A discussion on Lorentz preserving scalar fields in Lorentz violating theory

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Abstract. Lorentz symmetry can be preserved in effective higher derivative scalar field theories containing a constant vector that breaks Lorentz invariance of flat spacetime, through the choice of special field configurations. These fields do satisfy the equations of motion, yielding cubic dispersion relations analogous to those derived earlier. Moreover, the Lie algebra of the Lorentz group can be realised on these fields.

1. Introduction

Physicists have focussed a lot of their attention on theoretically and experimentally probing departures from Lorentz symmetry for the better part of the last couple of decades, [1, 2, 3, 4, 5, 6, 7, 8, 9]. This search was initiated by different theories of quantum gravity that hypothesise violation of Lorentz symmetry in flat spacetime at energies $E \gg M_{Pl}$, where M_{Pl} denotes the Planck mass. Deviations, suppressed by the Planck mass, from the standard special relativistic dispersion relation of free particles of mass m at high energies E ($M_{Pl} \gg E \gg m$) are considered to be signatures of quantum gravity induced Lorentz violation. Myers and Pospelov [6] constructed low energy effective actions of fields having spins 0, 1/2 and 1 that include new Planck suppressed dimension five operators, to account for these proposed corrections to the dispersion relations. In this report, we shall restrict ourselves to the case of a complex scalar field ϕ. The extended action of [6] is,

$$S_{MP_\phi} = \int d^4x \mathcal{L}_{MP_\phi} = \int d^4x \left[|\partial\phi|^2 - m^2|\phi|^2 \right] + \int d^4x \frac{i\kappa}{M_{Pl}} \phi^* \partial_n^3 \phi \, , \tag{1}$$

The first integral, let's call it S_S, is the usual action of a complex scalar field with mass m while the second integral, S_{V_S}, is the Lorentz violating contribution. Global Lorentz symmetry of flat spacetime is broken by the constant vector **n** which renders a preferred direction to the background. κ is a real, dimensionless parameter and $\mathbf{n} \cdot \partial \equiv \partial_n$. Choosing a Lorentz frame where $n^\mu = (1, \vec{0})$, corrections of $O(p^3)$ to the dispersion relation has been obtained in [6] in the limit $M_{Pl} \gg E \gg m$.

$$\omega^2 \simeq |\vec{p}|^2 + m^2 + \frac{\kappa}{M_{Pl}} |\vec{p}|^3 \, . \tag{2}$$

Although such deformed dispersion relations have attracted much attention, do they unequivocally imply Lorentz violation? We first explore the possibility that special field configurations exist for which the apparently Lorentz symmetry violating action of [6] may still be Lorentz *invariant* in close analogy to what happens in magnetic monopole theory [10]. In section 2 we consider the Nöther current corresponding to Lorentz transformation of the action (1). Demanding that this Nöther current be conserved leads us to the the result that the effective action remains Lorentz invariant if the fields are decomposed in a particular way. The initial absence of Lorentz symmetry in the action is transferred to the Lorentz non-invariant splitting of the fields. We further investigate in the same section, the dispersion relation that these special field configurations satisfy. This work is based on our more extensive study [11]

One must also have noticed that the additional term S_{V_S} in (1) contains third order derivatives of the field. Higher derivative Lagrangians are not new to physics [12]. Back in 1961, Ostrogradskii [13] had developed a canonical formalism for dealing with them. However, most of the extant literature in this field, [14, 15, 16] and references therein, study systems having finite number of degrees of freedom and higher time derivatives of the generalised coordinates. Section 3 will present a brief review of Ostrogradskii's technique generalised to field systems [17] and use it to study the canonical structure of $\mathcal{L}_{MP_\phi}$ with the Lorentz-preserving fields. This part of the problem is going to be dealt with in more detail in our forthcoming publication [18].

2. Lorentz invariant dynamics of Myers Pospelov scalar model
2.1. Finding the Lorentz preserving fields
Let an infinitesimal Lorentz transformation be applied to the action (1). Then $\delta_{\alpha\beta}S_S = 0$ while $\delta_{\alpha\beta}S_{V_S} = \int d^4x \phi^* n_{[\alpha}\partial_{\beta]}\partial_n^2\phi$. On the other hand, if the spacetime divergence of the corresponding Nöther current $\mathcal{J}$ is computed, we get $\partial_\mu \mathcal{J}^\mu_{\alpha\beta} = \phi^* n_{[\alpha}\partial_{\beta]}\partial_n^2\phi$.

Lorentz transformation will be a symmetry of the system if both $\delta_{\alpha\beta}S_{V_S} = 0$ and $\partial_\mu \mathcal{J}^\mu_{\alpha\beta} = 0$. This yields the condition $n_{[\alpha}\partial_{\beta]}\partial_n^2\Phi = 0$. A possible non-trivial solution is,

$$\phi(\mathbf{x}) = \phi_\parallel(\mathbf{x}_\parallel) + \phi_\perp(\mathbf{x}_\perp) \,, \tag{3}$$

where $\phi_\parallel$ and $\phi_\perp$ are arbitrary functions of their respective arguments $\mathbf{x}_\parallel$ and $\mathbf{x}_\perp$, defined by $\mathbf{x}_\parallel \equiv \frac{\mathbf{x}\cdot\mathbf{n}}{n^2}\mathbf{n}$ and $\mathbf{n}\cdot\mathbf{x}_\perp = 0$. So, $\mathbf{x} = \mathbf{x}_\parallel + \mathbf{x}_\perp$. Hence, the requirement of a Lorentz invariant action imposes a non-trivial restriction on the functional form of fields. Now, if we choose $\mathbf{n}$ to be timelike, we can align x^0 along $\mathbf{n}$. Then our condition (3) implies that when the full scalar field is a linear combination of a time-dependent, spatially homogeneous piece and a static spatially inhomogeneous piece, *the theory possesses Lorentz symmetry.*

2.2. Evaluation of dispersion relation
The scalar field $\phi(x)$ assumed to be given by (3) leads to the equation of motion $(\Box + m^2)\phi = \frac{i\kappa}{M_{Pl}}\partial_n^3\phi$ to be written as $(\nabla_\perp^2 + m^2)\phi_\perp = -(\nabla_\parallel^2 + m^2)\phi_\parallel + \frac{i\kappa}{M_{Pl}}\partial_n^3\phi_\parallel$, for $\mathbf{n}$ a unit vector (we have used the decomposition $\Box\phi = \nabla_\parallel^2\phi_\parallel + \nabla_\perp^2\phi_\perp$). By taking the simple ansätz $\phi_\perp \sim \exp(-\mathbf{k}_\perp \cdot \mathbf{x}_\perp)\,, \phi_\parallel \sim \exp(-i\,\mathbf{k}_\parallel \cdot \mathbf{x}_\parallel)$ and going over to the inertial frame characterised by $\mathbf{n} = (1, \vec{0})$, it can be shown that the dispersion relation of the complete scalar field $\phi(\mathbf{x})$ in the high energy regime $\mathcal{E} \simeq |\vec{k}| \gg m$ takes the simple form,

$$\mathcal{E}^2 \simeq |\vec{k}|^2 + \frac{\kappa}{M_{Pl}}\,|\vec{k}|^3, \tag{4}$$

provided the four momentum $\mathbf{k} = (\mathcal{E}, \vec{k})$. This is same as the dispersion relation (2) computed in [6]. Hence, observation of a deformed dispersion relation doesn't necessarily guarantee the existence of Lorentz violating dynamics.

3. Canonical structure of Lorentz preserving Lagrangian

For the rest of this report, we will study the canonical structure of Myers Pspelov theory in the presence of Lorentz preserving fields.

3.1. Canonical formalism in higher derivative field theories

Let us consider a system of scalar fields $\phi_a(\mathbf{x})$ described by the Lagrangian density $\mathcal{L}(\phi_a, \phi_{a,\rho_1}, \phi_{a,\rho_1\rho_2}, ..., \phi_{a,\rho_1...\rho_l})$. Here, $\phi_{a,\rho_1...\rho_j} \equiv \partial_{\rho_1}...\partial_{\rho_j}\phi_a$ and a labels the different fields. A canonical formalism of relativistic field theories requires us to 'foliate' spacetime into constant time, spacelike hypersurfaces. This in turn involves the separation of temporal and spatial derivatives of the fields[1]. It is most often possible to arrange terms in the Lagrangian such that "mixed" derivatives of the fields, as in $\partial_t^2 \partial_k \phi$, do not survive. This is true not only when the Lagrangian is Lorentz invariant [17] but also when the Lorentz violating action (1) is written in terms of the Lorentz preserving fields [18]. In such situations, the canonical momenta are given by

$$\pi^{a(j)} \equiv \sum_{i=j+1}^{l} (-d_t)^{i-(j+1)} \frac{\partial \mathcal{L}}{\partial \phi_{a(i)}}, \quad j = 0, ..., l-1 \ . \tag{5}$$

We have adopted the notation $\phi_{a(j)} \equiv d_t^j \phi_a$, $j = 0, ..., l$ and $\pi^{a(j)}$ are the momenta conjugate to $\phi_{a(j)}$ upto $j = l-1$. It is evident that we are now working in an enlarged phase space. The canonical variables satisfy the Poisson bracket $\{\phi_{a(i)}(t,\vec{x}), \pi^{b(j)}(t,\vec{x}')\} = \delta_a^b \, \delta_i^j \, \delta^{(3)}(\vec{x} - \vec{x}')$. We also wish to stress that higher derivative theories owe their peculiarities only to higher order time derivatives of the fields, spatial derivatives being rather benign.

3.2. Algebra of Lorentz preserving fields

Without loss of generality, we are going to work in a Lorentz frame defined by $\mathbf{n} = (1, \vec{0})$. This greatly simplifies the Lagrangian density:

$$\mathcal{L}_{MP_\phi} = \dot{\phi}_\parallel^* \dot{\phi}_\parallel - \vec{\nabla}\phi_\perp^* \cdot \vec{\nabla}\phi_\perp + \frac{i\kappa}{M_{Pl}}(\phi_\parallel^* + \phi_\perp^*)\dddot{\phi}_\parallel \ . \tag{6}$$

Here, we have neglected the masses of the fields as we are interested in behaviour of the system at energies much higher than the field masses. Eq.(6) has only higher order time derivatives of $\phi_\parallel(t)$ while $\vec{\nabla}\phi_\parallel = 0 = \partial_t\phi_\perp$. Thus all mixed derivatives in the sense described above will vanish. This permits us to safely use eq.(5) to determine the canonical momenta. We list them in table 1.

The Nöther charge $Q_{\alpha\beta} = \int d\sigma \, \sigma_{\rho_1} \mathcal{J}_{\alpha\beta}^{\rho_1}$, Σ being a three dimensional hypersurface. If we orient it orthogonal to the time axis then the Nöther charge may be written in terms of the canonical variables:

$$Q_{\alpha\beta} = \int d^3\vec{x} \left(\pi_\parallel^{(0)} \delta_{\alpha\beta} \phi_{\parallel(0)} + \pi_\parallel^{(1)} \delta_{\alpha\beta} \phi_{\parallel(1)} + \pi_\parallel^{(2)} \delta_{\alpha\beta} \phi_{\parallel(2)} + \pi_{\parallel*} \delta_{\alpha\beta} \phi_\parallel^* - x_{[\alpha} \delta_{\beta]}^0 \mathcal{L}_{MP_\phi} \right). \tag{7}$$

From here we can deduce the action of the Nöther charge on the Lorentz preserving fields and the algebra satisfied by them [18]. The results appear below in eqs. (8), (9), (10)

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[1] At this stage, it is imperative that we sort the indices. Latin letters from the middle of the alphabet set viz. i,j,k,l are being used as summation indices while those from the end like r,s,...,z will denote spatial components. Different fields will be labelled by the alphabets a,b,c,d. Greek letters are reserved for spacetime indices.

Generalised coordinate	Generalised momentum
$\phi_{\parallel(0)} \equiv \phi_{\parallel}$	$\pi_{\parallel(0)} = \dot{\phi}_{\parallel}^{*} + \frac{i\kappa}{M_{Pl}}\ddot{\phi}_{\parallel}^{*}$
$\phi_{\parallel(1)} \equiv \dot{\phi}_{\parallel}$	$\pi_{\parallel(1)} = -\frac{i\kappa}{M_{Pl}}\dot{\phi}_{\parallel}^{*}$
$\phi_{\parallel(2)} \equiv \ddot{\phi}_{\parallel}$	$\pi_{\parallel(2)} = \frac{i\kappa}{M_{Pl}}(\phi_{\parallel}^{*} + \phi_{\perp}^{*})$
$\phi_{\parallel(0)}^{*} \equiv \phi_{\parallel}^{*}$	$\pi_{\parallel*(0)} = \dot{\phi}_{\parallel}$
$\phi_{\perp(0)} \equiv \phi_{\perp}$	$\pi_{\perp(0)} = 0$
$\phi_{\perp(0)}^{*} \equiv \phi_{\perp}^{*}$	$\pi_{\perp*(0)} = 0$

Table 1. Canonically conjugate phase space variables

$$\{Q_{\alpha\beta}(t), \phi_{b(k)}(t,\vec{x})\} = -\delta_{\alpha\beta}\phi_{b(k)}(t,\vec{x}), \quad for \ \phi_b = \phi_{\parallel}, \phi_{\parallel}^{*} \ , \tag{8}$$

$$= 0, \quad for \ \phi_b = \phi_{\perp}, \phi_{\perp}^{*} \ , \tag{9}$$

$$\{Q_{\alpha\beta}(t), Q_{\rho\sigma}(t)\} = \eta_{\alpha\sigma}Q_{\beta\rho}(t) - \eta_{\alpha\rho}Q_{\beta\sigma}(t) + \eta_{\beta\rho}Q_{\alpha\sigma}(t) - \eta_{\beta\sigma}Q_{\alpha\rho}(t) \ . \tag{10}$$

This completes the demonstration that our Lorentz preserving fields do indeed provide a representation of the Lorentz Lie algebra.

4. Discussion
The special field configurations that we have obtained have aspects of intrinsic interest in cosmological scenarios. They may provide natural seeds for the growth of inhomogeneities in the early Universe [11]. An additional advantage is that these fields allow the use of standard Lorentz invariant formalism even in situations where Lorentz symmetry is violated.

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