On the Schwarzschild solution in TEGR

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Abstract. Conserved currents, superpotentials and charges for the Schwarzschild black hole in the Teleparallel Equivalent of General Relativity (TEGR) are constructed. We work in the covariant formalism and use the Noether machinery to construct conserved quantities that are covariant/invariant with respect to both coordinate and local Lorentz transformations. The constructed quantities depend on the vector field ξ and we consider two different possibilities, when ξ is chosen as either a timelike Killing vector or a four-velocity of an observer. We analyze and discuss the physical meaning of each choice in different frames: static and freely falling Lemaitre frame. Moreover, a new generalized free-falling frame with an arbitrary initial velocity at infinity is introduced. We derive the inertial spin connection for various tetrads in different frames and find that the "switching-off" gravity method leads to ambiguities.

1. Introduction

Teleparallel approach to gravity became increasingly popular in recent years [1-4]. Naturally, the most popular of these theories is the formulation of the ordinary general relativity itself known as the Teleparallel Equivalent of General Relativity (TEGR) [1,5,6]. The reason why TEGR is interesting, besides being the simplest model, is that one can construct coordinate covariant conserved quantities, in contrast with pseudotensors in the standard general relativity [7].

However, in the original formulation of TEGR, these coordinate covariant conserved quantities are not covariant with respect to local Lorentz rotations in a tangent (tetrad) space [8]. This problem has been solved in [9-11], where, using the language of differential forms, the authors have constructed conserved currents, superpotentials and charges, which are both coordinate covariant and invariant with respect to local Lorentz rotations. Later [12,13], an analogous method was developed in the framework of a more popular tensorial formalism. In this covariant formulation, we have to consider the so-called inertial spin connection (ISC) ${}^{\bullet}A^{a}{}_{b\mu}$, which is not a dynamical quantity, i.e. it does not affect the field equations of TEGR. Nevertheless, it has to be determined since it plays an important role as far as the action and conserved charges derived with it are concerned. One of the methods to determine ISC is a so-called procedure of "switching off" gravity that was originally suggested in [14] and applied later in many studies. In [12, 13] we have formalized this procedure in a more general way and outline it below.

Here, we apply the fully covariant formalism to analyze the Schwarzschild black hole solution in various frames, including the static and free-falling ones. There are some advantages of the fully covariant formalism with respect to non-covariant methods of constructing conserved charges. In particular, the covariant formalism allows us to work with arbitrary tetrads, what is often more convenient than finding the special class of proper tetrads in the non-covariant approach. We then show that the procedure of "switching off" gravity used to determine ISC is not unique and lead to a certain ambiguity since different tetrads can share the same ISC. We generalize our discussion to the case of free-falling observers with a non-zero initial velocity in which we get more physically expected results but the ambiguity still remains and depends on the initial velocity. We introduce a new notion named as a "gauge", which is a pair of a tetrad and a related ISC that correspond to the same physical situation.

These results have been presented at the Moscow PIRT 2021 conference [15] and our recent paper [16]. Compared to [16], we present our results here in a simplified form, such as we use only the static coordinates for our calculations, and focus on the main results.

Here we follow mostly notations used in the book [1]. The Latin indices mean the tetrad components, whereas the Greek indices mean the spacetime components. The tetrad indices are transformed to the spacetime ones and inversely by contracting with the tetrad vectors.

2. Covariant conserved quantities in TEGR

2.1. Lagrangian and field variables of TEGR

TEGR is formulated in tetrad formalism where the fundamental variables are components of the tetrad $h^a{}_\rho$ and the spin connection ${}^\bullet A^i{}_{k\nu}$. The tetrad is related with the metric $g_{\rho\sigma}$ as $g_{\rho\sigma} = \eta_{ab} h^a{}_\rho h^b{}_\sigma$, where η_{ab} is the Minkowski metric in the tangent space, and ${}^\bullet A^i{}_{k\nu}$, in the TEGR case, is the so-called inertial spin connection (ISC) constrained by

$$ {}^\bullet R^a{}_{b\mu\nu} = \partial_\mu {}^\bullet A^a{}_{b\nu} - \partial_\nu {}^\bullet A^a{}_{b\mu} + {}^\bullet A^a{}_{c\mu} {}^\bullet A^c{}_{b\nu} - {}^\bullet A^a{}_{c\nu} {}^\bullet A^c{}_{b\mu} \equiv 0 , $$

and can be written as ${}^\bullet A^a{}_{c\mu} = \Lambda^a{}_d(x) \partial_\mu \Lambda_c{}^d(x)$, where $\Lambda^a{}_b(x)$ is a matrix of a local Lorentz transformation.

We can then define the contortion tensor as a difference between ISC ${}^\bullet A^i{}_{k\nu}$ and the Levi-Civita spin connection (L-CSC) ${}^\circ A^i{}_{k\nu}$, i.e. ${}^\bullet K^i{}_{k\nu} = {}^\bullet A^i{}_{k\nu} - {}^\circ A^i{}_{k\nu}$, and write the TEGR Lagrangian as [1]

$$ {}^\bullet L = \frac{h}{2\kappa}\left({}^\bullet K^\rho{}_{\mu\nu} {}^\bullet K_\rho{}^{\nu\mu} - {}^\bullet K^\rho{}_{\mu\rho} {}^\bullet K^{\nu\mu}{}_\nu \right), \qquad (1) $$

where $h = \det h^a{}_\rho$ and $\kappa = 8\pi$ (in natural units).

This Lagrangian is equivalent to the Hilbert-Einstein Lagrangian up to a divergence. The variation of the Lagrangian (1) with respect to the tetrad leads to field equations equivalent to Einstein field equations from where the theory gets its name as the Teleparallel Equivalent of General Relativity (TEGR). The variation of the Lagrangian with respect to the spin connection turn out to be trivial [17] and hence ISC is not determined by any field equations. Nevertheless, it plays an important role since the Lagrangian and action do depend on ISC.

The theory defined in this way is covariant under simultaneous local Lorentz transformations of both the tetrad and ISC, i.e. $h'^a{}_\mu = \Lambda^a{}_b h^b{}_\mu$ and ${}^\bullet A'^a{}_{b\mu} = \Lambda^a{}_d {}^\bullet A^d{}_{c\mu} \Lambda_b{}^c + \Lambda^a{}_d \partial_\mu \Lambda_b{}^d$. Due to the particular pure-gauge form of ISC, we can always transform ISC to zero by a transformation with an appropriate $\Lambda^a{}_b(x)$. The tetrad which corresponds to a zero connection is called the *proper* tetrad, following the terminology of [14].

2.2. Covariant conserved quantities in TEGR

Considering invariance of the Lagrangian (1) under diffeomorphisms induced by an arbitrary displacement vector and applying the Noether theorem, one derives the differential conservation law [12,13]:

$$\partial_\rho {}^\bullet J^\rho(\xi) = \overset{\circ}{\nabla}_\rho {}^\bullet J^\rho(\xi) = 0 \quad (2)$$

for the Noether's current ${}^\bullet J^\rho(\xi)$, which is a vector density. Therefore the partial derivative ∂_ρ can be replaced by the Levi-Civita covariant derivative $\overset{\circ}{\nabla}_\rho$. The Klein-Noether identities allow us to express the current through the Noether's superpotential ${}^\bullet J^{\rho\sigma}(\xi)$, which is antisymmetric tensor density,

$${}^\bullet J^\rho(\xi) = \partial_\sigma {}^\bullet J^{\rho\sigma}(\xi) = \overset{\circ}{\nabla}_\sigma {}^\bullet J^{\rho\sigma}(\xi). \quad (3)$$

In the case of the TEGR Lagrangain (1), the explicit expression for the superpotential is

$${}^\bullet J^{\rho\sigma}(\xi) = \frac{h}{\kappa} {}^\bullet S_\mu{}^{\rho\sigma} \xi^\mu, \quad (4)$$

$${}^\bullet S_\mu{}^{\rho\sigma} = {}^\bullet K^{\rho\sigma}{}_\mu + \delta_\mu^\sigma {}^\bullet K^{\pi\rho}{}_\pi - \delta_\mu^\rho {}^\bullet K^{\pi\sigma}{}_\pi. \quad (5)$$

One has to note that both (4) and (5) are spacetime tensors antisymmetric in upper indices and invariant with respect to local Lorentz rotations. The conservation laws (2) and (3) allow us to construct integral conserved quantities defined on hypersurfaces Σ defined by $x^0 = const$. By the standard method in spherical coordinates, $r = x^1$, we obtain the charge

$$P(\xi) = \int_\Sigma dx^3 {}^\bullet J^0(\xi) = \int_{\partial\Sigma} dx^2 {}^\bullet J^{01}(\xi) = \frac{1}{\kappa}\int_{\partial\Sigma} dx^2 h {}^\bullet S_\mu{}^{01}\xi^\mu, \quad (6)$$

where the boundary in a surface integral can be defined by finite $r = r_0$ or $r \to \infty$.

2.3. "Switching off" gravity

In TEGR, ${}^\bullet A^i{}_{k\nu}$ is not a dynamical quantity and hence is not determined by any field equations. However, it does play an important role in definition of conserved charges and hence it needs to be determined. One of the methods to determine is the method of "switching off" gravity that we follow here [14,18,12,13]. This method is based on the idea that in absence of gravity the contortion tensor should vanish ${}^\bullet K^i{}_{k\nu} = {}^\bullet A^i{}_{k\nu} - {}^\circ A^i{}_{k\nu} \to 0$ and hence L-CSC reduces to ISC ${}^\circ A^i{}_{k\nu} \to {}^\bullet A^i{}_{k\nu}$. To define ${}^\bullet A^i{}_{k\nu}$ we follow the steps:

1) For a solution under consideration, we choose a tetrad and calculate L-CSC:
   ${}^\circ A^a{}_{b\sigma} = -h_b{}^\rho \overset{\circ}{\nabla}_\sigma h^a{}_\rho$.
2) From ${}^\circ A^a{}_{b\sigma}$ we construct the related curvature tensor ${}^\circ R^a{}_{b\gamma\delta}$.
3) We find some parameter that allows us to continuously "switch off" gravity and achieve ${}^\circ R^a{}_{b\gamma\delta} = 0$.
4) We then identify ${}^\bullet A^i{}_{k\nu}$ with this "switched-off" L-CSC ${}^\circ A^i{}_{k\nu}$.

It turns out that this method of determining ${}^\bullet A^i{}_{k\nu}$ is not unique since different tetrads can result in the same ISC. In this paper, we analyze and study this problem on the example of the Schwarzschild solution.

3. Mass of the Schwarzschild black hole and free-falling observers

3.1. Schwarzschild static gauge

The metric of the Schwarzschild solution is

$$ds^2 = -fdt^2 + f^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{7}$$

where $f = f(r) = 1 - 2M/r$. A convenient choice of a tetrad is the diagonal form:

$$^{B}h^{a}{}_{\mu} = diag\left(f^{1/2}, f^{-1/2}, r, r\sin\theta\right). \tag{8}$$

The non-zero components of L-CSC $^{\circ}A^{a}{}_{b\sigma}$ are

$$^{\circ}A^{0}{}_{10} = {}^{\circ}A^{1}{}_{00} = M/r^2, \ {}^{\circ}A^{1}{}_{22} = -{}^{\circ}A^{2}{}_{12} = -\left(1-2M/r\right)^{1/2},$$
$$^{\circ}A^{1}{}_{33} = -{}^{\circ}A^{3}{}_{13} = -\sin\theta\left(1-2M/r\right)^{1/2}, \ {}^{\circ}A^{2}{}_{33} = -{}^{\circ}A^{3}{}_{23} = -\cos\theta. \tag{9}$$

To obtain ISC $^{\bullet}A^{a}{}_{b\sigma}$ we "switch off" gravity by a simple requirement $M = 0$ that reduces (7) to a metric of Minkowski space, and we find

$$^{\bullet}A^{1}{}_{22} = -{}^{\bullet}A^{2}{}_{12} = -1, \quad {}^{\bullet}A^{1}{}_{33} = -{}^{\bullet}A^{3}{}_{13} = -\sin\theta, \quad {}^{\bullet}A^{2}{}_{33} = -{}^{\bullet}A^{3}{}_{23} = -\cos\theta. \tag{10}$$

The difference between (10) and (9) gives us the contortion $^{\bullet}K^{i}{}_{k\nu} = {}^{\bullet}A^{i}{}_{k\nu} - {}^{\circ}A^{i}{}_{k\nu}$, from where, using (5), we can find the non-zero components of $^{\bullet}S_{\mu}{}^{\rho\sigma}$:

$$^{\bullet}S_0{}^{01} = -{}^{\bullet}S_0{}^{10} = \frac{2}{r}\left[f - f^{1/2}\right], \quad {}^{\bullet}S_2{}^{12} = -{}^{\bullet}S_2{}^{21} = {}^{\bullet}S_3{}^{13} = -{}^{\bullet}S_3{}^{31} = -\frac{1}{2r}\left[f - 2f^{1/2}\right]. \tag{11}$$

To obtain the total mass of the Schwarzschild black hole we choose a displacement vector $\xi^{\alpha} = (-1, \ 0, \ 0, \ 0)$, which is both the velocity of the observer at infinity and also a Killing vector. Then using the formula (6) we find

$$P(\xi) = \frac{1}{\kappa}\lim_{r\to\infty}\int_{\partial\Sigma} dx^2 h\, {}^{\bullet}S_0{}^{01}\xi^0 = M, \tag{12}$$

where $\kappa = 8\pi$, $h = r^2\sin\theta$.

It is possible to find a local Lorentz transformation $\left(\Lambda_{Sch}\right)^{a}{}_{b}(x)$ that transforms ISC (10) to zero $^{\bullet}A^{a}{}_{c\mu} = 0$. Applying the same local Lorentz transformation to the tetrad (8), we find a new tetrad $^{A}h^{a}{}_{\mu} = \left(\Lambda_{Sch}\right)^{a}{}_{b}\,{}^{B}h^{b}{}_{\mu}$ that we call the proper tetrad [14]. By construction, the quantity (12) is both coordinate and locally Lorentz invariant. Thus, the result (12) is the same for the proper tetrad, as well as for any tetrad and ISC obtained from (8) and (10) by arbitrary coordinate transformations or simultaneous local Lorentz rotations. We call this set of pairs of tetrads and related ISCs as the *Schwarzschild static gauge*.

3.2. Lemaitre gauge

In order to describe a radially free-falling observer into the Schwarzschild black hole, it is natural to start with the Lemaitre coordinates $(\tau, \rho, \theta, \phi)$ that can be defined as [7]:

$$d\rho = dt + \frac{dr}{f\sqrt{1-f}}; \quad d\tau = dt + \frac{dr}{f}\sqrt{1-f}. \tag{13}$$

The metric (7) transforms to

$$ds^2 = -d\tau^2 + \left(1 - f(r)\right)d\rho^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{14}$$

where $r = r(\tau,\rho) = \left[3(\rho-\tau)/2\right]^{2/3}\left(2M\right)^{1/3}$. It is then possible to choose a diagonal tetrad for the metric (14) in these new coordinates (13) and follow the same method of determining ISC and calculating conserved charges as we did in the static case. This will result in all quantities being in the Lemaitre coordinates and we then have to transform them to the static coordinates since comparison with other gauges is more convenient when we work in the same coordinates. See for details our paper [16].

Here we follow a simpler method and analyze the free-falling case directly in the static coordinates. In order to obtain a tetrad representing a free-falling observer we consider a boosted tetrad ${}^{C}h^{a}{}_{\mu} = (\Lambda_{boost})^{a}{}_{b}\,{}^{B}h^{b}{}_{\mu}$, where

$$(\Lambda_{boost})^{a}{}_{b} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{15}$$

$\beta = \sqrt{2M/r}$ is a rapidity and $\gamma = 1\big/\sqrt{1-\beta^2}$. We can then follow the method of switching-off gravity to determine the spin connection. However, note that since the boost depends on M, when we "switch off" gravity $M = 0$, we obtain the same ISC as in the static case (10), i.e.

$$^{\bullet}A^{1}{}_{22} = -{}^{\bullet}A^{2}{}_{12} = -1,\quad {}^{\bullet}A^{1}{}_{33} = -{}^{\bullet}A^{3}{}_{13} = -\sin\theta,\quad {}^{\bullet}A^{2}{}_{33} = -{}^{\bullet}A^{3}{}_{23} = -\cos\theta. \tag{16}$$

This means that we can use the same $\left(\Lambda_{Sch}\right)^{a}{}_{b}(x)$ to obtain the proper tetrad in the free-falling case that we call ${}^{D}h^{a}{}_{\mu} = \left(\Lambda_{Sch}\right)^{a}{}_{b}\,{}^{C}h^{b}{}_{\mu}$.

We can now proceed to calculate the components of the superpotential (5) and we find that the non-zero components are:

$$^{\bullet}S_{0}{}^{01} = -{}^{\bullet}S_{0}{}^{10} = -\frac{4M}{r^2},\quad {}^{\bullet}S_{1}{}^{01} = -{}^{\bullet}S_{1}{}^{10} = -\frac{2}{rf}\sqrt{\frac{2M}{r}},$$
$$^{\bullet}S_{2}{}^{02} = {}^{\bullet}S_{3}{}^{03} = -{}^{\bullet}S_{2}{}^{20} = -{}^{\bullet}S_{3}{}^{30} = -\frac{1}{2rf}\sqrt{\frac{2M}{r}},\quad {}^{\bullet}S_{2}{}^{12} = {}^{\bullet}S_{3}{}^{13} = -{}^{\bullet}S_{2}{}^{21} = -{}^{\bullet}S_{3}{}^{31} = \frac{M}{r^2}. \tag{17}$$

Both the proper tetrad ${}^{D}h^{a}{}_{\mu}$ with zero ISC and the free-falling tetrad ${}^{C}h^{b}{}_{\mu}$ with ISC given by (16) will lead to the same superpotential. We call these combinations of tetrads and ISCs that lead equivalent results for the superpotential (17) as the *Lemaitre gauge.*

In order to calculate the Noether current and conserved charges we consider two distinct choices for the vector field ξ. The first choice is the vector field:

$$\tilde{\xi}^{\alpha} = \left(-f^{-1}, \sqrt{1-f}, 0, 0\right) \tag{18}$$

that represents the velocity of a free-falling observer. Using this vector field we find that the Noether current vanishes

$$^{\bullet}J^{\alpha}\left(\tilde{\xi}\right) = \left(0,0,0,0\right). \tag{19}$$

Naturally, it leads to a vanishing Noether charge (6). This corresponds to the equivalence principle since a free-falling observer is expected to measure a zero gravitational energy. We can also consider the second choice for the vector field ξ and choose it as a Killing vector $\xi^{\alpha} = \left(-1,\ 0,\ 0,\ 0\right)$. In this case we find that the Noether current is non-vanishing and results in the Noether conserved charge

$$P(\xi) = 2M\,. \tag{20}$$

It is rather difficult to understand this result that seems to be unphysical, what motivates us to support the first choice for the vector field ξ as the velocity of the observer. However, in the next section we demonstrate that it is possible to generalize this situation in a rather natural way by including an initial velocity and obtain a physically meaningful result using the Killing vector.

4. An arbitrary free-falling observer

A free-falling observer with 4-velocity (18) corresponds to the case when its velocity at infinity is zero. It is natural to generalize this by including an initial velocity for the observer. Solving the geodesic equation in a general form for a radially free-falling observer into the Schwarzschild black hole in the static coordinates, one obtains that the observer's velocity is given by [16]:

$$\tilde{\tilde{\xi}}^{\alpha} = \left(-ef^{-1}, \sqrt{e^2 - f}, 0, 0\right) \tag{21}$$

that coincides with (18) when $e = 1$. Here we consider only the case $e > 1$, for a discussion of the case $e < 1$ see [16].

We can then introduce freely falling observer's proper coordinates adopted to (21) and work in these coordinates, construct tetrads and calculate ISC, and then transform the results into static coordinates. See [16] for details. However, it is possible to work in the static coordinates only and find the tetrad representing the arbitrarily free-falling observer with the velocity (21) as

$$^{E}h^{a}{}_{\mu} = (\Lambda_{boost\_e})^{a}{}_{b}\, {}^{B}h^{b}{}_{\mu}, \tag{22}$$

where

$$(\Lambda_{boost\_e})^{a}{}_{b} = \begin{bmatrix} e\gamma & \gamma\sqrt{e^2-1+\beta^2} & 0 & 0 \\ \gamma\sqrt{e^2-1+\beta^2} & e\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

We can then use the method of "switching off" gravity and obtain ISC corresponding to (22):

$$^{\bullet}A^{0}{}_{22} = {}^{\bullet}A^{2}{}_{02} = -\sqrt{e^2-1}, \quad {}^{\bullet}A^{0}{}_{33} = {}^{\bullet}A^{0}{}_{33} = -\sin\theta\sqrt{e^2-1},$$

$$^{\bullet}A^{1}{}_{22} = -{}^{\bullet}A^{2}{}_{12} = -e, \quad {}^{\bullet}A^{1}{}_{33} = -{}^{\bullet}A^{3}{}_{13} = -e\sin\theta, \quad {}^{\bullet}A^{2}{}_{33} = -{}^{\bullet}A^{3}{}_{23} = -\cos\theta. \tag{24}$$

It is possible to find a local Lorentz transformation that transform this ISC to zero. Applying this transformation to the arbitrarily free-falling tetrad $^{E}h^{a}{}_{\mu}$ we obtain the proper tetrad $^{F}h^{a}{}_{\mu}$. Both $^{E}h^{a}{}_{\mu}$ with ISC (24) and the proper tetrad $^{F}h^{a}{}_{\mu}$ represent the same physical situation and hence we can call it the *generalized Lemaitre gauge* or *e*-gauge, for more details see [16].

Now we find the components $^{\bullet}S_{\mu}{}^{\rho\sigma}$ in the *e*-gauge in the static coordinates:

$$^{\bullet}S_{0}{}^{01} = -\frac{2}{r}\left(A_e - 1 + 2M/r\right), \quad {}^{\bullet}S_{1}{}^{01} = -\frac{2}{r}\frac{B_e}{1-2M/r}, \quad {}^{\bullet}S_{2}{}^{12} = {}^{\bullet}S_{3}{}^{13} = \frac{1}{r}\left(A_e - 1 + M/r\right),$$

$$^{\bullet}S_{2}{}^{02} = {}^{\bullet}S_{3}{}^{03} = -\frac{e}{r}\frac{A_e - 1 + M/r}{(1-2M/r)\sqrt{e^2-1+2M/r}}, \tag{25}$$

where one has to add components with opposite signs and antisymmetric in the last indices, and we have introduced a short-handed notation:

$$A_e \equiv e^2 - \sqrt{e^2-1}\sqrt{e^2-1+2M/r}, \quad A_{e=1} \equiv 1,$$
$$B_e \equiv e\left(\sqrt{e^2-1+2M/r} - \sqrt{e^2-1}\right), \quad B_{e=1} \equiv \sqrt{2M/r}. \tag{26}$$

The superpotential (25) reduces to (17) in the case $e = 1$.

We now proceed to calculate the conserved currents and charges using the Noether machinery presented here. We can again consider two choices for the vector field ξ. As the first choice, we consider the velocity of a free-falling observer (21), and using (25), we find the vanishing Noether current

$$\overset{\bullet}{J}{}^{\rho}\left(\tilde{\tilde{\xi}}\right) = \begin{pmatrix}0, & 0, & 0, & 0\end{pmatrix} \tag{27}$$

and, naturally, a vanishing Noether charge. We can understand this to be a consequence of the equivalence principle. The second choice is the timelike Killing vector $\xi^\alpha = \begin{pmatrix}-1, & 0, & 0, & 0\end{pmatrix}$ that leads to $P(\xi) = M$. This is in contrast with the Lemaitre gauge where we obtained the unphysical result of $2M$ in (20).

5. Discussion

In this paper, we have studied the Noether conserved currents and charges in the covariant formulation of TEGR in the case of the Schwarzschild black hole. We have considered three distinct physical situations by considering the static, Lemaitre, and generalized Lemaitre observers. In each situation, we have considered two choices of the vector ξ in the definition of the Noether conserved quantities: a) as a velocity of the observer, or b) as a time-like Killing vector. In the case of the static observer, both choices lead to the same result and hence we cannot effectively distinguish between them. The case of the Lemaitre free-falling observer revealed the difference and motivated us to suggest that ξ should be chosen as a velocity of the observer to satisfy the equivalence principle, while the other choice, in the charge calculations, leads to an unphysical result **2 *M***. However, we have found that when we generalize our observer by including some initial velocity, using the observer's proper vector we get the equivalence principle and using the time-like Killing vector we find the conserved charge to be equivalent to the mass of the black hole. Curiously, this happens for an arbitrarily small initial velocity.

The second problem studied here was the problem of calculating ISC, which is left undetermined by the field equations but plays an important role in a definition of conserved charges. We have considered here the method of "switching-off" gravity that is usually used in the TEGR literature. We have demonstrated a certain ambiguity and non-uniqueness in this method since both the static and Lemaitre tetrads correspond to the same ISC, despite representing physically distinct situations. By including the initial velocity this ambiguity seems to be removed for black hole mass calculations, but, in depth of structure, it remains creating the class of gauges depending on the initial velocity, and we distinct this class of ISCs for the generalized Lemaitre observers.

This issue of determining ISC, or, alternatively, the special class of preferred tetrads, is even more important in the case of modified gravity theories, such as the $f(T)$ gravity model, where it affects the field equations of the theory and hence becomes the matter of dynamics. Using our results here, it is possible to show that while the situation in TEGR and $f(T)$ gravity seem to share many similarities in the static case, the free-falling case reveals important differences and demonstrates that the so-called good tetrads in $f(T)$ gravity cannot be always constructed in analogy with the TEGR case [16].

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