Quantization of Lie groups and Lie algebras

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Last update: 30 May 2014

Notes and References

This is a typed copy of the LOMI preprint *Quantization of Lie groups and Lie algebras* by L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtajan.

Recommended for publication by the Scientific Council of Steklov Mathematical Institute, Leningrad Department (LOMI) 24, September, 1987.

Submitted to the volume, dedicated to Professor's M. Sato 60-th birthday.

Introduction

The Algebraic Bethe Ansatz, which is the essence of the quantum inverse scattering method, emerges as a natural development of the following different directions in mathematical physics: the inverse scattering method for solving nonlinear equations of evolution [GGK1967], quantum theory of magnets [Bet1931], the method of commuting transfer-matrices in classical statistical mechanics [Bax1982]] and factorizable scattering theory [Yan1967,Zam1979]. It was formulated in our papers [STF1979,TFa1979,Fad1984]. Two simple algebraic formulas lie in the foundation or the method:

$$RT1T2=T2\ T1R\ (*)$$

and

$$R12R13R23= R23R13R12.\ (**)$$

Their exact meaning will be explained in the next section. In the original context or the Algebraic Bethe Ansats T plays the role of the quantum monodromy matrix of the auxiliary linear problem and is a matrix with operator-valued entries whereas R is an ordinary "c-number" matrix. The second formula can be considered as a compatibility condition for the first one.

Realizations of the formulae (*) and (**) for particular models naturally led to new algebraic objects which can be viewed as deformations of Lie-algebraic structures [KRe1981,Skl1982,SklNONE,Skl1985]. V. Drinfeld has shown [Dri1985,Dri1986-3] that these constructions are adequately expressed in the language of Hopf algebras [Abe1980]. On this way he has obtained a deep generalization of his results of [KRe1981,Skl1982,SklNONE,Skl1985]. Part of these results were also obtained by M. Jimbo [Jim1986,Jim1986-2].

However, from our point of view, these authors did not use formula (*) to the full strength. We decided, using the experience gained in the analysis of concrete models, to look again at the basic constructions of deformations. Our aim is to show that one can naturally define the quantization (q-deformation) of simple Lie groups and Lie algebras using exclusively the main formulae (*) and (**). Following the spirit of non-commutative geometry [Con1986] we will quantize the algebra of functions

Fun(G) on a Lie group G instead of the group itself. The quantization of the universal enveloping algebra U(𝔤) of the Lie algebra 𝔤 will be based on a generalization of the relation

$$U(\mathfrak{g})=C_e^{-\infty}(G)$$

where $C_e^{-\infty}(G)$ is a subalgebra in $C^{-\infty}(G)$ of distributions with support in the unit element e of G.

We begin with some general definitions. After that we treat two important examples and finally we discuss our constructions from the point of view of deformation theory. In this paper we use a formal algebraic language and do not consider problems connected with topology and analysis. A detailed presentation of our results will be given elsewhere.

1. Quantum formal groups

Let V be an n-dimensional complex vector space (the reader can replace the field $\mathbb{C}$ by any field of characteristic zero). Consider a non-degenerate matrix $R\in Mat(V^{\otimes 2},\mathbb{C})$ satisfying the equation

$$R_{12}R_{13}R_{23}= R_{23}R_{13}R_{12}, \quad (1)$$

where the lower indices describe the imbedding of the matrix R into $Mat(V^{\otimes 3},\mathbb{C})$.

Definition 1. Let A=A(R) be an associative algebra over $\mathbb{C}$ with generators 1, t_{ij}, i,j=1,…,n, satisfying the following relations

$$RT_1T_2=T_2T_1R, \quad (2)$$

where $T_1=T\otimes I$, $T_2=I\otimes T\in Mat(V^{\otimes 2},A)$, $T=(t_{ij})_{i,j=1}^n\in Mat(V,\Lambda)$ and I is a unit matrix in $Mat(V,\mathbb{C})$. The algebra A(R) is called the algebra of functions on the quantum formal group corresponding to the matrix R.

In the case $R=I^{\otimes 2}$ the algebra A(R) is generated by the matrix elements of the group $GL(n,\mathbb{C})$ and is commutative.

Theorem 1. The algebra A is a bialgebra (a Hopf algebra) with comultiplioation $\Delta:A\to A\otimes A$

$$\Delta(1) = 1\otimes 1,\ \Delta(t_{ij}) = \sum_{k=1}^n t_{ik} \otimes t_{kj},\ i,j=1,\dots,n.$$

Let $A'=Hom(A,\mathbb{C})$ be the dual space to the algebra A. Comultiplication in A induces multiplication in A′:

$$(\ell_1 \ell_2,a)= (\ell_1 \ell_2)(a) =(\ell_1\otimes \ell_2) (\Delta(a)),$$

where $\ell_1, \ell_2\in A'$ and $a\in A$. Thus A′ has the structure of an associative algebra with unit 1′, where $1'(t_{ij})=\delta_{ij}$, i,j=1,…,n.

Definition 2. Let U(R) be the subalgebra in A(R)′ generated by elements 1′ and $\ell_{ij}^{(\pm)}$, i,j=1,…,n, where

$$(1',T_1\dots T_k) = I^{\otimes k},\ (L^{(+)},T_1\dots T_k) = R_1^{(+)}\dots R_k^{(+)},\ (L^{(-)},T_1\dots T_k) = R_1^{(-)}\dots R_k^{(-)}. \quad (3)$$

Here $L^{(\pm)}=(\ell_{ij}^{(\pm)})_{i,j=1}^n\in Mat(V,U)$,

$$T_i=I\otimes\dots\otimes \underbrace{T}_{i}\otimes\dots \otimes I\in Mat (V^{\otimes k},A),\ i=1,\dots,k.$$

The matrices $R_i^{(\pm)}\in Mat(V^{\otimes(k+1)},\mathbb{C})$ act nontrivially on factors number 0 and i in the tensor product $V^{\otimes(k+1)}$ and coincide there with the matrices $R^{(\pm)}$, where

$$R^{(+)}= PRP,\ R^{(-)}= R^{-1}; \quad (4)$$

here P is the permutation matrix in $V^{\otimes 2}$: $P(v\otimes w)=w\otimes v$ for $v,w\in V$. The left hand side of the formula (3) denotes the values of 1′ and of the matrices-functionals $L^{(\pm)}$ on the homogeneous elements of the algebra A of degree k. When k=0 the right hand side of the formula (3) is equal to I. The algebra U(R) is called the algebra of regular functionals on A(R).

Due to the equation (1) and

$$R_{12}R_{23}(-)\,R_{13}(-)=R_{13}(-)\,R_{23}(-)\,R_{12}$$

the definition 2 is consistent with the relations (2) in the algebra A.

Remark 1. The apparent doubling of the number of generators of the algebra U(R) in comparison with the algebra A(R) is explained as follows: due to the formula (3) some of the matrix elements of the matrices $L(\pm)$ are identical or equal to zero. In interesting examples (see below) the matrices $L(\pm)$ are of Borel type.

Theorem 2.

1) In the algebra U(R) the following relations take place:

$$R_{21}\,L_1(\pm)\,L_2(\pm)=L_2(\pm)\,L_1(\pm)\,R_{21},\; R_{21}L_1(+)\,L_2(-)=L_2(-)\,L_1(+)\,R_{21}, \quad (5)$$

where $R_{21}=PR_{12}P$ and $L_1(\pm)=L(\pm)\otimes 1,\ L_2(\pm)=I\otimes L(\pm)\in \mathrm{Mat}(V^{\otimes 2},A')$.

2) Multiplication in the algebra A(R) induces a comultiplication δ in U(R)

$$\delta(1')=1'\otimes 1',\ \delta(\ell_{ij}(\pm))=\sum_{k=1}^{n}\ell_{ik}(\pm)\otimes\ell_{kj}(\pm),\ i,j=1,\dots,n,$$

so that U(R) acquires a structure of a bialgebra.

The algebra U(R) can be considered as a quantization of the universal enveloping algebra, which is defined by the matrix R.

Let us also remark that in the framework of the scheme presented one can easily formulate the notion of quantum homogeneous spaces.

Definition 3. A subalgebra $B\subset A=A(R)$ which is a left coideal: $\Delta(B)\subset A\otimes B$ is called the algebra of functions on quantum homogeneous space associated with the matrix R.

Now we shall discuss concrete examples of the general construction presented, above.

2. A finite-dimensional example.

Let $V=\mathbb{C}^n$; a matrix R of the form [Jim1986-2]

$$R=\sum_{i\ne j,\,i,j=1}^{n} e_{ii}\otimes e_{jj}+q\sum_{i=1}^{n}e_{ii}\otimes e_{ii}+(q-q^{-1})\sum_{1\le j<i\le n} e_{ij}\otimes e_{ji}, \quad (6)$$

where $e_{ij}\in \mathrm{Mat}(\mathbb{C}^n)$ are matrix units and $q\in\mathbb{C}$, satisfies equation (1). It is natural to call the corresponding algebra A(R) the algebra of functions on the q-deformation of the group $GL(n,\mathbb{C})$ and denote it by $\mathrm{Fun}_q(GL(n,\mathbb{C}))$.

Theorem 3. The element

$$\det\nolimits_q T=\sum_{s\in S_n}(-q)^{\ell(s)}\,t_{1s_1}\dots t_{ns_n}$$

where summation goes over all elements s of the symmetric group S_n and $\ell(s)$ is the length of the element S, generates the center of the algebra $\mathrm{Fun}_q(GL(n,\mathbb{C}))$.

Definition 4. The quotient-algebra of $\mathrm{Fun}_q(GL(n,\mathbb{C}))$ defined by an additional relation $\det_q T=1$ is called the algebra of functions on the q-deformation of the group $SL(n,\mathbb{C})$ and is denoted by

Funq(SL(n,ℂ)).

Theorem 4. The algebra Funq(SL(n,ℂ)) has an antipode γ, which is given on the generators tij by:

$$\gamma(tij)= (-q)i-j\ t\sim ji,\ i,j=1,\ldots,n,$$

where

$$t\sim ij= \sum s\in Sn-1\ (-q)\ \ell\ (s)\ t1s1\ldots\ ti-1si-1\ ti+1si+1\ldots\ tnsn$$

and s=(s1,…,si-1,si+1,…,sn)=s(1,…,j-1,j+1,…,n). The antipode γ has the properties Tγ(T)=I and γ2(T)=DTD-1, where D=diag(1,q2,…,q2(n-1))∈Mat(ℂn).

In the case n=2 the matrix R is given explicitly by

$$R= (\ q000\ 0100\ 0q\text{-}q\text{-}110\ 000q\ )\ (7)$$

and the relations (2) reduce to the following simple formulae:

$$t11t12 = qt12t11,\ t12t21 = t21t12,\ t21t22 = qt22t21,\ t11t21 = qt21t11,\ t12t22 = qt22t12,\ t11t22\text{-}t22t11 = (q\text{-}q\text{-}1)\ t12t21$$

and

$$\det q\ T=t11t22\ \text{-}qt12t21.$$

In this case

$$\gamma(T)= (\ t22\text{-}q\text{-}1t12\ \text{-}qt21t11\ )\ .$$

Remark 2. When |q|=1 relations (2) admit the following *-anti-involution: tij*=tij, i,j=1,…,n. The algebra A(R) with this anti-involution is nothing but the algebra Funq(SL(n,ℝ)). In the case n=2 this algebra and the matrix R of the form (7) appeared for the first time in [FTa1986]. The subalgebra B⊂Funq(SL(n,ℝ)) generated by the elements 1 and ∑k=1ntiktjk, i,j=1,…,n, is the left coideal and may be called the algebra of functions on the q-deformation of the symmetric homogeneous space of rank n-1 for the group SL(n,ℝ). In the case n=2 we obtain the q-deformation of the Lobachevski plane.

Remark 3. When q∈ℝ the algebra Funq(SL(n,ℝ)) admits the following *-anti-involution: γ(tij)=tji*, i,j=1,…,n. The algebra Funq(SL(n,ℂ)) with this anti-involution is nothing but the algebra Funq(SU(n)). In the case n=2 this algebra was introduced in [VSoNONE,Wor1987].

Remark 4. The algebras Funq(G), where G is a simple Lie group, can be defined in the following way. For any simple group G there exists a corresponding matrix RG satisfying equation (1), which generalizes the matrix R of the form (6) for the case G=SL(n,ℂ). This matrix RG depends on the parameter q and, as q→1,

$$RG=I+(q\text{-}1)\ \tau G+O((q\text{-}1)2),$$

where

$$\tau G=\sum i\ \varrho(Hi)\otimes\varrho(Hi)2\ +\sum \alpha\in\Delta+\ \varrho(X\alpha)\otimes\varrho(X\text{-}\alpha).$$

Here ϱ is the vector representation of Lie algebra 𝔤, Hi, Xα its Cartan-Weyl basis and Δ+ the set of positive roots. The explicit form of the matrices RG can be extracted from [Jim1986-2], [Bas1985]. The corresponding algebra A(R) is defined by the relations (2) and an appropriate anti-involution compatible with them. It can be called the algebra of functions on the q-deformation of the Lie group G.

Let us discuss now the properties of the algebra U(R). It follows from the explicit form (6) of the matrix R and the definition 2 that the matrices-functionals L(+) and L(-) are, respectively, the upper- and lower-triangular matrices. Their diagonal parts are conjugated by the element S of the maximal length in the Weyl group of the Lie algebra 𝔰𝔩(n,ℂ):

$$\mathrm{diag}(L(+))= S\ \mathrm{diag}(L(\text{-}))\ S\text{-}1.$$

Theorem 5. The following equality holds:

$$U(R)=U_q(\mathfrak{sl}(n,\mathbb{C})),$$

where $U_q(\mathfrak{sl}(n,\mathbb{C}))$ is the q-deformation of the universal enveloping algebra $U(\mathfrak{sl}(n,\mathbb{C}))$ of the Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ introduced in [Dri1985] and [Jim1986]. The center of U(R) is generated by the elements

$$c_k=\sum_{s,s'\in S_n} (-q)^{\ell(s)+\ell(s')}\, \ell_{s_1s_1'}^{(+)}\dots \ell_{s_ks_k'}^{(+)}\, \ell_{s_{k+1}s_{k+1}'}^{(-)}\dots \ell_{s_ks_k'}^{(-)}.$$

Remark 5. It is instructive to compare the relations for the elements $\ell_{ij}^{(\pm)}$, $i,j=1,\dots,n$, which follow from (5), with those given in [Dri1985] and [Jim1986]. The elements $\ell_{ij}^{(\pm)}$ can be considered as a q-deformation of the Cartan-Weyl basis, whereas the elements $\ell_{ii+1}^{(+)}$, $\ell_{i-1i}^{(-)}$, $\ell_{ii}^{(+)}$ are the -deformation of the Chevalley basis. It was this basis that was used in [Dri1985] and [Jim1986]; the complicated relations between the elements of the q-deformation of the Chevalley basis presented in these papers follow from the simple formulae (3), (5) and (6).

Remark 6. It follows from the definition of the algebra $U_q(\mathfrak{sl}(n,\mathbb{C}))$ that it can be considered as the algebra $\mathrm{Fun}_q((G_+\times G_-)/H)$, where $G_\pm$ and H are, respectively, Borel and Cartan subgroups of the Lie group $SL(n,\mathbb{C})$. Moreover, in the general case the q-deformation $U_q(\mathfrak{g})$ of the universal enveloping algebra of simple Lie algebra $\mathfrak{g}$ can be considered as a quantization of the group $(G_+\times G_-)/H$. For the infinite-dimensional case (see the next section) this observation provides a key to the formulation of a quantum Riemann problem.

In the case n=2 we have the following explicit formulae:

$$L^{(+)} = \sqrt{q} \begin{pmatrix} e^{-hH/2} & hX \\ 0 & e^{hH/2} \end{pmatrix},\quad L^{(-)} = \frac{1}{\sqrt{q}} \begin{pmatrix} e^{hH/2} & 0 \\ hY & e^{-hH/2} \end{pmatrix},$$

where the generators 1 and $e^{\pm hH/2}$, X, Y of the algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$ satisfy the relations

$$e^{\pm\frac{hH}{2}}X= q^{\pm1}Xe^{\pm\frac{hH}{2}},\quad e^{\pm\frac{hH}{2}}Y= q^{\mp1}Ye^{\pm\frac{hH}{2}},\quad XY-YX=-\frac{(q-q^{-1})}{h^2} (e^{hH}-e^{-hH})$$

which appeared for the first time in [KRe1981].

3. An infinite-dimensional example

Replace in the general construction of section 1 the finite-dimensional vector space V by an infinite-dimensional $\mathbb{Z}$-graded vector space $\tilde V=\oplus_{n\in\mathbb{Z}}\lambda^nV=\oplus_{n\in\mathbb{Z}}V_n$, where λ is a formal variable (spectral parameter). Denote by $\mathbb{S}$ the shift operator (multiplication by λ) and consider as a matrix R an element $\tilde R\in \mathrm{Mat}(\tilde V^{\otimes2},\mathbb{C})$ satisfying equation (1) and commuting with the operator $\mathbb{S}\otimes\mathbb{S}$. Infinite-dimensional analogs of the algebras A(R) and U(R) - the algebras $A(\tilde R)$ and $U(\tilde R)$ are introduced as before by definitions 1 and 2; in addition the elements $\tilde T\in\mathrm{Mat}(\tilde V,A(\tilde R))$ and $\tilde L^{(\pm)}\in\mathrm{Mat}(\tilde V,U(\tilde R))$ commute with $\mathbb{S}$. Theorems 1 and 2 are valid for this case as well.

Let us discuss a meaningful example of this construction. Choose the matrix $\tilde R$ to be a matrix-valued function $R(\lambda,\mu)$ defined by the formula

$$R(\lambda,\mu)= \frac{\lambda q^{-1}R^{(+)}-\mu qR^{(-)}}{\lambda q^{-1}-\mu q},$$

where the matrices $R^{(\pm)}$ are given by (4). The role of the elements $\tilde T$ is now played by the infinite formal Laurent series

$$T(\lambda)=\sum_{m\in\mathbb{Z}} T_m\lambda^m$$

with the relations

$$R(\lambda,\mu)T_1(\lambda)\, T_2(\mu)=T_2(\mu)\, T_1(\lambda)R(\lambda,\mu).$$

The comultiplication in the algebra $A(\tilde R)$ is given by the formula

$$\Delta(t_{ij}(\lambda)) =\sum_{k=1}^n t_{ik} (\lambda)\otimes t_{kj}(\mu),\quad i,j=1,\dots,n.$$

The following result is the analog of Theorem 3.

Theorem 6. The element

$$\det_q T(\lambda) = \sum_{s \in S_n} (-q)^{\ell(s)} T_{1s_1}(\lambda) \ldots T_{ns_n}(\lambda q^{n-1})$$

generates the center of the algebra $A(\tilde{R})$.

Let us now briefly discuss the properties of the algebra $U(\tilde{R})$. It is generated by the formal Taylor series

$$L^{(\pm)}(\lambda) = \sum_{m \in \mathbb{Z}_+} L_m^{(\pm)} \lambda^{\pm m}$$

which, act on the elements of the algebra $A(\tilde{R})$ by the formulae (3). The relations in $U(\tilde{R})$ have the form

$$R(\lambda,\mu)\, L_1^{(\pm)}(\lambda)\, L_2^{(\pm)}(\mu) = L_2^{(\pm)}(\mu)\, L_1^{(\pm)}(\lambda)\, R(\lambda,\mu),\ R(\lambda,\mu)\, L_1^{(+)}(\lambda)\, L_2^{(-)}(\mu) = L_2^{(-)}(\mu)\, L_1^{(+)}(\lambda)\, R(\lambda,\mu).$$

Due to lack of space we shall not discuss here an interesting question about the connection of the algebra $U(\tilde{R})$ with the q-deformation of loop algebras, introduced in [Dri1986-3]. We shall only point out that the algebra $U(\tilde{R})$ has a natural limit when $q \to 1$. In this case the subalgebra generated by elements $L^{(+)}(\lambda)$ coincides with the Yangian $Y(\mathfrak{sl}(n,\mathbb{C}))$ introduced in the papers [Dri1985,KRe1986].

4. Deformation theory and quantum groups

Consider the contraction of the algebras $A(R)$ and $U(R)$ when $q \to 1$. For definiteness let us have in mind the above finite-dimensional example. The algebra $A(R)=Fun_q(G)$, when $q \to 1$, goes into the commutative algebra $Fun(G)$ with the Poisson structure given by the following formula

$$\{g \otimes, g\} = [\tau_G, g \otimes g]. \quad (8)$$

Here g_{ij}, $i,j=1,\ldots,n$ are the coordinate functions on the Lie group G. Passing from the Lie group G to its Lie algebra $\mathfrak{g}$, we obtain from (8) the Poisson structure on the Lie algebra

$$\{h \otimes, h\} = [\, \tau_G, h \otimes I + I \otimes h \,]\,. \quad (9)$$

Here $h=\sum_{i=1}^{\dim \mathfrak{g}} h_i X_i$, where X_i, $i=1,\ldots,\dim \mathfrak{g}$, form a basis of $\mathfrak{g}$. If we define $h^{(\pm)}=h_\pm + \frac{h_f}{2}$, where $h_\pm$ and h_f are respectively the nilpotent and Carton components of h, we can rewrite the formula (9) in the form

$$\{\, h^{(\pm)} \otimes, h^{(\pm)} \,\} = [\, \tau_G, h^{(\pm)} \otimes I + I \otimes h^{(\pm)} \,]\,,$$

$$\{\, h^{(\pm)} \otimes, h^{(\mp)} \,\} = 0. \quad (10)$$

(This Poison structure and its infinite-dimensional analogs were studied in [RFa1983] (see also [STS1985,FTa1987]). Thus the Lie algebra $\mathfrak{g}$ has the structure of a Lie bialgebra, where the cobracket $\mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is defined by the Poisson structure (10).

Analogously, the contraction $q \to 1$ of the algebra $U(R)$ leads to the Lie bialgebra structure on the dual vector space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$. The Lie bracket on $\mathfrak{g}^*$ is dual to the Poisson structure (10) and the cobracket $\mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$ is defined by the canonical Lie-Poisson structure on $\mathfrak{g}^*$.

This argument clarifies in what sense the algebras $A(R)$ and $U(R)$ determine deformations of the corresponding Lie group and Lie algebra. Moreover it shows how an additional structure is defined on these "classical" objects. Returning to the relations (*) and (**) we can now say that (*) constitutes a deformation of the Lie-algebraic defining relations with t_{ij} playing the role of generators, R being the array of "quantum" structure constants, and (**) generalizing the Jacoby Identity.

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