

Comparison of smoothening flows for the topological charge in QCD-like theories

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We investigate properties of the topological charge for several $SU(N_C)$ gauge field ensembles for $N_C = 4, 5, 6$ with a single fermion in the two-index antisymmetric representation, covering multiple lattice spacings at otherwise approximately constant physical parameters. Comparing the topological charge defined by the Wilson flow and the overimproved DBW2 flow we find that already at small flow times the latter stabilizes on discrete values. We provide evidence that as the lattice spacing is lowered the Wilson flow also separates into discrete sectors at earlier flow times. Adopting the DBW2 definition in the remainder of the analysis, we do not see any evidence of fractional topological charges, which could in principle appear at finite lattice spacing.

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I. INTRODUCTION AND MOTIVATIONS

We recently started exploring the spectrum of $SU(N_C)$ orientifold theories with one dynamical quark flavor in the two-index (anti)symmetric representation [1–5]. In the large N_C limit at fixed 't Hooft coupling, i.e., in the Corrigan-Ramond limit [6], the theory should approach super-Yang-Mills as first conjectured in Refs. [7,8]. Predictions for the ratio of masses of the lightest scalar and pseudoscalar mesons have been obtained using either effective field theory or string theory duality approaches [9,10] and such predictions can be compared to lattice results, as done in Ref. [11] using our result from Ref. [1].

For $N_C = 3$ the two-index antisymmetric representation coincides with the fundamental representation and hence standard codes optimized for QCD can be used. For $N_C \neq 3$ this is no longer the case, and with increasing N_C simulations become more costly. Naively, the cost of a matrix-vector multiplication grows quadratically with N_C , while for matrix-matrix multiplications the scaling is cubic. One therefore expects an overall scaling with N_C slightly faster than quadratic, assuming that matrix-vector multiplications dominate the cost. This is consistent with our findings in Ref. [3]. In addition, we are entering a largely unexplored territory concerning the choices of bare

(Lagrangian) as well as algorithmic parameters. To the best of our knowledge no other study exists where lattice simulations of the exact same theories considered here are discussed. One of our main concerns, given the cost of the simulations, is critical slowing down, known to become more severe as N_C is increased [12]. This is usually monitored by looking at the Monte-Carlo distribution of a slow (i.e., with long autocorrelation) observable, typically the topological charge.

The first goal of this paper is to select a numerical definition of the topological charge which is cheap and stable, in a sense that will be clarified below. Second, the study at hand is part of our wider program to determine the mass spectrum and properties of orientifold theories in the large N_C limit. Among such properties, here we focus on the behavior of the topological charge for different choices of N_C and investigate the existence of fractional values. Indeed, for the two-index antisymmetric representation, the winding number (i.e., the topological charge) of a gauge configuration is in general expected to be quantized in units of $1/(N_C - 2)$, see also Ref. [13] for a discussion of a single flavor in the adjoint representation. However, with periodic boundary conditions (as used here) and for sufficiently smooth configurations the topological charge should take integer values, as first shown in Ref. [14]. When different boundary conditions, such as 't Hooft twisted boundary conditions, are used fractional charges do instead survive in the continuum limit [15–17]. In our case we hence expect noninteger charges to appear as cutoff effects, if at all. A similar study was conducted in Ref. [18] for the $SU(2)$ case of a single flavor in the two-index

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TABLE I. Parameters of the ensembles used in this study. The temporal extent is fixed to be $T = 3L$ for all ensembles. Quoted uncertainties are statistical only.

Name	N_C	β	κ	L/a	t_0/a^2	aM_π^{conn}	$L/\sqrt{t_0}$	$M_\pi^{\text{conn}}\sqrt{t_0}$	N_{conf}
N4L12	4	7.1	0.15770	12	0.8691(12)	0.6481(6)	12.8720(90)	0.6042(7)	648
N4L14	4	7.2	0.15651	14	1.3641(17)	0.5101(5)	11.9868(75)	0.5958(7)	677
N4L16	4	7.3	0.15525	16	1.8592(19)	0.4350(4)	11.7344(60)	0.5931(6)	640
N4L18	4	7.4	0.15401	18	2.3885(21)	0.3949(3)	11.6470(51)	0.6103(5)	583
N5L12	5	11.3	0.16089	12	1.1171(11)	0.7571(3)	11.3536(55)	0.8002(5)	1081
N5L14	5	11.4	0.16026	14	1.4800(14)	0.6624(3)	11.5078(53)	0.8058(5)	704
N5L16a	5	11.5	0.15959	16	1.8422(15)	0.5581(3)	11.7884(49)	0.7575(5)	565
N5L16b	5	11.55	0.15929	16	2.0409(21)	0.5922(2)	11.1997(58)	0.8460(5)	469
N6L16	6	16.5	0.16200	16	1.33944(64)	0.8354(1)	13.8248(33)	0.9668(3)	742

symmetric representation (sextet model). The study was quenched, with the topological charge defined through the index-theorem [19] and the conclusion was indeed that configurations with fractional charge can occur, but disappear as the continuum limit is approached.

Here we determine the topological charge using a gluonic definition from smoothed (gradient-flowed) gauge fields. We systematically study the dependence of the charge on the amount of smoothing applied, as well as on the discretization scheme employed for the flow equations. Reference [20] performs a similar study but at a single lattice spacing in pure Yang-Mills for $SU(2)$. The findings in this work extend beyond that by varying the lattice spacing, including dynamical fermions and exploring different gauge groups.

For $N_C = 4$ and $N_C = 5$ we generated a number of ensembles with approximately tuned spatial extents and connected pseudoscalar (M_π^{conn}) masses, but varying lattice spacings. For $N_C = 6$ we generated a single ensemble with similar volume and mass and an intermediate lattice spacing. The generation and properties of these ensembles and the smoothing flows are described in more detail in Sec. II. In Sec. III we determine the lattice spacings and investigate the topological charge and susceptibility as a function of flow time and the choice of flow. Finally, in Sec. IV we summarize our findings and provide an outlook.

II. COMPUTATIONAL SETUP

A. Ensemble parameters

The gauge configurations were generated using the HiRep code package [21–24]. For the gluonic part of the action, we employ the tree-level improved Lüscher-Weisz (LW) gauge action [25]. The fermionic part contains a single Dirac flavor in the two-index antisymmetric representation. We use the Wilson formulation with a tree-level improved clover term, i.e., setting $c_{SW} = 1$. Since our setup contains a single Dirac flavor, the rational hybrid Monte-Carlo (RHMC) algorithm [26] is required. For the numerical integration of the RHMC trajectories, we adopt a 4th-order Omelyan integrator [27] with a trajectory length of $\tau = 2$

molecular dynamic units (MDUs). The number of integration steps in the fermionic force (20–26 for $N_C = 4$, 30–36 for $N_C = 5$, and 36 for $N_C = 6$) is tuned to ensure a high acceptance rate. In the gauge sector we also use a 4th-order Omelyan integrator with two steps per fermion update. To reduce autocorrelation, we save every fourth configuration, which results in a separation of 8 MDUs between consecutively saved gauge configurations. For thermalization, the first 200 trajectories are discarded. All gauge configurations were generated on GPUs (AMD MI250x) using the LUMI-G partition, while all measurements were carried out on CPUs. Table I provides an overview of the ensembles generated for this study. The relevant input files are provided in the arXiv submission.

Since the simulated masses are rather heavy (see Table I), based on our experience in Ref. [1], we do not expect these simulations to be affected by a sign problem. In the future we will supplement the ensembles in this work with ensembles at lighter-quark masses and perform extrapolations of observables to the massless limit. In this context we will perform a study of the severity of the sign problem for the lightest simulated quark masses, analogous to that of Ref. [1].

B. Flow definitions

The gradient flow is a standard tool in lattice field theory, consisting in evolving gauge fields in a fictitious 5th dimension typically referred to as *flow time* t . This evolution is governed by a gauge-covariant diffusion equation

$$\frac{dB_{\mu,x}(t)}{dt} = D_\nu G_{\mu\nu,x}(t), \quad (2.1)$$

with initial conditions $B_{\mu,x}(t=0) = A_\mu(x)$, $G_{\mu\nu,x}(t)$ being the field strength of the field $B_{\mu,x}(t)$ at flow time t . The core idea is that, as shown in the original paper [28], the flow leads to an effective smearing of the gauge configurations over a length scale $\sqrt{8t}$, suppressing UV fluctuations and

automatically rendering a large class of composite operators renormalized at positive flow times.

On the lattice, different discretizations of Eq. (2.1) are possible, which can be rewritten as

$$a^2 \left(\frac{d}{dt} U_{x,\mu}(t) \right) U_{\mu,x}(t)^\dagger = -g_0^2 \partial_{\mu,x} S_{\text{flow}}(U), \quad (2.2)$$

where $\partial_{\mu,x}$ is the Lie-algebra valued derivative with respect to $U_{\mu,x}(t)$. S_{flow} , also referred to as *kernel action*, represents some discretized version of the gauge action that is used to integrate the flow equations and which, in principle, can be different from the one used to generate the configurations. One possible choice is to consider $\mathcal{O}(a^2)$ -improved actions defined with an appropriate combination of 6-link Wilson loops, beside the usual 4-link (plaquette) loop [29]. Considering only 4 and 6-links Wilson loops lying on a plane, we can define the kernel action as

$$S_{\text{flow}} = c_0 S_{\text{plaquette}} + c_1 S_{\text{rectangle}}, \quad (2.3)$$

parametrized by 2 coefficients c_0 and c_1 constrained to satisfy

$$c_0 + 8c_1 = 1 \quad (2.4)$$

in order to ensure that in the classical continuum limit, S_{flow} recovers the standard Yang-Mills action. Among others, popular choices include: $c_1 = 0$ (the standard plaquette action, commonly referred to as *Wilson flow*), $c_1 = -0.311$ (also called Iwasaki [30]), $c_1 = -1/12$ (tree-level improved Lüscher-Weisz (LW) [25]) and $c_1 = -1.4088$ (DBW2 [31,32]). In this work, we will consider the case of Wilson and DBW2 flows. A study of all the four mentioned choices for c_1 in pure $SU(2)$ Yang-Mills appeared recently in Ref. [20]. In the same reference, the action for a one-instanton configuration is tree-level evaluated to $\mathcal{O}(a^2)$. The relative correction to the continuum value is given by $-(1 + 12c_1)/5 \times (a/\rho)^2$, ρ being the radius of the instanton. As a result, for $1 + 12c_1 > 0$ the flow-action is expected to favor (by lattice artefacts) small size instantons, whereas for $1 + 12c_1 < 0$ larger instantons should be preferred. The latter choice leads to *overimprovement* as first discussed along the lines above but in the framework of cooling in Ref. [33]. The combination $1 + 12c_1$ in particular corresponds to the coefficient ϵ in Ref. [33] when only plaquettes and rectangles are considered.

We modified the HiRep code to include the rectangle terms in the kernel action. All flow measurements were performed using a third-order Runge-Kutta integration scheme with a fixed step size of 0.01 in lattice units. We tested our HiRep-based implementation of the DBW2 flow by comparing our results to data obtained using the implementation in LatticeGPU [34] on a test configuration. The results agree to machine precision.

As mentioned above, the length scale over which the configurations are smeared is $\sqrt{8t}$, also for the case of overimproved actions, as discussed in Appendix B of Ref. [20]. The flow time t should therefore be chosen to satisfy

$$2a \ll \sqrt{8t} \ll \frac{L}{2}, \quad (2.5)$$

in order to sufficiently suppress discretization (left inequality) and finite volume (right inequality) effects. As a remark, in the initial stages of this study we also considered the Iwasaki-flow which displays qualitatively similar features to DBW2. However, for the volumes at hand we found that on a subset of configurations the topological charge reached a stable value only for t uncomfortably close to the right bound in the equation above.

III. ANALYSIS

A. Scale setting

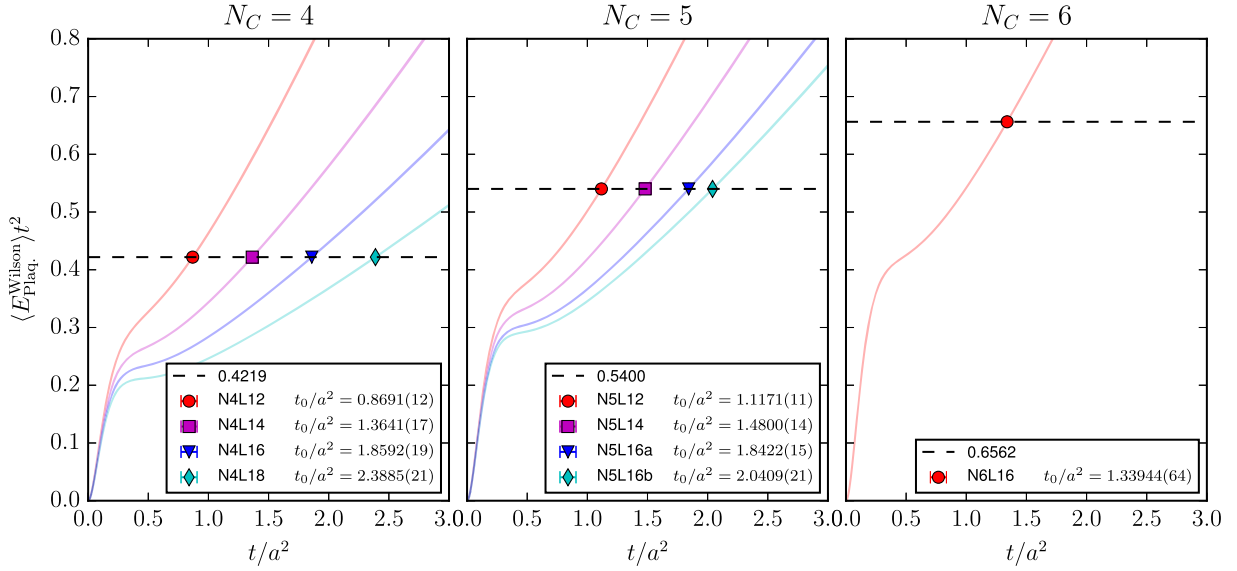
To set the scale of our simulations, we integrate the Wilson flow and compute the expectation value of the flowed energy density $\langle E_x(t) \rangle = \langle \text{tr} G_{\mu\nu,x}(t) G_{\mu\nu,x}(t) \rangle$, choosing the plaquette discretization E_{plaq} for the action density. The relative lattice scale is usually set by determining t_0 for which $\langle E_{\text{plaq}} \rangle t_0^2$ takes a particular value. For the case of $N_C = 3$ this value is typically chosen to be 0.3. Generalizing this for $N_C > 3$ while accounting for the leading N_C dependence of $\langle E \rangle$, and hence staying a fixed 't Hooft coupling, requires a rescaling by $(N_C^2 - 1)/N_C$ and so yields

$$\langle E_{\text{plaq}} \rangle t_0^2 = 0.3 \left(\frac{N_C^2 - 1}{N_C} \right) \frac{3}{8}. \quad (3.1)$$

This value can then be related to an absolute scale via the determination of a reference t_0^{ref} , obtained from a scale-setting fit. Since (to the best of our knowledge) no such determination exists for a single flavor ($N_f = 1$), for concreteness we use the average of the $N_f = 0$ [28] and $N_f = 2$ [35] results in QCD obtaining

$$\sqrt{8t_0^{\text{ref}}} = 0.45 \text{ fm}. \quad (3.2)$$

This yields lattice spacings $a \in [0.10, 0.17]$ fm ($[0.11, 0.15]$ fm) for the $N_C = 4$ ($N_C = 5$) ensembles and $a \sim 0.14$ fm for $N_C = 6$, as illustrated in Fig. 1. We stress that the absolute scale is not essential for this work, but useful to have an approximate estimate of the lattice spacings, masses and volumes under consideration. The values listed in Table I contain statistical uncertainties only which are estimated from a bootstrap analysis, not taking autocorrelations into account. Applying the Γ -method [36] instead leads to somewhat larger uncertainties, possibly


 FIG. 1. Determination of t_0/a^2 on all ensembles.

indicating the presence of autocorrelations in the data. In the worst case (on the N6L16 ensemble) the uncertainty estimate from the Γ -method is a factor 2.5 larger. Since in this work we do not present precision observables this is not expected to have any impact at this stage, but we will carefully investigate this in future studies.

B. Observables

For this work, the main observable of interest is the topological charge Q , defined in the classical continuum limit as the integral of the topological charge density $q(x)$

$$Q = \int d^4x q(x), \quad (3.3)$$

where q is defined as

$$q(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr}[F_{\mu\nu} F_{\rho\sigma}], \quad (3.4)$$

and $F_{\mu\nu}$ is the field strength tensor of the unflowed field. On the lattice, several discretizations of the topological charge density can be taken, all leading to Eq. (3.4) in the classical $a \rightarrow 0$ limit. One possible definition is¹

$$Q = \frac{1}{32\pi^2} \sum_x \epsilon_{\mu\nu\rho\sigma} \text{tr}[\hat{C}_{\mu\nu}(x) \hat{C}_{\rho\sigma}(x)], \quad (3.5)$$

where for concreteness \hat{C} is the clover discretization of the field strength which is used in our work. In principle,

¹For simplicity, we adopt the symbol Q to indicate both the continuum topological charge in Eq. (3.3) and its lattice discretized version in Eq. (3.5).

regardless of the discretization, the lattice topological charge must be properly renormalized. Alternatively, some smoothing techniques of the gauge configurations can be employed. Among others, popular choices include cooling [37–42], stout-smearing [43] and the gradient flow [28,44].² In practice, one also has to control the amount of smoothing applied to the gauge configurations. In the case of the gradient flow one needs to choose the flow time t_Q at which Q is computed. While Ref. [28] suggested to use $t_Q \simeq t_0$, we will explore the impact of choosing different kernel actions to integrate the flow as in Eq. (2.3) as well as the influence of t_Q . On the lattice, even after several smoothing steps are applied Q is not expected to take discrete values but rather to be distributed over real numbers clustered around peaks.³

The cumulants of the distribution of Q can be used to compute physical quantities, such as the *topological susceptibility* χ , defined as

$$\chi = \lim_{a \rightarrow 0} \lim_{V_4 \rightarrow \infty} \frac{\langle Q^2 \rangle}{V_4}, \quad (3.6)$$

where $V_4 = L^3 T$ is the 4-volume in physical units. Another observable that we will consider is the so-called *smoothness parameter* defined in Refs. [14,28] and adapted to our case as

$$h(p) = \text{ReTr} \left(\mathbb{1} - \prod_{(x,\mu) \in p} U_{x,\mu}(t) \right), \quad (3.7)$$

²Detailed comparisons between these methods can be found, e.g., in Refs. [42,45–47].

³This can be corrected for using rounded definitions, see, e.g., Refs. [12,48].

p being a given plaquette. We distinguish between

$$\begin{aligned} h_{\max} &= \max_p(h), \\ h_{\text{avg}} &= \text{avg}_p(h). \end{aligned} \quad (3.8)$$

Since the flow is a smoothening procedure, h_{avg} should monotonically decrease as a function of flow time. However, as we will see, h_{\max} is a local object and does not necessarily have a monotonic behavior. In order for the topological sector of gauge configurations to be well defined and characterizable, the smoothness parameter h_{\max} has to be smaller than a critical value of order 10^{-1} for the theories considered here [14,28]. Topological sectors cannot be connected by continuous deformations, and h_{\max} must exceed such a critical value when gauge configurations move among sectors. However, as discussed in Ref. [28], the occurrence of such configurations decreases proportionally to a^6 , which leads to the problem of topological freezing as the lattice spacing is reduced. This makes clear that studies of topological properties of a lattice gauge theory suffer a window problem. On the one side the lattice spacing should be fine enough such that configurations are sufficiently smooth to be classified according to a lattice version of the topological charge, while on the other side topological freezing should be avoided in order to have an adequate sampling of all the sectors.

C. Determination of the topological charge

At each positive flow time t we compute the topological charge for the Wilson-flowed and the DBW2-flowed configurations. Since we want to investigate the existence (or absence) of fractional topological charges, on a given configuration we desire a definition of Q which admits a window in flow time for which Q is insensitive of the exact choice of t_Q while adhering to the bounds in Eq. (2.5).

The top (bottom) panel of Fig. 2 shows the topological charge as defined by the Wilson (DBW2) flow as a function of flow time for our coarsest ensemble (N4L12). Each line corresponds to a different configuration. We observe that for this ensemble the Wilson flow does not admit such a definition, and instead find that even for large t_Q the topological charge fluctuates as a function of flow time. In contrast to this, the DBW2 flow does admit such an assignment. When instead considering our finest ensemble (N4L18), as shown in Fig. 3, we note that the rate of jumps between topological sectors decreases for both flows. For the Wilson flow, several configurations maintain an asymptotically stable value, but some jumps can still be seen even for the largest plotted flow times. For the DBW2 flow, the configurations quickly stabilize into discrete topological sectors.

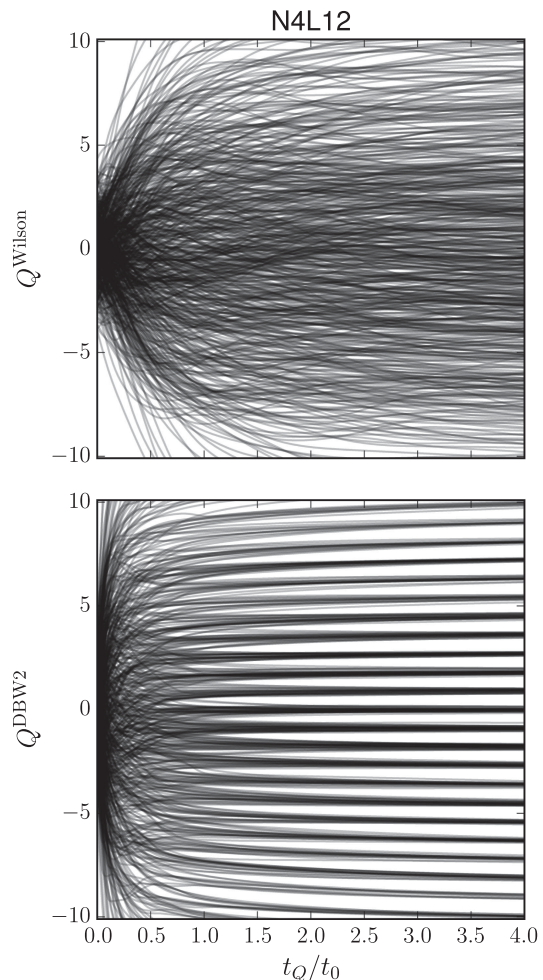


FIG. 2. Topological charge determined using the Wilson flow (top) and the DBW2 flow (bottom) on the coarsest $N_c = 4$ ensemble.

When rewriting the upper bound on t_Q in Eq. (2.5) above in terms of t_Q/t_0 we find

$$\frac{t_Q}{t_0} \ll \frac{1}{32} \frac{L^2}{t_0}. \quad (3.9)$$

Substituting the corresponding numbers from Table I we find the expected bounds to be $t_Q/t_0 \ll 5.18$ for N4L12 and $t_Q/t_0 \ll 4.24$ for N4L18. While we do not formally quantify this, by visual inspection of our data, e.g., in the top panel of Fig. 3, it appears that for the Wilson flow the number of jumps between Q -values on any given configuration decreases when comparing the interval $t_0 < t < 2t_0$ to $3t_0 < t < 4t_0$. Indeed for the latter interval a clear clustering around discrete values is visible, which is not the case for the first. In the following we explore the behavior as the flow time is increased further toward and beyond the upper bound in Eq. (3.9). While we are aware that this eventually introduces nonlocalities, we are curious about the qualitative features this induces. Figure 4 shows

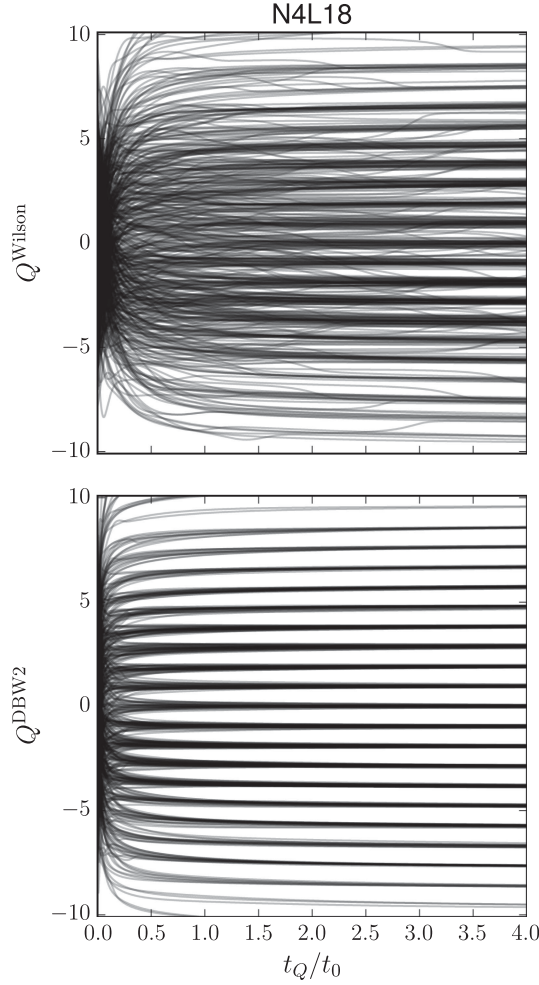


FIG. 3. Topological charge determined using the Wilson flow (top) and the DBW2 flow (bottom) on the finest $N_C = 4$ ensemble.

the topological charge Q , h_{\max} and h_{avg} as defined in Eq. (3.8) as a function of t_Q/t_0 for very long flow times on a representative configuration of the N4L12 ensemble. As expected for a smoothening flow, the value of h_{avg} monotonically decreases as a function of flow time for both choices of flow. For the DBW2 flow, this also holds for h_{\max} , indicating an approximately uniform smoothening, while for the Wilson flow this is not the case. Instead there are several occurrences where h_{\max}^{Wilson} suddenly increases before dropping again (note the logarithmic scale in the bottom panel). Since the average of h still decreases, this must be a local (at the lattice scale) effect. This observation is consistent with the expectation for non-overimproved actions, as they do not suppress small instantons, as discussed in Sec. II B and Ref. [33]. Turning our attention to the topological charge we find that Q^{DBW2} quickly becomes largely independent of flow time and slowly approaches an asymptotic value. We will discuss the fact that this value does not appear to be an

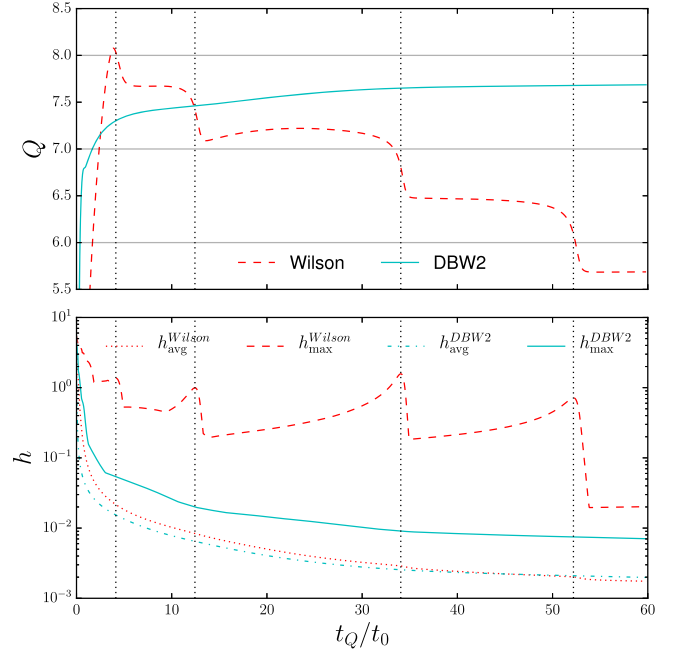


FIG. 4. Topological charge (top) as well as h_{avg} and h_{\max} (bottom) for the Wilson flow (red, dashed and dotted) and the DBW2 flow (cyan, solid and dash-dotted) as a function of very long flow times and on a single configuration of the N4L12 ensemble. The vertical dotted lines correspond to the location of local maxima in h_{\max}^{Wilson} .

integer in due course. The topological charge Q^{Wilson} instead displays metastable plateaus over some timescales, but continues to jump between different topological sectors even for very late times. In order to assess the existence of fractional topological charges, we hence prefer the DBW2 flow. From a closer look at the spikes in h_{\max}^{Wilson} , we find that they correlate with jumps in $Q^{\text{Wilson}}(t_Q)$ (as indicated by the vertical lines).

In Fig. 5 we depict the topological charge determined by the DBW2 flow configuration-by-configuration for flow times $t_Q = t_0, 2t_0, 4t_0$ and $8t_0$. As discussed before, we notice that discrete topological sectors form, but that they do not exactly coincide with integer values. Instead they appear to be multiples of a number close to, but slightly smaller than, unity. We observed in Figs. 2 and 3, that (within a given discrete sector) the topological charge continues to slowly grow in magnitude as a function of flow time. In Fig. 4 we saw that this trend continues even for very large flow times but still not reaching integer values. When comparing the N4L12 and the N4L18 ensembles in Fig. 5, we notice that with decreasing lattice spacing, the topological sectors approach integer values. Both of these effects contribute to the unit of discretization being smaller than one. In order to faithfully capture the discrete sectors when representing the data in histograms, in the following we use bin widths that reproduce this feature. In particular we will define the bin width ϵ such that

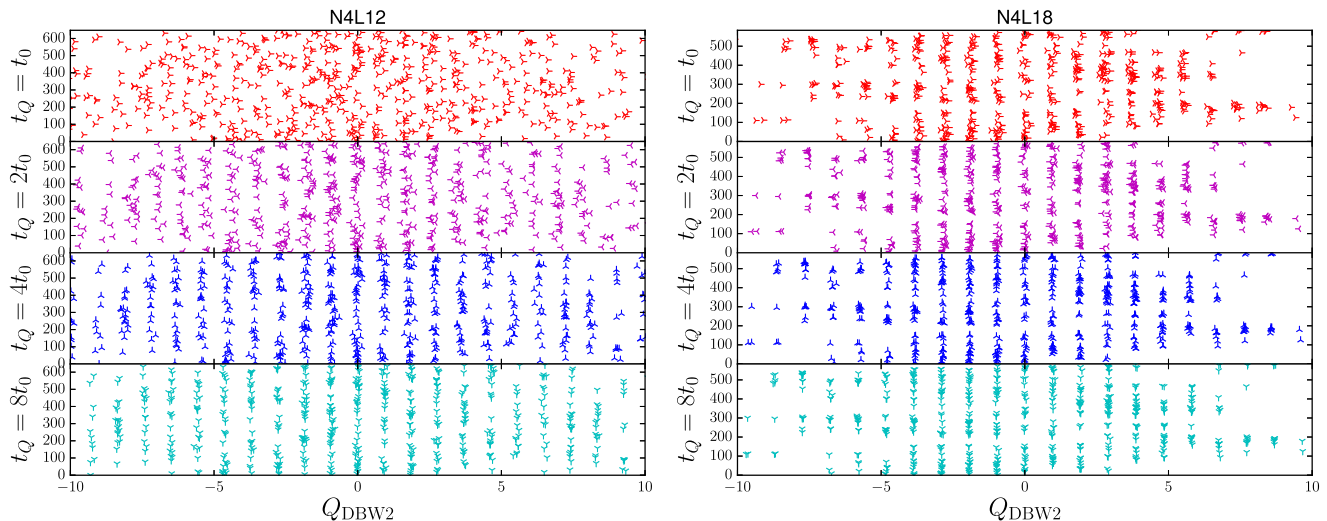


FIG. 5. Topological charge determination with the DBW2 flow for the N4L12 (left) and N4L18 (right) ensembles at 4 flow times (from top to bottom $t_Q = t_0$, $2t_0$, $4t_0$ and $8t_0$) plotted configuration by configuration.

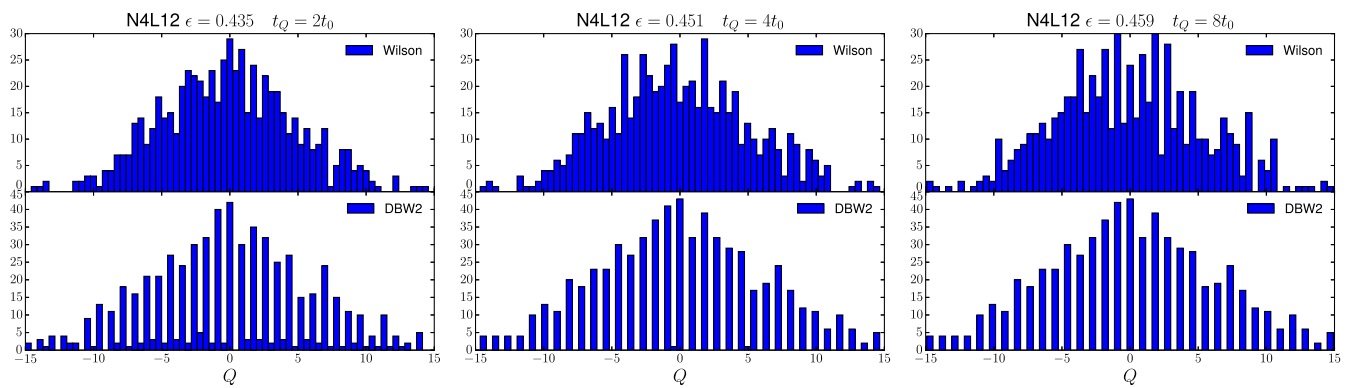


FIG. 6. Histograms of the topological charge for the Wilson flow (top) and the DBW2 flow (bottom) evaluated at flow time $t_Q = 2t_0$ (left), $t_Q = 4t_0$ (middle), and $t_Q = 8t_0$ (right) on the N4L12 ensemble.

2ϵ to a good approximation corresponds to the separation between adjacent topological sectors.

Figure 6 shows histograms of the topological charge evaluated at different flow times for the Wilson and the DBW2 flow on the N4L12 ensemble. The bin width is chosen to account for the quantized units of Q being different from one. In the left panel the histogram of Q is shown at $t_Q = 2t_0$. As can be inferred from Fig. 2 at this time the Wilson flow has not settled into any asymptotic value yet, leading to all bins in the central region being populated. The same figure indicates that for the DBW2 flow, the asymptotic sectors have been reached, i.e., on this ensemble no crossings between topological sectors occur for later flow times than $t_Q = 2t_0$. However, several configurations still slowly flow to those discrete values, leading to some contamination between the main bins. For the DBW2 flow this rapidly improves as the flow time is increased to $t_Q = 4t_0$ (middle) and $t_Q = 8t_0$ (right).

Contrary to this, even for this large flow time the Wilson flow displays results in the intermediate bins.

In Fig. 7 we present the histograms of the topological charge for all ensembles considered in this work. In each case the value of ϵ was tuned to ensure that the discrete bins are correctly captured. We find that Q does take discrete near-integer values in all cases and hence do not observe any fractional topological charges. We further find that the unit of discretization approaches 1 as the lattice spacing is reduced.

D. Dependence of gauge averages on flow smoothening

We now turn to assessing the impact of the effects discussed in the previous section on physical observables such as the topological susceptibility and related quantities. Similar studies of systematic effects associated with the smoothening radius—particularly in the context of comparisons between cooling and gradient flow—can be found

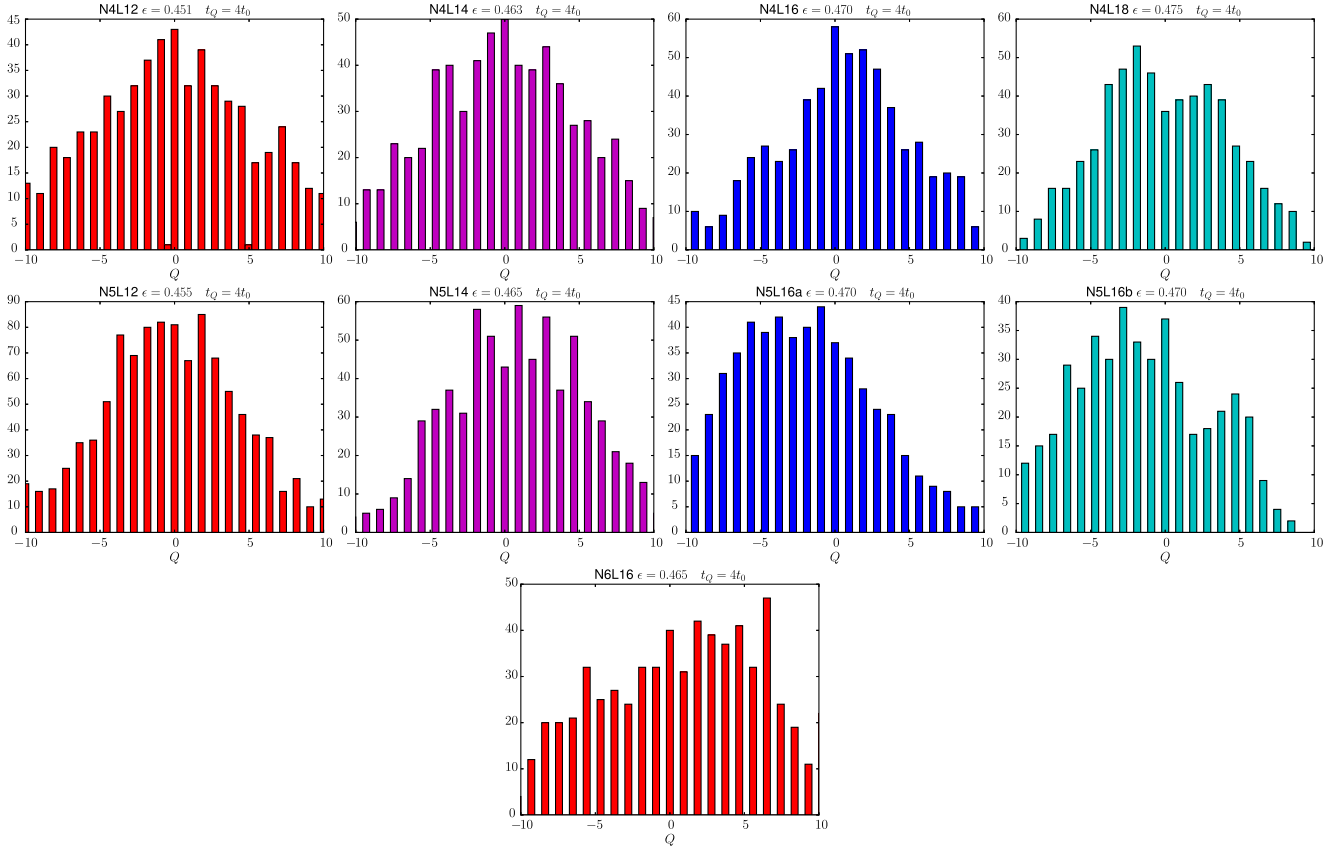


FIG. 7. Histograms of the topological charge defined via the DBW2 flow at $t = 4t_0$ for all ensembles considered in this study. From top to bottom N_C is increased from 4 to 6 and within each row the lattice spacing is reduced from left to right.

in Refs. [45–47,49]. In the same spirit, our goal is to investigate whether variations in the flow time t_Q or the choice of kernel action introduce significant artefacts on physical observables.

On our ensembles, i.e., at finite lattice spacing and volume, a direct computation of the topological susceptibility in the sense of Eq. (3.6), which requires the thermodynamic limit, is not possible. Nevertheless, we compute the quantity $\langle Q^2 \rangle / V$, which we refer to as the finite-volume topological susceptibility $\tilde{\chi}$, and compare the results obtained using the two flows. For each case, we consider several values of the flow time $t_Q \in \{t_0, 2t_0, 4t_0, 8t_0\}$ to assess the impact on the extracted values of $\tilde{\chi}$. We also determine the ratio of susceptibilities computed with different flows, defining

$$R = \frac{\tilde{\chi}^{\text{DBW2}}}{\tilde{\chi}^{\text{Wilson}}}, \quad (3.10)$$

which we expect to be less sensitive to slight mistunings of volumes and masses.

In Fig. 8 we plot this ratio as a function of the lattice spacing in units of $\sqrt{8t_0}$ for our ensembles with $N_C = 4$ and $N_C = 5$. We observe that the ratio approaches unity as

the lattice spacing is reduced, in line with our expectations. However, given the coarseness of our lattice spacings, a linear extrapolation in a is not sufficient, suggesting that higher powers of a are required to describe the scaling behavior. To investigate this, we present the same data in a log-log plot in Fig. 9, allowing us to extract the dominant power-law behavior in the parameter range covered in this work. A linear fit in this representation yields an effective scaling exponent between 3 and 4 for both choices of N_C , consistent across all t_Q .

IV. CONCLUSIONS AND OUTLOOK

We generated ensembles with different lattice spacings but approximately tuned spatial extents for $SU(4)$, $SU(5)$ and $SU(6)$, with one fermion in the two-index antisymmetric representation. On these ensembles we studied the behavior of different gradient-flows in the context of the topological charge. In order to investigate the presence of fractional topological charges we advocate the use of an overimproved action for the gradient flow as it quickly and unambiguously settles into discrete topological sectors. With such a prescription, the flow-time bounds defined in Eq. (2.5) can be satisfied and a topological charge (and functions of it, such as the topological susceptibility) can

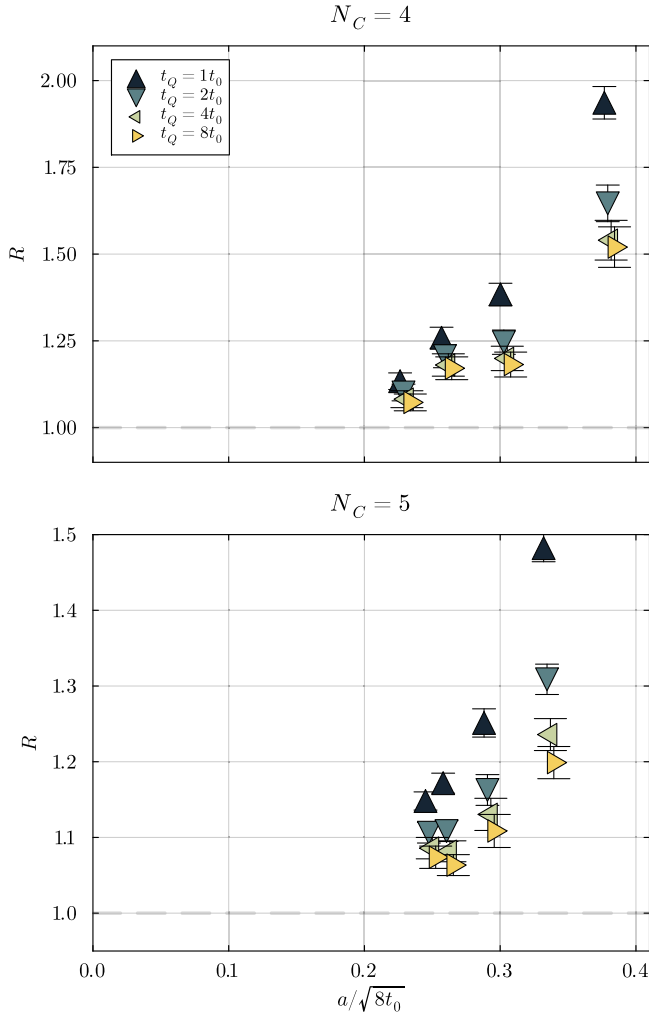


FIG. 8. Ratios of susceptibilities computed with Wilson and DBW2 as a function of the lattice spacings for $N_C = 4$ (upper panel) and $N_C = 5$ (lower panel). As in the caption, the color code reflects a different choice for t_Q . Point markers are slightly shifted in the x -axis to improve visibility.

easily be defined. While we do not observe any fractional topological charges, at finite lattice spacing we find the unit of discretization to be slightly below one. We expect the ratio of topological susceptibilities defined through the Wilson and the overimproved DBW2 flow to differ from unity by discretization effects. Our data confirm this and that the ratio approaches one as the lattice spacing is reduced. We expect these findings to be independent of the choice of fermion representation used in this work.

Having gained experience in the largely unexplored parameter space for these theories and defined and determined the topological charge on the ensembles at hand, we are continuing to generate a suite of ensembles for $N_C > 3$ which allows the computation of the mesonic spectrum as well as its extrapolation to the massless limit. This will provide further nonperturbative insight into the conjectured connection of these theories to super-Yang-Mills. One

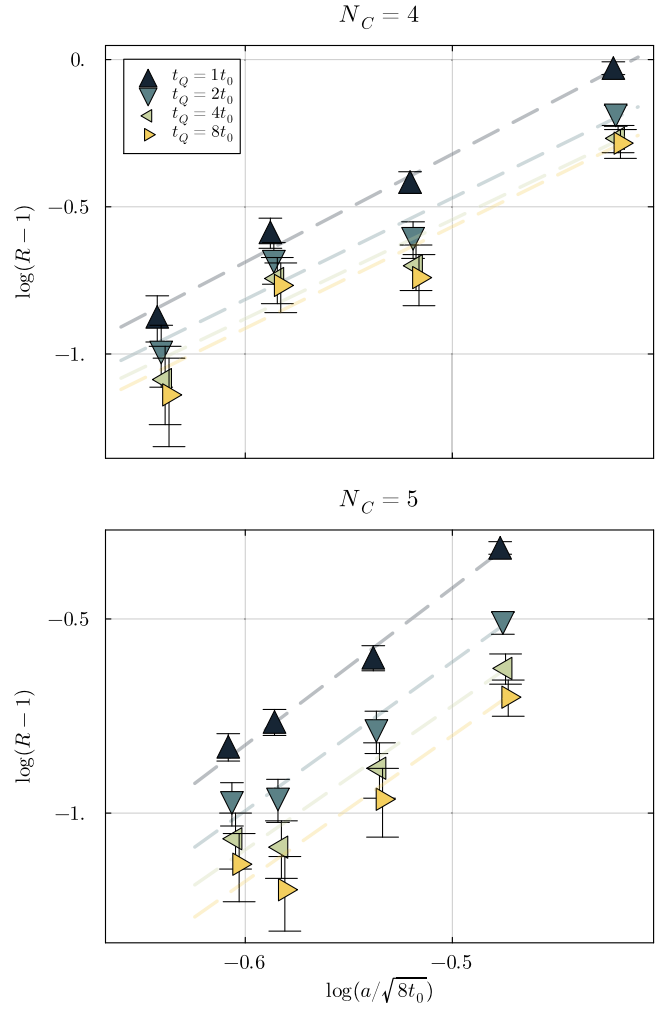


FIG. 9. Same as in Fig. 8, but in log-log scales. Dashed lines are linear fits to the rescaled points, in the caption, the exponents of a represent the fitted slopes of the lines.

crucial step to achieving this is to define a cost-efficient measurement strategy for the computation of disconnected diagrams in higher representations, which is ongoing work. Furthermore, the large- N_C limit of $\mathcal{N} = 1$ SUSY Yang-Mills has been explored for very large N_C using twisted volume reduction techniques [50–53]. In the future we plan to explore comparisons between such an approach and the method presented in this work.

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DATA AVAILABILITY

Some of the data that support the findings of this article are openly available [54]. Configurations ensembles are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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