

# Measuring the spin parameter of Kerr black holes and naked singularities

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## Abstract

The possibility that the existence of black holes as compact objects at the cores of galaxies has become high. The black hole is said to cast an apparent shape (or a shadow) as an optical appearance because of the strong gravitational field in the black hole itself. The apparent shape is of varied forms mainly depending on the spin parameter and the inclination angle of the object. In this paper, we investigate whether it is possible to determine the spin parameter and the inclination angle by observing the apparent shape, on the assumption that the intended compact object is the Kerr space-time. In particular, we define observables which definitely characterize the apparent shape. We find that one can estimate the spin parameter and the inclination angle of the Kerr black hole even when it is naked singularity, by measuring the observables. As a prerequisite, one needs only information of the mass of the object to say so. The result will become a realistic method which come a step closer to probe the Galactic centre using future advanced interferometers.

## 1 Introduction

It is widely believed that there exist black holes at the cores of many galaxies. Sagittarius A\*, which is the ultra compact radio source at the centre of the Milky Way, is highly likely to be a black hole. The possibility of direct observation of black holes by the future interferometers is becoming higher. However, the research of the observation of black holes in a theoretical approach is still lacking.

The theory of gravitational lensing has been developed in the weak field approximation and has been succeed to explain many physical and astronomical observations. The influence of the strong gravity of the black hole appears when the photon passes the vicinity of the origin of the gravity. In order to obtain a physical information of black holes holding the strong gravity, the direct imaging of the apparent shapes are important to be investigated [1, 2, 3, 4, 5, 6].

As a final state of a realistic gravitational collapse, the solutions of Einstein equations with strong gravity generally possess a space-time singularity. If the space-time singularity has appeared from physically reasonable initial conditions, the space-time singularity is hidden within the event horizon, so-called the cosmic censorship hypothesis, which was proposed by Penrose. It implies that a naked singularity which is a space-time singularity which is uncovered by event horizon does not existed. However, the proof for the hypothesis have not been successful up to the present date. So it is still one of the most important problem in general relativity. There is a possibility that the candidate of the black hole could be a naked singularity [7, 8].

To reach the information of the black holes such as the spin parameter and the inclination angle, we have to construct a method for a measurement of parameters with the flexibility to meet the case that the compact object is not to be a black hole but to be a naked singularity. The detail calculations of the following contents will be displayed in [9].

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## 2 Equations of geodesic motion

The Kerr space-time in Boyer-Lindquist coordinates [10] is given by

$$ds^2 = - \left( 1 - \frac{2Mr}{\rho^2} \right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 - \frac{4Mra \sin^2 \theta}{\rho^2} dt d\phi + \frac{A \sin^2 \theta}{\rho^2} d\phi^2, \quad (1)$$

where  $\Delta := r^2 - 2Mr + a^2$ ,  $\rho^2 := r^2 + a^2 \cos^2 \theta$ , and  $A := (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta$ .  $M$  and  $a$  are the mass and the spin parameter of the Kerr space-time, respectively. The regular horizon condition for Kerr space-time to be a black hole is  $|a| \leq M$ . In this paper, we deal with both cases that regular horizon condition is satisfied and not satisfied  $M < |a|$ , which the latter case is to be a naked singularity.

We can have the general solution of geodesic motion in the following first-order form [11],

$$\begin{aligned} \rho^2 \frac{dr}{d\lambda} &= \sigma_r \sqrt{R}, \quad R := [(r^2 + a^2) E - aL_z]^2 - \Delta[(L_z - aE)^2 + mr^2 + \mathcal{Q}], \\ \rho^2 \frac{d\theta}{d\lambda} &= \sigma_\theta \sqrt{\Theta}, \quad \Theta := \mathcal{Q} - [a^2(m^2 - E^2) + L_z^2 \csc^2 \theta] \cos^2 \theta \\ \rho^2 \frac{dt}{d\lambda} &= \frac{1}{\Delta} (AE - 2MraL_z), \\ \rho^2 \frac{d\phi}{d\lambda} &= \frac{1}{\Delta} [2MarE + L_z \csc^2 \theta (\rho^2 - 2Mr)]. \end{aligned} \quad (2)$$

$\lambda$  is the affine parameter and  $m$  is the mass of the particle.  $\sigma_r$  and  $\sigma_\theta$  are sign functions, which changes the signs independently at the turning point of the geodesic.  $E$  and  $L_z$  represent the energy and the angular momentum with respect to rotation axis of the space-time of the particle, respectively.  $\mathcal{Q}$  is Carter constant, which is the additional conserved quantity. For considering null geodesics  $m = 0$ ,  $E$  can be gauged away by rescaling of the affine parameter  $\lambda$  as

$$\tilde{R} := R/E^2, \quad \tilde{\Theta} := \Theta/E^2, \quad \xi := L_z/E, \quad \eta := \mathcal{Q}/E^2.$$

The conserved quantities  $\xi$  and  $\eta$  completely identify the null geodesic.

## 3 Measurement of the parameters

The strong gravitational field of the Kerr space-time are said to cast an apparent shape as an optical appearance. The apparent shapes of Kerr-Newman space-times were mutually complementary analyzed in [3, 6]. We would like to investigate, whether it is possible to estimate the spin parameter and the inclination angle by observing the apparent shape.

We assume that the observer is at the infinity of positive value  $r$  with inclination angle  $i$ . The celestial coordinates  $(\alpha, \beta)$  of the observer are defined as the apparent perpendicular distance of the image from the rotational axis and the apparent perpendicular distance of the image from its projection on the equatorial plane, respectively. The celestial coordinates have relations with conserved quantities  $\xi, \eta$  and the inclination angle  $i$  as,

$$\begin{aligned} \alpha(\xi, \eta; i) &:= \lim_{r \rightarrow \infty} \frac{rp^{(\varphi)}}{p^{(r)}} = -\xi \csc i, \\ \beta(\xi, \eta; i) &:= \lim_{r \rightarrow \infty} \frac{rp^{(\theta)}}{p^{(r)}} = \sqrt{\eta + a^2 \cos^2 i - \xi^2 \cot^2 i}, \end{aligned} \quad (3)$$

where  $(p^{(t)}, p^{(r)}, p^{(\theta)}, p^{(\phi)})$  are the tetrad components of the momentum of null geodesics with respect to locally nonrotating reference frames [12]. We define the apparent shape of the space-time as the region in the celestial coordinates which is the set of points coincide to the null geodesics that were launch from the infinity of positive value  $r$  toward the singularity and can not come back to the infinity of positive value  $r$  again. We assume that the light sources are at the infinity of positive value  $r$  with an abundant angular size.

(a) BHs

(b) NSs

Figure 1: Part (a) The contour map of the distorted rate  $\hat{d}$  (the green dashed curves) and the radius  $\hat{r}$  (the red solid curves). From this figure, we can estimate the spin parameter  $a$  of the Kerr black hole and the inclination angle  $i$  of the observer. Part (b) The contour map of the central angle  $\hat{c}$  (the green dashed curves) and the radius  $\hat{r}$  (the red solid curves). From this figure, we can estimate the spin parameter  $a$  of the Kerr naked singularity and the inclination angle  $i$  of the observer.

In the case of the Kerr black holes, we propose two observables that characterize the apparent shape, precisely. First we approximate the contour of the apparent shape by the circle passing through the three points which are at the extreme right, the upper limit, and the lower limit of the apparent shape in celestial coordinates. We define the radius  $\hat{r}$  of the apparent shape as the radius of the circle which we used for approximation. Calling into account the hollow of the extreme left of the apparent shape  $\Delta\hat{r}$  which is the maximal distance of the difference between the circle and the apparent shape, we define the distortion rate  $\hat{d}$  of the apparent shape as  $\hat{d} := \Delta\hat{r}/\hat{r}$ .

We can obtain the contour maps of the radius and the distortion rate and get them together to a single Fig. 1(a). If we can measure the radius  $\hat{r}$  and the distortion rate  $\hat{d}$  by observation, the spin parameter  $a$  and the inclination angle  $i$  could be restricted by putting Fig. 1(a) to practical use.

In the case of the Kerr naked singularities, the apparent shape is not always a single object. It consists of two isolated parts, at most. We have to consider which part of the apparent shape is important for observation. Based on this standpoint, the “arc” is one of an interesting object. In the sense of the capture region the “arc” is 1-dimensional object, which is measure zero, but in the realistic observation the neighborhood of the “arc” can be dark enough to be observed as a black “lunate surface”. We suppose a circle that passes both ends and the center of the “arc”, and prepare to define observables.

We propose here the radius  $\hat{r}$  and the central angle  $\hat{c}$  of the “arc”. The radius  $\hat{r}$  is defined as the radius of the circle which we supposed. The central angle  $\hat{c}$  is defined as the angular distance of the arc segment on the circle. The radius and the central angle feature the shape of the “arc”, which continuously changes by the spin parameter and the inclination angle. Fig. 1(b) is the contour map of the radius and the central angle, which enable one to estimate the spin parameter and the inclination angle by observing the two observables  $\hat{r}$  and  $\hat{c}$ .

## 4 Conclusion

We defined new observables and proposed a method for the estimation of the parameters of the Kerr space-time as the compact object. In fact, we have shown that one can estimate the spin parameter and the inclination angle of the Kerr space-time by measuring the observables of the apparent shape without degeneracy. However, the resolution capability of an observation equipment at present age makes a limitation for the measurement. We must hold up a hope for further progress in technology of future advanced interferometers.

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