^*M. Using this local frame to expand $\omega^i{}_j$ to obtain

$$\omega^i{}_j = \Gamma^i_{\mu j} dx^\mu \ , \tag{B.6}$$

where we shall call $\Gamma^i_{\mu j}$ the **connection one-form coefficients**. We should note that this object has indices of 2 types, 2 are indices that naturally take values in $\{1, ..., rank(E)\}$ while the other one takes values in $\{1, ..., n = dimM\}$.

As has already been stated, TM is of special interest to us in GR. If we consider a patch, as before, then we know $\{\partial_\mu\}$ forms a local frame for $TM|_U$ and so everything we've stated previously applies. However TM has the special property that its sections are the vector fields with respect to which we define covariant derivatives, and so if we have an affine connection ∇ we can consider

$$\nabla_{\partial_\mu} \partial_\nu = (\nabla \partial_\nu)(\partial_\mu) = (\omega^\rho{}_\nu \otimes \partial_\rho)(\partial_\mu) = \Gamma^\rho_{\mu\nu} \partial_\rho \ , \tag{B.7}$$

where in this context $\Gamma^\rho_{\mu\nu}$ are called **Christoffel coefficients**. It is in this form that connections are usually introduced in a first course on GR. Note it is slightly misleading as all of the indices on this object appear to

[142] On a patch $U \subset M$ a local frame is a set of local sections $e_1, .., e_r \in \Gamma(E|_U)$, with r the rank of E, that form a basis in all of the fibres contained in $E|_U = \pi^{-1}(U)$ with π the bundle projection map

[143] By using the inverse of the coordinate chart map ϕ to pushforward the partial derivatives defined on $\phi(U) \subset \mathbb{R}^n$.

be of the same type, however by (B.6) we see this is an illusion caused by the current context we're working in.

Often in GR coordinate basis methods are emphasised for performing explicit calculations. For example (B.1) on TM locally takes the coordinate basis form

$$R^{\sigma}{}_{\rho\mu\nu}\partial_\sigma = R(\partial_\mu, \partial_\nu)(\partial_\rho) = \big(\partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\tau_{\nu\rho}\Gamma^\sigma_{\mu\tau} - \Gamma^\tau_{\mu\rho}\Gamma^\sigma_{\nu\tau}\big)\partial_\sigma \ , \tag{B.8}$$

where we have relabelled the curvature as R to agree with the notation commonly used in the literature[144].

The notion of torsion is also usually encountered in a local form. Again in a coordinate basis, this gives

$$\tau^{\rho}{}_{\mu\nu}\partial_\rho = \tau(\partial_\mu, \partial_\nu) = (\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu})\partial_\rho \ , \tag{B.9}$$

which shows that locally torsion is a measure of the failure of the Christoffel coefficients to be symmetric in it's lower two indices. This only makes sense to discuss because in this context all indices run over the same range.

We now have both a metric, g, and connection, ∇, however as of yet there is no relationship between these. In GR this isn't the case as we work with **metric compatible connections** which obey

$$\nabla_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \ , \tag{B.10}$$

with $X, Y, Z \in \Gamma(TM)$ arbitrary. If we work locally, in a coordinate basis, this becomes

$$\nabla_{\partial_\rho}(g(\partial_\mu, \partial_\nu)) = g(\nabla_{\partial_\rho}\partial_\mu, \partial_\nu) + g(\partial_\mu, \nabla_{\partial_\rho}\partial_\nu) \implies \partial_\rho g_{\mu\nu} - \Gamma^\sigma_{\rho\mu} g_{\sigma\nu} - \Gamma^\sigma_{\rho\nu} g_{\sigma\mu} = 0 \ . \tag{B.11}$$

Not only is the connection used in GR metric compatible but it is also chosen to be torsion free. This further restriction may seem innocent initially, however this couldn't be further from the truth owing to the following theorem

Theorem B.1. *Let (M, g) be a manifold with Lorentzian metric. There exists a unique metric compatible torsion free affine connection ∇ called the **Levi-Civita connection**.*

In a patch $U \subset M$ with a local frame $\{\partial_\mu\}$ the Christoffel coefficients of the Levi-Civita connection take

[144]Note the final term in (B.1) doesn't contribute to this expression since in any coordinate basis $[\partial_\mu, \partial_\nu] = 0$.

the form

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \ , \tag{B.12}$$

where $g^{\rho\sigma}$ is the inverse of $g_{\mu\nu}$. The right hand side of this expression is the well known **Christoffel symbol** $\left\{ \begin{array}{c} \rho \\ \mu\nu \end{array} \right\}$.

So in summary in GR once the geometry of our spacetime has been determined the affine connection we use is also fixed uniquely and is intimately tuned to the geometry. Also whilst the theory is obviously insensitive to the choice of local frame coordinate basis methods are emphasised. This has the advantage of providing easily calculated local formulae like (B.8) and (B.12). However we might ask if there are other ways we can calculate such objects. We may also ask if there is any way to work with different affine connections. These questions were neatly answered by Cartan and form the basis of the next Section.

B.2 Non-coordinate bases: Cartan's moving frames

We are now in a position to describe the work of Cartan which uses non-coordinate bases to describe differential geometry. This will allow us to easily include torsion into our formalism and provides new calculation tools for us to use.

Consider a manifold with a Lorentzian metric (M, g). If we have a patch $U \subset M$ in which we have coordinates $\{x^\mu\}$ then we know we have the local frame $\{\partial_\mu\}$. However we can choose to work with any basis we like, for example $\{e_a\}$ $a \in \{1, .., n = dimM\}$ with $e_a \in \Gamma(TM|_U)$. If such a basis doesn't derive from a coordinate system then we call it a **non-coordinate basis**. Since we have two bases we can clearly expand one in terms of the other

$$e_a = e_a{}^\mu \partial_\mu \ , \tag{B.13}$$

where we call the components $e_a{}^\mu \in GL(n, \mathbb{R})$ **vielbeins**. Given such a non-coordinate basis we can define its dual frame $\{E^a\}$, which are elements of $\Gamma(T^*M|_U)$, and as such can be expanded as

$$E^a = E^a{}_\mu dx^\mu \ . \tag{B.14}$$

Since (B.13) and (B.14) are dual frames we have that

$$E^a(e_b) = \delta^a_b \implies E^a{}_\mu e_b{}^\mu = \delta^a_b \ . \tag{B.15}$$

Any tensor can be expanded in either of our local frames with the vielbeins and their inverse providing the objects require to perform the change of basis. For example for a vector $v \in \Gamma(TM)$, using (B.13), we have

$$v = v^a e_a = v^\mu \partial_\mu \implies v^\mu = e_a{}^\mu v^a \ . \tag{B.16}$$

So far everything we've discussed has been in general, but our manifold is equipped with a metric tensor. In such a case there is a special class of non-coordinate bases that prove to be particularly useful. These are so called **orthonormal frames**, $\{e_a\}$, which are bases such that in a patch on M the metric locally takes the form

$$g(e_a, e_b) = e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \eta_{ab} \ , \tag{B.17}$$

with η being the d-dimensional Minkowski metric (with a mostly plus signature) and $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$. So locally an orthonormal frame makes the tangent bundle fibres appear flat, however this requires a choice of basis that varies as we move around our patch in M. Due to this the vielbeins are in fact local functions $e_a{}^\mu \in C^\infty(U)$. Note we have been very careful to say this construction is local as it may be impossible to find a globally defined orthonormal frame[145].

It is well known that the Minkowski metric possesses the Lorentz group, $O(1, n-1)$, as its isometry group, meaning that

$$\Lambda^c{}_a \Lambda^d{}_b \eta_{cd} = \eta_{ab} \qquad \Lambda \in O(1, n-1) \ . \tag{B.18}$$

As a result of this, given any orthonormal frame $\{e_a\}$, any frame related to it by $\Lambda \in O(1, n-1)$ is also an orthonormal frame. If we wish to preserve the sign of the volume form on M then we need to restrict to $\Lambda \in SO(1, n-1)$, which we do from now on. Note that this Lorentz transformation can be different in different fibres and so varies over the manifold, hence (B.17) is insensitive to local Lorentz transformations. This point will be crucial when we come to discuss reformulations and extensions of GR, but for now it's just an interesting quirk of our new choice of basis.

We can now introduce an affine connection ∇ on (M, g) and then consider carrying out calculations in an orthonormal frame. Cartan's formulation provides a method to calculate the connection one-forms and the curvature of a given connection in a way that makes use of differential forms. A crucial difference compared

[145]If it is possible then TM is a trivial bundle

to a coordinate basis is that the Lie bracket of two orthonormal basis elements is non-zero

$$[e_a, e_b] = c_{ab}{}^{c} e_c \ , \tag{B.19}$$

where $c_{ab}{}^{c}$ are often called the **anholonomy coefficients**[146] as their being non-zero ensures there is no coordinate system from which the frame can be derived.

We can now re-express the torsion and curvature of our connection in an orthonormal frame as

$$\tau(e_a, e_b) = (\omega^{d}{}_{b}(e_a) - \omega^{d}{}_{a}(e_b) - c_{ab}{}^{d}) e_d \ , \tag{B.20}$$

$$\begin{aligned} R(e_a, e_b)(e_c) =& [e_a(\omega^{d}{}_{c}(e_b)) - e_b(\omega^{d}{}_{c}(e_a)) + \omega^{e}{}_{c}(e_b)\omega^{d}{}_{e}(e_a) \\ & - \omega^{e}{}_{c}(e_a)\omega^{d}{}_{e}(e_b) - c_{ab}{}^{e}\omega^{d}{}_{c}(e_e)] e_d \ . \end{aligned} \tag{B.21}$$

Note that these reduce to (B.9) and (B.8) respectively if a coordinate basis is used.

So far this has all just been a restatement of what we previously had but in a new basis. The critical insight of Cartan was that the connection one-forms obey the following relations

$$dE^a + \omega^a{}_b \wedge E^b = T^a \ , \tag{B.22}$$

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R^a{}_b \ , \tag{B.23}$$

where $T^a = \frac{1}{2}\tau^a{}_{bc} E^b \wedge E^c = \frac{1}{2} E^a(\tau(e_b, e_c)) E^b \wedge E^c$ and $R^a{}_b = \frac{1}{2} R^a{}_{bcd} E^c \wedge E^d = \frac{1}{2} E^a(R(e_c, e_d)(e_b)) E^c \wedge E^d$ are often called the **Torsion two-form** and **Curvature two-form** respectively. These equations are called the **first and second Cartan structure equations**[147]. If we take the exterior derivative of these equations and use them to substitute appropriately we obtain the **Bianchi identities**

$$dT^a + \omega^a{}_b \wedge T^b - R^a{}_b \wedge E^b = 0 \ , \tag{B.24}$$

$$dR^a{}_b - R^a{}_c \wedge \omega^c{}_b + \omega^a{}_c \wedge R^c{}_b = 0 \ . \tag{B.25}$$

The equations (B.22)-(B.25) now provide a method for calculating various quantities of interest using the

[146] This name derives from the fact that coordinate bases are sometimes called holonomic bases.

[147] To prove these act on e_b and e_c while recalling that $d\Omega(X, Y) = X(\Omega(Y)) - Y(\Omega(X)) - \Omega([X, Y])$ for $\Omega \in \Gamma(T^*M)$ and $X, Y \in \Gamma(TM)$ and then use (B.20)and (B.21).

language of differential forms, and as a result everything is manifestly diffeomorphism invariant.

Cartan's next insight was that if the affine connection being used was metric compatible then, in an orthonormal frame, (B.10) becomes

$$\nabla_{e_c}(g(e_a, e_b)) = g(\nabla_{e_c} e_a, e_b) + g(e_a, \nabla_{e_c} e_b) \implies \omega^d{}_a(e_c)\eta_{db} + \omega^d{}_b(e_c)\eta_{ad} = 0\ , \tag{B.26}$$

this is often stated as the connection one-forms being antisymmetric $\omega_{ba} = -\omega_{ab}$, where η has been used to lower an index, again this is only possible in the context of the TM. Now given a connection that is metric compatible and has a given torsion two-form the connection one-form coefficients are found to be

$$\omega^d{}_b(e_c)\eta_{da} = \frac{1}{2}\big((A^d{}_{bc} - \tau^d{}_{bc})\eta_{da} - (A^d{}_{ab} - \tau^d{}_{ab})\eta_{dc} + (A^d{}_{ca} - \tau^d{}_{ca})\eta_{db}\big)\ , \tag{B.27}$$

where $A^a{}_{bc} = (dE^a)_{bc}$ and, as such, is antisymmetric in the lower two indices. This is obtained from (B.22) and constitutes the unique solution of this equation for a given orthonormal frame and torsion two-form. Hence we now have a method to calculate the connection in our orthonormal basis and from there (B.23) allows us to calculate the curvature of our connection. Crucially we've now managed to obtain a formalism in to which torsion is naturally included.

B.3 The Cartan approach to gravity

In the preceding portions of this appendix we have mainly developed a mathematical framework that allows us to efficiently deal with non-coordinate bases and connections with torsion. However along the way we've had very little to say about physics, we now rectify this. We shall begin lightly by considering how to rewrite the theory of GR using the language of orthonormal frames.

In GR the metric is the basic fundamental field of the theory, however (B.13) shows that given a set of vielbeins the metric is a derived quantity. As such one might ask if it's possible to formulate an approach to gravity where the vielbein is the basic variable instead of the metric. At first glance we see[148] that g has $\frac{n(n+1)}{2}$ independent components while $e_a{}^\mu$ has n^2. This would seem to snooker us, however recall that (B.18) means that our vielbeins are only unique up to a local Lorentz transformation and so our theory possesses an additional gauge symmetry which can remove $dim(SO(1, d-1)) = \frac{n(n-1)}{2}$ degrees of freedom. As a result our basic variables in the two formulations have an equal number of degrees of freedom.

[148]Ignoring gauge degrees of freedom associated to diffeomorphism which affect both objects equally.

Since the metric, in a given orthonormal frame, takes the form

$$g = \eta_{ab} E^a \otimes E^b \ , \tag{B.28}$$

we can see that under a local Lorentz transformation $\Lambda^a{}_b$

$$E^a \to \tilde{E}^a = \Lambda^a{}_b E^b \to E^a{}_\mu \to \tilde{E}^a{}_\mu = \Lambda^a{}_b E^b{}_\mu \ , \tag{B.29}$$

hence (B.15) then tells us that

$$e_a \to \tilde{e}_a = e_b (\Lambda^{-1})^b{}_a \implies e_a{}^\mu \to \tilde{e}_a{}^\mu = e_b{}^\mu (\Lambda^{-1})^b{}_a \ , \tag{B.30}$$

with $(\Lambda^{-1})^b{}_a$ being the inverse of $\Lambda^a{}_b$. These rules then inform us as to how tensor components, in our orthonormal frame, transform under local Lorentz transformations. For example under Λ, using (B.29) and (B.30), a $(1,1)$ tensor has it components transform as

$$V = V^a{}_b e_a \otimes E^b = \tilde{V}^a{}_b \tilde{e}_a \otimes \tilde{E}^b \implies \tilde{V}^a{}_b = \Lambda^a{}_c (\Lambda^{-1})^d{}_b V^c{}_d \ . \tag{B.31}$$

Using (B.22) and (B.23) along with the fact that T^a transforms in a vector representation of the local Lorentz group, we find under the local transformation Λ

$$\omega^a{}_b \to \tilde{\omega}^a{}_b = \Lambda^a{}_c (\Lambda^{-1})^d{}_b \omega^c{}_d + \Lambda^a{}_c d(\Lambda^{-1})^c{}_b \ , \tag{B.32}$$

$$R^a{}_b \to \tilde{R}^a{}_b = \Lambda^a{}_c (\Lambda^{-1})^d{}_b R^c{}_d \ , \tag{B.33}$$

showing that the connection one-form transforms precisely as a connection should do, in the sense of Yang-Mills theory, while the curvature two-form transforms tensorially under local Lorentz transformations. Hence the theory we are beginning to write down is a form of gauge theory for the Lorentz group.

We can obviously expand the connection one-form form in any basis of T^*M we want. However whenever we use a coordinate basis we obtain

$$\omega^a{}_b = \omega_\mu{}^a{}_b dx^\mu \ , \tag{B.34}$$

where $\omega_\mu{}^a{}_b$ is called the **spin connection**[149], although we know it's just the local component form of

[149] This name is usually reserved for the torsion free case and hence, by (B.1), our connection is just the Levi-Civita connection

an underlying connection. By definition the spin connection behaves as the components of a one-form under general coordinate transformations, and as a connection under local Lorentz transformations. These transformation properties ensure that it allows us write a local expression for covariant derivatives of any tensor of the local Lorentz group. For example for $V \in \Gamma(TM \otimes T^*M)$ we have

$$\nabla_\mu V^a{}_b = \partial_\mu V^a{}_b + \omega_\mu{}^a{}_c V^c{}_b - \omega_\mu{}^c{}_b V^a{}_c \ , \tag{B.35}$$

with $\nabla_\mu = \nabla_{\partial_\mu}$.

If we now assume that the torsion of the spin connection is zero, hence we have the Levi-Civita connection- since we've already assumed the connection is metric compatible, we can then consider the action

$$S[E^a{}_\mu] = \frac{1}{2(n-2)!} \int_{M_n} \tilde{\epsilon}_{abc_1..c_{n-2}} R^{ab} \wedge E^{c_1} \wedge \ldots \wedge E^{c_{n-2}} \ , \tag{B.36}$$

where $\tilde{\epsilon}_{abc_1 \ldots c_{n-2}}$ is the Levi-Civita symbol[150] and R^{ab} is the curvature two-from associated to the Levi-Civita connection. Note that this action is invariant under local $SO(1, n-1)$ transformations. By going to a coordinate basis and using $\tilde{\epsilon}_{\mu\nu\sigma_1 \ldots \sigma_{n-2}} \tilde{\epsilon}^{\rho\tau\sigma_1 \ldots \sigma_{n-2}} = 2!(n-2)! \delta^\rho_{[\mu} \delta^\tau_{\nu]}$ we can show that (B.36) reduces to

$$S_{EH}[g] = \int_{M_n} R * 1 \ , \tag{B.37}$$

which is the usual form of the Einstein-Hilbert action.

B.4 Spinors in General Relativity

So we have now safely embedded GR into our new formalism but this all still seems like a lot of effort to expend to just obtain a nice rewriting of the theory. Fortunately the pay-off is far greater. So far we've only discussed bosonic fields but, as much as we may want it to be, the universe isn't just composed of bosons there are also fermions, and so far our treatment has completely glossed over them. In order to incorporate fermions[151] into our theory we need to be able to define spinor representations[152]. This seems

written in an unfamiliar way. Note the name spin connection comes from the fact that this object is used to define the covariant derivative of spinor fields.

[150] Our convention is that $\tilde{\epsilon}_{12 \ldots n} = 1$.

[151] This is a little bit of a slight of hand since we know fermions are meant to be represented by Grassmann valued (anticommuting) fields and the objects we construct here are built on $\mathbb{C}^m$ and so do not have this property. Really here fermion is being used as a shorthand for non-integer spin representations.

[152] As another technical detail in order to define spinors on a manifold we are looking for a bundle extension that means our manifold can support a spin structure [75]. This means we can extend our frame bundle transition functions from $SO(1, n-1)$ to $Spin(1, n-1)$ transition functions. A necessary and sufficient condition for this to succeed is the vanishing of the second Stiefel-Whitney class of the manifold.

like a formidable task in a curved space but now comes the magic of the orthonormal frame formulation, we've made our metric look like the Minkowski metric and we're very adept at dealing with spinors in flat space. So the idea is to set up everything using our orthonormal frame and then use the vielbein and its inverse to translate this to our curved space.

So the task is clear, we have to just follow what we do in flat space and make appropriate modifications to account for the local Lorentz invariance of our theory. For definiteness we shall deal with Dirac spinors[153] and as such the first step is to introduce Dirac matrices. In flat space these obey

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} , \tag{B.38}$$

where $\{A, B\} := AB + BA$ and is called the anticommutator, note we are suppressing spinor indices. Now γ^a are numerical matrices and from them we can define curved space Dirac matrices by

$$\gamma^\mu = e_a{}^\mu \gamma^a , \tag{B.39}$$

and as a result of (B.17) and (B.38) we find that these matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} , \tag{B.40}$$

which is exactly a curved space version of the flat space algebra. Our spinor ψ is meant to be in a spinor representation of the local Lorentz group but we know precisely how to construct this representation. At the algebra level, given the Dirac matrices the generators of this representation are

$$T^{ab} = -\frac{i}{4}[\gamma^a, \gamma^b] , \tag{B.41}$$

which can be shown to furnish a representation of the Lorentz algebra with the representation being called the Dirac representation. However, we want to be able to perform finite Lorentz transformations, and so we need to exponentiate this algebra representation to a group representation. So, given a Lorentz transformation Λ, described by the parameters $\alpha_{ab} = -\alpha_{ba}$, the Dirac representation of $Spin(1, n-1)$ is given by

$$\rho_{\frac{1}{2}}(\Lambda) = \exp(\frac{1}{2}\alpha_{ab}T^{ab}) , \tag{B.42}$$

with $\exp : spin(1, n-1) \to Spin(1, n-1)$ the usual exponential map between a Lie algebra and its associated

[153] Since these exist in all dimensions unlike Weyl or Majorana spinors.

Lie group. Under any Lorentz transformation our spinor now transforms as

$$\psi \to \tilde{\psi} = \rho_{\frac{1}{2}}(\Lambda)\psi \ . \tag{B.43}$$

With our spinor representation now safely in hand we can turn to trying to construct an action for our fermion. We know that this will involve derivatives of the fermion and hence we need a covariant derivative in order to be able to write down a local Lorentz invariant action. This needs to transform in the following way

$$\nabla_a \psi \to \rho_{\frac{1}{2}}(\Lambda)(\Lambda^{-1})^b{}_a \nabla_b \psi \ . \tag{B.44}$$

This may seem daunting to find but we've already found the connection we need, it's just given by the spin connection with its one-form index expressed in our orthonormal basis

$$\nabla_a \psi = \partial_a \psi + \frac{1}{2}\omega_{abc} T^{bc} \psi \tag{B.45}$$

where $\partial_a = e_a{}^\mu \partial_\mu$ and $\omega_{abc} = e_a{}^\mu \eta_{bd} \omega_\mu{}^d{}_c$. To show this obeys (B.44) use that if $d_{\frac{1}{2}}$ is the algebra representation associated to $\rho_{\frac{1}{2}}$ then

$$d_{\frac{1}{2}}(\omega_a) = \frac{1}{2}\omega_{abc} T^{bc} \to \rho_{\frac{1}{2}}(\Lambda) d_{\frac{1}{2}}(\omega_a) \rho_{\frac{1}{2}}(\Lambda)^{-1} - (\partial_a \rho_{\frac{1}{2}}(\Lambda)) \rho_{\frac{1}{2}}(\Lambda)^{-1} \ . \tag{B.46}$$

This now finally allows us to write down a curved space version of the Dirac action which is both invariant under local Lorentz transformations and general coordinate transformations

$$\int_{M_n} d^n x \ \det(E) \bar{\psi}(i\gamma^a \nabla_a + m)\psi \ , \tag{B.47}$$

where $\bar{\psi} = \psi^\dagger \gamma^0$, with γ^0 being the flat space gamma matrix, and $\det(E) = \det(E^a{}_\mu)$.

B.5 Torsion and fermions

In this Section we shall briefly review how torsion can enter into physical theories, for further details we recommend [44].

We're now starting to see the benefits of the orthonormal frame formalism as it has allowed us to efficiently incorporate spinors into GR. However we also promised that torsion would be included as well, but now we might ask why we'd want to do this, we've already been able to add fermions to our theory and that didn't require the inclusion of torsion, so why would we want to add it now? To understand this we have to switch approach a bit and consider the first order, also known as called the Palatini, formulation of GR. Fortunately (B.36) can easily be adapted to a first order formulation

$$S[E^a{}_\mu, \omega] = \frac{1}{2(n-2)!} \int \tilde{\epsilon}_{abc_1 \ldots c_{n-2}} R^{ab}[\omega] \wedge E^{c_1} \wedge \ldots \wedge E^{c_{n-2}} \ , \tag{B.48}$$

where now the connection is considered as an independent field, although it is still assumed to be metric compatible, and $R^{ab}[\omega]$ is given by (B.23) where one index has been raised by η^{ab}.

If we now consider the variation of this action with respect to ω, and recall that the variation of any connection is tensorial, then we have $\delta R^{ab} = d\delta\omega^{ab} + \omega^a{}_c \wedge \delta\omega^{cb} + \omega^b{}_c \wedge \delta\omega^{ac} = D(\delta\omega^{ab})$[154]. Using this and then an integration by parts, where we drop the boundary term $D(\tilde{\epsilon}_{abc_1 \ldots c_{n-2}} \delta\omega^{ab} \wedge E^{c_1} \wedge \ldots \wedge E^{c_{n-2}})$, we find that

$$\delta_\omega S = -\frac{1}{2(n-3)!} \int \tilde{\epsilon}_{abc_1 c_2 \ldots c_{n-2}} \delta\omega^{ab} \wedge D(E^{c_1}) \wedge E^{c_2} \wedge \ldots \wedge E^{c_{n-2}} \ , \tag{B.49}$$

where it should be noted that we have used $\tilde{\epsilon}_{abc_1 c_2 \ldots c_{n-2}} D(E^{c_1} \wedge E^{c_2} \wedge \ldots \wedge E^{c_{n-2}}) = (n-2) \tilde{\epsilon}_{abc_1 c_2 \ldots c_{n-2}} D(E^{c_1}) \wedge E^{c_2} \wedge \ldots \wedge E^{c_{n-2}}$. From which we can conclude the field equation for ω is

$$D(E^a) = dE^a + \omega^a{}_b \wedge E^b = 0 \ , \tag{B.50}$$

which is just (B.22) with vanishing torsion. This tells us that to solve this equation ω is the Levi-Civita connection as it's torsionless and metric compatible. So it turns out the Palatini formulation allows one to conclude, as a result of a field equation, that the preferred connection is the Levi-Civita connection. It should be noted that substituting this result in to the equation of motion for $E^a{}_\mu$ allows us to recover the Einstein equation, thus showing that the Palatini action is classically equivalent to the Einstein-Hilbert action.

However once fermions are included, such as by coupling (B.47) to (B.48), then the explicit connection in the fermionic term will lead to additional terms in (B.50) which are fermion bilinears, terms containing two fermionic fields. These terms can be shifted to the right hand side of the equation of motion and then lead to (B.22) with a non-zero torsion tensor. As a result we can see that the presence of fermions induces

[154] We are now using the language of differential forms to compactly present all of formulae. Here D refers to the covariant derivative written in form language as done in (B.5), but now we act on a two index object using the appropriate induced connection.

a non-zero torsion into physical systems. Obviously we could just start with the second order formulation of our theory and then the connection is anything we like. If we do this then for an action such as (B.36) coupled to (B.47) we have the variation

$$\delta S[E^a{}_\mu, \psi] = \int d^n x \ \det(E)\Big(\frac{\delta S}{\delta E^a{}_\mu}\delta E^a{}_\mu + \frac{\delta S}{\delta \psi}\delta\psi + \frac{\delta S}{\delta \omega_\mu{}^a{}_b}\big(\frac{\delta \omega_\mu{}^a{}_b}{\delta E^c{}_\nu}\delta E^c{}_\nu + \frac{\delta \omega_\mu{}^a{}_b}{\delta \psi}\delta\psi\big)\Big)\ , \tag{B.51}$$

in the case where there are no fermions the term $\frac{\delta S}{\delta \omega_\mu{}^a{}_b}$ vanishes identically if we choose $\omega_\mu{}^a{}_b$ to satisfy (B.50). As a result of this variations become much simpler to calculate. Hence we see that if we have fermions then it will be advantageous to emulate this. To this end we often use our freedom in selecting a connection to pick the one given by (B.27) with the torsion being the one induced by the fermions present in the system. Then in this case the term $\frac{\delta S}{\delta \omega_\mu{}^a{}_b}$ once more vanishes identically and variations of our action become simpler to compute. This computational ease is why we often choose to work with a connection with torsion. An example of the use of this formalism is when we attempt to show a theory is invariant under local supersymmetry, as is often done in Supergravity.

B.6 Connection one-forms and curvature two-forms for time dependent solutions

In Section 6 we provided a family of time dependent solutions to Type IIA supergravity. Here we give a list of the non-zero independent connection one-form and curvature two-form components. We use the notation of Section 6 throughout.

The connection one-forms in an orthonormal basis are

$$\begin{aligned}
&\omega_{0i} = -\frac{\dot{\alpha_1}}{\alpha_1}E^i, \quad \omega_{04} = -\frac{\dot{\alpha_3}}{\alpha_3}E^4, \quad \omega_{56} = A(\rho)E^7, \\
&\omega_{57} = -A(\rho)E^6, \quad \omega_{58} = -A(\rho)E^9, \quad \omega_{59} = A(\rho)E^8, \\
&\omega_{67} = -A(\rho)E^5 + B(\rho)E^6, \quad \omega_{68} = D(\rho)E^9, \quad \omega_{69} = -D(\rho)E^8, \\
&\omega_{78} = E(\rho)E^8, \quad \omega_{79} = E(\rho)E^9, \quad \omega_{89} = A(\rho)E^5 - D(\rho)E^6 + F(\rho,\theta)E^9, \\
&A(\rho) = \frac{g(\mathrm{sech}(2\rho))^2}{H(\rho)}, \quad D(\rho) = -\frac{g\sinh(2\rho)}{H^{\frac{1}{2}}(\cosh(2\rho))^{\frac{3}{2}}}, \\
&E(\rho) = -\frac{gH'(\rho)}{2H^{\frac{2}{3}}(\cosh(2\rho))^{\frac{1}{2}}} + D(\rho), \quad F(\rho,\theta) = -\frac{2g\cos(\theta)}{(H\cosh(2\rho))^{\frac{1}{2}}\sin(\theta)}, \\
&B(\rho) = \frac{gH'(\rho)}{2H^{\frac{3}{2}}(\cosh(2\rho))^{\frac{1}{2}}} + \frac{2g}{H^{\frac{1}{2}}\sinh(2\rho)}((\cosh(2\rho))^{\frac{1}{2}} - \frac{(\sinh(2\rho))^2}{2(\cosh(2\rho))^{\frac{3}{2}}}).
\end{aligned}$$

The curvature two-forms in an orthonormal basis are

$$R^0{}_i = \frac{\ddot{\alpha_1}}{\alpha_1} E^0 \wedge E^i, \qquad R^0{}_4 = \frac{\ddot{\alpha_3}}{\alpha_3} E^0 \wedge E^4, \qquad R^i_{\ j} = (\frac{\dot{\alpha_1}}{\alpha_1})^2 E^i \wedge E^j,$$

$$R^i{}_4 = \frac{\dot{\alpha_1}\dot{\alpha_3}}{\alpha_1\alpha_3} E^i \wedge E^4, \qquad R^5{}_6 = A^2 E^5 \wedge E^6,$$

$$R^5{}_7 = A^2 E^5 \wedge E^7 + \frac{gA'(\rho)}{(H\cosh(2\rho))^{\frac{1}{2}}} E^6 \wedge E^7 - 2A(E+D) E^8 \wedge E^9,$$

$$R^5{}_8 = A^2 E^5 \wedge E^8 + (A(D+E) - \frac{gA'(\rho)}{(H\cosh(2\rho))^{\frac{1}{2}}}) E^7 \wedge E^9 - A(E+D) E^6 \wedge E^8,$$

$$R^5{}_9 = A^2 E^5 \wedge E^9 + (\frac{gA'(\rho)}{(H\cosh(2\rho))^{\frac{1}{2}}} - A(D+E)) E^7 \wedge E^8 - A(D+E) E^6 \wedge E^9,$$

$$R^6{}_7 = \frac{gA'(\rho)}{(H\cosh(2\rho))^{\frac{1}{2}}} E^5 \wedge E^7 + (-3A^2 - B^2 - \frac{gB'(\rho)}{(H\cosh(2\rho))^{\frac{1}{2}}}) E^6 \wedge E^7$$
$$+ (2A^2 + 2D(B+E)) E^8 \wedge E^9,$$

$$R^6{}_8 = -A(D+E) E^5 \wedge E^8 + (BE + D^2) E^6 \wedge E^8$$
$$+ (-DE + A^2 + \frac{gD'(\rho}{(H\cosh(2\rho))^{\frac{1}{2}}}) E^7 \wedge E^9,$$

$$R^6{}_9 = -A(D+E) E^5 \wedge E^9 + (BE + D^2) E^6 \wedge E^9$$
$$+ (DE - A^2 - \frac{gD'(\rho}{(H\cosh(2\rho))^{\frac{1}{2}}}) E^7 \wedge E^8,$$

$$R^7{}_8 = A(D+E) E^5 \wedge E^9 - (A^2 + D(B+E)) E^6 \wedge E^9$$
$$+ (\frac{gE'(\rho)}{(H\cosh(2\rho))^{\frac{1}{2}}} - E^2) E^7 \wedge E^8,$$

$$R^7_9 = -A(D+E) E^5 \wedge E^8 + (A^2 + D(B+E)) E^6 \wedge E^8$$
$$+ (\frac{gE'(\rho}{(H\cosh(2\rho))^{\frac{1}{2}}} - E^2) E^7 \wedge E^9,$$

$$R^8{}_9 = -\frac{gA'(\rho)}{(H\cosh(2\rho))^{\frac{1}{2}}} E^5 \wedge E^7 + (2A^2 + DB + \frac{gD'(\rho)}{(H\cosh(2\rho))^{\frac{1}{2}}}) E^6 \wedge E^7$$
$$+ (\frac{g\partial_\rho F(\rho,\theta)}{(H\cosh(2\rho))^{\frac{1}{2}}} - FE) E^7 \wedge E^9 + (\frac{2g\partial_\theta F(\rho,\theta)}{(H\cosh(2\rho))^{\frac{1}{2}}} - 3A^2 - 3D^2 - E^2 - F^2) E^8 \wedge E^9.$$

where A, B, D, E, F are as in the connection one-form expressions.

C The Garfinkle-Vachaspati transformation

In this appendix an introduction to the solution generating technique of Garfinkle and Vachaspati (GV) [47],[46] [155] is given. The aim of this transformation is to take a solution to a set of field equations, (g, Φ), which includes gravity, with metric g, and some matter, Φ which can have any form e.g. scalar, vector, 2-form etc, into a new solution, $(\tilde{g}, \Phi)$. The new geometry, $\tilde{g}$, corresponds to the original geometry, g, but with a wave superimposed on top of it.

This technique has been employed to allow straight cosmic string, [47], and fundamental string, [23], solutions to be transformed into analogous solutions with a travelling wave on the respective strings. Here details of the application of the GV transformation to the M2 brane solution [37] of 11 dimensional supergravity will be discussed.

It may be of interest to the reader to note that generalisations of the technique, in which the matter content of the solution also transforms, have been explored in the literature in [73] and [13].

C.1 The M2 travelling wave solution

In this Section an overview of the application of the work of Garfinkle and Vachaspati to finding a travelling wave M2 brane solution to 11 dimensional supergravity is given. We provide details of their construction as it is applied to our current setting, however we hope this provides enough of the flavour of the general procedure to be easily applied by readers to other settings.

The bosonic dynamics of 11 dimensional supergravity, in Einstein frame variables, are determined by the action principle [18]

$$S[g, A_{[3]}] = \frac{1}{2\kappa_{11}^2} \int R *_{11} 1 - \frac{1}{2} F_{[4]} \wedge *_{11} F_{[4]} + \frac{1}{6} F_{[4]} \wedge F_{[4]} \wedge A_{[3]} \ , \tag{C.1}$$

where κ_{11} is the 11d gravitational coupling, g is the 11d metric, $*_{11}$ is the Hodge star associated to this metric and R is its Ricci scalar. $F_{[4]}$ is the field strength of the abelian 3-form gauge potential $A_{[3]}$, defined by $F_{[4]} = dA_{[3]}$.

[155] Be careful to be aware of the crucial difference in index position between equation (3.17) given in [47] and equation (9) in [46].

Variation of (C.1) leads to the field equations

$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{1}{12}(F_{MPQR}F_N{}^{PQR} - \frac{1}{8}g_{MN}F_{PQRS}F^{PQRS}) \ , \tag{C.2}$$

$$d *_{11} F_{[4]} + \frac{1}{2}F_{[4]} \wedge F_{[4]} = 0 \implies \partial_Q(\sqrt{-g}F^{QMNP}) + \frac{\tilde{\epsilon}^{MNPQR_1\ldots R_3R_4\ldots R_7}}{2!(4!)^2}F_{QR_1\ldots R_3}F_{R_4\ldots R_7} = 0 \ , \tag{C.3}$$

where R_{MN} is the Ricci tensor of g_{MN}, for $M \in \{1,\ldots,11\}$, $g = \det(g_{MN})$ and $\tilde{\epsilon}^{M_1\ldots M_{11}}$ is the numerical epsilon symbol.

The field equations (C.2) and (C.3) admit many solutions but of particular interest are those that correspond to extended objects such as the M2 brane[156] [37]

$$\begin{aligned} ds^2 &= (1+\frac{q}{r^6})^{-\frac{2}{3}}\eta_{\mu\nu}dx^\mu dx^\nu + (1+\frac{q}{r^6})^{\frac{1}{3}}\delta_{mn}dy^M dy^n \ , \\ A_{012} &= \pm(1+\frac{q}{r^6})^{-1} \qquad \mu \in \{0,1,2\} \qquad m \in \{3,\ldots,10\} \ , \\ r^2 &= \delta_{mn}y^m y^n \qquad q \in \mathbb{R} \ , \end{aligned} \tag{C.4}$$

where x^μ are coordinates on the 3d brane worldvolume and y^m are coordinates on the transverse space to the brane, which is 8 dimensional. A_{012} denotes the single, independent, non-zero component of the 3-form gauge field and δ_{mn} is the Kronecker delta in the transverse space.

For our purposes, following [23], it is useful to introduce light cone coordinates on the worldvolume defined by

$$u := x^0 - x^1 \qquad v := x^0 + x^1 \ , \tag{C.5}$$

In these coordinates (C.4) takes the form

$$\begin{aligned} ds^2 &= (1+\frac{q}{r^6})^{-\frac{2}{3}}(-dudv + (dx^2)^2) + (1+\frac{q}{r^6})^{\frac{1}{3}}\delta_{mn}dy^m dy^n \ , \\ A_{uv2} &= \pm\frac{1}{2}(1+\frac{q}{r^6})^{-1} \qquad \mu \in \{0,1,2\} \qquad m \in \{3,\ldots,10\} \ , \\ r^2 &= \delta_{mn}y^m y^n \qquad q \in \mathbb{R} \ . \end{aligned} \tag{C.6}$$

[156] For more details about the properties of this solution, such as sourcing by the supermembrane, see Section 1

In the coordinate patch described by (u, v, x^2, y^m), (C.6) has two obvious Killing vectors given by

$$k := \frac{\partial}{\partial u} \qquad m := \frac{\partial}{\partial v} \ . \tag{C.7}$$

To streamline the following discussion we shall focus on the Killing vector k, however what follows would work out equally well if m was used instead. It can be shown, by using (C.7) and (C.6), that k is a null vector[157]. Also if we construct the 1-form $k_{[1]} = g(k, \cdot)$ it can be shown to obey

$$k_{[1]} \wedge dk_{[1]} = 0 \ , \tag{C.8}$$

where d is the usual exterior derivative of forms. This result tells us that k is not only a Killing vector but also hypersurface orthogonal[158], a fact that will prove crucial later in the construction. It is also useful to note that the form of the M2 brane solution (C.6) means that

$$\mathcal{L}_k A_{[3]} = 0 \ , \tag{C.9}$$

with $\mathcal{L}_k$ denoting the Lie derivative with respect to the Killing vector k.

The results (C.8) and (C.9) are two of the crucial ingredients that allow us to apply the Garfinkle-Vachaspati transformation to the M2 brane solution. The heart of the transformation relies upon the properties of the Killing vector k which allow the metric in the M2 brane solution to be replaced with a new metric, whilst still ensuring the new combination of metric and 3-form solve (C.2) and (C.3). We shall now provide an overview of the construction outlined in [47].

Since the Garfinkle-Vachaspati transformation acts non-trivially on the metric it will have a non-trivial effect on covariant derivatives that occur in any equation. As a result it is useful to consider how the connection coefficients of two Levi-Civita connections, associated to two different metrics on a manifold, are related. It can be shown that if g and $\tilde{g}$ are metrics on a differentiable manifold, M, with associated Levi-Civita connection coefficients $\Gamma^{\mu}{}_{\nu\rho}$ and $S^{\mu}{}_{\nu\rho}$, respectively, then

[157] This means that $g(k, k) = 0$.

[158] With (C.8) serving as the definition for hypersurface orthogonality for 1 forms owing to the Frobenius theorem- see later for details.

$$\Gamma^\mu{}_{\nu\rho} = S^\mu{}_{\nu\rho} + W_{\nu\rho}{}^\mu \ ,$$
$$W_{\mu\nu}{}^\rho = \frac{1}{2}\tilde{g}^{\rho\lambda}(N_{\mu\nu\lambda} - N_{\nu\lambda\mu} - N_{\lambda\mu\nu}) \ , \quad \text{(C.10)}$$
$$N_{\mu\nu\rho} = \nabla_\rho \tilde{g}_{\mu\nu} \ ,$$

where ∇ is the Levi-Civita connection associated to the metric g. Using this result we can show that the covariant derivative of the Levi-Civita, $\tilde{\nabla}$, associated to $\tilde{g}$ can be related to that of g, exemplified here on a 1 form, as follows

$$\tilde{\nabla}_\mu \omega_\nu = \nabla_\mu \omega_\nu + W_{\mu\nu}{}^\rho \omega_\rho \ . \quad \text{(C.11)}$$

With these details in hand it is now possible to provide an overview of the Garfinkle-Vachaspati transformation.

To begin with we note that (C.8) implies, via Frobenius theorem[159], that

$$\nabla_\mu k_\nu = k_{[\nu}\nabla_{\mu]}A \ , \quad \text{(C.12)}$$

for some function A. It should be noted that the Killing equation [160] ensures that the left hand side of (C.12) is antisymmetric. Now contracting (C.12) with k^μ and using that k is a null Killing vector that is nowhere vanishing, it can be shown that

$$k^\mu \nabla_\mu A = 0 \ . \quad \text{(C.13)}$$

Garfinkle and Vachaspati proposed the Killing vector k should be used to deform the metric of the original solution to

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + F k_\mu k_\nu \ , \quad \text{(C.14)}$$

with F some function that is to be determined and where the index on the Killing vector has been lowered using g. This proposed form of the metric leads to an inverse metric given by

$$\tilde{g}^{\mu\nu} = g^{\mu\nu} - F k^\mu k^\nu \ , \quad \text{(C.15)}$$

[159]The form of the theorem used here states that if we have a nowhere vanishing 1-form, $n_{[1]}$, such that $n_{[1]} \wedge dn_{[1]} = 0$ everywhere then there exists functions f and g such that $n_{[1]} = g df$.

[160]Which tells us $\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0$.

where it is crucial to recall k is a null vector if we wish to check that (C.15) is the correct inverse. Using these expressions one can show, using that k is both null and a Killing vector, that

$$C^\rho{}_{\mu\nu} = -W_{\mu\nu}{}^\rho = k^\rho k_{(\mu}\nabla_{\nu)}F + Fk_\nu g^{\rho\sigma}\nabla_{[\mu}k_{\sigma]} + Fk_\mu g^{\rho\sigma}\nabla_{[\nu}k_{\sigma]} - \frac{1}{2}k_\mu k_\nu \nabla^\rho F \ . \tag{C.16}$$

We now use (C.12) to replace $\nabla_{[\mu}k_{\nu]}$ in (C.16) and obtain

$$C^\rho{}_{\mu\nu} = k^\rho(k_{(\mu}\nabla_{\nu)}F + Fk_{(\mu}\nabla_{\nu)}A) - \frac{1}{2}k_\mu k_\nu(\nabla^\rho F + 2F\nabla^\rho A) \ . \tag{C.17}$$

If one now assumes that F obeys the condition

$$k^\mu \nabla_\mu F = 0 \ , \tag{C.18}$$

then using (C.13), the fact that k is null and (C.18) in (C.17) we find that

$$C^\rho{}_{\rho\mu} = 0 \ , \tag{C.19}$$

which leads to the nice result that if we have a vector field V^μ or a tensor $Q^{\mu\nu_1\cdots\nu_{p-1}}$, that is antisymmetric in all of its indices, then

$$\begin{aligned} \tilde{\nabla}_\mu V^\mu =& \nabla_\mu V^\mu \ , \\ \tilde{\nabla}_\mu Q^{\mu\nu_1..\nu_{p-1}} =& \nabla_\mu Q^{\mu\nu_1..\nu_{p-1}} \ . \end{aligned} \tag{C.20}$$

We are now in a position to calculate the Ricci tensor, $\tilde{R}_{\mu\nu}$, associated to the Levi-Civita connection of $\tilde{g}$ in terms of quantities associated with the original metric, g. If we use (C.17) and (C.19) we find that

$$\begin{aligned} \tilde{R}_{\mu\nu} =& R_{\mu\nu} + \nabla_\rho C^\rho{}_{\mu\nu} \\ =& R_{\mu\nu} - \frac{1}{2}k_\mu k_\nu(\nabla^\rho\nabla_\rho F + 2F\nabla^\rho\nabla_\rho A + 2\nabla^\rho F\nabla_\rho A + F\nabla^\rho A\nabla_\rho A) \ . \end{aligned} \tag{C.21}$$

The fact that k is a Killing vector of the metric g and (C.12) means that

$$R_{\mu\nu}k^\nu = -\frac{1}{2}k_\mu \nabla^\nu\nabla_\nu A \ . \tag{C.22}$$

Now using $\tilde{g}$ to raise an index on (C.21) we find that

$$\tilde{R}^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}k^{\mu}k_{\nu}e^{-A}\nabla_{\rho}\nabla^{\rho}(e^{A}F) \ , \tag{C.23}$$

where the index on $R_{\mu\nu}$ is raised with $g^{\mu\nu}$ and (C.22) has been used. If we now chose F such that, in addition to (C.18), it obeys

$$\nabla_{\rho}\nabla^{\rho}(e^{A}F) = 0 \ , \tag{C.24}$$

then it follows that

$$\tilde{R}^{\mu}{}_{\nu} = g^{\mu\rho}R_{\rho\nu} \ , \tag{C.25}$$

and a consequence it can be shown that the Einstein tensors of g and $\tilde{g}$ obey the relationship

$$\tilde{g}^{\mu\rho}\tilde{G}_{\rho\nu} = g^{\mu\rho}G_{\rho\nu} \ , \tag{C.26}$$

where $\tilde{G}$ is the Einstein tensor of $\tilde{g}$ and G is the Einstein tensor of g. Due to (C.26) the left hand side of (C.2) remains unchanged, if we first raise an index, under the replacement of g by $\tilde{g}$. This is the critical insight of Garfinkle and Vachaspati and is at the heart of the solution generating transformation.

All that remains for us to do is consider how the index raised form field of the ansatz (C.6) changes. Unfortunately we can no longer use the work of Garfinkle and Vachaspati for this part of the calculation and instead have to rely upon a less elegant technique. To begin with it proves to be more efficient to use a spherical polar coordinate system in the transverse space for (C.6). If this is done the only non-zero, independent, component of the 4 form field strength can be shown to be

$$F_{uv2r} = -\partial_r A_{uv2} \ , \tag{C.27}$$

with A_{uv2} as in (C.6). From this one can show, by brute force, that for the M2 brane solution

$$\tilde{F}^{\mu\nu\rho\sigma} = \tilde{g}^{\mu\tau}\tilde{g}^{\nu\lambda}\tilde{g}^{\rho\chi}\tilde{g}^{\sigma\theta}F_{\tau\lambda\chi\theta} = g^{\mu\tau}g^{\nu\lambda}g^{\rho\chi}g^{\sigma\theta}F_{\tau\lambda\chi\theta} \ , \tag{C.28}$$

with this result we can show that if we consider the field configuration $(\tilde{g}_{\mu\nu}, A_{[3]})$, with $\tilde{g}_{\mu\nu}$ as in (C.14) and $A_{[3]}$ the 3 form appearing in (C.6), then it solves (C.2) and (C.3) as a result of the M2 brane solving these equations.

By direct calculation it can be shown that the $F_{[4]} \wedge F_{[4]}$ term vanishes for the $A_{[3]}$ of the M2 brane solution. As a result of this (C.3) reduces to

$$\partial_\sigma(\sqrt{-g}F^{\sigma\mu\nu\rho}) = 0 \qquad \Longrightarrow \qquad \nabla_\sigma F^{\sigma\mu\nu\rho} = 0 \ , \tag{C.29}$$

which is solved by (C.6). If we now instead consider this equation for our new field configuration $(\tilde{g}, A_{[3]})$ then we wish to evaluate

$$\tilde{\nabla}_\sigma \tilde{F}^{\sigma\mu\nu\rho} \ . \tag{C.30}$$

However, owing to (C.20) and (C.28) this equation is equivalent to (C.29) and hence the form equation is satisfied by our new configuration. If we now raise an index on (C.2) then we shall find that for our new configuration to be a solution we require that

$$\tilde{G}^\mu{}_\nu = \frac{1}{12}(\tilde{F}^{\mu\rho\lambda\sigma}F_{\nu\rho\lambda\sigma} - \frac{1}{8}\delta^\mu{}_\nu \tilde{F}^{\chi\rho\lambda\sigma}F_{\chi\rho\lambda\sigma}) \ , \tag{C.31}$$

where $\tilde{F}^{\mu\nu\lambda\sigma}$ indicates that indices have been raised by making use of the metric $\tilde{g}$. Due to (C.26) and (C.28) this simply reduces to the Einstein equation of the original metric g and hence the Einstein equation is also solved for our new configuration, thus ensuring it is a solution of 11 dimensional supergravity.

However, whilst we have claimed victory and declared that we have a solution we have yet to find it explicitly, we shall now rectify this. We begin by noting that (C.12) can be rewritten as

$$dk_{[1]} = k_{[1]} \wedge d(-A) \ , \tag{C.32}$$

where the minus sign arises to ensure agreement with (C.12). From this we find that

$$A = \log\big(B\big(1 + \frac{q}{r^6}\big)^{-\frac{2}{3}}\big) \ , \tag{C.33}$$

for B a real integration constant. From (C.7) and (C.33) we can see (C.13) clearly holds.

Then, since $k_{[1]} = -\frac{1}{2}(1 + \frac{q}{r^6})^{-\frac{2}{3}}du$, once (C.4) undergoes a Garfinkle Vachaspati transformation, as in (C.14), our new solution takes the form

$$\begin{aligned}
ds^2 &= (1+\frac{q}{r^6})^{-\frac{2}{3}}(-dudv+(dx^2)^2)+(1+\frac{q}{r^6})^{\frac{1}{3}}\delta_{mn}dy^m dy^n+\frac{F}{4}(1+\frac{q}{r^6})^{-\frac{4}{3}}du^2 \ , \\
A_{uv2} &= \pm\frac{1}{2}(1+\frac{q}{r^6})^{-1} \qquad \mu\in\{0,1,2\} \qquad m\in\{3,..,10\} \ , \\
r^2 &= \delta_{mn}y^m y^n \qquad q\in\mathbb{R} \ ,
\end{aligned} \tag{C.34}$$

where F is a function obeying (C.18) and (C.24). The first of these, given that in our coordinate system k takes the form given in (C.7), implies that

$$F = F(u, x^2, y^m) \ . \tag{C.35}$$

The second equation is more complicated to solve. However if we follow [46] and change to a spherical polar coordinate system in the transverse space, then only the metric in (C.6) is modified, and it becomes

$$ds^2 = (1+\frac{q}{r^6})^{-\frac{2}{3}}(-dudv+(dx^2)^2)+(1+\frac{q}{r^6})^{\frac{1}{3}}(dr^2+r^2d\Omega_7^2) \ , \tag{C.36}$$

where $d\Omega_7^2$ is the round metric on S^7.

Now recalling that (C.24) is an equation using the untransformed metric, and defining

$$\tilde{F} = \tilde{F}(u, x^2, r, \theta^\alpha) := e^A F \ , \tag{C.37}$$

with θ^α, $\alpha\in\{1,\ldots,7\}$, coordinates on a patch of the S^7 we can show that $\tilde{F}$ obeys the equation

$$g^{22}\frac{\partial^2\tilde{F}}{\partial(x^2)^2}+g^{rr}(\frac{\partial^2\tilde{F}}{\partial r^2}+\frac{7}{r}\frac{\partial\tilde{F}}{\partial r}+\frac{1}{r^2}D^2_{S^7}(\tilde{F}))=0 \ , \tag{C.38}$$

where g^{22} and g^{rr} are the appropriate components of the inverse of (C.36) and $D^2_{S^7}$ is the Laplacian associated to the round metric on S^7. Now proposing a separable solution to (C.38) of the form

$$\tilde{F} = f(u)X(x^2)P(r)Y(\theta^\alpha) \ , \tag{C.39}$$

we find the two equations

$$\frac{1}{X}\frac{d^2X}{d(x^2)^2}=\lambda, \qquad \frac{g^{rr}}{g^{22}PY}(Y(\frac{d^2P}{dr^2}+\frac{7}{r}\frac{dP}{dr})+\frac{P}{r^2}D^2_{S^7}(Y))=-\lambda \ , \tag{C.40}$$

with $\lambda \in \mathbb{R}$.

The (r, θ^α) problem can also be recast as a separation problem since g^{22} and g^{rr} are only functions of r. Then, making use of the well known eigenvalue problem on S^n, we have the eigenvalue problems

$$D^2_{S^7}(Y) = -l(l+6)Y, \qquad \frac{d^2P}{dr^2} + \frac{7}{r}\frac{dP}{dr} + (\lambda\frac{g^{22}}{g^{rr}} - \frac{l(l+6)}{r^2})P = 0 \;, \tag{C.41}$$

with $l \in \mathbb{N}_0$ and a direct calculation shows that

$$\frac{g^{22}}{g^{rr}} = 1 + \frac{q}{r^6} \;. \tag{C.42}$$

As a result we can see that the radial equation has an irregular singular point at $r = 0$. This rendered it unsolvable by the authors [161] except in the special case where $\lambda = 0$.

In the $\lambda = 0$ case the radial equation of (C.41) is solved by

$$P(r) = a_l r^l + b_l r^{-(l+6)} \;, \tag{C.43}$$

with $a_l, b_l \in \mathbb{R}$. Y is the spherical harmonic with eigenvalue $-l(l+6)$ and the X part of (C.40) is solved by

$$X(x^2) = \tilde{a}_l x^2 + \tilde{b}_l \;, \tag{C.44}$$

where $\tilde{a}_l, \tilde{b}_l \in \mathbb{R}$. Then summing over $l \in \mathbb{N}_0$ we obtain the most general solution to (C.40) and (C.41), in the case where $\lambda = 0$, which takes the form

$$ds^2 = (1 + \frac{q}{r^6})^{-\frac{2}{3}}(-dudv + d(x^2)^2 + (\tilde{a}x^2 + \tilde{b})\sum_{l \geq 0}(a_l(u)r^l + b_l(u)r^{-(l+6)})Y_l(\theta^\alpha)du^2) + (1 + \frac{q}{r^6})^{\frac{1}{3}}\delta_{mn}dy^m dy^n \;,$$

$$A_{uv2} = \pm\frac{1}{2}(1 + \frac{q}{r^6})^{-1} \qquad r^2 = \delta_{mn}y^m y^n \qquad q, a, b, \tilde{a}, \tilde{b} \in \mathbb{R} \;,$$

where the arbitrary function $f(u)$ associated to each of our separable solutions has been absorbed into what were previously the constants a_l and b_l.

It is now interesting to consider the case where $\tilde{a}$ is set to zero in (C.34). If we do this then the solution develops an isometry direction along the x^2 direction of the world volume which means it becomes admissible to the procedure of diagonal dimensional reduction, as discussed in Section 1. For this we take the reduction

[161] Attempts to use asymptotic methods, including the method of dominated balance, were used but didn't yield any results.

ansatz to be

$$d\hat{s}_{11}^2 = e^{-\frac{\phi}{6}} ds_{10}^2 + e^{\frac{4\phi}{3}}((dx^2)^2 + \mathcal{A}_{[1]})^2, \qquad \hat{A}_{[3]} = A_{[3]} + B_{[2]} \wedge dx^2 \ , \tag{C.45}$$

where $d\hat{s}_{11}^2$ and $\hat{A}$ are as in (C.34). This leads us to the 10 dimensional solution

$$\begin{aligned} ds_{10}^2 &= (1+\frac{q}{r^6})^{-\frac{3}{4}}(-dudv + \sum_{l\geq 0}(a_l(u)r^l + b_l(u)r^{-(l+6)})Y_l(\theta^\alpha)du^2) + (1+\frac{q}{r^6})^{-\frac{1}{4}}\delta_{mn}dy^m dy^n \ , \\ \mathcal{A}_{[1]} &= 0 \qquad B_{[2]} = \pm\frac{1}{2}(1+\frac{q}{r^6})^{-1} du \wedge dv \qquad e^{-2\phi} = 1 + \frac{q}{r^6} \ . \end{aligned} \tag{C.46}$$

If we now go to string frame using[162]

$$g^{Ein}_{\check{M}\check{N}} = e^{-\frac{\phi}{2}} g^{str}_{\check{M}\check{N}} \qquad \check{M} \in \{1, \ldots, 10\} \ , \tag{C.47}$$

then the metric in (C.46) becomes

$$ds^2_{string10} = (1+\frac{q}{r^6})^{-1}(-dudv + \sum_{l\geq 0}(a_l(u)r^l + b_l(u)r^{-(l+6)})Y_l(\theta^\alpha)du^2) + \delta_{mn}dy^m dy^n \ . \tag{C.48}$$

If we also perform a gauge transformation to $B_{[2]}$, to alter its asymptotic value [163], we see that (C.46) is a right moving version of the oscillating string solution presented in [23]. The left moving solution is obtained by instead implementing the same procedure but with respect to the killing vector m in (C.7).

So in conclusion we have demonstrated how the solution generating technique of [47] can be applied to the M2 brane solution of [37] and how this solution reduces to the oscillating string solution of [23]. It should be noted that we could attempt to apply this technique to the M5 brane solution of 11 dimensional supergravity [56] as well as other p-brane solutions of supergravity theories.

[162] This is specific to 10 dimensions.

[163] The precise gauge transformation is given by $B'[2] = B_{[2]} - \frac{1}{2}d(udv)$ with $B_{[2]}$ as in (C.46).

D Introduction to Legendre functions and their integrals

In Section 3 the solution of Crampton, Pope and Stelle, CPS, [17] was introduced and the problem of the gravitational spectrum around this background was discussed. In this appendix we shall show how this problem is related to the Legendre differential equation. This realisation leads to a method that allows the normalisation of the transverse wavefunctions appearing in the gravitational spectrum to be determined. We shall also provide details of a technique for the integration of a product of arbitrarily many of these transverse wavefunctions.

D.1 Introduction to the Crampton, Pope and Stelle transverse wavefunction problem and its relation to the Legendre equation

The solutions of CPS are a family of solutions, parametrised by $k \in \mathbb{R}^+$, to Type IIA Supergravity which in string frame[164] take the form

$$\begin{aligned}
ds^2_{10string} &= \eta_{\mu\nu}dx^\mu dx^\nu + dy^2 + \frac{1}{4g^2}\big(d\psi + \mathrm{sech}(2\rho)(d\chi + \cos\theta d\varphi)\big)^2 + \frac{H(\rho)}{g^2\cosh(2\rho)}ds^2_{EH},\\
ds^2_{EH} &= \cosh(2\rho)d\rho^2 + \frac{(\sinh(2\rho))^2}{4\cosh(2\rho)}(d\chi + \cos\theta d\varphi)^2 + \frac{\cosh(2\rho)}{4}(d\theta^2 + \sin^2\theta d\varphi^2),\\
e^\phi &= (H(\rho))^{\frac{1}{2}} \quad , \quad A_{(2)} = \frac{1}{4g^2}(d\chi + \mathrm{sech}(2\rho)d\psi)\wedge\big(d\chi + \cos\theta d\varphi\big),\\
H(\rho) &= \mathrm{sech}(2\rho) - k\log(\tanh(\rho)),
\end{aligned} \tag{D.1}$$

with $g \in \mathbb{R}$, $\mu \in \{0,\ldots,3\}$ parametrises indices for a four dimensional Minkowski space, called the world volume, and $y \in [0,2\pi)$, $\psi \in [0,4\pi)$, $\chi \in [0,2\pi)$, $\theta \in [0,\pi)$, $\phi \in [0,2\pi)$ and $\rho \in [0,\infty)$ denoting coordinates parametrising the space transverse to the world volume, called the transverse space. These solutions have many interesting properties, for details see Section 3, but of particular interest to us is the $k = 0$ solution.

Gravitational fluctuations around the $k = 0$ geometry, using the work of Bachas and Estes [4], B&E, can be considered. With the perturbation being chosen to take the form

$$H_{\mu\nu}(x,\rho) = h_{\mu\nu}(x)\xi(\rho) \quad , \quad \partial^\mu h_{\mu\nu} = 0 \quad , \quad h = \eta^{\mu\nu}h_{\mu\nu} = 0, \tag{D.2}$$

meaning that dependence on all but the ρ coordinate of the transverse space has been removed and only the

[164] String frame is a set of variables related to the usual Einstein frame variables by a conformal transformation of the metric. In ten dimensions this relationship is $g^{Ein}_{MN} = e^{-\frac{\phi}{2}}g^{str}_{MN}$ with $M \in \{0,\ldots,9\}$.

world volume metric has been perturbed. We call the function $\xi(\rho)$ appearing in this equation the transverse wavefunction of the perturbation. A perturbation of this form was found to obey a wave equation

$$\Box_{(4)} H_{\mu\nu} + g^2 \triangle_{rad} H_{\mu\nu} = 0, \tag{D.3}$$

where $\Box_{(4)} = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and

$$\triangle_{rad} = \frac{d^2}{d\rho^2} + 2\coth(2\rho)\frac{d}{d\rho} = \Big(\frac{d}{d\rho} + 2\coth(2\rho)\Big)\frac{d}{d\rho}, \tag{D.4}$$

(D.3), using (D.2), becomes a separable problem leading to the eigenvalue problems

$$\Box_{(4)} h_{\mu\nu} = \lambda h_{\mu\nu} \quad , \quad \triangle_{rad}\xi = -\Lambda\xi \quad \Lambda := \frac{\lambda}{g^2}. \tag{D.5}$$

Our main focus will be on the second of these.

The ξ eigenvalue problem can be converted to a Schödinger equation by the change of function

$$\Psi(\rho, \Lambda) = \sqrt{\sinh(2\rho)}\xi(\rho, \Lambda), \tag{D.6}$$

where we now give the specific eigenvalue as an additional argument to the function. Under this transformation the radial eigenvalue problem takes the form

$$-\frac{d^2\Psi(\Lambda)}{d\rho^2} + (2 - \frac{1}{\tanh^2(2\rho)})\Psi(\Lambda) = \Lambda\Psi(\Lambda), \tag{D.7}$$

where we suppress the ρ dependence of Ψ. This potential is an example of the famous Pöschl-Teller potential [87][165]. There are many excellent references on this type of potential but a few we found useful are [61, 7, 16].

Since the transverse wavefunction problem leads to a Schrödinger equation, we find ourselves firmly into the territory of Hilbert spaces. However, as we know, a Hilbert space is more than just a collection of state vectors, it also requires an inner product structure. So we may ask where this arises from in the current setting. As stated previously, (D.1) is a solution of Type IIA Supergravity in string frame variables. This

[165] A general Pöschl-Teller potential has the form $V(x) = -\frac{\sigma(\sigma+1)}{2}\,\mathrm{sech}^2(x) - \frac{\nu(\nu+1)}{2}\,\mathrm{cosech}^2(x)$ so the case at hand has $\frac{\sigma(\sigma+1)}{2} = 0$ and $\frac{\nu(\nu+1)}{2} = 1$.

theory is defined by the action

$$S=\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-\det(g^{(s)})}e^{-2\phi}(R(g^{(s)})-\frac{1}{12}H_{MNP}H^{MNP}+4\nabla_M\phi\nabla^M\phi), \tag{D.8}$$

where $M\in\{0,\ldots,9\}$, $g^{(s)}$ is the string frame metric, ϕ is the dilaton, $B_{[2]}$ is the NSNS two-form, with field strength $H_{[3]}=dB_{[2]}$, and ∇ is the covariant derivative associated to the Levi-Civita connection of $g^{(s)}$. It now becomes apparent that, if we perturb this action, using techniques similar to those detailed in Appendix A, all terms will come with a copy of $\int d^{10}x\sqrt{-\det(g^{(s)})}e^{-2\phi}$. This automatically gives us the inner product we're looking for. This inner product, due to the product manifold structure of (D.1), has the form

$$\begin{aligned}(\omega,\zeta)&=\int_{M_6}\sqrt{\det(g^{(s)})|_{M_6}}e^{-2\phi}\bar{\omega}\zeta \\ &=\int_0^\infty d\rho\int_0^\pi d\theta\int_0^{2\pi}d\varphi\int_0^{4\pi}d\psi\int_0^{2\pi}d\chi\int_0^{2\pi}dy\frac{\sinh(2\rho)\sin(\theta)}{16g^5}\bar{\omega}\zeta\ ,\end{aligned} \tag{D.9}$$

where $g^{(s)}|_{M_6}$ is the metric on the six dimensional transverse space of (D.1), $\bar{\omega}$ is the complex conjugate of ω and both ω and ζ have functional dependence on the full transverse space. In the case where the transverse wavefunction only depends on ρ, (D.9) reduces to

$$(\omega,\zeta)=\frac{4\pi^2}{g^5}\int_0^\infty d\rho\ \sinh(2\rho)\bar{\omega}(\rho)\zeta(\rho)\ . \tag{D.10}$$

This shows that for Ψ, as defined in (D.6), the correct inner product is given by the usual $L^2((0,\infty))$ inner product, up to an overall normalisation that we can easily deal with by a rescaling.

The Schödinger problem, (D.7), has particularly interesting behaviour as we approach the endpoints of the ρ interval. For example as $\rho\to\infty$

$$-\frac{d^2\Psi(\Lambda)}{d\rho^2}+(2-\Lambda-\frac{1}{\tanh^2(2\rho)})\Psi(\Lambda)=0\sim\frac{d^2\Psi(\Lambda)}{d\rho^2}+(\Lambda-1)\Psi(\Lambda)=0, \tag{D.11}$$

which admits scattering solutions[166] if $1<\Lambda$. Asymptotically these are given by

$$\Psi(\rho,\Lambda)\sim a(\Lambda)e^{i\sqrt{\Lambda-1}\rho}+b(\Lambda)e^{-i\sqrt{\Lambda-1}\rho}, \tag{D.12}$$

where $a(\Lambda)$ and $b(\Lambda)$ are arbitrary constants that are to be fixed by boundary conditions. Note that at large ρ these solutions are behaving as plane waves do, this indicates it's unlikely that they're normalisable in an

[166] These are solutions to the Schrödinger equation with positive energy, where we assume the asymptotic value of the potential is zero.

L^2 sense. While if $\Lambda < 1$ we have the potential for bound states[167]. From a physical point of view the case $\Lambda = 0$ is very attractive as, by (D.5), it will lead to a massless four dimensional graviton.

The other interesting piece of asymptotic behaviour is given by considering the $\rho \to 0$ asymptotic form of the equation

$$\begin{aligned}
&-\frac{d^2\Psi(\Lambda)}{d\rho^2} + (2 - \Lambda - \frac{1}{\tanh^2(2\rho)})\Psi(\Lambda) = 0 \\
&\sim -\frac{d^2\Psi(\Lambda)}{d\rho^2} + (2 - \Lambda - \frac{1}{4\rho^2})\Psi(\Lambda) = 0,
\end{aligned} \tag{D.13}$$

which is the famous $\frac{1}{\rho^2}$ potential of one dimensional quantum mechanics [67, 25]. An excellent review of this problem can be found in [41]. In [41] a careful analysis of the potential $\frac{\alpha}{\rho^2}$ shows that the value $\alpha = \frac{1}{4}$ is very special as it leads to a single normalisable bound state which must, for self-adjointness[168] of the Hamiltonian, obey the boundary condition

$$(\bar{\Psi}\frac{d\omega}{d\rho} - \omega\frac{d\bar{\Psi}}{dx}) \mid_0^\infty = 0, \tag{D.14}$$

where Ψ is our normalisable wavefunction and ω is any element in the self-adjoint domain of the Hamiltonian. This condition is easily dealt with at $\rho \to \infty$ but subtle as $\Psi \to 0$. Note that normalisability of the mode Ψ in fact forces the stronger boundary condition $\Psi \to 0$ as $\rho \to \infty$.

While the analysis of [41] tells us we have a single bound state in the case at hand it also tells us the eigenvalue of this bound state can be chosen arbitrarily[169]. Clearly one would like to select $\Lambda = 0$, however doing this by hand would be very artificial. As discussed in Section 3 CPS coupled an NS5-brane action [6] to the original system. The CPS family of solutions (D.1) is really a solution to this coupled system where $k \in \mathbb{R}^+$ represents the brane charge. By augmenting the eigenvalue problem (D.5) to include this source CPS showed that, for the system to be compatible with the NS5-brane source, the eigenvalue of the normalisable bound state is selected, for preserved supersymmetry reasons, to be $\Lambda = 0$. A simple form for

[167] These are solutions to the Schrödinger equation with negative energy, again assuming the potential is asymptotically zero.

[168] Recall operators are defined on subspaces of the full Hilbert space, $\mathcal{H}$, for example $A : D(A) \subset \mathcal{H} \to \mathcal{H}$. Then the adjoint of A, denoted A^*, is defined to have domain $D(A^*) = \{\omega \in \mathcal{H} | \exists \eta \in \mathcal{H}, \forall \psi \in D(A) : (\omega, A(\psi)) = (\eta, \psi)\}$ we then define the action of A^* by $\eta = A^*(\omega)$. An operator is then self-adjoint if $A = A^*$ which requires $D(A) = D(A^*)$, this subtlety of domains is often glossed over as, often, we deal with essentially self-adjoint operators. The domain on which an operator is self-adjoint is called its self-adjoint domain.

[169] This has to do with the Hamiltonian of the system not being essentially self-adjoint and hence there is no unique self-adjoint, there's in fact a one parameter family of self-adjoint extensions. The particular energy of the bound state is then intimately related to the parameter appearing in the self-adjoint extension we use, for a detailed analysis of this please see [41] for details of this relationship.

this normalisable wavefunction[170] was also found

$$\Psi(\rho,0) = \frac{2\sqrt{3}}{\pi}\sqrt{\sinh(2\rho)}\xi(\rho,0) \quad , \quad \xi(\rho,0) = \log(\tanh(\rho)), \tag{D.15}$$

where the wavefunction has been normalised so that its norm with respect to the $L^2((0,\infty))$ norm is one.

So far we've only recounted a tale already told by others, but now it's time to start our own chapter in this story. CPS attractively dealt with the bound state spectrum of the problem and also ascertained that there exists a continuum of scattering states separated from the single $\Lambda = 0$ state by a gap, which in the four dimensional graviton spectrum corresponds to a mass gap. For the case where the brane charge is set to zero[171] the spectrum has its next eigenvalue at $\Lambda = 1$. It turns out that the problem of solving

$$\frac{d^2\xi(\Lambda)}{d\rho^2} + 2\coth(2\rho)\frac{d\xi(\Lambda)}{d\rho} + \Lambda\xi(\Lambda) = 0, \tag{D.16}$$

for a scattering state is made much simpler if we perform the substitution $z = \cosh(2\rho)$, this leads to

$$(1-z^2)\frac{d^2\zeta(\Lambda)}{dz^2} - 2z\frac{d\zeta(\Lambda)}{dz} - \frac{\Lambda}{4}\zeta(\Lambda) = 0 \quad , \quad z \in (1,\infty), \tag{D.17}$$

where $\zeta(\Lambda) = \zeta(z,\Lambda) = \xi(\frac{1}{2}\cosh^{-1}(z),\Lambda)$. Upon defining $-\frac{\Lambda}{4} = \nu(\nu+1)$ this becomes Legendre's equation [1] with

$$\nu_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{1-\Lambda} \quad , \quad \nu_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{1-\Lambda}, \tag{D.18}$$

where we note that if $1 < \Lambda$ then ν_1 and ν_2 are complex numbers and are related by

$$\nu_1 = -\nu_2 - 1 \ . \tag{D.19}$$

It is this conversion of the scattering state equation to a Legendre equation, where (D.19) holds, that allows for its solution and for the overlap integrals of multiple scattering and bound states to be calculated. As stated in Section 1 such overlap integrals are required for one to be able to perform a Kaluza-Klein expansion of theories in a meaningful way. Finally note that under the transformation to the z variable the inner product (D.10) becomes

$$(\omega,\zeta) = \frac{2\pi^4}{g^5}\int_1^\infty dz\bar{\omega}(z)\zeta(z) \ . \tag{D.20}$$

[170] The full solution of (D.7) is in fact $\Psi(\rho,0) = \sqrt{\sinh(2\rho)}(C_1 + C_2\log(\tanh(\rho))$ but normalisability and normalisation to unity fix the two arbitrary constants C_1 and C_2.

[171] Which actually leads to a smooth decoupling of the brane source from the system see [17].

D.2 Introduction to Legendre functions

In the last Section we saw that the CPS transverse wavefunction problem could be recast into the form of a Legendre equation for each eigenvalue of the original problem. In this Section we shall provide an introduction to the solutions of this equation and the properties they possess, our main focus being on properties that aid our goal of performing overlap integrals of solutions to (D.16). The results collected here are in no way our own and have mostly been obtained from the Digital Library of Mathematical Functions [80] and [53].

The Legendre differential equation is

$$(1-z^2)\frac{d^2 f}{dz^2} - 2z\frac{df}{dz} + \nu(\nu+1)f = 0 \quad , \quad \nu \in \mathbb{C}, \tag{D.21}$$

which is a linear second order ODE with 3 regular singular points at $z \in \{1, -1, \infty\}$, which are branch points of the solutions of the equation. Any solution of this equation is called a Legendre function. This equation is usually first encountered when studying the quantum mechanical problem of a spherically symmetric potential in three dimensions [172]. The main difference between that and the current context is that in the spherically symmetric potential problem $\nu \in \mathbb{N}_0$ and the solutions to the equation are then polynomials. However in the current context $\nu \in \mathbb{C}$ and the solutions are usually described in terms of hypergeometric functions.

For our purposes we note that there are two useful pairs of bases for solutions to (D.21). The first pair is denoted by

$$P_\nu(z) \quad , \quad Q_\nu(z) \quad , \quad z \in (-1, 1), \tag{D.22}$$

these functions can be analytically continued to the cut plane $\mathbb{C} \setminus ((-\infty, 1] \cup [1, \infty))$ where the branch cuts are required in order to render these analytic continuations single valued. Of greater importance to us, given the range of z in (D.17), is the pair

$$\mathcal{P}_\nu(z) \quad , \quad \mathcal{Q}_\nu(z) \quad , \quad z \in \mathbb{C} \setminus (-\infty, 1]. \tag{D.23}$$

Where these are analytic continuations of a pair of functions originally defined for $z \in (1, \infty)$. The specific functional forms of these solutions won't be required for the calculation of overlap integrals but for those interested we suggest consulting [53] where their definition, in terms of hypergeometric functions, is given.

[172] See for example [55].

The functions appearing in (D.22) and (D.23) are often called Legendre functions of the first kind, P_ν and $\mathcal{P}_\nu$, and Legendre functions of the second kind, Q_ν and $\mathcal{Q}_\nu$.

Having identified (D.16) as a hidden Legendre equation we have now obtained the scattering solutions to it as they're just appropriate linear combinations of (D.23). So our problem of finding overlap integrals for our Schrödinger problem can neatly be restated as finding the overlap integrals of Legendre functions.

In order to calculate the required overlap integrals we shall need several relationships between Legendre functions and also their asymptotics as we approach the various branch points, which we now collect.

As $|z| \to 1^-$:

$$\begin{aligned} &P_\nu(z) \sim 1, \\ &Q_\nu(z) \sim \frac{1}{2}\log(\frac{2}{1-z}) - \gamma - \psi(\nu+1) \qquad \nu \notin \mathbb{Z}_{<0}, \end{aligned} \tag{D.24}$$

As $|z| \to 1^+$:

$$\begin{aligned} &\mathcal{P}_\nu(z) \sim 1, \\ &\mathcal{Q}_\nu(z) \sim -\frac{1}{2}\log(z-1) + \frac{1}{2}\log(2) - \gamma - \psi(\nu+1) \qquad \nu \notin \mathbb{Z}_{<0}, \end{aligned} \tag{D.25}$$

As $|z| \to \infty$:

$$\begin{aligned} &\mathcal{P}_\nu(z) \sim \frac{\Gamma(\nu+\frac{1}{2})}{\pi^{\frac{1}{2}}\Gamma(\nu+1)}(2z)^\nu \qquad Re(\nu) > -\frac{1}{2} \qquad \nu \notin \mathbb{Z}_{<0}, \\ &\mathcal{P}_\nu(z) \sim \frac{\Gamma(-\nu-\frac{1}{2})}{\pi^{\frac{1}{2}}\Gamma(-\nu)(2z)^{\nu+1}} \qquad Re(\nu) < -\frac{1}{2} \qquad \nu \notin \mathbb{N}_0, \\ &\mathcal{P}_{-\frac{1}{2}}(z) \sim \frac{1}{\Gamma(\frac{1}{2}}(\frac{2}{\pi z})^{\frac{1}{2}}\log(z), \\ &\mathcal{Q}_\nu(z) \sim \frac{\pi^{\frac{1}{2}}\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})(2z)^{\nu+1}} \qquad \nu \notin \{-1, -\frac{3}{2}, -2, -\frac{5}{2}, \ldots\}, \end{aligned} \tag{D.26}$$

where γ is the Euler-Mascheroni constant and $\psi(\sigma)$ is the digamma function[173]. For now the $z \to -1^+$ asymptotics have been omitted as we require the further results to be able to state them.

There also exist many useful relationships between Legendre functions which often go under the name of

[173]Which is defined as $\frac{d}{dz}\log(\Gamma(z))$.

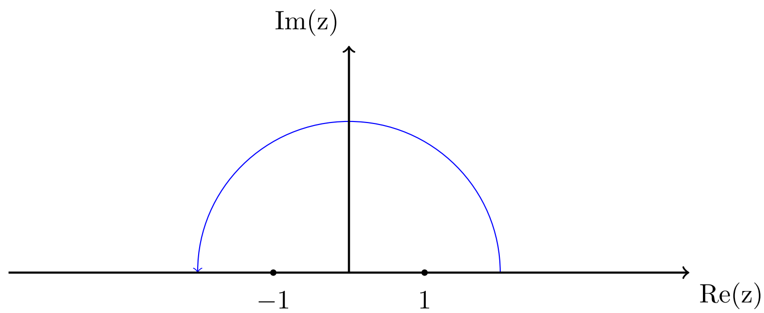

Figure 1: Contour enclosing branch points at 1 and -1.

connection formulae. Of particular use to us will be the specific relationships

$$P_\nu(-z) = -\frac{2}{\pi}\sin(\nu\pi)Q_\nu(z) + \cos(\nu\pi)P_\nu(z), \tag{D.27}$$

$$Q_\nu(-z) = -\frac{2}{\pi}\sin(\nu\pi)P_\nu(z) - \cos(\nu\pi)Q_\nu(z), \tag{D.28}$$

$$\cos(\nu\pi)\mathcal{P}_\nu(z) = -\frac{1}{\Gamma(-\nu)\Gamma(\nu+1)}\mathcal{Q}_\nu(z) + \frac{1}{\Gamma(\nu+1)^2}\mathcal{Q}_{-\nu-1}(z) \qquad \nu \notin \mathbb{Z}, \tag{D.29}$$

using (D.27) and (D.28) along with (D.24) the $z \to -1^+$ asymptotics can be obtained by direct substitution.

Almost any attempt to perform integrals along the axis is aided by an excursion into the complex plane, and our case is no exception. In order to use this we shall have to be able to relate our branches of the Legendre functions in (D.23) to other branches.

The first case to consider is what happens when we follow a contour of the type shown in figure 1, for which it can be shown that

$$\mathcal{P}_\nu(ze^{i\pi}) = e^{i\nu\pi}\mathcal{P}_\nu(z) - \frac{2}{\pi}\sin(\pi\nu)\mathcal{Q}_\nu(z) \quad , \quad \nu \notin \mathbb{Z}, \tag{D.30}$$

$$\mathcal{Q}_\nu(ze^{i\pi}) = -e^{-i\nu\pi}\mathcal{Q}_\nu(z), \tag{D.31}$$

where $\mathcal{P}_\nu(ze^{i\pi})$ and $\mathcal{Q}_\nu(ze^{i\pi})$ denote the branches obtained from (D.23) by following the contour in figure 1.

Also of interest to us will be the case of a contour of the form shown in figure 2 where we aim to find the value of the functions on the branch cut we've implemented. In this case we find that

$$P_\nu(x) = \mathcal{P}_\nu(x+i\epsilon) \quad , \quad Q_\nu(x) = \mathcal{Q}_\nu(x+i\epsilon) + \frac{i\pi}{2}\mathcal{P}_\nu(x+i\epsilon), \tag{D.32}$$

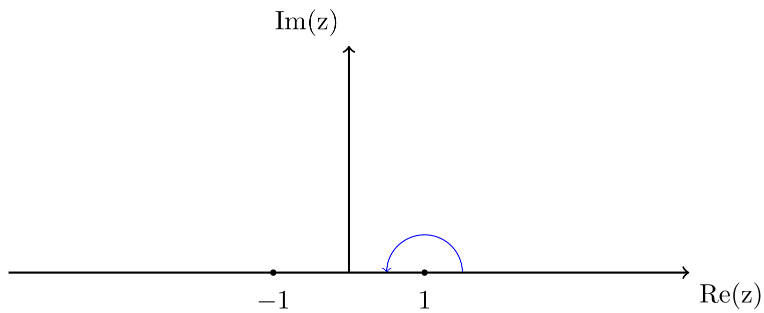

Figure 2: Contour enclosing branch point at 1 but not the branch point at -1.

where ϵ is positive and tends to zero so we're approaching the cut from above. Note a similar result can be obtained for a contour in the negative sense that approaches the branch cut in $\mathcal{P}_\nu$ and $\mathcal{Q}_\nu$.

Finally we change pace a little and consider an object called the $3j$-symbol. In all formulae to follow $2j_1, 2j_2, 2j_3 \in \mathbb{N}_0$. The $3j$-symbol is often denoted by

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} , \tag{D.33}$$

where $m_i \in \{-j_i, -j_i+1, \ldots, j_i-1, j_i\}$ $i \in \{1,2,3\}$. Whilst this may seem a bit daunting at first it's actually just a cousin of the Clebsch-Gordon coefficient, $(j_1, m_1, j_2, m_2 | j_1, j_2, j_3, m_3)$, of angular momentum fame

$$\Big(j_1, m_1, j_2, m_2 | j_1, j_2, j_3, m_3\Big) = (-1)^{j_1 - j_2 + m_3}(2j_3+1)^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} .$$

The $3j$-symbol is defined to be zero if either of the following conditions fails to hold

$$\forall i, j, k \in \{1,2,3\} \quad , \quad |j_i - j_j| \leqslant j_k \leqslant j_i + j_j, \tag{D.34}$$

$$m_1 + m_2 + m_3 = 0. \tag{D.35}$$

In cases where both (D.34) and (D.35) hold the $3j$-symbol becomes a finite sum, see [80] for the exact form. If all of the m_i are zero then this sum takes on a particularly simple form.

It can be shown that the $3j$-symbol has the following symmetry

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} . \tag{D.36}$$

If we define $J = j_1 + j_2 + j_3$ then it can be shown that odd permutations of the above result pick up a factor of $(-1)^J$.

In addition, the $3j$-symbol also obeys the very useful orthogonality relations

$$\sum_{m_1,m_2} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \delta_{j_3 j_3'} \delta_{m_3 m_3'} \ , \tag{D.37}$$

$$\sum_{j_3,m_3} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta_{m_1 m_1'} \delta_{m_2 m_2'} \ , \tag{D.38}$$

$$\sum_{j_1,j_2,j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 1 \ . \tag{D.39}$$

Given that we've been so focused on Legendre functions our little excursion into the world of the $3j$-symbol may seem a little off the path. However it turns out that if we have two Legendre polynomials then their product can neatly be expanded[174] in the following manner

$$P_{l_1}(x) P_{l_2}(x) = \sum_l (2l+1) \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix}^2 P_l(x) \ , \tag{D.40}$$

where l runs over values until (D.34) is violated and $l_1, l_2 \in \mathbb{N}_0$. This concludes the collection of results required for us to be able to calculate the integral of multiple Legendre functions.

D.3 Integrals of Legendre functions

Having assembled all of the players we can finally begin the task of calculating the form of integrals of Legendre functions. Recall we want these integrals as they can be used to construct the overlap integrals we believe are required for our EFT treatment of the CPS gravitational problem. Due to the form of the integral in (D.20) it's clear that the integral of Legendre functions we want is

$$J^{(m,n)}(\mu_1, \ldots, \mu_m | \nu_1, \ldots, \nu_n) := \int_1^\infty dz \, \mathcal{P}_{\mu_1}(z) \ldots \mathcal{P}_{\mu_m}(z) \mathcal{Q}_{\nu_1}(z) \ldots \mathcal{Q}_{\nu_n}(z) \ . \tag{D.41}$$

[174] As the Legendre polynomials form a basis, on the interval $(-1, 1)$, for precise details on the types of functions that the Legendre polynomials form a basis for please refer to the Section on orthogonal polynomials in [80].

however this integral is still a bit beyond us at this stage and we'll have to collect some more basic results to begin with.

The first thing to note is that [80] provides us with the following integrals

$$I^{(2,0)}(\mu_1,\mu_2) := \int_{-1}^{1} dx\ P_{\mu_1}(x)P_{\mu_2}(x) \qquad \mu_2 \notin \{\mu_1, -\mu_1-1\} \tag{D.42}$$
$$= \frac{4\sin(\mu_1\pi)\sin(\mu_2\pi)(H(\mu_1)-H(\mu_2)) + 2\pi\sin((\mu_2-\mu_1)\pi)}{\pi^2(\mu_2-\mu_1)(\mu_2+\mu_1+1)} \ ,$$

$$I^{(1,1)}(\mu|\nu) := \int_{-1}^{1} dx\ P_{\mu}(x)Q_{\nu}(x) \qquad Re(\mu), Re(\nu) > 0, \qquad \mu \neq \nu \tag{D.43}$$
$$= \frac{2\sin(\mu\pi)\cos(\nu\pi)(H(\mu)-H(\nu)) + \pi\cos((\nu-\mu)\pi) - \pi}{\pi(\nu-\mu)(\nu+\mu+1)} \ ,$$

$$I^{(0,2)}(\nu_1,\nu_2) := \int_{-1}^{1} dx\ Q_{\nu_1}(x)Q_{\nu_2}(x) \qquad \nu_2 \notin \{\nu_1, -\nu_1-1\} \qquad \nu_1 \notin \mathbb{Z}_{<0} \tag{D.44}$$
$$= \frac{(H(\nu_1)-H(\nu_2))(1+\cos(\nu_1\pi)\cos(\nu_2\pi)) + \frac{1}{2}\pi\sin((\nu_2-\nu_1)\pi)}{(\nu_2-\nu_1)(\nu_2+\nu_1+1)} \ ,$$

where $H(n) = \psi(n+1) - \gamma$ is the nth harmonic number.

Next let's take $m_1, m_2, m_3 \in \mathbb{N}_0$ and assume, without loss of generality, that $m_1 \leqslant m_2 \leqslant m_3$ and then consider the triple integral

$$I^{(3,0)}(m_1,m_2,m_3) := \int_{-1}^{1} dx\ P_{m_1}(x)P_{m_2}(x)P_{m_3}(x) \ . \tag{D.45}$$

To evaluate this integral use (D.40) to obtain

$$I^{(3,0)}(m_1,m_2,m_3) = \sum_{j=m_3-m_1}^{m_1+m_3} (2j+1) \begin{pmatrix} m_1 & m_3 & j \\ 0 & 0 & 0 \end{pmatrix}^2 \int_{-1}^{1} dx\ P_j(x)P_{m_2}(x) \ ,$$

which, using (D.42), yields

$$I^{(3,0)}(m_1,m_2,m_3) = 2 \begin{pmatrix} m_1 & m_3 & m_2 \\ 0 & 0 & 0 \end{pmatrix}^2 \ . \tag{D.46}$$

Note we may worry that this result is dependent on the pair of the m_i we choose to apply (D.40) to, but

(D.36) ensures this isn't the case.

We can now consider the next case where $m_1, m_2, m_3, m_4 \in \mathbb{N}_0$ and assume that $m_1 \leqslant m_2 \leqslant m_3 \leqslant m_4$. If we then consider the integral

$$I^{(4,0)}(m_1, m_2, m_3, m_4) := \int_{-1}^{1} dx\ P_{m_1}(x) P_{m_2}(x) P_{m_3}(x) P_{m_4}(x)\ , \tag{D.47}$$

using a similar method to before we can show that

$$I^{(4,0)}(m_1, m_2, m_3, m_4) = \int_{-1}^{1} dx \sum_{j=m_2-m_1}^{m_2+m_1} \sum_{l=m_4-m_3}^{m_4+m_3} (2j+1)(2l+1) \begin{pmatrix} m_1 & m_2 & j \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} m_4 & m_3 & l \\ 0 & 0 & 0 \end{pmatrix}^2 P_j(x) P_l(x)\ ,$$

and since the sums are finite we commute them with the integral and then perform the integral of the Legendre polynomials using (D.42), which gives us a δ_{jl}. This enables us to evaluate the integral as

$$I^{(4,0)}(m_1, m_2, m_3, m_4) = \sum_{j=m_2-m_1}^{m_2+m_1} \begin{pmatrix} m_1 & m_2 & j \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} m_4 & m_3 & j \\ 0 & 0 & 0 \end{pmatrix}^2 . \tag{D.48}$$

It now becomes transparent that if we have $m_1, \ldots, m_n \in \mathbb{N}_0$ then the integral

$$I^{(n,0)}(m_1, \ldots, m_n) := \int_{-1}^{1} dx\ P_{m_1}(x) \ldots P_{m_1}(x)\ , \tag{D.49}$$

can be obtained by similar methods.

Next, we'd like to look at evaluating integrals with Legendre functions instead of polynomials. To do this recall that on $(-1, 1)$ the set $\{P_n | n \in \mathbb{N}_0\}$ forms a complete basis, and so we can perform the following expansions

$$P_\mu(x) = \sum_{n=0}^{\infty} a_n(\mu) P_n(x) \quad , \quad a_n(\mu) = \frac{2n+1}{2} I^{(2,0)}(\mu, n), \tag{D.50}$$

$$Q_\nu(x) = \sum_{n=0}^{\infty} b_n(\nu) P_n(x) \quad , \quad b_n(\nu) = \frac{2n+1}{2} I^{(1,1)}(n|\nu). \tag{D.51}$$

Now consider an integral of the form

$$I^{(m,n)}(\mu_1, \ldots, \mu_m | \nu_1, \ldots, \nu_n) := \int_{-1}^{1} dx\ P_{\mu_1}(x) \ldots P_{\mu_m}(x) Q_{\nu_1}(x) \ldots Q_{\nu_n}(x)\ . \tag{D.52}$$

If we use the expansions (D.50) and (D.51) and then commute the integrals and infinite sums[175] we obtain

$$I^{(m,n)}(\mu_1,\ldots,\mu_m|\nu_1,\ldots,\nu_n) = \sum_{i_1,\ldots,i_m=0}^{\infty}\sum_{j_1,\ldots,j_n=0}^{\infty}(\frac{2i_1+1}{2})\ldots(\frac{2i_m+1}{2})(\frac{2j_1+1}{2})\ldots(\frac{2j_n+1}{2})$$

$$\times I^{(2,0)}(\mu_1,i_1)\ldots I^{(2,0)}(\mu_m,i_m)I^{(1,1)}(j_1|\nu_1)\ldots I^{(1,1)}(j_n|\nu_n) \tag{D.53}$$

$$\times\int_{-1}^{1}dx\; P_{i_1}(x)\ldots P_{i_n}(x)P_{j_1}(x)\ldots P_{j_n}(x)\;,$$

where the final integral is of the type given in (D.49).

So after all of our work we've finally managed to evaluate a collection of integrals of Legendre functions. However, they're not the ones we originally wanted. Unfortunately the basis argument we used to compute the last set of integrals won't work again as the domain of (D.41) isn't a region where this basis is admissible. However recall that $\mathcal{P}_\mu$ and $\mathcal{Q}_\nu$ are analytic on the entire complex plane except at $\{1,-1,\infty\}$ and along the branch cut $(-\infty,1]$. This suggests that a new approach using contour integrals may be of use. So let's consider a contour of the form shown in figure 3. This contour contains none of the poles of the Legendre functions and hence by Cauchy's theorem

$$\oint_C dz\; \mathcal{P}_{\mu_1}(z)\ldots\mathcal{P}_{\mu_m}(z)\mathcal{Q}_{\nu_1}(z)\ldots\mathcal{Q}_{\nu_n}(z) = 0\;. \tag{D.54}$$

It's now our goal to evaluate the pieces of this integral along each of the curves in figure 3.

To begin with consider

$$I_1 := \int_{\Gamma_\epsilon^+} dz\; \mathcal{P}_{\mu_1}(z)\ldots\mathcal{P}_{\mu_m}(z)\mathcal{Q}_{\nu_1}(z)\ldots\mathcal{Q}_{\nu_n}(z)\;. \tag{D.55}$$

If we set $z-1=\epsilon e^{i\theta}$, take the limit $\epsilon\to 0$ and use the asymptotic forms (D.25) then we find that

$$I_1 = \lim_{\epsilon\to 0} i\epsilon\int_{\pi}^{0} d\theta\; e^{i\theta}(-\frac{1}{2}\log(\epsilon e^{i\theta})+\frac{1}{2}\log(2)-\gamma-\psi(\nu_1+1))\ldots(-\frac{1}{2}\log(\epsilon e^{i\theta})+\frac{1}{2}\log(2)-\gamma-\psi(\nu_n+1))\;,$$

which evaluates to zero as $\lim_{\epsilon\to 0}\left(\epsilon(\log(\epsilon))^n\right)=0$. A similar treatment can be used to show that the integral

$$I_2 := \int_{\Gamma_\epsilon^-} dz\; \mathcal{P}_{\mu_1}(z)\ldots\mathcal{P}_{\mu_m}(z)\mathcal{Q}_{\nu_1}(z)\ldots\mathcal{Q}_{\nu_n}(z)\;, \tag{D.56}$$

[175] Which can be done due to uniform convergence of the expansions [80] - see the Section on orthogonal polynomials.

is also zero.

We next consider the final semicircular curve

$$I_3 := \int_{\Gamma_R} dz\ \mathcal{P}_{\mu_1}(z)\dots\mathcal{P}_{\mu_m}(z)\mathcal{Q}_{\nu_1}(z)\dots\mathcal{Q}_{\nu_n}(z)\ , \tag{D.57}$$

substituting $z = Re^{i\theta}$ and using (D.26) where $A = \{1, \dots, \alpha \leqslant m\}$ such that $\forall\tilde{\alpha} \in A,\ \ Re(\mu_{\tilde{\alpha}}) > -\frac{1}{2}$ and $B = \{\alpha+1, \dots, m\}$ such that $\forall\beta \in B\ \ Re(\mu_\beta) < -\frac{1}{2}$ shows that we get a factor of

$$R^{1+\mu_1+\dots+\mu_\alpha-(\mu_{\alpha+1}+1)-\dots-(\mu_m+1)-(\nu_1+1)-\dots-(\nu_n+1)}\ .$$

If we consider the $R \to \infty$ limit and

$$Re\Big(1+\mu_1+\dots+\mu_\alpha-(\mu_{\alpha+1}+1)-\dots-(\mu_m+1)-(\nu_1+1)-\dots-(\nu_n+1)\Big) < 0\ , \tag{D.58}$$

the contribution of this integral will tend to zero. Note this argument requires the following restrictions to hold $\forall\tilde{\alpha} \in A,\ \ \mu_{\tilde{\alpha}} \notin \mathbb{Z}_{<0}$, $\forall\beta \in B,\ \ \mu_\beta \notin \mathbb{N}_0$ and $\nu_1, \dots, \nu_n \notin \{-\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \dots\}$ and the special case of any of the $\mu_i = -\frac{1}{2}$ has not been included.

So in cases where (D.58) holds we've argued away all of the contributions from the semicircular contours in figure 3. The integral from $(1,\infty)$ is just (D.41) and hence is the integral we want. We now deal with the integral on $(-1,1)$ which is just slightly above the branch cut in our functions

$$I_4 := \int_{-1}^{1} dz\ \mathcal{P}_{\mu_1}(z)\dots\mathcal{P}_{\mu_m}(z)\mathcal{Q}_{\nu_1}(z)\dots\mathcal{Q}_{\nu_n}\ . \tag{D.59}$$

To evaluate this integral we make use of (D.32), which allows us to write (D.59) as

$$I_4 = \int_{-1}^{1} dz\ P_{\mu_1}(z)\dots P_{\mu_m}(z)\Big(Q_{\nu_1}(z) - \frac{i\pi}{2}P_{\nu_1}(z)\Big)\dots\Big(Q_{\nu_n}(z) - \frac{i\pi}{2}P_{\nu_n}(z)\Big)\ . \tag{D.60}$$

Now if we just multiply this out and commute the finite sums with our integrals, then we can use (D.53) to evaluate the integral as

$$\begin{aligned} I_4 =& I^{(m,n)}(\mu_1,\dots,\mu_m|\nu_1,\dots,\nu_n) - \frac{i\pi}{2}\sum_{i=1}^{n} I^{(m+1,n-1)}(\nu_i,\mu_1,\dots,\mu_m|\nu_1,\dots,\check{\nu_i},\dots,\nu_n) \\ &+ (\frac{-i\pi}{2})^2\sum_{\substack{i,j\\ i<j}} I^{(m+2,n-2)}(\nu_i,\nu_j,\mu_1,\dots,\mu_m|\nu_1,\dots,\check{\nu_i},\dots,\check{\nu_j},\dots,\nu_n) + \dots + (\frac{-i\pi}{2})^n I^{(m+n,0)}(\nu_1,\dots,\nu_n,\mu_1\dots,\mu_m|)\ , \end{aligned} \tag{D.61}$$

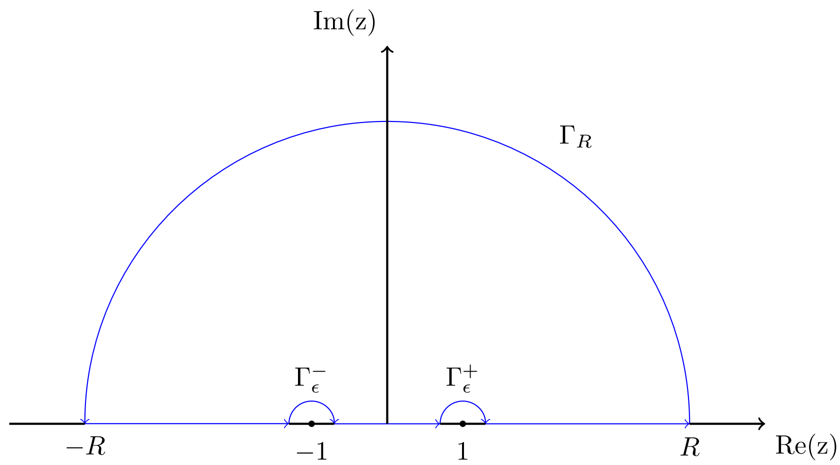

Figure 3: Contour C used for evaluation of integrals of multiple Legendre functions. Integrals along $(-\infty, 1]$ are evaluated in the limit of approaching the real axis from above.

where $\check{\nu}$ denotes that the index ν is omitted.

This leaves one final integral to evaluate

$$I_5 = \int_{-\infty}^{-1} dz\, \mathcal{P}_{\mu_1}(z) \dots \mathcal{P}_{\mu_m}(z) \mathcal{Q}_{\nu_1}(z) \dots \mathcal{Q}_{\nu_n}(z)\ . \tag{D.62}$$

We begin with the substitution $\tilde{z} = e^{-i\pi} z$, which leads to

$$I_5 = \int_{1}^{\infty} d\tilde{z}\, \mathcal{P}_{\mu_1}(e^{i\pi}\tilde{z}) \dots \mathcal{P}_{\mu_m}(e^{i\pi}\tilde{z}) \mathcal{Q}_{\nu_1}(e^{i\pi}\tilde{z}) \dots \mathcal{Q}_{\nu_n}(e^{i\pi}\tilde{z})\ .$$

We now see that we can use (D.30) and (D.31) to obtain

$$\begin{aligned} I_5 = \int_{1}^{\infty} d\tilde{z}\, \bigg(& \Big(e^{i\mu_1 \pi} \mathcal{P}_{\mu_1}(\tilde{z}) - \frac{2}{\pi} \sin(\pi\mu_1) \mathcal{Q}_{\mu_1}(\tilde{z}) \Big) \dots \Big(e^{i\mu_m \pi} \mathcal{P}_{\mu_m}(\tilde{z}) - \frac{2}{\pi} \sin(\pi\mu_m) \mathcal{Q}_{\mu_m}(\tilde{z}) \Big) \\ & \times \Big(- e^{-i\nu_1 \pi} \mathcal{Q}_{\nu_1}(\tilde{z}) \Big) \dots \Big(- e^{-i\nu_n \pi} \mathcal{Q}_{\nu_n}(\tilde{z}) \Big) \bigg)\ , \end{aligned}$$

which allows us to calculate the result

$$
\begin{aligned}
I_5 =&(-1)^n e^{i\pi(\mu_1+\ldots+\mu_m-\nu_1-\ldots-\nu_n)} J^{(m,n)}(\mu_1,\ldots,\mu_m|\nu_1,\ldots,\nu_n) \\
&+(-1)^n(-\frac{2}{\pi})\sum_{i=1}^{m}\sin(\mu_i\pi)e^{i\pi(\mu_1+\ldots+\check{\mu}_i+\ldots+\mu_m-\nu_1-\ldots-\nu_n)}J^{(m-1,n+1)}(\mu_1,\ldots,\check{\mu}_i,\ldots,\mu_m|\mu_i,\nu_1,\ldots,\nu_n) \\
&+(-1)^n(-\frac{2}{\pi})^2\sum_{\substack{i,j\\ i<j}}^{m}\sin(\mu_i\pi)\sin(\mu_j\pi)e^{i\pi(\mu_1+\ldots+\check{\mu}_i+\ldots+\check{\mu}_j\ldots+\mu_m-\nu_1-\ldots-\nu_n)}\times \\
&J^{(m-2,n+2)}(\mu_1,\ldots,\check{\mu}_i,\ldots,\check{\mu}_j,\ldots,\mu_m|\mu_i,\mu_j,\nu_1,\ldots,\nu_n) \\
&+\ldots+(-1)^n(-\frac{2}{\pi})^n e^{-i\pi(\nu_1+\ldots+\nu_n)}\prod_{i=1}^{m}\sin(\mu_i\pi)J^{(0,m+n)}(|\mu_1,\ldots,\mu_m,\nu_1,\ldots,\nu_n)\ .
\end{aligned}
\tag{D.63}
$$

Finally we have all of the contributions to the contour integral formally evaluated. So let's see which integrals these contours actually require in order for an explicit evaluation to be carried out

$$
\begin{aligned}
I_4 \quad &: \quad I^{(m,n)}, I^{(m+1,n-1)}, \ldots, I^{(m+n,0)}, \\
I_5 \quad &: \quad J^{(m,n)}, J^{(m-1,n+1)}, \ldots, J^{(0,m+n)}, \\
I_6 \quad &: \quad J^{(m,n)},
\end{aligned}
\tag{D.64}
$$

where we have dropped the explicit μ and ν labels for ease and the symbol : indicates the integral to the left is a linear combination of the integrals to the right. We firstly note that the terms required to evaluate I_4 are given by (D.53). Next we see that I_5 and I_6 both contain the integral we wish to evaluate, namely (D.41). However I_5 also contains other integrals of the form of (D.41) but with increasing numbers of $\mathcal{Q}_\nu$ functions. The trick is to note that if we do an integral of the form (D.41) with $m = 0$ and n arbitrary then I_5 will just contain $J^{(0,n)}$ and so can be combined easily with I_6 and hence can be evaluated using (D.54). Then we could perform the same sort of calculation but with $m = 1$ and then I_5 will require the integral $J^{(0,n+1)}$ to be evaluated but we've already done that in the previous step. So it seems that the integrals we require can be done in a recursive fashion. In order to make the resulting recursive formula more manageable we define

$$\hat{I}_5 =(-1)^n(-\frac{2}{\pi})\sum_{i=1}^{m}\sin(\mu_i\pi)e^{i\pi(\mu_1+\ldots+\check{\mu}_i+\ldots+\mu_m-\nu_1-\ldots-\nu_n)}J^{(m-1,n+1)}(\mu_1,\ldots,\check{\mu}_i,\ldots,\mu_m|\mu_i,\nu_1,\ldots,\nu_n)$$

$$+(-1)^n(-\frac{2}{\pi})^2\sum_{\substack{i,j\\ i<j}}^{m}\sin(\mu_i\pi)\sin(\mu_j\pi)e^{i\pi(\mu_1+\ldots+\check{\mu}_i+\ldots+\check{\mu}_j\ldots+\mu_m-\nu_1-\ldots-\nu_n)}\times$$
$$J^{(m-2,n+2)}(\mu_1,\ldots,\check{\mu}_i,\ldots,\check{\mu}_j,\ldots,\mu_m|\mu_i,\mu_j,\nu_1,\ldots,\nu_n)$$
$$+\ldots+(-1)^n(-\frac{2}{\pi})^n e^{-i\pi(\nu_1+\ldots+\nu_n)}\prod_{i=1}^{m}\sin(\mu_i\pi)J^{(0,m+n)}(|\mu_1,\ldots,\mu_m,\nu_1,\ldots,\nu_n)\ . \tag{D.65}$$

With this we finally obtain that

$$J^{(m,n)}(\mu_1,\ldots,\mu_m|\nu_1,\ldots,\nu_n)=-\frac{I_4+\hat{I}_5}{(-1)^n e^{i\pi(\mu_1+\ldots+\mu_m-\nu_1-\ldots-\nu_n)}+1}\ , \tag{D.66}$$

where we require (D.58) to hold and the restrictions given underneath it to also hold.

It can be seen that (D.66) cannot be used to evaluate

$$J^{(0,n)}(|0,\ldots,0)=\int_1^\infty dz\ \mathcal{Q}_0\ldots\mathcal{Q}_0\ ,$$

when n is an odd integer. Fortunately such integrals are already known and can be looked up [53]

$$J^{(0,n)}(|0,\ldots,0)=\frac{n!}{2^{n-1}}\zeta(n)\ , \tag{D.67}$$

with $\zeta(n)$ the Riemann zeta function. A simple proof of this comes from noting that[176]

$$\mathcal{Q}_0(z)=\frac{1}{2}\log\left(\frac{z+1}{z-1}\right)=\coth^{-1}(z)\quad,\quad |z|>1\ . \tag{D.68}$$

If we perform the substitution $z=\coth(t)$ we find that

$$\int_1^\infty dz\ \mathcal{Q}_0^n(z)=\int_0^\infty dt\ \frac{2^2e^{2t}t^n}{(e^{2t}-1)^2}\ ,$$

which, with a final substitution $y=2t$, can be noticed to be an integral representation of the zeta function.

[176] Note this means that $\xi_0(\rho(z))=-\frac{2\sqrt{3}}{\pi}\mathcal{Q}_0(z)$, where $z=\cosh(2\rho)$.

D.4 Normalising the scattering states

With our adventure into the world of integrals of Legendre functions over, now is a good moment to take stock and remember what our original goal was. We wanted to find scattering states to (D.16) which ensure our bound state (D.15) resides in the self-adjoint domain of our Hamiltonian, as such they must obey (D.14) as $\rho \to 0$. As a result of being in the self-adjoint domain the scattering states will necessarily be orthogonal to each other and the normalised bound state, since they all correspond to different eigenvalues. We might hope that (D.14) fixes the two arbitrary constants in our scattering state solutions however only the $\rho \to 0$ boundary is relevant for us as only at small ρ does our problem resemble a $\frac{1}{\rho^2}$ potential. This condition fixes the ratio of coefficients in our general solution but not the overall normalisation of the scattering states. In the bound state case normalisability, along with normalising to 1, was used to fix the constants appearing in the solution[177]. We might hope that the same would work for our scattering states, however, as we noted previously, the scattering states behave like plane waves as $\rho \to \infty$ and so we can see these functions are not in $L^2((0,\infty))$, and as such are not normalisable.

This may set off alarm bells in our head, however clearly the same issue afflicts plane waves, and we happily use those all the time. The resolution is that whilst these functions don't belong to our Hilbert space, as it stands, they do belong to an extension of it called a rigged Hilbert space. Whilst interesting the topic of rigged Hilbert spaces is a little too much for us to introduce here and so we instead recommend the interested reader to look at the pedagogical introduction to the subject presented in [27]. For us, the main point of using a rigged Hilbert space is that it allows distributions to be rigorously encompassed into the formalism. This allows us to make use of the Dirac delta distribution. We can now see the correct condition to use for our scattering states is that they be normalised to a Dirac delta distribution in the following manner

$$(\Psi(\Lambda), \Psi(\tilde{\Lambda})) = \delta(\Lambda - \tilde{\Lambda}) \ , \tag{D.69}$$

where $\delta(\Lambda - \tilde{\Lambda})$ is a Dirac delta distribution in the space of eigenvalues $1 < \Lambda, \tilde{\Lambda}$ and $(\Psi(\Lambda), \Psi(\tilde{\Lambda}))$ is the appropriate inner product for the functions Ψ as defined in (D.6). Our job in the rest of this Section shall be to appropriately normalise our scattering states so as to ensure (D.15) is in the self-adjoint domain of our Hamiltonian and (D.69) holds.

The mode corresponding to (D.15) in the Legendre approach to the problem is given by

$$\zeta(z, 0) = a(0)\mathcal{P}_0(z) + b(0)\mathcal{Q}_0(z) \ , \tag{D.70}$$

[177]This condition in fact leads to the solution $\xi_0 = c_1$ for $c_1 \in \mathbb{R}$ being set to zero as this mode is non-normalisable.

with $a(0)$ and $b(0)$ being arbitrary constants. Our boundary conditions are chosen in such a way that this mode must be normalisable and normalised to unity with respect to the inner product (D.20). Since $\mathcal{P}_0(z) = 1$ this instantly sets the constant $a(0)$ to zero, as was seen in the CPS treatment of the problem, and so the integral of interest becomes

$$\frac{2\pi^4}{g^5}\int_1^\infty dz\; b(0)^2\mathcal{Q}_0^2(z) = \frac{2\pi^4}{g^5}b(0)^2 J^{(0,2)}(|0,0)\;, \tag{D.71}$$

in order to evaluate this we need $J^{(0,2)}(|0,0)$, which from (D.66) is given by

$$J^{(0,2)}(|0,0) = -\frac{1}{2}\Big(I^{(0,2)}(|0,0) - \frac{i\pi}{2}I^{(1,1)}(0|0) + (\frac{-i\pi}{2})^2 I^{(2,0)}(0,0|)\Big) = \frac{\pi^2}{6}\;, \tag{D.72}$$

where we have used that $I^{(0,2)}(|0,0) = \frac{\pi^2}{6}$, $I^{(1,1)}(0|0) = 0$ and $I^{(2,0)}(0,0|) = 2$, which is given in [53]. Thus using (D.72) in (D.71) we obtain

$$\frac{2\pi^4}{g^5}\int_1^\infty dz\; \zeta(z,0)\zeta(z,0) = \frac{\pi^6}{3g^5}b(0)^2 \implies b(0) = \pm\frac{\sqrt{3}g^{\frac{5}{2}}}{\pi^3}\;. \tag{D.73}$$

This result completes the process of normalising the zero mode.

With the zero mode dealt with it's now time to move on to the scattering states. Just as with the bound state these can be described by linear combinations of the Legendre functions $\mathcal{P}_\nu$ and $\mathcal{Q}_\nu$. However owing to the identity

$$\cos(\nu\pi)\mathcal{P}_\nu(z) = -\frac{\mathcal{Q}_\nu(z)}{\Gamma(-\nu)\Gamma(\nu+1)} + \frac{\mathcal{Q}_{-1-\nu}(z)}{\Gamma(\nu+1)^2}\;, \tag{D.74}$$

we can instead use $\mathcal{Q}_\nu$ and $\mathcal{Q}_{-1-\nu}$ as a basis for our scattering states[178]. Note this has the nice property of removing the ambiguity in making a choice in (D.18) due to (D.19). With this if we recall that $z := \cosh(2\rho)$ then our scattering states can be written as

$$\xi(\rho,\Lambda) = a(\Lambda)\mathcal{Q}_{-\frac{1}{2}+\frac{i\omega}{2}}(\cosh(2\rho)) + b(\Lambda)\mathcal{Q}_{-\frac{1}{2}-\frac{i\omega}{2}}(\cosh(2\rho))\;, \tag{D.75}$$

$$\zeta(z,\Lambda) = a(\Lambda)\mathcal{Q}_{-\frac{1}{2}+\frac{i\omega}{2}}(z) + b(\Lambda)\mathcal{Q}_{-\frac{1}{2}-\frac{i\omega}{2}}(z)\;, \tag{D.76}$$

with $a(\Lambda)$, $b(\Lambda)$ arbitrary complex constants and where

$$\omega(\Lambda) = (\Lambda-1)^{\frac{1}{2}}\quad,\quad \Lambda > 1\;, \tag{D.77}$$

[178]Note this doesn't work for the solution to (D.17) with $\Lambda = 1$ as then, by (D.18), $\nu = \frac{1}{2}$ however this solution is neither normalisable nor delta function normalisable and so can be shown to not be admissible in the current setting.

In order for the scattering states to be in the self adjoint domain they must be orthogonal to the zero mode. This means we require

$$\frac{4\pi^2}{g^5}\int_0^\infty d\rho\ \sinh(2\rho)\xi(\rho,0)\xi(\rho,\Lambda) = \frac{2\pi^2}{g^5}\int_1^\infty dz\ \zeta(z,0)\zeta(z,\Lambda) = 0 \quad , \quad \Lambda > 1\ , \tag{D.78}$$

which using (D.76) and (D.70) leads to

$$b(0)\Big(\int_1^\infty dz\ a(\Lambda)\mathcal{Q}_0(z)\mathcal{Q}_\nu(z) + \int_1^\infty dz\ b(\Lambda)\mathcal{Q}_0(z)\mathcal{Q}_{\bar{\nu}}(z)\Big) = 0\ ,$$

where $\nu = -\frac{1}{2} + \frac{i\omega}{2}$ and $\bar{\nu} = -1-\nu$ is its complex conjugate. Since $b(0)$ is non zero we have

$$\frac{a(\Lambda)}{b(\Lambda)} = -\frac{J^{(0,2)}(|0,\bar{\nu})}{J^{(0,2)}(|0,\nu)}\ . \tag{D.79}$$

Now we can use (D.66) or [53] to evaluate the integrals appearing in this expression. Doing this we find that

$$J^{(0,2)}(|0,\sigma) = \frac{H(\sigma)-H(0)}{\sigma(\sigma+1)}\ ,$$

with $H(\sigma) = \psi(\sigma+1)+\gamma$ the harmonic number. In our case we note that $H(0) = 0$ and $\nu(\nu+1) = (\bar{\nu}+1)\bar{\nu}$ and hence (D.79) becomes

$$\frac{a(\Lambda)}{b(\Lambda)} = -\frac{H(\bar{\nu})}{H(\nu)}\ , \tag{D.80}$$

hence we have now been able to determine the ratio of the arbitrary constants in our solution, as we previously promised.

To go further we require more conditions, corresponding to completing the specification of boundary data. We demand our scattering states be real and to deal with this it's useful to recall that

$$\mathcal{Q}_\nu(z) = \frac{\pi^{\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\nu+1)}\int_0^\infty \frac{dt}{(z+(z^2-1)^{\frac{1}{2}}\cosh(t))^{\nu+1}} \quad , \quad Re(\nu+1) > 0\ , \tag{D.81}$$

Now since $\overline{\Gamma(z)} = \Gamma(\bar{z})\ \forall z \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}$ on the real axis we have $\overline{\mathcal{Q}_\nu(x)} = \mathcal{Q}_{\bar{\nu}}(x)$ and hence applying this to (D.76) we find that

$$a(\Lambda) = \overline{b(\Lambda)} \qquad \Lambda > 1\ . \tag{D.82}$$

We can now combine this with (D.79) and the fact that $\overline{H(\nu)} = H(\bar{\nu})$ to show that

$$a(\Lambda) = in(\Lambda)H(-\frac{1}{2} - \frac{i\omega(\Lambda)}{2}) \quad , \quad b(\Lambda) = -in(\Lambda)H(-\frac{1}{2} + \frac{i\omega(\Lambda)}{2}) \quad , \quad \Lambda > 1 \; , \tag{D.83}$$

with $n(\Lambda) \in \mathbb{R}$ an, as of yet, unfixed normalisation constant.

We've now ensured our scattering states are orthogonal to the zero mode and have also guaranteed that our solutions are real. However we've still not uniquely determined these states as we can see they contain $n(\Lambda)$. Fortunately we still have (D.69) to implement. It is easier to implement this condition using the Ψ form of our scattering states, given in (D.6), as it is in this form that the solutions take on the form of plane waves in the large ρ limit, see (D.12).

In order to make progress we require the asymptotic form of the scattering states. Fortunately this is not hard to obtain as one can use the hypergeometric definition of Legendre functions

$$\mathcal{Q}_\nu(z) = \frac{\pi^{\frac{1}{2}}\Gamma(\nu+1)}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})} z^{-\nu-1} \, {}_2F_1(\frac{1}{2}\nu+1, \frac{\nu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}) \; , \tag{D.84}$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function. Recalling that $z = \cosh(2\rho)$ we have $\sqrt{\sinh(2\rho)} = (z^2-1)^{\frac{1}{4}} = z^{\frac{1}{2}}(1-z^{-2})^{\frac{1}{4}}$ and hence

$$(z^2-1)^{\frac{1}{4}}\mathcal{Q}_{-\frac{1}{2}+\frac{i\omega}{2}}(z) \sim \frac{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}+\frac{i\omega}{2})}{2^{\frac{1}{2}+\frac{i\omega}{2}}\Gamma(1+\frac{i\omega}{2})} z^{-\frac{i\omega}{2}} \quad , \quad z \to \infty \; , \tag{D.85}$$

$$(z^2-1)^{\frac{1}{4}}\mathcal{Q}_{-\frac{1}{2}-\frac{i\omega}{2}}(z) \sim \frac{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{i\omega}{2})}{2^{\frac{1}{2}-\frac{i\omega}{2}}\Gamma(1-\frac{i\omega}{2})} z^{\frac{i\omega}{2}} \quad , \quad z \to \infty \; , \tag{D.86}$$

since

$$z = \cosh(2\rho) \sim \frac{e^{2\rho}}{2} \quad , \quad \rho \to \infty \; ,$$

using (D.6), (D.75) and (D.83) we find that

$$\Psi(\rho,\Lambda) \sim \frac{i\pi^{\frac{1}{2}}n(\Lambda)}{2^{\frac{1}{2}}}\Big(\frac{H(-\frac{1}{2}-\frac{i\omega}{2})\Gamma(\frac{1}{2}+\frac{i\omega}{2})}{\Gamma(1+\frac{i\omega}{2})}e^{-i\omega\rho} - \frac{H(-\frac{1}{2}+\frac{i\omega}{2})\Gamma(\frac{1}{2}-\frac{i\omega}{2})}{\Gamma(1-\frac{i\omega}{2})}e^{i\omega\rho}\Big) \quad , \quad \rho \to \infty \; ,$$

which, because $\overline{\Gamma(z)} = \Gamma(\bar{z})$ and $\overline{H(z)} = H(\bar{z})$, can be re-written as

$$\Psi(\rho,\Lambda) \sim \frac{i\pi^{\frac{1}{2}}n(\Lambda)}{2^{\frac{1}{2}}}\left|\frac{H(-\frac{1}{2}+\frac{i\omega}{2})\Gamma(\frac{1}{2}-\frac{i\omega}{2})}{\Gamma(1-\frac{i\omega}{2})}\right|\Big(-e^{i\omega\rho+i\Delta(\Lambda)} + e^{-i\omega\rho-i\Delta(\Lambda)}\Big) \quad , \quad \rho \to \infty \; ,$$

where we've denoted

$$\frac{H(-\frac{1}{2}+\frac{i\omega}{2})\Gamma(\frac{1}{2}-\frac{i\omega}{2})}{\Gamma(1-\frac{i\omega}{2})} = \left|\frac{H(-\frac{1}{2}+\frac{i\omega}{2})\Gamma(\frac{1}{2}-\frac{i\omega}{2})}{\Gamma(1-\frac{i\omega}{2})}\right| e^{i\Delta(\Lambda)} \ ,$$

which finally allows us to arrive at the expression

$$\Psi(\rho,\Lambda) \sim N(\Lambda)\sin(\omega(\Lambda)\rho+\Delta(\Lambda)), \quad \rho\to\infty, \quad N(\Lambda) := (2\pi)^{\frac{1}{2}} n(\Lambda)\left|\frac{H(-\frac{1}{2}+\frac{i\omega}{2})\Gamma(\frac{1}{2}-\frac{i\omega}{2})}{\Gamma(1-\frac{i\omega}{2})}\right| \ . \tag{D.87}$$

If we now take two copies of (D.7), one for Λ and another for $\tilde{\Lambda}$ with $\Lambda, \tilde{\Lambda} > 1$, and then multiply each of these equations by the eigenfunction not appearing in the equation it can be shown[179] that

$$\lim_{R\to\infty}\int_0^R d\rho\ \Psi(\Lambda,\rho)\Psi(\tilde{\Lambda},\rho) = \lim_{R\to\infty}\Big(\frac{\Psi(\Lambda,\rho)\Psi'(\tilde{\Lambda},\rho)-\Psi(\tilde{\Lambda},\rho)\Psi'(\Lambda,\rho)}{(\Lambda-\tilde{\Lambda})}\Big)\Big|_0^R \ . \tag{D.88}$$

The $\rho\to 0$ limit doesn't contribute, but the large ρ limit is non zero and, using (D.87), becomes

$$\frac{N(\Lambda)N(\tilde{\Lambda})\Big(\omega(\tilde{\Lambda})\sin\big(\omega(\Lambda)\rho+\Delta(\Lambda)\big)\cos\big(\omega(\tilde{\Lambda})\rho+\Delta(\tilde{\Lambda})\big)-\omega(\Lambda)\sin\big(\omega(\tilde{\Lambda})\rho+\Delta(\tilde{\Lambda})\big)\cos\big(\omega(\Lambda)\rho+\Delta(\Lambda)\big)\Big)}{(\omega^2(\Lambda)-\omega^2(\tilde{\Lambda}))} \ .$$

Using $\sin(a)\cos(b) = \frac{1}{2}\big(\sin(a+b)+\sin(a-b)\big)$ this becomes

$$\frac{N(\Lambda)N(\tilde{\Lambda})}{2}\Big(\frac{\sin\big((\omega+\tilde{\omega})R+\Delta(\omega)+\Delta(\tilde{\omega})\big)}{(\omega+\tilde{\omega})}+\frac{\sin\big((\omega-\tilde{\omega})R+\Delta(\omega)-\Delta(\tilde{\omega})\big)}{(\omega-\tilde{\omega})}\Big) \ ,$$

where $\tilde{\omega} = \omega(\tilde{\Lambda})$. If we now make use of double angle expansions this becomes

$$\begin{aligned}\frac{N(\Lambda)N(\tilde{(}\Lambda)}{2}\Big(&\frac{\sin\big((\omega+\tilde{\omega})R\big)\cos\big(\Delta(\omega)+\Delta(\tilde{\omega})\big)}{\omega+\tilde{\omega}}+\frac{\cos\big((\omega+\tilde{\omega})R\big)\sin\big(\Delta(\omega)+\Delta(\tilde{\omega})\big)}{\omega+\tilde{\omega}} \\ &+\frac{\sin\big((\omega-\tilde{\omega})R\big)\cos\big(\Delta(\omega)-\Delta(\tilde{\omega})\big)}{\omega-\tilde{\omega}}+\frac{\cos\big((\omega-\tilde{\omega})R\big)\sin\big(\Delta(\omega)-\Delta(\tilde{\omega})\big)}{\omega-\tilde{\omega}}\Big) \ .\end{aligned}$$

Using this result we can now evaluate the $R\to\infty$ limit in (D.88). Recalling that

$$\lim_{R\to\infty}\frac{\sin(xR)}{x} = \pi\delta(x) \quad , \quad \lim_{R\to\infty}\frac{\cos(xR)}{x} = 0 \ ,$$

[179]This calculation is done in a manner almost exactly analogous to the one presented in undergraduate courses where it is used to show that eigenvectors of hermitian matrices, corresponding to different eigenvalues, are orthogonal to each other.

we find

$$\int_0^\infty d\rho\ \Psi(\Lambda,\rho)\Psi(\tilde{\Lambda},\rho) = \frac{\pi N(\Lambda)N(\tilde{\Lambda})}{2}\Big(\cos\big(\Delta(\omega)+\Delta(\tilde{\omega})\big)\delta(\omega+\tilde{\omega}) + \cos\big(\Delta(\omega)-\Delta(\tilde{\omega})\big)\delta(\omega-\tilde{\omega})\Big)\ .$$

Since $\omega+\tilde{\omega} > 0$, because $\Lambda, \tilde{\Lambda} > 1$, the first delta distribution doesn't contribute, while the second only contributes when $\omega = \tilde{\omega}$ meaning the cosine factor is equal to unity. All of this means

$$\int_0^\infty d\rho\ \Psi(\Lambda,\rho)\Psi(\tilde{\Lambda},\rho) = \frac{\pi N(\Lambda)N(\tilde{\Lambda})}{2}\delta(\omega-\tilde{\omega})\ , \tag{D.89}$$

which is almost (D.69) except that $\delta(\Lambda-\tilde{\Lambda}) = \delta(\omega^2-\tilde{\omega}^2)$. However using standard results for delta distributions with arguments that are functions of several variables we find (D.69) becomes

$$\big(\Psi(\Lambda(\omega)),\Psi(\tilde{\Lambda}(\tilde{\omega}))\big) = \frac{2\pi^5 N(\Lambda(\omega))N(\tilde{\Lambda}(\tilde{\omega}))}{g^5}\delta(\omega-\tilde{\omega}) = \frac{1}{2(\omega^2+\tilde{\omega}^2)^{\frac{1}{2}}}\delta(\omega-\tilde{\omega})\ . \tag{D.90}$$

This finally gives us the normalisation constant

$$N(\Lambda(\omega))^2 = \frac{g^5}{4\sqrt{2}\pi^5\omega}\ . \tag{D.91}$$

From this $n(\Lambda)$ can be determined and hence finally all of our integration constants have been fixed, meaning we have specified enough boundary data for the Schrödinger problem.

So, in summary, in this Section we have identified the scattering states of the eigenvalue problem (D.16) as linear combinations of Legendre functions. Then by imposing reality, orthogonality to the zero mode, (D.15), and delta function normalisability, (D.69), the form of the scattering states has been shown to be uniquely fixed.