

Master constraint approach to quantum-reduced loop gravity

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Abstract

We introduce a master constraint operator on the kinematical Hilbert space of loop quantum gravity representing a set of gauge conditions which classically fix the densitized triad to be diagonal. We argue that the master constraint approach provides a natural and systematic way of carrying out the quantum gauge-fixing procedure which underlies the model known as quantum-reduced loop gravity. The Hilbert space of quantum-reduced loop gravity is obtained as a particular space of solutions of the gauge-fixing master constraint operator. We give a concise summary of the fundamental structure of the quantum-reduced framework, and consider several possible extensions thereof, for which the master constraint formulation provides a convenient starting point. In particular, we propose a generalization of the standard Hilbert space of quantum-reduced loop gravity, which may be relevant in the application of the quantum-reduced model to physical situations in which the Ashtekar connection is not diagonal.

Keywords: loop quantum gravity, quantum-reduced loop gravity, models of loop quantum gravity, master constraint, quantum gauge-fixing

1. Introduction

Quantum-reduced loop gravity is a physically motivated model of loop quantum gravity (LQG), which is designed to address the crucial question of extracting statements about concrete physical phenomena from the formalism of full LQG. From the practical standpoint, a key advantage of the quantum-reduced model is its simplified kinematical structure, which



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allows for physical applications of the model to be developed at the level of explicit calculations. Quantum-reduced loop gravity was introduced by Alesci and Cianfrani in [1, 2] and studied further in several subsequent works, e.g. [3–7]. The model has found applications in cosmology [8–11] and in the physics of black holes [12–14], and its relation with full LQG has been studied particularly in [15, 16].

On a technical level, quantum-reduced loop gravity is based on the idea of implementing in the quantum theory a gauge fixing to a diagonal densitized triad. The gauge conditions select a sector of the kinematical Hilbert space of full LQG, which serves as the Hilbert space of the quantum-reduced model. The kinematical structure of quantum-reduced loop gravity is then completed by the definition of the model’s basic operators, which can be seen as arising from the action of the corresponding operators of full LQG on the aforementioned Hilbert space [16]. A distinctive feature of the resulting framework is the considerable simplicity of its quantum states and operators, when compared with their counterparts in the full theory¹.

Thus, the choice of gauge which constitutes the fundamental premise of quantum-reduced loop gravity is motivated ultimately by practical considerations: Starting from the framework of full LQG, the program initiated by Alesci and Cianfrani is able to establish a model which, on one hand, enjoys a sufficiently simplified kinematical structure to enable concrete calculations about physics to be performed within the model, while on the other hand retaining a clear and definite connection with the formalism of the full theory.

In the original work of Alesci and Cianfrani [1, 2], the gauge conditions for a diagonal triad were enforced in the quantum theory through an intricate procedure involving $U(1)$ holonomies and the formalism of projected spin networks [19]. In the present article we propose the master constraint method, originally introduced to LQG by Thiemann [20–22] in connection with the Hamiltonian constraint, as a natural and straightforward approach for implementing the quantum gauge-fixing conditions underlying quantum-reduced loop gravity. The classical starting point for the construction of a master constraint operator is to assemble the gauge conditions fixing the densitized triad E_i^a to be diagonal, i.e. $E_i^a = 0$ for $a \neq i$, into a single master constraint functional M , such that the single condition $M = 0$ is equivalent to the gauge conditions being satisfied everywhere on the spatial manifold Σ . Upon quantization, the master constraint is promoted into a constraint operator, which represents the gauge conditions for a diagonal triad on the kinematical Hilbert space of LQG.

In the context of quantum-reduced loop gravity and quantum gauge-fixing, an advantage of the master constraint approach is that it results in the gauge conditions being formulated entirely in terms of a well-defined constraint operator. This allows one to largely avoid a certain level of confusion, which is present in much of the early literature of quantum-reduced loop gravity regarding various technical aspects of the framework. In a sense, the present article can therefore be seen as a complement to the earlier article [16], where the status of the fundamental operators of quantum-reduced loop gravity was clarified by establishing a clear relation between these operators and the corresponding operators of full LQG. The work presented in this article arguably accomplishes an analogous clarification for the quantum states which form the Hilbert space of quantum-reduced loop gravity.

¹ As a specific example we may mention the volume operator, which is important in LQG not only as a fundamental geometrical observable, but also as a key element in the formulation of the dynamics (the volume operator is involved in any standard definition of the Hamiltonian constraint operator in LQG). In full LQG, the matrix elements of the volume operator between basis states of the kinematical Hilbert space cannot be computed in explicit form in general, but only in certain simple enough special cases (see e.g. [17, 18]). In contrast, the volume operator in quantum-reduced loop gravity acts diagonally on the natural basis states of the Hilbert space of the model.

The material in this article is organized as follows. In section 2 we give a brief summary of the basic kinematical framework of LQG. This serves to establish our notation and conventions, and to introduce the necessary tools which will be used in the main part of the article. (For a more comprehensive introduction to LQG, we refer the reader to any of the standard references, e.g. [23–28].) In section 3 we define a master constraint operator representing the gauge conditions which fix the triad to be diagonal on the kinematical Hilbert space of LQG. We show that the states spanning the Hilbert space of quantum-reduced loop gravity can be recovered as solutions of our master constraint operator on cubical graphs, with the notion of solutions being understood in the generalized sense² utilized by Alesci and Cianfrani when originally introducing the quantum-reduced model.

Section 4 presents a concise but methodical overview of the fundamental structure of quantum-reduced loop gravity, recapitulating the results obtained in section 3 regarding the Hilbert space of the quantum-reduced model, and in the earlier article [16] regarding the basic operators of the model. When the Hilbert space of quantum-reduced loop gravity is viewed as a subspace of the kinematical Hilbert space of full LQG, operators of the quantum-reduced model arise in a natural way from the action of operators of full LQG on the reduced Hilbert space. A given operator of the full theory typically gives rise to a corresponding ‘reduced’ operator, which is obtained by discarding certain small terms generated by the action of the full operator on states in the reduced Hilbert space. The reduced operator is a well-defined operator in the framework of the quantum-reduced model, the Hilbert space of the model being preserved by the action of this operator, while from the point of view of full LQG, the reduced operator provides a good approximation of the action of the corresponding full theory operator on the reduced Hilbert space.

In section 5 we consider various possible generalizations of the standard formulation of quantum-reduced loop gravity. We discuss the question of $SU(2)$ gauge invariance in the quantum-reduced model, and present two ways in which this question could in principle be addressed. We also introduce an extension of the standard Hilbert space of quantum-reduced loop gravity, which is obtained by splitting each edge of a standard reduced basis state into two ‘half-edges’, which can carry arbitrary magnetic quantum numbers at their free ends. We argue that a generalization of the reduced Hilbert space (which may be the generalization proposed in section 5.2, or another, so far unknown generalization of a different kind) is likely needed in order to capture all the degrees of freedom contained in a general, not necessarily diagonal Ashtekar connection. Finally, the concluding section 6 summarizes the work presented in this article and briefly comments on possible directions for future work.

2. A brief review of LQG

2.1. The kinematical Hilbert space

The kinematical Hilbert space of LQG is formed by the so-called cylindrical functions. (More precisely, the kinematical Hilbert space is defined as the completion of the space of cylindrical functions with respect to the scalar product defined in equation (2.5) and in the subsequent

² Namely, since the master constraint operator represents a set of constraints which are non-commuting at the level of the quantum theory (although they are compatible with each other classically), one acknowledges that looking for solutions of the constraint equation as an exact equality would be too restrictive of a condition. Instead, (generalized) solutions are searched in the form of states which carry large spin quantum numbers, and which approximately satisfy the constraint equation in the regime of large spins.

text.) A function cylindrical with respect to a graph Γ is a complex-valued function of the form

$$\Psi_\Gamma(h_{e_1}, \dots, h_{e_N}). \quad (2.1)$$

The graph Γ consists of N edges e_1, \dots, e_N , which are assumed to be oriented and embedded in the spatial manifold Σ . The arguments of the cylindrical function (2.1) are $SU(2)$ -valued group elements, one for each edge of the graph.

The group elements h_e are typically referred to as holonomies, due to their origin as holonomies (path-ordered exponentials) of the Ashtekar–Barbero connection in the classical theory. In the quantum theory the holonomies are assumed to satisfy certain algebraic properties with respect to the edge label e , which are consistent with the classical interpretation of the holonomy as a parallel propagator. These properties are

$$h_{e^{-1}} = h_e^{-1}, \quad (2.2)$$

$$h_{e_2 \circ e_1} = h_{e_2} h_{e_1}, \quad (2.3)$$

$$h_p = \mathbb{1}, \quad (2.4)$$

where e^{-1} denotes the edge e taken with the opposite orientation, $e_2 \circ e_1$ denotes the combined edge obtained by taking e_1 followed by e_2 (it is assumed that the endpoint of e_1 coincides with the beginning point of e_2), and p denotes an edge consisting of a single point.

A scalar product on the space of cylindrical functions is given by the Ashtekar–Lewandowski scalar product [29, 30]. This is defined as follows. For two functions cylindrical with respect to the same graph Γ , we set

$$\langle \Psi_\Gamma | \Psi'_\Gamma \rangle = \int dh_1 \cdots dh_N \overline{\Psi_\Gamma(h_1, \dots, h_N)} \Psi'_\Gamma(h_1, \dots, h_N) \quad (2.5)$$

where dh denotes the normalized Haar measure of $SU(2)$. To extend the definition to two functions cylindrical with respect to two different graphs Γ_1 and Γ_2 , consider any graph Γ_{12} that contains both Γ_1 and Γ_2 as subgraphs. By introducing a trivial dependence on the group elements associated with the additional edges, each of the two functions in question can be seen as a cylindrical function on Γ_{12} (see e.g. [23]), and their scalar product is then defined by applying equation (2.5) on Γ_{12} . (The normalization of the Haar measure implies that the result is independent of the choice of the graph Γ_{12} .)

It follows from the Peter–Weyl theorem that a basis on the space of cylindrical functions can be constructed from the irreducible representation matrices of $SU(2)$ (Wigner matrices) $D_{mn}^{(j)}(h)$. An orthonormal basis on the space of functions cylindrical with respect to a graph Γ is given by the functions

$$\prod_{e \in \Gamma} \mathfrak{D}_{m_e n_e}^{(j_e)}(h_e) \quad (2.6)$$

where the quantum numbers $\{j_e, m_e, n_e\}$ range over all their possible values, and the notation

$$\mathfrak{D}_{mn}^{(j)}(h_e) = \sqrt{2j+1} D_{mn}^{(j)}(h_e) \quad (2.7)$$

has been introduced for the normalized³ matrix elements of the Wigner matrices.

³ In the sense of the Haar measure, i.e. $\int dh |\mathfrak{D}_{mn}^{(j)}(h)|^2 = 1$.

An important subspace of the space of all cylindrical functions is the space of functions invariant under local $SU(2)$ gauge transformations. (Classically such transformations correspond to internal rotations of the densitized triad E_i^a .) Under a gauge transformation given by a gauge function $g(x) \in SU(2)$, the holonomy transforms as

$$h_e \rightarrow g(t(e)) h_e g^{-1}(s(e)) \quad (2.8)$$

where $s(e)$ and $t(e)$ stand for the beginning point (source) and endpoint (target) of the edge e . The space of gauge invariant cylindrical functions on a graph Γ is spanned by cylindrical functions of the form

$$\left(\prod_{v \in \Gamma} \iota_v \right) \cdot \left(\prod_{e \in \Gamma} \mathfrak{D}^{(j_e)}(h_e) \right). \quad (2.9)$$

These are the well-known spin network functions of LQG [31, 32]. In the function (2.9), an invariant tensor of $SU(2)$ (intertwiner) ι_v of an appropriate index structure is associated with each node v of the graph, and the dot indicates a complete contraction of magnetic indices according to the pattern dictated by the graph. More precisely, suppose a node v contains M edges oriented inwards and carrying spins j_1, \dots, j_M , and N edges oriented outwards and carrying spins j'_1, \dots, j'_N . Then the tensors ι_v assigned to the node form a basis of the intertwiner space

$$\text{Inv}_{SU(2)} \left(\mathcal{H}_{j_1} \otimes \dots \otimes \mathcal{H}_{j_M} \otimes \mathcal{H}_{j'_1}^* \otimes \dots \otimes \mathcal{H}_{j'_N}^* \right), \quad (2.10)$$

where \mathcal{H}_j^* is the dual space, and $\text{Inv}_{SU(2)}$ denotes the $SU(2)$ -invariant subspace. The condition for ι_v to be invariant under $SU(2)$ is to be understood in an appropriate generalized sense, which takes into account the index structure of the intertwiner. Being an element of the space (2.10), ι_v is a tensor with M lower indices (in the representations j_1, \dots, j_M) and N upper indices (in the representations j'_1, \dots, j'_N), and is required to satisfy

$$D^{(j'_1)}(g^{-1}) \dots D^{(j'_N)}(g^{-1}) \iota_v D^{(j_1)}(g) \dots D^{(j_M)}(g) = \iota_v \quad (2.11)$$

where g is an arbitrary element of $SU(2)$, and the action of the representation matrices on ι_v is given by contraction of magnetic indices.

2.2. Kinematical operators

The elementary operators of LQG are the holonomy and flux operators. These are the quantum operators corresponding to holonomies of the Ashtekar–Barbero connection along one-dimensional curves, and fluxes of the densitized triad through two-dimensional surfaces.

The holonomy operator associated to an edge e acts on cylindrical functions by multiplication:

$$\widehat{D_{mn}^{(j)}(h_e)} \Psi_\Gamma(h_{e_1}, \dots, h_{e_N}) = D_{mn}^{(j)}(h_e) \Psi_\Gamma(h_{e_1}, \dots, h_{e_N}). \quad (2.12)$$

The form of the resulting state depends on whether the edge e is contained among the edges e_1, \dots, e_N of the graph Γ . For the work carried out in the present article, the relevant case is that e coincides with one of the edges e_1, \dots, e_N . In this case, the function on the right-hand

side of equation (2.12) is still a cylindrical function on the same graph Γ . The explicit form of this function in the basis (2.6) can be found by using the $SU(2)$ Clebsch–Gordan series

$$D_{m_1 n_1}^{(j_1)}(h_e) D_{m_2 n_2}^{(j_2)}(h_e) = \sum_j C_{m_1 m_2 m_1+m_2}^{(j_1 j_2 j)} C_{n_1 n_2 n_1+n_2}^{(j_1 j_2 j)} D_{m_1+m_2 n_1+n_2}^{(j)}(h_e) \quad (2.13)$$

to couple the holonomies on the edge e ; here $C_{m_1 m_2 m}^{(j_1 j_2 j)}$ are the Clebsch–Gordan coefficients of $SU(2)$. (In calculations where the phases of the Clebsch–Gordan coefficients are relevant, we assume that they have been fixed according to the Condon–Shortley convention.)

Consider then the flux operator associated to a surface S . The action of the flux operator on a holonomy $D^{(j)}(h_e)$ depends on the relative location of the edge e with respect to the surface S . Assuming for simplicity that there is a single point of intersection between the surface S and an edge e , the action of the flux operator is given by

$$\hat{E}_i(S) D^{(j)}(h_e) = i\nu(S, e) \times \begin{cases} \frac{1}{2} D^{(j)}(h_e) \tau_i^{(j)} & s(e) \text{ lies on } S \\ \frac{1}{2} \tau_i^{(j)} D^{(j)}(h_e) & t(e) \text{ lies on } S \\ D^{(j)}(h_{e_1}) \tau_i^{(j)} D^{(j)}(h_{e_2}) & S \text{ intersects } e \text{ at an interior point} \end{cases} \quad (2.14)$$

Here $\nu(S, e)$ denotes the relative orientation of the surface and the edge at the intersection point, and $\nu(S, e) = 0$ if e intersects S tangentially. In the case that S intersects e at an interior point, e_1 and e_2 denote the two segments into which e is divided by S . Moreover, $\tau_i^{(j)}$ are the anti-Hermitian generators of $SU(2)$ in the spin- j representation. The matrix elements of $\tau_i^{(j)}$ are given in terms of the matrix elements of the angular momentum operator as

$$\left(\tau_i^{(j)} \right)_{mn} = -i \langle jm | J_i | jn \rangle. \quad (2.15)$$

In the standard basis, which diagonalizes the z -component of angular momentum, we have the following explicit expressions:

$$\left(\tau_x^{(j)} \right)_{mn} = -iA_+(j, n) \delta_{m, n+1} - iA_-(j, n) \delta_{m, n-1} \quad (2.16)$$

$$\left(\tau_y^{(j)} \right)_{mn} = -A_+(j, n) \delta_{m, n+1} + A_-(j, n) \delta_{m, n-1} \quad (2.17)$$

$$\left(\tau_z^{(j)} \right)_{mn} = -im \delta_{mn} \quad (2.18)$$

where

$$A_{\pm}(j, m) = \frac{1}{2} \sqrt{j(j+1) - m(m \pm 1)}. \quad (2.19)$$

The action of the flux operator is extended on the basis states (2.6) by the Leibniz rule, i.e. each edge in the product (2.6) which intersects the surface S contributes a term of the form (2.14) to the action of the operator.

The volume operator [33, 34] is a quantization of the classical functional $\int d^3x \sqrt{|\det E|}$. When acting on a cylindrical function, the volume operator picks up contributions from the nodes of the graph, and its action has the generic form

$$\hat{V}|\Psi_\Gamma\rangle = \sum_{v \in \Gamma} \hat{V}_v|\Psi_\Gamma\rangle \quad (2.20)$$

with

$$\hat{V}_v = \sqrt{|\hat{q}_v|}. \quad (2.21)$$

Here \hat{q}_v is an operator which can be expressed in explicit form in terms of left- and right-invariant vector fields of $SU(2)$. In contrast, no such explicit expression is available for the operator \hat{V}_v , which must in general be defined implicitly through its spectral decomposition. In this article we will work exclusively with the Ashtekar–Lewandowski version of the volume operator [34], and will refer to this operator simply as the volume operator.

In the construction of operators in LQG, one often has to deal with negative powers of the volume element $\sqrt{|\det E|}$ appearing in classical functionals which one wishes to quantize. These factors cannot be quantized simply by using the inverse of the volume operator; the volume operator is not invertible, as its spectrum contains the eigenvalue zero. In the literature of LQG, this problem is frequently circumvented by using the so-called Tikhonov regularization of the inverse volume operator (examples of the use of this operator can be found e.g. in [35–37]). The regularized inverse volume operator can be defined as

$$\widehat{\mathcal{V}_v^{-1}} = \lim_{\epsilon \rightarrow 0} \frac{\hat{V}_v}{\hat{V}_v^2 + \epsilon^2}. \quad (2.22)$$

Note that the operator (2.22) shares the same eigenstates as the operator \hat{V}_v , and each eigenvalue of $\widehat{\mathcal{V}_v^{-1}}$ is the inverse of the corresponding eigenvalue of \hat{V}_v , except the zero eigenvalue of \hat{V}_v remains a zero eigenvalue of $\widehat{\mathcal{V}_v^{-1}}$. Hence, given a complete set of eigenstates of the volume operator, the spectral decomposition of the regularized inverse volume is given by

$$\widehat{\mathcal{V}_v^{-1}} = \sum_{\lambda \neq 0} \lambda^{-1} |\lambda\rangle \langle \lambda| \quad (2.23)$$

where $|\lambda\rangle$ denotes an eigenstate with eigenvalue λ : $\hat{V}_v|\lambda\rangle = \lambda|\lambda\rangle$.

Another operator which is often used in technical constructions in LQG is the so-called parallel transported flux operator (this operator is also known as the gauge covariant flux operator in the LQG literature). The parallel transported flux operator is a quantization of the classical functional

$$\tilde{E}_i(S, x_0) = -2 \text{Tr} \left(\tau_i \tilde{E}(S, x_0) \right) \quad (2.24)$$

where

$$\tilde{E}(S, x_0) = \int_S dS_a h_{x_0, x} E_i^a(x) \tau^i h_{x_0, x}^{-1} \quad (2.25)$$

is a matrix-valued generalized flux variable. In equation (2.25), $dS_a = \frac{1}{2} \epsilon_{abc} dx^b \wedge dx^c$ denotes the natural area element on the surface S , and $h_{x_0, x} \equiv h_{p(x_0, x)}$ are (classical) holonomies associated to a family of paths $p(x_0, x)$. For each point x on the surface, $p(x_0, x)$ is a path connecting x

to a fixed point x_0 , which may lie on S or outside of it. Apart from their endpoints being fixed, the paths $p(x_0, x)$ can otherwise be chosen in principle arbitrarily, with different choices leading to different, possibly inequivalent implementations of the parallel transported flux operator.

The action of the parallel transported flux operator on a holonomy is discussed e.g. in section 3.3 of [38]. For the purposes of the present article, it is enough to consider the case where the surface S intersects the edge e at a single interior point of e . In this case, the action of the operator can be expressed in the form

$$\hat{\tilde{E}}_i(S, x_0) D^{(j)}(h_e) = i\nu(S, e) D^{(j)}(h_{e_2}) D^{(j)}\left(h_{x_0, x_+}^{-1}\right) \tau_i^{(j)} D^{(j)}(h_{x_0, x_+}) D^{(j)}(h_{e_1}) \quad (2.26)$$

where x_+ denotes the point of intersection between e and S , and, as in equation (2.14), e_1 and e_2 are the two segments into which e is divided by the intersection with S . We can see that, just as the action of the standard flux operator inserts the generator $\tau_i^{(j)}$ at the point where the surface intersects the edge—see equation (2.14)—so does the action of the parallel transported flux operator insert the matrix

$$D^{(j)}\left(h_{x_0, x_+}^{-1}\right) \tau_i^{(j)} D^{(j)}(h_{x_0, x_+}), \quad (2.27)$$

which is simply the generator $\tau_i^{(j)}$ ‘rotated’ by the adjoint action of the $SU(2)$ group element h_{x_0, x_+} .

3. Master constraint for diagonal gauge

3.1. The master constraint approach

In this section we propose a constraint operator which represents the gauge conditions

$$E_i^a = 0 \quad \text{for } i \neq a \quad (3.1)$$

on the kinematical Hilbert space of LQG. (Our convention is that the spatial index takes the values $a = x, y, z$ while the internal index takes the values $i = 1, 2, 3$; however, the values 1, 2, 3 are identified respectively with x, y, z and are often used interchangeably with each other.) In the classical theory, the gauge conditions (3.1) can be realized by fixing (partially⁴) the gauge freedom corresponding to the Gauss and diffeomorphism constraints. The different gauge conditions classically commute among themselves, $\{E_i^a, E_j^b\} = 0$, which presents no obstruction against imposing the gauge (3.1). On the other hand, in the quantum theory the different components of the flux operator $\hat{E}_i(S)$ do not commute with each other in general. Hence the procedure of representing the gauge conditions (3.1) individually as constraint operators and solving them one by one is inherently ambiguous, and one would expect e.g. that the eventual space of solutions may depend on the order in which the individual constraints are imposed.

⁴ The gauge fixing is partial because there exist some residual diffeomorphisms which preserve the gauge (3.1). These are the diffeomorphisms which, when interpreted as coordinate transformations on the spatial manifold Σ , have the form

$$(x, y, z) \rightarrow (X(x), Y(y), Z(z))$$

where each new coordinate is a function of the corresponding old coordinate only.

To construct a constraint operator which treats the constraints (3.1) in a symmetric manner, we turn to the master constraint approach, which was introduced to LQG by Thiemann [20–22] as a method for dealing with the quantization of the Hamiltonian constraint. In this approach one begins by defining a master constraint functional M on the classical phase space, such that the condition $M=0$ is equivalent to the gauge conditions (3.1) being satisfied at every point of the spatial manifold Σ . A master constraint operator \hat{M} corresponding to the classical functional M is then constructed on the kinematical Hilbert space of LQG, and solutions of the constraint are defined as states satisfying the condition

$$\hat{M}|\Psi\rangle = 0. \quad (3.2)$$

In our case, it turns out that the condition (3.2) must be interpreted in a certain approximate sense, instead of imposing it as an exact equation. Since the different gauge conditions (3.1) do not commute with each other as quantum operators, one cannot expect to be able to find a large family of states which satisfy the constraint equation (3.2) exactly. Following the approach originally taken by Alesci and Cianfrani in their work on quantum-reduced loop gravity [2, 6, 15], we understand (generalized) solutions of the condition (3.2) to be states which are characterized by large values of the spin quantum numbers, and which approximately satisfy equation (3.2) in the large spin regime, such that the norm of the state $\hat{M}|\Psi\rangle$ approaches zero in the formal limit $j \rightarrow \infty$.

3.2. Definition of the master constraint operator

Consider the constraint function

$$\mu^2 = (E_2^x)^2 + (E_3^x)^2 + (E_1^y)^2 + (E_3^y)^2 + (E_1^z)^2 + (E_2^z)^2. \quad (3.3)$$

Clearly the condition $\mu^2(x) = 0$ is fulfilled if and only if all the gauge conditions (3.1) are satisfied at the point x . A constraint which encodes the gauge conditions at every point on the spatial manifold Σ can be obtained by integrating the constraint (3.3) over Σ . We divide μ^2 by \sqrt{q} , with $q = \det(q_{ab})$ being the determinant of the spatial metric, to construct an integrand of density weight +1. Integrating the resulting expression over Σ then gives rise to the master constraint functional

$$M = \int_{\Sigma} d^3x \frac{\mu^2}{\sqrt{q}}. \quad (3.4)$$

The integrand in equation (3.4) is a positive definite function. Therefore, in the classical theory, the single constraint equation $M=0$ is equivalent to the gauge conditions (3.1) being satisfied everywhere on Σ .

Our task now is to write down an operator corresponding to the classical functional (3.4) on the kinematical Hilbert space of LQG. To this end, it is useful to add and subtract the squared diagonal components of E_i^a in μ^2 , which allows us to rewrite equation (3.4) as

$$M = \int_{\Sigma} d^3x \sum_a \frac{\sum_i (E_i^a)^2 - (E_a^a)^2}{\sqrt{q}}. \quad (3.5)$$

To construct an operator corresponding to the expression (3.5), we make use of a system of cubical cells and surfaces analogous to the one employed e.g. in the construction of the Ashtekar–Lewandowski volume operator in [34]. The spatial manifold Σ is partitioned into

cubical cells, and for concreteness we let each cell have edge length ϵ and coordinate volume ϵ^3 . Inside any given cell \square , we set up an assembly of three surfaces $S^a(\square)$, $a = x, y, z$. Furthermore, we assume the existence of a fixed Cartesian background coordinate system, and we require that the faces of the cells \square and the surfaces $S^a(\square)$ are aligned with the coordinate directions defined by this coordinate system, such that the background coordinate $x^a = \text{const}$ on the surface $S^a(\square)$. When considering the action of the resulting quantum operator on a cylindrical function defined on a graph Γ , we assume that the partition is adapted to the graph, by which we mean that the following two conditions must be satisfied: (1) The parameter ϵ is taken small enough to ensure that each cell \square contains at most a single node of the graph; (2) If a cell \square contains a node of the graph, the node coincides with the single point of intersection between the surfaces $S^a(\square)$. For later use, denote this point of intersection by p_\square , i.e. $p_\square = S^x(\square) \cap S^y(\square) \cap S^z(\square)$.

With the above preparations in place, we now turn to defining an operator corresponding to the functional (3.5). The form of this operator is postulated as follows. We write the integral over Σ as a sum of integrals over the cells \square and, for each cell \square , quantize every instance of the densitized triad E_i^a as the flux operator $\hat{E}_i(S^a(\square))$, and the factor $1/\sqrt{q}$ as the regularized inverse volume operator $\widehat{\mathcal{V}_{p_\square}^{-1}}$ of equation (2.22) associated to the point p_\square . If the cell \square does not contain a node of the graph of the cylindrical function on which the operator acts, the action of $\widehat{\mathcal{V}_{p_\square}^{-1}}$ gives zero, due to the fact that the volume operator \hat{V}_v has a non-vanishing action only on nodes of valence three or higher. For a cell which contains a node, the action of $\widehat{\mathcal{V}_{p_\square}^{-1}}$ will in general be non-zero, since we have assumed that the point p_\square coincides with the node inside \square . Hence the action of the master constraint operator on a given cylindrical function will pick up non-trivial contributions only from cells which contain a node, i.e. it has the form

$$\hat{M}|\Psi_\Gamma\rangle = \sum_{v \in \Gamma} \hat{M}_v |\Psi_\Gamma\rangle. \quad (3.6)$$

Since the operator $\widehat{\mathcal{V}_v^{-1}}$ does not commute with the flux operator, a factor ordering has to be specified. We assume that the inverse volume operator is ordered to the left of the flux operators⁵. With this choice, the operator \hat{M}_v is given by the expression

$$\hat{M}_v = \widehat{\mathcal{V}_v^{-1}} \sum_a \left(\hat{A}(S^a(v))^2 - \hat{E}_a(S^a(v))^2 \right), \quad (3.7)$$

where we now use the notation $S^a(v)$ to denote the three surfaces placed within the cell containing the node v . In equation (3.7) the sum runs over $a = x, y, z$, and

$$\hat{A}(S) = \sqrt{\hat{E}_i(S) \hat{E}_i(S)} \quad (3.8)$$

is the area operator associated to the surface S . This concludes the definition of the master constraint operator corresponding to the classical master constraint (3.4).

⁵ The outcome of the derivation performed in section 3.3 is not affected by this choice. The states obtained there as solutions of the master constraint operator are also solutions of the constraint where the operator $\widehat{\mathcal{V}_v^{-1}}$ is ordered to the right.

3.3. A family of solutions

Following the work of Alesci and Cianfrani [2, 6, 15], we look for solutions of the master constraint operator defined by equations (3.6) and (3.7) among cylindrical functions based on cubical graphs. By a cubical graph we mean a graph whose nodes are six-valent, and whose edges are aligned with the Cartesian coordinate directions defined by the background coordinate system used in the construction of the master constraint operator in the preceding section. In the calculation which follows, we consider a fixed but arbitrary cubical graph; hence the analysis presented in this section applies to any cubical graph.

Since the operator \hat{M} acts locally at each node of the graph, we may focus our attention on a single node v . We label the edges incident on the node as indicated in figure 1, and we assume that the orientation of each edge e_i^\pm agrees with the positive direction of the corresponding background coordinate axis. Consider now the action of the operator \hat{M}_v on the generic basis state

$$\prod_{i=x,y,z} \mathfrak{D}_{m_i^+ n_i^+}^{(j_i^+)}(h_{e_i^+}) \mathfrak{D}_{m_i^- n_i^-}^{(j_i^-)}(h_{e_i^-}) \quad (3.9)$$

associated to the node v . For the moment, let us ignore the inverse volume operator in equation (3.7) and focus on the terms in the sum over a . Each of these terms will act only on two of the six holonomies in the state (3.9), since the edges corresponding to the remaining four holonomies are tangential to the surface $S^a(v)$. For instance, the term with $a = z$ will act only on the factor

$$\mathfrak{D}_{m_z^+ n_z^+}^{(j_z^+)}(h_{e_z^+}) \mathfrak{D}_{m_z^- n_z^-}^{(j_z^-)}(h_{e_z^-}) \quad (3.10)$$

while leaving the holonomies of the edges in the x - and y -directions untouched. Using equations (2.15)–(2.18), a straightforward calculation shows that the action of this term on the state (3.10) gives

$$\begin{aligned} & \left(\hat{A}(S^z(v))^2 - \hat{E}_z(S^z(v))^2 \right) \mathfrak{D}_{m_z^+ n_z^+}^{(j_z^+)}(h_{e_z^+}) \mathfrak{D}_{m_z^- n_z^-}^{(j_z^-)}(h_{e_z^-}) \\ &= \frac{1}{4} \left(j_z^+ (j_z^+ + 1) + j_z^- (j_z^- + 1) - (n_z^+)^2 - (m_z^-)^2 \right) \mathfrak{D}_{m_z^+ n_z^+}^{(j_z^+)}(h_{e_z^+}) \mathfrak{D}_{m_z^- n_z^-}^{(j_z^-)}(h_{e_z^-}) \\ &+ A_+(j_z^+, n_z^+) A_-(j_z^-, m_z^-) \mathfrak{D}_{m_z^+, n_z^+ + 1}^{(j_z^+)}(h_{e_z^+}) \mathfrak{D}_{m_z^- - 1, n_z^-}^{(j_z^-)}(h_{e_z^-}) \\ &+ A_-(j_z^+, n_z^+) A_+(j_z^-, m_z^-) \mathfrak{D}_{m_z^+, n_z^+ - 1}^{(j_z^+)}(h_{e_z^+}) \mathfrak{D}_{m_z^- + 1, n_z^-}^{(j_z^-)}(h_{e_z^-}). \end{aligned} \quad (3.11)$$

Note that, since the magnetic quantum numbers in a given representation j are constrained by $-j \leq m, n \leq j$, the coefficient of the diagonal term on the right-hand side is strictly positive, so the right-hand side as a whole cannot be equal to zero for any values of m_z^-, n_z^+ . This reflects the already anticipated fact that we should generally not expect to be able to find a large class of states satisfying the constraint equation $\hat{M}|\Psi\rangle = 0$ exactly; the best we can do is to look for states which solve the constraint in an appropriate approximate sense.

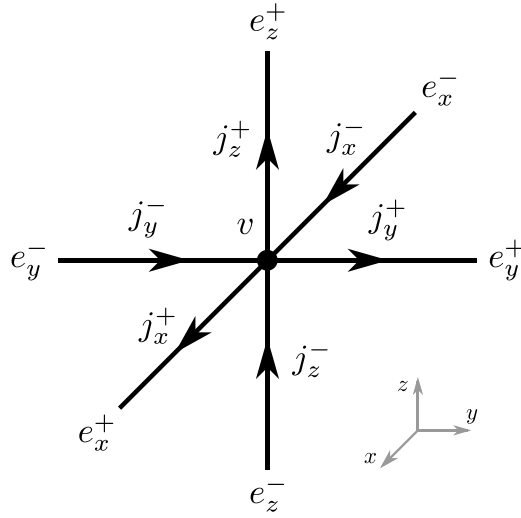


Figure 1. Labeling of the edges and spin quantum numbers for a generic node of a cubical spin network graph. The subscript $i = x, y, z$ in the notation e_i^\pm indicates that the edge is aligned with the x^i -coordinate direction of the background coordinate system. The superscript $-$ or $+$ shows whether the edge lies before or after the node v (in the sense defined by the positive direction of the x^i -coordinate axis).

Consider the choice

$$n_z^+ = j_z^+, \quad (3.12)$$

$$m_z^- = j_z^-. \quad (3.13)$$

This choice of the magnetic quantum numbers eliminates the off-diagonal terms on the right-hand side of equation (3.11), and cancels the terms quadratic in j in the coefficient of the diagonal term. Hence we are left with

$$\begin{aligned} & \left(\hat{A}(S^z(v))^2 - \hat{E}_z(S^z(v))^2 \right) \mathfrak{D}_{m_z^+ j_z^+}^{(j_z^+)}(h_{e_z^+}) \mathfrak{D}_{j_z^- n_z^-}^{(j_z^-)}(h_{e_z^-}) \\ &= \frac{1}{4} (j_z^+ + j_z^-) \mathfrak{D}_{m_z^+ j_z^+}^{(j_z^+)}(h_{e_z^+}) \mathfrak{D}_{j_z^- n_z^-}^{(j_z^-)}(h_{e_z^-}). \end{aligned} \quad (3.14)$$

(In fact, for fixed values of the spin quantum numbers j_z^\pm , the choice (3.12) and (3.13) leads to the smallest possible value of the norm of the state on the right-hand side of equation (3.11).)

Let us at this point address the terms with $a = x$ and $a = y$ in equation (3.7), which will act on the holonomies corresponding to $i = x$ and $i = y$ in the state (3.9). The above calculations were carried out in the standard basis, in which the generator $\tau_z^{(j)}$ is diagonal. These calculations will go through without modification for the x - and y -directions as well, provided that we do not use the standard basis, but an appropriately constructed basis which diagonalizes the corresponding generator $\tau_x^{(j)}$ or $\tau_y^{(j)}$. This basis is given by

$$|jm\rangle_i = D^{(j)}(g_i) |jm\rangle \quad (3.15)$$

where $i = x$ or y , and g_i denotes an $SU(2)$ element describing a rotation which rotates the z -axis into the i -axis. This rotation is of course not uniquely determined by the condition that the z -axis is rotated into the i -axis. We fix the ambiguity by requiring that the rotation corresponds to a cyclic permutation of the coordinate axes, so that the triple (x, y, z) is transformed into (y, z, x) or (z, x, y) . (This choice ensures that equations (2.16)–(2.18) are valid in the basis (3.15) simply after cyclically permuting the labels x, y and z , without the appearance of any additional phase factors.)

Introduce the notation

$$\mathfrak{D}_{mn}^{(j)}(h_e)_i = {}_i\langle jm | \mathfrak{D}^{(j)}(h_e) | jn \rangle_i \quad (3.16)$$

for the matrix elements of the normalized Wigner matrix $\mathfrak{D}^{(j)}(h_e) = \sqrt{2j+1} D^{(j)}(h_e)$ in the basis (3.15). Imposing the choice (3.12) and (3.13) on the holonomies in the x - and y -directions as well, with respect to the bases appropriate to these directions, we are led to consider the state

$$\prod_{i=x,y,z} \mathfrak{D}_{m_i^+ j_i^+}^{(j_i^+)}(h_{e_i^+})_i \mathfrak{D}_{j_i^- n_i^-}^{(j_i^-)}(h_{e_i^-})_i \quad (3.17)$$

(where $i = z$ refers to the standard basis). Denote this state by $|\Psi_v\rangle$. The calculation which produced equation (3.11) now shows that

$$\left(\hat{A}(S^a(v))^2 - \hat{E}_a(S^a(v))^2 \right) |\Psi_v\rangle = \frac{1}{4} (j_a^+ + j_a^-) |\Psi_v\rangle. \quad (3.18)$$

Hence the action of the entire operator (3.7) on the state (3.17) gives

$$\hat{M}_v |\Psi_v\rangle = \frac{1}{4} \sum_a (j_a^+ + j_a^-) \widehat{\mathcal{V}_v^{-1}} |\Psi_v\rangle. \quad (3.19)$$

Now the final step of our calculation is to find the action of the inverse volume operator $\widehat{\mathcal{V}_v^{-1}}$ on the state $|\Psi_v\rangle$. This can be deduced from the results obtained in [16], where the action of the volume operator \hat{V}_v on a state of the form (3.17) was derived under the assumption that all the spin quantum numbers characterizing the state are large:

$$j_i^\pm \gg 1. \quad (3.20)$$

The derivation was performed in two stages as follows. First, one considers the operator \hat{q}_v in equation (2.21). By direct calculation, it can be shown that

$$\hat{q}_v |\Psi_v\rangle = \lambda_v |\Psi_v\rangle + \mathcal{O}(j^2) \quad (3.21)$$

where

$$\lambda_v = \frac{1}{8} (j_x^+ + j_x^-) (j_y^+ + j_y^-) (j_z^+ + j_z^-) \quad (3.22)$$

and the term $\mathcal{O}(j^2)$ denotes a state which is not of the form (3.17), but whose norm depends quadratically on the spins. In contrast, the eigenvalue λ_v is of order $\mathcal{O}(j^3)$, so it follows from the assumption (3.20) that the second term on the right-hand side of equation (3.21) is small in comparison with the diagonal term. Due to the fact that the action of \hat{q}_v on the state $|\Psi_v\rangle$ is approximately diagonal, the action of the volume operator $\hat{V}_v = \sqrt{|\hat{q}_v|}$ on this state can be

deduced by applying standard perturbation theory to establish the relevant part of the spectral decomposition of the operator \hat{q}_v . In this way, as shown in detail in appendix B of [16], one finds

$$\hat{V}_v|\Psi_v\rangle = \sqrt{\lambda_v}|\Psi_v\rangle + \mathcal{O}(\sqrt{j}) \quad (3.23)$$

where the norm of the off-diagonal term is again small in comparison with the leading, diagonal term.

In the case presently at hand for us, the arguments presented in appendix B of [16] can equally well be applied to deduce the action of the operator $\widehat{\mathcal{V}}_v^{-1}$ on the state $|\Psi_v\rangle$, given the known action of \hat{q}_v on this state. The result is

$$\widehat{\mathcal{V}}_v^{-1}|\Psi_v\rangle = \lambda_v^{-1}|\Psi_v\rangle + \mathcal{O}(j^{-5/2}). \quad (3.24)$$

Going now back to equation (3.19), we finally establish that

$$\hat{M}_v|\Psi_v\rangle = \mathcal{O}(j^{-1/2})|\Psi_v\rangle + \mathcal{O}(j^{-3/2}), \quad (3.25)$$

which represents the conclusion of the calculations carried out in this section. We have found a family of states which, while not being exact solutions of the master constraint operator at the node v , do satisfy the constraint approximately in the regime of large spin quantum numbers, in the sense that the norm of the state $\hat{M}_v|\Psi_v\rangle$ approaches zero as the spins are taken larger and larger. This family of states is defined by equation (3.17), and is characterized by the property that the magnetic quantum number of each holonomy at v takes its maximum value with respect to the basis adapted to the direction of the corresponding edge.

The states given by equation (3.17) represent a set of approximate solutions of the gauge fixing constraint at the node v . To extend the solution to all nodes of a given cubical graph, it suffices to impose the conditions (3.12) and (3.13) at every node of the graph. In equation (3.17), this means that both magnetic indices of each holonomy are set to their maximal value. Finally, note that the calculations starting from equation (3.11) go through in exactly the same way if, instead of equations (3.12) and (3.13), we make the choice $n_z^+ = -j_z^+$ and $m_z^- = -j_z^-$, i.e. both magnetic indices now take their minimum value. (However, if one of n_z^+ and m_z^- is chosen to be maximal while the other is taken minimal, the off-diagonal terms in equation (3.11) are not completely eliminated, although the diagonal term behaves in the same way as before.) Hence we will allow both maximal and minimal values of the magnetic quantum numbers, but at any node v of the cubical graph, we require that both magnetic indices corresponding to a given coordinate direction must have the same sign. In this way we arrive at a space of (approximate) solutions of the master constraint operator (3.6), which is summarized in section 4.1 below.

4. Fundamental structure of quantum-reduced loop gravity

4.1. The reduced Hilbert space

We take the Hilbert space of the quantum-reduced model to be the space spanned by the states which were derived in section 3.3 as solutions of the gauge fixing master constraint operator. Thus, a basis on the reduced Hilbert space is given by the states

$$\prod_{e \in \Gamma} \mathfrak{D}_{\tau_e j_e \sigma_e j_e}^{(j_e)}(h_e)_{i_e}. \quad (4.1)$$

The graph Γ is a cubical graph, with its edges aligned along the coordinate directions defined by a fixed Cartesian background coordinate system. Note that the calculation of section 3.3 is valid for any cubical graph. Hence one is not necessarily restricted to working with just a single fixed graph; all cubical graphs (corresponding to a specific choice of the background coordinate directions) are included in the Hilbert space of quantum-reduced loop gravity.

All the spin quantum numbers labeling the state (4.1) are required to be large:

$$j_e \gg 1 \quad (4.2)$$

for every edge of the graph Γ . The labels σ_e and τ_e in the state (4.1) are sign factors, i.e. $\sigma_e = \pm 1$ and $\tau_e = \pm 1$. (The factors σ_e and τ_e belong respectively to the source and target of the edge e .) For any given edge, σ_e and τ_e can be chosen independently of each other; however, at each node of the graph the two labels associated with a given coordinate direction must have the same value, as described at the end of section 3.3. In other words, let $e_i^\pm(v)$ denote an edge which is incident on the node v and aligned in the x^i -coordinate direction, with the superscript indicating whether the edge lies before ($-$) or after ($+$) the node v (in the sense defined by the positive direction of the x^i -coordinate axis). Then we require that the labels σ_e and τ_e in the state (4.1) are constrained by the conditions

$$\sigma_{e_i^+(v)} = \tau_{e_i^-(v)}. \quad (4.3)$$

Regarding the orientation of the graph Γ , we assume that the orientation is fixed so that every edge of the graph is oriented towards the positive direction of the corresponding background coordinate axis. Note that there is no loss of generality in this assumption, due to the relation

$$D_{mn}^{(j)}(h_e^{-1}) = (-1)^{m-n} D_{-n-m}^{(j)}(h_e) \quad (4.4)$$

satisfied by the $SU(2)$ representation matrices. This relation shows that a state of the form (4.1) on a cubical graph of arbitrary orientation is equivalent to a state on a graph with the fixed orientation specified above, only with possibly modified values of some of the sign labels σ_e and τ_e .

Apart from the constraint (4.3), which has not been considered in the literature of quantum-reduced loop gravity so far, the reduced Hilbert space defined above agrees with the Hilbert space established in the articles originally introducing the quantum-reduced model, together with certain amendments proposed in later work. In particular, the states (4.1) with opposite signs of the magnetic quantum numbers (i.e. $\sigma_e = -\tau_e$) were not included in the Hilbert space considered originally in [1, 2], but such states were introduced later in [7], where the framework of quantum-reduced loop gravity was applied to the description of spherically symmetric spacetimes.

4.2. Action of LQG operators on the reduced Hilbert space

The relation between the operators of quantum-reduced loop gravity, as originally formulated in the literature of the quantum-reduced model, and the corresponding operators of full LQG has been examined in [16]. The calculations carried out in [16] reveal the following general picture: When an operator $\hat{\mathcal{O}}$ on the kinematical Hilbert space of LQG is applied on the basis states (4.1), the result typically (but not for all possible operators $\hat{\mathcal{O}}$) is of the schematic form

$$\hat{\mathcal{O}}|\Psi_0\rangle = f(j)|\Psi'_0\rangle + g(j)|\Phi_\perp\rangle. \quad (4.5)$$

Here $|\Psi_0\rangle$ and $|\Psi'_0\rangle$ are normalized states belonging to the reduced Hilbert space, while $|\Phi_\perp\rangle$ is a normalized state which is orthogonal to the reduced Hilbert space. Moreover, the coefficients $f(j)$ and $g(j)$ are such that, under the assumption of large j ,

$$f(j) \gg g(j). \quad (4.6)$$

Hence the conclusion is that, although the operator \hat{O} does not preserve the reduced Hilbert space exactly, the action of \hat{O} on a reduced state produces a state whose component outside of the reduced Hilbert space is much smaller than the component which lies within the reduced Hilbert space.

In the earlier literature on quantum-reduced loop gravity, the kinematical operators of the model are typically introduced as projections of the corresponding operators of full LQG down to the reduced Hilbert space (see e.g. [6]). Thus, the results obtained in [16] provide a justification for this procedure by showing that the terms which get projected out are small in comparison with the terms which are preserved by the projection.

We will now briefly review the results derived in [16] pertaining to the holonomy and flux operators, and extend these results to the basis states (4.1), in which arbitrary sign factors σ_e, τ_e are allowed (in [16] only the case $\sigma_e = \tau_e = +1$ was considered). For the flux operator, assuming the surface S intersects the edge e at one of its endpoints, equations (2.14) and (2.16)–(2.18) immediately yield

$$\hat{E}_i(S) \mathfrak{D}_{\tau_j \sigma_j}^{(j)}(h_e)_i = \nu(S, e) \times \begin{cases} \sigma \frac{j}{2} \mathfrak{D}_{\tau_j \sigma_j}^{(j)}(h_e)_i & s(e) \text{ lies on } S \\ \tau \frac{j}{2} \mathfrak{D}_{\tau_j \sigma_j}^{(j)}(h_e)_i & t(e) \text{ lies on } S \end{cases} \quad (4.7)$$

and

$$\hat{E}_k(S) \mathfrak{D}_{\tau_j \sigma_j}^{(j)}(h_e)_i = \mathcal{O}(\sqrt{j}) \quad (k \neq i). \quad (4.8)$$

Hence the dominant component in the action of the flux operator is the one where the internal index matches the direction of the edge on which the operator acts, and the action of this component on the reduced holonomy⁶ is diagonal. The remaining components of the flux operator act non-diagonally, producing states which do not belong to the reduced Hilbert space, but these components are suppressed by a factor of $j^{-1/2}$ relative to the diagonal component.

However, note that if the surface intersects the edge at an interior point, the action of the flux operator on a reduced holonomy gives

$$\hat{E}_k(S) D_{\tau_j \sigma_j}^{(j)}(h_e)_i = i\nu(S, e) \left(D^{(j)}(h_{e_2}) \tau_k^{(j)} D^{(j)}(h_{e_1}) \right)_{\tau_j \sigma_j}^{(i)}, \quad (4.9)$$

where the notation on the right-hand side indicates that the matrix element is taken in the basis $|jm\rangle_i$, i.e. $(M^{(j)})_{mn}^{(i)} = {}_i\langle jm|M^{(j)}|jn\rangle_i$. Here all the matrix elements of the generator $(\tau_i^{(j)})_{mn}$ enter the result, and the right-hand side of equation (4.9) does not reduce to any simple form even under the assumption of large j . The flux operator associated to a surface of this kind

⁶ We use the term ‘reduced holonomy’ to refer to the holonomies which make up the basis state (4.1). Thus, the characteristic property of a reduced holonomy associated to an edge e is that both magnetic quantum numbers of the holonomy have the maximal or minimal value with respect to the basis corresponding to the direction of the edge.

is therefore an example of an operator for which the result (4.5) does not hold. In quantum-reduced loop gravity one typically works only with flux operators whose surfaces intersect the cubical graph of the basis states (4.1) only at the nodes of the graph.

The action of the holonomy operator on a reduced holonomy is given by equation (2.13) as

$$\widehat{D_{mn}^{(s)}}(h_e) D_{\tau j \sigma j}^{(j)}(h_e) = \sum_k C_{\tau j m m'}^{(j s k)} C_{\sigma j n n'}^{(j s k)} D_{m' n'}^{(k)}(h_e), \quad (4.10)$$

where

$$m' = \tau j + m \quad (4.11)$$

$$n' = \sigma j + n \quad (4.12)$$

and we assume that $s = \mathcal{O}(1)$, while $j \gg 1$ as before. Under these assumptions, it was shown in [16] that the Clebsch–Gordan coefficients entering equation (4.10) behave as

$$C_{j m j+m}^{(j s j+m)} = 1 + \mathcal{O}\left(\frac{1}{j}\right) \quad (4.13)$$

and

$$C_{j m j+m}^{(j s k)} = \mathcal{O}\left(\frac{1}{\sqrt{j}}\right) \quad (k > j + m). \quad (4.14)$$

Using equations (4.13) and (4.14) in equation (4.10) one finds, considering first the case $\sigma = \tau = +1$,

$$\widehat{D_{mm}^{(s)}}(h_e) \mathfrak{D}_{jj}^{(j)}(h_e) = \mathfrak{D}_{j+m j+m}^{(j+m)}(h_e) + \mathcal{O}\left(\frac{1}{j}\right), \quad (4.15)$$

$$\widehat{D_{mn}^{(s)}}(h_e) \mathfrak{D}_{jj}^{(j)}(h_e) = \mathcal{O}\left(\frac{1}{\sqrt{j}}\right) \quad (m \neq n). \quad (4.16)$$

The case $\sigma = \tau = -1$ can be obtained from equations (4.15) and (4.16) with the help of the relation (4.4). We have

$$\widehat{D_{mm}^{(s)}}(h_e) \mathfrak{D}_{-j -j}^{(j)}(h_e) = \mathfrak{D}_{-j+m -j+m}^{(j-m)}(h_e) + \mathcal{O}\left(\frac{1}{j}\right), \quad (4.17)$$

$$\widehat{D_{mn}^{(s)}}(h_e) \mathfrak{D}_{-j -j}^{(j)}(h_e) = \mathcal{O}\left(\frac{1}{\sqrt{j}}\right) \quad (m \neq n). \quad (4.18)$$

Thus, the dominant terms in the action of the holonomy operator on a reduced holonomy with $\sigma = \tau$ are given only by the diagonal ($m = n$) components of the operator.

It remains to consider the cases $\sigma = +1, \tau = -1$ and $\sigma = -1, \tau = +1$. These can be deduced from the above results by using the symmetry properties of the Clebsch–Gordan coefficients, namely

$$C_{-m_1 -m_2 -m}^{(j_1 j_2 j)} = (-1)^{j_1+j_2-j} C_{m_1 m_2 m}^{(j_1 j_2 j)} \quad (4.19)$$

(see e.g. [39]). We find

$$\widehat{D_{m-m}^{(s)}}(h_e) \mathfrak{D}_{j-j}^{(j)}(h_e) = (-1)^{s-m} \mathfrak{D}_{j+m-j-m}^{(j+m)}(h_e) + \mathcal{O}\left(\frac{1}{j}\right) \quad (4.20)$$

$$\widehat{D_{m-m}^{(s)}}(h_e) \mathfrak{D}_{-j-j}^{(j)}(h_e) = (-1)^{s-m} \mathfrak{D}_{-j+m-j-m}^{(j-m)}(h_e) + \mathcal{O}\left(\frac{1}{j}\right) \quad (4.21)$$

and

$$\widehat{D_{mn}^{(s)}}(h_e) \mathfrak{D}_{\pm j \mp j}^{(j)}(h_e) = \mathcal{O}\left(\frac{1}{\sqrt{j}}\right) \quad (m \neq -n). \quad (4.22)$$

Hence, in the case of a reduced holonomy with $\sigma = -\tau$, the leading-order terms in the action of the holonomy operator arise from the ‘anti-diagonal’ ($m = -n$) components. In all the cases the leading term is again a reduced holonomy, i.e. each magnetic index is either maximal or minimal.

4.3. Reduced operators

The results summarized in the previous section suggest that operators of quantum-reduced loop gravity can be interpreted as arising from the action of the corresponding operators of full LQG on states in the reduced Hilbert space. (Note that the states (4.1) are elements of the kinematical Hilbert space of LQG.) From any operator $\hat{\mathcal{O}}$ whose action on the reduced Hilbert space has the form (4.5), we can obtain a reduced operator $\hat{\mathcal{O}}^R$ simply by discarding the second term on the right-hand side of equation (4.5), with equation (4.6) guaranteeing that the discarded term is small in comparison with the term which is kept. The action of the operator $\hat{\mathcal{O}}^R$ on the reduced Hilbert space is given by

$$\hat{\mathcal{O}}^R |\Psi_0\rangle = f(j) |\Psi'_0\rangle. \quad (4.23)$$

From the perspective of the quantum-reduced model, the reduced operator $\hat{\mathcal{O}}^R$ is therefore a well-defined operator on the reduced Hilbert space, this space being preserved by the action of the operator. On the other hand, comparing the action of the reduced operator to the original operator $\hat{\mathcal{O}}$, we can write

$$\frac{\|\hat{\mathcal{O}}|\Psi_0\rangle - \hat{\mathcal{O}}^R|\Psi_0\rangle\|}{\|\hat{\mathcal{O}}|\Psi_0\rangle\|} \ll 1 \quad (4.24)$$

which shows that from the perspective of the full theory, the operator $\hat{\mathcal{O}}^R$ provides a good approximation of the action of the full operator $\hat{\mathcal{O}}$ on states in the reduced Hilbert space.

The action of the reduced flux operator can be read off from equations (4.7) and (4.8) as

$$\hat{E}_k^R(S) \mathfrak{D}_{\tau j \sigma j}^{(j)}(h_e)_i = \delta_{ik} \nu(S, e) \varphi \frac{j}{2} \mathfrak{D}_{\tau j \sigma j}^{(j)}(h_e)_i, \quad (4.25)$$

where $\varphi = \sigma$ or $\varphi = \tau$ according to whether the surface S intersects the edge e at $s(e)$ or $t(e)$. The action of the reduced holonomy operator also follows from the results derived in the previous section. From the various cases considered there, we gather

$$\widehat{D_{mn}^{(s)}(h_e)_i}^R \mathfrak{D}_{\tau_j \sigma_j}^{(j)}(h_e)_i = \begin{cases} \delta_{mn} \mathfrak{D}_{j+m, j+m}^{(j+m)}(h_e)_i & (\tau, \sigma) = (++) \\ \delta_{mn} \mathfrak{D}_{-j+m, -j+m}^{(j-m)}(h_e)_i & (\tau, \sigma) = (--) \\ \delta_{m, -n} (-1)^{s-m} \mathfrak{D}_{j+m, -j-m}^{(j+m)}(h_e)_i & (\tau, \sigma) = (+-) \\ \delta_{m, -n} (-1)^{s-m} \mathfrak{D}_{-j+m, j-m}^{(j-m)}(h_e)_i & (\tau, \sigma) = (-+) \end{cases}. \quad (4.26)$$

It can be seen that in the cases where $\sigma = \tau$, the action of the reduced holonomy operator is essentially the multiplication law of the group $U(1)$, where the role of the $U(1)$ quantum number is played by the magnetic quantum number (and not the spin) carried by the holonomy operator.

Other LQG operators can be treated in the same way in order to derive their reduced counterparts. The example of the volume operator was analyzed in [16]. It was shown that the action of the reduced volume operator on the basis states (4.1) is diagonal, and is given by

$$\hat{V}_v^R \left(\prod_{e \in \Gamma} \mathfrak{D}_{\tau_e j_e \sigma_e j_e}^{(j_e)}(h_e)_{i_e} \right) = \lambda_v \left(\prod_{e \in \Gamma} \mathfrak{D}_{\tau_e j_e \sigma_e j_e}^{(j_e)}(h_e)_{i_e} \right) \quad (4.27)$$

where the eigenvalue is

$$\lambda_v = \sqrt{\frac{1}{8} (j_x^+ + j_x^-) (j_y^+ + j_y^-) (j_z^+ + j_z^-)} \quad (4.28)$$

with j_i^\pm denoting the spins on the six edges incident on v , as illustrated in figure 1. Note that the reduced volume operator is not constructed by starting with the reduced flux operators defined by equation (4.25) and using them to form a volume operator. Instead, we evaluate the action of the volume operator of the full theory on the reduced Hilbert space and, extracting the term of leading order in the spins, find that this term is an element of the reduced Hilbert space—in other words, the action of the operator is of the form (4.5). The reduced volume operator of equation (4.27) is then obtained by keeping only the leading term and discarding the terms of lower order in j .

To conclude our discussion of reduced operators, let us consider the Hamiltonian constraint. In order to obtain a well-defined Hamiltonian operator for the quantum-reduced model, the operator must be constructed in such a way that the terms of leading order in j produced by the action of the operator on states in the reduced Hilbert space are again elements of the reduced Hilbert space. This condition is necessary (and sufficient) to ensure that the reduced Hilbert space is preserved by the resulting reduced Hamiltonian, which is obtained by truncating the full action of the Hamiltonian at leading order in j . (If the space spanned by the states (4.1) is interpreted as a space of quantum states where the gauge conditions encoded in the master constraint operator (3.6) are satisfied, then the above condition can, loosely speaking, be interpreted as the condition for the chosen gauge to be preserved under the quantum dynamics generated by the reduced Hamiltonian.)

A concrete example of a Hamiltonian operator which satisfies the condition formulated above can be obtained as follows. As the classical starting point for the construction, we take the classical Hamiltonian constraint in the form

$$C(N) = \frac{1}{\beta^2} \int_{\Sigma} d^3x N \left(\frac{\epsilon_{ij}^k E_i^a E_j^b F_{ab}^k}{\sqrt{|\det E|}} + (1 + \beta^2) \sqrt{|\det E|} {}^{(3)}R \right). \quad (4.29)$$

Here β is the Barbero–Immirzi parameter, N is the lapse function, F_{ab}^i is the curvature of the Ashtekar connection A_a^i , and ${}^{(3)}R$ is the Ricci scalar of the spatial manifold Σ . For the quantization of the Lorentzian part of the Hamiltonian, represented in equation (4.29) by the term involving the Ricci scalar, we refer to the work carried out in [38, 40]. In this pair of articles, an operator representing the integrated Ricci scalar was first constructed for arbitrary kinematical states of LQG based on cubical graphs, and then it was established that the reduced Hilbert space is preserved by the action of this operator at leading order in j . To deal with the quantization of the Euclidean part of the Hamiltonian, we follow the construction in [41] which, when adapted to the quantization of the Euclidean part of the constraint (4.29), gives rise to the following operator (up to a constant numerical factor, which is omitted):

$$\hat{C}_E(N) = \sum_{v \in \Gamma} N(v) \widehat{\mathcal{V}_v^{-1}} \sum_{e \nparallel e' \text{ at } v} \epsilon^{ijk} \text{Tr} \left(\tau_k^{(s)} D^{(s)} \left(h_{\alpha_{ee'}} \right) \right) \hat{J}_i^{(v,e)} \hat{J}_j^{(v,e')}. \quad (4.30)$$

Here the inner sum runs over pairs of edges whose tangent vectors at v are linearly independent, and the operator $\hat{J}_i^{(v,e)}$ is defined as

$$\hat{J}_i^{(v,e)} D_{mn}^{(j)}(h_e) = \begin{cases} i D_{mn'}^{(j)}(h_e) \left(\tau_i^{(j)} \right)_{m'n} & v = s(e) \\ -i \left(\tau_i^{(j)} \right)_{mn'} D_{m'n}^{(j)}(h_e) & v = t(e) \end{cases} \quad (4.31)$$

However, we depart from [41] when it comes to specifying the loop $\alpha_{ee'}$, which arises from the regularization of the curvature F_{ab}^i . In our case, the operator (4.30) is applied on a cubical graph, and we take $\alpha_{ee'}$ to be the minimal closed loop formed by the edges e , e' and by two other edges of the cubical graph. Now a straightforward calculation can be performed to show that when the operator (4.30) acts on the basis states (4.1), the term of leading order in j does belong to the reduced Hilbert space. Therefore the construction sketched above provides a well-defined Hamiltonian constraint operator for quantum-reduced loop gravity.

5. Possible extensions and generalizations

5.1. $SU(2)$ gauge invariance

The basis states (4.1) are not invariant under the local $SU(2)$ gauge transformations given by equation (2.8). Hence, when working with a model whose state space is spanned by these states, one essentially takes the point of view that since the gauge freedom represented by the Gauss constraint is completely fixed by the gauge choice (3.1), there should be no trace of $SU(2)$ gauge invariance left in the quantum theory after the gauge fixing constraint has been imposed. Here we will not try to address the conceptual question of whether the correct way to deal with the Gauss constraint in quantum-reduced loop gravity is simply to ignore it, or whether it should be taken into account in some non-trivial way. Instead, we will take this question as a purely technical one, and propose two possible ways in which the Gauss constraint could be incorporated into the framework considered so far in this article.

Following the extended master constraint proposal of Thiemann [20], we can include the Gauss constraint⁷ as a part of the master constraint by considering the extended master constraint

$$M_{\text{ext}} = M + M_{\text{Gauss}}, \quad (5.1)$$

where M is the master constraint for diagonal gauge from section 3.2, and the master Gauss constraint is defined as

$$M_{\text{Gauss}} = \int_{\Sigma} d^3x \frac{G_i G_i}{\sqrt{q}} \quad (5.2)$$

with

$$G_i = \mathcal{D}_a E_i^a = \partial_a E_i^a + \epsilon_{ij}^k A_a^j E_k^a. \quad (5.3)$$

In order to promote the classical functional (5.2) into an operator, we use a construction similar (but not identical) to the one introduced in [38] to quantize covariant derivatives of the densitized triad in terms of finite differences of parallel transported flux operators on a cubical graph.

Consider, as in section 3.2, a partition of the spatial manifold into cubical cells adapted to a given cubical graph. For any node v of the graph, let $S_{\pm}^a(v)$ ($a = x, y, z$) denote the six faces of the cell containing v , such that the surface $S_{\pm}^a(v)$ lies in the coordinate plane $x^a = \text{const.}$ and the subscript indicates whether the surface is located before or after the node v in the positive direction of the x^a -coordinate axis (as in the notation for the edges in figure 1). Now the covariant derivative $\mathcal{D}_a E_i^a$ at v (no summation over a is understood) can be approximated by the discretized variable

$$\Delta_a E_i^a(v) = \tilde{E}_i(S_+^a(v), v) - \tilde{E}_i(S_-^a(v), v) \quad (5.4)$$

where $\tilde{E}(S, v)$ is the parallel transported flux variable discussed in section 2.2. We assume that the parallel transport from any point x on the surface $S_{\pm}^a(v)$ to the node v is taken first along a straight line from x to the point at which the surface intersects an edge e of the cubical graph, and then from the intersection point to v along e .

We then define

$$\hat{M}_{\text{Gauss}} = \sum_v \widehat{\mathcal{V}_v^{-1}} \sum_{a,i} \left(\widehat{\Delta_a E_i^a(v)} \right)^2 \quad (5.5)$$

as the operator representing the master Gauss constraint (5.2). The action of the operator $\widehat{\Delta_a E_i^a(v)}$ (no sum over a) on the basis states (4.1) can be computed with the help of equation (2.26). The result is

$$\widehat{\Delta_a E_i^a(v)} \left(\prod_{e \in \Gamma} \mathfrak{D}_{\tau_{ej_e} \sigma_{ej_e}}^{(j_e)}(h_e)_{i_e} \right) = \begin{cases} (j_i^+ - j_i^-) \left(\prod_{e \in \Gamma} \mathfrak{D}_{\tau_{ej_e} \sigma_{ej_e}}^{(j_e)}(h_e)_{i_e} \right) & (i = a) \\ \mathcal{O}(\sqrt{j}) & (i \neq a) \end{cases} \quad (5.6)$$

⁷ In principle there is no obstruction against treating the spatial diffeomorphism constraint in the same way. An operator representing the master diffeomorphism constraint on a cubical graph can be constructed in a relatively straightforward manner, but the form of this operator is much more complicated than the master Gauss constraint considered in this section, and finding a space of solutions of the constraint in explicit form is a non-trivial task, which we have so far not accomplished.

where the notation of figure 1 is again used for the spins at the node v . It follows that the constraint equation $\hat{M}_{\text{Gauss}}|\Psi\rangle = 0$ is satisfied exactly by basis states fulfilling the condition $j_i^+ = j_i^-$ at every node. However, since solutions of the extended master constraint operator can again be found only in the approximate sense discussed in section 3.3, there is room for this condition to be relaxed accordingly. Thus, we consider solutions of the extended master constraint to be the basis states (4.1) where the spins are constrained by the conditions

$$|j_i^+ - j_i^-| \ll j_i^\pm \quad (5.7)$$

which state that the difference between two consecutive spins along a given ‘line’ of edges must be small compared to the spins themselves.

Another possible way of addressing gauge invariance would be to apply $SU(2)$ group averaging (see e.g. [42]) in order to obtain a set of gauge invariant states. Since the gauge conditions do not commute with the Gauss constraint, it should be expected that the outcome of the construction will depend on the order of imposing the gauge-fixing constraint and implementing $SU(2)$ gauge invariance. (Note that this question does not arise in the extended master constraint approach discussed above, where the gauge conditions and the Gauss constraint are both combined into a single constraint operator.) We propose that the natural order of performing these operations is suggested by the fact that the master constraint operator defined by equations (3.6) and (3.7) is not gauge invariant, i.e. the gauge invariant Hilbert space is not preserved by the action of this operator. Hence it does not seem clear how this operator could be imposed as a constraint on the gauge invariant Hilbert space in a consistent manner. On the other hand, performing an $SU(2)$ group averaging on the Hilbert space spanned by the states (4.1) is, from the technical standpoint, a well-defined and unambiguous procedure, acting as a projection onto the $SU(2)$ invariant subspace. For this reason we will only consider the order where the gauge-fixing constraint is solved first, and group averaging is then applied to the resulting basis states (4.1).

Let us look at the effect of group averaging on a given node v of a reduced basis state. Here it is convenient to depart temporarily from the convention introduced in section 4.1 concerning the orientation of the edges, and instead use equation (4.4) to arrange for all the edges incident on v to have an outgoing orientation. Then the relevant part of the state (4.1) has the form⁸

$$\prod_{i=x,y,z} \mathfrak{D}^{(j_i^+)}(h_{e_i^+}) |j_i^+, j_i^+\rangle_i \mathfrak{D}^{(j_i^-)}(h_{\tilde{e}_i^-}) |j_i^-, -j_i^-\rangle_i, \quad (5.8)$$

where \tilde{e}_i^- stands for the edge e_i^- with a reversed orientation. Applying the group averaging operation to the state (5.8) now gives

$$\int dg \prod_{i=x,y,z} \mathfrak{D}^{(j_i^+)}(h_{e_i^+} g) |j_i^+, j_i^+\rangle_i \mathfrak{D}^{(j_i^-)}(h_{\tilde{e}_i^-} g) |j_i^-, -j_i^-\rangle_i \quad (5.9)$$

where the integral is taken over the $SU(2)$ group manifold. Due to the invariance of the Haar measure, the state (5.9) is manifestly invariant under $SU(2)$ gauge transformations acting at the node v .

⁸ The expression (5.8) is obtained by writing $\mathfrak{D}_{\tau_j \sigma_j}^{(j)}(h_e) = {}_i \langle j, \tau_j | \mathfrak{D}^{(j)}(h_e) | j, \sigma_j \rangle_i$ and omitting the bra vector ${}_i \langle j, \tau_j |$, which is associated not with the node v but with the target node of the edge e . For concreteness, we also assume that all the sign factors in the state (4.1) are positive at v (before adjusting the orientations): $\sigma_{e_i^+}(v) = \tau_{\tilde{e}_i^-}(v) = +1$.

It is helpful at this point to recall the definition of the coherent states of angular momentum [43, 44]. The coherent state $|j\vec{n}\rangle$ is constructed from the state of highest weight $|jj\rangle$ by applying a $SU(2)$ rotation $g(\vec{n})$ which rotates the z -axis into the direction of the unit vector $\vec{n} \in \mathbb{R}^3$: $|j\vec{n}\rangle = D^{(j)}(g(\vec{n}))|jj\rangle$. A closely related object is the Livine–Speziale coherent intertwiner, which was introduced in [45] as the $SU(2)$ -invariant projection of the tensor product state $|j_1\vec{n}_1\rangle \cdots |j_N\vec{n}_N\rangle$:

$$|j_1 \cdots j_N; \vec{n}_1 \cdots \vec{n}_N\rangle = \int dg D^{(j_1)}(g) |j_1\vec{n}_1\rangle \cdots D^{(j_N)}(g) |j_N\vec{n}_N\rangle. \quad (5.10)$$

Referring now back to equation (5.9), note that the states $|j, \pm j\rangle_i$, as defined in equation (3.15), are simply the coherent states $|j\vec{n}\rangle$ corresponding to the choice $\vec{n} = \pm \hat{e}_i$ (with $\hat{e}_x, \hat{e}_y, \hat{e}_z$ being the unit vectors along the background coordinate directions). Hence we see that the group integration in equation (5.9) combines with the representation matrices and the state vectors, i.e. $D^{(j_i^\pm)}(g)|j_i^\pm, \pm j_i^\pm\rangle_i$, to produce precisely the coherent intertwiner (5.10). The conclusion emerging from the calculation is therefore as follows: The group averaging of the reduced basis state (4.1) results in a gauge invariant spin network state of the general form (2.9), where each node carries a six-valent Livine–Speziale intertwiner, and the unit vectors $\vec{n}_1, \dots, \vec{n}_6$ characterizing the intertwiner are the six face normal vectors of a cube.

Here a crucial question, to which we do not currently have a conclusive answer, has to do with the property summarized in equation (4.5), whereby the reduced Hilbert space is preserved by the action of the operator $\hat{\mathcal{O}}$ at leading order in j . It is *a priori* not clear whether this property continues to be valid after the group averaging operation has been applied. Assuming that $\hat{\mathcal{O}}$ is a gauge invariant operator, and letting $\hat{\mathbb{P}}_G$ denote the group averaging operator (equivalently, the projection operator onto the gauge invariant Hilbert space), we obtain from equation (4.5)

$$\hat{\mathcal{O}}\hat{\mathbb{P}}_G|\Psi_0\rangle = f(j)\hat{\mathbb{P}}_G|\Psi'_0\rangle + g(j)\hat{\mathbb{P}}_G|\Phi_\perp\rangle. \quad (5.11)$$

However, the state vectors appearing in this equation are no longer normalized (as a projection operator, $\hat{\mathbb{P}}_G$ does not preserve the norm of the state on which it acts, unless this state is already gauge invariant) and a more careful analysis would be needed in order to establish whether the second term on the right-hand side of equation (5.11) is still much smaller than the first term.

5.2. A generalized basis of reduced spin network states

The space spanned by the states (4.1) is clearly not the most general set of solutions of the gauge-fixing master constraint introduced in section 3.2, although it is a natural set of solutions of the constraint operator, and up to details related to the sign factors σ_e, τ_e , it coincides with the reduced Hilbert space considered in earlier literature of quantum-reduced loop gravity. However, the following considerations suggest that this space may not be general enough with a view towards the possible physical applications of the quantum-reduced model. In the classical theory, the gauge conditions $E_i^a = 0$ for $i \neq a$ generally do not restrict any components of the conjugate variable A_a^i to be vanishing. There exist possible classical configurations in which the densitized triad is diagonal but all components of the Ashtekar connection are non-zero. On the other hand, equation (4.10) shows that in the action of the reduced holonomy operator $\widehat{D_{mn}^{(s)}(h_e)^R}$ on the basis (4.1), only the diagonal ($m = n$) and anti-diagonal ($m = -n$) components of the operator give a non-vanishing result. This does not seem to be consistent with the classical expectation that all components of the connection A_a^i could generally be

non-vanishing, since for an arbitrary value of the connection A_a^i , all matrix elements of the classical holonomy $D_{mn}^{(s)}(h_e[A])$ will in general be non-zero.

One could ask whether this problem might have to do with the restriction to cubical graphs, whose edges are aligned in the background coordinate directions. Considering that one can find many examples of states in the Hilbert space of full LQG which are defined on cubical graphs but on which all components of the holonomy operator act non-trivially, it seems unlikely that the vanishing action of the off-diagonal components on the reduced Hilbert space could be explained entirely by the choice of graph. Nevertheless, it is plausible that this issue could be addressed by looking for solutions of the gauge-fixing master constraint on other types of graphs, on which the action of all components of the holonomy would possibly be non-vanishing. This idea will not be pursued further in the present article, but we leave it as a potential topic for future work. (To the author's best knowledge, there have not been any attempts so far in the literature to extend the framework of quantum-reduced loop gravity to non-cubical graphs.)

Instead, we will propose an extension of the standard reduced Hilbert space described in section 4.1, in which the system of cubical graphs is not entirely discarded, but is merely modified in a way which allows for all components of the reduced holonomy operator to have a non-trivial action on the extended Hilbert space. In searching for such an extension, we would like to leave the structure present at the nodes of the standard basis states (4.1) unchanged, since this structure essentially follows from the considerations of section 3.3, and hence it seems difficult to envision any substantial modification of this structure while ensuring that the resulting states continue to be solutions of the gauge-fixing master constraint. This leaves the possibility of introducing some additional structure at the interior of the edges of the graphs on which the standard basis states are defined.

To this end, consider the basis state (4.1) and focus on the reduced holonomy corresponding to a given edge e . Divide the edge into two segments e_1, e_2 (so that $e = e_2 \circ e_1$) and replace the reduced holonomy $\mathfrak{D}_{\tau_{ej_e} \sigma_{ej_e}}^{(j_e)}(h_e)_i$ with a product of two holonomies associated with the two segments as follows:

$$\mathfrak{D}_{\tau_{ej_e} \sigma_{ej_e}}^{(j_e)}(h_e)_i \rightarrow \mathfrak{D}_{\tau_{ej_e} m_e}^{(j_e)}(h_{e_2})_i \mathfrak{D}_{m'_e \sigma_{ej'_e}}^{(j'_e)}(h_{e_1})_i. \quad (5.12)$$

Here m_e and m'_e are new quantum numbers characterizing the generalized state, and the spins j_e and j'_e may generally be different from each other. From the perspective of standard LQG, it may seem somewhat unnatural to assume that the reduced holonomies associated to the two segments can carry different spin quantum numbers. Nevertheless, here it is in fact necessary to allow for this possibility. If the spins on the two segments were required to match, then the Hilbert space spanned by the resulting states would not be closed under the action of the reduced holonomy operator (see equation (5.16) below).

Applying the above procedure to every edge e in the basis states (4.1), we arrive at the generalized basis states

$$\prod_{e \in \Gamma} \mathfrak{D}_{\tau_{ej_e} m_e}^{(j_e)}(h_{e_2})_i \mathfrak{D}_{m'_e \sigma_{ej'_e}}^{(j'_e)}(h_{e_1})_i. \quad (5.13)$$

In these states, the magnetic quantum numbers associated with every proper six-valent node of the cubical graph Γ are τ_{ej_e} and $\sigma_{ej'_e}$, while the quantum numbers m_e and m'_e are located at a 'degenerate' two-valent node formed by the edges e_1 and e_2 (see figure 2 for a pictorial illustration). When the master constraint operator defined by equations (3.6) and (3.7) is applied on the states (5.13), the operator acts non-trivially only on the six-valent nodes; its action on

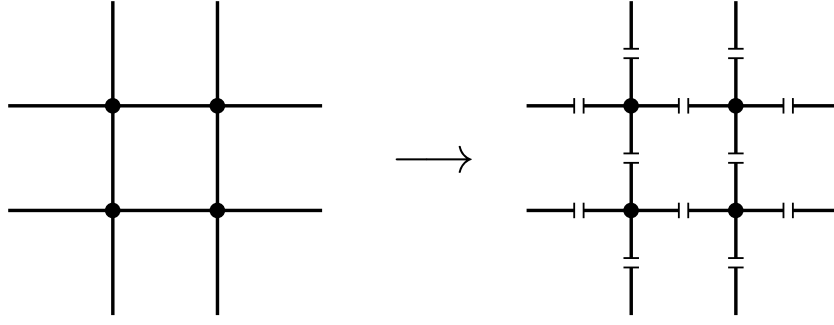


Figure 2. Construction of the generalized basis states. (Third dimension suppressed for clarity.) Each edge of the cubical graph is split into two segments, and the reduced holonomy is replaced with a pair of holonomies, which may carry arbitrary magnetic quantum numbers at the end which is not attached to a proper, non-degenerate node of the graph. Thus, a solid circle in the drawing represents a node where the values of all magnetic indices are maximal or minimal with respect to the appropriate coordinate direction, while a truncated line represents an arbitrary value of the magnetic index.

the two-valent nodes vanishes due to the presence of the inverse volume operator $\widehat{\mathcal{V}_v^{-1}}$. Since the structure at the six-valent nodes of the generalized basis states (5.13) has not been altered relative to the standard basis states (4.1), the generalized states therefore continue to be valid solutions of the gauge-fixing master constraint, as long as the conditions

$$j_e \gg 1 \quad (5.14)$$

$$j'_e \gg 1 \quad (5.15)$$

are fulfilled for each edge e , and the constraint (4.3) on the sign factors is satisfied at every node.

The action of the holonomy operator on the generalized states (5.13) can be found by writing the holonomy along the edge e as $h_e = h_{e_2} h_{e_1}$, and then applying equation (2.13) separately to each of the two segments. Using also equations (4.13) and (4.14), one finds (we display the result only for the case $\sigma_e = \tau_e = +1$; the other cases are not different in any essential way)

$$\begin{aligned} & \widehat{D_{mn}^{(s)}}(h_e)_i \mathfrak{D}_{jm}^{(j)}(h_{e_2})_i \mathfrak{D}_{m'j'}^{(j')}(h_{e_1})_i \\ &= \sum_{\mu} C_{m \mu m+\mu}^{(j \ s \ j+n)} C_{m' \mu m'+\mu}^{(j' \ s \ j'+n')} \mathfrak{D}_{j+n \ m+\mu}^{(j+n)}(h_{e_2})_i \mathfrak{D}_{m'+\mu \ j'+n'}^{(j'+n')}(h_{e_1})_i + \mathcal{O}(j^{-1/2}). \end{aligned} \quad (5.16)$$

The term of leading order in j on the right-hand side of equation (5.16) is a linear combination of states of the form

$$\mathfrak{D}_{jm}^{(j)}(h_{e_2})_i \mathfrak{D}_{m'j'}^{(j')}(h_{e_1})_i \quad (5.17)$$

so the space of such states is preserved by the action of the reduced holonomy operator obtained from equation (5.16). Moreover, for any values of m and n , the reduced holonomy operator $\widehat{D_{mn}^{(s)}}(h_e)_i^R$ will generally have a non-vanishing action on the basis states (5.13).

In the terminology of cylindrical functions, the construction proposed in this section can be seen as a particular kind of refinement of the graph on which the standard basis states (4.1)

are defined. The standard basis states are cylindrical functions on a regular cubical graph containing only six-valent nodes. In contrast, the generalized basis states of equation (5.13) are cylindrical functions on a finer graph, which is obtained by taking a regular cubical graph and introducing a two-valent node at an interior point of every edge. Note that the standard basis states can also be seen as cylindrical functions on this finer graph. (This is achieved by assigning a Kronecker delta at each two-valent node as an intertwiner which contracts the magnetic indices of the holonomies at the node and thus joins the two segments into one whole edge.) Hence the original Hilbert space spanned by the states (4.1) is a proper subspace of the generalized Hilbert space spanned by the states (5.13), so in this sense the generalized Hilbert space is indeed a genuine extension of the standard reduced Hilbert space.

6. Conclusions

In this article we constructed and analyzed a constraint operator representing a set of gauge conditions which fix the densitized triad to be diagonal on the kinematical Hilbert space of LQG. A gauge fixing to a diagonal triad at the level of the quantum theory is the basic premise of quantum-reduced loop gravity, a physically motivated model of LQG proposed by Alesci and Cianfrani in [1, 2]. Hence the work presented in this article can be seen as re-examining the foundations of quantum-reduced loop gravity from an alternative and arguably more systematic standpoint.

In order to construct a constraint operator encoding the desired gauge conditions, we turned to the master constraint method introduced to LQG by Thiemann [20, 21] as a technique for the quantization of the Hamiltonian constraint. The gauge conditions aggregated in the resulting master constraint operator do not commute among themselves as quantum operators, despite being compatible with each other in the classical theory. To search for solutions of a constraint of this kind, we followed the approach originally taken by Alesci and Cianfrani. Instead of requiring the constraint equation to be satisfied exactly, one looks for solutions of the constraint among kinematical states characterized by large spin quantum numbers, with the constraint equation being interpreted as a condition to be satisfied approximately in the regime of large spins (the equality becoming exact if one formally takes the limit $j \rightarrow \infty$). By deriving a set of solutions of the constraint operator (understood in the above generalized sense) on a cubical graph, we confirmed that the Hilbert space of quantum-reduced loop gravity is recovered in the form considered in earlier literature, up to minor technical details regarding the signs of the magnetic quantum numbers.

Having established a space of solutions of the gauge fixing master constraint, we proceeded to examine the action of the basic operators of LQG on this space. We summarized the central results of the earlier article [16], where the relation between the operators of quantum-reduced loop gravity and those of full LQG was analyzed, and extended these results to the more general set of states considered in the present article. These calculations show that the Hilbert space of quantum-reduced loop gravity, when viewed as a subspace of the kinematical Hilbert space of full LQG, is approximately preserved by the action of many fundamental operators of LQG. More precisely, the action of a LQG operator on a state in the reduced Hilbert space typically produces a state which lies predominantly in the reduced Hilbert space, in the sense that the component of this state orthogonal to the reduced Hilbert space is much smaller than the component which belongs to the reduced Hilbert space. Operators for quantum-reduced loop gravity can therefore be derived from the corresponding operators of full LQG by dropping the small terms which do not belong to the reduced Hilbert space. From the perspective of the quantum-reduced model, the reduced operator obtained in this way is a well-defined

operator on the Hilbert space of the model, whereas from the point of view of the full theory the action of the reduced operator can be seen as a good approximation of the action of the full operator on the reduced Hilbert space.

Expressing the gauge conditions for a diagonal triad in terms of a precisely formulated constraint operator provides an opportune starting point for considering various possible generalizations of the construction presented in this article. We showed that the master constraint for diagonal gauge can be straightforwardly extended to include the Gauss constraint, making it possible to implement a notion of $SU(2)$ gauge invariance on the reduced Hilbert space. Another way to address the question of gauge invariance would be to directly apply a $SU(2)$ group averaging map on the reduced basis states. However, for the time being it is not known whether the group averaging operation preserves the decomposition of the action of operators on the reduced Hilbert space into a large term belonging to the reduced Hilbert space and a small term lying outside of this space.

We further proposed a basis of generalized reduced states, making use of the fact that the master constraint operator (in both its non-extended and extended forms) acts only on the nodes of a given graph, so there is room to modify the definition of the standard reduced basis states away from the nodes. The need to look for a generalization of the standard reduced Hilbert space was motivated by the observation that only the diagonal and ‘anti-diagonal’ components of the reduced holonomy operator have a non-vanishing action on the standard reduced states. This feature appears to be in conflict with the classical expectation that fixing the off-diagonal components of the densitized triad to zero should not necessarily constrain any components of the Ashtekar connection to vanish. For this reason, it is unclear whether the standard formulation of the reduced Hilbert space can correctly capture all the degrees of freedom contained in the Ashtekar connection in physical situations where off-diagonal components of the connection are relevant. On the space spanned by the generalized reduced states, all components of the reduced holonomy operator have a generically non-vanishing action.

Developing a clearer understanding of the physical interpretation of the generalized reduced states and investigating their possible relation with other ideas discussed in the literature of LQG and related fields will make an interesting topic for future work. The splitting of a whole edge of a reduced state into two segments carrying arbitrary magnetic quantum numbers at their ‘free’ ends seems to bear at least a superficial resemblance to the discussion of the link entanglement of spin network states in [46]. Moreover, if the notion of splitting an edge in the construction of the generalized reduced states is taken in a very literal sense, and is interpreted to mean that the connectivity information encoded in the graph of the reduced state is also discarded, one would end up with a collection of individual, disconnected six-valent nodes with open-ended edges. This picture is very similar to the so-called condensate states studied in the group field theory approach to quantum gravity (see e.g. [47, 48]).

Data availability statement

No new data were created or analysed in this study.

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