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Three-loop HQET vertex diagrams for B^0–$\bar{B}^0$ mixing

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Abstract: Three-loop vertex diagrams in HQET needed for sum rules for B^0–$\bar{B}^0$ mixing are considered. They depend on two residual energies. An algorithm of reduction of these diagrams to master integrals has been constructed. All master integrals are calculated exactly in d dimensions; their ε expansions are also obtained.

Keywords: NLO Computations, B-Physics.

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1. Introduction

The mass difference Δm in B^0–$\bar{B}^0$ mixing is determined in the Standard Model by the matrix element $<\bar{B}^0|Q(\mu)|B^0>$ of the four-quark operator

$$Q(\mu) = J_\alpha J^\alpha , \quad J^\alpha = \bar{b}_L \gamma^\alpha d_L \tag{1.1}$$

(see, e.g., [1]). This matrix element is traditionally written as

$$<\bar{B}^0|Q(\mu)|B^0> = 2\left(1 + \frac{1}{N_c}\right) <\bar{B}^0|J_\alpha|0><0|J^\alpha|B^0> B(\mu) , \tag{1.2}$$

where N_c is the number of colours. Here the first part of the right-hand side is the value of the matrix element according to the naive factorization prescription (this part does not depend on μ), and $B(\mu)$ describes violation of this prescription. The hadronic parameter $B(\mu)$ can only be obtained by using some non-perturbative method, such as lattice simulations (see, e.g., [2]) or QCD sum rules [3 − 5].

In the QCD sum rules approach, the correlator $<jQj>$ is investigated, where j is a current with $<B^0|j|0> \neq 0$ (axial or pseudoscalar). Contributions to the theoretical expression for this correlator can be subdivided into two groups:

$$<jQj> = 2\left(1 + \frac{1}{N_c}\right) <jJ_\alpha><J^\alpha j> + <jQj>_{\rm nf} . \tag{1.3}$$

The first term includes the leading perturbative contribution plus all corrections (perturbative, vacuum condensates) to the two two-point correlators $<jJ_\alpha>$, $<J^\alpha j>$ separately. It just gives the square of the sum rule for f_B^2. Only the second, non-factorizable part

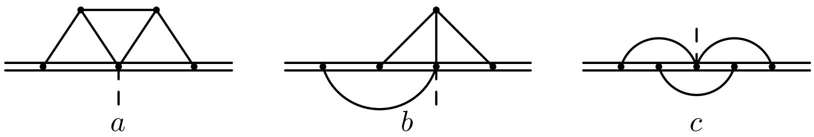


Figure 1: Generic topologies.

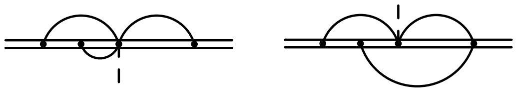


Figure 2: Diagrams to which the topology c reduces.

contributes to the sum rule for $B(\mu)-1$. Non-factorizable perturbative contributions first appear at three loops (one gluon is exchanged between the two two-point correlators). In general, their calculation is a very difficult three-loop problem with three energy scales (m_b^2, p_1^2, p_2^2; we suppose that $q^2=0$) which cannot be solved at present. Several terms of the expansion in p_1^2, p_2^2 have been obtained [4] (this is a much easier single-scale problem). There are also non-factorizable terms due to vacuum condensates.

It is also possible to consider sum rules in the HQET framework (see, e.g., [6, 7]). The QCD operators Q, j can be expressed via HQET operators; matching coefficients are calculable series in $\alpha_s(m_b)$. Correlators of HQET operators don't involve the scale m_b. Therefore, no large logarithms appear in perturbative corrections. On the other hand, derivation and analysis of HQET sum rules for $1/m_b$ corrections is difficult (though not impossible). Calculations in HQET are technically easier. In particular, three-loop diagrams describing the leading perturbative contribution to the sum rules for $B-1$ involve only two scales — two residual energies. Here we present the method for calculating such diagrams. Calculation of this perturbative contribution is very desirable, because it allows one to control the μ-dependence of $B(\mu)-1$.

2. Reduction

Non-factorizable three-loop diagrams belong to three topologies (figure 1). Four HQET denominators in figure 1c are linearly dependent; therefore, one heavy line can be killed, and this diagram reduces to those in figure 2, which are particular cases of figure 1b.

Let the incoming and outgoing residual momenta be $p_{1,2}$. The scalar integrals depend only on the residual energies $\omega_{1,2}=p_{1,2}\cdot v$, where v is the heavy-quark velocity. In the case $\omega_1=\omega_2$ they reduce to single-scale HQET integrals [8] (see also [9, 10]).

We need to consider two topologies. The first one is (figure 3)

$$I_a(n_i;m_j;\omega_1,\omega_2)=\frac{1}{(i\pi^{d/2})^3}\int\frac{\prod_j N_j^{m_j}\,d^dk_1\,d^dk_2\,d^dk_3}{\prod_i D_i^{n_i}}\,, \tag{2.1}$$

$$\begin{aligned}
&D_1=-2(k_1\cdot v+\omega_1)\,, && D_2=-2(k_2\cdot v+\omega_2)\,, && D_3=-k_1^2\,, && D_4=-k_2^2\,,\\
&D_5=-k_3^2\,, && D_6=-(k_3-k_1)^2\,, && D_7=-(k_3-k_2)^2\,,\\
&N_1=-2k_3\cdot v\,, && N_2=-(k_1-k_2)^2\,,
\end{aligned}$$

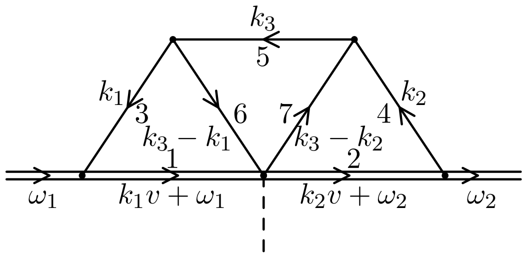


Figure 3: Topology 1.

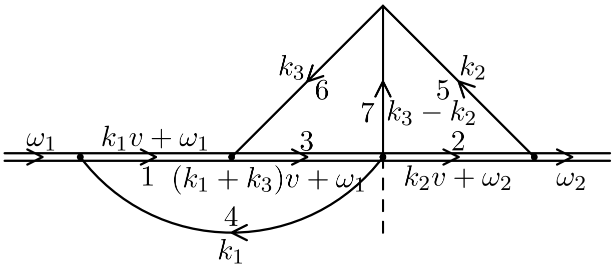
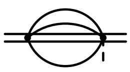
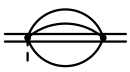
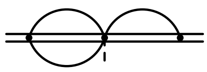
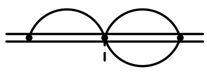
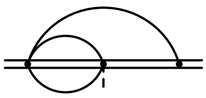
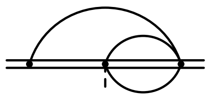
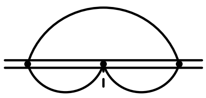
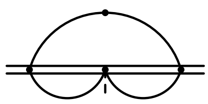


Figure 4: Topology 2.

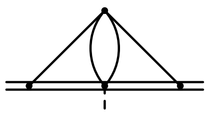
where $-i0$ is assumed in all denominators, n_i and m_j are integer, and $m_j \geq 0$. They can be reduced to master integrals using integration by parts [11]. A Mathematica program (R.N. Lee, unpublished, based on [12]) has succeeded in constructing an algorithm to reduce these scalar integrals to the following simple master integrals:

 ,  ,  ,  ,

$M_1(\omega_1, \omega_2) =$  , $M_1(\omega_2, \omega_1) =$  ,

$M_2 =$  , $M_2' =$  , (2.2)

and one difficult integral:

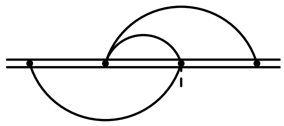
$M_3 =$  . (2.3)

The second topology is (figure 4)

$$I_b(n_i; m_j; \omega_1, \omega_2) = \frac{1}{(i\pi^{d/2})^3} \int \frac{\prod_j N_j^{m_j} \, d^d k_1 \, d^d k_2 \, d^d k_3}{\prod_i D_i^{n_i}} \,, \tag{2.4}$$

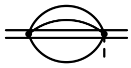
$$D_1 = -2(k_1 \cdot v + \omega_1)\,, \quad D_2 = -2(k_2 \cdot v + \omega_2)\,, \quad D_3 = -2((k_1 + k_3) \cdot v + \omega_1)\,,$$
$$D_4 = -k_1^2\,, \qquad D_5 = -k_2^2\,, \qquad D_6 = -k_3^2\,, \qquad D_7 = -(k_3 - k_2)^2\,,$$
$$N_1 = -(k_1 - k_3)^2\,, \qquad N_2 = -(k_1 - k_2)^2\,.$$

The same program has succeeded in constructing an algorithm to reduce these scalar integrals to the same simple master integrals (2.2) and one difficult integral

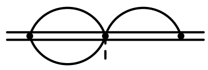
$$M_4 = \quad$$  $$. \tag{2.5}$$

3. Simple master integrals

We consider the below-threshold region $\omega_{1,2} < 0$; expressions for other regions can be obtained by analytical continuation. The simplest master integrals are single-scale, or products of single-scale integrals:



$$= I_3(-2\omega_1)^{3d-7}\,, \tag{3.1}$$

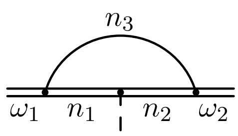


$$= I_1 I_2(-2\omega_1)^{2d-5}(-2\omega_2)^{d-3}\,, \tag{3.2}$$

where the n-loop HQET sunset is

$$I_n = \Gamma(2n+1-nd)\Gamma^n\left(\tfrac{d}{2}-1\right)\,. \tag{3.3}$$

Several master integrals reduce to the one-loop vertex with two residual energies



$$= I(n_1,n_2,n_3;\omega_1,\omega_2) = \frac{1}{i\pi^{d/2}}\int\frac{d^d k}{D_1^{n_1}D_2^{n_2}D_3^{n_3}}\,, \tag{3.4}$$

$$D_1 = -2(k\cdot v+\omega_1)\,,\quad D_2 = -2(k\cdot v+\omega_2)\,,\quad D_3 = -k^2\,.$$

It is [13]

$$\begin{aligned} I(n_1,n_2,n_3;\omega_1,\omega_2) &= I(n_1+n_2,n_3)\,{}_2F_1\left(\left.\begin{array}{c} n_1, n_1+n_2+2n_3-d \\ n_1+n_2 \end{array}\right| 1-\frac{1}{x}\right) \\ &\quad\times(-2\omega_2)^{d-n_1-n_2-2n_3}\,, \end{aligned} \tag{3.5}$$

where the HQET two-point integral is

$$I(n_1,n_2) = \frac{\Gamma(n_1+2n_2-d)\Gamma\left(\frac{d}{2}-n_2\right)}{\Gamma(n_1)\Gamma(n_2)}\,, \tag{3.6}$$

and

$$x = \frac{\omega_2}{\omega_1}\,. \tag{3.7}$$

Naturally,

$$\begin{aligned} I(n_1,n_2,n_3;\omega_1,\omega_2) &= I(n_2,n_1,n_3;\omega_2,\omega_1)\,, \\ I(n_1,n_2,n_3;\omega,\omega) &= I(n_1+n_2,n_3)(-2\omega)^{d-n_1-n_2-2n_3}\,. \end{aligned}$$

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Later we shall also need

$$
\begin{aligned}
I(n_1,n_2,n_3;\omega,0) &= I_0(n_1,n_2,n_3)(-2\omega)^{d-n_1-n_2-2n_3}\,,\\
I_0(n_1,n_2,n_3) &= \frac{\Gamma\left(\frac{d}{2}-n_3\right)\Gamma(d-n_2-2n_3)\Gamma(n_1+n_2+2n_3-d)}{\Gamma(n_1)\Gamma(n_3)\Gamma(d-2n_3)}\,.
\end{aligned}
\tag{3.8}
$$

Several ways to derive (3.5) are discussed in [10].

Using this integral, we easily obtain

$$M_1(\omega_1,\omega_2) = I_2 I(5-2d,1,1;\omega_1,\omega_2) = I_2 I(1,5-2d,1;\omega_2,\omega_1)\,, \tag{3.9}$$

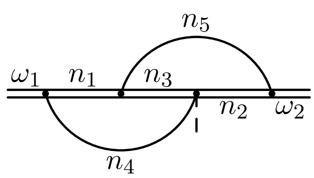
$$M_2(\omega_1,\omega_2) = I_1^2 I(3-d,3-d,1;\omega_1,\omega_2)\,, \tag{3.10}$$

$$M_2'(\omega_1,\omega_2) = I_1^2 I(3-d,3-d,2;\omega_1,\omega_2)\,. \tag{3.11}$$

4. Master integral M_4

We were able to calculate a more general integral

$$J(n_1,n_2,n_3,n_4,n_5;\omega_1,\omega_2) = $$



. (4.1)

Substituting (3.5) for the left one-loop vertex subdiagram, we have

$$
\begin{aligned}
&\frac{I(n_1+n_3,n_4)}{i\pi^{d/2}}(-2\omega_1)^{d-n_1-n_3-2n_4}\\
&\times\int\frac{dk_0\,d^{d-1}\vec{k}}{(-k^2)^{n_5}(-2(k_0+\omega_2))^{n_2}}\,{}_2F_1\left(\left.\begin{array}{c}n_1,n_1+n_3+2n_4-d\\ n_1+n_3\end{array}\right|-\frac{k_0}{\omega_1}\right)\,.
\end{aligned}
$$

Then we perform Wick rotation $k_0 = ik_{E0}$ and take the $d^{d-1}\vec{k}$ integral. The integrand has a cut from 0 to $+i\infty$; we deform the integration contour around this cut ($k_{E0} = i(-\omega_2)z$):

$$
\begin{aligned}
&\frac{I(n_1+n_3,n_4)\Gamma\left(n_5-\frac{d-1}{2}\right)}{\pi^{1/2}2^{d-2n_5-1}\Gamma(n_5)}\cos\left[\pi\left(\frac{d}{2}-n_5\right)\right](-2\omega_1)^{d-n_1-n_3-2n_4}(-2\omega_2)^{d-n_2-2n_5}\\
&\times\int_0^\infty\frac{dz\,z^{d-2n_5-1}}{(z+1)^{n_2}}\,{}_2F_1\left(\left.\begin{array}{c}n_1,n_1+n_3+2n_4-d\\ n_1+n_3\end{array}\right|-xz\right)\,.
\end{aligned}
$$

This integral can be calculated in terms of two ${}_3F_2$ functions, and we arrive at

$$
\begin{aligned}
J(n_1,n_2,n_3,n_4,n_5;\omega_1,\omega_2) &= \frac{\Gamma\left(\frac{d}{2}-n_4\right)\Gamma\left(\frac{d}{2}-n_5\right)}{\Gamma(n_4)\Gamma(n_5)}\times\left[\frac{\Gamma(n_1+n_3+2n_4-d)\Gamma(n_2+2n_5-d)}{\Gamma(n_2)\Gamma(n_1+n_3)}\right.\\
&\times{}_3F_2\left(\left.\begin{array}{c}n_3,n_1+n_3+2n_4-d,d-2n_5\\ n_1+n_3,d-n_2-2n_5+1\end{array}\right|x\right)x^{d-n_2-2n_5}\\
&+\frac{\Gamma(d-n_2-2n_5)\Gamma(n_2+n_3+2n_5-d)\Gamma(n_1+n_2+n_3+2n_4+2n_5-2d)}{\Gamma(n_3)\Gamma(d-2n_5)\Gamma(n_1+n_2+n_3+2n_5-d)}\\
&\left.\times{}_3F_2\left(\left.\begin{array}{c}n_2,n_2+n_3+2n_5-d,n_1+n_2+n_3+2n_4+2n_5-2d\\ n_2+2n_5-d+1,n_1+n_2+n_3+2n_5-d\end{array}\right|x\right)\right]\\
&\times(-2\omega_1)^{2d-n_1-n_2-n_3-2n_4-2n_5}\,.
\end{aligned}
\tag{4.2}
$$

Trivial cases are reproduced:

$$J(n_1,n_2,0,n_4,n_5;\omega_1,\omega_2) = I(n_1,n_4)I(n_2,n_5)(-2\omega_1)^{d-n_1-2n_4}(-2\omega_2)^{d-n_2-2n_5}\,,$$
$$J(n_1,0,n_3,n_4,n_5;\omega_1,\omega_2) = I(n_3,n_5)I(n_1+n_3+2n_5-d,n_4)(-2\omega_1)^{2d-n_1-n_3-2n_4-2n_5}\,.$$

At $\omega_1 = \omega_2$, the single-scale integral [8, 9] is reproduced (its derivation is also discussed in [10]).

Now it is easy to write down the master integral (2.5)

$$M_4(\omega_1,\omega_2) = I_1 J(1,1,3-d,1,1;\omega_1,\omega_2)\,. \tag{4.3}$$

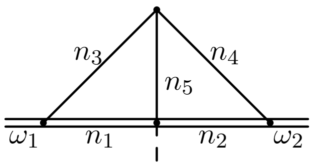
Note that the first ${}_3F_2$ function in eq. (4.2) turns into ${}_2F_1$ when one substitutes $n_2 = 1$ in order to obtain M_4.

5. Master integral M_3

This integral can be expressed as

$$M_3(\omega_1,\omega_2) = G_1 I(1,1,1,1,2-\tfrac{d}{2};\omega_1,\omega_2) \tag{5.1}$$

via the two-loop integral

$$I(n_1,n_2,n_3,n_4,n_5;\omega_1,\omega_2) =$$


$$= \frac{1}{(i\pi^{d/2})^2}\int\frac{d^dk_1\,d^dk_2}{\prod_i D_i^{n_i}}\,, \tag{5.2}$$

$$D_1 = -2(k_1\cdot v+\omega_1)\,,\qquad D_2 = -2(k_2\cdot v+\omega_2)\,,$$
$$D_3 = -k_1^2\,,\qquad D_4 = -k_2^2\,,\qquad D_5 = -(k_1-k_2)^2$$

with non-integer n_5, where

$$G_n = \frac{\Gamma\left(n+1-n\frac{d}{2}\right)\Gamma^{n+1}\left(\frac{d}{2}-1\right)}{\Gamma\left((n+1)\left(\frac{d}{2}-1\right)\right)} \tag{5.3}$$

is the n-loop massless sunset.

In order to express M_3 in closed form, we can use the method of differential equations [14, 15]. The differential equation for this master integral can be obtained by differentiating it with respect to ω_1 and then applying the reduction rules obtained by the Mathematica program. It reads

$$\omega_1\frac{\partial M_3(\omega_1,\omega_2)}{\partial\omega_1} = \frac{3d-10}{2}M_3(\omega_1,\omega_2)+H(\omega_1,\omega_2)\,, \tag{5.4}$$
$$H(\omega_1,\omega_2) = \frac{2d-5}{2(\omega_1-\omega_2)^2}M_1(\omega_2,\omega_1)-\frac{(3d-7)[(3d-8)\omega_1-(5d-14)\omega_2]}{8(d-3)\omega_2^2(\omega_1-\omega_2)^2}I_3(-2\omega_2)^{3d-7}$$
$$-\frac{2d-5}{2(\omega_1-\omega_2)^2}M_1(\omega_1,\omega_2)+\frac{(3d-7)[(3d-8)\omega_2-(5d-14)\omega_1]}{8(d-3)\omega_1^2(\omega_1-\omega_2)^2}I_3(-2\omega_1)^{3d-7}\,.$$

Using the explicit expressions for the simple master integrals, it is easy to check that singularities at $\omega_1 = \omega_2$ cancel in H separately on the second and third lines in eq. (5.4).

The general solution of this differential equation has the form

$$M_3(\omega_1, \omega_2) = M_0(\omega_1) \left[C + \int_{-\infty}^{\omega_1} d\omega\, M_0^{-1}(\omega) H(\omega, \omega_2) \right],$$

where

$$M_0(\omega) = (-2\omega)^{3d/2-5}$$

is the solution of the homogeneous part of the equation (5.4). In order to fix the constant C, we consider the asymptotics of $M_3(\omega_1, \omega_2)$ when $\omega_1 \to -\infty$ [16]. Using the method of expansion by regions (see [17]), it is easy to determine that there is no $\mathcal{O}(\omega_1^{3d/2-5})$ term in the asymptotics. Thus, $C = 0$, and we obtain

$$\begin{aligned} M_3(\omega_1, \omega_2) =\ & 2(-2\omega_1)^{3d/2-5}(-2\omega_2)^{3d/2-5}\Gamma^3\left(\tfrac{d}{2}-1\right)\Gamma(8-3d) \int_{1/x}^{\infty} \frac{dy}{(y-1)^2} \\ & \times \left\{ y^{4-3d/2} \left[{}_2F_1\left(\left. \begin{matrix} 1, 8-3d \\ 6-2d \end{matrix} \right| 1-y \right) - 1 - \frac{8-3d}{6-2d}(1-y) \right] \right. \\ & \left. - y^{3d/2-4} \left[{}_2F_1\left(\left. \begin{matrix} 1, 8-3d \\ 6-2d \end{matrix} \right| 1-\frac{1}{y} \right) - 1 - \frac{8-3d}{6-2d}\left(1-\frac{1}{y}\right) \right] \right\}. \end{aligned} \tag{5.5}$$

Note that the rational terms in brackets are the two first terms of expansion of the corresponding ${}_2F_1$ with respect to its argument. Now, using the parametrization

$$\begin{aligned} & {}_2F_1\left(\left. \begin{matrix} 1, 8-3d \\ 6-2d \end{matrix} \right| 1-t \right) - 1 - \frac{8-3d}{6-2d}(1-t) \\ & = \frac{\Gamma(6-2d)}{\Gamma(8-3d)\Gamma(d-2)} \int_0^{\infty} ds\, s^{7-3d}(1+s)^{2d-5} \left(\frac{1}{1+st} - \frac{1}{1+s} - \frac{s(1-t)}{(1+s)^2} \right), \end{aligned} \tag{5.6}$$

we can take the integrals first over y and then over s. Finally, we obtain

$$\begin{aligned} M_3(\omega_1, \omega_2) =\ & 4(-2\omega_1)^{3d-10}\Gamma^3\left(\tfrac{d}{2}-1\right) \\ & \times \left[\frac{\Gamma(8-3d)}{2(d-3)} x^{3d-9}\, {}_3F_2\left(\left. \begin{matrix} 1, d-2, \frac{3}{2}d-4 \\ \frac{3}{2}d-3, 3d-8 \end{matrix} \right| x \right) \right. \\ & + \frac{3\Gamma(9-3d)}{2(d-3)(3d-10)}\, {}_3F_2\left(\left. \begin{matrix} 1, 10-3d, 5-\frac{3}{2}d \\ 6-\frac{3}{2}d, 4-d \end{matrix} \right| x \right) \\ & + \frac{\pi\Gamma(6-2d)}{(3d-10)\Gamma(d-2)\sin(3\pi d)}\, {}_2F_1\left(\left. \begin{matrix} 5-\frac{3}{2}d, 7-2d \\ 6-\frac{3}{2}d \end{matrix} \right| x \right) \\ & \left. + \frac{\pi\Gamma(6-2d)}{(d-4)\Gamma(d-2)\sin(\pi d)} x^{d-3}\, {}_2F_1\left(\left. \begin{matrix} 2-\frac{d}{2}, 7-2d \\ 3-\frac{d}{2} \end{matrix} \right| x \right) \right]. \end{aligned} \tag{5.7}$$

It follows from the analyticity of $M_3(\omega_1, \omega_2)$ in the region $\omega_{1,2} < 0$ that the above expression is analytical in the interval $x \in (0, +\infty)$. In particular, branching singularities at $x = 1$ cancel.

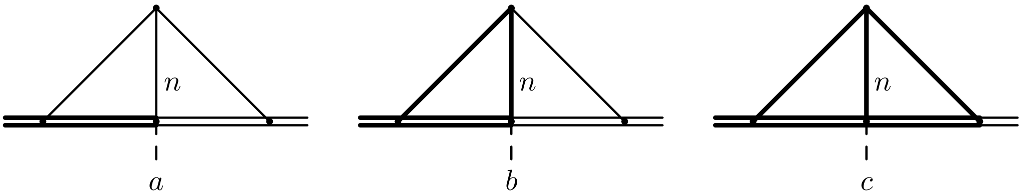


Figure 5: Regions: thick lines are hard (momenta $\sim \omega_1$), thin lines are soft (momenta $\sim \omega_2$).

The integral M_3 is a symmetric function of its arguments. This symmetry can be made explicit if we rewrite the integral over y of the terms in the last line of (5.5) as follows:

$$\int_{1/x}^{\infty} dy = \int_0^{\infty} dy - \int_x^{\infty} d(1/y) y^2 \,,$$

and make the replacement $y \to 1/y$ in the second integral. Then, using the same parametrization (5.6), we obtain

$$\begin{aligned}
M_3(\omega_1,\omega_2) &= (-2\omega_1)^{3d/2-5}(-2\omega_2)^{3d/2-5}\Gamma^3(d/2-1) \\
&\quad\times \left[\frac{\Gamma\left(\frac{3}{2}d-4\right)\Gamma^2\left(5-\frac{3}{2}d\right)\Gamma\left(2-\frac{d}{2}\right)}{(d-3)\Gamma(d-2)} \right. \\
&\qquad + 2\frac{\Gamma(8-3d)}{d-3} x^{4-3d/2}\, {}_3F_2\left(\left. \begin{array}{c} 1, d-2, \frac{3}{2}d-4 \\ \frac{3}{2}d-3, 3d-8 \end{array} \right| \frac{1}{x} \right) \\
&\qquad + \frac{4\pi\Gamma(6-2d)x^{3d/2-5}}{(3d-10)\Gamma(d-2)\sin(3\pi d)}\, {}_2F_1\left(\left. \begin{array}{c} 5-\frac{3}{2}d, 7-2d \\ 6-\frac{3}{2}d \end{array} \right| \frac{1}{x} \right) \\
&\qquad + 2\frac{\Gamma(8-3d)}{d-3} x^{3d/2-4}\, {}_3F_2\left(\left. \begin{array}{c} 1, d-2, \frac{3}{2}d-4 \\ \frac{3}{2}d-3, 3d-8 \end{array} \right| x \right) \\
&\qquad \left. + \frac{4\pi\Gamma(6-2d)x^{5-3d/2}}{(3d-10)\Gamma(d-2)\sin(3\pi d)}\, {}_2F_1\left(\left. \begin{array}{c} 5-\frac{3}{2}d, 7-2d \\ 6-\frac{3}{2}d \end{array} \right| x \right) \right] . \qquad (5.8)
\end{aligned}$$

We have performed two crucial checks of the above expressions for M_3. The first check is due to the fact that at $\omega_1 = \omega_2$ the integral M_3 reduces to the known single-scale integral [18]. Though our representations do not literally coincide with those in [18], we have been able to check the perfect numerical agreement.

The asymptotics of M_3 (5.7) at $x \to 0$ can be also obtained by using the method of regions [17] for $I(1,1,1,1,n;\omega_1,\omega_2)$ with $n = 2 - d/2$ (5.1). There are 3 regions shown in figure 5. The region a gives the first term in (5.7); b — the fourth term; and c — the second and the third ones (this is clear from the powers of $\omega_{1,2}$).

In the region a, we expand $1/D_1$ (5.2) in $k_1 \cdot v$. Then we calculate the left massless loop (lines 3 and 5) with the numerator $(k_1 \cdot v)^l$ (see, e.g., [17], eqs. (A.11), (A.12)). In the numerator of the remaining HQET integral, powers of $2k_2 \cdot v$ may be replaced by powers of $-2\omega_2$, because integrals in which the denominator D_2 cancels are zero. We obtain a series in x whose coefficients are finite sums. We have checked that a few terms in this series agree with the expansion of the first term in (5.7).

In the region b, we expand $1/D_5^n$ (5.2) in k_2. Then we calculate the left (hard) HQET loop with a numerator (see [8], eq. (2.13)), and finally the right (soft) HQET loop (it also has numerators). The coefficients of the resulting series are finite sums. We have checked that a few terms in this series agree with the expansion of the fourth term in (5.7).

In the region c, we expand $1/D_2$ (5.2) in ω_2:

$$(-2\omega_1)^{2d-2n-6} \sum_{l=0}^{\infty} I_0(1, l+1, 1, 1, n)(-x)^l \,,$$

where

$$I(n_1, n_2, n_3, n_4, n_5; \omega, 0) = I_0(n_1, n_2, n_3, n_4, n_5)(-2\omega)^{2d-n_1-n_2-2n_3-2n_4-2n_5} \,.$$

Using integration by parts, we obtain

$$I_0(1, l+1, 1, 1, n) = n \frac{I_0(1, l+1, 1, 0, n+1) - I_0(1, l+1, 0, 1, n+1)}{d-n-l-3} \,,$$

where

$$\begin{aligned} I_0(n_1, n_2, 0, n_4, n_5) &= I(n_1, n_5) I_0(n_1 + 2n_5 - d, n_2, n_4) \,, \\ I_0(n_1, n_2, n_3, 0, n_5) &= I(n_2, n_5) I_0(n_1, n_2 + 2n_5 - d, n_3) \end{aligned}$$

(see (3.8)). The contribution of the region c is thus

$$\begin{aligned} &\frac{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}-n-1\right)\Gamma(2n+6-2d)}{(d-n-3)\Gamma(n)} \\ &\times \left[\frac{\Gamma(2d-2n-5)\Gamma(2n+3-d)}{\Gamma(d-2)} \, {}_2F_1\left(\left. \begin{array}{c} n+3-d, 2n+3-d \\ n+4-d \end{array} \right| x \right) \right. \\ &\qquad \left. - \frac{1}{d-3} \, {}_3F_2\left(\left. \begin{array}{c} 1, n+3-d, 2n+6-2d \\ 4-d, n+4-d \end{array} \right| x \right) \right] . \end{aligned}$$

Substituting $n = 2 - d/2$ and multiplying by G_1 (see (5.1)), we reproduce the second and the third terms in (5.7).

6. Conclusion

We have considered scalar loop integrals needed for the perturbative part of HQET sum rules for $B-1$. The sum rules will be considered in a future publication. The width difference $\Delta\Gamma$ involves matrix elements of four-quark operators similar to (1.1) but with different Dirac structures. In higher orders in $1/m_b$, similar operators involving derivatives appear. Matrix elements of such operators can also be estimated using HQET sum rules (operators with derivatives are very difficult for lattice simulations).

More general classes of three-loop HQET vertex diagrams can be analyzed using the same method. Master integrals calculated here will be useful for such an analysis.

We are grateful to A.A. Pivovarov for discussions of HQET sum rules for B^0–$\bar{B}^0$ mixing.

A. Expansions in ε

We use the Mathematica package HypExp [19] to expand the master integrals in ε ($d = 4 - 2\varepsilon$):

$$
\begin{aligned}
M_1 &= \frac{\Gamma^3(1-\varepsilon)\Gamma(1+6\varepsilon)}{72\varepsilon^2(1-2\varepsilon)(1-3\varepsilon)(2-3\varepsilon)(3-4\varepsilon)(1-6\varepsilon)}\Big\{3x(1-x)^3 \\
&\quad - \frac{1}{2}\Big[36x(1-x)^3\log x - 6 + 71x - 141x^2 + 105x^3 - 27x^4\Big]\varepsilon \\
&\quad - \frac{1}{2}(1-x)\Big[18x(1-x)^2\left(8L(x) - 4\log^2 x - 9\log x\right) - 4 + 63x - 78x^2 + 21x^3\Big]\varepsilon^2 \\
&\quad + (1-x)\Big[9x(1-x)^2\big(48\,\mathrm{Li}_3(1-x) + 16\,\mathrm{Li}_3(1-x^{-1}) - 4\log^3 x + 36L(x) \\
&\qquad - 18\log^2 x + 7\log x\big) + 2(2 - 54x + 69x^2 - 18x^3)\Big]\varepsilon^3 + \cdots\Big\}(-2\omega_1)^{4-6\varepsilon}, \qquad \text{(A.1)}
\end{aligned}
$$

$$
\begin{aligned}
M_2 &= \frac{(1-4\varepsilon)\Gamma^3(1-\varepsilon)\Gamma^2(1+2\varepsilon)\Gamma(1+6\varepsilon)}{36\varepsilon^2(1-2\varepsilon)^2(1-3\varepsilon)(2-3\varepsilon)(1-6\varepsilon)\Gamma(1+4\varepsilon)}\Big\{x^2 - \frac{1}{2}(1-x)^2(1+x^2) \\
&\quad - \frac{3}{2}(1-x)^2\Big[(1-x)(1+x)\log x + x\Big]\varepsilon \\
&\quad - \frac{3}{4}\Big[8(1-x)^3(1+x)L(x) + (1 - 2x - 2x^3 + x^4)\log^2 x \\
&\qquad - 2x(1-x)(1+x)\log x + 8x(1-x)^2\Big]\varepsilon^2 \\
&\quad + \frac{1}{4}\Big[96x^3(2-x)\,\mathrm{Li}_3(1-x) - 96(1-2x)\,\mathrm{Li}_3(1-x^{-1}) \\
&\qquad + 24(1 - 2x - 2x^3 + x^4)L(x)\log x + (1-x)^3(1+x)\log^3 x \\
&\qquad + 24x(1-x)(1+x)(L(x) + \log x) \\
&\qquad - 3x(1-x)^2\left(3\log^2 x + 32\right)\Big]\varepsilon^3 + \cdots\Big\}(-2\omega_1)^{4-6\varepsilon}x^{-3\varepsilon}, \qquad \text{(A.2)}
\end{aligned}
$$

$$
\begin{aligned}
M_2' &= -\frac{(1-4\varepsilon)\Gamma^3(1-\varepsilon)\Gamma^2(1+2\varepsilon)\Gamma(1+6\varepsilon)}{6\varepsilon^3(1-2\varepsilon)(1-3\varepsilon)(1-6\varepsilon)\Gamma(1+4\varepsilon)}\Big\{x + \frac{1}{2}(1-x)^2\varepsilon \\
&\quad + \frac{1}{2}\Big[3x\log^2 x + 3(1-x)(1+x)\log x + 4(1-x)^2\Big]\varepsilon^2 \\
&\quad + \frac{1}{4}\Big[48x\left(2\,\mathrm{Li}_3(1-x) + 2\,\mathrm{Li}_3(1-x^{-1}) - L(x)\log x\right) \\
&\qquad + 24(1-x)(1+x)(L(x) + \log x) + (1-x)^2\left(3\log^2 x + 32\right)\Big]\varepsilon^3 \\
&\quad + \cdots\Big\}(-2\omega_1)^{2-6\varepsilon}x^{-3\varepsilon}, \qquad \text{(A.3)}
\end{aligned}
$$

$$
\begin{aligned}
M_3 = {} & \frac{\Gamma^3(1-\varepsilon)\Gamma(1+6\varepsilon)}{36\varepsilon^3(1-2\varepsilon)^2(1-3\varepsilon)(2-3\varepsilon)(1-6\varepsilon)} \biggl\{ 6x - 3(1+13x+x^2)\varepsilon \\
& - \frac{1}{2}\Bigl[2x\left(9\log^2 x + 16\pi^2\right) + 18(1-x)(1+x)\log x - 9(1+x)^2\Bigr]\varepsilon^2 \\
& - \frac{1}{2}\Bigl[48x\left(12\,\mathrm{Li}_3(1-x) + 12\,\mathrm{Li}_3(1-x^{-1}) - 6L(x)\log x - 28\zeta(3) - 5\pi^2\right) \\
& \quad + 9(1-x)(1+x)\left(16L(x) - 3\log x\right) - 9(1+13x+x^2)\log^2 x \\
& \quad - 12(1-15x+x^2)\Bigr]\varepsilon^3 + \cdots \biggr\} (-2\omega_1)^{2-6\varepsilon} x^{-3\varepsilon}\,, \qquad \text{(A.4)}
\end{aligned}
$$

$$
\begin{aligned}
M_4 = {} & \frac{\Gamma^3(1-\varepsilon)\Gamma(1+6\varepsilon)}{24\varepsilon^3(1-2\varepsilon)^3(1-3\varepsilon)(1-6\varepsilon)} \biggl\{ x^2 - \frac{1}{2}x\Bigl[6x\log x - 1 + 18x + x^2\Bigr]\varepsilon \\
& + \frac{1}{6}\Bigl[6x^2\left(3\log^2 x - 2\pi^2\right) - 18x(1-9x-x^2)\log x + 2 - 45x + 96x^2 + 15x^3\Bigr]\varepsilon^2 \\
& - \Bigl[2x^2\left(24\,\mathrm{Li}_3(1-x) + 24\,\mathrm{Li}_3(1-x^{-1}) - 12L(x)\log x - 4\pi^2\log x - 60\zeta(3) - 9\pi^2\right) \\
& \quad + 12x(1-x)(1+x)L(x) - 3x(2-9x-2x^2)\log^2 x - 3x(5-18x-5x^2)\log x \\
& \quad + x(7+2x-x^2)\Bigr]\varepsilon^3 + \cdots \biggr\} (-2\omega_1)^{3-6\varepsilon}\,, \qquad \text{(A.5)}
\end{aligned}
$$

where

$$
L(x) = -L(x^{-1}) = \mathrm{Li}_2(1-x) + \frac{1}{4}\log^2 x\,.
$$

As it was mentioned above, all the master integrals are analytical in $x \in (0,+\infty)$, and hence the coefficients in the expansions (A.1)–(A.5) are analytical, too. It is easy to see that M_2, M_2', M_3 are symmetric with respect to $\omega_1 \leftrightarrow \omega_2$. The series (A.4), (A.5) at $\omega_1 = \omega_2$ coincide with the expansions of the single-scale integrals [18, 8, 9].

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