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A Geometric Construction of the Exceptional Lie Algebras F_4 and E_8

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Abstract We present a geometric construction of the exceptional Lie algebras F_4 and E_8 starting from the round spheres S^8 and S^{15}, respectively, inspired by the construction of the Killing superalgebra of a supersymmetric supergravity background.

1 Introduction

The Killing–Cartan classification of simple Lie algebras over the complex numbers is well known: there are four infinite families $A_{n\geq 1}$, $B_{n\geq 2}$, $C_{n\geq 3}$ and $D_{n\geq 4}$, with the range of ranks chosen to avoid any overlaps, and five exceptional cases G_2, F_4, E_6, E_7 and E_8. Whereas the classical series (*A*-*D*) correspond to matrix Lie algebras, and indeed their compact real forms are the Lie algebras of the special unitary groups over $\mathbb{C}$ (*A*), $\mathbb{H}$ (*B*) and $\mathbb{R}$ (*C* and *D*), the exceptional series do not have such classical descriptions; although they can be understood in terms of more exotic algebraic structures such as octonions and Jordan algebras. There is, however, a uniform construction of all exceptional Lie algebras (except for G_2) using spin groups and their spinor representations, described in Adams' posthumous notes on exceptional Lie groups (1) and, for the special case of E_8, also in (2). This construction, once suitably geometrised, is very familiar to practitioners of supergravity. The purpose of this note is to present this geometrisation, perhaps as an invitation for differential geometers to think about supergravity.

Indeed in supergravity there is a geometric construction which associates a Lie superalgebra to any supersymmetric supergravity background: typically a lorentzian spin manifold with extra geometric data and with a notion of privileged spinor fields, called Killing spinors. The resulting superalgebra is called the Killing superalgebra because it is constructed out of these Killing spinors and Killing vectors.

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The Killing superalgebra for general ten- and eleven-dimensional supergravities is constructed in (3; 4). In this note we will apply this construction not to supergravity backgrounds, but to riemannian manifolds without any additional structure. The relevant notion of Killing spinor is then that of a *geometric Killing spinor*: a nonzero section ε of the spinor bundle satisfying

$$\nabla_X \varepsilon = \frac{1}{2} X \cdot \varepsilon,$$

where X is any vector field and the dot means the Clifford action. We will apply this construction to the unit spheres $S^7 \subset \mathbb{R}^8$, $S^8 \subset \mathbb{R}^9$ and $S^{15} \subset \mathbb{R}^{16}$ and in this way obtain the compact real Lie algebras $\mathfrak{so}_9$, $\mathfrak{f}_4$ and $\mathfrak{e}_8$, respectively. It is curious that these three spheres are linked by the exceptional Hopf fibration which defines the octonionic projective line,

$$\begin{array}{ccc} S^7 & \longrightarrow & S^{15} \\ & & \downarrow \\ & & S^8 \end{array}$$

and it is natural to wonder whether their Killing superalgebras are similarly related. We will not answer this question here.

This note is organised as follows. In Sect. 2 we briefly review the relevant notions of Clifford algebras, spin groups and their spinorial representations. In Sect. 3 we define the Killing superalgebra after introducing the basic notions of Killing spinors and Bär's cone construction. In Sect. 4 we construct the Killing superalgebras of the round spheres S^7, S^8 and S^{15} and show that they are isomorphic to the compact real Lie algebras $\mathfrak{so}_9$, $\mathfrak{f}_4$ and $\mathfrak{e}_8$, respectively. Finally in Sect. 5 we discuss some open questions motivated by the results presented here.

2 Spinorial Algebra

In this section we start with some algebraic preliminaries on euclidean Clifford algebras and spinors in order to set the notation. We will be sketchy, but fuller treatments can be found, for example, in (5; 6; 7).

2.1 Clifford algebras and Clifford modules

Let V be a finite-dimensional real vector space with a positive-definite euclidean inner product $\langle -, - \rangle$. The **Clifford algebra** $\mathrm{C}\ell(V)$ is the associated algebra with unit generated by V and the identity $\mathbf{1}$ subject to the Clifford relations

$$v^2 = -\langle v, v \rangle \mathbf{1} \tag{1}$$

for all $v \in V$. More formally, the Clifford algebra is the quotient of the tensor algebra of V by the two-sided ideal generated by the Clifford relations. Since the Clifford relations—having terms of degree 0 and degree 2—are not homogeneous in the natural grading of the tensor algebra, $\mathrm{C}\ell(V)$ is not graded but only filtered.

Table 1 Clifford algebras $\mathrm{C}\ell_n$, where $\mathbb{K}(m)$ denotes the algebra of $m \times m$ matrices with entries in $\mathbb{K}$.

n	0	1	2	3	4	5	6	7
$\mathrm{C}\ell_n$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$

The associated graded algebra is the exterior algebra ΛV, to which it is isomorphic as a vector space. Nevertheless since the terms in (1) have even degree, $\mathrm{C}\ell(V)$ is $\mathbb{Z}_2$ graded

$$\mathrm{C}\ell(V) = \mathrm{C}\ell(V)_0 \oplus \mathrm{C}\ell(V)_1, \tag{2}$$

with vector-space isomorphisms $\mathrm{C}\ell(V)_0 \cong \Lambda^{\mathrm{even}}V$ and $\mathrm{C}\ell(V)_1 \cong \Lambda^{\mathrm{odd}}V$. These isomorphisms can be seen explicitly as follows. Relative to an orthonormal basis e_i for V, the Clifford relations become

$$e_i e_j + e_j e_i = -2\delta_{ij}\mathbf{1}, \tag{3}$$

which shows that up to terms in lower order we may always antisymmetrise any product $e_{i_1}e_{i_2}\dots e_{i_k}$ in $\mathrm{C}\ell(V)$ without ever changing the parity. The Clifford algebra of $\mathbb{R}^n$ generated by $\mathbf{1}$ and e_i subject to (3) is denoted $\mathrm{C}\ell_n$. As a real associative algebra with unit it is isomorphic to one or two copies of matrix algebras, as shown in Table 1 for $n \leq 7$. The higher values of n are obtained by Bott periodicity $\mathrm{C}\ell_{n+8} \cong \mathrm{C}\ell_n \otimes \mathbb{R}(16)$, where $\mathbb{R}(16)$ is the algebra of 16×16 real matrices.

Since matrix algebras have a unique irreducible representation (up to isomorphism), we can easily read off the irreducible representations of $\mathrm{C}\ell_n$ from the table. We see, in particular, that if n is even there is a unique irreducible representation, which is real for $n \equiv 0,6 \pmod 8$ or quaternionic when $n \equiv 2,4 \pmod 8$; whereas if n is odd there are two inequivalent irreducible representations, which are real when $n \equiv 7 \pmod 8$ and quaternionic when $n \equiv 3 \pmod 8$, and form a complex conjugate pair for $n \equiv 1 \pmod 4$. These two inequivalent Clifford modules are distinguished by the action of $\omega := e_1 e_2 \cdots e_n$, which for n odd is central in $\mathrm{C}\ell_n$. This element obeys $\omega^2 = (-1)^{n(n+1)/2}\mathbf{1}$, whence it is a complex structure for $n \equiv 1 \pmod 4$, in agreement with the table. The dimension of one such irreducible **Clifford module**, relative to either $\mathbb{R}$ if real or $\mathbb{C}$ if not, is $2^{\lfloor n/2 \rfloor}$. We will use the notation $\mathfrak{M}$ for the unique irreducible Clifford module in even dimension, $\mathfrak{M}_\pm$ for the irreducible Clifford modules for $n \equiv 3 \pmod 4$. For $n \equiv 1 \pmod 4$ we will let $\mathfrak{M}$ denote the irreducible Clifford module on which ω acts like $+i$ and let $\overline{\mathfrak{M}}$ denote the irreducible module on which ω acts like $-i$.

2.2 The spin group and spinor modules

The Clifford algebra $\mathrm{C}\ell(V)$ admits a natural Lie algebra structure via the Clifford commutator. The map $\Lambda^2 V \to \mathrm{C}\ell(V)$ given by

$$e_i \wedge e_j \mapsto -\frac{1}{2} e_i e_j, \tag{4}$$

for $i < j$, induces a Lie algebra homomorphism $\rho : \mathfrak{so}(V) \to \mathrm{C}\ell(V)$. Moreover the action of $\mathfrak{so}(V)$ on V is realised by the Clifford commutator, so that if $A \in \mathfrak{so}(V)$ and $v \in V$, then

$$A(v) = \rho(A)v - v\rho(A) \in \mathrm{C}\ell(V). \tag{5}$$

Exponentiating the image of ρ in $\mathrm{C}\ell(V)$ we obtain a connected Lie group called $\mathrm{Spin}(V)$. The subspace $V \subset \mathrm{C}\ell(V)$ is closed under conjugation by $\mathrm{Spin}(V)$ whence we obtain a map $\mathrm{Spin}(V) \to \mathrm{SO}(V)$, whose kernel is the central subgroup consisting of $\pm\mathbf{1}$.

Restricting an irreducible Clifford module $\mathfrak{M}$ (or $\mathfrak{M}_\pm$) to $\mathrm{Spin}(V)$ we obtain a **spinor module**, which may or may not remain irreducible. Since $\mathrm{Spin}_n \subset (\mathrm{C}\ell_n)_0 \cong \mathrm{C}\ell_{n-1}$, we can immediately infer the type of spinor module from Table 1. If $n \equiv 1 \pmod 8$, $\mathfrak{M} \cong \mathfrak{S} \otimes \mathbb{C}$ is the complexification of the unique irreducible real spinor module $\mathfrak{S}$, whereas if $n \equiv 5 \pmod 8$, $\mathfrak{M} \cong \mathfrak{S}$, but $\mathfrak{S}$ possesses a Spin_n-invariant quaternionic structure, whence $\overline{\mathfrak{M}} \cong \mathfrak{S}$ as well. For $n \equiv 3 \pmod 4$, $\mathfrak{M}_\pm \cong \mathfrak{S}$. For odd n, the spinor module is real if $n \equiv 1, 7 \pmod 8$ and quaternionic otherwise and its dimension (over $\mathbb{R}$ if real and over $\mathbb{C}$ otherwise) is again $2^{(n-1)/2}$. For even n, the unique irreducible Clifford module decomposes (perhaps after complexification) into two inequivalent Spin_n modules, called **half-** (or **chiral**) **spinor** modules. They are denoted $\mathfrak{S}_\pm$ if $n \equiv 0 \pmod 4$ and $\mathfrak{S}$ and $\overline{\mathfrak{S}}$ if $n \equiv 2 \pmod 4$. They are real if $n \equiv 0 \pmod 8$, quaternionic if $n \equiv 4 \pmod 8$ and complex otherwise. If $n \equiv 6 \pmod 8$ then it is the complexification of $\mathfrak{M}$ which decomposes $\mathfrak{M} \otimes \mathbb{C} \cong \mathfrak{S} \oplus \overline{\mathfrak{S}}$. In all cases, the dimension, computed relative to the appropriate field for the type, is $2^{(n-2)/2}$.

2.3 Spinor inner products

The Clifford algebra $\mathrm{C}\ell(V)$ has a natural antiautomorphism defined by $-\mathrm{id}_V$ on V. On a given irreducible Clifford module $\mathfrak{M}$ (or $\mathfrak{M}_\pm$) there always exists an inner product $(-,-)$ which realises this automorphism; that is, such that

$$(v \cdot \varepsilon_1, \varepsilon_2) = -(\varepsilon_1, v \cdot \varepsilon_2), \tag{6}$$

for all $v \in V$ and $\varepsilon_i \in \mathfrak{M}$. It follows that $(-,-)$ is $\mathrm{Spin}(V)$-invariant; indeed,

$$(e_i e_j \cdot \varepsilon_1, \varepsilon_2) = -(\varepsilon_1, e_i e_j \cdot \varepsilon_2). \tag{7}$$

In positive-definite signature, $(-,-)$ is either symmetric or hermitian, depending on the type of representation, and positive-definite (7).

The transpose of the Clifford action $V \otimes \mathfrak{M} \to \mathfrak{M}$ relative to the above inner product on $\mathfrak{M}$ and the euclidean inner product $\langle -,- \rangle$ on V, defines a map which we suggestively denote by $[-,-] : \mathfrak{M} \otimes \mathfrak{M} \to V$. Explicitly, we have that for all $v \in V$ and $\varepsilon_i \in \mathfrak{M}$,

$$\langle [\varepsilon_1, \varepsilon_2], v \rangle = (\varepsilon_1, v \cdot \varepsilon_2). \tag{8}$$

3 The Killing Superalgebra

In this section we will define the Killing superalgebra of a riemannian spin manifold admitting Killing spinors.

3.1 Spin manifolds

Let (M,g) be an n-dimensional riemannian manifold and let $\mathrm{O}(M)$ denote the bundle of orthonormal frames. It is a principal O_n-bundle. If the manifold is orientable, we can restrict ourselves consistently to oriented orthonormal frames. In this case, the subbundle $\mathrm{SO}(M)$ of oriented orthonormal frames is a principal SO_n-bundle. The obstruction to orientability is measured by the first Stiefel–Whitney class $w_1 \in H^1(M;\mathbb{Z}_2)$. If (M,g) is orientable one can ask whether there is a principal Spin_n-bundle $\mathrm{Spin}(M)$ lifting the oriented orthonormal frame bundle $\mathrm{SO}(M)$; that is, admitting a bundle map $\mathrm{Spin}(M) \to \mathrm{SO}(M)$ covering the identity and restricting fibrewise to the natural homomorphism $\mathrm{Spin}_n \to \mathrm{SO}_n$. The obstruction to the existence of such a lift is measured by the second Stiefel–Whitney class $w_2 \in H^2(M;\mathbb{Z}_2)$ and, if it vanishes, the manifold (M,g) is said to be spin. Spin structures $\mathrm{Spin}(M)$ on M need not be unique: they are measured by $H^1(M;\mathbb{Z}_2) = \mathrm{Hom}(\pi_1 M, \mathbb{Z}_2)$, which we can understand as assigning a sign (consistently) to every noncontractible loop. In this section we will assume our manifolds to be spin and that a choice of spin structure has been made. The main examples in this note are spheres, which are spin—indeed, the total space of the spin bundle of S^n is the Lie group Spin_{n+1}—and, since they are simply-connected, have a unique spin structure.

If $\mathfrak{M}$ is a $\mathrm{C}\ell_n$-module, then it is also a (perhaps reducible) Spin_n-module and we may form the **spinor bundle**

$$S(M) := \mathrm{Spin}(M) \times_{\mathrm{Spin}_n} \mathfrak{M}$$

over M as an associated vector bundle to the spin bundle. Furthermore we have a fibrewise action of the Clifford bundle $\mathrm{C}\ell(TM)$ on $S(M)$. The spinor inner products globalise to give an inner product on $S(M)$.

The Levi-Cività connection on the orthonormal frame bundle of (M,g) induces a connection on $\mathrm{Spin}(M)$ and hence on any associated vector bundle. In particular we have a **spin connection** on $S(M)$:

$$\nabla : \Gamma(S(M)) \to \Omega^1(M;S(M)),$$

allowing us to write down interesting equations on spinors. One such equation is the Killing spinor equation, which is the subject of the next section. A classic treatise on this equation is (8).

3.2 Killing spinors

Throughout this section we will let (M^n,g) be a spin manifold with chosen spinor bundle $S(M)$ on which we have a fibrewise action of the Clifford bundle $\mathrm{C}\ell(TM)$

and a Spin_n-invariant inner product which in addition satisfies Eq. (6). A nonzero $\varepsilon \in \Gamma(S(M))$ is said to be a **(real) Killing spinor** if for all vector fields X,

$$\nabla_X \varepsilon = \lambda X \cdot \varepsilon, \tag{9}$$

where $\lambda \in \mathbb{R}$ is the **Killing constant**. The origin of the name is that if ε_i, $i = 1,2$, are Killing spinors, then the vector field $V := [\varepsilon_1, \varepsilon_2]$ defined by Eq. (8) is a Killing vector. Indeed, for all vector fields X, Y,

$$\begin{aligned} g(\nabla_X V, Y) &= (\nabla_X \varepsilon_1, Y \cdot \varepsilon_2) + (\varepsilon_1, Y \cdot \nabla_X \varepsilon_2) && \text{(by definition of } \nabla) \\ &= \lambda\,(X \cdot \varepsilon_1, Y \cdot \varepsilon_2) + \lambda\,(\varepsilon_1, Y \cdot X \cdot \varepsilon_2) && \text{(using Eq (9))} \\ &= -\lambda\,(\varepsilon_1, X \cdot Y \cdot \varepsilon_2) + \lambda\,(\varepsilon_1, Y \cdot X \cdot \varepsilon_2), && \text{(using Eq. (6))} \end{aligned}$$

which is manifestly skewsymmetric in X, Y, whence we conclude that

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0,$$

which is one form of Killing's equation.

3.3 The cone construction

The problem of determining which riemannian manifolds admit real Killing spinors was the subject of much research until it was elegantly solved by Bär (9) via the cone construction. We will assume that the Killing constant λ has been set to $\pm\frac{1}{2}$ by rescaling the metric, if necessary. Let $(\overline{M}, \overline{g})$ denote the **(deleted) cone** over M, defined by $\overline{M} = \mathbb{R}^+ \times M$ and $\overline{g} = dr^2 + r^2 g$, where $r > 0$ is the coordinate on $\mathbb{R}^+$. Bär observed that there is a one-to-one correspondence between Killing spinors on M and parallel spinors on the cone $\overline{M}$. More precisely, if $n = \dim M$ is even, there is an isomorphism between Killing spinors on M with Killing constant $\pm\frac{1}{2}$ and parallel spinors on $\overline{M}$; the choice of sign having to do with the choice of embedding $\mathrm{C}\ell_n \subset \mathrm{C}\ell_{n+1}$. If on the other hand n is odd, then the space of Killing spinors on M with Killing constant $\pm\frac{1}{2}$ is isomorphic to the space of parallel half-spinors on $\overline{M}$, the chirality depending on the sign of the Killing constant. Together with a theorem of Gallot (10) which says that the cone of a complete manifold is either flat or irreducible, the above observation reduces the problem of determining the complete riemannian manifolds admitting real Killing spinors to a holonomy problem which was solved by Wang in (11). If (M, g) is not complete, its cone may be reducible, but if so it can be shown to be locally a product of subcones and applying Bär's results to each of the subcones allows one to write local forms for the metrics on M in terms of (double) warped products (12).

For example, in the case of $M = S^n$, the cone is $\overline{M} = \mathbb{R}^{n+1} \setminus \{0\}$, but the metric extends smoothly to the origin. The space of parallel (half-)spinors on $\mathbb{R}^{n+1}$ is isomorphic to the relevant (half-)spinor representation of Spin_{n+1}.

3.4 The Killing superalgebra

To a riemannian manifold admitting real Killing spinors we may associate an algebraic structure called the Killing superalgebra which extends the Lie algebra of isometries in the following way. The underlying vector space is $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$, where $\mathfrak{k}_0$ is the Lie algebra of isometries and $\mathfrak{k}_1$ is the space of Killing spinors with $\lambda = \frac{1}{2}$. (There is a similar story for $\lambda = -\frac{1}{2}$.) The bracket on $\mathfrak{k}$ consists of three pieces: the Lie bracket on $\mathfrak{k}_0$, a map $\mathfrak{k}_0 \otimes \mathfrak{k}_1 \to \mathfrak{k}_1$ and a map $\mathfrak{k}_1 \otimes \mathfrak{k}_1 \to \mathfrak{k}_0$. Depending on dimension and signature, the latter map may be symmetric or antisymmetric, whence the resulting bracket might correspond (if the Jacobi identity is satisfied) to a Lie algebra or a Lie superalgebra. In the riemannian examples in this section we will recover Lie algebras, but in the lorentzian examples common in supergravity the similar construction leads to Lie superalgebras. Let us now define these maps.

The map $\mathfrak{k}_1 \otimes \mathfrak{k}_1 \to \mathfrak{k}_0$ is induced from the algebraic map $[-,-]$ in Eq. (8), which explains the notation. As we saw before the image indeed consists of Killing vector fields.

The map $\mathfrak{k}_0 \otimes \mathfrak{k}_1 \to \mathfrak{k}_1$ is given by the spinorial Lie derivative of Lichnerowicz and Kosmann(-Schwarzbach) (13) and which we now define. If X is a vector field on M, then let $A_X : TM \to TM$ denote the endomorphism of the tangent bundle defined by $A_X Y = -\nabla_Y X$, for ∇ the Levi-Cività connection. The vector field X is Killing if and only if A_X is skewsymmetric relative to the metric; that is, if and only if $A_X \in \mathfrak{so}(TM)$. Let $\rho : \mathfrak{so}(TM) \to \mathrm{End}(S(M))$ denote the spin representation and define the **spinorial Lie derivative** along a Killing vector X by

$$\mathscr{L}_X = \nabla_X + \rho(A_X). \tag{10}$$

In fact, this Lie derivative makes sense on sections of any vector bundle associated to the orthonormal frame bundle provided that we substitute ρ by the relevant representation. For instance, on the tangent bundle itself, we have

$$\mathscr{L}_X Y = \nabla_X Y + A_X Y = \nabla_X Y - \nabla_Y X = [X,Y],$$

as expected. The spinorial Lie derivative satisfies the following properties for all Killing vectors X, Y, spinors ε, functions f and arbitrary vector fields Z:

- $\mathscr{L}_X$ is a derivation, so that

$$\mathscr{L}_X(f\varepsilon) = X(f)\varepsilon + f\mathscr{L}_X\varepsilon; \tag{11}$$

- $X \mapsto \mathscr{L}_X$ is a representation of the Lie algebra of Killing vector fields:

$$\mathscr{L}_X\mathscr{L}_Y - \mathscr{L}_Y\mathscr{L}_X = \mathscr{L}_{[X,Y]}\ ; \tag{12}$$

- $\mathscr{L}_X$ is compatible with Clifford multiplication:

$$\mathscr{L}_X(Z\cdot\varepsilon) = [X,Z]\cdot\varepsilon + Z\cdot\mathscr{L}_X\varepsilon; \tag{13}$$

- and $\mathscr{L}_X$ preserves the Levi-Cività connection:

$$\mathscr{L}_X\nabla_Z - \nabla_Y\mathscr{L}_X = \nabla_{[X,Z]}. \tag{14}$$

It follows from Eqs. (13) and (14) that the Lie derivative of a Killing spinor along a Killing vector is again a Killing spinor. Indeed, let ε be a Killing spinor and let X be a Killing vector. We have for all vector fields Y that

$$\begin{aligned}\nabla_Y \mathscr{L}_X \varepsilon &= \mathscr{L}_X \nabla_Y \varepsilon - \nabla_{[X,Y]}\varepsilon && \text{(using (14))}\\ &= \lambda \mathscr{L}_X (Y\cdot\varepsilon) - \lambda [X,Y]\cdot\varepsilon && \text{(since } \varepsilon \text{ is Killing)}\\ &= \lambda Y\cdot \mathscr{L}_X\varepsilon, && \text{(using (13))}\end{aligned}$$

as advertised. We define $[-,-]:\mathfrak{k}_0\otimes\mathfrak{k}_1\to\mathfrak{k}_1$ by $[X,\varepsilon]:=\mathscr{L}_X\varepsilon$.

Of course, the existence of a bracket is not enough to conclude that $\mathfrak{k}$ is Lie (super)
algebra: one must also check the Jacobi identity. The Jacobi identity is the vanishing of a tensor in $\mathfrak{k}\otimes\Lambda^3\mathfrak{k}^*$. Since $\mathfrak{k}=\mathfrak{k}_0\oplus\mathfrak{k}_1$ and the bracket respects the $\mathbb{Z}_2$ grading, there are four components to the Jacobi identity. The component in $\mathfrak{k}_0\otimes\Lambda^3\mathfrak{k}_0$ vanishes due to the Jacobi identity of the Lie algebra $\mathfrak{k}_0$. The component in $\mathfrak{k}_1\otimes\Lambda^2\mathfrak{k}_0^*\otimes\mathfrak{k}_1^*$ vanishes because of the fact that $\mathfrak{k}_1$ is a representation of $\mathfrak{k}_0$; indeed, this identity says that if $X,Y\in\mathfrak{k}_0$ and $\varepsilon\in\mathfrak{k}_1$, then

$$[X,[Y,\varepsilon]]-[Y,[X,\varepsilon]]=[[X,Y],\varepsilon],$$

which is precisely Eq. (12). The component in $\mathfrak{k}_0\otimes\Lambda^2\mathfrak{k}_1^*\otimes\mathfrak{k}_0^*$ vanishes because the bracket $\mathfrak{k}_1\otimes\mathfrak{k}_1\to\mathfrak{k}_0$ is $\mathfrak{k}_0$-equivariant. Indeed, if $X\in\mathfrak{k}_0$ and $\varepsilon_i\in\mathfrak{k}_1$ for $i=1,2$, then for all vector fields Y,

$$\begin{aligned}g([X,[\varepsilon_1,\varepsilon_2]],Y) &= g(\mathscr{L}_X[\varepsilon_1,\varepsilon_2],Y)\\ &= Xg([\varepsilon_1,\varepsilon_2],Y) - g([\varepsilon_1,\varepsilon_2],\mathscr{L}_X Y) && \text{(since } X \text{ is Killing)}\\ &= X(\varepsilon_1,Y\cdot\varepsilon_2) - (\varepsilon_1,\mathscr{L}_X Y\cdot\varepsilon_2)\\ &= (\mathscr{L}_X\varepsilon_1,Y\cdot\varepsilon_2)+(\varepsilon_1,\mathscr{L}_X(Y\cdot\varepsilon_2)) - (\varepsilon_1,\mathscr{L}_X Y\cdot\varepsilon_2)\\ &= (\mathscr{L}_X\varepsilon_1,Y\cdot\varepsilon_2)+(\varepsilon_1,Y\cdot\mathscr{L}_X\varepsilon_2) && \text{(using (13))}\\ &= g([[X,\varepsilon_1],\varepsilon_2],Y)+g([\varepsilon_1,[X,\varepsilon_2]],Y).\end{aligned}$$

The final component of the Jacobi identity lives in the $\mathfrak{k}_0$-invariant subspace of $\mathfrak{k}_1\otimes\Lambda^3\mathfrak{k}_1^*$. This identity does not seem to follow formally from the construction, but requires a case-by-case argument. In some cases it follows because there simply are no $\mathfrak{k}_0$-invariant tensors in $\mathfrak{k}_1\otimes\Lambda^3\mathfrak{k}_1^*$, but this is not universal and in many cases one needs to perform an explicit calculation. Luckily, for the examples in this note, the representation-theoretic argument will suffice.

3.5 Equivariance of the cone construction

In order to calculate or simply identify the Killing superalgebras it is often convenient to work in the cone. This requires understanding how to lift the calculation of the Lie derivative of a Killing spinor along a Killing vector to the cone. In (14) it is shown that the cone construction is equivariant under the action of the isometry group of (M,g). We will work at the level of the Lie algebra. Every Killing

vector on (M,g) defines a Killing vector on the cone $(\overline{M},\overline{g})$. Generically there are no other Killing vectors on the cone, except in the case when (M,g) is the round sphere and hence the cone is flat. Let X be a Killing vector on (M,g) and let $\overline{X}$ denote its lift to a Killing vector on the cone. Similarly let ε be a Killing spinor on (M,g) and let $\overline{\varepsilon}$ denote the parallel spinor on the cone to which it lifts. Then it is proved in (14) that

$$\mathscr{L}_{\overline{X}}\overline{\varepsilon} = \overline{\mathscr{L}_X \varepsilon},$$

which suggests a way to calculate the bracket $[-,-] : \mathfrak{k}_0 \otimes \mathfrak{k}_1 \to \mathfrak{k}_1$:

- we lift the Killing vectors in $\mathfrak{k}_0$ and the Killing spinors in $\mathfrak{k}_1$ to Killing vectors and parallel spinors, respectively, on the cone;
- we compute the spinorial Lie derivative there; and
- we restrict the result to a Killing spinor on (M,g).

Although somewhat circuitous, this procedure has the added benefit that the Lie derivative of a parallel spinor is an algebraic operation:

$$\mathscr{L}_{\overline{X}}\overline{\varepsilon} = \rho(A_{\overline{X}})\overline{\varepsilon}.$$

Since parallel spinors are determined by their value at any one point, we can work at a point and we see that the above formula corresponds to the restriction of the spin representation of $\mathfrak{so}_{n+1}$ to the subalgebra corresponding to the image of $\mathfrak{k}_0$ in $\mathfrak{so}_{n+1}$, acting on the subspace of the spinor module which is invariant under the holonomy algebra of the cone. For the case of the round spheres which will occupy us in this paper, the holonomy algebra is trivial and the isometries act linearly in the cone, whence $A_{\overline{X}} = -\overline{\nabla}\overline{X}$ is actually constant. Therefore the above action is precisely the standard action of $\mathfrak{k}_0 = \mathfrak{so}_{n+1}$ on the relevant spinor module.

There is no need to lift the bracket $\mathfrak{k}_1 \otimes \mathfrak{k}_1 \to \mathfrak{k}_0$ to the cone, but it is possible to do this as well. The only point to notice is that in the cone we do not square parallel spinors to parallel vectors, but to parallel 2-forms, which are constructed out of the lifts of the Killing vectors on (M,g).

4 The Killing Superalgebras of S^7, S^8 and S^{15}

In this section we will exhibit the Killing superalgebras of some low-dimensional spheres S^n, for $n = 7, 8, 15$, and will show that they are Lie algebras isomorphic to $\mathfrak{so}_9$, $\mathfrak{f}_4$ and $\mathfrak{e}_8$, respectively. The strategy is to exploit the equivariance of the cone construction to show that these Killing algebras are isomorphic to the Lie algebras constructed in (1).

4.1 $\mathfrak{k}(S^7) \cong \mathfrak{so}_9$

The isometry Lie algebra of the unit sphere in $\mathbb{R}^8$ is $\mathfrak{so}_8$, acting via linear vector fields on $\mathbb{R}^8$ which are tangent to the sphere. The 7-sphere admits the maximal number of Killing spinors of either sign of the Killing constant, which here is 8. Lifting them to the cone, we have $\mathfrak{so}_8$ acting on the positive chirality spinor module $\mathfrak{S}_+$ which is real and eight-dimensional. The Killing superalgebra is thus

$\mathfrak{k} = \mathfrak{so}_8 \oplus \mathfrak{S}_+$ with the following brackets: $\mathfrak{so}_8 \subset \mathfrak{k}$ is a Lie subalgebra, $\mathfrak{so}_8 \otimes \mathfrak{S}_+ \to \mathfrak{S}_+$ is the standard action and the map $\Lambda^2\mathfrak{S}_+ \to \mathfrak{so}_8$ is the transpose of the previous map relative to the inner products on both vectors and spinors. The map is skewsymmetric as shown because the spinor inner product is symmetric. Therefore we will obtain a Lie algebra. Observe that triality says that $\Lambda^2\mathfrak{S}_+ \cong \Lambda^2 V$, so that this map is actually an isomorphism in this case. The Jacobi identity requires the vanishing of a trilinear map $\Lambda^3\mathfrak{k} \to \mathfrak{k}$. The only component which is in question is the one in $\Lambda^3\mathfrak{S}_+ \to \mathfrak{S}_+$. Using the inner product on $\mathfrak{S}_+$ we may identify this with an $\mathfrak{so}_8$-invariant element in $\mathfrak{S}_+ \otimes \Lambda^3\mathfrak{S}_+$, but it may be shown the only such element is the zero map. Indeed, letting $\mathfrak{S}_+$, $\mathfrak{S}_-$ and V have Dynkin indices $[0001]$, $[0010]$ and $[1000]$, respectively, we find that $\Lambda^3\mathfrak{S}_+$ is irreducible with Dynkin index $[1010]$, corresponding to the 56-dimensional kernel of the Clifford multiplication $V \otimes \mathfrak{S}_- \to \mathfrak{S}_+$. Finally, a roots-and-weights calculation shows that

$$\mathfrak{S}_+ \otimes \Lambda^3\mathfrak{S}_+ \cong [0020] \oplus [0100] \oplus [1011] \oplus [2000],$$

whence there is no nontrivial invariant subspace. The Lie algebra structure just defined on $\mathfrak{k}$ is 36-dimensional and coincides with $\mathfrak{so}_9$.

4.2 $\mathfrak{k}(S^8) \cong \mathfrak{f}_4$

The isometry Lie algebra of the unit sphere in $\mathbb{R}^9$ is $\mathfrak{so}_9$, acting via linear vector fields on $\mathbb{R}^9$ which are tangent to the sphere. The 8-sphere admits the maximal number of Killing spinors of either sign of the Killing constant, which here is 16. Lifting them to the cone, we have $\mathfrak{so}_9$ acting on the spinor module $\mathfrak{S}$ which is real and sixteen-dimensional. The Killing superalgebra is $\mathfrak{k} = \mathfrak{so}_9 \oplus \mathfrak{S}$ with the following brackets: $\mathfrak{so}_9$ is a Lie subalgebra, $\mathfrak{so}_9 \otimes \mathfrak{S} \to \mathfrak{S}$ is the standard action of $\mathfrak{so}_9$ on its spinor representation, and $\Lambda^2\mathfrak{S} \to \mathfrak{so}_9$ is the transpose of the standard action using the inner products on vectors and spinors. Since the spinor inner product is symmetric, the map is skewsymmetric as shown. This means that we will obtain a Lie algebra. The only nontrivial component of the Jacobi identity lives in the subspace of $\mathfrak{so}_9$-equivariant maps $\Lambda^2\mathfrak{S} \to \mathfrak{S}$, or using the inner product, an $\mathfrak{so}_9$-invariant element of $\mathfrak{S} \otimes \Lambda^3\mathfrak{S}$. However one can check that there are no such invariants. Indeed, since $\mathfrak{S}$ has Dynkin index $[0001]$, a roots-and-weights calculation shows that

$$\Lambda^3\mathfrak{S} \cong [0101] \oplus [1001], \tag{15}$$

where the representations on the right-hand side have dimensions 432 and 128, respectively. Indeed, $[1001]$ is the kernel of the Clifford multiplication $V \otimes \mathfrak{S} \to \mathfrak{S}$. Tensoring the first with $\mathfrak{S}$ we obtain

$$\begin{aligned}[0101] \otimes [0001] \cong\ & [0002] \oplus [0010] \oplus [0100] \oplus [0102] \\ & \oplus [0110] \oplus [0200] \oplus [1002] \oplus [1010] \oplus [1100],\end{aligned}$$

whereas tensoring the second with $\mathfrak{S}$ we obtain

$$\begin{aligned}[1001] \otimes [0001] =\ & [0002] \oplus [0010] \oplus [0100] \oplus [1000] \oplus [1002] \\ & \oplus [1010] \oplus [1100] \oplus [2000].\end{aligned}$$

It is plain that there are no invariants in either expression. The resulting Lie algebra has dimension $36+16=52$ and can be shown (1) to be a compact real form of $\mathfrak{f}_4$. Unlike the case of $\mathfrak{so}_9$ in Sect. 4.1, here $\Lambda^2\mathfrak{S}\to\mathfrak{so}_9$ is not an isomorphism: indeed $\Lambda^2\mathfrak{S}\cong\Lambda^2V\oplus\Lambda^3V$.

4.3 $\mathfrak{k}(S^{15})\cong\mathfrak{e}_8$

The isometry Lie algebra of the unit sphere in $\mathbb{R}^{16}$ is $\mathfrak{so}_{16}$. The 15-sphere admits the maximal number of Killing spinors of either sign of the Killing constant, which here is 128. Lifting them to the cone, we have $\mathfrak{so}_{16}$ acting on the spinor module $\mathfrak{S}_+$ which is real and 128-dimensional. The Killing superalgebra is $\mathfrak{k}=\mathfrak{so}_{16}\oplus\mathfrak{S}_+$ with the following brackets: $\Lambda^2\mathfrak{so}_{16}\to\mathfrak{so}_{16}$ is the Lie bracket, $\mathfrak{so}_{16}\otimes\mathfrak{S}_+\to\mathfrak{S}_+$ is the action of $\mathfrak{so}_{16}$ on its half-spinor representation and $\Lambda^2\mathfrak{S}_+\to\mathfrak{so}_{16}$ the transpose map using the inner products. As before, since the spinor inner product is symmetric, the map is skewsymmetric as shown. This means that we will obtain a Lie algebra. The resulting bracket can be seen to satisfy the Jacobi identity. Indeed, the only nontrivial component of the Jacobi identity defines an $\mathfrak{so}_{16}$-equivariant map $\Lambda^3\mathfrak{S}_+\to\mathfrak{S}_+$. Since the inner product is non-degenerate on $\mathfrak{S}_+$, we can think of this as an $\mathfrak{so}_{16}$-invariant element of $\mathfrak{S}_+\otimes\Lambda^3\mathfrak{S}_+$, but we can see that no such nontrivial element exists. Indeed, letting $[00000001]$ denote the Dynkin index of $\mathfrak{S}_+$, we find that

$$\Lambda^3\mathfrak{S}_+\cong[00001001]\oplus[01000010]\oplus[10000001],$$

whence tensoring each of the modules in the right-hand side with $\mathfrak{S}_+$ we obtain

$$\begin{aligned}[00001001]\otimes[00000001]=\,&[00000011]\oplus[00001000]\oplus[00001002]\oplus[00001100]\\&\oplus[00010011]\oplus[00011000]\oplus[00100002]\oplus[00100100]\\&\oplus[01000011]\oplus[01001000]\oplus[10000002]\oplus[10000100],\end{aligned}$$

$$\begin{aligned}[00000001]\otimes[01000010]=\,&[00000011]\oplus[00001000]\oplus[00100000]\oplus[01000011]\\&\oplus[01001000]\oplus[01100000]\oplus[10000020]\oplus[10000100]\\&\oplus[10010000]\oplus[11000000],\end{aligned}$$

and

$$\begin{aligned}[00000001]\otimes[10000001]=\,&[00000011]\oplus[00001000]\oplus[00100000]\oplus[10000000]\\&\oplus[10000002]\oplus[10000100]\oplus[10010000]\oplus[11000000].\end{aligned}$$

In all cases we see that there is a nonzero invariant element. The resulting Lie algebra has dimension $120+128=248$ and can be shown (1; 2) to be isomorphic to the compact real form of $\mathfrak{e}_8$. Choosing $i\mathfrak{S}_+$ instead of $\mathfrak{S}_+$, we obtain the maximally split real form of $\mathfrak{e}_8$ which has been the focus of recent attention (15). Notice that again $\Lambda^2\mathfrak{S}_+\to\mathfrak{so}_{16}$ is not an isomorphism, instead $\Lambda^2\mathfrak{S}_+\cong\Lambda^2V\oplus\Lambda^6V$.

This construction of $\mathfrak{e}_8$ is also explained in (2, 6.A), where the nontrivial component of the Jacobi identity is proved combinatorially using Fierz identities.

5 Conclusion

We have seen that a notion arising from supergravity, namely the Killing superalgebra, when applied in a classical context, yields a geometric construction of the exceptional Lie algebras of type F_4 and E_8. This was accomplished by using Bär's cone construction to relate the Killing superalgebra to the well-known construction of these algebras using spin groups and their spinor representations.

There are a number of things left to explore in relation to the construction presented in this paper, some of which we are actively considering:

- *Further riemannian examples?* The three examples considered here are of the following general form: $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$, where $\mathfrak{k}_0$ is a Lie subalgebra, $\mathfrak{k}_1$ an $\mathfrak{k}_0$-module, where there are $\mathfrak{k}_0$-invariant positive-definite inner products on $\mathfrak{k}_0$ and $\mathfrak{k}_1$ and hence on $\mathfrak{k}$ by declaring $\mathfrak{k}_0$ and $\mathfrak{k}_1$ to be orthogonal. The bracket $\Lambda^2\mathfrak{k}_1 \to \mathfrak{k}_0$ is defined precisely by the condition that the resulting inner product on $\mathfrak{k}$ be ad-invariant. All but one component of the Jacobi identity of $\mathfrak{k}$ vanish. If Jacobi is satisfied, then we obtain a Lie algebra with a symmetric split and a positive-definite ad-invariant scalar product. This means we have a riemannian symmetric space and in fact the nontrivial Jacobi identity is the algebraic Bianchi identity for the would-be curvature tensor. We may therefore read off the possible such constructions from the list of symmetric spaces whose isotropy representation is spinorial, in which case the only examples are the above ones and the ones involving the exceptional Lie algebras E_6 and E_7, about which there is more below. At any rate, we have looked explicitly at riemannian spheres in dimension ≤ 40 which could give rise to Lie algebras, and have checked that the nontrivial Jacobi identity cannot follow trivially from representation theory. It is therefore doubtful that other examples exist of *precisely* this construction in riemannian signature.
- *Killing superalgebras of "spheres" in arbitrary signature.* Considering other signatures (and hence possibly also imaginary Killing spinors) might provide geometric realisations of Lie superalgebras.
- *A similar construction for the remaining exceptional Lie algebras.* In the case of E_6 and E_7, $\mathfrak{k}_0$ also contains "R-symmetries" which do not act geometrically on the manifold. Understanding these cases should help to understand conformal Killing superalgebras. There does not seem to be a construction of G_2 using only spinors.
- *Of which structure on S^{15} is E_8 the automorphism group?* The existence of a Lie group is most naturally explained as automorphisms of some structure. The construction of E_8 out of the 15-sphere suggests that there ought to be some structure on S^{15} of which E_8 is the automorphism group. This may also provide a simple proof of the Jacobi identity without resorting to Fierz or roots-and-weights combinatorics.

I hope to report answers to some of these questions in the near future.

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