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Raychaudhuri Equation, Geometrical Flows and Geometrical Entropy

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Abstract: The Raychaudhuri equation is derived by assuming geometric flow in space–time M of $n+1$ dimensions. The equation turns into a harmonic oscillator form under suitable transformations. Thereby, a relation between geometrical entropy and mean geodesic deviation is established. This has a connection to chaos theory where the trajectories diverge exponentially. We discuss its application to cosmology and black holes. Thus, we establish a connection between chaos theory and general relativity.

Keywords: Raychaudhuri equation; chaos theory; Kaluza Klein theory; Kaluza Klein cosmology; geometrical flow; geometrical entropy; Riccati equation

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1. Introduction

In general relativity the motion of nearby bits of matter is described by the celebrated Raychaudhuri equation or the Landau–Raychaudhuri equation [1,2]. It shows a general validation that gravitation should be a universally attractive interaction between any two bits of matter in general relativity and also in Newton's theory of gravity. This equation was formulated by Raychaudhuri and Landau independently in 1954 [3,4]. Later it became a fundamental lemma in proving the famous Hawking–Penrose singularity theorems and in studying exact solutions of Einsteins equations in general relativity [5,6].

Saurya Das has proposed in the quantum theory a Raychaudhuri equation where the usual classical trajectories are replaced by Bohmian trajectories [7]. Bohmian trajectories do not converge and thus the issue of geodesic incompleteness, singularities such as big bang or big crunch can be avoided [8,9]. In this paper we treat the classical geometrical flow as a dynamical system in such a way that the Raychaudhuri equation becomes the equation of motion and that the action can be used to quantize the dynamical system. The asymmetry of the Raychaudhuri equation then leads to a characterization of the instabilities of the geodesic flow. Classical chaos is essentially characterized by the exponential divergence of neighboring trajectories inducing a high degree of instability in the orbits with respect to initial conditions.

The Raychaudhuri equation is the basis for deriving the singularity theorems. The study is expected to show the effect such a quantization will have on the geometrical flow, and as part of the process it can be shown that a quantum space–time is non-singular. The existence of a conjugate point is a necessary condition for the occurrence of singularities [9]. However it is possible to demonstrate that conjugate points cannot arise because of the quantum effects. An intriguing result obtained was that the Raychaudhuri equation can be written in a harmonic oscillator form under suitable transformations. Here a new quantity called geometrical entropy $S = ln\chi(x)$ can be defined where $\chi(x)$ represents the distance between two nearby geodesics. We have expressed the above equations in terms of the entropy which by transformation to a Ricatti-type equation becomes similar to the Jacobi equation. We have recently proved that the geodesic deviation equation of Jacobi becomes unitarily equivalent to that of a harmonic oscillator. In this way, a connection between general relativity and chaos theory is established [10–12].

The connection can be further investigated by the addition of gauge fields in the metric. Here too the Raychaudhuri equation and the geodesic equation acquire the harmonic oscillator-form under suitable transformations. However, the convergence and divergence criteria get modified by the effect of the gauge field. In this case the particles deviate from the geodesics. A point to be noted when adding a gauge field into the picture is that the particle no longer follows a geodesic. According to the work by S.G.Rajeev [13], the Riemannian geometry is a particular case of Hamiltonian Mechanics. He explores the links between Riemannian geometry and Hamiltonian Mechanics by changing the form of the Hamiltonian through the addition of a scalar field or vector field and investigates the corresponding change in the geometry(change in curvature and Ricci tensor).

2. Raychaudhuri Equation from Geometric Flow

We study the congruence of a test particle moving on an n+1 dimensional space–time M. We use the proper time (τ) for this particle as a dynamical foliation parameter so as to foliate the space–time into topology $T \times R$. Here T is a Riemannian Manifold with a metric $g_{\alpha\beta}$ that projects any vector field into the manifold. We also define $H(T)$, a hyper-surface in the transverse manifold that the world-lines intersect at time τ. The volume of that hyper-surface is given by:

$$Vol = \int_{\sigma_\tau} \sqrt{det\, g} d^n x \tag{1}$$

we consider the velocity field of the test particle in the congruence to be normal to the n-dimensional transverse manifold $H(T)$. The gradient of velocity is a second rank tensor having three parts: the symmetric traceless part, the antisymmetric part and the trace. The three parts define the shear, the rotation and the expansion of the flow. We consider the cross-sectional hypersurface σ_τ as a dynamical system. We define the volume of the cross-sectional hypersurface [14–17]

$$\rho(\tau) = 2 \int_{\sigma_\tau} \sqrt{det\, g} d^n x \tag{2}$$

as the dynamical degree of freedom.

We define the dynamical evolution of the metric as

$$\partial_\tau g_{\alpha\beta} = \theta_{\alpha\beta} = 2\sigma_{\alpha\beta} + \frac{2}{n} g_{\alpha\beta}\theta \tag{3}$$

Multiplying both sides by $\sqrt{det\, g}$ and using $\delta(det\, g) = g\, g_{\alpha\beta}\, \delta g^{\alpha\beta}$, we get a very important result

$$\dot{\rho} = \frac{2}{n}\rho\,\theta \tag{4}$$

Using the Lagrangian $L = (\frac{n}{4}\frac{1}{\rho}\dot{\rho}^2 - \rho(\mathcal{R} - \dot{\xi}^{\alpha}_{;\alpha}) - V_\sigma(\rho))$, we define the action

$$S(\rho, \dot{\rho}) = \int d\tau (\frac{n}{4}\frac{1}{\rho}\dot{\rho}^2 - \rho(\mathcal{R} - \dot{\xi}^{\alpha}_{;\alpha}) - V_\sigma(\rho)) \tag{5}$$

where $\mathcal{R}$ is the Raychaudhuri scalar, $\mathcal{R} := R_{\mu\nu}\xi^\mu\xi^\nu$, and $V_\sigma(\rho)$ is the shear potential that satisfies the equation

$$\frac{V_\sigma(\rho)}{\partial \rho} = 2\sigma^2 \tag{6}$$

We can express the canonical conjugate momentum as

$$\Pi = \frac{\delta L}{\delta \dot{\rho}} = \frac{n}{2}\,\rho^{-1}\dot{\rho}, \;= \frac{n}{2}\,\rho^{-1}(\frac{2}{n}\rho\,\theta) = \theta.$$

Thus, as one would expect, the expansion parameter is the conjugate momentum to the dynamical degree of freedom $\rho(\sigma)$. We proceed further by computing the variation

$$\frac{\delta L}{\delta \rho} = -\frac{n}{4}\rho^{-2}\dot{\rho}^2 - 2\sigma^2 - \mathcal{R} + \dot{\xi}^{\alpha}_{;\alpha} = \frac{n}{4}\left[\rho^{-2}\left(\frac{4}{n^2}\rho^2\theta^2\right)\right] - 2\sigma^2 - \mathcal{R} + \dot{\xi}^{\alpha}_{;\alpha}$$
$$= -\frac{1}{n}\,\theta^2 - 2\sigma^2 - \mathcal{R} + \dot{\xi}^{\alpha}_{;\alpha}$$

Using the Euler-Lagrange equation, we can rewrite the Raychaudhuri equation as

$$\frac{d\theta}{d\tau} = \frac{\delta L}{\delta \rho} \tag{7}$$

$$\dot{\theta} = -\frac{1}{n}\,\theta^2 - 2\sigma^2 - \mathcal{R} + \dot{\xi}^{\alpha}_{;\alpha} \tag{8}$$

We can define the Hamiltonian as

$$H = \frac{1}{n}\rho\,\theta^2 + (\mathcal{R} - \dot{\xi}^{\alpha}_{;\alpha})\rho + V_\sigma(\rho). \tag{9}$$

Thus the derivation of Raychaudhuri equation without the acceleration term can be found.

3. Raychaudhuri Equation in Harmonic Oscillator Form

Let us consider Raychaudhuri equation without the acceleration term ($\dot{\xi}^{\alpha}_{\alpha}$ set to zero).

$$\frac{d\theta}{d\tau} + \frac{\theta^2}{3} + \sigma^2 = -R_{\alpha\beta}\xi^\alpha\xi^\beta \tag{10}$$

In order for the LHS to be negative it must fulfill the condition $\frac{d\theta}{d\tau} < -\frac{1}{\theta^2}$ which finally leads to the inequality

$$\frac{1}{\theta(\tau)} \geq \frac{1}{\theta_0} + \frac{1}{\theta\tau} \tag{11}$$

One can infer that any initially converging hyper-surface-orthogonal congruence must continue to converge and within a finite proper time $\tau \leq -3\theta_0^{-1}$ must lead to crossing of the geodesics. Since the Strong Energy Condition(SEC) causes gravitation to be attractive, matter obeying the SEC cannot cause geodesic deviation, on the other-hand it will increase the rate of convergence. Since entropy is defined as the average convergence/divergence of the geodesics in a congruence, the SEC will cause further decrease in entropy. If we set $\theta = 3\chi'/\chi$ the Raychaudhuri equation is transformed to

$$\frac{\partial^2\chi}{\partial\tau^2} + \frac{1}{3}\left(R_{\alpha\beta}\chi^\alpha\chi^\beta + \sigma^2\right)\chi = 0, \tag{12}$$

which is a harmonic oscillator equation [18].

As pointed out above, θ may be identified with the derivative of the entropy, so that the entropy will be of the form $S = ln\chi$. Here, χ may be identified with an effective or average geodesic deviation.

Recently, Kar and Sengupta [18] have shown that the condition for geodesic convergence is the existence of zeroes in χ at finite values of the affine parameter(τ), and they argue that convergence occurs if

$$R_{\alpha\beta}\xi^{\alpha}\xi^{\beta} + \sigma^2 - \omega^2 \geq 0 \tag{13}$$

Most of the physical matter fields satisfy the strong energy conditions which state that for all time like vectors U, the inequality holds

$$T_{\mu\nu}U^{\mu}U^{\nu} \geq \frac{1}{2}Tg_{\mu\nu}U^{\mu}U^{\nu} \tag{14}$$

It follows, when(SEC) holds the term $R_{\mu\nu}U^{\mu}U^{\nu}$ is always positive. Furthermore, note that the shear and the rotation are spatial vectors and consequently $\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0$ and $\omega_{\mu\nu}\omega^{\mu\nu} \geq 0$. As mentioned above $\omega_{\mu\nu} = 0$ is zero if and only if the congruence is hyper-surface orthogonal. If that is satisfied the Raychaudhuri equation simplifies to the form

$$\frac{d\theta}{d\tau} + \frac{1}{3}\tau^2 + \sigma^2 = -R_{\mu\nu}U^{\mu}U^{\nu} \tag{15}$$

In order for the left hand side to be negative it must fulfill the condition $\frac{d\theta}{d\tau} < -\frac{1}{3}\theta^2$ which finally leads to the inequality

$$\frac{1}{\theta(\tau)} \geq \frac{1}{\theta_0} + \frac{1}{3}\tau \tag{16}$$

If we set $\theta = \frac{3\chi'}{\chi}$ the Raychaudhuri equation is transformed to

$$\frac{d^2\chi}{d\tau^2} + \frac{1}{3}(R_{\mu\nu}U^{\mu}U^{\nu} + \sigma^2 - \omega^2)\chi = 0 \tag{17}$$

which is a harmonic oscillator equation. We have recently proved that the geodesic deviation equation of Jacobi is unitarily equivalent to that of harmonic oscillator. The expansions rate of growth of the cross-sectional area orthogonal to the bundle of geodesics. Increase/decrease of this area is same as that of divergence/convergence of the geodesics. The average growth of the cross-sectional area is the same as that of the geodesics. The average growth of the cross-sectional area is compatible with the average geodesic deviation. Kar and the Sengupta [18] have shown that the condition for geodesic convergence is the existence of zeros in $\ln\chi$ at finite values of the affine parameter, and they argue that convergence occurs if $R_{\mu\nu}U^{\mu}U^{\nu} + \sigma^2 - \omega^2$. Here shear increases convergence and rotation obstructs convergence.

4. Raychaudhuri Equation in Harmonic Oscillator Form: With the Acceleration Term

The Raychaudhuri equation in harmonic oscillator form can be written as

$$\frac{\partial^2\chi}{\partial\tau^2} + \frac{1}{3}(\sigma^2 + R_{\alpha\beta}\xi^{\alpha}\xi^{\beta} - \dot{\xi}^{\alpha}_{;\alpha})\chi = 0 \tag{18}$$

and convergence occurs if:

$$\sigma^2 + R_{\alpha\beta}\xi^{\alpha}\xi^{\beta} - \dot{\xi}^{\alpha}_{;\alpha} \geq 0 \tag{19}$$

This clearly shows that the velocity field has a significant role in the convergence or divergence of world-lines.

Let us study this effect in more detail. The acceleration term causes the particle to deviate from geodesic. Therefore it has a logarithmic relation to entropy which increases when geodesics diverge.

To give a clear picture we consider the Kaluza Klein cosmology. The Kaluza Klein metric is given by, $g_{AB}, A, B = 0, 1, 2, 3, 5$ with the electromagnetic potential

$$g_{AB} = \begin{bmatrix} g_{\alpha\beta} + \alpha^2\, g_{55} A_\alpha A_\beta & \alpha_0 g_{55} A_\alpha \\ \alpha_0 g_{55} A_\alpha & g_{55} \end{bmatrix} \tag{20}$$

where $g_{\alpha\beta}$ is the 4-dimensional metric and A_α is the electromagnetic potential. Now the space–time interval becomes

$$dS^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta - g_{55}\, (dx^5 + \alpha_0 A_\alpha dx^\alpha)^2. \tag{21}$$

We also have

$$g^{AB} = \begin{bmatrix} g^{\alpha\beta} & -\alpha_0 g^{\alpha\beta} A_\alpha \\ -\alpha_0 g^{\alpha\beta} A_\alpha & 1/g_{55} + \alpha_0^2\, g^{\alpha\beta} A_\alpha A_\beta \end{bmatrix}.$$

This provides a space–time with electromagnetism and gravity unified. The geodesic equation in five-dimensional space–time,

$$\frac{d^2 z^A}{dS^2} + \Gamma^A_{BC} \frac{dz^B}{dS} \frac{dz^C}{dS} = 0 \tag{22}$$

can be transformed by applying cylindrical condition on the metric as

$$\frac{d^2 z^\alpha}{dS^2} + \Gamma^\alpha_{\beta\lambda}\, \frac{dz^\beta}{dS}\, \frac{dz^\lambda}{dS} = a\alpha_0 F_{\alpha\beta} \frac{dz^\beta}{dS} + \frac{1}{2} \frac{a^2}{g_{55}^2} g^{\alpha\lambda} (\partial_\lambda g_{55}), \tag{23}$$

where

$$a = g_{5\alpha} \frac{dz^\alpha}{dS} + g_{55} \frac{dz^5}{dS},$$

a is a constant along the $5 - D$ world line. In our case $g_{5\alpha}$ is non zero since we have included electromagnetic fields. We assume that

$$\frac{g_{5\alpha}}{g_{55}} = \alpha_0 A_\alpha(x)$$

where α_0 is determined by

$$\alpha_0 = \frac{q}{amc}.$$

Here, the electromagnetic potential emerges out of $g_{5\alpha}$. Let us now consider Raychaudhuri equation in four dimensions.

$$\frac{d\theta}{dS} = -\frac{\theta^2}{3} - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - R_{\alpha\beta}\xi^\alpha \xi^\beta + \dot{\xi}^\alpha_{;\alpha} \tag{24}$$

where $\xi^\alpha = \dfrac{dz^\alpha}{dS}$, $2\sigma^2 = \sigma_{\alpha\beta}\sigma^{\alpha\beta}$, $2\omega^2 = \omega_{\alpha\beta}\omega^{\alpha\beta}$. The cosmological constant Λ is set to zero. Since

$$\dot{\xi^\alpha} = \frac{d^2 z^\alpha}{dS^2} + \Gamma^\alpha_{\alpha\beta} u^\alpha u^\beta,$$

the last term in Equation (8) can be written as

$$\dot{\xi}^\alpha_{;\alpha} = -\frac{a^2}{2}\, g^{\alpha\rho} D_\alpha (\partial_\rho \frac{1}{g_{55}}) + a\alpha_0 D_\mu (F_{\mu\beta} \frac{dz^\beta}{dS}) \tag{25}$$

with D_μ as the covariant derivative. The vorticity ω and σ induces expansion and contraction respectively. It is useful to note that $-D_\alpha(\partial_\rho \frac{1}{g_{55}})$ is positive for static spherically symmetric space–time in five dimensions without electromagnetism. The additional term,

$-D_\alpha(\partial_\rho \frac{1}{g_{55}})$ is positive for static spherically symmetric space–time in five dimensions without electromagnetism. Considering a case with $R_{\alpha\beta}\xi^\alpha\xi^\beta > 0$ and $\omega = 0$ we get

$$\frac{d\theta}{dS} \leq \frac{1}{4}\theta^2 - \frac{a^2}{2}\, g^{\mu\rho} D_\mu(\partial_\rho \frac{1}{g_{55}}) + a\alpha_0 D_\mu(F_{\mu\beta}\frac{dz^\beta}{dS}) \tag{26}$$

For a static spherical symmetric metric, $-\frac{a^2}{2}\, g^{\mu\rho} D_\mu(\partial_\rho \frac{1}{g_{55}})$ is always positive [19]. This indicates that a scalar field will always defocus world lines. Thus in Kaluza Klein cosmology, scalar field always creates a defocus of worldlines and we get a bouncing model of universe [20].

5. Conclusions

The formalism that we have developed can be applied to any physical system where the equation for geometrical flow is valid. This can also be applied to cosmology with a scalar field.

The physical significance of geometrical entropy is that, it represents the chaotic behavior of world-lines that tend to converge or diverge. This can be observed in cosmology where geodesics try to converge near big-bang singularity. However, scalar fields try to inhibit the convergence and causes divergence. Thus there is a possibility of bouncing model of the universe in classical theory. Further studies are possible in charged black holes.

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