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$\wp$-adic quantum mechanics, the Dirac equation, and the violation of Einstein causality

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Abstract

This article studies the breaking of the Lorentz symmetry at the Planck length in quantum mechanics. We use three-dimensional $\wp$-adic vectors as position variables, while the time remains a real number. In this setting, the Planck length is $1/\wp$, where $\wp$ is a prime number, and the Lorentz symmetry is naturally broken. In the framework of the Dirac-von Neumann formalism for quantum mechanics, we introduce a new $\wp$-adic Dirac equation that predicts the existence of particles and antiparticles and charge conjugation like the standard one. The discreteness of the $\wp$-adic space imposes substantial restrictions on the solutions of the new equation. This equation admits localized solutions, which is impossible in the standard case. We show that an isolated quantum system whose evolution is controlled by the $\wp$-adic Dirac equation does not satisfy the Einstein causality, which means that the speed of light is not the upper limit for the speed at which conventional matter or energy can travel through space. The new $\wp$-adic Dirac equation is not intended to replace the standard one; it should be understood as a new version (or a limit) of the classical equation at the Planck length scale.

Keywords: p-adic numbers, quantum mechanics, Dirac equation, breaking of the Lorentz symmetry, Einstein causality

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1. Introduction

This article revolves around the breaking of the Lorentz symmetry, models of quantum mechanics (QM) at the Planck scale, and the Volovich conjecture on the $\wp$-adic nature of space-time at the Planck scale. The Lorentz symmetry is one of the most essential symmetries of the quantum field theory. While the validity of this symmetry continues to be verified with a high degree of precision [1], in the last thirty-five years, the experimental and theoretical studies of the Lorentz breaking symmetry have been an area of intense research, see, e.g. the reviews [2, 3] and the references therein. The quantum-gravity perspective is considered in [4–8]. When both QM and general relativity are considered, there should be a Planck scale limitation to the localization of a space-time point. This naturally leads one to consider discretized space-times.

In the Dirac-von Neumann formulation of QM, the states of a quantum system are vectors of an abstract complex Hilbert space $\mathcal{H}$, and the observables correspond to linear self-adjoint operators in $\mathcal{H}$, [9–12]. A particular choice of space $\mathcal{H}$ goes beyond the mathematical formulation and belongs to the domain of physical practice and intuition, [12, chapter 1, section 1.1]. Let $\Psi_0 \in \mathcal{H}$ be the state at time $t = 0$ of a specific quantum system. Then, at any time t, the system is represented by the vector $\Psi(t) = \boldsymbol{e}^{-\mathrm{i}t\boldsymbol{H}}\Psi_0$, $t \geqslant 0$, where $\boldsymbol{H}$ is the observable associated with the total energy of the system. It is crucial to mention that this description of the evolution of quantum states requires real-time ($t \in \mathbb{R}$). Nowadays, we do not have a convenient notion of unitary semigroup with $\wp$-adic time.

By $\wp$-adic QM, where $\wp$ is a fixed prime number, we mean QM with $\mathcal{H} = L^2(\mathbb{Q}_\wp^3)$, where $\mathbb{Q}_\wp$ is the field of $\wp$-adic numbers. Since the time and the position are not interchangeable ($t \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{Q}_\wp^3$), such theory is not Lorentz-invariant. The geometry of $\mathbb{Q}_\wp^3$ radically differs from that of $\mathbb{R}^3$. The $\wp$-adic space $\mathbb{Q}_\wp^3$ is a completely disconnected topological space, while $\mathbb{R}^3$ is an arcwise connected topological space. Intuitively, $\mathbb{Q}_\wp^3$ has discrete geometry, while $\mathbb{R}^3$ has a continuous one; see [13] for a further discussion. In addition, there is no algebraic and topological isomorphism between $\mathbb{Q}_\wp^m$ and $\mathbb{R}^m$, because as topological fields $\mathbb{Q}_\wp$ is not isomorphic to $\mathbb{R}$, [14, chapter I, sections 3, 4]. We propose the $\wp$-adic QM as a model of QM at the Planck length.

The $\wp$-adic QM is a model of the standard QM in the space $\mathbb{Q}_\wp^3$. This space is invariant under the transformations of the form $\boldsymbol{x} \to \boldsymbol{a} + \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{A} \in GL_3(\mathbb{Q}_\wp)$. This group is the substitute of the Poincaré group in the $\wp$-adic framework. In particular, this group contains scale transformations of the type $\boldsymbol{x} \to \boldsymbol{a} + \wp^L\boldsymbol{x}$, where $\boldsymbol{a} \in \mathbb{Q}_\wp^3$, and $L \in \mathbb{Z}$. Given two different points $\boldsymbol{x},\boldsymbol{y} \in \mathbb{Q}_\wp^3$, after a suitable scale transformation, one may assume that $\boldsymbol{x} = \boldsymbol{a} + \wp\boldsymbol{b} \neq \boldsymbol{0}$, with $\boldsymbol{a},\boldsymbol{b} \in \{0,1,\ldots,\wp-1\}^3$, $\boldsymbol{y} = \boldsymbol{0}$. Then, the distance between these points is $\left\|\boldsymbol{a} + \wp^L\boldsymbol{b}\right\|_\wp$, and this quantity can take two values, 1 or $\wp^{-1}$, which means that the Planck length is exactly $\wp^{-1}$, see also [13, section 5]. It is relevant to reemphasize that the existence of a Planck length is invalid if we replace $\mathbb{Q}_\wp^3$ with $\mathbb{R}^3$, and also that the $\wp$-adic QM is not a replacement of the standard QM, but a model of the standard one at the Planck length.

In the last forty years $\wp$-adic QM has been studied intensively; see, e.g. [15–42], among many available references. There are several different types of $\wp$-adic QM. In particular, the one in which the time is $\wp$-adic, e.g. [18], radically differs from the one considered here. If the time is $\wp$-adic, we cannot use the classical theory of semigroups; then, a new theory for the computation of quantum expectation values is required. Using real-time allows us to do calculations using the standard axioms of QM, with the same physical interpretations but in a discrete space.

In the 1930 s Bronstein showed that the general relativity and QM imply that the uncertainty Δx of any length measurement satisfies

$$\Delta x \geqslant L_{\text{Planck}}, \tag{1.1}$$

where L_{Planck} is the Planck length, [43]. This inequality establishes an absolute limitation on length measurements, so the Planck length is the smallest possible distance that can, in principle, be measured. Below the Planck scale there are no intervals just points. The choice of $\mathbb{R}$ as a model the unidimensional space is not compatible with inequality (1.1) because, $\mathbb{R}$ contains intervals with arbitrarily small length. On the other hand, there are no intervals in $\mathbb{Q}_{\wp}$, i.e. the non-trivial connected subsets are points. So $\mathbb{Q}_{\wp}$ is the prototype of a discrete space with a very rich mathematical structure. This idea is a reformulation of the Volovich conjecture on the $\wp$-adic nature of the space at the Planck scale, [44]. It is relevant to mention that another interpretation of Bronstein's inequality drives to quantum gravity, [45].

Since 1925, all the parents of the QM have been aware of the need to abandon the classical notion of continuous space-time in studying phenomena at the quantum scale. The reader may consult [46] for an in-depth historical review. For instance, according to [46], Einstein and Born believed that the traditional concept of space-time of macroscopic physics cannot simply be transferred to quantum physics. But all the efforts (especially from Born and Jordan) to construct physical theories with discrete space-times failed.

The theoretical study of quantum models admitting Lorentz symmetry breaking is relevant, [2, 3]. Here, we review some ideas presented in [8] directly related to this discussion. Based on the inequality $E^{-1}\Delta E \gtrsim 1$, where ΔE is the uncertainty in energy measurement, where $\hbar = c = 1$ so the Planck length is $\sqrt{G}$, in [8] is argued that the uncertainty in the energy of a particle is more significant than its rest mass and this makes the concept of particle unclear. The Planck length imposes a resolution limit for relativistic QM: localizing a particle with better accuracy than its Compton wavelength is impossible. These features are reflected in the difficulty of defining a position operator (in the sense of Newton and Wigner). This operator, whose eigenvalues give the positions of a certain particle, has the following property. If a particle is localized in a certain region at a certain time, then at any arbitrarily close instant of time, there is a non-zero probability of finding it anywhere; therefore, the particle would travel faster than light. Finally, the author argues, without giving additional details, that this paradox disappears when, instead of dealing with a single particle, one allows the possibility of particle creation and annihilation.

The simplest quantum model of the phenomena just described is the $\wp$-adic Dirac equation. This article introduces a such equation which shares many properties with the standard one. In particular, the new equation also predicts the existence of pairs of particles and antiparticles and a charge conjugation symmetry. The $\wp$-adic Dirac spinors depend on the standard Pauli-Dirac matrices. The new equation is a version of the standard Dirac Equation at the Planck scale, where the breaking of the Lorentz symmetry naturally occurs. In this framework, the Einstein causality is not valid.

The derivation of the $\wp$-adic Dirac equation is based on the fact that the plane wave solutions of the standard Dirac equation have natural analogs when one considers the position and momenta as elements of a metric, locally compact topological group; the construction of these analogs does not require Lorentz invariance, just a version of the relativistic energy formula, with $c = 1$. This last normalization, or a similar one, is essential because, in the new theory, the speed of light is not the upper bound for the speed at which conventional matter or energy can travel through space. We warn the reader that our relativistic energy formula with $c = 1$ is not a substitute for the classical one. It provides the proper geometric restriction for the existence of $\wp$-adic plane waves. We do not discuss the Planck units in the space-time $\mathbb{R}\times\mathbb{Q}_{\wp}^{3}$; instead, we use 'natural units' to simplify the discussion of our model.

Our $\wp$-adic Dirac equation has the form

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(t,\boldsymbol{x})=(\boldsymbol{\alpha}\cdot\nabla_{\wp}+\beta m)\Psi(t,\boldsymbol{x}),t\in\mathbb{R},\boldsymbol{x}\in\mathbb{Q}_{\wp}^{3},\tag{1.2}$$

where $\boldsymbol{\alpha}$, $\cdot$, m has the standard meaning,

$$\Psi^{T}(t,\boldsymbol{x})=\begin{bmatrix}\Psi_{1}(t,\boldsymbol{x}) & \Psi_{2}(t,\boldsymbol{x}) & \Psi_{3}(t,\boldsymbol{x}) & \Psi_{4}(t,\boldsymbol{x})\end{bmatrix}\in\mathbb{C}^{4},$$

$\nabla_{\wp}^{T}=\begin{bmatrix}\boldsymbol{D}_{x_1} & \boldsymbol{D}_{x_2} & \boldsymbol{D}_{x_3}\end{bmatrix}$, where $\boldsymbol{D}_{x_k}$ denotes the Taibleson–Vladimirov operator. We start with an ansatz that describes the $\wp$-adic counterparts of the classical plane waves in terms of the classical Pauli–Dirac matrices and a version of the relativistic energy formula, with $c=1$. Using these particular solutions, we derive a new $\wp$-adic Dirac equation. It turns out that the $\wp$-adic Dirac equation shares many properties with the usual one. Indeed, we use several results from [11, chapter 1]; our notation follows closely the one used in this reference.

The geometry of space $\mathbb{Q}_{\wp}^{3}$ imposes substantial restrictions on the solutions of (1.2). The $\wp$-adic Dirac equation admits space-localized planes waves $\Psi_{\boldsymbol{rnj}}(t,\boldsymbol{x})$ for any time $t\geqslant 0$, which is, supp $\Psi_{\boldsymbol{rnj}}(t,\cdot)$ is contained in a compact subset of $\mathbb{Q}_{\wp}^{3}$; see theorem 6.1. This phenomenon does not occur in the standard case; see, e.g. [11, section 1.8, corollary 1.7]. On the other hand, we compute the transition probability from a localized state at time $t=0$ to another localized state at $t>0$, assuming that the space supports of the states are arbitrarily far away. It turns out that this transition probability is greater than zero for any time $t\in(0,\epsilon)$, for arbitrarily small ϵ; see theorem 9.1. Since this probability is nonzero for some arbitrarily small t, the system has a nonzero probability of getting between the mentioned localized states arbitrarily shortly, thereby propagating with superluminal speed.

The concept of discrete space-time is not uniform in the literature, and thus, many different QM over discrete spaces occur; see, e.g. [47, 48]. For instance, in [48, section 3], the author considers discrete the space-time of type $\mathbb{Z}^{m}\subset\mathbb{R}^{m}$, i.e. the space-time is a lattice of the standard Euclidean space. This approach is not convenient here because the discreteness of $\mathbb{Z}^{m}$ is a relative property in $\mathbb{R}^{m}$; this choice does not change the Poincaré group of $\mathbb{R}^{m}$, and the discrete space is not invariant under all these symmetries. Finally, the discussion about the Planck length requires a mathematical framework where the notion of length measurements using rational numbers may be formulated. For all these reasons, the approach considered in [47, 48] is not useful here.

The article is organized as follows. Section 2 gives a quick review of the results of $\wp$-the analysis required here. In section 3, we derive the $\wp$-adic Dirac equation. The section 4 is dedicated to studying the spectrum of the $\wp$-adic Dirac operator. The charge conjugation is studied in section 5, while section 6 is dedicated to showing the existence of localized particles and antiparticles. The semigroup attached to the free Dirac operator is studied in section 7. The position and momenta operators are considered in sections 8 and 9 is dedicated to the violation of Einstein causality. In the last section, we give some final comments and conclusions.

2. Basic facts on $\wp$-adic analysis

In this section we fix the notation and collect some basic results on $\wp$-adic analysis that we will use through the article. For a detailed exposition on $\wp$-adic analysis the reader may consult [27, 49–51].

2.1. The field of $\wp$-adic numbers

Along this article $\wp$ denotes a prime number. The field of $\wp-$adic numbers $\mathbb{Q}_\wp$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $\wp-$adic norm $|\cdot|_\wp$, which is defined as

$$|x|_\wp = \begin{cases} 0 & \text{if } x = 0 \\ \wp^{-\gamma} & \text{if } x = \wp^\gamma \dfrac{a}{b}, \end{cases}$$

where a and b are integers coprime with $\wp$. The integer $\gamma = \mathrm{ord}_\wp(x) := \mathrm{ord}(x)$, with $\mathrm{ord}(0) := +\infty$, is called the $\wp-$*adic order of* x. We extend the $\wp-$adic norm to $\mathbb{Q}_\wp^N$ by taking

$$||\boldsymbol{x}||_\wp := \max_{1\leqslant i\leqslant N} |x_i|_\wp, \qquad \text{for } \boldsymbol{x} = (x_1,\ldots,x_N) \in \mathbb{Q}_\wp^N.$$

By defining $\mathrm{ord}(\boldsymbol{x}) = \min_{1\leqslant i\leqslant N}\{\mathrm{ord}(x_i)\}$, we have $||\boldsymbol{x}||_\wp = \wp^{-\mathrm{ord}(\boldsymbol{x})}$. The metric space $\left(\mathbb{Q}_\wp^N, ||\cdot||_\wp\right)$ is a complete ultrametric space. As a topological space $\mathbb{Q}_\wp$ is homeomorphic to a Cantor-like subset of the real line; see, e.g. [27, 49].

Any $\wp-$adic number $x \neq 0$ has a unique expansion of the form

$$x = \wp^{\mathrm{ord}(x)} \sum_{j=0}^{\infty} x_j \wp^j, \tag{2.1}$$

where $x_j \in \{0,1,\ldots,\wp-1\}$ and $x_0 \neq 0$. In addition, any $\boldsymbol{x} \in \mathbb{Q}_\wp^N \smallsetminus \{0\}$ can be represented uniquely as $\boldsymbol{x} = \wp^{\mathrm{ord}(\boldsymbol{x})}\boldsymbol{v}$, where $\|\boldsymbol{v}\|_\wp = 1$.

2.2. Topology of $\mathbb{Q}_\wp^N$

For $r \in \mathbb{Z}$, denote by $B_r^N(\boldsymbol{a}) = \{\boldsymbol{x} \in \mathbb{Q}_\wp^N; ||\boldsymbol{x} - \boldsymbol{a}||_\wp \leqslant \wp^r\}$ the ball of radius $\wp^r$ with center at $\boldsymbol{a} = (a_1,\ldots,a_N) \in \mathbb{Q}_\wp^N$, and take $B_r^N(\boldsymbol{0}) := B_r^N$. Note that $B_r^N(\boldsymbol{a}) = B_r(a_1) \times \cdots \times B_r(a_N)$, where $B_r(a_i) := \{x \in \mathbb{Q}_\wp; |x_i - a_i|_\wp \leqslant \wp^r\}$ is the one-dimensional ball of radius $\wp^r$ with center at $a_i \in \mathbb{Q}_\wp$. The ball B_0^N equals the product of N copies of $B_0 = \mathbb{Z}_\wp$, the ring of $\wp-$adic integers. A polydisc is a set of the form

$$B_{r_1}(a_1) \times \cdots \times B_{r_N}(a_N).$$

We denote by $S_r^N(\boldsymbol{a}) = \{\boldsymbol{x} \in \mathbb{Q}_\wp^N; ||\boldsymbol{x} - \boldsymbol{a}||_\wp = \wp^r\}$ the sphere of radius $\wp^r$ with center at $\boldsymbol{a} = (a_1,\ldots,a_N) \in \mathbb{Q}_\wp^N$, and take $S_r^N(\boldsymbol{0}) := S_r^N$. We notice that $S_0^1 = \mathbb{Z}_\wp^\times$ (the group of units of $\mathbb{Z}_\wp$), but $(\mathbb{Z}_\wp^\times)^N \subsetneq S_0^N$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_\wp^N$. In addition, two balls in $\mathbb{Q}_\wp^N$ are either disjoint or one is contained in the other.

As a topological space $(\mathbb{Q}_\wp^N, ||\cdot||_\wp)$ is totally disconnected, i.e. the only connected subsets of $\mathbb{Q}_\wp^N$ are the empty set and the points. A subset of $\mathbb{Q}_\wp^N$ is compact if and only if it is closed and bounded in $\mathbb{Q}_\wp^N$; see, e.g. [27, section 1.3], or [49, section 1.8]. The balls and spheres are compact subsets. Thus $(\mathbb{Q}_\wp^N, ||\cdot||_\wp)$ is a locally compact topological space.

Notation 1. We will use $\Omega(\wp^{-r}||\boldsymbol{x} - \boldsymbol{a}||_\wp)$ to denote the characteristic function of the ball $B_r^N(\boldsymbol{a}) = \boldsymbol{a} + \wp^{-r}\mathbb{Z}_\wp^N$, where

$$\mathbb{Z}_\wp^N = \left\{\boldsymbol{x} \in \mathbb{Q}_\wp^N; \|\boldsymbol{x}\|_\wp \leqslant 1\right\}$$

is the N-dimensional unit ball. For more general sets, we will use the notation 1_A for the characteristic function of set A.

2.3. The Haar measure

Since $(\mathbb{Q}_{\wp}^{N},+)$ is a locally compact topological group, there exists a Haar measure $\mathrm{d}^{N}\boldsymbol{x}$, which is invariant under translations, i.e. $\mathrm{d}^{N}(\boldsymbol{x}+\boldsymbol{a})=\mathrm{d}^{N}\boldsymbol{x}$, [52]. If we normalize this measure by the condition $\int_{\mathbb{Z}_{\wp}^{N}}\mathrm{d}^{N}\boldsymbol{x}=1$, then $\mathrm{d}^{N}\boldsymbol{x}$ is unique.

2.4. The Bruhat-Schwartz space

A complex-valued function φ defined on $\mathbb{Q}_{\wp}^{N}$ is called locally constant if for any $\boldsymbol{x}\in\mathbb{Q}_{\wp}^{N}$ there exist an integer $l(\boldsymbol{x})\in\mathbb{Z}$ such that

$$\varphi(\boldsymbol{x}+\boldsymbol{x}')=\varphi(\boldsymbol{x})\text{ for any }\boldsymbol{x}'\in B_{l(\boldsymbol{x})}^{N}. \tag{2.2}$$

A function $\varphi:\mathbb{Q}_{\wp}^{N}\to\mathbb{C}$ is called a Bruhat–Schwartz function (or a test function) if it is locally constant with compact support. Any test function can be represented as a linear combination, with complex coefficients, of characteristic functions of balls. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_{\wp}^{N})$. For $\varphi\in\mathcal{D}(\mathbb{Q}_{\wp}^{N})$, the largest number $l=l(\varphi)$ satisfying (2.2) is called the exponent of local constancy (or the parameter of constancy) of φ.

2.5. L^{ρ} spaces

Given $\rho\in[1,\infty)$, we denote by $L^{\rho}\left(\mathbb{Q}_{\wp}^{N}\right):=L^{\rho}\left(\mathbb{Q}_{\wp}^{N},\mathrm{d}^{N}\boldsymbol{x}\right)$, the $\mathbb{C}-$vector space of all the complex valued functions g satisfying

$$\|g\|_{\rho}=\left(\int_{\mathbb{Q}_{\wp}^{N}}|g(\boldsymbol{x})|^{\rho}\,\mathrm{d}^{N}\boldsymbol{x}\right)^{\frac{1}{\rho}}<\infty,$$

where $\mathrm{d}^{N}\boldsymbol{x}$ is the normalized Haar measure on $\left(\mathbb{Q}_{\wp}^{N},+\right)$.

If U is an open subset of $\mathbb{Q}_{\wp}^{N}$, $\mathcal{D}(U)$ denotes the $\mathbb{C}$-vector space of test functions with supports contained in U, then $\mathcal{D}(U)$ is dense in

$$L^{\rho}(U)=\left\{\varphi:U\to\mathbb{C};\|\varphi\|_{\rho}=\left\{\int_{U}|\varphi(\boldsymbol{x})|^{\rho}\,\mathrm{d}^{N}\boldsymbol{x}\right\}^{\frac{1}{\rho}}<\infty\right\},$$

for $1\leqslant\rho<\infty$; see, e.g. [49, section 4.3].

2.6. The Fourier transform

By using expansion (2.1), we define the fractional part of $x\in\mathbb{Q}_{p}$, denoted $\{x\}_{p}$, as the rational number

$$\{x\}_{\wp}=\begin{cases}0 & \text{if}\quad x=0\text{ or }\mathrm{ord}(x)\geqslant 0\\ \wp^{\mathrm{ord}(x)}\sum_{j=0}^{-\mathrm{ord}(x)-1}x_{j}\wp^{j} & \text{if}\quad \mathrm{ord}(x)<0.\end{cases}$$

We now set $\chi_{\wp}(y):=\exp(2\pi\mathrm{i}\{y\}_{\wp})$ for $y\in\mathbb{Q}_{\wp}$. The map $\chi_{\wp}(\cdot)$ is an additive character on $\mathbb{Q}_{\wp}$, i.e. a continuous map from $(\mathbb{Q}_{\wp},+)$ into S (the unit circle considered as multiplicative group) satisfying $\chi_{\wp}(x_{0}+x_{1})=\chi_{\wp}(x_{0})\chi_{p}(x_{1})$, $x_{0},x_{1}\in\mathbb{Q}_{\wp}$. The additive characters of $\mathbb{Q}_{\wp}$ form an Abelian group which is isomorphic to $(\mathbb{Q}_{\wp},+)$. The isomorphism is given by $\xi\to\chi_{\wp}(\xi x)$; see, e.g. [49, section 2.3].

Set $\boldsymbol{p}\cdot\boldsymbol{x} := \sum_{j=1}^{N} p_j x_j$, for $\boldsymbol{p} = (p_1,\ldots,p_N)$, $\boldsymbol{x} = (x_1,\ldots,x_N) \in \mathbb{Q}_{\wp}^N$. The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_{\wp}^N)$ is defined as

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{Q}_{\wp}^N} \chi_{\wp}(\boldsymbol{p}\cdot\boldsymbol{x})\,\varphi(\boldsymbol{x})\,\mathrm{d}^N\boldsymbol{x} \quad \text{for } \boldsymbol{p} \in \mathbb{Q}_{\wp}^N,$$

where $\mathrm{d}^N\boldsymbol{x}$ is the normalized Haar measure on $\mathbb{Q}_{\wp}^N$. The Fourier transform is a linear isomorphism from $\mathcal{D}(\mathbb{Q}_{\wp}^N)$ onto itself satisfying

$$(\mathcal{F}(\mathcal{F}\varphi))(\boldsymbol{x}) = \varphi(-\boldsymbol{x}); \tag{2.3}$$

see, e.g. [49, section 4.8]. We also use the notation $\mathcal{F}_{\boldsymbol{x}\to\boldsymbol{p}}\varphi$ and $\widehat{\varphi}$ for the Fourier transform of φ.

The Fourier transform extends to L^2. If $f \in L^2(\mathbb{Q}_{\wp}^N)$, its Fourier transform is defined as

$$(\mathcal{F}f)(\boldsymbol{p}) = \lim_{k\to\infty} \int_{||x||_{\wp}\leqslant \wp^k} \chi_{\wp}(\boldsymbol{p}\cdot\boldsymbol{x}) f(x)\,\mathrm{d}^N\boldsymbol{x}, \quad \text{for } \boldsymbol{p} \in \mathbb{Q}_{\wp}^N,$$

where the limit is taken in $L^2(\mathbb{Q}_{\wp}^N)$. We recall that the Fourier transform is unitary on $L^2(\mathbb{Q}_{\wp}^N)$, i.e. $||f||_2 = ||\mathcal{F}f||_2$ for $f \in L^2(\mathbb{Q}_{\wp}^N)$ and that (2.3) is also valid in $L^2(\mathbb{Q}_{\wp}^N)$; see, e.g. [51, chapter III, section 2].

2.7. Distributions

The $\mathbb{C}$-vector space $\mathcal{D}'(\mathbb{Q}_{\wp}^N)$ of all continuous linear functionals on $\mathcal{D}(\mathbb{Q}_{\wp}^N)$ is called the Bruhat–Schwartz space of distributions. Every linear functional on $\mathcal{D}(\mathbb{Q}_{\wp}^N)$ is continuous, i.e. $\mathcal{D}'(\mathbb{Q}_{\wp}^N)$ agrees with the algebraic dual of $\mathcal{D}(\mathbb{Q}_{\wp}^N)$; see, e.g. [27, chapter 1, VI.3, lemma].

We endow $\mathcal{D}'(\mathbb{Q}_{\wp}^N)$ with the weak topology, i.e. a sequence $\{T_j\}_{j\in\mathbb{N}}$ in $\mathcal{D}'(\mathbb{Q}_{\wp}^N)$ converges to T if $\lim_{j\to\infty} T_j(\varphi) = T(\varphi)$ for any $\varphi \in \mathcal{D}(\mathbb{Q}_{\wp}^N)$. The map

$$\begin{array}{lll} \mathcal{D}'\left(\mathbb{Q}_{\wp}^N\right)\times\mathcal{D}\left(\mathbb{Q}_{\wp}^N\right) & \to & \mathbb{C} \\ (T,\varphi) & \to & T(\varphi) \end{array}$$

is a bilinear form which is continuous in T and φ separately. We call this map the pairing between $\mathcal{D}'(\mathbb{Q}_{\wp}^N)$ and $\mathcal{D}(\mathbb{Q}_{\wp}^N)$. From now on we will use (T,φ) instead of $T(\varphi)$.

Every f in L^1_{loc} defines a distribution $f \in \mathcal{D}'(\mathbb{Q}_{\wp}^N)$ by the formula

$$(f,\varphi) = \int_{\mathbb{Q}_{\wp}^N} f(\boldsymbol{x})\,\varphi(\boldsymbol{x})\,\mathrm{d}^N\boldsymbol{x}.$$

2.8. The Fourier transform of a distribution

The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in \mathcal{D}'(\mathbb{Q}_{\wp}^N)$ is defined by

$$(\mathcal{F}[T],\varphi) = (T,\mathcal{F}[\varphi]) \text{ for all } \varphi \in \mathcal{D}\left(\mathbb{Q}_{\wp}^N\right).$$

The Fourier transform $T \to \mathcal{F}[T]$ is a linear and continuous isomorphism from $\mathcal{D}'(\mathbb{Q}_{\wp}^N)$ onto $\mathcal{D}'(\mathbb{Q}_{\wp}^N)$. Furthermore, $T(\varphi) = \mathcal{F}[\mathcal{F}[T](-\varphi)]$.

2.9. The Taibleson-Vladimirov operator

We denote by $\boldsymbol{D}_z$ the Taibleson–Vladimirov derivative, where $z \in \mathbb{Q}_\wp$, which is defined as

$$(\boldsymbol{D}_z\varphi)(z) = \frac{1-\wp}{1-\wp^{-2}} \int_{\mathbb{Q}_\wp} \frac{\varphi(z-y)-\varphi(z)}{|y|_\wp^2}, \text{ for } \varphi \in \mathcal{D}(\mathbb{Q}_\wp).$$

$\boldsymbol{D}_z$ is an unbounded operator with a dense domain in $L^2(\mathbb{Q}_\wp)$.

We denote by $C(\mathbb{Q}_\wp, \mathbb{C})$ the $\mathbb{C}$-vector space of continuous $\mathbb{C}$-valued functions defined on $\mathbb{Q}_\wp$. We use the symbol $\Omega(t)$ to denote the characteristic function of the interval $[0,1]$.

The Taibleson–Vladimirov derivative $\boldsymbol{D}_{x_i}$ is a pseudo-differential operator of the form

$$\begin{array}{cccc} \boldsymbol{D}_{x_i}: & \mathcal{D}(\mathbb{Q}_\wp) & \to & C(\mathbb{Q}_\wp,\mathbb{C}) \cap L^2(\mathbb{Q}_\wp) \\ & \varphi & \to & (\boldsymbol{D}_{x_i}\varphi)(x_i) = \mathcal{F}^{-1}_{p_i \to x_i}\left\{|p_i|_p \mathcal{F}_{x_i \to p_i}\varphi\right\}; \end{array} \tag{2.4}$$

see, e.g. [27, chapter 2, section IX] and [50, section 2.2].

The set of functions $\{\psi_{rnj}\}$ defined as

$$\psi_{rnj}(x_i) = \wp^{\frac{-r}{2}} \chi_\wp\left(\wp^{-1} j (\wp^r x_i - n)\right) \Omega\left(|\wp^r x_i - n|_p\right), \tag{2.5}$$

where $r \in \mathbb{Z}$, $j \in \{1, \cdots, \wp - 1\}$, and n runs through a fixed set of representatives of $\mathbb{Q}_\wp/\mathbb{Z}_\wp$, is an orthonormal basis of $L^2(\mathbb{Q}_\wp)$ consisting of eigenvectors of operator $\boldsymbol{D}_{x_i}$:

$$\boldsymbol{D}_{x_i}\psi_{rnj}(x_i) = \wp^{1-r}\psi_{rnj}(x_i) \text{ for any } r, n, j; \tag{2.6}$$

see, e.g. [53, theorem 3.29] and [49, theorem 9.4.2]. Notice that $\psi_{rnj}(x_i)$ is supported on the ball

$$B_r\left(\wp^{-r}n\right) = \wp^{-r}n + \wp^{-r}\mathbb{Z}_\wp = \left\{z \in \mathbb{Q}_\wp; \left|z - \wp^{-r}n\right| \leqslant \wp^r\right\}. \tag{2.7}$$

3. $\wp$-adic Pseudo-differential equations of Dirac type

3.1. A $\wp$-adic version of the Dirac equation

We denote the Pauli matrices as

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $i = \sqrt{-1} \in \mathbb{C}$, and the 4×4 Dirac matrices

$$\beta = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix}, \ \alpha_k = \begin{bmatrix} \mathbf{0} & \sigma_k \\ \sigma_k & \mathbf{0} \end{bmatrix}, \text{ for } k = 1,2,3,$$

where $\mathbf{1}$ denotes the 2×2 identity matrix, and $\mathbf{0}$ denotes the 2×2 zero matrix.

We set

$$\nabla_\wp := \begin{bmatrix} \boldsymbol{D}_{x_1} \\ \boldsymbol{D}_{x_2} \\ \boldsymbol{D}_{x_3} \end{bmatrix},$$

and

$$\boldsymbol{\alpha} \cdot \nabla_\wp := \sum_{k=1}^{3} \alpha_k \boldsymbol{D}_{x_k}, \text{ and } \boldsymbol{\sigma} \cdot \nabla_\wp := \sum_{k=1}^{3} \sigma_k \boldsymbol{D}_{x_k}.$$

We suppress Einstein's convention because we need just a version of the 'relativistic energy formula.'

We define the free Hamiltonian as the operator

$$\boldsymbol{H}_0 := \boldsymbol{\alpha}\cdot\nabla_{\wp} + \beta m = \begin{bmatrix} m\mathbf{1} & \boldsymbol{\sigma}\cdot\nabla_{\wp} \\ \boldsymbol{\sigma}\cdot\nabla_{\wp} & -m\mathbf{1} \end{bmatrix}. \tag{3.1}$$

We assume that the constant $m \in \mathbb{R}$ (the mass) has a similar meaning as in the standard QM.

The matrix-valued operator $\boldsymbol{H}_0$ acts on functions

$$\phi(\boldsymbol{x}) = \begin{bmatrix} \phi_1(\boldsymbol{x}) \\ \vdots \\ \phi_4(\boldsymbol{x}) \end{bmatrix} \in \mathcal{D}(\mathbb{Q}_{\wp}) \bigoplus \mathcal{D}(\mathbb{Q}_{\wp}) \bigoplus \mathcal{D}(\mathbb{Q}_{\wp}) \bigoplus \mathcal{D}(\mathbb{Q}_{\wp}) := \mathcal{D}(\mathbb{Q}_{\wp}) \bigotimes \mathbb{C}^4.$$

We denote by $\Psi(t,\boldsymbol{x})$ a vector-valued wavefunction, where $t \in \mathbb{R}$, $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{Q}_{\wp}^3$. defined as

$$\Psi(t,\boldsymbol{x}) = \begin{bmatrix} \Psi_1(t,\boldsymbol{x}) \\ \vdots \\ \Psi_4(t,\boldsymbol{x}) \end{bmatrix} \in \mathbb{C}^4.$$

Our $\wp$-adic version of the Dirac equation has the form

$$i\frac{\partial}{\partial t}\Psi(t,\boldsymbol{x}) = \boldsymbol{H}_0 \Psi(t,\boldsymbol{x}). \tag{3.2}$$

This version of the Dirac equation can be derived using the original Dirac's argument. Starting with the relativistic energy formula (with $c = 1$)

$$E^2 = \left(p_x^2 + p_y^2 + p_z^2\right) + m^2 =: p_{\mathbb{R}}^2 + m^2,$$

with $p_x, p_y, p_z \in \mathbb{R}$, and using the following adhoc quantization scheme

$$E \to i\frac{\partial}{\partial t} \quad \text{and} \quad p_{\mathbb{R}} \to \nabla_{\wp}, \tag{3.3}$$

one formally obtains

$$i\frac{\partial}{\partial t}\Psi(t,\boldsymbol{x}) = \sqrt{\boldsymbol{D}_{x_1}^2 + \boldsymbol{D}_{x_2}^2 + \boldsymbol{D}_{x_3}^2 + m^2}\,\Psi(t,\boldsymbol{x}).$$

Now, the square root is computed using Dirac's original reasoning. These calculations are embedded in the demonstration of proposition 3.1.

Remark 1. As discussed in the introduction, the Planck length implies that localizing a particle with better accuracy than its Compton wavelength (λ_{Compton}) is impossible. Then, the notion of particle does not make sense. Instead of this notion, we introduce the idea of localized particle, but for the sake of simplicity, we will use the word particle. The localization property means that the position function is a locally constant with an exponent of local constancy controlled by λ_{Compton}, which implies that the assertion the position of a particle is $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{Q}_{\wp}^3$ means that the particle is in a ball of radius controlled by λ_{Compton}. We will use the expressions localized states or localized waves to mean that they are functions with compact support.

3.2. Plane waves

Given $\boldsymbol{x} = (x_1, x_2, x_3), \boldsymbol{p} = (p_1, p_2, p_3) \in \mathbb{Q}_\wp^3$, we set

$$\left|\underline{\boldsymbol{p}}\right|_\wp := \left(|p_1|_\wp, |p_2|_\wp, |p_3|_\wp\right) \in \mathbb{R}^3, \text{ and } \left|\underline{\boldsymbol{p}}\right|_\wp^2 := |p_1|_\wp^2 + |p_2|_\wp^2 + |p_3|_\wp^2.$$

We recall that $\|\boldsymbol{p}\|_\wp = \max_{1\leqslant i\leqslant 3}\left\{|p_i|_\wp\right\}$, and that $\boldsymbol{p}\cdot\boldsymbol{x} = \sum_{i=1}^3 p_i x_i$.

A $\wp$-adic counterpart of the Dirac equation is worthwhile if it predicts the existence of particles and antiparticles with spin-$\frac{1}{2}$ on a space-time of the form $\mathbb{R}\times\mathbb{Q}_\wp^3$. The following definition describes our anzatz for $\wp$-adic plane waves.

Definition 1. By a plane wave, we mean a function of the form

$$\Psi(t, \boldsymbol{x}) = \mathrm{e}^{-\mathrm{i}Et}\chi_\wp(\boldsymbol{p}\cdot\boldsymbol{x})\, w(\boldsymbol{p}), \tag{3.4}$$

where

$$E^2 = \left(|p_1|_\wp^2 + |p_2|_\wp^2 + |p_3|_\wp^2\right) + m^2 = \left|\underline{\boldsymbol{p}}\right|_\wp^2 + m^2,$$

and

$$w(\boldsymbol{p}) = \begin{bmatrix} w_1(\boldsymbol{p}) \\ \vdots \\ w_4(\boldsymbol{p}) \end{bmatrix} \in \mathbb{C}^4. \tag{3.5}$$

The functions $w(\boldsymbol{p})$ are 'radial,' i.e. $w(\boldsymbol{p}) = w\left(\left|\underline{\boldsymbol{p}}\right|_\wp\right)$, and they have the form

$$w_1(\boldsymbol{p}) = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \frac{\boldsymbol{\sigma}\cdot\left|\underline{\boldsymbol{p}}\right|_\wp}{E+m}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}, w_2(\boldsymbol{p}) = \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \frac{\boldsymbol{\sigma}\cdot\left|\underline{\boldsymbol{p}}\right|_\wp}{E+m}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}, \tag{3.6}$$

$$w_3(\boldsymbol{p}) = \begin{bmatrix} \frac{-\boldsymbol{\sigma}\cdot\left|\underline{\boldsymbol{p}}\right|_\wp}{E+m}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}, w_4(\boldsymbol{p}) = \begin{bmatrix} \frac{-\boldsymbol{\sigma}\cdot\left|\underline{\boldsymbol{p}}\right|_\wp}{E+m}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}. \tag{3.7}$$

The $\wp$-adic plane waves described above are the natural counterparts of the standard ones; see, e.g. [54, 55]. The plane waves for the ordinary Dirac equation have the form

$$\Psi(t, \boldsymbol{x}_\mathbb{R}) = \left(\mathrm{e}^{-\mathrm{i}Et}\prod_{k=1}^3 \mathrm{e}^{\left(\mathrm{i}p_\mathbb{R}^k x_\mathbb{R}^k\right)}\right) w(\boldsymbol{p}_\mathbb{R}), \tag{3.8}$$

where $\boldsymbol{x}_\mathbb{R} = \left(x_\mathbb{R}^1, x_\mathbb{R}^2, x_\mathbb{R}^3,\right), \boldsymbol{p}_\mathbb{R} = \left(p_\mathbb{R}^1, p_\mathbb{R}^2, p_\mathbb{R}^3,\right) \in \mathbb{R}^3$. The term $\exp(-\mathrm{i}Et)$ is not affected by the hypothesis of the discreteness of space; for this reason it appears in (3.4). This choice implies that a version relativistic energy for formula for E should be valid in the $\wp$-adic framework. For the other terms in (3.8), we use the correspondence

$$\exp\left(2\pi - \mathrm{i}p_\mathbb{R}^k x_\mathbb{R}^k\right) \to \exp\left(2\pi - \mathrm{i}\{p_k x_k\}_\wp\right), \tag{3.9}$$

where $p_k, x_k \in \mathbb{Q}_\wp$. Since $p_k x_k$ should be dimensionless quantity, we require a constant $h_\wp = 1$ with dimension ML^2T^{-1}, so that $\frac{p_k x_k}{h_\wp}$ be a $\wp$-adic number. In addition, we need the formula

$$\boldsymbol{D}_{x_i}\chi_\wp(p_i x_i) = \boldsymbol{D}_{x_i}\exp\left(2\pi - \mathrm{i}\{p_i x_i\}_\wp\right) = |p_i|_\wp\,\chi_\wp(p_i x_i), \tag{3.10}$$

see [27, chapter 2, section IX, example 4]. Finally, we need a $\wp$-adic counterpart for the term $w(\boldsymbol{p}_\mathbb{R})$ in (3.8). Assuming that the Dirac bispinors are the correct description of particles/anti-particles with spin-$\frac{1}{2}$, one is naturally driven to use the correspondence

$$\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}_\mathbb{R}}{E+m} \to \frac{\boldsymbol{\sigma}\cdot|\underline{\boldsymbol{p}}|_\wp}{E+m}.$$

Proposition 3.1. *The $\wp$-adic Dirac equation admits plane waves of type (3.4)–(3.7) as solutions.*

Proof. The demonstration is just a variation of the classical calculation showing the existence of plane waves for the Dirac equation. By replacing $\Psi(t,\boldsymbol{x})$, see (3.4), in (3.2), and using $\frac{\partial}{\partial t}\Psi(t,\boldsymbol{x}) = -\mathrm{i}E\Psi(t,\boldsymbol{x})$, and formula (3.10), one obtains that

$$Ew(\boldsymbol{p}) = \left(\boldsymbol{\alpha}\cdot|\underline{\boldsymbol{p}}|_\wp + \beta m\right)w(\boldsymbol{p}). \tag{3.11}$$

Which is a system of linear equations in the variables $w_1(\boldsymbol{p}),\ldots,w_4(\boldsymbol{p})$ with coefficients in $\mathbb{C}\left[|p_1|_\wp, |p_2|_\wp, |p_3|_\wp\right]$, more precisely,

$$\begin{bmatrix} -E+m & 0 & |p_3|_\wp & |p_1|_\wp - \mathrm{i}|p_2|_\wp \\ 0 & -E+m & |p_1|_\wp + \mathrm{i}|p_2|_\wp & -|p_3|_\wp \\ |p_3|_\wp & |p_1|_\wp - \mathrm{i}|p_2|_\wp & -E-m & 0 \\ |p_1|_\wp + \mathrm{i}|p_2|_\wp & -|p_3|_\wp & 0 & -E-m \end{bmatrix}$$
$$\times\begin{bmatrix} w_1(\boldsymbol{p}) \\ w_2(\boldsymbol{p}) \\ w_3(\boldsymbol{p}) \\ w_4(\boldsymbol{p}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The condition for non-trivial solutions for $w(\boldsymbol{p})$ is that the determinant of this system vanishes:

$$\left(m^2 + |\underline{\boldsymbol{p}}|_\wp^2 - E^2\right)^2 = 0.$$

The calculation of the determinant is the same as in the classical case. Then, necessarily

$$E = \pm\sqrt{m^2 + |\underline{\boldsymbol{p}}|_\wp^2}.$$

We now consider (3.11) as an eigenvalue/eigenvector problem in the ring $\mathbb{C}\left[|p_1|_\wp, |p_2|_\wp, |p_3|_\wp\right]$. The solution of this problem follows the classical reasoning and drives to the announced solutions; see, e.g. [54, 55]. □

4. The free Dirac operator and its spectrum

4.1. Some function spaces

This section uses a notation similar to the one used in [11, chapter 1] to compare the standard and the $\wp$-adic Dirac operators quickly. Furthermore, we use several results and calculations in [11, chapter 1]. We set

$$\begin{aligned}\mathfrak{H} &:= L^2\left(\mathbb{Q}_\wp\right)\bigoplus L^2\left(\mathbb{Q}_\wp\right)\bigoplus L^2\left(\mathbb{Q}_\wp\right)\bigoplus L^2\left(\mathbb{Q}_\wp\right)\\ &= L^2\left(\mathbb{Q}_\wp\right)\bigotimes\mathbb{C}^4 = L^2\left(\mathbb{Q}_\wp\right)^4,\end{aligned}$$

and identify the elements of $\mathfrak{H}$ with column vectors of the form

$$\psi(\boldsymbol{x}) = \begin{bmatrix}\psi_1(\boldsymbol{x})\\ \vdots\\ \psi_4(\boldsymbol{x})\end{bmatrix}.$$

The inner product is given by

$$(\psi(\boldsymbol{x}),\phi(\boldsymbol{x})) = \int_{\mathbb{Q}_\wp^3}\sum_{i=1}^4\psi_i(\boldsymbol{x})\overline{\phi}_i(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x},$$

where the bar denotes the complex conjugate, and the norm is given by

$$\|\psi\| = \sqrt{\sum_{i=1}^4\int_{\mathbb{Q}_\wp^3}|\psi_i(\boldsymbol{x})|^2\,\mathrm{d}^3\boldsymbol{x}}.$$

Given an integrable function $\psi\in\mathfrak{H}$ its Fourier transform is defined as

$$(\mathcal{F}\psi)(\boldsymbol{p}) = \widehat{\psi}(\boldsymbol{p}) = \begin{bmatrix}\widehat{\psi}_1(\boldsymbol{p})\\ \vdots\\ \widehat{\psi}_4(\boldsymbol{p})\end{bmatrix},$$

where

$$(\mathcal{F}\psi_i)(\boldsymbol{p}) = \widehat{\psi}_i(\boldsymbol{p}) = \int_{\mathbb{Q}_\wp^3}\chi_\wp(\boldsymbol{p}\cdot\boldsymbol{x})\,\psi_i(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x},\text{ for } i = 1,2,3,4.$$

The Fourier transform extends to a uniquely defined operator (denote as $\mathcal{F}$) in the Hilbert space $\mathfrak{H}$.

We now introduce a $\wp$-adic analogue of the first Sobolev space. We define the scalar product

$$(\psi,\phi)_{H^1} = \int_{\mathbb{Q}_\wp^3}\sum_{i=1}^4\widehat{\psi}_i(\boldsymbol{p})\left(\max\left(1,\|\boldsymbol{p}\|_\wp\right)\right)^{\frac{1}{2}}\widehat{\phi}_i(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p},$$

and the corresponding norm $\|\psi\|_{H^1} = \sqrt{(\psi,\psi)_{H^1}}$. We define

$$H^1\left(\mathbb{Q}_\wp^3\right) = \overline{\left(\mathcal{D}\left(\mathbb{Q}_\wp\right)\bigotimes\mathbb{C}^4,\|\cdot\|_{H^1}\right)},$$

where the bar de notes the completion of $\mathcal{D}(\mathbb{Q}_\wp)\bigotimes\mathbb{C}^4$ with respect to the distance induced by $\|\cdot\|_{H^1}$. It also verifies that

$$H^1\left(\mathbb{Q}_\wp^3\right) = \{\psi \in \mathfrak{H}; \|\psi\|_{H^1} < \infty\}.$$

The results about the space $H^1\left(\mathbb{Q}_\wp^3\right)$ are just variations of well-known results about $\wp$-adic Sobolev spaces; see, e.g. [53, section 10.2], [56], and the references therein.

The Hamiltonian $\boldsymbol{H}_0$ is a matrix operator on $\boldsymbol{D}_{x_1}, \boldsymbol{D}_{x_2}, \boldsymbol{D}_{x_3}$. From now on, we consider the Taibleson–Vladimirov derivative as a pseudo-differential operator, see (2.4). With the above notation, it follows that the mapping

$$\begin{array}{ccc} H^1\left(\mathbb{Q}_\wp^3\right) & \rightarrow & \mathfrak{H} \\ & & \\ \psi & \rightarrow & \boldsymbol{H}_0\psi \end{array}$$

is a well-defined bounded, linear operator.

The operator $\left(\boldsymbol{H}_0, H^1\left(\mathbb{Q}_\wp^3\right)\right)$ is self-adjoint, and $\left(\boldsymbol{H}_0, \mathcal{D}\left(\mathbb{Q}_\wp^3\right)\bigotimes\mathbb{C}^4\right)$ is essentially self-adjoint, see [50, theorem 3.2], [57, proposition 7]. Then, by Stone's theorem the initial value problem

$$\begin{cases} i\frac{\partial}{\partial t}\Psi(t,\boldsymbol{x}) = \boldsymbol{H}_0\Psi(t,\boldsymbol{x}), \quad t \geqslant 0, \quad \boldsymbol{x} \in \mathbb{Q}_\wp^3 \\ \\ \Psi(0,\boldsymbol{x}) = \Psi_0(\boldsymbol{x}), \end{cases}$$

has a unique solution given by

$$\Psi(t,\boldsymbol{x}) = \mathrm{e}^{-\mathrm{i}t\boldsymbol{H}_0}\Psi_0(\boldsymbol{x}).$$

4.2. The free Dirac operator in the Fourier space

The results presented in this section are analogs of the results of the standard Dirac operator. In particular, the calculations used here are the same as the ones given in [11, section 1.4.1]. The Hamiltonian $\boldsymbol{H}_0$ is a matrix pseudo-differential operator on $\boldsymbol{D}_{x_1}, \boldsymbol{D}_{x_2}, \boldsymbol{D}_{x_3}$ defined on $H^1\left(\mathbb{Q}_\wp^3\right) \subset \mathfrak{H}$. Any such operator is transformed via $\mathcal{F}$ into a matrix multiplication operator in $\mathfrak{H}$. In the case of $\boldsymbol{H}_0$, we have

$$(\boldsymbol{H}_0\phi)(\boldsymbol{x}) = \mathcal{F}^{-1}_{\boldsymbol{p}\rightarrow\boldsymbol{x}}\left(h(\boldsymbol{p})\mathcal{F}_{\boldsymbol{x}\rightarrow\boldsymbol{p}}\phi\right), \text{for } \phi \in H^1\left(\mathbb{Q}_\wp^3\right), \tag{4.1}$$

where

$$h(\boldsymbol{p}) := \begin{bmatrix} m\mathbf{1} & \boldsymbol{\sigma}\cdot|\underline{\boldsymbol{p}}|_\wp \\ \boldsymbol{\sigma}\cdot|\underline{\boldsymbol{p}}|_\wp & -m\mathbf{1} \end{bmatrix}.$$

The matrix $h(\boldsymbol{p}) = h\left(|\underline{\boldsymbol{p}}|_\wp\right)$ is a 4×4 Hermitian matrix which has the eigenvalues

$$\begin{aligned} \lambda_1\left(|\underline{\boldsymbol{p}}|_\wp\right) &= \lambda_2\left(|\underline{\boldsymbol{p}}|_\wp\right) = -\lambda_3\left(|\underline{\boldsymbol{p}}|_\wp\right) = -\lambda_4\left(|\underline{\boldsymbol{p}}|_\wp\right) \\ &=: \lambda\left(|\underline{\boldsymbol{p}}|_\wp\right) = \sqrt{\left(|p_1|_\wp^2 + |p_2|_\wp^2 + |p_3|_\wp^2\right) + m^2}. \end{aligned}$$

We also use the notation $\lambda(\boldsymbol{p}) = \lambda\left(|\underline{\boldsymbol{p}}|_\wp\right)$.

The unitary transformation $u(\boldsymbol{p}) = u\left(\left|\underline{\boldsymbol{p}}\right|_{\wp}\right)$, which diagonalizes $h(\boldsymbol{p})$ is

$$u(\boldsymbol{p}) = \frac{(m+\lambda(\boldsymbol{p}))\mathbf{1} + \beta\,\boldsymbol{\alpha}\cdot|\boldsymbol{p}|_{\wp}}{\sqrt{2\lambda(\boldsymbol{p})(m+\lambda(\boldsymbol{p}))}} = a_{+}(\boldsymbol{p})\mathbf{1} + a_{-}(\boldsymbol{p})\beta\frac{\boldsymbol{\alpha}\cdot\left|\underline{\boldsymbol{p}}\right|_{\wp}}{\sqrt{|p_1|_{\wp}^2+|p_2|_{\wp}^2+|p_3|_{\wp}^2}},$$

$$u^{-1}(\boldsymbol{p}) = a_{+}(\boldsymbol{p})\mathbf{1} - a_{-}(\boldsymbol{p})\beta\frac{\boldsymbol{\alpha}\cdot\left|\underline{\boldsymbol{p}}\right|_{\wp}}{\sqrt{|p_1|_{\wp}^2+|p_2|_{\wp}^2+|p_3|_{\wp}^2}}$$

where $\mathbf{1}$ is the 4×4 matrix identity,

$$a_{\pm}(\boldsymbol{p}) = \frac{1}{\sqrt{2}}\sqrt{1\pm\frac{m}{\lambda(\boldsymbol{p})}},$$

and the diagonal form of $h(\boldsymbol{p})$ is

$$u^{-1}(\boldsymbol{p})\,h(\boldsymbol{p})\,u(\boldsymbol{p}) = \beta\lambda(\boldsymbol{p}). \tag{4.2}$$

By using (4.1) and (4.2), the unitary transformation

$$\mathcal{W} := u\mathcal{F} : \mathfrak{H} \to \mathfrak{H}$$

converts the $\wp$-adic Dirac operator $\boldsymbol{H}_0$ into a multiplication operator by the diagonal matrix $\beta\lambda(\boldsymbol{p})$,

$$\boldsymbol{H}_0 = \mathcal{W}^{-1}\beta\lambda(\boldsymbol{p})\mathcal{W} \tag{4.3}$$

in $\mathfrak{H}$.

4.3. The spectrum of $\boldsymbol{H}_0$

In the Hilbert space $\mathcal{W}\mathfrak{H}$ the $\wp$-adic Dirac operator is diagonal, see (4.3). The upper two components of the wavefunctions belong to positive energies, while the lower two components belong to the negative energies. Following, Thaller's book [11, section 1.4.2], we introduce the subspaces of positive energies $\mathfrak{H}_{\text{pos}} \subset \mathfrak{H}$ spanned by vectors ψ_{pos}, and negative energies $\mathfrak{H}_{\text{neg}} \subset \mathfrak{H}$ spanned by vectors ψ_{neg}, where

$$\psi_{\text{pos}} = \mathcal{W}^{-1}\frac{1}{2}(\mathbf{1}+\beta)\mathcal{W}\psi,\ \psi_{\text{neg}} = \mathcal{W}^{-1}\frac{1}{2}(\mathbf{1}-\beta)\mathcal{W}\psi,\ \psi\in\mathfrak{H},$$

where $\mathbf{1}$ is the 4×4 identity matrix. Since $(\mathbf{1}+\beta)(\mathbf{1}-\beta) = \mathbf{0}$, $\mathfrak{H}_{\text{pos}}$ is orthogonal to $\mathfrak{H}_{\text{neg}}$, then

$$\mathfrak{H} = \mathfrak{H}_{\text{pos}}\bigoplus\mathfrak{H}_{\text{neg}}. \tag{4.4}$$

Taking

$$\phi_{\pm} := \frac{1}{2}(\mathbf{1}\pm\beta)\mathcal{W}\psi, \tag{4.5}$$

we have

$$(\psi_{\text{pos}}, \boldsymbol{H}_0\psi_{\text{pos}}) = \left(\mathcal{W}^{-1}\phi_{+}, \mathcal{W}^{-1}\beta\lambda(\boldsymbol{p})\phi_{+}\right) = (\phi_{+}, \beta\lambda(\boldsymbol{p})\phi_{+}) = (\phi_{+}, \lambda(\boldsymbol{p})\phi_{+}) > 0,$$

which means that $\mathfrak{H}_{\text{pos}}$ is invariant under $\boldsymbol{H}_0$. Similarly, one shows that $\mathfrak{H}_{\text{neg}}$ is invariant under $\boldsymbol{H}_0$. The orthogonal projection operators onto the positive/negative energy subspaces are given by

$$\mathcal{P}_{\substack{\text{pos}\\\text{neg}}} = \mathcal{W}^{-1}\frac{1}{2}(\mathbf{1}\pm\beta)\mathcal{W} = \frac{1}{2}\left(\mathbf{1}\pm\frac{\boldsymbol{H}_0}{|\boldsymbol{H}_0|}\right), \tag{4.6}$$

where $\mathbf{1}$ is the identity operator on $\mathfrak{H}$, and $|\boldsymbol{H}_0|$ is the pseudo-differential operator on $\mathfrak{H}$ with symbol

$$\sqrt{\left(|p_1|_\wp^2 + |p_2|_\wp^2 + |p_3|_\wp^2\right) + m^2}\mathbf{1}.$$

We identify $|\boldsymbol{H}_0|$ with the operator $\sqrt{\boldsymbol{H}_0^2} = \sqrt{\left(\boldsymbol{D}_{x_1}^2 + \boldsymbol{D}_{x_2}^2 + \boldsymbol{D}_{x_3}^2\right) + m^2}$. Like in the standard case, we have

$$\boldsymbol{H}_0\psi_{\substack{\text{pos}\\\text{neg}}} = \pm|\boldsymbol{H}_0|\,\psi_{\substack{\text{pos}\\\text{neg}}},$$

and if we define $\operatorname{sgn}\boldsymbol{H}_0 = \frac{\boldsymbol{H}_0}{|\boldsymbol{H}_0|}$, then $\boldsymbol{H}_0 = |\boldsymbol{H}_0|\operatorname{sgn}\boldsymbol{H}_0$, which is polar decomposition of $\boldsymbol{H}_0$.

Again, following the classical case, we define the Foldy–Wouthuysen transformation as

$$\mathcal{U}_{\text{FW}} = \mathcal{F}^{-1}\mathcal{W}.$$

It transforms the free Dirac operator into the pseudo-differential operator

$$\begin{aligned}\mathcal{U}_{\text{FW}}\boldsymbol{H}_0\mathcal{U}_{\text{FW}}^{-1} &= \begin{bmatrix}\sqrt{\left(\boldsymbol{D}_{x_1}^2 + \boldsymbol{D}_{x_2}^2 + \boldsymbol{D}_{x_3}^2\right) + m^2} & \mathbf{0}\\ \mathbf{0} & -\sqrt{\left(\boldsymbol{D}_{x_1}^2 + \boldsymbol{D}_{x_2}^2 + \boldsymbol{D}_{x_3}^2\right) + m^2}\end{bmatrix}\\ &= \beta\,|\boldsymbol{H}_0|.\end{aligned}$$

We interpret this formula as the fact that the free Dirac equation is unitarily equivalent to a pair of (two component) square-root Klein–Gordon equations.

Theorem 4.1. *The free Dirac operator is essentially self-adjoint on the dense domain* $\mathcal{D}(\mathbb{Q}_\wp)\bigotimes\mathbb{C}^4$ *and self-adjoint in the Sobolev space* $H^1\left(\mathbb{Q}_\wp^3\right)$ *Its spectrum* $\sigma(\boldsymbol{H}_0)$ *is the union of the essential range of the functions* $\pm\lambda(\boldsymbol{p}) : \mathbb{Q}_\wp^3 \to \mathbb{R}$.

Remark 2. We denote by $\sigma\left(\boldsymbol{H}_0^{\text{Arch}}\right)$ the spectrum of the standard free Dirac operator, by [11, theorem 1.1],

$$\sigma\left(\boldsymbol{H}_0^{\text{Arch}}\right) = (-\infty,-m]\cup[m,\infty).$$

Then $\sigma(\boldsymbol{H}_0)\subset\sigma\left(\boldsymbol{H}_0^{\text{Arch}}\right)$. Notice that we use the normalization $c=1$ in the Archimedean case too.

Proof. By (4.3) the spectrum of $\boldsymbol{H}_0$ equals the spectrum of the multiplication operator $\beta\lambda(\boldsymbol{p})$, which is essential range of the functions $\pm\lambda(\boldsymbol{p})$; see [58, section VII.2]. □

5. Charge conjugation

Following the standard case, see [11, section 1.4.6], the $\wp$-adic Dirac operator for a charge $e\in\mathbb{R}$ in an external electromagnetic field $(\phi,\boldsymbol{A})\in\mathbb{R}\times\mathbb{R}^3$ is given by

$$\boldsymbol{H}(e) := \boldsymbol{\alpha}\cdot(\nabla_\wp - e\boldsymbol{A}(t,\boldsymbol{x})) + \beta m + e\phi(t,\boldsymbol{x})\mathbf{1}.$$

We define the charge conjugation $\mathcal{C}$ as the antiunitary transformation

$$\mathcal{C}\Psi = U_{\mathcal{C}}\overline{\Psi},$$

where $U_{\mathcal{C}} = -\mathrm{i}\beta\alpha_2$ is a 4×4 unitary matrix.

Lemma 5.1. *With the above notation, if $\Psi(t,\boldsymbol{x})$ is a solution of*

$$i\frac{\partial}{\partial t}\Psi(t,\boldsymbol{x}) = \boldsymbol{H}(e)\Psi(t,\boldsymbol{x}), \tag{5.1}$$

then

$$i\frac{\partial}{\partial t}\mathcal{C}\Psi(t,\boldsymbol{x}) = \boldsymbol{H}(-e)\mathcal{C}\Psi(t,\boldsymbol{x}).$$

Moreover, $\mathcal{C}^{-1}\boldsymbol{H}(e)\mathcal{C} = -\boldsymbol{H}(-e)$.

Proof. Taking the complex conjugate in (5.1), then multiplying by $-\mathrm{i}\beta\alpha_2$, and using that $\overline{\alpha}_1 = \alpha_1$, $\overline{\alpha}_2 = -\alpha_2$, $\overline{\alpha}_3 = \alpha_3$, $\alpha_2^2 = \mathbf{1}$, $\beta\alpha_k = \alpha_k\beta$, and $\alpha_k\alpha_j = -\alpha_j\alpha_k$ for $k\neq j$, one gets that

$$\begin{aligned}-i\frac{\partial}{\partial t}\mathcal{C}\Psi(t,\boldsymbol{x}) &= -\boldsymbol{\alpha}\cdot\nabla_{\wp}\mathcal{C}\Psi(t,\boldsymbol{x}) + e\boldsymbol{\alpha}\cdot\boldsymbol{A}\mathcal{C}\Psi(t,\boldsymbol{x}) \\ &\quad - \beta m\mathcal{C}\Psi(t,\boldsymbol{x}) - e\phi(t,\boldsymbol{x})\mathcal{C}\Psi(t,\boldsymbol{x}).\end{aligned}$$

The announced formulas follow from this calculation. □

Then negative energy subspace of $\boldsymbol{H}(e)$ is connected via a symmetry transformation with the positive energy subspace of the Dirac operator $\boldsymbol{H}(-e)$ for a particle with opposite charge (antiparticle, positron). For $\mathcal{C}\psi(\boldsymbol{x})$ in the positive energy subspace of $\boldsymbol{H}(-e)$, by interpreting $|\mathcal{C}\psi(\boldsymbol{x})|^2$ as *a position probability density*, the equality

$$|\mathcal{C}\psi(\boldsymbol{x})|^2 = |\psi(\boldsymbol{x})|^2$$

shows that the motion of a negative energy electron state ψ is indistinguishable from that of a positive energy positron. Then, one obtains the interpretation:

a state $\psi\in\mathfrak{H}_{\mathrm{neg}}$ describes an antiparticle with positive energy.

Here, we do not discuss the problem that the Hilbert space $\mathfrak{H}$ contains states which are superpositions of positive and negative energy states, [11, section 1.4.6].

6. Localized particles and antiparticles

By using that $\{\psi_{rnj}\}_{rnj}$ is an orthonormal basis of $L^2(\mathbb{Q}_{\wp})$ and some well-known results, we have

$$L^2(\mathbb{Q}_{\wp},\mathrm{d}x_1)\otimes L^2(\mathbb{Q}_{\wp},\mathrm{d}x_2)\otimes L^2(\mathbb{Q}_{\wp},\mathrm{d}x_3) = L^2\left(\mathbb{Q}_{\wp}^3,\mathrm{d}^3\boldsymbol{x}\right),$$

where $\otimes$ denotes the tensor product of Hilbert spaces. Furthermore,

$$\psi_{\boldsymbol{rnj}}(\boldsymbol{x}) = \prod_{i=1}^{3}\psi_{r_i n_i j_i}(x_i), \tag{6.1}$$

where $\boldsymbol{r} = (r_1,r_2,r_3)\in\mathbb{Z}^3$, $\boldsymbol{n} = (n_1,n_2,n_3)\in(\mathbb{Q}_{\wp}/\mathbb{Z}_{\wp})^3$, $\boldsymbol{j} = (j_1,j_2,j_3)\in\{1,\cdots,\wp-1\}^3$, is an orthonormal basis for $L^2\left(\mathbb{Q}_{\wp}^3,\mathrm{d}^3\boldsymbol{x}\right)$, see, e.g. [58, chapter II, proposition 2, theorem II.10-(a)].

Definition 2. By a space-localized plane wave, we mean a function of the form

$$\Psi_{\boldsymbol{rnj}}(t,\boldsymbol{x}) = \mathrm{e}^{-\frac{iE}{\hbar}t}\psi_{\boldsymbol{rnj}}(\boldsymbol{x})\, w_{\boldsymbol{r}}\left(\wp^{(1-r_1)},\wp^{(1-r_2)},\wp^{(1-r_3)}\right),$$

where $\boldsymbol{r}=(r_1,r_2,r_3)\in\mathbb{Z}^3$, $\boldsymbol{n}=(n_1,n_2,n_3)\in(\mathbb{Q}_\wp/\mathbb{Z}_\wp)^3$, $\boldsymbol{j}=(j_1,j_2,j_3)\in\{1,\cdots,\wp-1\}^3$,

$$E^2=\left(\wp^{2(1-r_1)}+\wp^{2(1-r_2)}+\wp^{2(1-r_3)}\right)+m^2,$$

and $w_{\boldsymbol{r}}\left(\wp^{(1-r_1)},\wp^{(1-r_2)},\wp^{(1-r_3)}\right)=w\left(\wp^{(1-r_1)},\wp^{(1-r_2)},\wp^{(1-r_3)}\right)$, with $w\left(|p_1|_\wp,|p_2|_\wp,|p_3|_\wp\right)$ defined as (3.5)–(3.7).

The term space-localized means that $\Psi_{\boldsymbol{rnj}}(t,\cdot)$ has support on the polydisc

$$B_{r_1}\left(\wp^{-r_1}n_1\right)\times B_{r_2}\left(\wp^{-r_2}n_2\right)\times B_{r_3}\left(\wp^{-r_3}n_3\right). \tag{6.2}$$

Notice that the support of $\Psi_{\boldsymbol{rnj}}(t,\boldsymbol{x})$ is

$$[0,\infty)\times B_{r_1}\left(\wp^{-r_1}n_1\right)\times B_{r_2}\left(\wp^{-r_2}n_2\right)\times B_{r_3}\left(\wp^{-r_3}n_3\right).$$

Theorem 6.1. *(i) The localized plane wave $\Psi_{\boldsymbol{rnj}}(t,\boldsymbol{x})$ is a solution of the $\wp$-adic Dirac equation for any $\boldsymbol{r}$, $\boldsymbol{n}$, $\boldsymbol{j}$. (ii) Set*

$$\Psi^{\pm}_{\boldsymbol{rnj}}(t,\boldsymbol{x})=\mathcal{W}^{-1}\frac{1}{2}(\boldsymbol{1}\pm\beta)\,\mathcal{W}\Psi_{\boldsymbol{rnj}}(t,\boldsymbol{x}).$$

Then, $\Psi^{+}_{\boldsymbol{rnj}}(t,\boldsymbol{x})$ is a particle, resp. $\Psi^{-}_{\boldsymbol{rnj}}(t,\boldsymbol{x})$ is an antiparticle, space-localized in the polydisc (6.2). In particular,

$$\Psi^{+}_{\boldsymbol{rnj}}(0,\boldsymbol{x})\in\mathfrak{H}_{pos},\ \Psi^{-}_{\boldsymbol{rnj}}(0,\boldsymbol{x})\in\mathfrak{H}_{neg}.$$

Proof. (i) It follows from (2.6), by using the calculations done in the proof of proposition 3.1. (ii) It follows from (4.4)–(4.5). □

Remark 3. In the standard case, a wavefunction with positive energy cannot be initially localized in a proper subset of $\mathbb{R}^3$. Any wavefunction with positive energy has to be spread over all space ($\mathbb{R}^3$) at all times. In more precise form, for any state $\psi\in\mathfrak{H}_{\mathrm{pos}}$ (or $\mathfrak{H}_{\mathrm{neg}}$), the support of ψ is $\mathbb{R}^3$; see [11, corollary 1.7]. The falsity of this result implies the violation of Einstein causality; see [11, section 1.8.2]. Then, theorem 6.1 provides a strong indication that Einstein causality is not valid in a discrete space ($\mathbb{Q}^3_\wp$); this result will be established later.

7. The free time evolution

We define the Lizorkin space of test functions of second kind as

$$\mathcal{L}\left(\mathbb{Q}^3_\wp\right)=\left\{\varphi\in\mathcal{D}\left(\mathbb{Q}^3_\wp\right);\int_{\mathbb{Q}^3_\wp}\varphi(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x}=0\right\}=\left\{\varphi\in\mathcal{D}\left(\mathbb{Q}^3_\wp\right);\widehat{\varphi}(\boldsymbol{0})=0\right\}.$$

This space in dense in $L^2\left(\mathbb{Q}^3_\wp\right)$; see [49, theorem 7.4.3].

Theorem 7.1. *With the above notation, for $\psi \in \mathcal{L}\left(\mathbb{Q}_\wp^3\right) \bigotimes \mathbb{C}^4$,*

$$\left(e^{-i\boldsymbol{H}_0 t}\mathcal{P}_{\substack{pos\\neg}}\psi\right)(\boldsymbol{x}) = \int\limits_{\mathbb{Q}_\wp^3}\left\{\int\limits_{\mathbb{Q}_\wp^3}\chi_\wp\left((\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{p}\right)e^{\mp\lambda(\boldsymbol{p})t}\left(\frac{\pm\lambda(\boldsymbol{p})+\boldsymbol{h}(\boldsymbol{p})}{2\lambda(\boldsymbol{p})}\right)d^3\boldsymbol{p}\right\}\psi(\boldsymbol{y})\,d^3\boldsymbol{y}$$
$$= \int\limits_{\mathbb{Q}_\wp^3}\left\{\chi_\wp(-\boldsymbol{x}\cdot\boldsymbol{p})\,e^{\mp\lambda(\boldsymbol{p})t}\left(\frac{\pm\lambda(\boldsymbol{p})+\boldsymbol{h}(\boldsymbol{p})}{2\lambda(\boldsymbol{p})}\right)\widehat{\psi}(\boldsymbol{p})\,d^3\boldsymbol{p}\right\}.$$

Proof. Like in the standard case, see proof of theorem 1.2 in [11], by taking $t_\pm := t \mp \mathrm{i}\epsilon$, we have

$$\lim_{\epsilon\to 0}\mathrm{e}^{-\mathrm{i}\boldsymbol{H}_0 t_\pm}\phi_\pm = \mathrm{e}^{-\mathrm{i}\boldsymbol{H}_0 t}\phi_\pm \text{for } \phi_\pm \in \mathfrak{H}_{\substack{\mathrm{pos}\\\mathrm{neg}}}.$$

Then, by using (4.2) and (4.6),

$$\left(\mathrm{e}^{-\mathrm{i}\boldsymbol{H}_0 t_\pm}\mathcal{P}_{\substack{\mathrm{pos}\\\mathrm{neg}}}\psi\right)(\boldsymbol{x}) = \mathcal{F}^{-1}_{\boldsymbol{p}\to\boldsymbol{x}}\left(\mathrm{e}^{\mp\mathrm{i}\lambda(\boldsymbol{p})t_\pm}\frac{1}{2}\left(\boldsymbol{1}\pm\frac{\boldsymbol{h}(\boldsymbol{p})}{\lambda(\boldsymbol{p})}\right)\mathcal{F}_{\boldsymbol{x}\to\boldsymbol{p}}\psi\right)(\boldsymbol{x}), \tag{7.1}$$

where $\mathcal{F}$ is the Fourier transform in $\mathfrak{H} = L^2\left(\mathbb{Q}_\wp^3\right)\bigotimes\mathbb{C}^4$. Since $\psi \in \mathcal{L}\left(\mathbb{Q}_\wp^3\right)\bigotimes\mathbb{C}^4$, and $\mathcal{F}_{\boldsymbol{x}\to\boldsymbol{p}}\psi$ is a test function (i.e. an element of $\mathcal{L}\left(\mathbb{Q}_\wp^3\right)\bigotimes\mathbb{C}^4$) satisfying $\mathcal{F}_{\boldsymbol{x}\to\boldsymbol{p}}\psi(\boldsymbol{0}) = 0$ in some ball B_l^N around the origin, then, the function

$$\left(\boldsymbol{1}\pm\frac{\boldsymbol{h}(\boldsymbol{p})}{\lambda(\boldsymbol{p})}\right)\mathcal{F}_{\boldsymbol{x}\to\boldsymbol{p}}\psi(\boldsymbol{p}) = 0 \text{for } \boldsymbol{p} \in B_l^N,$$

is continuous on the support of $\mathcal{F}_{\boldsymbol{x}\to\boldsymbol{p}}\psi$, which is a compact subset. Thus, the function

$$\mathrm{e}^{\mp\mathrm{i}\lambda(\boldsymbol{p})t_\pm}\frac{1}{2}\left(\boldsymbol{1}\pm\frac{\boldsymbol{h}(\boldsymbol{p})}{\lambda(\boldsymbol{p})}\right)\mathcal{F}_{\boldsymbol{x}\to\boldsymbol{p}}\psi$$

is integrable. Consequently, we can rewrite (7.1) as follows:

$$\left(\mathrm{e}^{-\mathrm{i}\boldsymbol{H}_0 t_\pm}\mathcal{P}_{\substack{\mathrm{pos}\\\mathrm{neg}}}\psi\right)(\boldsymbol{x}) = \int\limits_{\mathbb{Q}_\wp^3}\left\{\int\limits_{\mathbb{Q}_\wp^3}\chi_\wp\left((\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{p}\right)\mathrm{e}^{\mp\mathrm{i}\lambda(\boldsymbol{p})t_\pm}\frac{1}{2}\left(\boldsymbol{1}\pm\frac{\boldsymbol{h}(\boldsymbol{p})}{\lambda(\boldsymbol{p})}\right)\mathrm{d}^3\boldsymbol{p}\right\}\psi(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}$$
$$= \int\limits_{\mathbb{Q}_\wp^3}\chi_\wp(-\boldsymbol{x}\cdot\boldsymbol{p})\,\mathrm{e}^{\mp\mathrm{i}\lambda(\boldsymbol{p})t_\pm}\frac{1}{2}\left(\boldsymbol{1}\pm\frac{\boldsymbol{h}(\boldsymbol{p})}{\lambda(\boldsymbol{p})}\right)\widehat{\psi}(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p}.$$

In order to compute the limit $\epsilon \to 0$, we first observe that if

$$\widehat{\psi}(\boldsymbol{p}) = \left[\widehat{\psi}_1(\boldsymbol{p}),\ldots,\widehat{\psi}_4(\boldsymbol{p})\right]^T,$$

then

$$\chi_{\wp}(-\boldsymbol{x}\cdot\boldsymbol{p})\,\mathrm{e}^{\mp\mathrm{i}\lambda(\boldsymbol{p})t_{\pm}}\frac{1}{2}\left(\mathbf{1}\pm\frac{\boldsymbol{h}(\boldsymbol{p})}{\lambda(\boldsymbol{p})}\right)\widehat{\psi}(\boldsymbol{p})$$
$$=\chi_{\wp}(-\boldsymbol{y}\cdot\boldsymbol{p})\,\mathrm{e}^{\mp\mathrm{i}\lambda(\boldsymbol{p})t_{\pm}}\begin{bmatrix}\frac{1}{2}\widehat{\psi}_1(\boldsymbol{p})+\frac{1}{\lambda(\boldsymbol{p})}\sum_{j=1}^{4}A_j^1(\boldsymbol{p})\,\widehat{\psi}_j(\boldsymbol{p})\\ \vdots\\ \frac{1}{2}\widehat{\psi}_4(\boldsymbol{p})+\frac{1}{\lambda(\boldsymbol{p})}\sum_{j=1}^{4}A_j^4(\boldsymbol{p})\,\widehat{\psi}_j(\boldsymbol{p})\end{bmatrix},$$

where the functions $A_j^k(\boldsymbol{p})$ are continuous. Now, taking

$$S:=\bigcup_{j=1}^{4}\mathrm{supp}\left(\widehat{\psi}_j\right),$$

which is compact subset of $\mathbb{Q}_{\wp}^3$, we have

$$\left|\chi_{\wp}(-\boldsymbol{y}\cdot\boldsymbol{p})\,\mathrm{e}^{\mp\mathrm{i}\lambda(\boldsymbol{p})t_{\pm}}\left(\frac{1}{2}\widehat{\psi}_k(\boldsymbol{p})+\frac{1}{\lambda(\boldsymbol{p})}\sum_{j=1}^{4}A_j^k(\boldsymbol{p})\,\widehat{\psi}_j(\boldsymbol{p})\right)\right|$$
$$\leqslant \mathrm{e}^{-\epsilon|\lambda(\boldsymbol{p})|}\left|\frac{1}{2}\widehat{\psi}_k(\boldsymbol{p})+\frac{1}{\lambda(\boldsymbol{p})}\sum_{j=1}^{4}A_j^k(\boldsymbol{p})\,\widehat{\psi}_j(\boldsymbol{p})\right|\leqslant C\sum_{j=1}^{4}\left|\widehat{\psi}_j(\boldsymbol{p})\right|,$$

where

$$C:=\frac{1}{2}+\max_{1\leqslant j,k\leqslant 4}\left\{\sup_{\boldsymbol{p}\in S\smallsetminus B_l^N}\frac{A_j^k(\boldsymbol{p})}{\lambda(\boldsymbol{p})}\right\}.$$

Since $\sum_{j=1}^{4}\left|\widehat{\psi}_j(\boldsymbol{p})\right|$ is an integrable function, by using the dominated convergence theorem, we conclude that

$$\lim_{\epsilon\to 0}\left(\mathrm{e}^{-\mathrm{i}\boldsymbol{H}_0 t_{\pm}}\mathcal{P}_{\substack{\mathrm{pos}\\ \mathrm{neg}}}\psi\right)(\boldsymbol{x})=\int_{\mathbb{Q}_{\wp}^3}\left\{\int_{\mathbb{Q}_{\wp}^3}\chi_{\wp}((\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{p})\,\mathrm{e}^{\mp\mathrm{i}\lambda(\boldsymbol{p})t}\frac{1}{2}\left(\mathbf{1}\pm\frac{\boldsymbol{h}(\boldsymbol{p})}{\lambda(\boldsymbol{p})}\right)\mathrm{d}^3\boldsymbol{p}\right\}\psi(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}$$
$$=\int_{\mathbb{Q}_{\wp}^3}\chi_{\wp}(-\boldsymbol{x}\cdot\boldsymbol{p})\,\mathrm{e}^{\mp\mathrm{i}\lambda(\boldsymbol{p})t}\frac{1}{2}\left(\mathbf{1}\pm\frac{\boldsymbol{h}(\boldsymbol{p})}{\lambda(\boldsymbol{p})}\right)\widehat{\psi}(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p}.$$

□

8. Position and momentum operators

Like the standard equation, the $\wp$-adic Dirac equation predicts the existence of particles/antiparticles of spin-$\frac{1}{2}$; for this reason, we propose $\boldsymbol{H}_0$ as the $\wp$-adic counterpart of the operator for the energy of a free electron. The definition of the self-adjoint operators for other observables is a highly non-trivial problem.

The position of a particle corresponds to a point $\boldsymbol{x}=(x_1,x_2,x_3)\in\mathbb{Q}_\wp^3$. The Fourier transform sends a function $f(\boldsymbol{x})$ to a function $\widehat{f}(\boldsymbol{p}),\boldsymbol{p}=(p_1,p_2,p_3)\in\mathbb{Q}_\wp^3$. For this reason, we identify $\boldsymbol{p}$ with the momentum of the particle. We use the quantization $p_k\rightarrow\boldsymbol{D}_{x_k}$, $k=1,2,3$, where $\boldsymbol{D}_{x_k}$ is a pseudo-differential operator with symbol $|p_k|_\wp$. Here, there is an important difference with standard QM. Now, it is natural to take the position operator as the multiplication by $|x_k|_\wp$, more precisely,

$$\mathrm{Dom}\left(|x_k|_\wp\right)=\left\{\psi\in\mathfrak{H};\int_{\mathbb{Q}_\wp^3}\sum_{j=1}^{4}|x_k|_\wp\,|\psi_j(\boldsymbol{x})|\,\mathrm{d}^3\boldsymbol{x}<\infty\right\},$$

and

$$\left(|x_k|_\wp\,\psi\right)(\boldsymbol{x})=\begin{bmatrix}|x_k|_\wp\,\psi_1(\boldsymbol{x})\\ \vdots\\ |x_k|_\wp\,\psi_4(\boldsymbol{x})\end{bmatrix},$$

$k=1,2,3$. The position operator $|x_k|_\wp$, $k=1,2,3$, is self-adjoint. The spectrum of $|x_k|_\wp$ is the essential range of the function

$$|x_k|_\wp:\mathbb{Q}_\wp\rightarrow\mathbb{Q},$$

which is the set $\{\wp^m;m\in\mathbb{Z}\}$.

8.1. Spectral projections for the position operator

For $\lambda\in\mathbb{R}$, we define $m_\lambda\in\mathbb{Z}$ as the unique integer number satisfying

$$\wp^{m_\lambda}\leqslant\lambda<\wp^{m_\lambda+1}.$$

We also set

$$\Omega_\lambda^{\cdot}(t)=\begin{cases}1 & -\infty<t\leqslant\lambda\\ 0 & \text{otherwise.}\end{cases}$$

Then, $\Omega_\lambda^{\cdot}:\mathbb{R}\rightarrow\mathbb{R}$ is a Borel measurable, bounded, function. Now,

$$\Omega_\lambda^{\cdot}\circ|x_k|_\wp=\begin{cases}1 & |x_k|_\wp\leqslant\wp^{m_\lambda}\\ 0 & \text{otherwise.}\end{cases}=\Omega\left(\wp^{-m_\lambda}|x_k|_\wp\right),$$

which is the characteristic function of the ball $\wp^{-m_\lambda}\mathbb{Z}_p$. The spectral projection of $|x_k|_\wp$ is the operator

$$\boldsymbol{E}(B_{m_\lambda}):\psi(\boldsymbol{x})\rightarrow\Omega\left(\wp^{-m_\lambda}|x_k|_\wp\right)\psi(\boldsymbol{x}).$$

By the functional calculus,

$$\boldsymbol{E}(B_{m_\lambda})=\boldsymbol{E}(B_{m_\lambda})^2,\ \ \boldsymbol{E}(B_{m_\lambda})^*=\boldsymbol{E}(B_{m_\lambda}),$$

which means that $\boldsymbol{E}(B_{m_\lambda})$ is an orthogonal projection.

Given a polydisc $B_{m_{\lambda_1}}\times B_{m_{\lambda_2}}\times B_{m_{\lambda_3}}$, we define the operator

$$\boldsymbol{E}\left(B_{m_{\lambda_1}}\times B_{m_{\lambda_2}}\times B_{m_{\lambda_3}}\right)=\boldsymbol{E}\left(B_{m_{\lambda_1}}\right)\boldsymbol{E}\left(B_{m_{\lambda_2}}\right)\boldsymbol{E}\left(B_{m_{\lambda_3}}\right).$$

Then the probability of finding the particle in the region $B_{m_{\lambda_1}} \times B_{m_{\lambda_2}} \times B_{m_{\lambda_3}}$ is

$$\left(\psi, \boldsymbol{E}\left(B_{m_{\lambda_1}} \times B_{m_{\lambda_2}} \times B_{m_{\lambda_3}}\right)\psi\right) = \int\limits_{B_{m_{\lambda_1}} \times B_{m_{\lambda_2}} \times B_{m_{\lambda_3}}} |\psi(\boldsymbol{x})|^2 \, \mathrm{d}^3\boldsymbol{x}$$
$$= \int\limits_{B_{m_{\lambda_1}} \times B_{m_{\lambda_2}} \times B_{m_{\lambda_3}}} \sum_{j=1}^{4} |\psi_j(\boldsymbol{x})|^2 \, \mathrm{d}^3\boldsymbol{x}.$$

Consequently, $|\psi(\boldsymbol{x})|^2$ can be interpreted as the position probability density, but, this interpretation holds true only for region of the form $B_{m_{\lambda_1}} \times B_{m_{\lambda_2}} \times B_{m_{\lambda_3}}$. We do not know if this interpretation is valid for arbitrary Borel subsets of $\mathbb{Q}_{\wp}^3$. In the standard case the region $B_{m_{\lambda_1}} \times B_{m_{\lambda_2}} \times B_{m_{\lambda_3}}$ can be replaced by an arbitrary Borel subset of $\mathbb{R}^3$.

9. The violation of Einstein causality

In the $\wp$-adic framework, the Dirac equation predicts the existence of localized particles/anti-particles; see theorem 6.1. We now consider a single particle and assume that it has the property of being localized in some region of $\mathbb{Q}_{\wp}^3$. To show that this property is an observable, we construct a self-adjoint operator $\boldsymbol{\Pi}_B$ in $\mathfrak{H}$ which describes the two possibilities of being localized within B or outside B. Thus, $\boldsymbol{\Pi}_B$ should have only two eigenvalues 1 (within B), and 0 (outside of B), and consequently, it is a projector operator.

Let $B \subseteq \mathbb{Q}_{\wp}^3$ be a Borel subset. We set

$$L^2(B) = \left\{ f : B \to \mathbb{C}; \|f\|_{2,B} := \sqrt{\int\limits_{B} |f(\boldsymbol{x})|^2 \, \mathrm{d}^3\boldsymbol{x}} < \infty \right\}.$$

By extending any function from $L^2(B)$ as zero outside of B, we have a continuous embedding

$$\begin{array}{lll} 1_B : L^2(B) & \to L^2\left(\mathbb{Q}_{\wp}^3\right) \\ \quad f & \to 1_B f, \end{array}$$

where 1_B is the characteristic function of B. We set

$$\mathfrak{H}(B) := L^2(B) \bigotimes \mathbb{C}^4 = L^2(B) \bigoplus L^2(B) \bigoplus L^2(B) \bigoplus L^2(B).$$

Then, $\mathfrak{H}(B)$ is a closed subspace of $\mathfrak{H}$, and thus $\mathfrak{H}(B)$ has an orthogonal complement $\mathfrak{H}^{\perp}(B)$, i.e.

$$\mathfrak{H} = \mathfrak{H}(B) \bigoplus \mathfrak{H}^{\perp}(B).$$

We set $\boldsymbol{\Pi}_B$ as the projection $\mathfrak{H} \to \mathfrak{H}(B)$. Notice that $\boldsymbol{\Pi}_B(\psi) = 1_B\psi$, and that $\boldsymbol{\Pi}_B$ is a bounded, self-adjoint operator, which is an extension of operator $\boldsymbol{E}(B_{m_{\lambda_1}} \times B_{m_{\lambda_2}} \times B_{m_{\lambda_3}})$. We use the notation $\boldsymbol{\Pi}_B$ instead of $\boldsymbol{E}(B)$ to emphasize that the construction of $\boldsymbol{\Pi}_B$ is not based on the spectral theorem. We interpret $\boldsymbol{\Pi}_B$ as the property of a system to be localized in B. If the state of a system is $\psi \in \mathfrak{H}$, $\|\psi\| = 1$, then the probability of finding the system in state $\boldsymbol{\Pi}_B\psi$ localized in B is $(\psi, \boldsymbol{\Pi}_B\psi)$.

Lemma 9.1. *Set* $\phi(x_i) := \wp^{\frac{R_0}{2}} \Omega\left(\wp^{R_0} |x_i|_\wp\right)$, *where* $R_0 \in \mathbb{Z}$. *With this notation the following assertions hold true:*

(i)

$$\Omega\left(\wp^{R_0} |x_i|_\wp\right) \psi_{rnj}(x_i) = \begin{cases} \psi_{rnj}(x_i) & \textit{if } n\wp^{-r} \in \wp^{R_0}\mathbb{Z}_\wp,\ r \leqslant -R_0 \\ \wp^{-\frac{r}{2}} \Omega\left(\wp^{R_0} |x_i|_\wp\right) & \textit{if } n\wp^{-r} \in \wp^{-r}\mathbb{Z}_\wp,\ r \geqslant -R_0+1 \\ 0 & \textit{if } n\wp^{-r} \notin \wp^{-r}\mathbb{Z}_\wp,\ r \geqslant -R_0+1. \end{cases} \tag{9.1}$$

(ii) *The Fourier expansion of* $\phi(x_i)$ *respect to the basis* $\{\psi_{rnj}(x_i)\}_{rnj}$ *is given by*

$$\phi(x_i) = \wp^{-\frac{R_0}{2}} \sum_{r \geqslant -R_0+1} \sum_j \wp^{-\frac{r}{2}} \psi_{r0j}(x),$$

where the support of $\psi_{r0j}(x_i)$ *is* $\wp^{-r}\mathbb{Z}_\wp$.

(iii) *The Fourier expansion of* $\Omega\left(\wp^{R_0} \|\boldsymbol{x}\|_\wp\right) \subset \mathbb{Q}_\wp^3$ *in the basis* $\{\psi_{\boldsymbol{rnj}}(\boldsymbol{x})\}_{\boldsymbol{rnj}}$ *is*

$$\wp^{\frac{3R_0}{2}} \Omega\left(\wp^{R_0} \|\boldsymbol{x}\|_\wp\right) = \wp^{-\frac{3R_0}{2}} \sum_{\substack{r_1 \geqslant -R_0+1 \\ r_2 \geqslant -R_0+1 \\ r_3 \geqslant -R_0+1}} \sum_{j_1, j_2, j_3} \wp^{-\frac{(r_1+r_2+r_3)}{2}} \psi_{\boldsymbol{r0j}}(\boldsymbol{x}),$$

where $\boldsymbol{r} = (r_1, r_2, r_3), \boldsymbol{j} = (j_1, j_2, j_3)$.

Proof. (i) The formula is well-known by the experts. For the sake of completeness, we review it here. Recall that supp $\psi_{rnj} = n\wp^{-r} + \wp^{-r}\mathbb{Z}_p$. The cases that appear in (9.1) corresponds to

$$n\wp^{-r} + \wp^{-r}\mathbb{Z}_p \subseteq \wp^{R_0}\mathbb{Z}_p, \tag{9.2}$$

$$n\wp^{-r} + \wp^{-r}\mathbb{Z}_p \supsetneq \wp^{R_0}\mathbb{Z}_p, \tag{9.3}$$

$$n\wp^{-r} + \wp^{-r}\mathbb{Z}_p \cap \wp^{R_0}\mathbb{Z}_p = \varnothing. \tag{9.4}$$

In the first case, $n\wp^{-r} \in \wp^{R_0}\mathbb{Z}_p$, and thus $\wp^{-r}\mathbb{Z}_p \subseteq \wp^{R_0}\mathbb{Z}_p$, which equivalent to $-r \geqslant R_0$. Conversely $-r \geqslant R_0$ (i.e. $\wp^{-r}\mathbb{Z}_p \subseteq \wp^{R_0}\mathbb{Z}_p$) and $n\wp^{-r} \in \wp^{R_0}\mathbb{Z}_p$ imply (9.2). In the second case, $0 \in n\wp^{-r} + \wp^{-r}\mathbb{Z}$, and since any point of a ball is its center, we have $n\wp^{-r} + \wp^{-r}\mathbb{Z} = \wp^{-r}\mathbb{Z}$, which implies that $n\wp^{-r} \in \wp^{-r}\mathbb{Z}$. Now, the condition, $\wp^{-r}\mathbb{Z}_p \supsetneq \wp^{R_0}\mathbb{Z}_p$ is equivalent to $R_0 > -r$. The converse assertion can be easily verified. The last case follows from the first two cases.

(ii) The Fourier expansion of $\phi(x_i)$ in the basis $\{\psi_{rnj}\}_{rnj}$ is given by

$$\phi(x_i) = \sum_{rnj} C_{rnj} \psi_{rnj}(x_i), \text{ with } C_{rnj} = \int_{\mathbb{Q}_\wp} \phi(x_i) \psi_{rnj}(x_i) \mathrm{d}x_i.$$

Then, the coefficient C_{rnj} depends on restriction of $\psi_{rnj}(x_i)$ to the ball $\wp^{R_0}\mathbb{Z}_\wp$. The condition $np^{-r} \in \wp^{-r}\mathbb{Z}_\wp$, $r \geqslant -R_0+1$ in the second in (9.1), is equivalent to $n=0$, $r \geqslant -R_0+1$. If the support of $\psi_{rnj}(x_i) \subseteq \wp^{R_0}\mathbb{Z}_p$, this case corresponds to the first line in (9.1), by using that

$$\int\limits_{\mathbb{Q}_\wp} \psi_{rnj}(x_i)\, \mathrm{d}x_i = 0,$$

we have $C_{rnj} = 0$. If the support of $\psi_{rnj}(x_i) \supsetneqq \wp^{R_0}\mathbb{Z}_p$, this case corresponds to the second line in (9.1), then

$$C_{r0j} = \wp^{\frac{R_0}{2}} \int\limits_{\mathbb{Q}_\wp} \wp^{-\frac{r}{2}} \Omega\left(\wp^{R_0}|x_i|_\wp\right) \mathrm{d}x_i = \wp^{-\frac{R_0}{2}-\frac{r}{2}},\ r \geqslant -R_0+1.$$

Therefore,

$$\phi(x_i) = \sum_{r \geqslant -R_0+1} \sum_k \wp^{-\frac{R_0}{2}-\frac{r}{2}} \psi_{r0j}(x_i).$$

(iii) It follows directly form (ii).

□

We set

$$\boldsymbol{\wp}^{1-\boldsymbol{r}} := \left(\wp^{1-r_1}, \wp^{1-r_2}, \wp^{1-r_3}\right),\ \boldsymbol{C}^T = \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \end{bmatrix} \in \mathbb{C}^4. \tag{9.5}$$

Lemma 9.2. (i) *With the above notation, it verifies that*

$$\left(e^{-i\boldsymbol{H}_0 t}\mathcal{P}_{\substack{pos\\neg}}\boldsymbol{C}\psi_{\boldsymbol{rnj}}\right)(\boldsymbol{x}) = e^{\mp\lambda(\wp^{1-\boldsymbol{r}})t}\left(\frac{\pm\lambda\left(\boldsymbol{\wp}^{1-\boldsymbol{r}}\right) + \boldsymbol{h}\left(\boldsymbol{\wp}^{1-\boldsymbol{r}}\right)}{2\lambda\left(\boldsymbol{\wp}^{1-\boldsymbol{r}}\right)}\right)\boldsymbol{C}\psi_{\boldsymbol{rnj}}(\boldsymbol{x}).$$

(ii) *Take $\psi(\boldsymbol{x}) = \sum_{\boldsymbol{rnj}} \psi_{\boldsymbol{rnj}}(\boldsymbol{x})$ and $\boldsymbol{C}$ as in (9.5), such that*
$\psi(\boldsymbol{x})\boldsymbol{C} \in \mathfrak{H}_{\substack{pos\\neg}}$.

Then

$$\left(e^{-i\boldsymbol{H}_0 t}\psi\boldsymbol{C}\right)(\boldsymbol{x}) = \sum_{\boldsymbol{rnj}} e^{\mp\lambda(\wp^{1-\boldsymbol{r}})t}\left(\frac{\pm\lambda\left(\boldsymbol{\wp}^{1-\boldsymbol{r}}\right) + \boldsymbol{h}\left(\boldsymbol{\wp}^{1-\boldsymbol{r}}\right)}{2\lambda\left(\boldsymbol{\wp}^{1-\boldsymbol{r}}\right)}\right)\boldsymbol{C}\psi_{\boldsymbol{rnj}}(\boldsymbol{x}).$$

Proof. The Fourier transform of $\psi_{r_i n_i j_i}(x_i)$ is

$$\widehat{\psi}_{r_i n_i j_i}(p_i) = \wp^{\frac{r_i}{2}} \chi_\wp\left(\wp^{-r_i} n_i p_i\right) \Omega\left(\left|\wp^{-r_i} p_i + \wp^{-1} j_i\right|_p\right),$$

and then

$$\widehat{\psi}_{\boldsymbol{rnj}}(\boldsymbol{p}) = \prod_{i=1}^{3} \widehat{\psi}_{r_i n_i j_i}(p_i).$$

Now, for a radial function $\mathfrak{a}\left(\left|\underline{\boldsymbol{p}}\right|_\wp\right) = \mathfrak{a}\left(|p_1|_\wp, |p_2|_\wp, |p_3|_\wp\right)$, it verifies that

$$\begin{aligned}\mathfrak{a}\left(\left|\underline{\boldsymbol{p}}\right|_\wp\right)\widehat{\psi}_{\boldsymbol{rnj}}(\boldsymbol{p}) &= \mathfrak{a}\left(\left|\wp^{r_1-1} j_1\right|_\wp, \left|\wp^{r_2-1} j_2\right|_\wp, \left|\wp^{r_3-1} j_3\right|_\wp\right)\widehat{\psi}_{\boldsymbol{rnj}}(\boldsymbol{p}) \\ &= \mathfrak{a}\left(\wp^{1-r_1}, \wp^{1-r_2}, \wp^{1-r_3}\right)\widehat{\psi}_{\boldsymbol{rnj}}(\boldsymbol{p}) = \mathfrak{a}\left(\boldsymbol{\wp}^{1-\boldsymbol{r}}\right)\widehat{\psi}_{\boldsymbol{rnj}}(\boldsymbol{p}).\end{aligned}$$

By using this formulae, we have

$$\mathrm{e}^{\mp \mathrm{i}\lambda(\boldsymbol{p})t}\frac{1}{2}\left(\mathbf{1}\pm\frac{\boldsymbol{h}(\boldsymbol{p})}{\lambda(\boldsymbol{p})}\right)\boldsymbol{C}\widehat{\psi}_{\boldsymbol{rnj}}(\boldsymbol{p})=\mathrm{e}^{\mp\lambda(\wp^{1-r})t}\left(\frac{\pm\lambda(\wp^{1-r})+\boldsymbol{h}(\wp^{1-r})}{2\lambda(\wp^{1-r})}\right)\boldsymbol{C}\widehat{\psi}_{\boldsymbol{rnj}}(\boldsymbol{p}).$$

The result follows from this formula by theorem 7.1.

(ii) It follows from the first part. □

Theorem 9.1. *Take* $\boldsymbol{a}^T=\begin{bmatrix} a_1 & a_2 & a_3 & a_4\end{bmatrix}\in\mathbb{R}^4$, *with* $a_i>0$, $i=1,2,3,4$,

$$|\boldsymbol{a}|=\sqrt{|a_1|^2+|a_2|^2+|a_3|^2+|a_4|^2}\neq 0,$$

and $L\in\mathbb{Z}$, *and the normalized state* $\psi_0^+, \|\psi_0^+\|=1$, *defined as*

$$\psi_0^+(\boldsymbol{x})=\mathcal{P}_{pos}\left(\frac{\wp^{\frac{3L}{2}}}{|\boldsymbol{a}|}\Omega\left(\wp^L\|\boldsymbol{x}\|_\wp\right)\begin{bmatrix}a_1\\a_2\\a_3\\a_4\end{bmatrix}\right). \tag{9.6}$$

We fix the ball

$$B^3_{l_0}\left(\wp^{-l_0}\boldsymbol{b}\right)=\wp^{-l_0}\boldsymbol{b}+\wp^{-l_0}\mathbb{Z}^3_\wp,$$

where $l_0\in\mathbb{Z}$ *and* $\boldsymbol{b}=(b_1,b_2,b_3)\in(\mathbb{Q}_\wp/\mathbb{Z}_\wp)^3$ *satisfy*

$$\wp^{-l_0}b_i\notin\wp^{-l_0}\mathbb{Z}_\wp,\ i=1,2,3, \text{ and } l_0\geqslant -L+1.$$

The distance between the balls $B^3_{-L}=\wp^L\mathbb{Z}^3_\wp$ *and* $B^3_{l_0}\left(\wp^{-l_0}\boldsymbol{b}\right)$ *is* $\wp^{l_0}\|\boldsymbol{b}\|_\wp>0$. *Then*

$$\left(e^{-it\boldsymbol{H}_0}\psi_0^+,\boldsymbol{\Pi}_{B^3_{l_0}\left(\wp^{-l_0}\boldsymbol{b}\right)}e^{-it\boldsymbol{H}_0}\psi_0^+\right)>0 \text{for any } t\in(0,\infty).$$

Remark 4. The result is also valid if we replace $\mathcal{P}_{\mathrm{pos}}$ by $\mathcal{P}_{\mathrm{neg}}$ in (9.6). By lemma 9.1(i),

$$B^3_{-L}\cap B^3_{l_0}\left(\wp^{-l_0}\boldsymbol{b}\right)=\varnothing.$$

Now take $\boldsymbol{y}\in\mathbb{Z}^3_\wp$, with $\|\boldsymbol{y}\|_\wp>\wp^{-L}$, i.e. $\boldsymbol{y}\notin B^3_{-L}$, by using the ultrametric property of $\|\cdot\|_\wp$ we have $\|\boldsymbol{x}-\boldsymbol{y}\|_\wp=\max\left\{\|\boldsymbol{x}\|_\wp,\|\boldsymbol{y}\|_\wp\right\}=\|\boldsymbol{y}\|_\wp$ for any $\boldsymbol{x}\in B^3_{-L}$, then

$$\operatorname{dist}\left(B^3_{-L},\boldsymbol{y}\right)=\inf_{\boldsymbol{x}\in B^3_{-L}}\|\boldsymbol{x}-\boldsymbol{y}\|_\wp=\|\boldsymbol{y}\|_\wp.$$

Now the distance between the balls B^3_{-L}, $B^3_{l_0}\left(\wp^{-l_0}\boldsymbol{b}\right)$ is given by

$$\begin{aligned}\operatorname{dist}\left(B^3_{-L},B^3_{l_0}\left(\wp^{-l_0}\boldsymbol{b}\right)\right)&=\inf_{\substack{\boldsymbol{x}\in B^3_{-L}\\ \boldsymbol{y}\in B^3_{l_0}\left(\wp^{-l_0}\boldsymbol{b}\right)}}\|\boldsymbol{x}-\boldsymbol{y}\|_\wp=\inf_{\boldsymbol{y}\in B^3_{l_0}\left(\wp^{-l_0}\boldsymbol{b}\right)}\left(\inf_{\boldsymbol{x}\in B^3_{-L}}\|\boldsymbol{x}-\boldsymbol{y}\|_\wp\right)\\&=\inf_{\boldsymbol{y}\in B^3_{l_0}\left(\wp^{-l_0}\boldsymbol{b}\right)}\left(\|\boldsymbol{y}\|_\wp\right)=\left\|\wp^{-l_0}\boldsymbol{b}\right\|_\wp=\wp^{l_0}\|\boldsymbol{b}\|_\wp.\end{aligned}$$

Proof. By lemmas 9.1 and 9.2, with $L = R_0$, and

$$\left(\frac{\lambda\left(\wp^{1-\boldsymbol{r}}\right)+\boldsymbol{h}\left(\wp^{1-\boldsymbol{r}}\right)}{2\lambda\left(\wp^{1-\boldsymbol{r}}\right)}\right)\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

$$= \frac{1}{2\lambda\left(\wp^{1-\boldsymbol{r}}\right)}\begin{bmatrix} a_1\lambda\left(\wp^{1-\boldsymbol{r}}\right)+c^2ma_1+ca_3\wp_3^{-r_3+1}+a_4\left(c\wp_1^{-r_1+1}-\mathrm{i}c\wp_2^{-r_2+1}\right) \\ a_2\lambda\left(\wp^{1-\boldsymbol{r}}\right)+c^2ma_2-ca_4\wp_3^{-r_3+1}+a_3\left(c\wp_1^{-r_1+1}+\mathrm{i}c\wp_2^{-r_2+1}\right) \\ a_3\lambda\left(\wp^{1-\boldsymbol{r}}\right)-c^2ma_3+ca_1\wp_3^{-r_3+1}+a_2\left(c\wp_1^{-r_1+1}-\mathrm{i}c\wp_2^{-r_2+1}\right) \\ a_4\lambda\left(\wp^{1-\boldsymbol{r}}\right)-c^2ma_4-ca_2\wp_3^{-r_3+1}+a_1\left(c\wp_1^{-r_1+1}+\mathrm{i}c\wp_2^{-r_2+1}\right) \end{bmatrix}$$

$$=: \begin{bmatrix} A_1\left(\wp^{1-\boldsymbol{r}}\right) \\ A_2\left(\wp^{1-\boldsymbol{r}}\right) \\ A_3\left(\wp^{1-\boldsymbol{r}}\right) \\ A_4\left(\wp^{1-\boldsymbol{r}}\right) \end{bmatrix},$$

we have

$$\mathrm{e}^{-\mathrm{i}t\boldsymbol{H}_0}\psi_0^+(\boldsymbol{x}) = \wp^{-\frac{3L}{2}}\sum_{\substack{r_1\geqslant -L+1\\ r_2\geqslant -L+1\\ r_3\geqslant -L+1}}\sum_{j_1,j_2,j_3}\wp^{-\frac{(r_1+r_2+r_3)}{2}}\mathrm{e}^{-\lambda\left(\wp^{1-\boldsymbol{r}}\right)t}\psi_{\boldsymbol{r0j}}(\boldsymbol{x})\begin{bmatrix} A_1\left(\wp^{1-\boldsymbol{r}}\right) \\ A_2\left(\wp^{1-\boldsymbol{r}}\right) \\ A_3\left(\wp^{1-\boldsymbol{r}}\right) \\ A_4\left(\wp^{1-\boldsymbol{r}}\right) \end{bmatrix},$$

where $\boldsymbol{r} = (r_1, r_2, r_3)$. We now compute

$$\boldsymbol{\Pi}_{B^3_{l_0}\left(\wp^{-l_0}\boldsymbol{b}\right)}\mathrm{e}^{-\mathrm{i}t\boldsymbol{H}_0}\psi_0^+ =$$

$$\wp^{-\frac{3L}{2}}\sum_{\substack{r_1\geqslant -L+1\\ r_2\geqslant -L+1\\ r_3\geqslant -L+1}}\sum_{j_1,j_2,j_3}\wp^{-\frac{(r_1+r_2+r_3)}{2}}\mathrm{e}^{-\lambda\left(\wp^{1-\boldsymbol{r}}\right)t}\Omega\left(\left\|\wp^{l_0}\boldsymbol{x}-\boldsymbol{b}\right\|_\wp\right)\psi_{\boldsymbol{r0j}}(\boldsymbol{x})\begin{bmatrix} A_1\left(\wp^{1-\boldsymbol{r}}\right) \\ A_2\left(\wp^{1-\boldsymbol{r}}\right) \\ A_3\left(\wp^{1-\boldsymbol{r}}\right) \\ A_4\left(\wp^{1-\boldsymbol{r}}\right) \end{bmatrix}.$$

Now, since

$$\mathrm{supp}\psi_{\boldsymbol{r0j}}(\boldsymbol{x}) = \wp^{-r_1}\mathbb{Z}_\wp\times\wp^{-r_2}\mathbb{Z}_\wp\times\wp^{-r_3}\mathbb{Z}_\wp,$$

by taking $l_0 \geqslant -r_j$, since $\mathrm{ord}(b_j)\geqslant 0$, we have $-l_0+\mathrm{ord}(b_j)\geqslant -r_j$ and

$$\wp^{-l_0}b_j+\wp^{-l_0}\mathbb{Z}_\wp\subseteq\wp^{-r_j}\mathbb{Z}_\wp^3, \text{for } j=1,2,3.$$

Therefore

$$\wp^{-l_0}\boldsymbol{b}+\wp^{-l_0}\mathbb{Z}_\wp^3\subseteq\mathrm{supp}\psi_{\boldsymbol{r0j}}(\boldsymbol{x})\,\text{for } r_j\geqslant -\min\{L-1,l_0\}, j=1,2,3,$$

and

$$\boldsymbol{\Pi}_{B^3_{l_0}\left(\wp^{-l_0}\boldsymbol{b}\right)} \mathrm{e}^{-\mathrm{i}t\boldsymbol{H}_0}\psi_0^+ = \wp^{-\frac{3L}{2}} \sum_{\substack{r_1 \geqslant -\min\{L-1,l_0\} \\ r_2 \geqslant -\min\{L-1,l_0\} \\ r_3 \geqslant -\min\{L-1,l_0\}}} \sum_{j_1,j_2,j_3} \wp^{-\frac{(r_1+r_2+r_3)}{2}} \mathrm{e}^{-\lambda\left(\wp^{1-\boldsymbol{r}}\right)t} \psi_{\boldsymbol{r}\boldsymbol{0}\boldsymbol{j}}(\boldsymbol{x}) \begin{bmatrix} A_1\left(\wp^{1-\boldsymbol{r}}\right) \\ A_2\left(\wp^{1-\boldsymbol{r}}\right) \\ A_3\left(\wp^{1-\boldsymbol{r}}\right) \\ A_4\left(\wp^{1-\boldsymbol{r}}\right) \end{bmatrix}.$$

Finally,

$$\begin{aligned}
&\left(\mathrm{e}^{-\mathrm{i}t\boldsymbol{H}_0}\psi_0^+, \boldsymbol{\Pi}_{B^3_{l_0}\left(\wp^{-l_0}\boldsymbol{b}\right)} \mathrm{e}^{-\mathrm{i}t\boldsymbol{H}_0}\psi_0^+\right) \\
&= \sum_{k=1}^{4} \wp^{-\frac{3L}{2}} \sum_{\substack{r_1 \geqslant -\min\{L-1,l_0\} \\ r_2 \geqslant -\min\{L-1,l_0\} \\ r_3 \geqslant -\min\{L-1,l_0\}}} \sum_{j_1,j_2,j_3} \wp^{-\frac{(r_1+r_2+r_3)}{2}} \mathrm{e}^{-\lambda\left(\wp^{1-\boldsymbol{r}}\right)t} \int_{\mathbb{Q}_\wp^3} |\psi_{\boldsymbol{r}\boldsymbol{0}\boldsymbol{j}}(\boldsymbol{x})|^2 \left|A_k\left(\wp^{1-\boldsymbol{r}}\right)\right|^2 \mathrm{d}^3\boldsymbol{x} \\
&= \wp^{-\frac{3L}{2}} \sum_{k=1}^{4} \sum_{\substack{r_1 \geqslant -\min\{L-1,l_0\} \\ r_2 \geqslant -\min\{L-1,l_0\} \\ r_3 \geqslant -\min\{L-1,l_0\}}} \sum_{j_1,j_2,j_3} \wp^{-\frac{(r_1+r_2+r_3)}{2}} \mathrm{e}^{-\lambda\left(\wp^{1-\boldsymbol{r}}\right)t} \left|A_k\left(\wp^{1-\boldsymbol{r}}\right)\right|^2 > 0,
\end{aligned}$$

for any $t \in (0,\infty)$.

□

The Einstein causality requires a finite propagation speed for all physical particles. In the standard case, any solution of the Dirac equation propagates slower than the speed of light. This requires that the support of any state $\psi \in \mathfrak{H}_{\mathrm{pos}} \cup \mathfrak{H}_{\mathrm{neg}}$ be the whole $\mathbb{R}^3$; see [11, section 1.8.2]. By theorem 9.1, the transition probability from a localized state in a ball B^3_{-L} to a state localized in a ball $B^3_{l_0}\left(\wp^{-l_0}\boldsymbol{b}\right)$, which is arbitrary far away from B^3_{-L}, is positive for any time $t \in (0,\epsilon)$, where ϵ is arbitrarily small. Then, the system has a non-zero probability of getting from B^3_{-L} to $B^3_{l_0}\left(\wp^{-l_0}\boldsymbol{b}\right)$ in an arbitrary short time, thereby propagation with superluminal speed. This feature is a consequence of the discrete nature of the space $\mathbb{Q}_\wp^3$.

10. Conclusions

Although the validity of the Lorentz symmetry has been experimentally proven with great precision [1], there is a consensus within the community of quantum-spacetime phenomenology (particularly in the quantum-gravity community) that the breaking of Lorentz symmetry occurs at the Planck scale, [3]. The breakdown of this symmetry has been investigated in connection with many other physical phenomena, [2].

The study of the limit of QM at the Planck length is a central scientific problem connected to the unification of general relativity and QM. Since 1990, it has been recognized that the Planck length has substantial implications in QM. Among them, the localization of a particle with better accuracy than its Compton wavelength is impossible. Then, it is necessary to abandon the notion of particles in favor of localized particles. Also, it was recognized that the discreteness of space-time may imply the possibility that particles would travel faster than light. However, it was pointed out that this paradox disappears when allowing the creation and annihilation of particles, [8].

The $\wp$-adic Dirac equation is the simplest model where the abovementioned matters can be discussed in a precise mathematical form. The new model uses $\mathbb{R} \times \mathbb{Q}_\wp^3$ as space-time, with $t \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{Q}_{\wp}^3$, and thus the Lorentz symmetry is naturally broken. The space $\mathbb{Q}_{\wp}^3$ is a completely disconnected topological space, which means that the only connected subsets are points or the empty set, i.e. the there are no 'intervals.' This space is a self-similar set admitting a group of symmetries of the form $\boldsymbol{x} \to \boldsymbol{a} + \boldsymbol{A}\boldsymbol{x}$, $\boldsymbol{a} \in \mathbb{Q}_{\wp}^3$, $\boldsymbol{A} \in GL_3(\mathbb{Q}_{\wp}^3)$. The action of this group on the space $\mathbb{Q}_{\wp}^3$ imposes a Planck length of exactly $\wp^{-1}$, which is independent of the speed of light. In contrast, the classical model does not have a Planck length because, in $\mathbb{R}$, the Archimedean axiom implies the existence of arbitrary small segments.

The two types of Dirac equations have common properties; particularly, both predict the existence of pairs of particle and antiparticles. However, in the $\wp$-adic case, the geometry of $\mathbb{Q}_{\wp}^3$ allows localized states (particles) to exist. In contrast, this possibility is ruled out in the standard case since it implies the violation of Einstein's causality. Our main result states the violation of the Einstein causality in the space-time $\mathbb{R} \times \mathbb{Q}_{\wp}^3$ in the Dirac-von Neumann formalism of QM.

The $\wp$-adic nature of space-time was conjectured by Volovich in 1980 s, [44]. Quantum mechanical theories with $\wp$-adic space and time are possible, see, e.g. [23–31, 59–61], and the references therein. The assumption that the time is $\wp$-adic requires abandoning the QM in the sense of Dirac-von Neumann because the evolution of quantum states is based on the theory of semigroups, which uses real-time. This, in turn implies that we cannot compute transition probabilities in classical way. Since the study of the Einstein causality is based on computing transition probabilities, here we do not use a $\wp$-adic time.

Finally, this work indicates the difficulty of unifying QM and gravity. Assuming as space-time $\mathbb{R} \times \mathbb{R}^3$, QM and general relativity together imply that

$$L_{\text{Planck}} = \sqrt{\frac{\hbar G}{c^3}} \text{and Einstein causality principle.} \tag{10.1}$$

Interpreting the Bronstein inequality $\Delta x \geqslant L_{\text{Planck}}$ as the non-existence of intervals below the Planck length and assuming as space-time $\mathbb{R} \times \mathbb{Q}_{\wp}^3$, QM implies that

$$L_{\text{Planck}} = \wp^{-1} \text{and the violation of Einstein causality principle.} \tag{10.2}$$

Now, due to the non-existence of topological and algebraic isomorphism between $\mathbb{R}$ and $\mathbb{Q}_{\wp}$, the conclusions (10.1) and (10.2) cannot 'mixed'. This fact evokes the old idea of the adelic nature of space, [27].

Data availability statement

There are no new data attached to this manuscript. The data that support the findings of this study are available upon reasonable request from the authors.

Conflict of interest

I have no conflicts of interest to disclose.

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References

[1] Kostelecký V A and Russell N 2011 Data tables for Lorentz and CPT violation *Rev. Mod. Phys.* **83** 11

[2] Mariz T, Nascimento J R and Petrov A 2022 *Lorentz Symmetry Breaking—Classical and Quantum Aspects* (*Springerbriefs in Physics*) (Springer)

[3] Amelino-Camelia G 2013 Quantum-spacetime phenomenology *Living Rev. Relativ.* **16** 5

[4] Doplicher S, Fredenhagen K and Roberts J E 1994 Space-time quantization induced by classical gravity *Phys. Lett.* B **331** 39–44

[5] Ahluwalia D V 1994 Quantum measurements, gravitation and locality *Phys. Lett.* B **339** 301–3

[6] Ng Y J and van Dam H 1994 Limit to space-time measurement *Mod. Phys. Lett.* A **9** 335–40

[7] Amelino-Camelia G 1994 Limits on the measurability of space-time distances in the semiclassical approximation of quantum gravity *Mod. Phys. Lett.* A **9** 3415–22

[8] Garay L J 1995 Quantum gravity and minimum length *Int. J. Mod. Phys.* A **10** 145–66

[9] Dirac P A M 1958 *The Principles of Quantum Mechanics* 1st edn 1930. 4th Edn (Oxford University Press)

[10] von Neumann J 2018 *Mathematical Foundations of Quantum Mechanics* 1st edn 1932 (Princeton University Press)

[11] Thaller B 1992 *The Dirac equation* (*Texts Monogr. Phys.*) (Springer)

[12] Berezin F A and Shubin M A 1991 *The SchröDinger equation* (*Math. Appl. (Soviet Ser.)*) vol 66 (Kluwer Academic Publishers Group)

[13] Zúñiga-Galindo W A 2023 The *p*-Adic Schrödinger equation and the two-slit experiment in quantum mechanics (arXiv:2308.01283v2 [quant-ph])

[14] Weil A 1967 *Basic Number Theory* (Springer)

[15] Beltrametti E G and Cassinelli G 1972 Quantum mechanics and *p*-adic numbers *Found. Phys.* **2** 1–7

[16] Vladimirov V S and Volovich I V 1988 p-adic quantum mechanics *Sov. Phys. Dokl.* **33** 669–70

[17] Vladimirov V S and Volovich I V 1989 A vacuum state in p-adic quantum mechanics *Phys. Lett.* B **217** 411–5

[18] Vladimirov V S and Volovich I V 1989 *p*-adic quantum mechanics *Commun. Math. Phys.* **123** 659–76

[19] Meurice Y 1989 Quantum mechanics with *p*-adic numbers *Int. J. Mod. Phys.* A **4** 5133–47

[20] Zelenov E I 1989 *p*-adic quantum mechanics for $p = 2$ *Theor. Math. Phys.* **80** 848–56

[21] Ruelle P, Thiran E, Verstegen D and Weyers J 1989 Quantum mechanics on *p*-adic fields *J. Math. Phys.* **30** 2854–74

[22] Vladimirov V S, Volovich I V and Zelenov E I 1990 Spectral theory in *p*-adic quantum mechanics and representation theory *Sov. Math. Dokl.* **41** 40–44

[23] Khrennikov A Y 1991 *p*-adic quantum mechanics with *p*-adic valued functions *J. Math. Phys.* **32** 932–7

[24] Zelenov E I 1991 *p*-adic quantum mechanics and coherent states *Theor. Math. Phys.* **86** 143–51

[25] Zelenov E I 1991 *p*-adic quantum mechanics and coherent states. II. Oscillator eigenfunctions *Theor. Math. Phys.* **86** 258–65

[26] Kochubei Anatoly N 1993 A Schrödinger-type equation over the field of *p*-adic numbers *J. Math. Phys.* **34** 3420–8

[27] Vladimirov V S, Volovich I V and Zelenov E I 1994 *p*-Adic Analysis and Mathematical Physics (World Scientific)

[28] Cianci R and Khrennikov A 1994 *p*-adic numbers and renormalization of eigenfunctions in quantum mechanics *Phys. Lett.* B **328** 109–12

[29] Kochubei Anatoly N 1996 *p*-Adic commutation relations *J. Phys. A: Math. Gen.* **29** 6375–8

[30] Varadarajan V S 1997 Path integrals for a class of *p*-adic Schrödinger equations *Lett. Math. Phys.* **39** 97–106

[31] Khrennikov A 1998 Non-Kolmogorov probabilistic models with *p*-adic probabilities and foundations of quantum mechanics *Programs and Probability* vol 42 (Birkhäuser Boston, Inc.) pp 275–303

[32] Albeverio S, Cianci R, De Grande-De Kimpe N and Khrennikov A 1999 *p*-adic probability and an interpretation of negative probabilities in quantum mechanics *Russ. J. Math. Phys.* **6** 1–19

[33] Dimitrijević D, Djordjević G S and Dragovich B 2000 On Schrödinger-type equation on *p*-adic spaces *Bulgar. J. Phys.* **27** 50–53

[34] Dragovich B 2004 *p*-adic and adelic quantum mechanics *Proc. Steklov Inst. Math.* **245** 64–77

[35] Varadarajan V S 2004 Arithmetic quantum physics: why, what and whither *Trudy Mat. Inst. Steklova* **245** 273–80

[36] Vourdas A 2008 Quantum mechanics on *p*-adic numbers *J. Phys. A: Math. Theor.* **41** 455303

[37] Dragovich B, Khrennikov A Y, Kozyrev S V and Volovich I V 2009 On *p*-adic mathematical physics. *p*-adic numbers ultrametric *Anal. Appl.* **1** 1–17

[38] Zelenov E I 2014 *p*-adic model of quantum mechanics and quantum channels *Proc. Steklov Inst. Math.* **285** 132–44

[39] Vourdas A 2017 *Finite and Profinite Quantum Systems*(*Quantum Science and Technology*) (Springer)

[40] Zúñiga-Galindo W A 2022 Non-Archimedean quantum mechanics via quantum groups *Nucl. Phys.* B **985** 116021

[41] Anashin V 2023 Free choice in quantum theory: a *p*-adic view *Entropy* **25** 830

[42] Aniello P, Mancini S and Parisi V 2023 Trace class operators and states in *p*-adic quantum mechanics *J. Math. Phys.* **64** 053506

[43] Bronstein M 2012 Republication of: quantum theory of weak gravitational fields *Gen. Relativ. Gravit.* **44** 267–83

[44] Volovich I V 2010 Number theory as the ultimate physical theory. *p*-Adic Numbers *Ultrametric Anal. Appl.* **2** 77–87

[45] Rovelli C and Vidotto F 2015 *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory* (Cambridge University Press)

[46] Capellmann H 2021 Space-time in quantum theory *Found. Phys.* **51** 44

[47] Šť ovíček P and Tolar J 1984 Quantum mechanics in a discrete space-time *Rep. Math. Phys.* **20** 157–70

[48] Gudder Stanley P 1988 Quantum graphic dynamics *Found. Phys.* **18** 751–76

[49] Albeverio S, Khrennikov A Y and Shelkovich V M 2010 *Theory of p-Adic Distributions: Linear and Nonlinear Nodels* (*London Mathematical Society Lecture Note Series* vol 370) (Cambridge University Press)

[50] Kochubei A N 2001 *Pseudo-Differential Equations and Stochastics Over Non-Archimedean Fields* (Marcel Dekker)

[51] Taibleson M H 1975 *Fourier Analysis on Local Fields.* (Princeton University Press)

[52] Halmos P 1950 *Measure Theory* (D. Van Nostrand Company Inc.)

[53] Khrennikov A, Kozyrev S and Zúñiga-Galindo W A 2018 *Ultrametric Equations and its Applications* (*Encyclopedia of Mathematics and its Applications* vol 168) (Cambridge University Press)

[54] Bjorken James D and Drell Sidney D 1964 *Relativistic Quantum Mechanics.* (McGraw-Hill Book Co.)

[55] Greiner W 2000 *Relativistic Quantum Mechanics.* (Springer)

[56] Chacón-Cortés L F, Garcia-Bibiano C A and Zúñiga-Galindo W A 2022 Local well-posedness of the Cauchy problem for a p -adic Nagumo-type equation, *p*-Adic Numbers *Ultrametric Anal. Appl.* **14** 279–96

[57] Zúñiga-Galindo W A 2016 *Pseudodifferential Equations Over Non-Archimedean Spaces* (*Lecture Notes in Mathematics* vol 2174) (Springer)

[58] Michael R and Barry S 1980 *Methods of Modern Mathematical Physics. I. Functional Aanalysis* 2nd edn (Academic, Inc.)

[59] Zúñiga-Galindo W A 2022 Non-Archimedean statistical field theory *Rev. Math. Phys.* **34** 8

[60] Mendoza-Martínez M L, Vallejo J A and Zúñiga-Galindo W A 2019 Acausal quantum theory for non-Archimedean scalar fields *Rev. Math. Phys.* **31** 1950011

[61] Arroyo-Ortiz E and Zúñiga-Galindo W A 2019 Construction of p -adic covariant quantum fields in the framework of white noise analysis *Rep. Math. Phys.* **84** 1–34