Theory and phenomenology of area-metric gravity

by

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This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of contributions

This thesis contains original work in form of published articles and results to be published.

Chapter 2 contains elements motivated by the articles [1, 2], co-authored with Bianca Dittrich and Pei-Ming Ho. Chapter 3 contains explicit material and generalised content from the articles [1, 3], co-authored with Bianca Dittrich and Kirill Krasnov. Chapter 4 contains results to be published [4, 5] in collaboration with Bianca Dittrich, Astrid Eichhorn, Breno Giacchini, Aaron Held and Marc Schiffer. In the main part of this thesis, these contents will be used without further citation.

The articles [6–14], authored or co-authored with Niayesh Afshordi, Bianca Dittrich, Aaron Held, Stefano Liberati, Alessia Platania, Dongxue Qu, Shouryya Ray, Marc Schiffer and Matt Visser, are not explictly part of this thesis.

Abstract

Area metrics are generalised geometric structures to describe spacetime. They feature additional non-metric degrees of freedom beyond the metric degrees of freedom of classical gravity at low energies. As such, area-metric gravity is a candidate effective field theory for the continuum limit of loop quantum gravity and spin foams which accounts for the extended gravitational configuration space of these approaches in their semiclassical regime.

On these grounds, following a bottom-up approach, we construct area-metric gravity perturbatively guided by the principle of general covariance. The most general local and diffeomorphism-invariant action quadratic in area-metric perturbations and of second order in derivatives contains four free parameters. These are the two masses of the right-handed and left-handed non-metric degrees of freedom, and the two interaction couplings between these and the metric degrees of freedom of the area metric. Linearised area-metric gravity violates parity for generic values of these parameters.

The effective metric actions obtained after integrating out the non-metric degrees of freedom are quasi-local linearised Einstein-Weyl actions. For special choices of couplings, the spin-2 propagator is ghostfree and exhibits only the pole associated with the massless graviton. The corresponding two-parameter subclass of linearised area-metric actions is characterised by a shift symmetry in the kinetic term. The physical spectrum of these theories consists of two massless transverse-traceless modes and five additional massive modes. The Hamiltonian dynamics mixes the two massless transverse-traceless modes in the linear polarisation basis as a result of parity violation in the original area-metric Lagrangian.

Extending the analysis of area-metric actions, we show that modified Plebanski theories provide a natural framework for non-linear area-metric gravity. In these theories a subset of the simplicity constraints on the bivector field in the original Plebanski action is replaced by a potential. This mechanism may be viewed as a continuum analogue of the weak imposition of second-class constraints in the spin-foam path integral. The Immirzi parameter γ, defined as the inverse coupling in front of the Holst action, and appearing in the commutator between second-class constraints in the quantum theory, is identified as a parity-violating coupling in area-metric gravity. Different from classical metric gravity, γ enters the dynamical equations of motion in area-metric gravity.

Based on these results, we proceed to analyse aspects concerning the phenomenological viability of area-metric gravity as a quantum and classical effective field theory.

Considering area-metric gravity as a local quantum effective field theory in a regime below the cutoff scale of a fundamental theory of quantum gravity, we evaluate its renormalisation-group flow towards the infrared regime. The non-metric degrees of freedom generically decouple as a result of their heavy masses at low energies. However, simultaneously growing interaction couplings between these and the metric degrees of freedom may leave an imprint in the low-energy effective action for the metric. In addition, parity violation at high energies is dynamically enhanced at low energies. The flow of the Immirzi parameter exhibits fixed points and zero and

infinite γ.

Finally, we consider non-linear quasi-local Einstein-Weyl gravity as a classical effective field theory for the metric degrees of freedom in area-metric gravity. After localising the action and restriction to static spherical symmetry, we show that solutions in the weak-field regime are characterised by an effective mass parameter which reduces to the mass of the spin-2 ghost in local Einstein-Weyl gravity when the non-locality in the original action is taken to zero. Additionally, we derive a regular Frobenius solution family at the radial center as the first step towards a future classification of Frobenius solutions around a generic expansion point.

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Dedication

To my parents.

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1 Introduction

One of the key questions in quantum gravity is

What are the fundamental gravitational degrees of freedom?

The tentative answer differs accross approaches to quantum gravity. Which degrees of freedom are considered as fundamental depends on the physical assumptions about the quantum nature of spacetime and the underlying mathematical framework.

In approaches to quantum gravity based on quantum field theory the fundamental gravitational degrees of freedom are encoded in the metric. Prominent examples are asymptotic safety [15–17], or quadratic [18, 19] and non-local [20, 21] quantum gravity. The concept of metric degrees of freedom as fundamental applies also to approaches based on discrete Regge geometry [22, 23]. Therein the gravitational degrees of freedom are encoded in geometric quantities of a spacetime triangulation. An example is quantum Regge calculus [24, 25], where the configuration variables are given by the edge lengths of the simplicial building blocks. Causal dynamical triangulations [26–28], on the other hand, consist of standardised simplices such that the degrees of freedom are fully determined by the prescription of how these are glued together to form the triangulation. In these approaches discretisation serves as a regularisation of the gravitational path integral and has to be removed in a continuum limit.

By contrast, there are approaches to quantum gravity in which the spacetime metric, and more generally the notion of metric degrees of freedom, are expected to emerge from more fundamental fields or geometric structures at the microscopic level. Such an emergent spacetime scenario can be understood and realised in many distinct ways. String theory and holography [29–32] are examples in which the physical spacetime metric is derived from more fundamental continuum physics at high energies. Several other approaches to quantum gravity postulate a fundamentally discrete nature of spacetime. An example is causal set theory [33, 34], which envisages to reconstruct continuum spacetime from sums over fundamental sets of causally ordered spacetime points. Another example is the canonical quantisation framework of loop quantum gravity [35–37]. In this approach, the quantum building blocks of spacetime arise from the dynamical evolution of quantum states of the geometry of space. The transition between two such states is described covariantly by spin-foam models [38–40]. The fundamental gravitational degrees of freedom in this framework can be associated with geometric data of a triangulation consisting of quantum four-simplices.

Spin foams are covariant path integrals for quantum gravity based on discrete quantum simplicial geometries. The underlying gravitational configuration space is larger than the configuration space of quantum Regge calculus [41–44]. This enlargement in the semiclassical regime is described by twisted simplicial geometries [45–48].

A twisted simplex is obtained by gluing five classical tetrahedra along pairs of shared triangles such that only the triangle areas are identified but their shapes may differ. This fuzzy gluing reflects the quantum uncertainty encoded in the commutator between geometric operators of a quantum tetrahedron and results in a shape mismatched simplex configuration. This configuration is characterised by a fundamental set of twenty geometric degrees of freedom. A natural choice are the ten triangle areas and ten dihedral angles in the twisted simplex [42, 43, 48]. This geometric data set can be mapped onto the algebraically independent components of a discrete area metric [49]. Thereby the gravitational degrees of freedom of spin foams in the semiclassical regime are encoded in an area metric.

An area metric [50–55] is a rank-4 tensor with the same algebraic symmetries as a curvature tensor,

$$G_{\mu\nu\rho\sigma} = G_{\rho\sigma\mu\nu} = -G_{\rho\sigma\nu\mu}\,, \tag{1.1}$$

$$G_{\mu[\nu\rho\sigma]} = 0\,. \tag{1.2}$$

Geometrically, such a tensor measures areas and dihedral angles between planes. Every length metric induces an area metric defined by

$$(G_g)_{\mu\nu\rho\sigma} = g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\,. \tag{1.3}$$

However, a generic area metric in four spacetime dimensions has twenty degrees of freedom and therefore ten more degrees of freedom than a length metric. Thus area metrics represent a finer geometric structure to describe quantum and classical spacetime. We will occasionally refer to a length metric simply as a *metric*, but will always refer to an area metric explicitly as *area metric*.

The discrete area metric associated to a twisted simplex σ reduces to one that is induced by a discrete length metric, only when the shapes of the triangles shared between pairs of tetrahedra in the simplex match,

$$G(\sigma) \;\;\rightarrow\;\; \text{shape matching} \;\;\rightarrow\;\; G_{g(\sigma)}\,. \tag{1.4}$$

This is illustrated in figure 1.1. In the limit of shape matching the twisted simplex becomes a classical simplex, and is formally completely characterised by its ten triangle areas or its ten edge lengths. These provide a geometric data set which can be mapped onto the degrees of freedom of a discrete length metric associated to the classical simplex.

Realising shape matching in the twisted simplex thus provides the microscopic mechanism by which the non-metric degrees of freedom of its intrinsic area metric freeze out and metric degrees of freedom emerge at the fundamental level. Formally, this limit can be understood as a reduction of the semiclassical configuration space of loop quantum gravity and spin foams, to the configuration space of quantum Regge calculus. Understanding how this reduction is

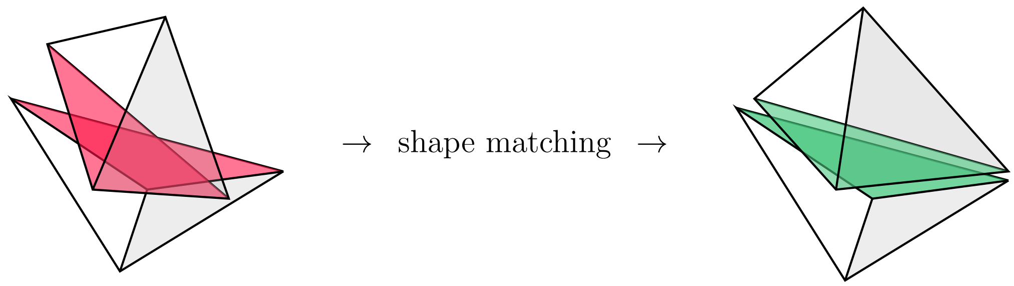

Figure 1.1: A twisted simplex consists of five classical tetrahedra, glued along pairs of shared triangles such that only the triangle areas are identified, but their shapes may differ. The resulting shape mismatched simplex configuration is characterised by twenty geometric degrees of freedom associated with a discrete area metric. In the limit of shape matching the twisted simplex becomes a classical simplex, and is completely characterised by ten geometric degrees of freedom associated with a discrete length metric. This limit represents a reduction of the extended semiclassical configuration space of loop quantum gravity and spin foams, to the configuration space of quantum Regge calculus.

realised dynamically and gives rise to the classical metric in a continuum limit, is one of the key challenges. This challenge entails ultimately matching the extended gravitational configuration space of spin foams with the degrees of freedom in an effective field theory.

A central component towards establishing the effective continuum dynamics of spin foams is the Area-Regge action [56, 57] which describes the dynamics of spin foams in the semiclassical regime [58–65]. For a triangulation consisting of four-simplices with triangles t this action is defined as

$$S_{\text{Area-Regge}}[a_t] = \sum_t a_t \epsilon_t(a_t)\,, \tag{1.5}$$

where the configuration variables are given by the triangle areas a_t and ϵ_t denotes a generalised deficit angle at a given triangle. The equations of motion derived from this action impose that classical configurations have vanishing generalised deficit angles [66–68],

$$\epsilon_t(a_t) = 0\,. \tag{1.6}$$

For a geometric understanding of the configurations described by this condition it is instructive to compare the Area-Regge action with the Length-Regge action [22, 23]. The latter is a discretisation of the Einstein action for general relativity defined in the same way as in (1.5), with configuration variables given by the edge lengths l_e of the simplices. In Length-Regge calculus the deficit angles $\epsilon_t(l_e)$ provide a measure for the curvature concentrated on triangles. A naive generalisation of this geometric interpretation suggests that the equations of motion of Area-Regge calculus are solved by flat configurations, and in particular do not admit curved solutions. [1]

The previous conclusion is, however, incorrect as the map $a_t(l_e)$ between areas and lengths in a triangulation is generically not invertible. As a result, the deficit angles in the Area-Regge action acquire a different geometric interpretation.

A single four-simplex can be described by the areas of its ten triangles, or equivalently by the lengths of its ten edges, such that the map between areas and lengths is in principle locally invertible [68]. However, in triangulations consisting of more than one four-simplex, a gluing of simplices entails identifying fewer area variables than length variables. Corresponding area-length constraints [42, 63–65] have to be implemented in the Area-Regge action to guarantee that solutions to the equations of motion reproduce solutions to the Length-Regge equations of motion. The geometric role of the area-length constraints is to ensure a gluing of simplices such that the shapes of triangle areas of the tetrahedra, shared between simplices in the triangulation, match. Without these constraints, the Area-Regge action describes the dynamics of more degrees of freedom than the Length-Regge action [68, 76, 77]. The geometric interpretation of the generalised deficit angles in Area-Regge calculus is a combined measure of curvature and shape mismatching of triangles,

$$\epsilon_t(a_t) \quad \leftrightarrow \quad \text{curvature} + \text{shape } \textit{mis}\text{matching}\,, \tag{1.7}$$

where the latter may be attributed to torsion [42–44, 76, 79].

The degrees of freedom of linearised Area-Regge calculus on a regular lattice can be identified with ten massless and ten massive modes [76, 77]. These can be arranged into a macroscopic area metric inherent to the lattice. The massless modes represent the metric degrees of freedom, whereas the massive modes represent the non-metric degrees of freedom of the area metric and are associated with shape mismatching. The Area-Regge action can thereby be interpreted as an action for an area metric,

$$S_{\text{Area-Regge}}[a_t] \;\rightarrow\; S[G] \quad \text{where} \quad G \;\leftrightarrow\; 10 \text{ massless d.o.f. } (g) + 10 \text{ massive d.o.f.}\,. \tag{1.8}$$

Integrating out the massive modes results in an effective action for the metric. In the lattice continuum limit this action reproduces the linearised Einstein-Hilbert action to leading order, and additionally features a subleading correction quadratic in the linearised Weyl tensor [77]. Schematically, the effective action can be written as

$$S_{\text{eff}}[g] = S_{\text{EH}}[g] - \int \text{Weyl}^2\,. \tag{1.9}$$

These are first indications that the continuum effective metric dynamics derived from spin foams reproduces general relativity with corrections originating from additional massive degrees

1. A similar issue lies at the core of the so-called flatness problem in spin foams [69–73], which asserts that spin-foam amplitudes in the semiclassical regime are peaked on flat configurations and suppress configurations with curvature. There is, however, compelling evidence against this scenario [63–65, 74–78].

of freedom contained in a more fundamental action for an area metric. These non-metric degrees of freedom in the Area-Regge action are absent when area-length constraints are imposed sharply to ensure that the areas arise from a consistent assignment of lengths in the triangulation and the equations of motion reproduce the dynamics of Length-Regge calculus.

However, in the quantum theory, the algebra of area-length constraints is second-class [42–44], with non-vanishing commutators proportional to the Immirzi parameter γ [80, 81],

$$\text{area-length constraints}: \quad [\cdot\,,\cdot] \;\sim\; \gamma\,. \tag{1.10}$$

As second-class constraints, the area-length constraints cannot be implemented sharply in a spin-foam path integral based on the Area-Regge action, as this would violate the quantum uncertainty principle. Implementing these constraints weakly leads to an extended gravitational configuration space parametrised by area metrics. The Immirzi parameter γ can be associated with fluctuations of the non-metric degrees of freedom in this area-metric configuration space.

The mechanism of weak imposition of second-class constraints in the spin-foam path integral can be mimicked in the classical continuum action through a potential for the non-metric degrees of freedom. Spin foams are based on the Plebanski formulation of general relativity [82–87]. Plebanski theory can be understood as a topological field theory, known as BF theory [38], supplemented by the so-called simplicity constraints, [1]

$$\text{Plebanski} \;=\; \text{BF} + \text{simplicity constraints}\,. \tag{1.11}$$

Spin-foam quantisation proceeds by discretising BF theory first, and subsequently implementing the simplicity constraints into the quantum theory. The second-class area-length constraints, which have to be implemented weakly in a path integral, can be associated with a subset of the simplicity constraints in the discretised simplicial phase space [41–44, 79]. This suggests that the continuum effective dynamics derived from the Area-Regge action [76, 77] can be reproduced by weakening an analogue subset of the simplicity constraints in the continuum theory. In practice, this amounts to imposing only a subset of the simplicity constraints in the Plebanski action sharply and replacing the rest by a potential,

$$\text{simplicity constraints} \quad\to\quad \text{subset (sharp)} + \text{potential (weak)}\,, \tag{1.12}$$

as in so-called modified Plebanski theories [88–103]. Subclasses of modified Plebanski theories thereby provide a natural framework for the continuum dynamics of an area metric.

1. The acronym BF stands for the bivector field B and the curvature $F(\omega)$ of the connection ω, in terms of which the BF action is defined. The simplicity constraints in the Plebanski action inherit their name by requiring that the bivector field be a simple bivector.

Thesis outline and results

The primary objective of this thesis is to analyse theoretical and phenomenological aspects of area-metric gravity as a candidate effective field theory for the continuum limit of loop quantum gravity and spin foams. However, the approach presented in this thesis is based on first principles and does not make assumptions about the quantum-gravitational origin of area metrics.

Chapter 2 introduces the concept of an area metric.

Section 2.1 defines the area-metric tensor and illustrates its geometric function as a measuring tool for areas and dihedral angles. Additionally, this section states the area metric induced by a length metric and the Gilkey decomposition of a generic area metric into a sum of induced area metrics. Section 2.2 demonstrates the counting of the degrees of freedom of an area metric in general spacetime dimensions. This number identifies area-metric backgrounds as generalised geometric structures to describe spacetime in four and higher dimensions. Moreover, this section provides physical and mathematical motivation for imposing the cyclicity condition on the area-metric tensor, which is assumed everywhere in this thesis. In section 2.3, we construct scalar densities from the area metric and thereby emphasise an infinite-dimensional freedom in the definition of a covariant area-metric volume element. These are combined aspects of the mathematical challenge of defining differential geometry on area-metric backgrounds.

Chapter 3 derives actions for area metrics.

In the absence of an established mathematical framework for area-metric differential geometry, in section 3.1, we follow a bottom-up approach and construct area-metric gravity perturbatively guided by the principle of general covariance. To that end, subsection 3.1.1 reviews the derivation of the linearised Einstein-Hilbert action as the unique local and diffeomorphism-invariant kinetic action for a spin-2 field. In subsection 3.1.2, we apply the analogue procedure to derive the most general local and diffeomorphism-invariant second-order action for area-metric perturbations around an area-metric background induced by the flat length metric. The eight-dimensional tensor basis forming the Lagrangian ansatz is derived in appendix section A.1, in two distinct ways. Appendix subsection A.1.1 derives the kinetic basis elements by making use of the tensor algebra package `xTras`, whereas appendix subsection A.1.2 makes use of the decomposition of the area-metric perturbation into SO(4) irreducible components and subsequently applies representation coupling theory. As the first main result, we find that diffeomorphism-invariant linearised area-metric gravity is characterised by four free parameters. These are the two interaction couplings between the length-metric degrees of freedom, and the selfdual and anti-selfdual Weyl components representing the non-metric degrees of freedom of the area-metric perturbation. The other two parameters are the masses of the non-metric degrees of freedom. The area-metric Lagrangian violates parity symmetry for generic values of these parameters. In subsection 3.1.3, we proceed to derive effective actions for the length-metric perturbation by integrating out the non-metric degrees of freedom from the area-metric action. These effective actions are identified as a quasi-local modification of linearised Einstein-Weyl gravity. Special choices of couplings are discussed, for which the spin-2 propagator features less than the general number three of poles. In particular, a two-parameter subclass of area-metric actions is identified, for which the propagator does not exhibit additional poles beyond the massless graviton pole. As a result, these effective metric actions are ghostfree in an expansion around flat Minkowski background. The underlying area-metric actions are characterised by a 5-parameter shift symmetry in the kinetic term which is identified explicitly in subsection 3.1.4. An analysis of the linearised equations of motion derived from such types of actions for the length-metric perturbation and for the non-metric fields, embedded non-locally into the space of symmetric transverse-traceless tensors, is used to provide a first indication that the physical spectrum of these theories consists of two massless transverse-traceless modes and five additional massive modes. This conclusion is confirmed through a Hamiltonian analysis based on the local fields in appendix section A.2. To that end, appendix subsection A.2.1 reviews the Hamiltonian formulation of linearised general relativity. In appendix subsections A.2.2 and A.2.3, we perform the $3+1$ decomposition of the area-metric Lagrangian and derive the Hamiltonian reduced by the second-class constraints. The Hamiltonian equations of motion are stated in appendix subsection A.2.4. To solve these, we apply a mode decomposition of symmetric spatial tensors into transverse-traceless and longitudinal-traceless modes, as well as a trace mode. The dynamical equations are solved in appendix subsections A.2.5 and A.2.6, separately in the circular polarisation basis and in the linear polarisation basis. Independent of the choice of basis, we find that the spectrum of energy eigenvalues describes two massless and two massive transverse-traceless propagating modes, as well as three longitudinal-traceless massive propagating modes. The classical dynamics to the considered order is stable. Additionally, we observe that the dynamics of the massless transverse-traceless modes, associated with the shifted spatial metric subject to the Hamiltonian and diffeomorphism constraints, exhibits a mixing between the $+$ and $\times$ modes in the linear polarisation basis. This mixing effect is a manifestation of parity violation in the area-metric Lagrangian. This concludes our analysis of linearised area-metric gravity.

In section 3.2, we derive implicit non-linear area-metric actions. Concretely, we show that subclasses of modified non-chiral Plebanski theories provide a non-perturbative framework for area metrics. To that end, subsection 3.2.1 reviews the non-chiral Plebanski action and emphasises the role of the simplicity constraints in recovering general relativity. Subsequently, we introduce modified non-chiral Plebanski theories, in which all or a subset of the simplicity constraints on the bivector field is replaced by a potential. We supplement the action by the Holst term with coupling given by the inverse Immirzi parameter γ. In subsection 3.2.2, we identify a subset of ten simplicity constraints to be imposed on the bivector field in the non-chiral Plebanski action, in order to reduce its thirty gauge-invariant degrees of freedom to twenty as required for a generic area metric. This identification makes use of the right-left handed splitting of the Lie algebra $\mathrm{so}(4) = \mathrm{su}(2)_+ \oplus \mathrm{su}(2)_-$, and the subsequent parametrisation of each $\mathrm{SU}(2)_\pm$ bivector field in terms of a spacetime tetrad and a unimodular internal frame field. As the main input, we impose that the right-handed and left-handed length-metric geometries defined by the two spacetime tetrads, coincide. Thereby, after the connection has been integrated out from the action, we arrive at an implicit action for an area metric. The degrees of freedom of this area metric are encoded in a length metric and two unimodular internal metrics, each of which represents a set of five spacetime scalars. A potential in the form of mass terms for these fields is added to the action. In subsection 3.2.3, we invert the relation between these fields and the length metric on the one side, and the area metric on the other side, to linear order in area-metric perturbations around a background induced by the flat length metric. To that end, we introduce projectors onto the length-metric and non-metric components of the area-metric perturbation. The linearised area-metric Lagrangian exhibits a 5-parameter shift symmetry in the kinetic term. The two interaction couplings between the length-metric perturbation and the non-metric fields involve the Immirzi parameter γ, which is thereby identified as a parity-violating coupling in area-metric gravity. For identical mass parameters of the non-metric degrees of freedom, the effective quasi-local Einstein-Weyl action for the length-metric fluctuations features a ghostfree spin-2 propagator with no additional poles beyond the massless graviton pole.

Chapter 4 analyses phenomenological aspects of area-metric gravity.

In section 4.1, area-metric gravity is treated at a local quantum effective field theory. To that end, we assume an area-metric description of spacetime at an ultraviolet (UV) cutoff scale Λ_{UV}, below the scale of a fundamental theory of quantum gravity. Therefrom, we analyse the renormalisation-group (RG) flow of area-metric gravity towards the infrared (IR) regime of low energies. Subsection 4.1.1 introduces the relevant concepts from the functional RG, such as the flow equation for the effective average action and the beta functions of the dimensionless couplings. In the presence of quantum fluctuations, the latter acquire additional contributions beyond the term determined by their canonical mass dimension. The sign of these non-canonical terms allows us to determine if a given coupling is driven to large or small values by the RG flow towards the IR. Starting from the ansatz for the effective average action defined in subsection 4.1.2, we analyse the RG flow of the masses of the non-metric degrees of freedom, and of the interaction couplings between these and the length-metric degrees of freedom in area-metric gravity. Concretely, we focus on the RG flow induced by gravitational length-metric fluctuations, and for simplicity consider only momentum-independent three-point vertices. The corresponding scalar invariants are derived in appendix section A.3, using SO(4) coupling representation theory. The structural form of the area-metric propagator is illustrated in appendix section A.4. On these grounds, in subsection 4.1.3, we analyse the mechanism of decoupling of the non-metric degrees of freedom as a phenomenological viability constraint on area-metric gravity. As a result, we find that decoupling due to large-growing masses, which freeze the dynamics of the non-metric degrees of freedom, is a generic phenomenon. However, we also find that the interaction couplings between these fields and the length metric are driven towards large values by the RG flow. This may result in a residual effect of the non-metric degrees of freedom in the low-energy effective action for the length metric. An analogous observation applies to the RG flow of parity-violating couplings in area-metric gravity, which is analysed in subsection 4.1.4. We find that any departure from the parity-symmetric subspace of area-metric gravity in the UV is dynamically enhanced towards the IR. The RG flow of some of the vertex couplings may result in an IR Landau pole. Therefore, parity symmetry of the low-energy theory does not emerge under the RG flow, and requires a fine-tuning of exactly parity-symmetric initial conditions in the UV. As a special parity-violating coupling in area-metric gravity, we analyse the RG flow of the Immirzi parameter γ in subsection 4.1.5. Its beta function exhibits fixed points at zero and infinite γ. At the former fixed point, γ is always marginally irrelevant. These results suggest a scenario in which an RG trajectory starts at exact parity symmetry in the UV, and ends with maximal parity violation in the IR. This concludes our analysis of area-metric gravity as a quantum effective field theory.

In section 4.2, we consider quasi-local Einstein-Weyl gravity as a classical effective field theory for the length-metric degrees of freedom in area-metric gravity. Subsection 4.2.1 states the original quasi-local action, whereas subsection 4.2.2 derives the covariant equations of motion from the action localised by means of an additional tensor field with Weyl symmetries. In subsection 4.2.3, we introduce a static spherically symmetric ansatz for the metric and the additional field. The symmetries of this ansatz reduce the equations of motion down to three algebraically independent, coupled second-order non-linear differential equations which are stated in appendix section A.5. The general solution to these equations in the weak-field regime is derived in subsection 4.2.4. Asymptotically flat solutions are described by two free parameters apart from a global time-rescaling. One of these parameters is the Arnowitt-Deser-Misner mass, and the other one a charge which mediates a Yukawa interaction. The falloff of the Yukawa terms is controlled by an effective mass defined as a combination of parameters appearing in the quasi-local Weyl-squared term in the action. In the limit in which the parameter in front of the covariant Laplacian in the original action is set to zero, this effective mass parameter coincides with the mass of the spin-2 ghost in local Einstein-Weyl gravity. Finally, in subsection 4.2.5, we derive a regular Frobenius solution family at the radial center. In addition, we observe that the singular Frobenius solution families of local Einstein-Weyl gravity are removed. These results provide first insights into the effects stemming from a covariant d'Alembert operator in combination with a mass term, in the form of an inverse operator, in an Einstein-Weyl action. Additionally, they set the stage for a numerical construction of regular solutions through the application of shooting methods.

Chapter 5 concludes with a dicussion of future directions in area-metric gravity.

2 Area metrics

2.1 Definition

This section introduces the concept of an area metric by analogy with the notion of a length metric [104, 105]. A length metric g at a point $p \in M$ on a d-dimensional manifold M is a symmetric non-degenerate tensor of rank $(0, 2)$, which defines an inner product on the tangent space T_pM. In other words, the bilinear map

$$g \,:\, T_pM \times T_pM \to \mathbb{R}\,, \quad (v_1, v_2) \mapsto g(v_1, v_2) \tag{2.1}$$

satisfies $g(v_1, v_2) = g(v_2, v_1)$ for all $v_1, v_2 \in T_pM$, and $g(v, v_1) = 0$ for all $v_1 \in T_pM$ if and only if $v = 0$. In Euclidean signature this inner product is required to be positive-definite. In this case, a length metric assigns a norm associated with a squared length $l_v^2 \equiv g(v, v)$ to each vector $v \in T_pM$, and a two-dimensional angle between two vectors $v_1, v_2 \in T_pM$ through $g(v_1, v_2) = l_{v_1} l_{v_2} \cos(\angle_{2d} v_1, v_2)$. In a tangent basis $\{v_\mu \,|\, \mu = 0, ..., d-1\}$, the conditions of symmetry and non-degeneracy can be phrased as conditions on the matrix $(g_{\mu\nu})$ of length-metric components $g_{\mu\nu} = g(v_\mu, v_\nu)$,

$$\begin{aligned} \text{symmetry} \,:&\quad g_{\mu\nu} = g_{\nu\mu}\,, \\ \text{non-degeneracy} \,:&\quad \det(g_{\mu\nu}) \neq 0\,. \end{aligned}$$

A length metric in d dimensions has

$$\frac{d(d+1)}{2} \tag{2.2}$$

algebraically independent components. These can be deduced entirely from the squared lengths of the basis vectors v_μ, and the squared lengths of pairwise distinct basis vectors $v_{\mu\nu} \equiv v_\mu + v_\nu$ labeled by $\mu \neq \nu$. These are given by

$$\begin{aligned} l_{v_\mu}^2 &= g(v_\mu, v_\mu) = g_{\mu\mu}\,, \\ l_{v_{\mu\nu}}^2 &= g(v_\mu + v_\nu, v_\mu + v_\nu) = l_{v_\mu}^2 + l_{v_\nu}^2 + 2g_{\mu\nu}\,. \end{aligned}$$

The last equality follows from bilinearity of the map (2.1). Measuring the squared lengths of all basis vectors determines the diagonal components of the metric. Therefrom, by measuring the

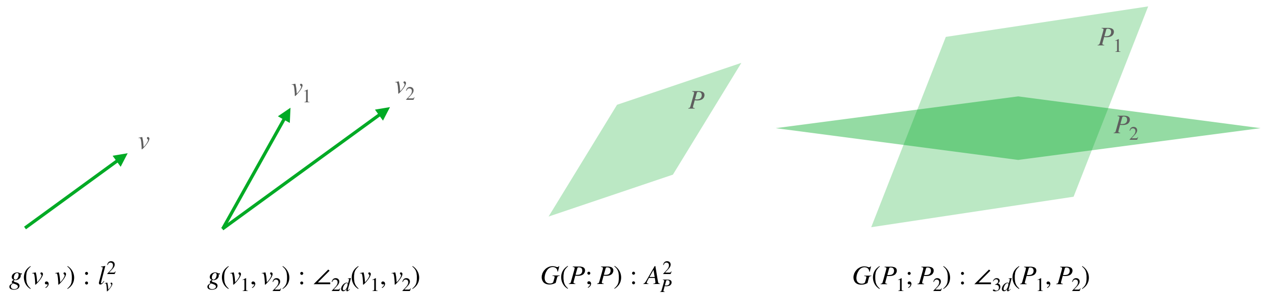

Figure 2.1: A length metric g associates a squared length l_v^2 to a vector v, and a two-dimensional angle $\angle_{2d}(v_1, v_2)$ between two vectors v_1 and v_2. An area metric G associates a squared area A_P^2 to a parallelogram P, and a three-dimensional dihedral angle $\angle_{3d}(P_1, P_2)$ beween two intersecting planes P_1 and P_2.

squared lengths of sums of pairwise distinct basis vectors, the off-diagonal components can be inferred. It is in particular not necessary to measure angles in order to reconstruct the entire metric. Figure 2.1 illustrates the geometric quantities computed by a length metric.

The previous notions allow us to introduce the definition of an area metric [50–55] by analogy with the one of a length metric. An area metric G at a point $p \in M$ on a d-dimensional manifold M is a non-degenerate tensor of rank $(0, 4)$ with the same symmetries as an algebraic curvature tensor. In other words, the multilinear map

$$G \,:\, T_pM \times T_pM \times T_pM \times T_pM \to \mathbb{R}\,, \quad (v_1, v_2, v_3, v_4) \mapsto G(v_1, v_2, v_3, v_4) \tag{2.3}$$

obeys the exchange symmetries

$$G(v_1, v_2, v_3, v_4) = G(v_3, v_4, v_1, v_2) = -G(v_3, v_4, v_2, v_1)\,, \tag{2.4}$$

and the cyclicity condition, or algebraic Bianchi identity,

$$G(v, v_1, v_2, v_3) + G(v, v_2, v_3, v_1) + G(v, v_3, v_1, v_2) = 0\,, \tag{2.5}$$

for all $v_1, v_2, v_3, v_4, v \in T_pM$. We will occasionally refer to a tensor satisfying the symmetries (2.4), but not the cyclicity condition (2.5), as an acyclic area metric [51–54].

The antisymmetry within the first and second pair of entries, and symmetry under the exchange of these, imply that an area metric G provides a symmetric bilinear form on the space $\Lambda^2 T_pM$ of bivectors,

$$G \,:\, \Lambda^2 T_pM \times \Lambda^2 T_pM \to \mathbb{R}\,, \quad (B_1, B_2) \mapsto G(B_1; B_2)\,, \tag{2.6}$$

with the definition

$$G(v_1 \wedge v_2; w_1 \wedge w_2) \equiv G(v_1, v_2, w_1, w_2)\,, \tag{2.7}$$

and extension by linearity. [1] The condition of non-degeneracy can be formulated analogously as for a length metric. It requires the existence of a map $\hat{G}^{-1} : \Lambda^2 T_p^* M \to \Lambda^2 T_p M$, inverse to $\hat{G} : \Lambda^2 T_p M \to \Lambda^2 T_p^* M$ and defined by $\hat{G}(B_1)(B_2) \equiv G(B_1; B_2)$, such that $\hat{G}^{-1} \circ \hat{G} = \mathrm{id}_{\Lambda^2 T_p M}$.

A generic bivector can be decomposed into a finite sum of simple bivectors. A bivector $P \in \Lambda^2 T_p M$ is called simple if it can be written as the wedge product of two vectors. This condition is equivalent to the quadratic condition $P \wedge P = 0$. The set of simple bivectors

$$A^2 T_p M = \left\{ P \in \Lambda^2 T_p M \,|\, P \wedge P = 0 \right\} \tag{2.8}$$

is hence no longer a vector space, and instead defines a variety embedded in $\Lambda^2 T_p M$. In other words, $A^2 T_p M$ is a polynomial subset of the vector space $\Lambda^2 T_p M$, which can be identified with the space of oriented areas over $T_p M$. These are obtained from the vector space of parallelograms $T_p M \oplus T_p M$, by defining an equivalence relation which identifies two parallelograms if they can be transformed into each other by an $\mathrm{SL}(2,\mathbb{R})$ transformation. Given a parallelogram (v_1, v_2) spanned by two vectors v_1 and v_2 and describing an oriented area $v_1 \wedge v_2$, the parallelogram (ω_1, ω_2) spanned by the two vectors $\omega_1 = a v_1 + b v_2$ and $\omega_2 = c v_1 + d v_2$ is said to be equivalent to (v_1, v_2) if it describes the same oriented area, such that $w_1 \wedge w_2 = v_1 \wedge v_2$. This is the case if $ad - bc = 1$ holds. The area metric does not distinguish between two such parallelograms and returns the same norm for these,

$$G(w_1 \wedge w_2; w_1 \wedge w_2) = (ad - bc)^2 G(v_1 \wedge v_2; v_1 \wedge v_2)\,. \tag{2.9}$$

This is one of the properties which motivates calling G an area metric. Another one follows from the observation that any length metric g induces an area metric G_g defined by

$$G_g(v_1, v_2, v_3, v_4) \equiv g(v_1, v_3) g(v_2, v_4) - g(v_1, v_4) g(v_2, v_3)\,. \tag{2.10}$$

The norm of a simple bivector $P = v_1 \wedge v_2 \in \Lambda^2 T_p M$, computed by the induced area metric G_g, coincides with the squared area A_P^2 of the parallelogram (v_1, v_2) spanned by the two vectors v_1 and v_2, and measured in the underlying Euclidean metric geometry,

$$\begin{aligned} G_g(P;P) &\equiv G_g(v_1, v_2, v_1, v_2) = l_{v_1}^2 l_{v_2}^2 \left(1 - \cos^2(\angle_{2d} v_1, v_2)\right) \\ &= \left(l_{v_1} l_{v_2} \sin(\angle_{2d} v_1, v_2)\right)^2 = A_P^2\,. \end{aligned} \tag{2.11}$$

Furthermore, G_g computes a three-dimensional dihedral angle between two planes spanned by two simple bivectors $P_1 = v_1 \wedge v$ and $P_2 = v \wedge v_2$, which intersect along the line defined by the

1. This definition absorbs a factor of $\frac{1}{4}$ arising from the convention for the wedge product, $v_1 \wedge v_2 \equiv \frac{1}{2!}(v_1 \otimes v_2 - v_2 \otimes v_1)$.

vector v, [1]

$$G_g(P_1;P_2) \equiv G_g(v_1,v,v_2,v) = l_{v_1}l_{v_2}l_v^2\left(\cos(\angle_{2d}v_1,v_2) - \prod_{i=1,2}\cos(\angle_{2d}v_i,v)\right) = A_{P_1}A_{P_2}\cos(\angle_{3d}P_1,P_2)\,. \tag{2.13}$$

The previous properties motivate calling G an area metric, even if it is not induced by a length metric. Figure 2.1 illustrates the geometric quantities computed by an area metric.

In the following, area metrics will be denoted in the tensor index notation. Given a tangent basis $\{v_\mu \,|\, \mu = 0,\ldots,d-1\}$, let $G_{\mu\nu\rho\sigma} = G(v_\mu,v_\nu,v_\rho,v_\sigma)$ denote the components of the area metric in this basis. The exchange symmetries (2.4) and cyclicity condition (2.5) are

$$G_{\mu\nu\rho\sigma} = G_{\rho\sigma\mu\nu} = -G_{\rho\sigma\nu\mu} \qquad \text{and} \qquad G_{\mu\alpha\beta\gamma} + G_{\mu\beta\gamma\alpha} + G_{\mu\gamma\alpha\beta} = 0\,. \tag{2.14}$$

The cyclicity condition is equivalent to the statement that the total antisymmetrisation over the last three indices vanishes, $G_{\mu[\alpha\beta\gamma]} = 0$. [2]

The inverse area metric is denoted with upper indices $G^{\mu\nu\rho\sigma}$ and defined by the condition

$$G_{\mu\nu\alpha\beta}G^{\alpha\beta\rho\sigma} = \frac{1}{2}\left(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\sigma_\mu\delta^\rho_\nu\right) \equiv \delta_{\mu\nu}{}^{\rho\sigma}\,, \tag{2.16}$$

with the Kronecker delta δ^μ_ν being one if $\mu=\nu$ and zero otherwise. The expression involving the generalised delta on the right hand side represents the identity on the space of bivectors. With the previous definition, G and G^{-1} can be used to raise and lower bivector indices.

The area metric G_g induced by a length metric g is given by

$$(G_g)_{\mu\nu\rho\sigma} = g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\,. \tag{2.17}$$

As an algebraic curvature map, a general area metric G admits a Gilkey decomposition into a

1. Consider three faces of a polyhedron, which share a vertex (i) and contain edges (ai), (bi) and (ci). The cosine of the three-dimensional dihedral angle between the faces $P_1 = (aic)$ and $P_2 = (bic)$ can be expressed in terms of two-dimensional angles as [41]

$$\cos(\angle_{3d}P_1,P_2) = \frac{\cos(\angle_{2d}aib) - \cos(\angle_{2d}aic)\cos(\angle_{2d}bic))}{\sin(\angle_{2d}aic)\sin(\angle_{2d}bic)}\,. \tag{2.12}$$

2. In $d=4$ spacetime dimensions, after using the first set of symmetries in (2.14), the cyclicity condition on G becomes equivalent to requiring this tensor to have no totally antisymmetric component. Thus, the cyclicity condition in $d=4$ can be expressed as

$$G_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = 0\,. \tag{2.15}$$

Here $\epsilon^{\mu\nu\rho\sigma}$ denotes the totally antisymmetric four-dimensional Levi-Civita symbol, normalised to $\tilde{\epsilon}^{0123} = +1$ and with tensor-density weight $+1$.

finite sum of area metrics G_{g^I} induced by length metrics g^I, in the form [51, 106, 107]

$$G = \sum_{I=1}^{N} \sigma_I G_{g^I} \qquad \text{where} \qquad \sigma_I = \pm 1\,. \tag{2.18}$$

The number N of metrics is bounded from above by [50, 108]

$$N_{\max} = \frac{d(d+1)}{2}\,, \tag{2.19}$$

where d denotes the spacetime dimension. A Gilkey decomposition of the area metric is non-unique, and a constructive algorithm to achieve such a decomposition is not known.

2.2 Degrees of freedom and cyclicity

In Petrov notation, an area metric can be associated with a $D \times D$ symmetric matrix (G_{AB}), where $A, B, ... = [\mu\nu]$ are D-dimensional indices, and $D = \binom{d}{2} = \frac{d(d-1)}{2}$ is the dimension of the space of bivectors $\Lambda^2 T_p M$. [1] The condition of non-degeneracy of G translates as

$$\text{non-degeneracy} \; : \; \det(G_{AB}) \neq 0\,.$$

An area metric in d dimensions has

$$\frac{D(D+1)}{2} \tag{2.20}$$

algebraically independent components prior to imposing the cyclicity condition (2.5). This amounts to 1, 6, and 21 degrees of freedom for an acyclic area metric in $d = 2$, 3, and 4 spacetime dimensions. The cyclicity condition (2.5) represents a non-trivial identity only in $d \geq 4$ dimensions, whereas an area metric in $d < 4$ dimensions is automatically cyclic as a result of the exchange symmetries (2.4). There are no area metrics in $d = 1$ dimensions, as the symmetries (2.4) would set any component of G to zero. In $d = 2$ dimensions, an area metric has only one independent component and defines a coarser geometric structure than a length metric. This can be understood intuitively by the fact that the area of a triangle is not sufficient to determine its shape. In $d = 3$ dimensions, an area metric has the same number 6 of independent components as a length metric. In this case, it can be shown that every area metric is induced by a length metric [54]. However, in $d \geq 4$ dimensions, an area metric contains more degrees of freedom than a length metric, and cyclicity is not implied by the index exchange

1. Applying the summation convention over repeated indices, a factor of $\frac{1}{2}$ has to be taken into account for each index contraction when translating between matrix and tensor expressions. For example, $G_{AC}G^{CB} \hat{=} \frac{1}{2} G_{\mu\nu\alpha\beta} G^{\alpha\beta\rho\sigma}$, because the contraction of a Petrov index C corresponds to a sum over ordered antisymmetric pairs of tangent-space indices, whereas the double contraction of a pair of antisymmetric tensor indices $[\alpha\beta]$ corresponds to an unordered sum, see e.g. [54].

symmetries of G. The number of degrees of freedom encoded in a cyclic area metric follows from (2.20) by subtracting the number $\binom{d}{4}$ of conditions represented by equation (2.4). This results in

$$\frac{1}{12}d^2(d-1)(d+1) \tag{2.21}$$

algebraically independent degrees of freedom for a cyclic area metric in d spacetime dimensions. This is the same number as the number of independent components of the Riemann curvature tensor, or more generally any algebraic curvature map in d dimensions. The Gilkey decomposition theorem (2.18) illustrates that area-metric backgrounds in $d \geq 4$ can be viewed as multi-metric backgrounds, where a priori none of the metrics is distinguished.

In the remainder of this section, we will provide physical and mathematical motivation for the cyclicity condition (2.5) as part of the definition of an area metric. To that end, consider the basis of tangent vectors $\{v_\mu \,|\, \mu = 0, ..., d-1\}$, in which the components of the area metric are denoted by $G_{\mu\nu\rho\sigma} = G(v_\mu, v_\nu, v_\rho, v_\sigma)$. The algebraically independent components of an area metric satisfying the cyclicity condition (2.5) can be deduced entirely from the squared areas of parallelograms spanned by basis vectors and by sums of basis vectors. Let us define the parallelograms

$$\begin{aligned} P_{\mu\nu} &= v_\mu \wedge v_\nu \,, && (2.22)\\ P_{\mu\nu\rho} &= v_\mu \wedge (v_\nu + v_\rho) = P_{\mu\nu} + P_{\mu\rho} \,, && (2.23)\\ P_{\mu\nu\rho\sigma} &= (v_\mu + v_\nu) \wedge (v_\rho + v_\sigma) = P_{\mu\rho\sigma} + P_{\nu\rho\sigma} \,, && (2.24) \end{aligned}$$

for distinct μ, ν, ρ and σ. Their squared areas computed by the area metric G can be expressed as

$$\begin{aligned} A^2_{P_{\mu\nu}} &\equiv G(P_{\mu\nu}; P_{\mu\nu}) = G_{\mu\nu\mu\nu} \,, && (2.25)\\ A^2_{P_{\mu\nu\rho}} &\equiv G(P_{\mu\nu\rho}; P_{\mu\nu\rho}) = A^2_{P_{\mu\nu}} + A^2_{P_{\mu\rho}} + 2G_{\mu\nu\mu\rho} \,, && (2.26)\\ A^2_{P_{\mu\nu\rho\sigma}} &\equiv G(P_{\mu\nu\rho\sigma}; P_{\mu\nu\rho\sigma}) = A^2_{P_{\mu\rho\sigma}} + A^2_{P_{\nu\rho\sigma}} + A^2_{P_{\rho\mu\nu}} + A^2_{P_{\sigma\mu\nu}} \\ &- A^2_{P_{\rho\mu}} - A^2_{P_{\rho\nu}} - A^2_{P_{\sigma\mu}} - A^2_{P_{\sigma\nu}} + 2(G_{\mu\rho\nu\sigma} + G_{\mu\sigma\nu\rho}) \,. && (2.27) \end{aligned}$$

Measuring the areas of the parallelograms (2.22)–(2.24) only allows us to determine the tensor

$$\tilde{G}_{\mu\nu\rho\sigma} \equiv G_{\mu\rho\nu\sigma} + G_{\mu\sigma\nu\rho} \,. \tag{2.28}$$

It is straightforward to verify that $\tilde{G}$ defines a cyclic area metric, whose non-vanishing components with one or two equal indices are related to those of the original area metric G by $\tilde{G}_{\mu\nu\mu\rho} = -G_{\mu\nu\mu\rho}$ and $\tilde{G}_{\mu\nu\mu\nu} = -G_{\mu\nu\mu\nu}$. However, it is not possible to reconstruct the full information encoded in the mixed components of the original area metric, unless G is cyclic. Specifically, the left hand side of the second equation in (2.14) can not be expressed entirely in terms of $\tilde{G}$. The cyclicity condition sets this expression to zero such that the area metric G can be recovered from $\tilde{G}$.

Another motivation for imposing cyclicity of the area metric follows from the observation that only the tensor $\tilde{G}_g$ in (2.28), for an induced area metric G_g, is necessary to define the area of a two-dimensional surface Σ embedded in a d-dimensional spacetime equipped with a metric g [55]. To see this, let $x^\mu(\xi)$ with $\mu = 0, ..., d-1$ denote coordinates on spacetime and $\xi^i = (\xi^0, \xi^1)$ intrinsic coordinates on Σ. The induced metric h_{ij} on Σ is given by

$$h_{ij}(\xi) = g_{\mu\nu}(x)\partial_i x^\mu(\xi)\partial_j x^\nu(\xi)\,. \tag{2.29}$$

Its determinant gives rise to the infinitesimal area element $\mathrm{d}A$ on Σ, from which the area of Σ is defined as

$$\mathrm{Area}(\Sigma) \equiv \int_\Sigma \mathrm{d}A = \int_\Sigma \mathrm{d}^2\xi \sqrt{|\det(h)|}\,. \tag{2.30}$$

The determinant of the matrix (h_{ij}) can be expressed in terms of the two-dimensional Levi-Civita symbol $\tilde{\epsilon}^{ij}$ by

$$\det(h) = \frac{1}{2!}\tilde{\epsilon}^{ij}\tilde{\epsilon}^{kl}h_{ik}h_{jl} = \frac{1}{4}\tilde{\epsilon}^{ij}\tilde{\epsilon}^{kl}\partial_i x^\mu \partial_j x^\nu \partial_k x^\rho \partial_l x^\sigma (G_g)_{\mu\nu\rho\sigma} = \frac{1}{2}\dot{x}^\mu \dot{x}^\nu x'^\rho x'^\sigma \left(\tilde{G}_g\right)_{\mu\nu\rho\sigma}, \tag{2.31}$$

where primes and dots denote partial derivatives with respect to ξ^0 and ξ^1. Equation (2.31) illustrates that only the tensor $\tilde{G}_g$ is necessary to define the area of Σ according to (2.30). One may object that this is little of a motivation for cyclicity, as a metric-induced area metric G_g is already cyclic by definition. However, promoting G_g to a generic area metric G in (2.31) defines the area of Σ with respect to a general, area-metric background instead of a length-metric background. In this case, (2.31) illustrates that the definition of $\mathrm{Area}(\Sigma)$ in (2.30) requires only the tensor $\tilde{G}$ in (2.28). This observation is of particular relevance for area metrics as generalised backgrounds for the target space of the worldsheet of a string, in which case (2.30) defines a generalised Nambu-Goto action.

2.3 Scalar densities

In this section we construct scalar densities from the area metric, and therefrom area-metric volume elements. To that end, we remind the reader that the canonical volume element associated with a length metric in a local coordinate basis is given by

$$\mathrm{d}\mathcal{V}_g = |\det(g)|^{\frac{1}{2}}\,\mathrm{d}^d x\;, \tag{2.32}$$

where the determinant of the $d \times d$ metric $(g_{\mu\nu})$ can be expressed as

$$\det(g) = \frac{1}{d!}\epsilon^{\mu_0\dots\mu_{d-1}}\epsilon^{\nu_0\dots\nu_{d-1}}g_{\mu_0\nu_0}\dots g_{\mu_{d-1}\nu_{d-1}}\,. \tag{2.33}$$

The metric $g_{\mu\nu}$ is a tensor density of weight zero, whereas $\epsilon^{\mu_0\cdots\mu_{d-1}}$ is a tensor density of weight $+1$. Therefore $\det(g)$ is a scalar density of weight $+2$. Taking the square root implies that $\mathrm{d}\mathcal{V}_g$ is a scalar density of weight $+1$, and thereby provides a generally covariant volume element for a length-metric action, which is unique up to a scalar multiple. Intuitively, this statement can be rephrased as the observation that the symmetric rank-2 metric tensor does not allow for building any non-trivial contractions with itself or with multiples of the Levi-Civita density, apart from the determinant of its associated $d \times d$ matrix. This does not apply to the rank-4 area-metric tensor, as we will see next.

An area metric G in d dimensions gives rise to a canonical volume element defined by [50, 51, 53–55]

$$\mathrm{d}\mathcal{V}_G^{(0)} = |\det(G)|^{\frac{1}{2d-2}}\,\mathrm{d}^d x\ , \tag{2.34}$$

where $\det(G)$ denotes the determinant of the $D \times D$ symmetric Petrov matrix (G_{AB}). The quantity $\mathrm{d}\mathcal{V}_G^{(0)}$ can be identified as a scalar density of weight $+1$ by relating the determinant of the Petrov matrix of an induced area metric G_g to the determinant of the corresponding metric g, via [51, 53, 54]

$$\det(G_g) = \det(g)^{d-1}\,. \tag{2.35}$$

The metric determinant is a scalar density of weight $+2$, such that the area-metric determinant transforms as a scalar density of weight $2d-2$. Therefore $\mathrm{d}\mathcal{V}_G^{(0)}$ in (2.34) is a scalar density of weight $+1$, and provides a generally covariant volume element for an area-metric action.

In the following, we will focus on $d = 4$ spacetime dimensions and illustrate that there are different and inequivalent ways of defining an area-metric volume element. Moreover, we will derive a formula for the determinant of the 6×6 Petrov matrix (G_{AB}) of a general area metric G, in terms of four-dimensional Levi-Civita densities.

Let us introduce contractions of n powers of an area metric $G_{\mu\nu\rho\sigma}$ in $d = 4$ dimensions, with n powers of the Levi-Civita density $\epsilon^{\mu\nu\rho\sigma}$, defined by

$$I_n(G) \equiv 2^{-2n}\, G_{\alpha_1\alpha_2\alpha_3\alpha_4}\cdots G_{\alpha_{4n-3}\alpha_{4n-2}\alpha_{4n-1}\alpha_{4n}}\epsilon^{\alpha_3\alpha_4\alpha_5\alpha_6}\cdots\epsilon^{\alpha_{4n-1}\alpha_{4n}\alpha_1\alpha_2}\,. \tag{2.36}$$

Adopting the matrix notation and considering the 6×6 matrices (ϵ_{AB}) and (G_{AB}), the contractions $I_n(G)$ translate as

$$I_n(G) = \mathrm{Tr}[(\epsilon G)^n]\,. \tag{2.37}$$

The trace of a matrix is the sum of its eigenvalues, and the trace of a matrix to the nth power is the sum of the eigenvalues to their nth power. Let λ_i for $i = 1, \ldots, 6$ denote the six eigenvalues of the matrix $(\tilde{\epsilon}G)$. Then (2.37) becomes

$$I_n(G) = \sum_{i=1}^{6} \lambda_i^n = p_n(\lambda_1, ..., \lambda_6) \,, \tag{2.38}$$

where $p_n(\lambda_1, ..., \lambda_6)$ are the power-sum symmetric polynomials of degree n in the six variables λ_i. In particular, the six eigenvalues λ_i of the matrix (ϵG) are sufficient to characterise all $I_n(G)$.

The area metric $G_{\mu\nu\rho\sigma}$ and $\epsilon^{\mu\nu\rho\sigma}$ are tensor densities of weight zero and $+1$, respectively. Thus, the contractions $I_n(G)$ defined in (2.36) transform as scalar densities of weight $+n$. We can obtain scalar densities of weight $+1$ by considering the contractions [1]

$$\mathcal{I}_n(G) \equiv |I_n(G)|^{\frac{1}{n}} \,. \tag{2.39}$$

This result suggests an infinite-dimensional ambiguity in the definition of a generally covariant area-metric volume element. For example, such a volume element can be built by forming linear combinations of invariants $\mathcal{I}_n$ in the form

$$\mathrm{d}\mathcal{V}_G = \left| \sum_{n\in\mathbb{N}} \alpha_n \mathcal{I}_n(G) \right| \mathrm{d}^4x \,. \tag{2.40}$$

One may impose as algebraic constraint that the area-metric volume element reduces to the length-metric volume element, for an area metric which is induced by a length metric. Such a constraint can be interpreted physically in terms of the zeroth-order contribution to the derivative expansion of an action for an area metric. For example, in a low-energy limit, in which area-metric gravity is expected to reduce to classical length-metric gravity, this constraint would entail that the non-derivative term in the expansion of an area-metric action reduces to the non-derivative term in the expansion of a metric action, and therefore to the cosmological constant term,

$$\int \mathrm{d}\mathcal{V}_G \quad \xrightarrow{G=G_g} \quad \Lambda \int \mathrm{d}^4x \, \sqrt{|\det(g)|} \,. \tag{2.41}$$

A necessary condition for an area metric G in $d = 4$ dimensions to be induced by a metric g, is

$$\frac{1}{4^2} G_{\mu\nu\alpha\beta}\epsilon^{\alpha\beta\gamma\delta} G_{\gamma\delta\lambda\tau}\epsilon^{\lambda\tau\rho\sigma} = \pm\frac{1}{2}|\det(G)|^{\frac{1}{3}} \delta_{\mu\nu}{}^{\rho\sigma} \,, \tag{2.42}$$

where the sign on the right hand side depends on the sign of $\det(G)$. Contracting both sides of (2.42) with $\delta_{\rho\sigma}{}^{\mu\nu}$ produces a multiple of the invariant $I_2(G)$ on the left hand side. Moreover, when G is induced by a metric g, the right hand side becomes proportional to $\det(G)^{1/3} = \det(g)$ after using (2.35). Therefore, $I_2(G_g)$ is proportional to $\det(g)$.

More generally, the matrix (ϵG_g) for an induced area metric G_g, for any signature of the

1. An analogue of the volume element $\mathcal{I}_2$ has been constructed for general spacetime dimension d in [55].

inducing metric g, has three eigenvalues equal to $+\det(g)^{1/2}$ and three eigenvalues equal to $-\det(g)^{1/2}$. From (2.38), it thereby follows that

$$I_n(G_g) = \begin{cases} 0 & \text{for } n \text{ odd}\,, \\ 6 \cdot \det(g)^{\frac{n}{2}} & \text{for } n \text{ even}\,. \end{cases} \tag{2.43}$$

With this identity, the constraint expressed in (2.41) becomes

$$\mathrm{d}\mathcal{V}_G \bigg|_{G=G_g} = \left|\sum_{n\in\mathbb{N}} \alpha_{2n} \mathcal{I}_{2n}(G_g)\right| = \left|\sum_{n\in\mathbb{N}} \alpha_{2n}\right| 6^{\frac{1}{2n}} \sqrt{|\det(g)|} \stackrel{!}{=} \Lambda \sqrt{|\det(g)|}\,, \tag{2.44}$$

and can be interpreted as a restriction on the couplings α_{2n} in an area-metric action, to resum into the coupling Λ in a length-metric action. It should be emphasised that this constraint and the condition (2.42) are purely algebraic. In a physical context, it is more reasonable to expect that there exists a dynamical decoupling mechanism, through which in the low-energy limit of area-metric gravity the degrees of freedom of the area metric, which are not associated to length-metric degrees of freedom, are switched off. Such a mechanism is analysed in subsection 4.1.3.

Alternatively to the invariants $\mathcal{I}_n(G)$, it is possible to build polynomials from the $I_k(G)$ for different k such that each term has weight $+n$. By subsequently taking the nth square root, we again obtain densities of weight $+1$. For example, the determinant $\det(G)$ of the 6×6 area-metric Petrov matrix (G_{AB}) can be expressed in this form. To see this, we use that $\det(\epsilon) = -1$ implies

$$\det(G) = -\det(\epsilon)\det(G) = -\det(\epsilon G) = -\prod_{i=1}^{6} \lambda_i \equiv -e_6(\lambda_1, ..., \lambda_6)\,, \tag{2.45}$$

where $e_6(\lambda_1, ..., \lambda_6)$ denotes the elementary symmetric polynomial of degree 6 in the six variables λ_i. The Newton identities, which relate the power-sum symmetric polynomials p_n to the elementary symmetric polynomials e_n [109], allow us to express the polynomial e_6 through the determinant formula

$$e_6 = \frac{1}{6!} \det \begin{pmatrix} p_1 & 1 & 0 & \cdots & \cdots \\ p_2 & p_1 & 2 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \vdots \\ p_5 & p_4 & \cdots & p_1 & 5 \\ p_6 & p_5 & \cdots & p_2 & p_1 \end{pmatrix}\,. \tag{2.46}$$

Using (2.45) and the identity $p_1 = I_1(G) = 0$, which follows from the cyclicity condition (2.15), we can state a formula for the determinant of the area-metric Petrov matrix (G_{AB}) in terms of contractions between the area-metric tensor $G_{\mu\nu\rho\sigma}$ and four-dimensional Levi-Civita densities

$\epsilon^{\mu\nu\rho\sigma}$, encoded in the invariants $I_n(G)$ in (2.36) and (2.38). The determinant of the 6×6 matrix (G_{AB}) is given by

$$\det(G) = \frac{1}{6} I_6 - \frac{1}{8} I_4 I_2 - \frac{1}{18} {I_3}^2 + \frac{1}{48} {I_2}^3 \,. \tag{2.47}$$

In general, the infinite-dimensional freedom in the definition of a covariant area-metric volume element is only one of multiple aspects which illustrate the complexity of geometry based on area metrics. This complexity is a result of the higher-rank tensor structure of area metrics and the additional degrees of freedom they encode in spacetime dimensions $d \geq 4$. A limited number of works have attempted to construct area connections and therefrom curvature tensors [51, 53, 55]. However, these constructions rely on a Gilkey decomposition of the area metric into a sum of area metrics induced by length metrics. As such a decomposition is non-unique, there are many choices of area connections which satisfy the conditions of area metricity and torsion-freeness. Altogether, therefore, area-metric differential geometry has not been rigorously established to date.

3 Actions for area metrics

A non-perturbative formulation of area-metric gravity requires the mathematical framework of differential geometry on area-metric backgrounds. In the absence of an established non-perturbative formulation of area-metric gravity, in section 3.1, we will follow a bottom-up approach and construct area-metric gravity perturbatively guided by the principle of general covariance. Subsequently, in section 3.2, we will show that actions for gravity based on bivector variables provide a natural, albeit implicit, framework for non-perturbative area-metric gravity.

3.1 Linearised area-metric gravity

General covariance is a fundamental principle in theoretical physics and profoundly constrains the landscape of gravitational theories. According to Lovelock's theorem [110, 111], any second-order equations of motion derived from a diffeomorphism-invariant Lagrangian for the metric tensor in $d = 4$ dimensions, are equivalent to Einstein's field equations. This establishes the uniqueness of the Einstein-Hilbert action for general relativity,

$$S_{\mathrm{EH}} = \frac{1}{2\kappa^2} \int \mathrm{d}^4x \, \sqrt{-g} R \,, \tag{3.1}$$

as a local second-order theory for the metric $g_{\mu\nu}$. [1] The global rescaling parameter in front of the action is determined by the gravitational coupling $\kappa^2 = 8\pi G = m_{\mathrm{Pl}}^{-2}$, where G is Newton's constant and m_{Pl} is the reduced Planck mass in Planck units, $c = \hbar = 1$.

Remarkably, the essence of this uniqueness theorem can be observed already at the level of a local, Lorentz- and gauge-invariant kinetic Lagrangian for a symmetric rank-2 tensor. When the gauge transformations are identified as linearised diffeomorphisms, the resulting action coincides with the Einstein-Hilbert action linearised around flat Minkowski background, up to a global rescaling. This construction is reviewed in the next subsection 3.1.1. Our main goal will be to extend this construction, and derive analogously the most general local diffeomorphism-invariant kinetic Lagrangian for a rank-4 tensor associated with the perturbations of an area metric around a background induced by the flat Minkowski metric. This is the main content of

1. The uniqueness holds up to Lovelock invariants. In $d = 4$ dimensions, the only non-vanishing Lovelock invariant is the Gauss-Bonnet term, which reduces to a total derivative and thereby does not contribute to the equations of motion [112].

subsection 3.1.2. Different from length-metric gravity, we will see that this action depends on several free parameters. One may take the viewpoint that the derived area-metric actions represent linearisations of a non-linear theory of area metrics. This non-linear completion is, however, unknown and anticipatedly non-unique in view of the non-uniqueness of the linearised theory. In subsection 3.1.3, we will derive linearised effective actions for a subset of the degrees of freedom encoded in the area-metric perturbation, which can be associated with length-metric perturbations. Finally, in subsection 3.1.4, we will consider a two-parameter subclass of linearised area-metric actions with shift-symmetric kinetic term, for which the effective length-metric actions are ghostfree.

3.1.1 Uniqueness of the Einstein-Hilbert action for a massless spin-2 field

Consider a symmetric rank-2 tensor $h_{\mu\nu}$ in flat spacetime, with Minkowski metric denoted by $(\eta_{\mu\nu})$ and signature convention $(-,+,+,+)$. The most general kinetic term for $h_{\mu\nu}$ is built from four possible local contributions, with a priori independent dimensionless real coefficients a_i, in the form

$$S = \int \mathrm{d}^4x\, (a_1\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + a_2\partial^\mu h_{\mu\rho}\partial_\nu h^{\nu\rho} + a_3\partial_\mu h^{\mu\nu}\partial_\nu h + a_4\partial^\mu h\partial_\mu h)\,, \tag{3.2}$$

where $h = \eta_{\mu\nu}h^{\mu\nu}$ denotes the trace of $h_{\mu\nu}$. In this expression $h_{\mu\nu}$ is a bosonic field with canonical mass dimension one.

Let us consider the gauge transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\,, \tag{3.3}$$

generated by a vector field $\xi^\mu(x)$. Imposing invariance of the action (3.2) under this transformation, up to boundary terms, fixes three of the free parameters a_i as functions of the remaining one and explicitly sets $a_3 = -2a_4 = -a_2 = 2a_1$. It is standard convention to fix the overall normalisation by choosing $a_1 = -\frac{1}{2} < 0$, such that a transverse-traceless wave carries positive energy. This leads to the Fierz-Pauli action [113] for a massless spin-2 particle,

$$S = \int \mathrm{d}^4x \left(-\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \partial^\mu h_{\mu\rho}\partial_\nu h^{\nu\rho} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial^\mu h\partial_\mu h\right). \tag{3.4}$$

A mass term for $h_{\mu\nu}$ would explicitly break the gauge symmetry.

The expression in (3.4) coincides with the expansion of the Einstein-Hilbert action (3.1) at second order in the dimensionless field $\delta g_{\mu\nu} = 2\kappa h_{\mu\nu}$ representing perturbations of the metric $g_{\mu\nu}$ around flat Minkowski background,

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}\,. \tag{3.5}$$

Thereby the local gauge symmetry (3.3) is recognised as the linearised version of full diffeomorphism invariance of the non-linear theory of general relativity.

In summary, after integrating by parts, the unique local and Lorentz-invariant kinetic Lagrangian for $h_{\mu\nu}$ in the momentum-space representation, obtained by replacing $\partial_\mu \to ip_\mu$ and $\Box \equiv \eta_{\mu\nu}\partial^\mu\partial^\nu \to -p^2$, is the Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\mathrm{EH}} = \frac{1}{2} h^{\mu\nu}(-p)\hat{\mathcal{E}}_{\mu\nu}{}^{\rho\sigma}(p) h_{\rho\sigma}(p)\,. \tag{3.6}$$

Here, $\hat{\mathcal{E}}$ denotes the Fierz-Pauli operator

$$\hat{\mathcal{E}}_{\mu\nu}{}^{\rho\sigma} = \left(-p^2\right)\left(P^{(2)}{}_{\mu\nu}{}^{\rho\sigma} - 2P^{(0)}{}_{\mu\nu}{}^{\rho\sigma}\right), \tag{3.7}$$

where $p^2 = \eta_{\mu\nu}p^\mu p^\nu$ is the square of the Minkowski four-momentum vector. $P^{(2)}$ and $P^{(0)}$ are the projectors onto the transverse-traceless component $h^{TT}_{\mu\nu}$ and the scalar component s of the tensor $h_{\mu\nu}$, defined by

$$P^{(2)}{}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}\left(T_\mu{}^\rho T_\nu{}^\sigma + T_\mu{}^\sigma T_\nu{}^\rho\right) - \frac{1}{3}T_{\mu\nu}T^{\rho\sigma} \quad \text{and} \quad P^{(0)}{}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{3}T_{\mu\nu}T^{\rho\sigma}\,, \tag{3.8}$$

where $T_\mu{}^\nu$ is the projector onto the transverse component of a vector,

$$T_\mu{}^\nu = \delta_\mu{}^\nu - \frac{p_\mu p^\nu}{p^2}\,. \tag{3.9}$$

In all expressions indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$. [1]

An arbitrary kinetic Lagrangian of the form (3.2) with untuned coefficients a_i would have led to higher-derivative terms for the longitudinal components of $h_{\mu\nu}$, parametrised by the vector field ξ_μ in the decomposition (3.10), and thereby to an Ostrogradsky ghost [115]. From a contrary point of view, requiring the kinetic Lagrangian for $h_{\mu\nu}$ to propagate no ghost automatically leads to invariance under the gauge transformation (3.3), see e.g. [116].

1. A generic symmetric rank-2 tensor $h_{\mu\nu}$ in d-dimensional Minkowski spacetime can be decomposed into four irreducible representations with respect to the Lorentz group,

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + i(p_\mu\xi_\nu + p_\mu\xi_\nu) + \frac{1}{d}T_{\mu\nu}s + \frac{1}{d}L_{\mu\nu}\omega\,. \tag{3.10}$$

Here $h^{TT}_{\mu\nu}$ is a symmetric transverse-traceless tensor, ξ^μ is a transverse vector, and s and ω are scalars. $T_\mu{}^\nu$ defined in (3.9) projects onto the transverse component of a vector, whereas $L_\mu{}^\nu$ is the projector onto the longitudinal component of a vector,

$$L_\mu{}^\nu = \frac{p_\mu p^\nu}{p^2}\,. \tag{3.11}$$

Thus, in the decomposition (3.10), the first term is transverse and traceless, the second is traceless but not transverse, the third is transverse but not traceless, and the last is neither transverse nor traceless, see e.g. [114].

Equation (3.7) signalises that the scalar mode s in the decomposition (3.10) of $h_{\mu\nu}$ exhibits a kinetic term with the opposite sign, compared to the kinetic term of the transverse-traceless mode $h^{TT}_{\mu\nu}$. This is the same type of potentially dangerous sign as for the kinetic term of the trace mode h in (3.4). The overall sign choice $a_1 < 0$ ensures that a transverse-traceless wave carries positive energy. This can be verified by replacing $h_{\mu\nu} \to h^{TT}_{\mu\nu}$ in (3.4) and computing the Hamiltonian to see that it is positive definite. On the other hand, this choice of global sign implies a negative Hamiltonian associated with the action (3.2) for a pure-trace field $\frac{1}{4}\eta_{\mu\nu}h$. A classical instability is avoided by the fact that this mode does not propagate in general relativity, and can be eliminated by a residual gauge transformation. The only physical degrees of freedom of linearised general relativity are encoded in the transverse-traceless tensor $h^{TT}_{\mu\nu}$ associated with the massless spin-2 graviton. To illustrate the previous statements, appendix A.2.1 reviews the Hamiltonian formulation of linearised general relativity.

3.1.2 Diffeomorphism-invariant local actions for area-metric perturbations

In this subsection, we will take a bottom-up approach and construct area-metric gravity perturbatively. To that end, we will apply the procedure of the previous subsection to derive the most general local and diffeomorphism-invariant kinetic Lagrangian for a rank-4 tensor $a_{\mu\nu\rho\sigma}$ associated with the perturbations of an area metric around a background induced by the flat Minkowski metric. An analogous viewpoint lies at the core of the constructive-gravity programme [117–121], whose implementation differs from the approach presented here, but whose results are in agreement with the area-metric actions stated at the end of this subsection.

Our starting point is an expansion of the area metric around a background induced by the flat Minkowski metric $\eta_{\mu\nu}$,

$$G_{\mu\nu\rho\sigma} = (G_\eta)_{\mu\nu\rho\sigma} + \delta G_{\mu\nu\rho\sigma} = \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho} + \delta G_{\mu\nu\rho\sigma}\,. \tag{3.12}$$

The bosonic field $a_{\mu\nu\rho\sigma}$ with canonical mass dimension one arises from the dimensionless tensor $\delta G_{\mu\nu\rho\sigma}$ of area-metric perturbations through a global rescaling, $\delta G_{\mu\nu\rho\sigma} = 2\kappa a_{\mu\nu\rho\sigma}$. This tensor has the same algebraic symmetries (2.14) as the area metric itself. The cyclicity condition in four dimensions can be stated as

$$a_{\alpha\mu\nu\rho}\epsilon^{\beta\mu\nu\rho} = 0\,, \tag{3.13}$$

for all α, β, and is equivalent to the condition $a_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = 0$ after taking into account the index exchange symmetries of the area metric.

The most general kinetic term for a in momentum space is built from eight local contributions, as derived in appendix A.1. These can be combined with a priori independent dimensionless real coefficients b_i into a Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{kin}} &= b_1 a_{\mu\nu\rho\sigma}a^{\mu\nu\rho\sigma}p^2 + b_2 a_{\mu\nu}{}^{\mu\rho}a_{\sigma\rho}{}^{\sigma\nu}p^2 + b_3 a_{\mu\nu}{}^{\mu\nu}a_{\rho\sigma}{}^{\rho\sigma}p^2 + b_4 a_{\mu\nu}{}^{\rho\sigma}a^{\mu\nu\lambda\tau}\epsilon_{\lambda\tau\rho\sigma}p^2 \\ &+ b_5 a_{\mu\rho\nu\sigma}a^{\rho\lambda\sigma}{}_{\lambda}p^\mu p^\nu + b_6 a_{\mu}{}^{\rho}{}_{\nu\rho}a^{\sigma\lambda}{}_{\sigma\lambda}p^\mu p^\nu + b_7 a_{\mu}{}^{\rho}{}_{\rho\sigma}a_{\lambda\nu}{}^{\sigma\lambda}p^\mu p^\nu \\ &+ b_8 a_{\mu}{}^{\rho\sigma\lambda}a_{\rho}{}^{\tau}{}_{\tau}{}^{\kappa}\epsilon_{\nu\sigma\kappa\lambda}p^\mu p^\nu\,. \end{aligned} \tag{3.14}$$

Further insight into the algebraic structure of the individual contributions can be gained by decomposing $a_{\mu\nu\rho\sigma}$ into a triple of tensors

$$a_{\mu\nu\rho\sigma} \;\leftrightarrow\; \left(h, \hat{h}_{\mu\nu}, \omega_{\mu\nu\rho\sigma}\right) \qquad \text{where} \qquad \hat{h}_{\mu\nu}\eta^{\mu\nu} = 0 \quad \text{and} \quad \omega_{\mu\nu\rho\sigma}\eta^{\mu\rho} = 0\,. \tag{3.15}$$

This reparametrisation is analogous to the Ricci-Weyl decomposition of the Riemann curvature tensor. Here, h is a scalar proportional to the trace of a, and $\hat{h}$ is a symmetric and traceless tensor with nine independent degrees of freedom. The remaining ten degrees of freedom of the area-metric perturbation are contained in the tensor ω, which is fully traceless and plays the role of the Weyl tensor for a. In terms of these components, we can write

$$\begin{aligned} a_{\mu\nu\rho\sigma} &= \eta_{\mu[\rho}\eta_{\sigma]\nu}h + 2\left(\eta_{\mu[\rho}\hat{h}_{\sigma]\nu} - \eta_{\nu[\rho}\hat{h}_{\sigma]\mu}\right) + \omega_{\mu\nu\rho\sigma} \\ &= 2\left(\eta_{\mu[\rho}h_{\sigma]\nu} - \eta_{\nu[\rho}h_{\sigma]\mu}\right) + \omega_{\mu\nu\rho\sigma}\,, \end{aligned} \tag{3.16}$$

where in the second line h and $\hat{h}_{\mu\nu}$ are combined into the symmetric tensor

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}h\,. \tag{3.17}$$

This tensor is a priori not related to the tensor of lenght-metric perturbations in the discussion of the previous subsection. Only later will we identify these two tensors, in order to reproduce the linearised Einstein-Hilbert action as one of the contributions to the general gauge-invariant Lagrangian for area-metric perturbations.

To make the explicit form of the gauge transformation corresponding to linearised diffeomorphisms transparent, we introduce a further reparametrisation of the Weyl component ω of the area-metric perturbation. In $d = 4$ dimensions, this tensor can be decomposed as

$$\omega_{\mu\nu\rho\sigma} = \omega^{+}_{\mu\nu\rho\sigma} + \omega^{-}_{\mu\nu\rho\sigma}\,, \tag{3.18}$$

where $\omega^{\pm}_{\mu\nu\rho\sigma}$ are selfdual and anti-selfdual, respectively, on the first and second index pairs separately with respect to the Hodge star operator defined by the Minkowski background metric $\eta_{\mu\nu}$. This duality condition reads

$$\frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}\omega^{\pm}_{\alpha\beta\rho\sigma} = \pm i\omega^{\pm}_{\mu\nu\rho\sigma}\,, \tag{3.19}$$

where on the left hand side the expression involving ϵ with two lower indices is a shorthand notation for $\epsilon_{\mu\nu}{}^{\alpha\beta} = \eta_{\mu\rho}\eta_{\nu\sigma}\epsilon^{\rho\sigma\alpha\beta}$. The fields $\omega^{\pm}_{\mu\nu\rho\sigma}$ are generically complex, with components related by conjugation, $\omega^{+}_{\mu\nu\rho\sigma} = \overline{\omega^{-}_{\mu\nu\rho\sigma}}$. Moreover, they are orthogonal in the sense that $\omega^{+}_{\mu\nu\alpha\beta}\omega^{-\alpha\beta\rho\sigma}$ vanishes. The 10 real degrees of freedom of ω split into $5+5$ degrees of freedom, which can be associated with the real and imaginary parts of the selfdual component ω^{+}. It is convenient to trade $\omega^{\pm}_{\mu\nu\rho\sigma}$ for a pair of symmetric and traceless matrices $\chi^{\pm}_{ab}$ of spacetime scalars, with internal indices $a, b, ... = 1, 2, 3$ raised and lowered with the internal metric δ_{ab}. This reparametrisation is defined by projectors

$$\mathbb{P}^{\pm ab}{}_{\mu\nu\rho\sigma} = P^{\pm(a}_{\mu\nu}P^{\pm b)}_{\rho\sigma} - \frac{1}{3}\delta^{ab}P^{\pm c}_{\mu\nu}P^{\pm d}_{\rho\sigma}\delta_{cd} \quad \text{where} \quad P^{\pm a}_{\mu\nu} \equiv \frac{1}{2}\Sigma^{\pm a}_{\mu\nu}(\eta)\,. \tag{3.20}$$

Here, $\Sigma^{\pm a}_{\mu\nu}(\eta)$ represent the selfdual and anti-selfdual Plebanski 2-forms constructed from the Minkowski background tetrad η^{I}_{μ}. Explicitly, they are given by

$$\Sigma^{\pm a}_{\mu\nu}(\eta) = \mp i\big(\eta^0_\mu\eta^a_\nu - \eta^0_\nu\eta^a_\mu\big) + \epsilon^a{}_{bc}\eta^b_\mu\eta^c_\nu = \pm i\big(\delta^0_\mu\delta^a_\nu - \delta^0_\nu\delta^a_\mu\big) + \epsilon^a{}_{bc}\delta^b_\mu\delta^c_\nu\,, \tag{3.21}$$

and satisfy

$$\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\Sigma^{\pm a}_{\rho\sigma} = \pm i\Sigma^{\pm a}_{\mu\nu}\,. \tag{3.22}$$

The projectors $\mathbb{P}^{\pm}$ defined in (3.20) are traceless in their internal indices. The contraction of two projectors in their spacetime indices yields the identity on the space of symmetric and traceless tensors χ_{ab} defined on the 3-dimensional internal space,

$$\mathbb{P}^{\pm ab}{}_{\mu\nu\rho\sigma}\mathbb{P}^{\pm cd\,\mu\nu\rho\sigma} = \delta^{a(c}\delta^{d)b} - \frac{1}{3}\delta^{ab}\delta^{cd} \equiv \mathbb{I}^{abcd}\,. \tag{3.23}$$

Using these projectors, we can construct a pair of symmetric and traceless fields $\chi^{\pm}_{ab}$ defined by

$$\chi^{\pm ab} = \frac{1}{2}\mathbb{P}^{\pm ab}{}_{\mu\nu\rho\sigma}\omega^{\pm\mu\nu\rho\sigma} \qquad \Leftrightarrow \qquad \omega^{\pm}_{\mu\nu\rho\sigma} = 2\mathbb{P}^{\pm ab}{}_{\mu\nu\rho\sigma}\chi^{\pm}_{ab}\,. \tag{3.24}$$

Similarly as $\omega^{\pm}_{\mu\nu\rho\sigma}$, the scalar fields $\chi^{\pm}_{ab}$ are complex conjugates $\chi^{+}_{ab} = \overline{\chi^{-}_{ab}}$ and orthogonal, as $\chi^{+}_{ab}\chi^{-ab}$ vanishes.

So far, we have parametrised the 20 degrees of freedom of the field $a_{\mu\nu\rho\sigma}$ into $10+5+5$ degrees of freedom, via

$$a_{\mu\nu\rho\sigma} \;\;\leftrightarrow\;\; \big(h_{\mu\nu}, \chi^{+}_{ab}, \chi^{-}_{ab}\big)\,. \tag{3.25}$$

Now, let us consider the gauge transformation associated with linearised diffeomorphisms generated by a vector field $\xi^\mu(x)$. The symmetric tensor $h_{\mu\nu}$ transforms as in (3.3), whereas the spacetime scalars $\chi^{\pm}_{ab}$ are left invariant,

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + p_\mu \xi_\nu + p_\nu \xi_\mu \,, \tag{3.26}$$
$$\chi^{\pm}_{ab} \rightarrow \chi^{\pm}_{ab} \,. \tag{3.27}$$

In view of (3.16), this transformation is equivalent to a gauge transformation of $a_{\mu\nu\rho\sigma}$ given by

$$\begin{aligned} a_{\mu\nu\rho\sigma} \rightarrow a_{\mu\nu\rho\sigma} &+ \eta_{\mu\rho}(p_\nu \xi_\sigma + p_\sigma \xi_\nu) - \eta_{\mu\sigma}(p_\nu \xi_\rho + p_\rho \xi_\nu) \\ &- \eta_{\nu\rho}(p_\mu \xi_\sigma + p_\sigma \xi_\mu) + \eta_{\nu\sigma}(p_\mu \xi_\rho + p_\rho \xi_\mu) \,. \end{aligned} \tag{3.28}$$

Requiring the kinetic Lagrangian (3.14) to be invariant under this gauge transformation fixes three of the eight free parameters b_i as functions of the remaining ones, and explicitly imposes

$$b_5 = -2(4b_1 + 3b_2 + 6b_3) \,, \tag{3.29}$$
$$b_6 = \frac{4}{3}(2b_1 + b_2) \,, \tag{3.30}$$
$$b_7 = -2(2b_1 + b_2) \,. \tag{3.31}$$

Altogether, the most general local and diffeomorphism-invariant kinetic Lagrangian for area-metric perturbations is given by

$$\begin{aligned} \mathcal{L}_{\text{kin}} = \; & b_1 a_{\mu\nu\rho\sigma} a^{\mu\nu\rho\sigma} p^2 + b_2 a_{\mu\nu}{}^{\mu\rho} a_{\sigma\rho}{}^{\sigma\nu} p^2 + b_3 a_{\mu\nu}{}^{\mu\nu} a_{\rho\sigma}{}^{\rho\sigma} p^2 + b_4 a_{\mu\nu}{}^{\rho\sigma} a^{\mu\nu\lambda\tau} \epsilon_{\lambda\tau\rho\sigma} p^2 \\ - \; & 2(4b_1 + 3b_2 + 6b_3) a_{\mu\rho\nu\sigma} a^{\rho\lambda\sigma}{}_\lambda p^\mu p^\nu + \frac{4}{3}(2b_1 + b_2) a_\mu{}^\rho{}_{\nu\rho} a^{\sigma\lambda}{}_{\sigma\lambda} p^\mu p^\nu \\ - \; & 2(2b_1 + b_2) a_\mu{}^\rho{}_{\rho\sigma} a_{\lambda\nu}{}^{\sigma\lambda} p^\mu p^\nu + b_8 a_\mu{}^{\rho\sigma\lambda} a_\rho{}^\tau{}_\tau{}^\kappa \epsilon_{\nu\sigma\kappa\lambda} p^\mu p^\nu \,. \end{aligned} \tag{3.32}$$

It is parametrised by five real parameters b_i, for $i = 1, 2, 3, 4, 8$. Notably, the parity-violating couplings b_4 and b_8, in front of the two contractions involving the Levi-Civita density in the original ansatz (3.14), have remained unconstrained by the requirement of gauge invariance. This can be understood by rewriting the corresponding contractions in terms of the decomposition (3.16), with $\omega = \omega^+ + \omega^-$, to see that they are respectively proportional to $(\omega^{+2} - \omega^{-2})p^2$ and $h(\omega^+ - \omega^-)pp$, where in the last expression $\omega^\pm$ couples only to the traceless parts of the symmetric tensors h and pp. Appendix subsection A.1.2 contains further details on the kinetic terms which define the ansatz (3.14), translated into the parametrisation of a in terms of the tensors h and $\omega^\pm$. This rewriting, together with the expression of $\omega^\pm$ via the scalars $\chi^\pm$ in (3.24), shows that the two parity-violating contractions $aa\epsilon p^2$ and $aa\epsilon pp$ are by themselves already invariant under the transformations (3.26) and (3.27), and therefore must appear with a free parameter in the gauge-invariant Lagrangian (3.32). This will become further evident below, using an embedding of the scalars $\chi^\pm$ into the space of symmetric transverse-traceless tensors on spacetime.

Rewriting the gauge-invariant Lagrangian (3.32) in terms of the decomposition (3.16) of a

into h and $\omega = \omega^+ + \omega^-$, after making use of the duality condition (3.19) for $\omega^\pm$, leads to

$$\begin{aligned}\mathcal{L}_{\text{kin}} &= A\Big[-\frac{1}{2}h_{\mu\nu}h^{\mu\nu}p^2 + h_{\mu\rho}h_\nu{}^\rho p^\mu p^\nu - h_{\mu\nu}hp^\mu p^\nu + \frac{1}{2}h^2p^2\Big] \\ &- \frac{1}{2}\sum_\pm\Big[\alpha_\pm h^{\rho\sigma}\omega^\pm_{\mu\rho\nu\sigma}p^\mu p^\nu + \frac{1}{8}\beta_\pm\omega^\pm_{\mu\nu\rho\sigma}\omega^{\pm\mu\nu\rho\sigma}p^2\Big]\,. \end{aligned} \tag{3.33}$$

The parameters A, $\alpha_\pm$ and $\beta_\pm$ are given in terms of the original real coupling constants b_i by

$$A = 16b_{123}\,, \tag{3.34}$$

$$\alpha_\pm = -8(b'_{123} \pm ib_8)\,, \tag{3.35}$$

$$\beta_\pm = -16(b_1 \pm 2ib_4)\,, \tag{3.36}$$

where $b_{123} = b_1 + b_2 + 3b_3$ and $b'_{123} = -4b_1 - 3b_2 - 6b_3$. The couplings $\alpha_\pm$ and $\beta_\pm$, which appear in front of the interaction term and the term quadratic in $\omega^\pm$ in (3.33), are in general complex and respectively related by conjugation,

$$\alpha_+ = \overline{\alpha_-} \quad\text{and}\quad \beta_+ = \overline{\beta_-}\,. \tag{3.37}$$

Together with the field relation $\omega^+ = \overline{\omega^-}$, this guarantees that the Lagrangian (3.33) is real. This must be the case, given that, in the original expression (3.32), the field a and couplings b_i are real.

From the result (3.33) for the gauge-invariant kinetic Lagrangian, the following observations can be extracted.

First of all, the term quadratic in $h_{\mu\nu}$ is uniquely determined by the linearised Einstein-Hilbert Lagrangian (3.4), up to the global rescaling parameter A. This is not surprising, as invariance under the identical gauge transformations (3.3) and (3.26) was imposed on this sector. We may without loss of generality fix the global rescaling of the Lagrangian (3.33) by setting $A \equiv 1$, in order to reproduce the linearised Einstein-Hilbert action for $h_{\mu\nu}$ as one of the contributions to the gauge-invariant Lagrangian for area-metric perturbations. This still leaves four real or, equivalently, two complex unconstrained parameters, associated with the couplings $\alpha_\pm$ and $\beta_\pm$. This result is in stark contrast to the unique local and gauge-invariant kinetic Lagrangian for a symmetric rank-2 tensor $h_{\mu\nu}$, reviewed in the previous subsection. In fact, the condition of linearised diffeomorphism invariance is not more powerful in providing constraints on the free parameters in the area-metric Lagrangian ansatz (3.14), than it is for the length-metric Lagrangian ansatz (3.2). In both cases, imposing linearised diffeomorphism invariance fixes three of the initial free parameters, and in the area-metric Lagrangian constrains only the hh sector. The couplings in front of the interaction terms $h\omega^\pm pp$ and kinetic terms $\omega^{\pm 2}p^2$ are, respectively, related by complex conjugation, as a result of the purely algebraic duality condition (3.19), but are not affected by the condition of gauge invariance under (3.26) and (3.27). This observation can be further strengthened by noticing that the fields $\omega^\pm$, being fully traceless, couple only to the traceless part $\hat{h}_{\mu\nu}$ of the tensor $h_{\mu\nu}$ defined in (3.17). We can construct a pair of symmetric transverse-traceless rank-2 spacetime tensors $\chi^{\pm}_{\mu\nu}$, defined from $\omega^{\pm}$ by

$$\chi^{\pm}_{\mu\nu} \equiv \omega^{\pm}_{\mu\rho\nu\sigma}\frac{p^\rho p^\sigma}{p^2} = 4P^{\pm a}_{\mu\rho}P^{\pm b}_{\nu\sigma}\frac{p^\rho p^\sigma}{p^2}\chi^{\pm}_{ab} \quad \text{with} \quad \chi^{\pm}_{\mu\nu}p^\mu = 0 \quad \text{and} \quad \chi^{\pm\mu}_{\mu} = 0\,, \tag{3.38}$$

such that the spin-2 projector acts as the identity on these fields,

$$P^{(2)}{}_{\mu\nu}{}^{\rho\sigma}\chi^{\pm}_{\rho\sigma} = \chi^{\pm}_{\mu\nu}\,. \tag{3.39}$$

This implies that the $\chi^{\pm}_{\mu\nu}$ couple only to the transverse-traceless part $h^{TT}_{\mu\nu}$ of the tensor $h_{\mu\nu}$ in the interaction terms $h^{\rho\sigma}\omega^{\pm}_{\mu\rho\nu\sigma}p^\mu p^\nu = h_{\mu\nu}\chi^{\pm\mu\nu}p^2$, which are therefore invariant under linearised diffeomorphisms acting as $h^{TT}_{\mu\nu} \to h^{TT}_{\mu\nu}$ and $\chi^{\pm}_{\mu\nu} \to \chi^{\pm}_{\mu\nu}$. Moreover, from the definition (3.38), it follows that the squares of the various fields, used to describe the non-metric degrees of freedom of the area-metric perturbation, are related by

$$\chi^{\pm}_{\mu\nu}\chi^{\pm\mu\nu} = \chi^{\pm}_{ab}\chi^{\pm ab} = \frac{1}{4}\omega^{\pm}_{\mu\nu\rho\sigma}\omega^{\pm\mu\nu\rho\sigma}\,, \tag{3.40}$$

where the last identity follows from the property (3.23) of the projectors $\mathbb{P}^{\pm}$. The kinetic terms $\omega^{\pm}_{\mu\nu\rho\sigma}\omega^{\pm\mu\nu\rho\sigma}p^2 = 4\chi^{\pm}_{\mu\nu}\chi^{\pm\mu\nu}p^2$ are therefore also invariant under linearised diffeomorphisms. This explains why imposing linearised diffeomorphism invariance on the original kinetic Lagrangian ansatz for area-metric perturbations (3.14) constraints only the hh sector. Indeed, there are no other algebraically allowed contractions of the form $h\chi^{\pm}$, $\chi^{\pm}\chi^{\pm}$ or $\chi^{\pm}\chi^{\mp}$, coupled to two powers of the momentum, that could have appeared in the original ansatz (3.14). Appendix A.1.2 contains a representation-theoretic explanation for this statement.

We can complete the kinetic Lagrangian (3.33) by adding a mass term for the non-metric components of the area-metric perturbation,

$$\mathcal{L}_{\text{mass}} = -\frac{1}{16}\sum_{\pm} m^2_{\pm}\omega^{\pm}_{\mu\nu\rho\sigma}\omega^{\mu\nu\rho\sigma} = -\frac{1}{4}\sum_{\pm} m^2_{\pm}\chi^{\pm}_{\mu\nu}\chi^{\mu\nu}\,, \tag{3.41}$$

where the mass parameters $m^2_{\pm}$ are complex conjugated, $m^2_{+} = \overline{m^2_{-}}$. This term does not spoil the invariance under gauge transformations (3.27), in contrast to a mass term for the tensor $h_{\mu\nu}$. It is well-known that adding a general mass term $-\frac{1}{4}m^2(h_{\mu\nu}h^{\mu\nu} - \lambda h^2)$ with a dimensionless parameter λ in the Einstein-Hilbert action (3.4), to yield the Fierz-Pauli action for a massive spin-2 particle [113], explicitly breaks the gauge symmetry under linearised diffeomorphisms (3.26).

In summary, combining the kinetic Lagrangian in (3.33) with the mass Lagrangian (3.41), the most general local diffeomorphism-invariant quadratic Lagrangian for area-metric perturbations can be expressed as

$$\begin{aligned}\mathcal{L} &\equiv \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{mass}} \\ &= A\mathcal{L}_{\text{EH}}(h_{\mu\nu}) - \frac{1}{2}\sum_{\pm}\left[\alpha_{\pm} h_{\mu\nu}\chi^{\pm\mu\nu}p^2 + \frac{1}{2}\beta_{\pm}\chi^{\pm}_{\mu\nu}\chi^{\pm\mu\nu}p^2 + \frac{1}{2}m^2_{\pm}\chi^{\pm}_{\mu\nu}\chi^{\pm\mu\nu}\right].\end{aligned} \tag{3.42}$$

This Lagrangian violates parity under the condition

$$\text{parity violation} \quad \Leftrightarrow \quad \text{Im}[\gamma_{\pm}] \neq 0 \ \ (\text{i.e. } \gamma_+ \neq \gamma_-) \quad \forall \text{ couplings } \gamma_{\pm} \in \mathcal{L}\,, \tag{3.43}$$

where we remind the reader of the relation $\gamma_+ = \overline{\gamma_-}$.

Before closing this subsection, let us comment on a peculiarity of the Lagrangian (3.42), which arises because $\chi^+ = \overline{\chi^-}$ are complex conjugate fields. Decomposing these fields and the couplings $\alpha_{\pm}$ and $\beta_{\pm}$ into their real and imaginary parts,

$$\chi^{\pm}_{\mu\nu} \equiv \chi^1_{\mu\nu} \pm i\chi^2_{\mu\nu} \qquad \text{and} \qquad (\alpha_{\pm}, \beta_{\pm}) \equiv (\alpha_1 \pm i\alpha_2, \beta_1 \pm i\beta_2)\,, \tag{3.44}$$

allows us to write the interaction term in (3.42) as

$$-\frac{1}{2}\sum_{\pm}\alpha_{\pm}h_{\mu\nu}\chi^{\pm\mu\nu}p^2 = -h_{\mu\nu}\text{Re}\big[\alpha_+\chi^{+\mu\nu}\big]p^2 = -h_{\mu\nu}\big(\alpha_1\chi^{1\mu\nu} - \alpha_2\chi^{2\mu\nu}\big)p^2, \tag{3.45}$$

whereas the kinetic term for $\chi^{\pm}_{\mu\nu}$ becomes

$$\begin{aligned}-\frac{1}{4}\sum_{\pm}\beta_{\pm}\chi^{\pm}_{\mu\nu}\chi^{\pm\mu\nu}p^2 &= -\frac{1}{2}\text{Re}\big[\beta_+\chi^{+}_{\mu\nu}\chi^{+\mu\nu}\big]p^2 \\ &= -\frac{1}{2}\beta_1\big(\chi^1_{\mu\nu}\chi^{1\mu\nu} - \chi^2_{\mu\nu}\chi^{2\mu\nu}\big)p^2 + \beta_2\chi^1_{\mu\nu}\chi^{2\mu\nu}p^2\,,\end{aligned} \tag{3.46}$$

and similarly for the mass term. The Lagrangian contribution quadratic $\chi^{\pm}$, rewritten in terms of the real fields $\chi^{1,2}$, has an indefinite sign and also generically mixes these two fields. The origin of the mixing term, with coupling constant $\beta_2 = 32b_4$, can be traced back to the parity-violating term in the first line of (3.32). The other parity-violating term in the third line of (3.32) is manifested in a non-zero coupling constant $\alpha_2 = 8b_8$, for the coupling of the field χ^2 to h. Such an indefiniteness of the kinetic and mass terms for $\chi^{\pm}$ does not occur in Euclidean signature, where these fields and all couplings in the Lagrangian are real. This can be traced back to the duality condition (3.19), which in Euclidean signature becomes an equation involving the real eigenvalues ± 1, instead of $\pm i$, on the right hand side.

The local diffeomorphism-invariant quadratic Lagrangian for area-metric perturbations (3.42) is the main result of this subsection. In the next subsection 3.1.3, we will derive effective actions for the tensor $h_{\mu\nu}$ parametrising the area-metric fluctuations induced by length-metric fluctuations. Subsequently, in subsection 3.1.4, we will reconsider the area-metric Lagrangian for a special choice of couplings, resulting in a shift symmetry of the kinetic term.

3.1.3 Effective linearised length-metric actions and spin-2 propagator

In this subsection, we derive effective actions for the symmetric tensor field $h_{\mu\nu}$. In the linearised area-metric Lagrangian (3.42), the contribution quadratic in this field is given by the linearised Einstein-Hilbert action. On these grounds, $h_{\mu\nu}$ can be interpreted as representing length-metric fluctuations, which induce area-metric fluctuations corresponding to the trace terms in the decomposition (3.16).

To obtain effective actions for $h_{\mu\nu}$, we integrate out the Weyl components of the area-metric perturbation, associated with the fields $\chi^{\pm}$. Their contribution to the gauge-invariant Lagrangian (3.42) is

$$\mathcal{L} \supset -\frac{1}{2}\alpha_{\pm} h^{TT}_{\mu\nu}\chi^{\pm\mu\nu}p^2 - \frac{1}{4}\beta_{\pm}\chi^{\pm}_{\mu\nu}\chi^{\pm\mu\nu}p^2 - \frac{1}{4}m^2_{\pm}\chi^{\pm}_{\mu\nu}\chi^{\pm\mu\nu}\,, \tag{3.47}$$

where we have used that $\chi^{\pm}_{\mu\nu}$ couples only to the transverse-traceless component $h^{TT}_{\mu\nu}$ of the length-metric perturbation. The solution to the equations of motion for the fields $\chi^{\pm}$,

$$p^2\big(\alpha_{\pm}h^{TT\mu\nu} + \beta_{\pm}\chi^{\pm\mu\nu}\big) + m^2_{\pm}\chi^{\pm\mu\nu} = 0\,, \tag{3.48}$$

can be expressed in the form

$$\chi^{\pm}_{\mu\nu} = \frac{\rho_{\pm}}{\alpha_{\pm}}\frac{1}{-p^2 - M^2_{\pm}}P^{(2)\mu\nu}{}_{\rho\sigma}h^{\rho\sigma}p^2\,, \tag{3.49}$$

with parameters $\rho_{\pm}$ and $M^2_{\pm}$ defined by

$$\rho_{\pm} = \frac{\alpha^2_{\pm}}{\beta_{\pm}} \qquad \text{and} \qquad M^2_{\pm} = \frac{m^2_{\pm}}{\beta_{\pm}}\,. \tag{3.50}$$

Next, we insert (3.49) into the Lagrangian (3.42) and make use of the identity [1]

$$h^{\mu\nu}P^{(2)}{}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma}p^4 = 2\,{}^{(1)}C_{\mu\nu\rho\sigma}{}^{(1)}C^{\mu\nu\rho\sigma}\,, \tag{3.56}$$

where ${}^{(1)}C_{\mu\nu\rho\sigma}$ denotes the perturbation of the Weyl tensor for the metric $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$ at first order in $h_{\mu\nu}$. Thereby, we arrive at an effective Lagrangian for the length-metric perturba-

1. The perturbation of the Weyl tensor $\delta C_{\mu\nu\rho\sigma}$ at first order in the perturbation $\delta g_{\mu\nu}$ of the metric, expanded around flat space, $g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$, is given by

$$\delta C_{\mu\nu\rho\sigma} = \delta R_{\mu\nu\rho\sigma} - \frac{1}{2}(\eta_{\mu\rho}\delta R_{\nu\sigma} - \eta_{\mu\sigma}\delta R_{\nu\rho} - \eta_{\nu\rho}\delta R_{\mu\sigma} + \eta_{\nu\sigma}\delta R_{\mu\rho}) + \frac{1}{6}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho})\delta R\,, \tag{3.51}$$

where the first-order perturbations of the Riemann tensor $\delta R_{\mu\nu\rho\sigma}$, of the Ricci tensor $\delta R_{\mu\nu}$, and of the Ricci tions,

$$\mathcal{L}_{\text{eff}}(h_{\mu\nu}) = A\mathcal{L}_{\text{EH}}(h_{\mu\nu}) - \frac{1}{2}\sum_{\pm} \rho_{\pm}{}^{(1)}C_{\mu\nu\rho\sigma}\frac{1}{-p^2 - M_{\pm}^2}{}^{(1)}C^{\mu\nu\rho\sigma} . \tag{3.57}$$

This Lagrangian is manifestly real, as $\rho_+ = \overline{\rho_-}$ and $M_+^2 = \overline{M_-^2}$.

The expression in (3.57) represents a linearised Einstein-Weyl Lagrangian characterised by four real or, equivalently, two complex free parameters encoded in $\rho_\pm$ and $M_\pm^2$. We will refer to Lagrangians containing inverse derivative operators, which are analytic at $p^2 = 0$, as quasi-local. By contrast, non-local Lagrangians contain operators which are not expandable as a power series at $p^2 = 0$. The degree of non-locality in the quasi-local effective Lagrangian (3.57) is thus controlled by the effective mass squares $M_\pm^2$. In the limit $M_\pm^2 \to 0$, these Lagrangians become genuinely non-local and reduce to an Einstein-Weyl subclass of generalised non-local quadratic-curvature gravity [122]. In the opposite limit of large mass $M_\pm^2 \gg p^2$, they reduce to the Einstein-Weyl subclass of local quadratic-curvature gravity [123].

The quasi-local Weyl-squared term, beyond the Einstein-Hilbert Lagrangian, affects the propagator $\mathcal{P}^{(2)}$ for the transverse-traceless mode $h_{\mu\nu}^{TT}$. The form of the kinetic operator for this mode can be inferred from the expression for the linearised Einstein-Hilbert Lagrangian (3.6), together with the identity (3.56). Omitting tensorial structures, it is given by

$$\left(\mathcal{P}^{(2)}\right)^{-1} = -\frac{A}{2}p^2 - \frac{1}{4}\sum_{\pm}\rho_{\pm}\frac{p^4}{-p^2 - M_\pm^2} = -\frac{p^2}{2}\left(A + \frac{1}{2}\sum_{\pm}\frac{\rho_\pm p^2}{-p^2 - M_\pm^2}\right) . \tag{3.58}$$

Poles in the spin-2 propagator $\mathcal{P}^{(2)}$, defined by $\mathcal{P}^{(2)}\left(\mathcal{P}^{(2)}\right)^{-1} = 1$, correspond to zeros of the inverse propagator $\left(\mathcal{P}^{(2)}\right)^{-1}$. From (3.58), we immediately recognise the pole at $p^2 = 0$ associated with the massless graviton. For generic couplings $\rho_\pm$ and $M_\pm^2$, additional poles are described by the roots of the equation

$$2A\left(-p^2 - M_+^2\right)\left(-p^2 - M_-^2\right) + \left(-p^2 - M_-^2\right)\rho_+ p^2 + \left(-p^2 - M_+^2\right)\rho_- p^2 = 0 . \tag{3.59}$$

scalar δR, are given by

$$\delta R_{\mu\nu\rho\sigma} = \frac{1}{2}(p_\mu p_\rho h_{\sigma\nu} - p_\mu p_\sigma h_{\rho\nu} + p_\nu p_\sigma h_{\rho\mu} - p_\nu p_\rho h_{\sigma\mu}) , \tag{3.52}$$

$$\delta R_{\mu\nu} = \frac{1}{2}\left(p^2 h_{\mu\nu} - p_\mu p_\nu h^\rho{}_\nu - p_\mu p_\rho h^\rho{}_\mu + p_\mu p_\nu h\right) , \tag{3.53}$$

$$\delta R = p^2 h - p_\mu p_\nu h^{\mu\nu} . \tag{3.54}$$

The square of the Weyl perturbation then satisfies

$$\delta g^{\mu\nu} {P^{(2)}}_{\mu\nu}{}^{\rho\sigma} \delta g_{\rho\sigma} p^4 = 2\delta C_{\mu\nu\rho\sigma}\delta C^{\mu\nu\rho\sigma} , \tag{3.55}$$

where $P^{(2)}$ is the spin-2 projector defined in (3.8). Thus, the relation (3.56) holds with ${}^{(1)}C^{\mu\nu\rho\sigma} = 2\kappa\delta C_{\mu\nu\rho\sigma}$.

The left hand side is a quadratic polynomial in p^2 with two zeros in general. It is instructive to distinguish a few special cases.

Suppose $M_\pm^2 \gg p^2$, such that the effective Lagrangian (3.57) reduces to the Lagrangian for linearised local Einstein-Weyl gravity [123]. Then the kinetic term in equation (3.58) simplifies to

$$\left(\mathcal{P}^{(2)}\right)^{-1}\Big|_{\mathcal{O}\left(\frac{p^2}{M_\pm^2}\right)} = -\frac{p^2}{2}\left(A - \mathrm{Re}\left[\frac{\rho_+}{M_+^2}\right]p^2\right) \quad \Rightarrow \quad i\mathcal{P}^{(2)}\left(p^2\right) = \frac{2}{A}\left[\frac{i}{-p^2} - \frac{i}{-p^2 + \frac{A}{\mathrm{Re}\left[\frac{\rho_+}{M_+^2}\right]}}\right]. \tag{3.60}$$

Thus, we recover the well-known result that linearised Einstein-Weyl gravity propagates an additional massive spin-2 particle beyond the massless graviton [123]. The squared mass of this particle is identified as $M^2 = -A\mathrm{Re}\left[\rho_+/M_+^2\right]^{-1}$. The negative sign in front of its contribution to the propagator, relative to the contribution of the massless graviton, indicates that this additional massive spin-2 degree of freedom is a ghost if $M^2 > 0$, and a tachyon if $M^2 < 0$.

Next, suppose that the masses of the fields $\chi^\pm$ vanish, such that $M_\pm^2 = 0$. In this case, the effective Lagrangian (3.57) reduces to a linearised non-local version of Einstein-Weyl gravity [122]. The kinetic term (3.58) simplifies to

$$\left(\mathcal{P}^{(2)}\right)^{-1}\Big|_{M_\pm^2=0} = -\frac{p^2}{2}(A - \mathrm{Re}[\rho_+]) \quad \Rightarrow \quad i\mathcal{P}^{(2)}\left(p^2\right) = \frac{2}{A - \mathrm{Re}[\rho_+]}\frac{i}{-p^2}. \tag{3.61}$$

Thus, in this case, the spectrum consists only of a massless spin-2 particle, unless $2\mathrm{Re}[\rho_+] = \rho_+ + \rho_- = 2A$. If the last equality holds, the kinetic term for the spin-2 mode becomes degenerate. Concretely, this relation manifests a shift symmetry of the massless gauge-invariant Lagrangian for area-metric perturbations (3.42), which will be analysed in the next subsection 3.1.4. Using (3.6) and (3.56), we see that in this special case the effective Lagrangian (3.57) becomes

$$\begin{aligned} \mathcal{L}_{\mathrm{eff}}(h_{\mu\nu}) &= A\left[\mathcal{L}_{\mathrm{EH}}(h_{\mu\nu}) - {}^{(1)}C_{\mu\nu\rho\sigma}\frac{1}{-p^2}{}^{(1)}C^{\mu\nu\rho\sigma}\right] \\ &= A\left[h^{\mu\nu}\frac{(-p^2)}{2}\left(P^{(2)}{}_{\mu\nu}{}^{\rho\sigma} - 2P^{(0)}{}_{\mu\nu}{}^{\rho\sigma}\right)h_{\rho\sigma} - \frac{(-p^2)}{2}h^{\mu\nu}P^{(2)}{}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma}\right] \\ &= -Ah^{\mu\nu}\left(-p^2\right)P^{(0)}{}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma}\,. \end{aligned} \tag{3.62}$$

The additional Weyl-squared contribution, beyond the Einstein-Hilbert term, increases the gauge symmetry of the effective Lagrangian and eliminates the propagating massless spin-2 mode from the spectrum, such that only the non-propagating scalar mode is left. It should be emphasised that this conclusion is made here only for the linearised theory.

Finally, let us consider the special case when the identity $\rho_+ + \rho_- = 2A$ holds, but the mass parameters are non-zero and identical, $M_\pm^2 = M^2$. Under these assumptions, the kinetic term in (3.58) simplifies to

$$\left(\mathcal{P}^{(2)}\right)^{-1}\Big|_{\rho_+ + \rho_- = 2A\,,\, M_\pm^2 = M^2} = \frac{p^2}{2}\frac{AM^2}{-p^2 - M^2} \qquad \Rightarrow \qquad i\mathcal{P}^{(2)}\left(p^2\right) = \frac{2}{A}\left[\frac{i}{-p^2} - \frac{i}{-M^2}\right]. \tag{3.63}$$

Notably, in this case, the spin-2 propagator does not exhibit any additional poles beyond the massless graviton pole, and only receives a constant contribution proportional to $1/M^2$. This contact term, in a quantum field theory, will generically lead to a higher degree of divergencies in loop integrals. Moreover, this term has a wrong sign, such that one may anticipate negative-energy modes in the spectrum of the linearised local area-metric Lagrangian (3.42), when $\rho_+ + \rho_- = 2A$ and $M_\pm^2 = M^2$ holds. This class of actions will be analysed in detail in the next subsection 3.1.4, where we will show that the first relation implies a shift symmetry in the kinetic term, which is exact only in the absence of a mass term. Mass terms for the $\chi^\pm$ fields break this shift symmetry and thereby allow for local propagating degrees of freedom. The nature of these degrees of freedom can be anticipated by considering the equations of motion for the fields h and $\chi^\pm$, and is established explicitly through a Hamiltonian analysis in appendix A.2. Altogether, these analyses show that the two-parameter linearised area-metric actions, with the above conditions on the couplings and masses, propagate five massive modes, in addition to a massless spin-2 mode. Nevertheless, the effective Lagrangian for the length-metric perturbations $h_{\mu\nu}$ in (3.57),

$$\mathcal{L}_{\text{eff}}(h_{\mu\nu}) = A\left[\mathcal{L}_{\text{EH}}(h_{\mu\nu}) - {}^{(1)}C_{\mu\nu\rho\sigma}\frac{1}{-p^2 - M^2}{}^{(1)}C^{\mu\nu\rho\sigma}\right], \tag{3.64}$$

is ghostfree.

3.1.4 Subclasses of area-metric actions with shift-symmetric kinetic term

The gauge-invariant Lagrangian for area-metric perturbations (3.42) exhibits an additional shift symmetry in the kinetic term, for special values of the couplings $\alpha_\pm$ and $\beta_\pm$. To see this, it is useful to rewrite the Lagrangian in the form

$$\mathcal{L} = \sum_\pm \mathcal{L}_\pm\,, \tag{3.65}$$

where

$$\mathcal{L}_\pm = \frac{A}{2}\mathcal{L}_{\text{EH}}(h_{\mu\nu}) - \frac{1}{2}\alpha_\pm h_{\mu\nu}\chi^{\pm\mu\nu}p^2 - \frac{1}{4}\beta_\pm\chi^\pm_{\mu\nu}\chi^{\pm\mu\nu}p^2 - \frac{1}{4}m_\pm^2\chi^\pm_{\mu\nu}\chi^{\pm\mu\nu}\,. \tag{3.66}$$

Using (3.39) and ${P^{(0)}}_{\mu\nu}{}^{\rho\sigma}\chi^{\pm}_{\rho\sigma} = 0$, together with the Einstein-Hilbert action (3.6) expressed in terms of the Fierz-Pauli operator (3.7), we observe that the following sum of squares,

$$\frac{1}{2}\sum_{\pm}\left[\frac{\alpha_\pm}{\sqrt{\beta_\pm}}h^{\mu\nu} + \sqrt{\beta_\pm}\chi^{\pm\mu\nu}\right]\frac{(-p^2)}{2}\left({P^{(2)}}_{\mu\nu}{}^{\rho\sigma} - 2{P^{(0)}}_{\mu\nu}{}^{\rho\sigma}\right)\left[\frac{\alpha_\pm}{\sqrt{\beta_\pm}}h_{\rho\sigma} + \sqrt{\beta_\pm}\chi^{\pm}_{\rho\sigma}\right], \tag{3.67}$$

simplifies to

$$\frac{1}{2}(\rho_+ + \rho_-)\mathcal{L}_{\mathrm{EH}}(h_{\mu\nu}) - \frac{1}{2}\sum_{\pm}\left[\alpha_\pm h_{\mu\nu}\chi^{\pm\mu\nu}p^2 + \frac{1}{2}\beta_\pm\chi^{\pm}_{\mu\nu}\chi^{\pm\mu\nu}p^2\right], \tag{3.68}$$

where $\rho_\pm$ are defined in (3.50). This allows us to identify an additional shift symmetry, beyond the linearised diffeomorphism symmetry, in the kinetic term of the Lagrangian (3.65), for special values of the couplings satisfying

$$\rho_+ + \rho_- = 2A\,. \tag{3.69}$$

This condition implies $\mathrm{Re}[\rho_\pm] = A$, but $\rho_\pm$ can still have a non-zero imaginary part. We have seen previously, that such a non-zero imaginary part signalises parity violation in the area-metric Lagrangian. When the condition (3.69) is satisfied, the kinetic term in the Lagrangian (3.32) can be written as

$$\mathcal{L}\Big|_{\sum_\pm \rho_\pm = 2A\,,\, m^2_\pm = 0} = \frac{A}{2}\sum_{\pm}\mathcal{L}_{\mathrm{EH}}\big(h^{\pm}_{\mu\nu}\big)\,, \tag{3.70}$$

with fields $h^\pm$ defined by

$$h^{\pm}_{\mu\nu} = \frac{\alpha_\pm}{\sqrt{\beta_\pm}}h_{\mu\nu} + \sqrt{\beta_\pm}\chi^{\pm}_{\mu\nu}\,. \tag{3.71}$$

The shift symmetry in the kinetic term of (3.65), when (3.69) holds, can be expressed in view of (3.67) explicitly. It is parametrised by five degrees of freedom encoded in a real symmetric transverse-traceless rank-2 tensor $\tau_{\mu\nu}$, via [1]

$$h_{\mu\nu} \;\to\; h_{\mu\nu} + \tau_{\mu\nu}\,, \tag{3.72}$$

$$\chi^{\pm}_{\mu\nu} \;\to\; \chi^{\pm}_{\mu\nu} - \frac{\alpha_\pm}{\beta_\pm}\tau_{\mu\nu}\,. \tag{3.73}$$

This transformation maps h into another real field and preserves the relation $\chi^+ = \overline{\chi^-}$. The consequences of this shift symmetry for the spin-2 propagator in the effective Lagrangian for $h_{\mu\nu}$ were observed in the previous subsection. When the fields $\chi^\pm$ are massless, this shift symmetry is exact and leads to a degenerate kinetic term in the effective length-metric Lagrangian. In this case, any propagating degrees of freedom therein are eliminated, see e.g. (3.62). Mass terms for

1. Thus, in particular, $\tau_{\mu\nu}$ satisfies ${P^{(2)}}_{\mu\nu}{}^{\rho\sigma}\tau_{\rho\sigma} = \tau_{\mu\nu}$.

$\chi^{\pm}$ in the area-metric Lagrangian break this shift symmetry, and allow for propagating degrees of freedom in the effective Lagrangian for $h_{\mu\nu}$. When the two mass parameters $M_{\pm}^2$, defined in (3.50), are identical, the effective Lagrangian for $h_{\mu\nu}$ is given by (3.64). The associated spin-2 propagator (3.63) is ghostfree and exhibits only the massless graviton pole.

In the remaining part of this subsection, we will shed light on the nature of the above degeneracy and on the role of the mass terms in identifying what types of degrees of freedom are propagated in the local area-metric Lagrangian. To that end, we will consider analogue Lagrangians as functions of $h_{\mu\nu}$ and the fields $\chi^{\pm}_{\mu\nu}$, defined from $\chi^{\pm}_{ab}$ through the non-local embedding (3.38). These considerations are supplemented by an explicit Hamiltonian analysis of the corresponding two-parameter subclass of linearised area-metric theories, parametrised in terms of the local fields $h_{\mu\nu}$ and $\chi^{\pm}_{ab}$, in appendix A.2.

First, let us consider the chiral Lagrangians $\mathcal{L}_{\pm}$ defined in (3.65), separately with the condition $\rho_{\pm} = A$. By an analogous computation as in going from (3.67) to (3.68), it follows that

$$\mathcal{L}_{\pm}\Big|_{\rho_{\pm}=A} = \frac{A}{2}\mathcal{L}_{\mathrm{EH}}\big(h^{\pm}_{\mu\nu}\big) - \frac{1}{2}m_{\pm}^2\chi^{\pm}_{\mu\nu}\chi^{\pm\mu\nu}\,. \tag{3.74}$$

When the field $\chi^{\pm}$ is massless, the last expression reduces to a multiple of the linearised Einstein-Hilbert action for the shifted metric $h^{\pm}$ defined in (3.71). In Euclidean signature, such a massless chiral Lagrangian arises in a linearisation of the effective action, obtained after integrating out the connection from an su(2) topological BF theory, and can be seen to propagate no local degrees of freedom [99, 102]. This is due to a 5-parameter shift symmetry, analogous to (3.72) and (3.73). However, a few cautions should be mentioned.

This first comment concerns the choice of signature. Throughout the previous discussions, we assumed Lorentzian signature and therefore complex fields $\chi^{\pm}$. In particular, $h^{\pm}$ defined in (3.71) are complex fields. In fact, the relation $\mathcal{L}_{+} = \overline{\mathcal{L}_{-}}$ in (3.65), with $\mathcal{L}_{\pm}$ themselves complex, is essential in order to guarantee that the full non-chiral Lagrangian $\mathcal{L}$ is real. Therefore, in a Lorentzian chiral version of the theory (3.65) with Lagrangian $\mathcal{L}_{+}$ or $\mathcal{L}_{-}$, defined as in (3.74), reality conditions would still need to be imposed. This issue does not arise in Euclidean signature, where the fields $\chi^{\pm}$ as well as the couplings $\alpha_{\pm}$ and $\beta_{\pm}$ are real. The reason is that, in Euclidean signature, the eigenvalues of the Hodge star operator acting on 2-forms are given by ± 1, instead of $\pm i$ in Lorentzian signature. This is also the reason, why in Euclidean signature the Lagrangian quadratic in $\chi^{\pm}$ does not encounter the previously mentioned indefiniteness in sign.

The shift symmetry of the chiral Lagrangians $\mathcal{L}_{\pm}$, with $\rho_{\pm} = A$, is exact only in the absence of a mass term for $\chi^{\pm}$, and is broken when the mass parameters $m_{\pm}^2$ of these fields are non-zero. When this is the case, $\mathcal{L}_{\pm}$, according to (3.74), can be written as the linearised Einstein-Hilbert action for the field $h^{\pm}$ in (3.71) with an additional auxiliary non-propagating massive field $\chi^{\pm}$. This observation in Euclidean signature suggests that the linearised theory in Lorentzian signature, with reality conditions imposed, propagates only two local degrees of freedom of a massless graviton as in general relativity. However, the field redefinition required to arrive at this conclusion is non-local. This becomes clear from the definition of the $\chi^{\pm}$ fields in equation (3.38), which implies

$$h_{\mu\nu} \rightarrow h^{\pm}_{\mu\nu} = \frac{\alpha_{\pm}}{\sqrt{\beta_{\pm}}} h_{\mu\nu} + \sqrt{\beta_{\pm}} \omega^{\pm}_{\rho\mu\sigma\nu} \frac{p^{\rho} p^{\sigma}}{p^2} \, . \tag{3.75}$$

Nevertheless, so-called modified chiral or selfdual Plebanski theories [94, 95, 98–101] indicate that there are entire classes of theories, closely related to general relativity, with only two local propagating degrees of freedom.

In the remainder of this subsection, we consider the equations of motion derived from the shift-symmetric subclass of area-metric Lagrangians with identical effective mass parameters for the non-metric fields, using the parametrisation of area-metric perturbations in terms of $h_{\mu\nu}$ and $\chi^{\pm}_{\mu\nu}$. These allow us to elucidate on the number of propagating degrees of freedom described by these theories. However, it should be emphasised that the fields $\chi^{\pm}_{\mu\nu}$ arise from $\chi^{\pm}_{ab}$ through the non-local embedding (3.38). Therefore, any tentative conclusions made about the counting of degrees of freedom are in addition supported and confirmed explicitly through a Hamiltonian analysis based on the local fields, $h_{\mu\nu}$ and $\chi^{\pm}_{ab}$, in appendix A.2.

According to the result derived in subsection 3.1.2, the most general diffeomorphism-invariant second-order quadratic action for area-metric perturbations, around a background induced by the flat Minkowski metric, can be written as

$$\begin{aligned} S \;=\; & \int \mathrm{d}^4 x \Bigg(A \Big[-\frac{1}{2} \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} + \partial^\mu h_{\mu\rho} \partial_\nu h^{\nu\rho} - \partial_\mu h^{\mu\nu} \partial_\nu h + \frac{1}{2} \partial^\mu h \partial_\mu h \Big] \\ & - \frac{1}{2} \sum_{\pm} \Big[\alpha_\pm \partial_\alpha h_{\mu\nu} \partial^\alpha \chi^{\pm\mu\nu} + \frac{1}{2} \beta_\pm \partial_\alpha \chi^{\pm}_{\mu\nu} \partial^\alpha \chi^{\pm\mu\nu} + \frac{1}{2} m_\pm^2 \chi^{\pm}_{\mu\nu} \chi^{\pm\mu\nu} \Big] \Bigg) \, . \end{aligned} \tag{3.76}$$

The equations of motion for $h_{\mu\nu}$ and $\chi^{\pm}_{\mu\nu}$, in transverse-traceless gauge for $h_{\mu\nu}$, can be expressed as

$$2A \Box h_{\mu\nu} + \sum_{\pm} \alpha_\pm \Box \chi^{\pm}_{\mu\nu} \;=\; 0 \, , \tag{3.77}$$

$$\rho_\pm \Box h_{\mu\nu} + \alpha_\pm \Box \chi^{\pm}_{\mu\nu} - M_\pm^2 \alpha_\pm \chi^{\pm}_{\mu\nu} \;=\; 0 \, , \tag{3.78}$$

where we remind the reader of the coupling combinations

$$\rho_\pm = \frac{\alpha_\pm^2}{\beta_\pm} \qquad \text{and} \qquad M_\pm^2 = \frac{m_\pm^2}{\beta_\pm} \, . \tag{3.79}$$

Subtracting the two equations (3.78) from equation (3.77), we arrive at

$$(2A - \rho_+ - \rho_-)\Box h_{\mu\nu} + \sum_{\pm} M_\pm^2 \alpha_\pm \chi^\pm_{\mu\nu} \;=\; 0\,. \tag{3.80}$$

Now, let us assume that the relation $2A = \rho_+ + \rho_-$ in (3.69), for a shift-symmetric kinetic term, is satisfied. In this case, the $\Box h_{\mu\nu}$ part drops out and we obtain a relation between χ^+ and χ^-,

$$M_-^2 \alpha_- \chi^-_{\mu\nu} + M_+^2 \alpha_+ \chi^+_{\mu\nu} = 0\,. \tag{3.81}$$

This relation suggests that five degrees of freedom are redundant in the description, as anticipated based on the 5-parameter shift symmetry of the Lagrangian under the transformations (3.72) and (3.73). Inserting (3.81) into the equation of motion for χ^- in (3.78), and therein replacing $\Box\chi^+_{\mu\nu}$ eliminated from the equation of motion for χ^+, leads to

$$\left(\frac{M_+^2}{M_-^2}\rho_+ + \rho_-\right)\Box h_{\mu\nu} + M_+^2\alpha_+\left(1 - \frac{M_+^2}{M_-^2}\right)\chi^+_{\mu\nu} = 0\,. \tag{3.82}$$

If additionally $M_\pm^2 = M^2$ holds, we are left with

$$2A\Box h_{\mu\nu} = 0\,. \tag{3.83}$$

This is the usual massless Klein-Gordon equation for the transverse-traceless component of the metric fluctuation in linearised general relativity, with solution given by plane waves.

Finally, there are the equations of motion for $\chi^\pm$ in (3.78). When (3.83) holds, their sum and difference become

$$\big(\Box - M^2\big)\big(\alpha_+\chi^+_{\mu\nu} + \alpha_-\chi^-_{\mu\nu}\big) = 2\big(\Box - M^2\big)\chi^1_{\mu\nu} \;=\; 0\,, \tag{3.84}$$

$$\big(\Box - M^2\big)\big(\alpha_+\chi^+_{\mu\nu} - \alpha_-\chi^-_{\mu\nu}\big) = 2i\big(\Box - M^2\big)\chi^2_{\mu\nu} \;=\; 0\,, \tag{3.85}$$

where the two real fields $\chi^{1,2}$ are defined by the real and imaginary parts of $\alpha_+\chi^+$,

$$\alpha_\pm\chi^\pm = \chi^1_{\mu\nu} \pm i\chi^2_{\mu\nu}\,. \tag{3.86}$$

However, the relation (3.81), with $M_\pm^2 = M^2$, implies $\chi^1 = 0$. Therefore, we are left with a massless Klein-Gordon equation for $h_{\mu\nu}$ (3.83), and a massive Klein-Gordon equation for $\chi^2_{\mu\nu}$,

$$\big(\Box - M^2\big)\chi^2_{\mu\nu} = 0\,. \tag{3.87}$$

Altogether, the previous observations suggest that the physical spectrum of linearised area-metric actions, with shift-symmetric kinetic term and identical effective mass parameters for the non-metric fields, consists of two massless transverse-traceless modes and five additional massive modes. This expectation is confirmed explicitly through a Hamiltonian analysis, based on the local fields $h_{\mu\nu}$ and $\chi^\pm_{ab}$, in appendix A.2.

3.2 Area-metric actions from non-chiral modified Plebanski theories

In this section we identify a concrete framework for non-linear area-metric gravity. This framework underlines the physical relevance of the two-parameter subclass of linearised area-metric actions which lead to a ghostfree effective length-metric action. For simplicity, the analysis in this entire section is presented in Euclidean signature. The derivation is inspired by the Plebanski formulation of general relativity which involves as fundamental variables a connection and a Lie-algebra valued 2-form known as the B-field. Plebanski gravity has been originally formulated in terms of chiral or self-dual variables [82–84]. However, the Plebanski action can also be defined using non-chiral variables [85–87, 96, 124]. In both cases, the action can be viewed as a topological BF theory [38], supplemented by constraints on the B-field. The latter break the shift symmetry of the topological action, and thereby lead to propagating degrees of freedom. Concretely, the role of the so-called simplicity constraints is to reduce the degrees of freedom of the B-field, to the degrees of freedom of a tetrad, which consequently leads to the Einstein-Cartan action. The latter is equivalent to the Einstein action for general relativity for non-degenerate tetrads, after solving the equation of motion for the connection.

In modified Plebanski theories [88–103], the Lagrange multiplier, which imposes the simplicity constraints in the Plebanski action, is promoted to an auxiliary field with a potential. Thereby, the equations of motion for this field do not represent constraints on the B-field anymore, but algebraic equations which fix the components of this field as functions of the B-field. Integrating out the connection and the auxiliary field leads to an effective action with a potential for the B-field. Despite the modified dynamics due to non-trivial curvature of the connection, canonical analyses of modified self-dual Plebanski actions reveal that these theories propagate only two local degrees of freedom and thereby can be viewed as neighbors of general relativity [94, 95, 98–100]. By contrast, the physical content of modified non-chiral Plebanski theories differs in general substantially from that of general relativity, and for non-degenerate potentials consists of eight propagating degrees of freedom [97, 102, 103].

In this section we show that subclasses of modified non-chiral Plebanski theories with partially degenerate potential provide a non-perturbative framework for area-metric gravity. To that end, we first construct an area metric from the B-field. The entire analysis is performed in Euclidean signature, with local gauge group SO(4). The B-field contains more gauge-invariant degrees of freedom than required for an area metric. These have to be reduced by a suitable subset. To achieve this reduction, we use the left-right handed splitting of the Lie algebra, and the parametrization of an SU(2) B-field in terms of a spacetime tetrad and a unimodular internal frame field [99, 125]. Thereby the gauge-invariant content of the SO(4) B-field in the non-chiral Plebanski action can be described in terms of two length metrics and two unimodular internal metrics which represent a set of spacetime scalars. Identifying the two length metrics with each other leads to the correct number of degrees of freedom for a generic area metric. Crucially, this identification entails imposing only a subset of the simplicity constraints in the original non-chiral Plebanski action. We supplement this action by the Holst term [126], with coupling given by the inverse of the so-called Immirzi parameter γ [80, 81]. Additionally, we introduce a potential in the form of mass terms for the unconstrained components of the B-field. When the connection has been integrated out, the resulting effective action is a second-order non-linear action describing the dynamics of a length metric and ten spacetime scalars as implicit functions of the area metric. The expansion of this action at second order in the fluctuations of these fields reproduces the subclass of linearised area-metric actions with shift-symmetric kinetic term. The coupling between the length-metric and non-metric degrees of freedom of the area-metric perturbation is determined by the Immirzi parameter γ, which is thereby identified as a parity-breaking coupling in area-metric gravity.

The conventions in this section are as follows. Greek letters $\mu, \nu, ... = 0, 1, 2, 3$ label Euclidean spacetime indices. The expressions $\epsilon^{\mu\nu\rho\sigma}$ and $\epsilon_{\mu\nu\rho\sigma}$ denote the totally antisymmetric Levi-Civita symbol with tensor-density weight ± 1, respectively. The spacetime metric is not used to raise or lower indices of these densities. Latin capital letters $I, J, ... = 0, 1, 2, 3$ label internal SO(4) group indices, whereas antisymmetric index pairs $[IJ]$ provide labels for the so(4) basis. Latin lowercase letters $a, b, ... = 1, 2, 3$ label a basis of the su(2) algebra. Moreover, we use the bilinear forms $\delta_{IJKL} = \frac{1}{2}(\delta_{IK}\delta_{JL} - \delta_{IL}\delta_{JK})$ and $\frac{1}{2}\epsilon_{IJKL} = \frac{1}{2}\epsilon_{KLIJ}$ on the Lie algebra so(4). Here, ϵ_{IJKL} is a completely antisymmetric tensor taking values ± 1, depending on the orientation chosen for the internal space. This orientation can be fixed by tying it to the orientation of the spacetime manifold, such that for a tetrad e^I_μ the following relation holds, $\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu e^K_\rho e^L_\sigma = 24\det(e)$, see e.g. [127]. Finally, the Hodge star operator associated with the flat internal metric δ_{IJ} is defined by $(\star)^{IJ}_{KL} = \frac{1}{2}\epsilon^{IJ}_{KL}$ and squares to the identity.

3.2.1 Non-chiral modified Plebanski action

This subsection provides a review of non-chiral Plebanski theory [85–87, 96, 124], which can be understood as a topological BF theory [38], supplemented by constraints. Subsequently, we consider modified non-chiral Plebanski theories [97, 102, 103], in which all or a subset of these constraints are replaced by a potential.

The fundamental variables in the non-chiral Plebanski action are a connection 1-form ω^{IJ}_μ, with associated curvature 2-form denoted by $F^{IJ}_{\mu\nu}(\omega)$, and a 2-form $B^{IJ}_{\mu\nu}$, referred to as the B-field and valued in the Lie algebra so(4) of the local gauge group SO(4). The action is given by [1]

$$S_{\text{Plebanski}}[\omega, B, \phi] = S_{\text{BF}}[\omega, B] - \frac{1}{2}\int \phi_{IJKL}B^{IJ} \wedge B^{KL} \quad \text{where} \tag{3.88}$$

$$S_{\text{BF}}[\omega, B] = \int \delta_{IJKL}B^{IJ} \wedge F^{KL}(\omega)\,. \tag{3.89}$$

Here ϕ_{IJKL} is a Lagrange multiplier satisfying

$$\phi_{IJKL} = \phi_{KLIJ} = -\phi_{KLJI} \quad \text{and} \quad \phi_{IJKL}\epsilon^{IJKL} = 0\,. \tag{3.90}$$

1. In this section, the prefactor $\frac{1}{2\kappa^2} = \frac{1}{16\pi G}$ in front of the Plebanski action is set to one, unless otherwise stated.

The equations of motion, obtained by varying (3.88) with respect to ω and B, are

$$\mathrm{d}_\omega B^{IJ} \equiv \mathrm{d}B^{IJ} + [\omega, B]^{IJ} = 0 \qquad \text{and} \qquad F^{IJ}(\omega) = \phi^{IJKL} B_{KL} \,. \tag{3.91}$$

Variation with respect to the Lagrange multiplier ϕ imposes the so-called simplicity constraints [1]

$$B^{IJ} \wedge B^{KL} = \mathcal{V} \epsilon^{IJKL} \qquad \text{where} \qquad \mathcal{V} = \frac{1}{4!} B^{IJ} \wedge B^{KL} \epsilon_{IJKL} \,. \tag{3.92}$$

The Plebanski action (3.88) without the Lagrange multiplier imposing the simplicity constraints reduces to the BF action (3.89). The latter describes a topological theory with no local propagating degrees of freedom. This is due to its invariance under the shift transformation $\delta_\alpha B^{IJ} = \mathrm{d}_\omega \alpha^{IJ}$ and $\delta_\alpha \omega^{IJ} = 0$, parametrised by a Lie algebra valued 1-form α^{IJ}, as well as its invariance under gauge transformations. The first equation of motion in (3.91) is known as the Gauss law and imposes compatibility of the connection ω with the bivector field B. The second equation in (3.91) imposes zero curvature $F(\omega) = 0$, and therefore a flat connection. The shift and gauge symmetries of the BF action identify all solutions to the equations of motion as locally equivalent.

Let us now consider the full Plebanski action (3.88) with the Lagrange multiplier imposing the simplicity constraints (3.92). From (3.90), it follows that ϕ_{IJKL} has 20 independent components. Therefore, equation (3.92) represents 20 constraints on the a priori 36 algebraically independent components of the B-field. There are four independent solutions sectors to (3.92), parametrised by a spacetime tetrad e^I_μ, and given by

$$B^{IJ} = \pm e^I \wedge e^J \qquad \text{and} \qquad B^{IJ} = \pm \frac{1}{2} \epsilon^{IJ}{}_{KL} e^K \wedge e^L \,. \tag{3.93}$$

Thus, the simplicity constraints reduce the B-field to a simple bivector determined by the wedge product of two tetrads.

The first set of solutions in (3.93) is known as the topological sector, and results in an action with no local propagating degrees of freedom. This can be seen by inserting this set of solutions for the B-field into the Plebanski action, to arrive at

$$S[\omega, e] = \pm \int e^I \wedge e^J \wedge F_{IJ}(\omega) \,, \tag{3.94}$$

which is proportional to the Holst action [126]. For non-degenerate tetrads, $\det(e) \neq 0$, the connection can be integrated out by solving its equation of motion, $\mathrm{d}_\omega e^I = 0$. The latter is known as the first Cartan structure equation and imposes zero torsion. The solution is the torsion-free spin connection, $\omega^{IJ}_\mu(e) = e^I_\nu \nabla_\mu e^{\nu J}$, whose curvature is related to the Riemann tensor through the second Cartan structure equation, $R_{\mu\nu\rho\sigma}(e) = e_{I\rho} e_{J\sigma} F^{IJ}_{\mu\nu}(\omega(e))$. As a result, the expression in (3.94) becomes proportional to $\tilde{\epsilon}^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$, which vanishes due to the algebraic Bianchi identity. Thus, this action is topological, as the field equations vanish identically.

The second set of solutions in (3.93) defines the gravitational sector. When inserted into the

1. The 4-form $\mathcal{V}$ can be determined by contracting the first equation in (3.92) with ϵ_{IJKL} on both sides.

Plebanski action, these solutions lead to a multiple of the Einstein-Cartan action

$$S[\omega, e] = \pm\frac{1}{2}\int \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}(\omega)\,. \tag{3.95}$$

For non-degenerate tetrads, the connection onshell is given by the torsion-free spin connection $\omega(e)$, and in this case (3.95) coincides with the first-order Palatini action for general relativity. Making use of the second Cartan structure equation leads to the Einstein action

$$S[e] = \pm\int \det(e) R(e)\,. \tag{3.96}$$

In summary, the role of the 20 simplicity constraints in recovering classical gravity from the Plebanski action, is to reduce the 36 algebraically independent components of the B-field to 16 components of a tetrad e^μ_I. This, after an SO(4) gauge reduction, leads to the 10 degrees of freedom of the metric $g_{\mu\nu}$ in general relativity.

In the following, we consider modified non-chiral Plebanski theories [97, 102, 103]. Therein, the non-chiral Plebanski action (3.88) is supplemented by an additional term, which represents a potential $V(\phi)$ for the field ϕ_{IJKL}. The resulting action can be written as

$$\begin{aligned} S_{\text{modified Plebanski}}[\omega, B, \phi] &= S_{\text{Plebanski}}[\omega, B, \phi] + S_{\text{Potential}}[B, \phi]\,, \qquad \text{where} \qquad (3.97)\\ S_{\text{Potential}}[B, \phi] &= -\frac{1}{12}\int V(\phi)\epsilon_{IJKL} B^{IJ} \wedge B^{KL}\,. \qquad (3.98)\end{aligned}$$

The equations of motion, obtained after variation with respect to ω and B, are

$$\mathrm{d}_\omega B^{IJ} \equiv \mathrm{d}B^{IJ} + [\omega, B]^{IJ} = 0 \qquad \text{and} \qquad F^{IJ}(\omega) = \phi^{IJKL} B_{KL} + \frac{1}{6} V(\phi)\epsilon^{IJKL} B_{KL}\,. \tag{3.99}$$

Thus, the additional potential in the action does not alter the Gauss law, but introduces non-trivial curvature. Moreover, the equations of motion for the auxiliary field ϕ are now

$$B^{IJ} \wedge B^{KL} = \mathcal{V}\left(\epsilon^{IJKL} - 4\frac{\partial V(\phi)}{\partial \phi_{IJKL}}\right), \tag{3.100}$$

where $\mathcal{V}$ is defined in (3.92). The previous equation does not represent constraints on the B-field anymore, as it does sfor the original Plebanski action. Instead, these are 20 algebraic equations which can be solved for $\phi(B)$ provided the function $V(\phi)$ admits a non-degenerate Hessian. Thereby the auxiliary field ϕ can be eliminated from the modified Plebanski action (3.97). This, in turn, leads to an effective potential $V(B)$ of the B-field. However, in view of (3.99), the resulting curvature is in general a complicated function for the B-field, and thus the dynamics of modified non-chiral Plebanski theories differs in general from the dynamics of the non-chiral Plebanski action. For non-degenerate potentials $V(\phi)$, these theories describe eight propagating degrees of freedom [97, 102, 103].

More generally, the potential $V(\phi)$ can have constant directions or, equivalently, one may consider adding a potential only for a subset of the components of the Lagrange multiplier field ϕ_{IJKL} in the original Plebanski action (3.88). In this case, the part of the simplicity constraints (3.92) on B corresponding to the constant directions of $V(\phi)$ is still present, whereas the remaining components of B acquire a potential $V(B)$, after solving the analogue of (3.100) in the form $\phi(B)$. Examples for partially degenerate potentials, resulting in a bimetric theory of gravity, are considered in [102, 103].

The previous observation is the main ingredient, which allows us to interpret a subclass of modified non-chiral Plebanski actions as non-linear actions for an area metric. To that end, our goal is to find a subset of simplicity constraints, to be imposed on the B-field, in oder to reduce its gauge-invariant degrees of freedom to the correct number required for a general area metric. The explicit construction of such an area metric is performed in the next subsection.

We close this subsection with a final remark. Generically, there are many different possibilities for partially degenerate potentials. These can be classified in terms of the decomposition of the field ϕ in the Plebanski action (3.88) into irreducible representations under the action of the group SO(4). The field ϕ_{IJKL}, obeying the symmetries (3.90), decomposes into the SO(4) irreducibles

$$\phi \;\in\; (0,0) \oplus (1,1) \oplus (2,0) \oplus (0,2)\,. \tag{3.101}$$

Here, $(0,0)$ describes one scalar degree of freedom proportional to the trace of ϕ, whereas $(1,1)$ can be associated with a symmetric and traceless rank-2 tensor with 9 degrees of freedom. These two components can be thought of as the Ricci scalar and the traceless part of the Ricci tensor for ϕ_{IJKL}, by analogy to the Ricci-Weyl decomposition of the Riemann tensor. The remaining two representations, $(2,0)$ and $(0,2)$, describe the selfdual and anti-selfdual Weyl components of ϕ_{IJKL}. These can be arranged into two symmetric and traceless rank-2 tensors on the internal spaces $\mathrm{su}(2)_\pm$, according to the Lie algebra isomorphism $\mathrm{so}(4) = \mathrm{su}(2)_+ \oplus \mathrm{su}(2)_-$.

3.2.2 Area metric from B-field via reduction of degrees of freedom

In this subsection, we construct an area metric from the SO(4) B-field and thereby provide an interpretation of a subclass of modified non-chiral Plebanski theories as non-linear actions for an area metric. Given an SO(4) B-field $B^{IJ}_{\mu\nu}$, an area metric can be defined by

$$G_{\mu\nu\rho\sigma}(B) \equiv B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma}\delta_{IJKL} - \frac{1}{4!}\big(B^{IJ}_{\alpha\beta}B^{KL}_{\gamma\delta}\delta_{IJKL}\epsilon^{\alpha\beta\gamma\delta}\big)\epsilon_{\mu\nu\rho\sigma}\,, \tag{3.102}$$

where the last term ensures $G_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = 0$, such that G is cyclic. Modulo this last term, the B-field can be thought of as a frame field for the area metric, similarly as a tetrad for the length metric. However, $B^{IJ}_{\mu\nu}$ contains more gauge-invariant degrees of freedom than a generic area metric. To derive an action describing the dynamics of an area metric, the gauge-invariant content of the B-field has to be reduced to the number of degrees of freedom contained in a generic area metric. This reduction can be achieved by imposing only a subset of the simplicity constraints on the B-field in the original non-chiral Plebanski action.

To achieve the reduction of B-field to area-metric degrees of freedom, we make use of the right-left handed splitting of the Lie algebra, according to the isomorphism $\mathrm{so}(4) = \mathrm{su}(2)_+ \oplus \mathrm{su}(2)_-$. This isomorphism decomposes the space of so(4)-valued 2-forms into right-handed and left-handed eigenspaces $\mathrm{su}(2)_\pm$, with respect to the internal Hodge star operator $(\star)^{IJ}_{KL}$. Accordingly, the SO(4) B-field can be written as

$$B^{IJ}_{\mu\nu} = B^{+IJ}_{\mu\nu} + B^{-IJ}_{\mu\nu} = P^{+IJ}_{a} B^{+a}_{\mu\nu} + P^{-IJ}_{a} B^{-a}_{\mu\nu} \qquad \text{where} \qquad P^{\pm IJ}_{a} \equiv \pm\delta^{IJ}_{0a} + \frac{1}{2}\epsilon^{IJ}_{0a}\,. \tag{3.103}$$

We remind the reader that Latin letters $a, b, ... = 1, 2, 3$ provide labels for the $\mathrm{su}(2)_\pm$ bases. The fields $B^{\pm a}_{\mu\nu}$ are $\mathrm{su}(2)_\pm$-valued 2-forms. Together with the $\mathrm{su}(2)_\pm$-connections $\omega^{\pm a}_{\mu}$, defined by the right-handed and left-handed projections of the so(4)-connection ω^{IJ}_{μ}, these fields represent the configuration variables in the $\mathrm{SU}(2)_\pm$ BF action.

A general $\mathrm{su}(2)_\pm$-valued 2-form $B^{\pm a}_{\mu}$ can be parametrised in terms of a tetrad $e^{\pm I}_{\mu}$ and a unimodular internal frame field $b^{\pm a}_{b}$ with $\det(b^\pm) = 1$ [99, 125]. Specifically, we can write

$$B^{\pm a}_{\mu\nu}\big(e^\pm, b^\pm\big) = \sigma_\pm b^{\pm a}_{b} \Sigma^{\pm b}_{\mu\nu}(e^\pm)\,, \tag{3.104}$$

where $\sigma_\pm = \pm 1$ is a sign introduced for later convenience, and $\Sigma^{\pm a}_{\mu\nu}$ are the Plebanski 2-forms associated with a given non-degenerate tetrad e^I_μ. Their definition in Euclidean signature is

$$\Sigma^{\pm a}_{\mu\nu}(e) = \pm\big(e^0_\mu e^a_\nu - e^0_\nu e^a_\mu\big) + \epsilon^{a}{}_{bc} e^b_\mu e^c_\nu = 2\delta^{ab}\delta_{IJKL} P^{\pm IJ}_{b} e^K_\mu e^L_\nu\,. \tag{3.105}$$

The Plebanski 2-forms are selfdual and anti-selfdual, respectively, with respect to the Hodge star operator defined by the spacetime metric $g_{\mu\nu} = e^I_\mu e^J_\nu \delta_{IJ}$. This property is expressed as

$$\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\Sigma^{\pm a}_{\rho\sigma} = \pm\Sigma^{\pm a}_{\mu\nu}\,, \tag{3.106}$$

where the expression involving ϵ on the left hand side stands for $\epsilon_{\mu\nu}{}^{\rho\sigma} = \sqrt{g} g^{\alpha\rho} g^{\beta\sigma}\epsilon_{\mu\nu\alpha\beta}$.

The algebraically independent components and gauge-invariant content of the SO(4) B-field are contained in the reparametrisation (3.103) and (3.104) as follows.

The field $B^{IJ}_{\mu\nu}$, as an so(4)-valued 2-form, has 36 independent components. In (3.103), this field is decomposed into two fields $B^{\pm a}_{\mu\nu}$, which are $\mathrm{su}(2)_\pm$-valued 2-forms with 18 independent components, respectively. According to (3.104), each of these fields $B^{\pm a}_{\mu\nu}$ can be parametrised in terms of the Plebanski 2-forms $\Sigma^{a\pm}_{\mu\nu}(e^\pm)$ of a tetrad $e^{\pm I}_{\mu}$, and a unimodular internal frame field $b^{\pm a}_{b}$. It should be noted that $\Sigma^{a\pm}_{\mu\nu}(e^\pm)$, defined in (3.105), depends only on 13 of the 16 components of the tetrad $e^{\pm I}_{\mu}$, as the internal direction $e^{\pm 0}$ has been fixed. For non-degenerate tetrads $e^{\pm I}_{\mu}$, the Plebanski 2-forms $\Sigma^{\pm a}_{\mu\nu}(e^\pm)$ form a basis of the 3-dimensional spaces of right-handed and left-handed 2-forms [84, 94, 99]. Therefore, $B^{\pm a}_{\mu\nu}$ can always be decomposed as a linear combination in the form $B^{\pm a}_{\mu\nu} = b^{\pm a}_{b}\Sigma^{\pm b}_{\mu\nu}(e^\pm)$, with a priori 9 independent coefficients $b^{\pm a}_{b}$. This decomposition is defined only up to an $\mathrm{SU}(2)_\pm$ rotation acting on the contracted index. Moreover, in this linear decomposition, a global rescaling of the tetrad, $e^{\pm I}_{\mu} \to \Omega e^{\pm I}_{\mu}$, can be compensated by a global rescaling of the internal frame field, $b^{\pm a}_{b} \to \Omega^{-2} b^{\pm a}_{b}$. This freedom of rescaling represents one degree of freedom, which can be absorbed by imposing the unimodularity condition, $\det(b^{\pm}) = 1$, on the 3×3 matrix with entries $b^{\pm a}_{b}$. Altogether, we recover the correct number of $13+9-3-1 = 18$ algebraically independent components of $B^{\pm a}_{\mu\nu}$.

Next, let us focus on the gauge-invariant content of the SO(4) B-field. Six of the 36 algebraically independent components of $B^{IJ}_{\mu\nu}$ are gauge degrees of freedom, associated with an SO(4) rotation. Therefore, $B^{IJ}_{\mu\nu}$ contains only 30 gauge-invariant degrees of freedom. Similarly, 3 of the 18 algebraically independent components of $B^{\pm a}_{\mu\nu}$ are gauge degrees of freedom, associated with an SO(3) rotation. Therefore, each of the fields $B^{\pm a}_{\mu\nu}$ contains only 15 gauge-invariant degrees of freedom. Out of these, 10 are encoded in the right or left metric associated with the tetrad $e^{\pm I}_{\mu}$, and defined by

$$g^{\pm}_{\mu\nu} = e^{\pm I}_{\mu} e^{\pm J}_{\nu} \delta_{IJ} \,. \tag{3.107}$$

This metric belongs to the same conformal class as the so-called Urbantke metric for the right-handed or left-handed sector, respectively. Given a triple of su(2)-valued 2-forms $B^{i}_{\mu\nu}$, the Urbantke metric $g^{U}_{\mu\nu}$ is defined by [99, 102, 125]

$$\sqrt{g^{U}}\, g^{U}_{\mu\nu} = \frac{1}{12} \epsilon_{abc} \epsilon^{\alpha\beta\gamma\delta} B^{a}_{\mu\alpha} B^{b}_{\beta\gamma} B^{c}_{\delta\nu} \,. \tag{3.108}$$

The previous statement thus follows by noticing that

$$\begin{aligned} \pm\frac{1}{12} \epsilon_{abc} \epsilon^{\alpha\beta\gamma\delta} B^{\pm a}_{\mu\alpha} B^{\pm b}_{\beta\gamma} B^{\pm c}_{\delta\nu} &= \pm\frac{\sigma_{\pm}}{12} \det(b) \epsilon_{abc} \epsilon^{\alpha\beta\gamma\delta} \Sigma^{\pm a}_{\mu\alpha}(e^{\pm}) \Sigma^{\pm b}_{\beta\gamma}(e^{\pm}) \Sigma^{\pm c}_{\delta\nu}(e^{\pm}) \\ &= \sigma_{\pm} \sqrt{g^{\pm}}\, g^{\pm}_{\mu\nu} \,. \end{aligned} \tag{3.109}$$

The other 5 gauge-invariant degrees of freedom of $B^{\pm a}_{\mu\nu}$ can be arranged into a unimodular internal metric $q^{\pm}_{ab}$, constructed from the frame field $b^{\pm a}_{b}$, and defined by [99, 102]

$$q^{\pm}_{ab} = b^{\pm c}_{a} b^{\pm d}_{b} \delta_{cd} \,. \tag{3.110}$$

As a symmetric 3×3 matrix with determinant $\det(q^{\pm}) = 1$, each field $q^{\pm}_{ab}$ has indeed 5 independent components. With this definition, it follows [1]

$$\pm\frac{1}{8} B^{\pm a}_{\mu\nu} B^{\pm b}_{\rho\sigma} \delta_{ac} \delta_{bd} \epsilon^{\mu\nu\rho\sigma} = \sqrt{g^{\pm}}\, q^{\pm}_{cd} \,. \tag{3.112}$$

Altogether, the 36 algebraically independent degrees of freedom of the SO(4) B-field in the non-chiral Plebanski action can be identified with the $10+10+8+8$ degrees of freedom of two

1. This identity follows from the definition of the Plebanski 2-forms $\Sigma^{\pm}_{\mu\nu}$ in (3.105), which implies

$$\Sigma^{a}_{\mu\nu}(e) \Sigma^{b}_{\rho\sigma}(e) g^{\mu\rho} g^{\nu\sigma} = 4\delta^{ab} \,. \tag{3.111}$$

length metrics, $g^{\pm}_{\mu\nu}$, and two unimodular internal frame fields, $b_b^{\pm a}$. Moreover, the 30 gauge-invariant degrees of freedom of the SO(4) B-field can be identified with the $10+10+5+5$ degrees of freedom of the two length metrics $g^{\pm}_{\mu\nu}$, and the two unimodular internal metrics $q^{\pm}_{ab}$.

In summary, it holds

$$\text{before SO(4) gauge reduction (36 d.o.f.)}: \quad B^{IJ}_{\mu\nu} \quad \leftrightarrow \quad \left(g^{+}_{\mu\nu}, g^{-}_{\mu\nu}, b_b^{+a}, b_b^{-a}\right), \tag{3.113}$$

$$\text{after SO(4) gauge reduction (30 d.o.f.)}: \quad B^{IJ}_{\mu\nu} \quad \leftrightarrow \quad \left(g^{+}_{\mu\nu}, g^{-}_{\mu\nu}, q^{+}_{ab}, q^{-}_{ab}\right). \tag{3.114}$$

Previously, we have seen that the role of the 20 simplicity constraints in the non-chiral Plebanski action is to reduce the 30 gauge-invariant degrees of freedom of the B-field to 10 degrees of freedom for classical gravity formulated in terms of a metric. The solutions to the simplicity constraints (3.93), expressed in terms of the parametrisation (3.104), are equivalent to the conditions

$$e_\mu^{+I} = e_\mu^{-I} \equiv e_\mu^I \qquad \text{and} \qquad b_b^{+a} = b_b^{-a} \equiv \delta_b^a\,, \tag{3.115}$$

as well as $\sigma_+ = \sigma_-$ for the topological sector, and $\sigma_+ = -\sigma_-$ for the gravitational sector. Concretely, the simplicity constraints identify the two length metrics $g^{\pm}_{\mu\nu}$ with each other and freeze the internal metrics $q^{\pm}_{ab}$ to the identity on the internal space. As a result, for classical general relativity the following identities hold [102],

$$g^{+}_{\mu\nu} = g^{-}_{\mu\nu} \equiv g_{\mu\nu} \qquad \text{and} \qquad q^{+}_{ab} = q^{-}_{ab} = \delta_{ab}\,. \tag{3.116}$$

These identities can be used to simplify the expression for the area metric $G_{\mu\nu\rho\sigma}$ defined from the SO(4) B-field $B^{IJ}_{\mu\nu}$ in (3.102). To that end, we note that, using the splitting (3.103), this area metric can be written as [1]

$$G_{\mu\nu\rho\sigma}(B) = \sum_{\pm} G^{\pm}_{\mu\nu\rho\sigma}\left(B^{\pm}\right), \tag{3.118}$$

with the right and left area metrics $G^{\pm}_{\mu\nu\rho\sigma}$ defined from the SU(2) B-fields $B^{\pm}_{\mu\nu}$ by

$$G^{\pm}_{\mu\nu\rho\sigma}\left(B^{\pm}\right) \equiv \frac{1}{2}\left[B^{\pm a}_{\mu\nu} B^{\pm b}_{\rho\sigma}\delta_{ab} - \frac{1}{4!}\left(B^{\pm a}_{\alpha\beta} B^{\pm b}_{\gamma\delta}\delta_{ab}\epsilon^{\alpha\beta\gamma\delta}\right)\epsilon_{\mu\nu\rho\sigma}\right]. \tag{3.119}$$

Since $G^{\pm}_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = 0$, these area metrics are cyclic, as is $G_{\mu\nu\rho\sigma}$ itself. Moreover, they are invariant under an $\mathrm{SU}(2)_{\pm}$ rotation acting on the internal index of $B^{\pm a}_{\mu\nu}$. $G^{\pm}_{\mu\nu\rho\sigma}$ can be expressed in terms of the parametrisation of $B^{\pm a}_{\mu\nu}$ in (3.104), with the definition of $q^{\pm}_{ab}$ in (3.110). Using

1. This rewriting uses the following identities for the projectors $P_a^{\pm IJ}$ defined in (3.103),

$$P_a^{\pm IJ} P_b^{\pm KL}\delta_{IJKL} = \delta_{ab} \qquad \text{and} \qquad P_a^{\pm IJ} P_b^{\mp KL}\delta_{IJKL} = 0\,. \tag{3.117}$$

the identity (3.112), the result is

$$G^{\pm}_{\mu\nu\rho\sigma}(e^{\pm},q^{\pm}) = \frac{1}{2}\left[\Sigma^{\pm a}_{\mu\nu}(e^{\pm})\Sigma^{\pm b}_{\rho\sigma}(e^{\pm})q^{\pm}_{ab} \mp \frac{1}{3}\sqrt{g^{\pm}}\,\mathrm{Tr}\big(q^{\pm}\big)\tilde{\epsilon}_{\mu\nu\rho\sigma}\right], \tag{3.120}$$

where $\mathrm{Tr}(q^{\pm}) = q^{\pm}_{ab}\delta^{ab}$.

The right and left area metrics $G^{\pm}$, defined in (3.119), do not provide a convenient parametrisation for the gauge-invariant content of $B^{\pm a}_{\mu\nu}$, which respectively consists of only 15 gauge-invariant degrees of freedom. Instead, they can be used to reformulate the 20 simplicity constraints for the gravitational solution sector in the non-chiral Plebanski action, in terms of 20 algebraically independent equations encoded in the condition [86]

$$G^{+}_{\mu\nu\rho\sigma} = G^{-}_{\mu\nu\rho\sigma}\,. \tag{3.121}$$

This is indeed the relation obtained by inserting the identities (3.116) into (3.120). Moreover, as a result, the area metric G in (3.118) reduces to the area metric G_g induced by the length metric $g_{\mu\nu} = e^I_\mu e^J_\nu \delta_{IJ}$. Thus, we obtain [1]

$$G_{\mu\nu\rho\sigma} = \sum_{\pm} G^{\pm}_{\mu\nu\rho\sigma}(e,\delta) = \frac{1}{2}\sum_{\pm}\Sigma^{\pm a}_{\mu\nu}(e)\Sigma^{\pm b}_{\rho\sigma}(e)\delta_{ab} = g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\rho\nu} = \left(G_g\right)_{\mu\nu\rho\sigma}\,. \tag{3.123}$$

Altogether, for general relativity the right and left area metrics (3.119) coincide and their sum (3.118) evaluates to the area metric induced by the length metric as the fundamental variable in classical metric gravity.

Now, let us return to our original goal of deriving an action which describes the dynamics of a generic area metric, starting from the non-chiral Plebanski action. To that end, it is necessary to weaken a subset of the simplicity constraints on the B-field. In the remainder of this subsection we will use the reparametrisation of the SO(4) B-field, in the form (3.114), in order to reduce its 30 gauge-invariant degrees of freedom to 20 degrees of freedom for a generic area metric. On these grounds, the resulting subclass of modified Plebanski theories can be viewed as non-linear implicit theories for an area metric.

An immediate possible choice for achieving the above reduction, is to impose that the right

1. This equality arises by using that the Plebanski 2-forms $\Sigma^{\pm a}_{\mu\nu}$ defined in (3.105) satisfy

$$\Sigma^{\pm a}_{\mu\nu}(e)\Sigma^{\pm b}_{\rho\sigma}(e)\delta_{ab} = g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\rho\nu} \pm \frac{1}{\sqrt{g}}\epsilon_{\mu\nu\rho\sigma}\,. \tag{3.122}$$

and left length metrics coincide, [1]

$$g^+_{\mu\nu} = g^-_{\mu\nu} \equiv g_{\mu\nu}\,. \tag{3.125}$$

As a result, we anticipate to arrive at an implicit action for the dynamics of an area metric (3.118) given by

$$G_{\mu\nu\rho\sigma} = \sum_{\pm} G^{\pm}_{\mu\nu\rho\sigma}(e, q^{\pm})\,. \tag{3.126}$$

Here, e^I_μ is a tetrad for the metric $g_{\mu\nu}$.

Altogether, the following picture for a non-linear area-metric theory arises. The starting point is the modified non-chiral Plebanski action (3.97), together with a Holst term [126],

$$S_{\text{modified Plebanski + Holst}}[\omega, B, \phi] = \int P^{\gamma}_{IJKL} B^{IJ} \wedge F^{KL}(\omega) + F_{IJKL}(\phi) B^{IJ} \wedge B^{KL}\,, \tag{3.127}$$

where

$$P^{\gamma}_{IJKL}(\phi) = \delta_{IJKL} + \frac{1}{2\gamma}\epsilon_{IJKL} \qquad \text{and} \qquad F_{IJKL}(\phi) = -\frac{1}{2}\left(\phi_{IJKL} + \frac{1}{6}V(\phi)\epsilon_{IJKL}\right). \tag{3.128}$$

Here, γ denotes the Immirzi parameter [80, 81], whose inverse is proportional to the coupling in front of the Holst action.

In the action (3.127), the right-left splitting of the Lie algebra, and accordingly of the B-field (3.103), can be used to decompose the equations of motion for the connection ω into two independent $\mathrm{su}(2)_\pm$ Gauss laws. Their solution can be expressed in terms of the parametrization (3.104) of the $\mathrm{SU}_\pm(2)$ B-fields as $\omega^\pm(B^\pm(e^\pm, b^\pm))$, see e.g. [99, 102]. Additionally, the auxiliary field ϕ can be integrated out from the action, by solving its equations of motion (3.100) in the form $\phi(B) = \phi(e^+, e^-, b^+, b^-)$. As a result, the action becomes an effective second-order action for the fields $e^\pm$ and $q^\pm$, whose explicit expression for a generic SO(4)-invariant quadratic potential can be found in [102, 103]. Imposing identical right and left tetrads $e^+ = e^- \equiv e$, in order to satisfy the constraint (3.125), leads to an action describing the dynamics of the metric g and scalar fields $q^\pm$. These encode the degrees of freedom of the area metric G in (3.126).

1. Another possible choice is to leave the two right and left length metrics independent, and freeze the dynamics of the internal scalars by setting them equal to the identity on the internal space,

$$q^+_{ab} = q^-_{ab} = \delta_{ab}\,. \tag{3.124}$$

This leads to a bimetric theory of gravity, which has been shown to propagate a massless and a massive spin-2 particle, as well as a massive scalar ghost mode in an expansion around a biflat background [102, 103]. Here, we will not be concerned with this theory further.

These steps can be summarised as

$$\begin{aligned} S_{\text{modified Plebanski + Holst}}[\omega, B, \phi] \quad &\rightarrow \quad S_{\text{eff}}[\omega(B), B, \phi(B)]\Big|_{B=B(B^+(e^+,b^+),B^-(e^-,b^-))} \\ \rightarrow \; S[e^+, e^-, q^+, q^-] \quad &\rightarrow \quad S[e, q^+, q^-] \;\equiv\; S[G]\,. \end{aligned} \tag{3.129}$$

The resulting action depends on the area metric G only implicitly. Expressing $g = g(G)$ and $q^{\pm} = q^{\pm}(G)$ would require the inversion of the defining relation (3.126), which represents 20 polynomial equations. An exact inversion is not presently available. Nevertheless, this relation can be inverted perturbatively around a fixed background configuration for the area metric. In the next subsection, we will consider the inversion to linear order around an area-metric background induced by the flat length metric. Thereby, we will arrive at a subclass of second-order quadratic actions for area-metric perturbations, which can be identified as the Euclidean version of the subclass of area-metric actions with shift-symmetric kinetic term, analysed in subsection 3.1.4.

3.2.3 Perturbative inversion of implicit non-linear area-metric actions

Whereas the general solution $g(G)$ and $q^{\pm}(G)$ to the polynomial equations (3.126) cannot be stated explicitly, the inversion can be done perturbatively order by order around a given background configuration for the area metric. To that end, we assume a decomposition $G = G_0 + \delta G$ of the area metric, into a background G_0 and perturbations around this background δG. Accordingly, we can expand $g = g_0 + \delta g$, where $\delta g = \delta g^{(1)} + \delta g^{(2)} + \ldots$, and $q^{\pm} = q_0^{\pm} + \delta q^{\pm}$, where $\delta q^{\pm} = \delta q^{\pm(1)} + \delta q^{\pm(2)} + \ldots$. Based on these expansions, the relation

$$G_0 + \delta G = G(g, q^{\pm})\,, \tag{3.130}$$

with the right hand side given by (3.126), can be solved order by order in δG, for $\delta g^{(n)}(\delta G^n)$ and $\delta q^{\pm(n)}(\delta G^n)$.

In the following, we consider an expansion of the area metric around a configuration induced by the flat Euclidean metric $\delta_{\mu\nu}$, and therewith invert (3.130) to linear order in perturbations. Thus, let us introduce the perturbation of the area metric by

$$G_{\mu\nu\rho\sigma} \equiv (G_\delta)_{\mu\nu\rho\sigma} + \delta G_{\mu\nu\rho\sigma} = \delta_{\mu\rho}\delta_{\sigma\nu} - \delta_{\mu\sigma}\delta_{\rho\nu} + \delta G_{\mu\nu\rho\sigma}\,, \tag{3.131}$$

and the perturbations around flat space of the tetrad and the length metric, as well as of the internal metrics by

$$e^I_\mu \;=\; \delta^I_\mu + \delta e^I_\mu \quad \Rightarrow \quad g_{\mu\nu} = \delta_{\mu\nu} + \delta g_{\mu\nu} \quad \text{where} \quad \delta g_{\mu\nu} = \delta e^I_\mu \delta_{I\nu} + \delta e^I_\nu \delta_{I\mu}\,, \tag{3.132}$$

$$q^{\pm}_{ab} \;=\; \delta_{ab} + \delta q^{\pm}_{ab}\,. \tag{3.133}$$

From now on, spacetime and internal indices are raised and lowered with the flat Euclidean spacetime metric $\delta_{\mu\nu}$ and the internal metric δ_{ab}, respectively. The unimodularity condition

$\det(q^{\pm}) = 1$ implies, to first order in the perturbations of the internal metrics, $\mathrm{Tr}(\delta q^{\pm}) = \delta q^{\pm}_{ab}\delta^{ab} = 0$. To match with the conventions of the previous section, at this point we formally reintroduce the gravitational coupling κ and denote the perturbations of the area metric by $\delta G_{\mu\nu\rho\sigma} = 2\kappa a_{\mu\nu\rho\sigma}$, and the perturbations of the length metric and internal metric by $\delta g_{\mu\nu} = 2\kappa h_{\mu\nu}$ and $\delta q^{\pm}_{ab} = 2\kappa\chi^{\pm}_{ab}$. Thus, a, h and $\chi^{\pm}$ are bosonic fields with canonical mass dimension one.

Expanding the area metric in (3.126) to first order in $h_{\mu\nu}$ and $\chi^{\pm}_{ij}$ allows us to identify the area-metric perturbation $a_{\mu\nu\rho\sigma}$ on the left hand side of (3.130) as

$$\begin{aligned} a_{\mu\nu\rho\sigma} &\equiv \mathbb{L}^{\alpha\beta}{}_{\mu\nu\rho\sigma} h_{\alpha\beta} + 2\mathbb{P}^{+ab}{}_{\mu\nu\rho\sigma}\chi^{+}_{ab} + 2\mathbb{P}^{-ab}{}_{\mu\nu\rho\sigma}\chi^{-}_{ab} && (3.134)\\ &\equiv 2\big(\delta_{\mu[\rho}h_{\sigma]\nu} - \delta_{\nu[\rho}h_{\sigma]\mu}\big) + \omega^{+}_{\mu\nu\rho\sigma} + \omega^{-}_{\mu\nu\rho\sigma}\,, && (3.135)\end{aligned}$$

with

$$\begin{aligned} \mathbb{L}^{\alpha\beta}{}_{\mu\nu\rho\sigma} &\equiv 2\Big(\delta_{\mu[\rho}\delta^{(\alpha}_{\sigma]}\delta^{\beta)}_{\nu} - \delta_{\nu[\rho}\delta^{(\alpha}_{\sigma]}\delta^{\beta)}_{\mu}\Big)\,, && (3.136)\\ \mathbb{P}^{\pm ab}{}_{\mu\nu\rho\sigma} &\equiv P^{\pm(a}_{\mu\nu}P^{\pm b)}_{\rho\sigma} - \frac{1}{3}\delta^{ab}P^{\pm c}_{\mu\nu}P^{\pm d}_{\rho\sigma}\delta_{cd} \quad \text{where} \quad P^{\pm a}_{\mu\nu} \equiv \frac{1}{2}\Sigma^{\pm a}_{\mu\nu}(\delta)\,. && (3.137)\end{aligned}$$

The Plebanski 2-forms $\Sigma^{\pm a}_{\mu\nu}(\delta)$ of the flat Euclidean tetrad are defined as in (3.105). The projectors $\mathbb{P}^{\pm ab}$ are traceless in their internal indices. These and $\mathbb{L}^{\alpha\beta}$ satisfy the projector and orthogonality relations

$$\begin{aligned} \mathbb{L}^{\alpha\beta}\cdot\mathbb{L}^{\gamma\delta} &= 8\mathbb{I}^{\alpha\beta\gamma\delta} + 4\delta_{\alpha\beta}\delta^{\gamma\delta} \quad \text{where} \quad \mathbb{I}^{\alpha\beta\gamma\delta} \equiv \delta^{\alpha(\gamma}\delta^{\delta)\beta}\,, && (3.138)\\ \mathbb{P}^{\pm ab}\cdot\mathbb{P}^{\pm cd} &= \mathbb{I}^{abcd} \quad \text{where} \quad \mathbb{I}^{abcd} \equiv \delta^{a(c}\delta^{d)b} - \frac{1}{3}\delta^{ab}\delta^{cd}\,, && (3.139)\\ \mathbb{L}^{\alpha\beta}\cdot\mathbb{P}^{\pm ab} &= \mathbb{P}^{\pm ab}\cdot\mathbb{P}^{\pm cd} = 0\,. && (3.140)\end{aligned}$$

In the previous expressions, we have used the notation $A\cdot B \equiv A_{\mu\nu\rho\sigma}B^{\mu\nu\rho\sigma}$. The tensors $\mathbb{I}^{\alpha\beta\gamma\delta}$ and $\mathbb{I}^{abcd}$ define, respectively, the identity on the space of symmetric rank-2 tensors $t_{\alpha\beta}$ on spacetime, and the identity on the space of symmetric and traceless rank-2 tensors t_{ab} on the internal space.

The expression in (3.135) is of the same form as the algebraic decomposition of the field $a_{\mu\nu\rho\sigma}$ according to (3.16) and (3.25). Moreover, the fields $\chi^{\pm}_{ab}$ and $\omega^{\pm}_{\mu\nu\rho\sigma}$ are related to each other in the same way as in (3.24). They represent the selfdual and anti-selfdual components of the area-metric perturbation. However, in this section, we have used the Euclidean projectors $\mathbb{P}^{\pm}$ defined in (3.137). These differ from their Lorentzian analogues (3.20), in that they involve the Plebanski 2-forms constructed from the flat Euclidean tetrad δ^{I}_{μ}, instead of the flat Minkowski tetrad η^{I}_{μ}. In Euclidean signature, the projectors $\mathbb{P}^{\pm}$, and in particular the fields $\chi^{\pm}_{ab}$ and $\omega^{\pm}_{\mu\nu\rho\sigma}$, are real. The selfduality and anti-selfduality property in Lorentzian signature (3.19) translates to Euclidean signature as

$$\frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}\omega^{\pm}_{\alpha\beta\rho\sigma} = \pm\omega^{\pm}_{\mu\nu\rho\sigma}\,. \tag{3.141}$$

Similarly, the area-metric perturbation induced by the length-metric perturbation $h_{\mu\nu}$ involves the quantity $\mathbb{L}^{\alpha\beta}$, defined in equation (3.137) from the Euclidean spacetime metric $\delta_{\mu\nu}$, which in Lorentzian signature would be defined from the Minkowski metric $\eta_{\mu\nu}$. Formally, one may define a Wick rotation from the Euclidean to the Lorentzian theory by the replacement $\delta^0_\mu \to \eta^0_\mu = i\delta^0_\mu$ in the Plebanski-2 forms, while leaving spatial components of the tetrad unchanged.

Using the definitions of $\mathbb{L}^{\alpha\beta}$ and $\mathbb{P}^{\pm ab}$ in (3.136)–(3.137), and the relations (3.138)–(3.140), we can explicitly invert the relation (3.130) to linear order in the area-metric perturbation. The result is an expression for the length-metric perturbation $h_{\mu\nu}$ and internal metric perturbations $\chi^{\pm}_{ab}$, as functions of the area-metric perturbation $a_{\mu\nu\rho\sigma}$,

$$h_{\alpha\beta} = \mathbb{K}_{\alpha\beta}\cdot a \quad \text{where} \quad (\mathbb{K}_{\alpha\beta})_{\mu\nu\rho\sigma} \equiv \frac{1}{8}\mathbb{I}_{\alpha\beta\gamma\delta}\mathbb{L}^{\gamma\delta}{}_{\mu\nu\rho\sigma} - \frac{1}{12}\delta_{\alpha\beta}\delta_{\mu[\rho}\delta_{\sigma]\nu}\,, \tag{3.142}$$

$$\chi^{\pm}_{ab} = \frac{1}{2}\mathbb{P}^{\pm}_{ab}\cdot\omega^{\pm} = \frac{1}{2}\mathbb{P}^{\pm}_{ab}\cdot a \quad \Leftrightarrow \quad \omega^{\pm}_{\mu\nu\rho\sigma} = 2\mathbb{P}^{\pm ab}{}_{\mu\nu\rho\sigma}\chi^{\pm}_{ab} = \mathbb{P}^{\pm ab}_{\mu\nu\rho\sigma}\mathbb{I}_{abcd}\mathbb{P}^{\pm cd}\cdot a\,. \tag{3.143}$$

For later reference, we note that the separate parts of the area-metric perturbation in (3.134) and (3.135) can be written in terms of projectors Π^{L} and $\Pi^{\pm}$ onto the length-metric and the selfdual and anti-selfdual components as

$$a_{\mu\nu\rho\sigma} = \Pi^{\mathrm{L}}_{\mu\nu\rho\sigma}\cdot a + \Pi^{+}_{\mu\nu\rho\sigma}\cdot a + \Pi^{-}_{\mu\nu\rho\sigma}\cdot a\,. \tag{3.144}$$

The projectors are defined explicitly by

$$\begin{aligned}
\Pi^{\mathrm{L}}_{\mu\nu\rho\sigma,\alpha\beta\gamma\delta} &\equiv \mathbb{L}^{\lambda\tau}{}_{\mu\nu\rho\sigma}(\mathbb{K}_{\lambda\tau})_{\alpha\beta\gamma\delta} \qquad (3.145)\\
&= 2\big(\mathbb{A}^{+}_{\mu\nu\alpha\beta}\mathbb{A}^{-}_{\rho\sigma\gamma\delta} + \mathbb{A}^{+}_{\mu\nu\gamma\delta}\mathbb{A}^{-}_{\rho\sigma\alpha\beta}\big) + 2\big(\mathbb{A}^{-}_{\mu\nu\alpha\beta}\mathbb{A}^{+}_{\rho\sigma\gamma\delta} + \mathbb{A}^{-}_{\mu\nu\gamma\delta}\mathbb{A}^{+}_{\rho\sigma\alpha\beta}\big)\\
&+ \frac{2}{3}\mathbb{A}^{\mathrm{S}}_{\mu\nu\rho\sigma}\mathbb{A}^{\mathrm{S}}_{\alpha\beta\gamma\delta}\,,\\
\Pi^{\pm}_{\mu\nu\rho\sigma,\alpha\beta\gamma\delta} &\equiv \mathbb{P}^{\pm ab}_{\mu\nu\rho\sigma}\mathbb{I}_{abcd}\mathbb{P}^{\pm cd}_{\alpha\beta\gamma\delta} \qquad (3.146)\\
&= 2\big(\mathbb{A}^{\pm}_{\mu\nu\alpha\beta}\mathbb{A}^{\pm}_{\rho\sigma\gamma\delta} + \mathbb{A}^{\pm}_{\mu\nu\gamma\delta}\mathbb{A}^{\pm}_{\rho\sigma\alpha\beta}\big) - \frac{4}{3}\mathbb{A}^{\pm}_{\mu\nu\rho\sigma}\mathbb{A}^{\pm}_{\alpha\beta\gamma\delta}\,,
\end{aligned}$$

where [1]

$$\mathbb{A}^{\pm}_{\mu\nu\rho\sigma} \equiv \frac{1}{8}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) \pm \frac{1}{8}\epsilon_{\mu\nu\rho\sigma}\,, \tag{3.148}$$

$$\mathbb{A}^{\mathrm{S}}_{\mu\nu\rho\sigma} \equiv \mathbb{A}^{+}_{\mu\nu\rho\sigma} + \mathbb{A}^{-}_{\mu\nu\rho\sigma} = \frac{1}{4}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})\,, \tag{3.149}$$

$$\mathbb{A}^{\mathrm{D}}_{\mu\nu\rho\sigma} \equiv \mathbb{A}^{+}_{\mu\nu\rho\sigma} - \mathbb{A}^{-}_{\mu\nu\rho\sigma} = \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\,. \tag{3.150}$$

According to (3.144), by construction it holds

$$\mathbb{L}^{\lambda\tau}{}_{\mu\nu\rho\sigma} h_{\lambda\tau} = \Pi^{\mathrm{L}}_{\mu\nu\rho\sigma} \cdot a\,, \tag{3.151}$$

$$\omega^{\pm}_{\mu\nu\rho\sigma} = \Pi^{\pm}_{\mu\nu\rho\sigma} \cdot a\,. \tag{3.152}$$

The projectors $\Pi^{\pm}$ act as the identity on the tensors $\omega^{\pm}$, and together with Π^{L} sum to the identity on the space of cyclic area metrics,

$$\begin{aligned}\mathbb{I}_{\mu\nu\rho\sigma,\alpha\beta\gamma\delta} &\equiv 2\big(\mathbb{A}^{\mathrm{S}}_{\mu\nu\alpha\beta}\mathbb{A}^{\mathrm{S}}_{\rho\sigma\gamma\delta} + \mathbb{A}^{\mathrm{S}}_{\mu\nu\gamma\delta}\mathbb{A}^{\mathrm{S}}_{\rho\sigma\alpha\beta}\big) - \frac{2}{3}\mathbb{A}^{\mathrm{D}}_{\mu\nu\rho\sigma}\mathbb{A}^{\mathrm{D}}_{\alpha\beta\gamma\delta} \\ &= \Pi^{\mathrm{L}}_{\mu\nu\rho\sigma,\alpha\beta\gamma\delta} + \Pi^{+}_{\mu\nu\rho\sigma,\alpha\beta\gamma\delta} + \Pi^{-}_{\mu\nu\rho\sigma,\alpha\beta\gamma\delta}\,.\end{aligned} \tag{3.153}$$

The previous definitions provide all ingredients to explicitly state the linearisation, around flat space, of the implicit non-linear area-metric action described at the end of the previous subsection. The starting point are the $\mathrm{SU}_{\pm}(2)$ BF actions, where the connection has been integrated out using the parametrisation (3.103) for the $\mathrm{SU}_{\pm}(2)$ B-fields in terms of the spacetime tetrads $e^{\pm}$ and the internal frame fields $b^{\pm}$ [99, 102]. Linearising these actions around a flat background configuration for the spacetime metrics $g^{\pm}_{\mu\nu}$ and the internal metrics $q^{\pm}_{ab}$, leads to the effective SO(4) BF action as a functional of the length-metric perturbation $h^{\pm}_{\mu\nu}$ and internal metric perturbations $\chi^{\pm}_{ab}$ [102, 103]. In order to arrive at a quadratic action for area-metric perturbations, we identify $h^{+}_{\mu\nu} = h^{-}_{\mu\nu} \equiv h_{\mu\nu}$. Additionally, we consider a potential for the non-metric degrees of freedom of the area-metric perturbation, associated with the perturbations of the internal metrics. Assuming SO(4) invariance of the original potential $V(\phi)$ in the modified non-chiral Plebanski action, the only possibility at quadratic order are mass terms for $\chi^{\pm}_{ab}$ [102, 103]. This can be understood in view of the decomposition of the field ϕ_{IJKL} into irreducible representations of SO(4), according to (3.101). To obtain a singlet quadratic in the fields $\omega^{\pm}$, we have to consider the tensor product of each of the representations $(0,2)$ and $(2,0)$ with itself.

The resulting Lagrangian is a function of $h_{\mu\nu}$ and $\chi^{\pm}_{ab}$, and can be written in momentum space

1. This rewriting uses the property (3.122) of the Plebanski 2-forms. According to this identity, or alternatively by direct computation, it follows that the projectors $P^{\pm a}_{\mu\nu} = \frac{1}{2}\Sigma^{\pm}_{\mu\nu}(\delta)$ satisfy

$$P^{\pm a}_{\mu\nu} P^{\pm b}_{\rho\sigma}\delta_{ab} = \frac{1}{4}(\delta_{\mu\rho}\delta_{\sigma\nu} - \delta_{\mu\sigma}\delta_{\rho\nu}) \pm \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\,. \tag{3.147}$$

as

$$\mathcal{L}\big(h_{\mu\nu}, \chi^+_{ab}, \chi^-_{ab}\big) = \mathcal{L}_{\mathrm{BF}}\big(h_{\mu\nu}, \chi^+_{ab}, \chi^-_{ab}\big) + \mathcal{L}_{\mathrm{mass}}\big(\chi^+_{ab}, \chi^-_{ab}\big) \tag{3.154}$$

$$\equiv \frac{1}{2}\sum_{\pm}\left[\gamma_\pm \mathcal{L}_{\mathrm{EH}}\big(h_{\mu\nu} + \chi^\pm_{\mu\nu}\big) + \frac{1}{2}m^2_\pm \chi^\pm_{\mu\nu}\chi^{\pm\mu\nu}\right], \tag{3.155}$$

where

$$\gamma_\pm = 1 \pm \frac{1}{\gamma}\,. \tag{3.156}$$

The structure of the Lagrangian (3.154) can be understood by the shift symmetry of the topological BF actions for each of the selfdual and anti-selfdual sectors separately. This shift symmetry eliminates the transverse-traceless propagating modes in the Einstein-Hilbert action. Adding a mass term to the self-dual or anti-selfdual Lagrangian breaks this shift symmetry, such that each of these Lagrangians on its own features two transverse-traceless propagating degrees of freedom, encoded in the shifted metric $H^\pm_{\mu\nu} = h_{\mu\nu} + \chi^\pm_{\mu\nu}$, and five massive non-propagating modes, encoded in $\chi^\pm_{\mu\nu}$. Combinining these two sectors into the non-chiral Lagrangian (3.155) similarly results in propagating degrees of freedom. However, the final Lagrangian still features a remnant 5-parameter shift symmetry in the kinetic term, as discussed in subsection 3.1.4 and demonstrated below.

For completeness, we note that the linearised Einstein-Hilbert action is defined by the Fierz-Pauli operator $\hat{\mathcal{E}}$ in Euclidean signature as

$$\mathcal{L}_{\mathrm{EH}}(h_{\mu\nu}) = \frac{1}{2}h^{\mu\nu}\hat{\mathcal{E}}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} \equiv \frac{p^2}{2}h^{\mu\nu}\Big(P^{(2)}{}_{\mu\nu}{}^{\rho\sigma} - 2P^{(0)}{}_{\mu\nu}{}^{\rho\sigma}\Big)h_{\rho\sigma}\,, \tag{3.157}$$

with $p^2 = \delta_{\mu\nu}p^\mu p^\nu$ denoting the square of the Euclidean four-momentum vector. The spin-2 and spin-0 projectors $P^{(2)}$ and $P^{(0)}$ are defined as in (3.8), with indices raised and lowered with the Euclidean metric $\delta_{\mu\nu}$.

The fields $\chi^\pm_{\mu\nu}$ in (3.155) are real symmetric transverse-traceless rank-2 tensors on spacetime, defined from $\chi^\pm_{ab}$ in the same way as in (3.38), using the projectors $P^{\pm a}_{\mu\nu}$ in Euclidean signature. As a consequence, these fields represent an isometric embedding of $\chi^\pm_{ab}$ into the space of symmetric transverse-traceless tensors on spacetime, such that the identity (3.40) holds in Euclidean signature as well. For convenience, we display these definitions and identities again,

$$\chi^\pm_{\mu\nu} \equiv \omega^\pm_{\mu\rho\nu\sigma}\frac{p^\rho p^\sigma}{p^2} = 4P^{\pm a}_{\mu\rho}P^{\pm b}_{\nu\sigma}\frac{p^\rho p^\sigma}{p^2}\chi^\pm_{ab} \quad \text{with} \quad \chi^\pm_{\mu\nu}p^\mu = 0 \quad \text{and} \quad \chi^{\pm\mu}_{\mu} = 0\,, \tag{3.158}$$

$$\chi^\pm_{\mu\nu}\chi^{\pm\mu\nu} = \chi^\pm_{ab}\chi^{\pm ab} = \frac{1}{4}\omega^\pm_{\mu\nu\rho\sigma}\omega^{\pm\mu\nu\rho\sigma}\,. \tag{3.159}$$

In Euclidean signature, the Lagrangian quadratic in $\chi^\pm_{\mu\nu}$ is manifestly positive definite, as these fields are real. Moreover, all couplings $c_\pm$ in (3.155) are real, and parity is violated if $c_+ \neq c_-$.

The Lagrangian (3.155) can be identified as the Euclidean version of the special class of area-metric Lagrangians with shift-symmetric kinetic term, analysed in subsection 3.1.4. To see this, we use that $\chi^{\pm}_{\mu\nu}$ are annihilated by the spin-0 projector, whereas the spin-2 projector acts as the identity on these fields. This allows us to write the expression in (3.155) as

$$\begin{aligned}\mathcal{L}(h_{\mu\nu}, \chi^{+}_{\mu\nu}, \chi^{-}_{\mu\nu}) &= \frac{1}{2}(\gamma_+ + \gamma_-)\mathcal{L}_{\mathrm{EH}}(h_{\mu\nu}) \\ &+ \frac{1}{2}\sum_{\pm}\left[\gamma_{\pm}h_{\mu\nu}\chi^{\pm\mu\nu}p^2 + \frac{1}{2}\gamma_{\pm}\chi^{\pm}_{\mu\nu}\chi^{\pm\mu\nu}p^2 + \frac{1}{2}m^2_{\pm}\chi^{\pm}_{\mu\nu}\chi^{\pm\mu\nu}\right]. \qquad (3.160)\end{aligned}$$

The kinetic part is of the same form as in the Lagrangian (3.68), with the identification $(\alpha_{\pm}, \beta_{\pm}) \leftrightarrow -\gamma_{\pm}$, and consequently $\rho_{\pm} \leftrightarrow -\gamma_{\pm}$. In particular, the condition (3.69) for a 5-parameter shift symmetry of the kinetic term, with $A \equiv -1$ in Euclidean signature, is automatically satisfied for linearised area-metric actions derived from modified non-chiral Plebanski theories. Indeed, the definition of $\gamma_{\pm}$ in (3.156) implies

$$\gamma_+ + \gamma_- = 2\,. \qquad (3.161)$$

Thus, we can conclude that the Immirzi parameter γ is a parity-violating coupling in area-metric actions derived from modified non-chiral Plebanski theories. Concretely, γ enters the dynamical equations of motion derived from the linearised area-metric Lagrangian (3.160). This is in contrast to classical length-metric gravity, where γ does not affect the equations of motion. Moreover, in view of the Hamiltonian analysis of these area-metric theories in Lorentzian signature, in appendix A.2, we can conclude that γ is responsible for a parity-violating mixing effect between the $+$ and $\times$ modes of the transverse-traceless components of the shifted spatial metric, subject to the Hamiltonian and diffeomorphism constaints, see subsection A.2.6 and in particular equation (A.150).

The discussion of the 5-parameter shift symmetry and explicit formulae in subsection 3.1.4, as well as the derivation of the effective actions for the field $h_{\mu\nu}$ in subsection 3.1.3, extend straightforwardly to the Lagrangian (3.160) in Euclidean signature. The effective action for $h_{\mu\nu}$ is given by

$$\mathcal{L}_{\mathrm{eff}}(h_{\mu\nu}) = \mathcal{L}_{\mathrm{EH}}(h_{\mu\nu}) - \frac{1}{2}\sum_{\pm}\gamma_{\pm}{}^{(1)}C_{\mu\nu\rho\sigma}\frac{1}{p^2 + M^2_{\pm}}{}^{(1)}C^{\mu\nu\rho\sigma} \qquad \text{where} \qquad M^2_{\pm} \equiv \frac{m^2_{\pm}}{\gamma_{\pm}}\,. \qquad (3.162)$$

If the effective mass parameters $M^2_{\pm} = M^2$ are identical, this action simplifies and leads to a ghostfree propagator for the spin-2 mode, [1]

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}}(h_{\mu\nu}) &= \mathcal{L}_{\mathrm{EH}}(h_{\mu\nu}) - {}^{(1)}C_{\mu\nu\rho\sigma}\frac{1}{p^2+M^2}{}^{(1)}C^{\mu\nu\rho\sigma}\,, &(3.163)\\ \mathcal{P}^{(2)}(p^2) &= 2\left[\frac{1}{p^2}+\frac{1}{M^2}\right]. &(3.164)\end{aligned}$$

Finally, for completeness, we close this subsection by stating the result for the linearised area-metric Lagrangian in (3.160) explicity in terms of the inversion of equation (3.135), with $h = \mathbb{K}\cdot a$ and $\omega^{\pm} = \Pi^{\pm}\cdot a$ as in (3.142) and (3.143). Defining

$$\mathbb{E}^{\pm}_{\mu\nu,\alpha\beta\gamma\delta} \equiv \Pi^{\pm}_{\mu\rho\nu\sigma,\alpha\beta\gamma\delta}\frac{p^\rho p^\sigma}{p^2}\,, \tag{3.165}$$

and using the definition of $\chi^{\pm}_{\mu\nu}$ in (3.158) and identity (3.159), we arrive at

$$\mathcal{L}(a) = a\left[-\frac{1}{2}\mathbb{K}^{\mathrm{T}}\hat{\mathcal{E}}\mathbb{K} + \frac{1}{4}\sum_{\pm}\gamma_{\pm}\big(\mathbb{E}^{\pm\mathrm{T}}\mathbb{K}+\mathbb{K}^{\mathrm{T}}\mathbb{E}^{\pm}\big)p^2 + \frac{1}{2}\big(\gamma_{\pm}p^2+m_{\pm}^2\big)\Pi^{\pm}\right]a\,. \tag{3.166}$$

This terminates our analysis of area-metric actions in the framework of non-chiral modified Plebanski theories.

1. It should be noted that the condition $M_+^2 = M_-^2$ requires $m_+^2 = \frac{\gamma_+}{\gamma_-}m_-^2$, but not necessarily $\gamma_+ = \gamma_-$. Therefore, an effective action of the form (3.162) can arise even in the presence of a finite Immirzi parameter γ in the area-metric action (3.160).

4 Phenomenology of area metrics

Area-metric gravity in four spacetime dimensions contains more degrees of freedom than classical gravity. These degrees of freedom can be associated with the ten massless degrees of freedom of a length metric, and ten additional massive degrees of freedom which parametrise the deviation of the area metric from one that is induced by a length metric. Thereby, area-metric gravity can be viewed as an effective theory for a length metric coupled to a special type of matter fields. We assume this effective field theory to be valid at an energy scale below the scale of a fundamental theory of quantum gravity. Candidate approaches to quantum gravity, in which area metrics appear as effective semiclassical descriptions of spacetime, are loop quantum gravity and spin foams [1, 3, 49, 76, 77], as well as string theory [2, 50–53, 55].

For area metrics to represent a viable geometric framework for gravity in an intermediate energy regime, the area-metric effective field theory has to satisfy theoretical and phenomenological consistency constraints. These should ensure a low-energy limit compatible with general relativity, which is experimentally exceedingly well tested in the strong-field regime [128–130]. Such a constraint on area-metric gravity can be analysed both classically, and from the perspective of quantum field theory. In this chapter we consider separate aspects of this constraint as criteria to assess the viability of area-metric gravity.

4.1 RG flows in perturbative area-metric gravity

In this section, area-metric gravity is treated as a local quantum effective field theory. To that end, we assume a description of spacetime in terms of an area metric at a given UV cutoff scale Λ_{UV}. Starting from this regime, we analyse the properties of the RG flow of area-metric gravity from the UV towards to IR. The relevant concepts from the functional RG are summarised in subsection 4.1.2, whereas the truncation ansatz for the effective action is defined in subsection 4.1.2.

Compatibility of area-metric gravity with metric gravity at low energies requires the decoupling of the non-metric degrees of freedom of the area metric. This constraint stems from the absence of associated types of matter fields in the standard model of particle physics, and the absence of any observational indications for these. One possibility for the extra fields to decouple is as a result of a large-growing mass, which freezes the dynamics of these fields in the IR. This scenario is illustrated in figure 4.1, and can be additionally enhanced by a dynamical suppression of the interaction couplings between these fields and the length metric. Subsection 4.1.3 analyses these two distinct mechanisms of decoupling.

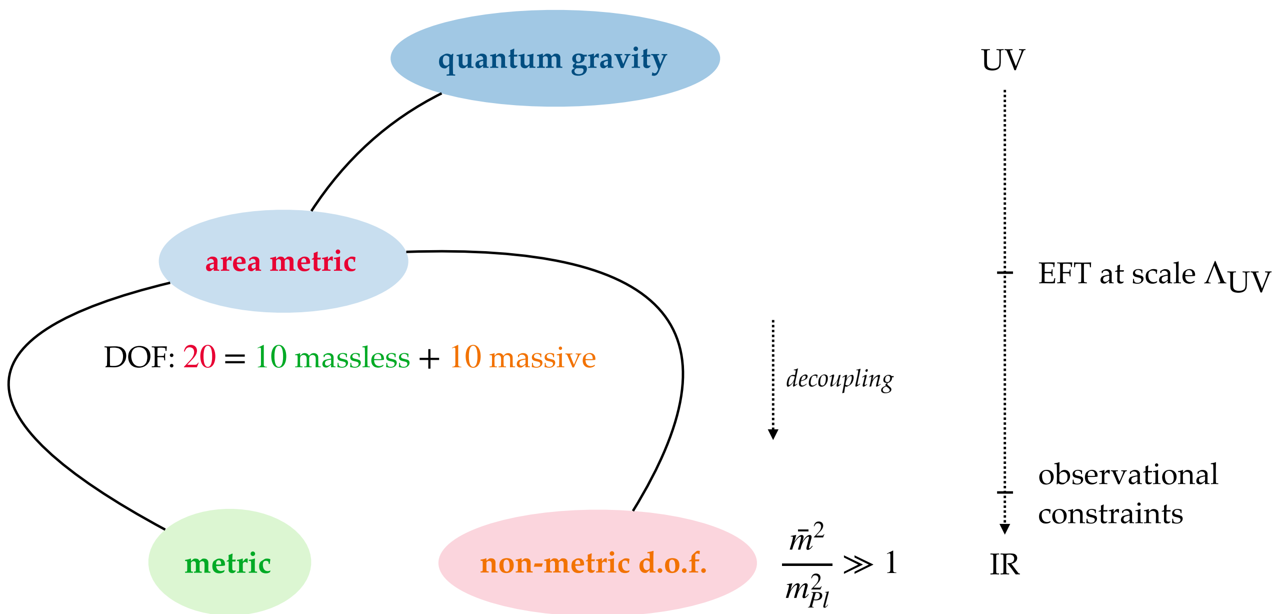

Figure 4.1: Scenario of area-metric gravity as an effective field theory at an energy scale Λ_{UV}, below the scale of a fundamental theory of quantum gravity. Compatibility of area-metric gravity with IR physics requires the decoupling of the non-metric degrees of freedom from the length-metric degrees of freedom of the area metric at low energies.

A further aspect to determine the IR consistency of area-metric gravity are constraints on gravitational parity violation, see e.g. [131, 132]. Parity violation is a prediction of various extensions to general relativity, such as Chern-Simons theory [133, 134], and variants thereof [135–139], or chiral extensions of ghostfree scalar-tensor theories [140]. Other examples are symmetric teleparallel gravity [141] and Hořava-Lifshitz gravity [142, 143]. Such parity-violating extensions of general relativity can arise as low-energy effective field theories of some high-energy UV completion such as string theory, where parity violation is triggered by anomaly cancellation [144, 145], or extensions of loop quantum gravity, in which the Immirzi parameter is promoted to a dynamical field [146–148].

Similarly, a generic ansatz for the effective action for area-metric perturbations contains parity-violating couplings. These must be dynamically suppressed under the RG flow towards lower scales, in order to guarantee compatibility with observational constraints on parity violation in pure-gravitational systems. On these grounds, in subsection 4.1.4, we analyse directions away from the parity-symmetric theory space of area-metric gravity, as well as the possibility of parity invariance as an emergent symmetry. Such a scenario starts with a parity-violating area-metric theory in the UV, and ends with a parity-invariant theory in the IR. Additionally, as a special parity-violating coupling in area-metric gravity, we analyse the RG flow of the Immirzi parameter γ in subsection 4.1.5.

4.1.1 Brief overview of the functional renormalisation group

The functional RG is a non-perturbative framework for deriving the quantum effective action Γ, which governs the dynamics in a quantum field theory. The latter can be computed formally by solving the Euclidean path integral of the quantum field theory. This amounts to integrating out all quantum fluctuations of the fields at once. By contrast, the functional RG is based on the Wilsonian picture of renormalisation, which consists of integrating out quantum fluctuations step by step [149–152]. This process, known as coarse graining, starts from many microscopic degrees of freedom in the UV, and ends with fewer macroscopic degrees of freedom in the IR.

The central object in the functional RG is a scale-dependent effective action Γ_k, known as the effective average action (EAA) [153]. The functional Γ_k depends on a variable RG mass scale k, and interpolates between the classical action $S = \Gamma_{k=\Lambda_{\mathrm{UV}}}$ at a UV cutoff scale Λ_{UV} and the quantum effective action $\Gamma \equiv \Gamma_{k\to 0}$ in the IR.

The scale dependence of Γ_k can be derived from an exact functional differential equation [154, 155],

$$k\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\partial_k \mathcal{R}_k\right]. \tag{4.1}$$

Here, $\Gamma_k^{(2)}$ is the matrix of second functional derivatives of Γ_k with respect to the quantum fields at fixed background, and $\mathcal{R}_k$ is a regulator. The regulator acts as an IR cutoff by suppressing low-momentum modes, such that only quantum fluctuations with momenta $p^2 \gtrsim k^2$ are included in Γ_k. The generalised trace involves sums and integrals over fields and momenta.

The flow equation was originally formulated for scalar fields [154, 155], and can be generalised to fermions [156, 157], and gauge fields, including gravity [158, 159]. Reviews and introductions to the functional RG can be found in [160–168]. The following is a collection of terminologies and concepts necessary to follow the discussion in the next subsections.

A solution to the flow equation (4.1) can be interpreted as a trajectory between the bare action and the quantum effective action in theory space. The latter is the infinite-dimensional space spanned by all operators $\mathcal{O}_i$ which are compatible with the symmetries of the theory. An action functional in theory space can be expanded in the basis of operators $\{\mathcal{O}_i\}$,

$$\Gamma_k = \sum_i \bar{c}_i(k)\mathcal{O}_i\,, \tag{4.2}$$

with coordinates given by couplings $\bar{c}_i(k)$. Taking the scale derivative of (4.2) yields

$$k\partial_k\Gamma_k = \sum_i \beta_{\bar{c}_i}\mathcal{O}_i\,, \tag{4.3}$$

where $\beta_{\bar{c}_i} = k\partial_k \bar{c}_i(k)$ are the beta functions of the dimensionful couplings $\bar{c}_i$. These depend in general on all couplings $\{\bar{c}_j\}$, as well as on the RG scale parameter k. According to (4.3), expanding the trace on the right hand side in the flow equation (4.2), in the basis of operators $\{\mathcal{O}_i\}$, allows for extracting the beta functions $\beta_{\bar{c}_i}$ of the couplings $\bar{c}_i$ as the coefficients in front of the operators $\mathcal{O}_i$.

As Γ_k is dimensionless, each coupling $\bar{c}_i$ has canonical mass dimension $d_{\bar{c}_i}$ given by the negative mass dimension of the operator $\mathcal{O}_i$. The corresponding dimensionless couplings c_i are obtained by multiplication with appropriate powers of the RG scale parameter k,

$$c_i(k) = \bar{c}_i(k) k^{-d_{\bar{c}_i}} \,. \tag{4.4}$$

It is standard to work with the dimensionless couplings $\{c_i\}$ as coordinates in theory space. Their beta functions are given by

$$\beta_{c_i}(c_j) = k\partial_k c_i(k) = -d_{\bar{c}_i} c_i + k^{-d_{\bar{c}_i}} \beta_{\bar{c}_i} \,. \tag{4.5}$$

They contain a canonical term and a contribution that is induced by quantum fluctuations. Different from the beta functions for the dimensionful couplings, there is no explicit dependence on the RG scale parameter k. Therefore, dealing with the flow equation (4.1) replaces the challenge of solving a path integral, by the task of solving a coupled system of autonomous differential equations. The flow equation is formally exact as an infinite-dimensional integro-differential equation, but not exactly solvable in practice. Instead, Γ_k can be expanded in systematic truncations and its flow analysed within the truncated theory space.

Fixed points of the RG flow are of particular interest, as there the theory becomes quantum scale-invariant. At a fixed point $\vec{c}_*$, all beta functions of the dimensionless couplings vanish,

$$\beta_{c_i}(\vec{c}_*) = 0 \quad \forall\, i \,. \tag{4.6}$$

Therefore the dimensionless couplings are constant at a fixed point, whereas the dimensionful couplings follow a simple power-law scaling according to their canonical dimension,

$$\bar{c}_i(k) = c_{i*} k^{d_{\bar{c}_i}} \,. \tag{4.7}$$

A fixed point $\vec{c}_*$ is called Gaussian, or free, if all couplings at the fixed point vanish, and non-Gaussian, or interacting, if at least one coupling is non-zero at the fixed point.

Generically, a fixed point is a point in theory space where RG trajectories of the flow equation start or end. In the first case it is called a UV fixed point and in the second case an IR fixed point. Some fixed points can only serve as either UV or IR fixed points, but generically a fixed point can be both. The properties of a fixed point can be deduced by linearising the beta functions β_{c_i} around the fixed point. This results in a set of linear differential equations,

$$\beta_{c_i}(\vec{c}) = \underbrace{\beta_{c_i}(\vec{c}_*)}_{=0} + \sum_j B_{ij}(\vec{c}_*)(c_j - c_{j*}) + \mathcal{O}\big((c_j - c_{j*})^2\big) \,, \tag{4.8}$$

where B_{ij} are the entries of the stability matrix defined by

$$B_{ij}(\vec{c}) = \frac{\partial \beta_{c_i}(\vec{c})}{\partial c_j} \,. \tag{4.9}$$

The eigenvalues of the stability matrix are denoted by $-\theta_I$, and corresponding eigenvectors by $\vec{B}_I$. The flow in the eigendirection I can be analysed by writing $\vec{c}(k) = \vec{c}_* + b_I(k)\vec{B}_I$, such that the linearised flow equation (4.8) reduces to

$$k\partial_k b_I(k)\vec{B}_I = -\theta_I b_I(k)\vec{B}_I \qquad \Rightarrow \qquad b_I(k) = b_{I0}\left(\frac{k}{k_0}\right)^{-\theta_I} . \tag{4.10}$$

The integration in the last step assumes the initial condition $b_I(k_0) = b_{I0}$, where k_0 is a reference scale usually set to the initial scale Λ_{UV}. The general solution to (4.8) is

$$\vec{c}(k) = \vec{c}_* + \sum_I b_{I0}\,\vec{b}_I\left(\frac{k}{k_0}\right)^{-\theta_I} . \tag{4.11}$$

The critical exponents θ_I are the negative eigenvalues of the stability matrix. The integration constants $\{b_{I0}\}$ are the free parameters of the theory. In principle, all of these infinitely many constants must be known in order to characterise an RG trajectory, which enters the linearised regime, exactly. However, in the presence of a fixed point, flowing towards the IR by lowering k/k_0, the contributions from the eigenvectors $\vec{b}_I$ which feature a negative critical exponent $\theta_I < 0$ are suppressed. In turn, the associated parameters b_{I0} become irrelevant for the values of the couplings at a scale $k \gg k_0$. The corresponding directions in the space of couplings are called irrelevant, or IR attractive, or UV repulsive. Conversely, contributions from eigenvectors $\vec{b}_I$ which feature a positive critical exponent $\theta_I > 0$, grow towards the IR. Thus, the IR values of couplings depend on the associated parameters b_{I0}. These directions are called relevant, or IR repulsive, or UV attractive. If a critical exponent $\theta_I = 0$ vanishes, the corresponding direction is called marginal. In this case, higher-order contributions in the expansion (4.8) have to be taken into account in order to determine if this direction is marginally relevant or irrelevant.

Generically, when the beta functions are real, the eigenvalues of the stability matrix can occur as complex conjugate pairs. Thus, the critical exponents θ_I can be complex. In this case, their imaginary part is responsible for a spiralling of the flow around the fixed point, whereas their real part determines if the direction is attractive or repulsive,

$$\mathrm{Re}[\theta_I] \;>\; 0 \qquad \Leftrightarrow \qquad \text{relevant (IR repulsive or UV attractive)}\,, \tag{4.12}$$
$$\mathrm{Re}[\theta_I] \;<\; 0 \qquad \Leftrightarrow \qquad \text{irrelevant (IR attractive or UV repulsive)}\,, \tag{4.13}$$
$$\mathrm{Re}[\theta_I] \;=\; 0 \qquad \Leftrightarrow \qquad \text{marginal}\,. \tag{4.14}$$

For a Gaussian fixed point, the linearised beta functions are given by

$$\beta_{c_i} = -d_{\bar{c}_i} c_i + \mathcal{O}\big(\vec{c}^2\big)\,, \tag{4.15}$$

such that the stability matrix is diagonal with entries determined by the negative canonical dimensions of the couplings,

$$B_{ij} = -d_{\bar{c}_i}\delta_{ij}\,. \tag{4.16}$$

In particular, in this case the eigendirections align with the couplings and the critical exponents are given by $\theta_I = d_{\bar{c}_i}$. As a consequence, in the perturbative regime near the Gaussian fixed point, the RG flow drives higher-order couplings to zero, while couplings with positive canonical dimension are relevant.

For a non-Gaussian fixed point, the situation is more complicated. In this case, the beta functions receive an anomalous contribution beyond the canonical term, which arises from quantum fluctuations. This contribution can render canonically relevant couplings irrelevant and vice versa.

4.1.2 Truncation ansatz for the effective average action

In this subsection, we define an ansatz for the effective average action Γ_k which allows us to address the question of decoupling of the non-metric degrees of freedom in area-metric gravity. In the absence of a non-perturbative action for area-metrics, we construct this ansatz perturbatively. Our starting point is the most general diffeomorphism-invariant action for area-metric perturbations around a background configuration induced by the flat length metric. This action was derived in subsection 3.1.2 at quadratic order in area-metric perturbations and momenta, and is a functional of the fields $h_{\mu\nu}$ and $\omega^{\pm}_{\mu\nu\rho\sigma}$ associated with the length-metric and the non-metric degrees of freedom of the area-metric perturbation.

For our analysis of RG flows in area-metric gravity, we consider a simple approximation which only takes into account momentum-independent contributions at third order in the perturbations of the area metric, and specifically focus on the flow of masses of $\omega^{\pm}$ which is induced by gravitational fluctuations h. Under these assumptions, there are only two types of interaction vertices to consider, as derived in appendix A.3. The corresponding scalar invariants are given by

$$h\omega^{\pm}_{\mu\nu\rho\sigma}\omega^{\pm\mu\nu\rho\sigma} \qquad \text{and} \qquad h^{\mu\nu}h^{\rho\sigma}\omega^{\pm}_{\mu\rho\nu\sigma}\,. \tag{4.17}$$

The couplings in front of these terms in a Lagrangian for area-metric perturbations may in principle depend on each other, when the condition of diffeomorphism invariance is taken into account at higher orders. In the lack of knowledge about such a possible constraint, we consider these terms with independent couplings $\bar{\lambda}'_{1\pm}$ and $\bar{\lambda}'_{2\pm}$ in the ansatz for Γ_k. Combining the result for the general quadratic second-order action for area-metric perturbations, derived in subsection 3.1.2, with the two terms in (4.17), defines our ansatz for the EAA,

$$\Gamma_k \equiv S_k^{(2)}[h_{\mu\nu}] + \int \mathrm{d}^4x \sum_{\pm} \Big[\bar{\alpha}'_{\pm} \partial^{\mu} h^{\rho\sigma} \partial^{\nu} \omega^{\pm}_{\mu\rho\nu\sigma} + \frac{1}{2} \partial_{\alpha} \omega^{\pm}_{\mu\nu\rho\sigma} \partial^{\alpha} \omega^{\pm\mu\nu\rho\sigma} + \frac{1}{2} \bar{m}^2_{\pm} \omega^{\pm}_{\mu\nu\rho\sigma} \omega^{\pm\mu\nu\rho\sigma} + \bar{\lambda}'_{1\pm} h \omega^{\pm}_{\mu\nu\rho\sigma} \omega^{\pm\,\mu\nu\rho\sigma} + \bar{\lambda}'_{2\pm} h^{\mu\nu} h^{\rho\sigma} \omega^{\pm}_{\mu\rho\nu\sigma} \Big] . \tag{4.18}$$

Here, we have denoted interaction couplings between h and $\omega^{\pm}$ with a prime, in order to reserve the notation without a prime for a redefined version of these couplings following a field rescaling detailed below.

The first contribution in Γ_k, quadratic in $h_{\mu\nu}$, is given by the second-order term in an expansion of the Einstein-Hilbert action around the flat Euclidean background, [1]

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu} \, , \tag{4.19}$$

together with a term which gauge-fixes the metric fluctuations $h_{\mu\nu}$ with respect to the flat background,

$$\begin{aligned} S_k^{(2)}[h_{\mu\nu}] &\equiv S_{\mathrm{EH}}^{(2)}[h_{\mu\nu}] + S_{\mathrm{gf}}^{(2)}[h_{\mu\nu}] \\ &= -\frac{1}{16\pi G} \int \mathrm{d}^4x \, \sqrt{g} R \Big|_{\mathcal{O}(h^2)} + \frac{1}{32\pi G \alpha_h} \int \mathrm{d}^4x \left(\partial^{\nu} h_{\nu\mu} - \frac{1+\beta_h}{4} \partial_{\mu} h \right)^2 . \end{aligned} \tag{4.20}$$

In the previous expression G is the dimensionful Newton coupling, whereas α_h and β_h are gauge parameters. All results stated in the following apply in the limit $\beta_h \to \alpha_h \to 0$. Our analysis targets the RG flow of the masses for $\omega^{\pm}$, and of the interaction couplings between $\omega^{\pm}$ and h. In particular, we do not evaluate the scale dependence of couplings in the pure length-metric sector, and for this reason do not consider Faddeev-Popov ghost terms. The structure of the regularised propagator entering the flow equation (4.1) for the effective action (4.18) is illustrated in appendix A.4.

The field h in (4.18) denotes the dimensionless metric perturbation. After expanding the Einstein-Hilbert action at second order in the perturbation, we promote this field to a standard graviton field with canonical mass dimension one by rescaling

$$h_{\mu\nu} \to \sqrt{16\pi G Z_h} h_{\mu\nu} \tag{4.21}$$

everywhere in Γ_k. Here, we have introduced the wave-function renormalisation Z_h of h. The

1. It should be noted that in the flat-background expansion of the metric (4.19), the tensor $h_{\mu\nu}$ represents the dimensionless metric perturbations. By contrast, in section 3.1, the tensor $h_{\mu\nu}$ was already considered to be a bosonic field with canonical mass dimension one, and was related to the perturbation of the length metric by $\delta g_{\mu\nu} = 2\kappa h_{\mu\nu} = 16\pi G h_{\mu\nu}$ where G is the dimensionful Newton coupling. Both descriptions are equivalent when rescaling $h_{\mu\nu} \to \sqrt{16\pi G} h_{\mu\nu}$ in the Taylor expansion of S_{EH} in (4.20) at second order in $h_{\mu\nu}$, and keeping in mind that this expansion generates an additional factor of $\frac{1}{2!}$.

fields $\omega^{\pm}$ in Γ_k are already canonically normalised. Consequently, we only introduce the wave-function renormalisation $Z_{\omega^\pm}$ of $\omega^{\pm}$ by rescaling

$$\omega^{\pm}_{\mu\nu\rho\sigma} \to \sqrt{Z_{\omega^\pm}}\,\omega^{\pm}_{\mu\nu\rho\sigma} \tag{4.22}$$

in the effective action. As a consequence of the rescalings (4.21) and (4.22), we redefine all couplings in front of the terms in Γ_k, which contain both h and $\omega^{\pm}$, so as to absorb all numerical factors and powers of G. Concretely, as an intermediate step we introduce

$$\begin{aligned}
\bar{\alpha}_{\pm} &= \sqrt{16\pi G}\,\bar{\alpha}'_{\pm}\,, && (4.23)\\
\bar{\lambda}_{1\pm} &= \sqrt{16\pi G}\,\bar{\lambda}'_{1\pm}\,, && (4.24)\\
\bar{\lambda}_{2\pm} &= 16\pi G\,\bar{\lambda}'_{2\pm}\,. && (4.25)
\end{aligned}$$

The redefined couplings $\bar{\alpha}_{\pm}$ are dimensionless, whereas $\bar{\lambda}_{1\pm}$ and $\bar{\lambda}_{2\pm}$ have canonical dimension one. From now on, we focus on the dimensionless versions of these couplings and of the Newton coupling,

$$\begin{aligned}
g &= Gk^2\,, && (4.26)\\
\alpha_{\pm} &= \bar{\alpha}_{\pm}\,, && (4.27)\\
m^2_{\pm} &= \bar{m}^2_{\pm}k^{-2}\,, && (4.28)\\
\lambda_{1,2\pm} &= \bar{\lambda}_{1,2\pm}k^{-1}\,. && (4.29)
\end{aligned}$$

4.1.3 Decoupling of non-metric degrees of freedom

In this subsection, we consider the decoupling of the non-metric degrees of freedom of the area metric. Decoupling can occur as a result of a dynamically growing mass, and can be enhanced further through a dynamical suppression of the interaction couplings between these fields and the length metric. To that end, we analyse the RG flow of these parameters separately.

In the absence of interactions, the dimensionful mass squares $\bar{m}^2_{\pm}$ are constant and the dimensionless mass squares scale as $m^2_{\pm} \propto k^{-2}$. With this canonical scaling, large physical masses of the non-metric degrees of freedom in the IR can be achieved, provided the initial conditions for the flow of $m^2_{\pm}$ in the UV are chosen appropriately. A natural choice is $m^2_{\pm}(\Lambda_{\mathrm{UV}}) = m^2_{\pm\mathrm{UV}} = \mathcal{O}(1)$, where Λ_{UV} denotes the cutoff scale of the area-metric effective field theory. However, in the following we are interested in determining whether gravitational fluctuations can generically drive the masses of the non-metric degrees of freedom further towards relevance, compared to their canonical scaling. If this is the case, the non-metric degrees of freedom decouple in the IR, irrespective of the initial conditions set in the UV.

In the presence of quantum fluctuations, the beta functions for $m^2_{\pm}$ receive a non-canonical contribution, see e.g. (4.5). As a result, the dimensionless mass squares scale as

$$m_\pm^2 \propto k^{\Delta_{m_\pm^2}} \qquad \text{where} \qquad \Delta_{m_\pm^2} = -2 + \eta_{m_\pm^2} \tag{4.30}$$

are the scaling dimensions, which differ from their canonical value -2 by the anomalous dimensions $\eta_{m_\pm^2}$. In a regime $k \in (k_{\mathrm{IR}}, \Lambda_{\mathrm{UV}})$, where this anomalous scaling holds, the dimensionful mass squares are given by

$$\bar{m}_\pm^2(k) = m_\pm^2(k)k^2 = \left[m_{\pm\mathrm{UV}}^2\left(\frac{k}{\Lambda_{\mathrm{UV}}}\right)^{\Delta_{m_\pm^2}}\right]k^2 = m_{\pm\mathrm{UV}}^2 \cdot \Lambda_{\mathrm{UV}}^2 \cdot \left(\frac{k}{\Lambda_{\mathrm{UV}}}\right)^{\eta_{m_\pm^2}}. \tag{4.31}$$

A negative or positive anomalous dimension $\eta_{m_\pm^2}$ implies that the dimensionful mass squares at the scale k_{IR} are larger, or smaler, respectively, than their value at the scale Λ_{UV}, by a factor of $(k_{\mathrm{IR}}/\Lambda_{\mathrm{UV}})^{\eta_{m_\pm^2}}$. For ordinary scalar fields, the anomalous scaling dimension of the mass induced by quantum-gravitational fluctuations is generically positive [169–171]. Analogous observations hold for spin-$\frac{1}{2}$ fermions [172]. Therefore, the canonical growth of the dimensionless masses of these fields is counteracted by quantum fluctuations, such that their dimensionful masses becomes lighter in the IR. For tensor fields such as $\omega^\pm$, an analogous effect has not yet been investigated. It is, however, central for the phenomenological viability of area-metric gravity that quantum-gravitational fluctuations support the canonical growth of their masses, and thereby allow for a decoupling of these fields at low energies.

To determine whether gravitational fluctuations can make the masses of the non-metric degrees of freedom of the area metric more relevant, we consider their beta functions. These, and other beta functions, are generically complicated non-polynomial functions of all couplings. On these grounds, we resort to expansions of beta functions around particular values of couplings.

First, we expand the beta functions $\beta_{m_\pm^2}$ for small $m_\pm^2$ and small couplings $c \in \{\alpha_\pm, \lambda_{1,2\pm}\}$. The leading term in this expansion is of order zero in the mass,

$$\beta_{m_\pm^2} = -\frac{2\lambda_{1\pm}^2}{3\pi^2} + \frac{7\lambda_{2\pm}^2}{192\pi^2} + \mathcal{O}\big(m_\pm^2, c^2c'^2, c^2c'c''\big)\,. \tag{4.32}$$

Thus, masses for the non-metric degrees of freedom are automatically generated by quantum fluctuations, even if the initial conditions set these to zero in the UV. Moreover, if we assume that $\lambda_{1\pm}$ and $\lambda_{2\pm}$ are of approximately the same magnitude, then the first term in (4.32) is dominant. The sign of this term is negative for any non-zero values of $\lambda_{1\pm}$. Therefore, the masses of the non-metric degrees of freedom grow under the RG flow towards the IR.

As a next step, we assume initial conditions at an intermediate scale, at which the masses have already grown to large positive values. The primary question is whether quantum fluctuations support this growth further. To answer this question, we consider an expansion of the beta functions for large $m_\pm^2$,

$$\beta_{m_\pm^2} = \left(-2 - \frac{3\lambda_{2\pm}^2}{128\pi^2}\right) m_\pm^2 + \mathcal{O}\left(m_\pm^0\right) . \tag{4.33}$$

The non-canonical contribution has the same sign as the canonical term. The anomalous scaling dimension $\eta_{m_\pm^2}$ can be read off from the expression in brackets, after subtracting the negative canonical dimension -2. Since $\eta_{m_\pm^2}$ is negative for non-zero $\lambda_{2\pm}$, we can conclude that the dimensionful masses of the non-metric degrees of freedom are protected from decreasing under the flow, and continue to grow towards the IR as a result of quantum-gravitational fluctuations. This provides a robust decoupling mechanism for the non-metric degrees of freedom in area-metric gravity.

In the remaining part of this subsection, we extend the previous considerations for the decoupling of the non-metric degrees of freedom, by additionally taking into account the vertex couplings between these and the length metric. To that end, we analyse whether these couplings can be suppressed dynamically under the flow towards the IR, assuming initial conditions in a regime where the masses of the non-metric degrees of freedom are already large.

In the absence of interactions, the dimensionful couplings $\bar{\lambda}_{1,2\pm}$ are constant, whereas the dimensionless couplings $\lambda_{1,2\pm}$ scale as $\lambda_{1,2\pm} \propto k^{-1}$, and are therefore canonically relevant. According to this canonical scaling, positive or negative initial values $\lambda_{1,2\pm}(\Lambda_{\mathrm{UV}}) = \lambda_{1,2\pm\mathrm{UV}}$ in the UV result in a growth of these couplings to large positive or large negative values in the IR. In the following, we analyse whether gravitational fluctuations can counteract this growth.

In the presence of quantum fluctuations, the beta functions for $\lambda_{1,2\pm}$ receive a non-canonical contribution, see e.g. (4.5). As a result, the dimensionless couplings $\lambda_{1,2\pm}$ scale as

$$\lambda_{1,2\pm} \propto k^{\Delta_{\lambda_{1,2\pm}}} \qquad \text{where} \qquad \Delta_{\lambda_{1,2\pm}} = -1 + \eta_{\lambda_{1,2\pm}} \tag{4.34}$$

are the scaling dimensions, which differ from their canonical value -1 by the anomalous dimensions $\eta_{\lambda_{1,2\pm}}$. In a regime $k \in (k_{\mathrm{IR}}, \Lambda_{\mathrm{UV}})$, where this anomalous scaling holds, the dimensionful couplings are given by

$$\bar{\lambda}_{1,2\pm}(k) = \lambda_{1,2\pm}(k) k = \left[\lambda_{1,2\pm\mathrm{UV}} \left(\frac{k}{\Lambda_{\mathrm{UV}}}\right)^{\Delta_{\lambda_{1,2\pm}}}\right] k = \lambda_{1,2\pm\mathrm{UV}} \cdot \Lambda_{\mathrm{UV}} \cdot \left(\frac{k}{\Lambda_{\mathrm{UV}}}\right)^{\eta_{\lambda_{1,2\pm}}} . \tag{4.35}$$

A negative or positive anomalous dimension $\eta_{\lambda_{1,2\pm}}$ implies that the dimensionful couplings at the scale k_{IR} are larger, or smaler, respectively, in magnitude than their value at the scale Λ_{UV}, by a factor of $(k_{\mathrm{IR}}/\Lambda_{\mathrm{UV}})^{\eta_{\lambda_{1,2\pm}}}$.

To determine whether gravitational fluctuations can drive the vertex couplings towards irrelevance relative to their canonical scaling, we consider their beta functions. Their expansion for

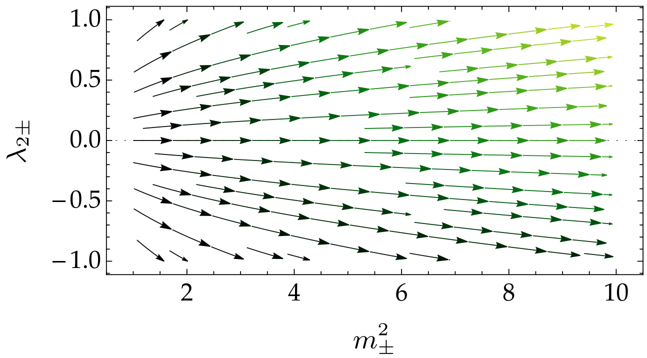

Figure 4.2: Flow towards the IR in the plane spanned by $m_\pm^2$ and $\lambda_{2\pm}$, according to the beta functions (4.33) and (4.37). Gravitational fluctuations drive the masses $m_\pm^2$ to large values in the IR even faster than dictated by their canonical scaling. This provides a robust decoupling mechanism for the non-metric degrees of freedom in area-metric gravity at low energies. Simoultaneously, the interaction couplings $\lambda_{2\pm}$ between these fields and the length metric also grow dynamically towards the IR, relative to their canonical scaling. This enhancement may result in a residual effect of the non-metric degrees of freedom in the low-energy effective action for the metric degrees of freedom in area-metric gravity.

large masses $m_\pm^2$ is given by

$$\beta_{\lambda_{1\pm}} = \left(-1 - \frac{3\lambda_{2\pm}^2}{128\pi^2}\right)\lambda_{1\pm} + \frac{7\sqrt{\pi g}\lambda_{2\pm}^2}{96\pi^2} + \mathcal{O}\left(m_\pm^{-2}\right), \tag{4.36}$$

$$\beta_{\lambda_{2\pm}} = \left(-1 - \frac{3\lambda_{2\pm}^2}{256\pi^2}\right)\lambda_{2\pm} + \mathcal{O}\left(m_\pm^{-2}\right). \tag{4.37}$$

For a suppression of the couplings $\lambda_{1,2\pm}$, relative to their canonical scaling, the non-canonical contribution to their beta function has to be of opposite sign to the canonical term.

The sign of the non-canonical term in the beta function for $\lambda_{1\pm}$ for non-zero $\lambda_{2\pm}$ is

$$\mathrm{sign}(-9\lambda_{1\pm} + 28\sqrt{\pi g}) = \begin{cases} \mathrm{sign}(\lambda_{1\pm}), & \text{for} \quad \lambda_{1\pm} \in (-\infty, 0) \cup \left(\frac{28}{9}\sqrt{\pi g}, \infty\right), \\ -\mathrm{sign}(\lambda_{1\pm}), & \text{for} \quad \lambda_{1\pm} \in \left(0, \frac{28}{9}\sqrt{\pi g}\right). \end{cases} \tag{4.38}$$

In the first case, $\lambda_{1\pm}$ cannot be driven towards irrelevance in the regime of large $m_\pm^2$, and instead grows in magnitude towards the IR even faster than canonically. An anomalous scaling opposite to canonical scaling can only be achieved for positive $\lambda_{1\pm} < \frac{28}{9}\sqrt{\pi g}$.

Next, we consider the non-canonical term in the beta function for the couplings $\lambda_{2\pm}$. The sign of this term is

$$\mathrm{sign}\left(-\lambda_{2\pm}^3\right) = \mathrm{sign}(-\lambda_{2\pm}). \tag{4.39}$$

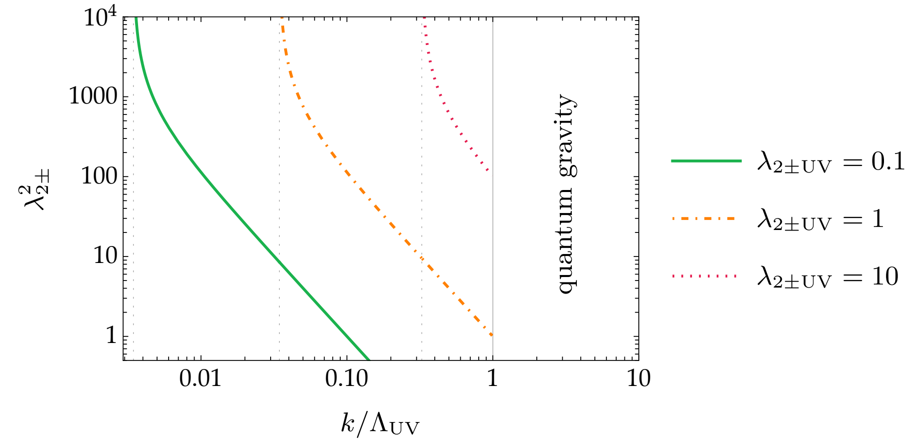

Figure 4.3: Vertex couplings $\lambda^2_{\pm 2}$ as functions of the scale parameter $k/\Lambda_{\rm UV}$ for different initial conditions $\lambda_{2\pm\rm UV} \equiv \lambda_{2\pm}(\Lambda_{\rm UV})$ in the UV. Under the RG flow towards the IR, these couplings increase in magnitude and reach a strong-coupling regime signalled by a Landau pole in (4.40).

Therefore, $\lambda_{2\pm}$ becomes even more relevant than dictated by its canonical scaling. Figure 4.2 shows the RG flow towards the IR in the plane spanned by $m^2_\pm$ and $\lambda_{2\pm}$, according to the combined beta functions (4.33) and (4.37).

The beta functions $\beta_{\lambda_{2\pm}}$ in (4.37) can be integrated analytically to leading order with the initial conditions $\lambda_{2\pm}(\Lambda_{\rm UV}) = \lambda_{2\pm\rm UV}$. The result is

$$\lambda^2_{2\pm}(k) = -\frac{\lambda^2_{2+\rm UV}}{\frac{3\lambda^2_{2+\rm UV}}{256\pi^2} - \left(1 + \frac{3\lambda^2_{2+\rm UV}}{256\pi^2}\right)\left(\frac{k}{\Lambda_{\rm UV}}\right)^2}\,. \tag{4.40}$$

As anticipated, the couplings $\lambda_{2\pm}$ increase in magnitude under the flow towards the IR and reach a strong-coupling regime signalled by a Landau pole in the above expression. Figure 4.3 shows $\lambda^2_{2\pm}$ as functions of the scale parameter $k/\Lambda_{\rm UV}$ for different choices of initial conditions in the UV.

Altogether, the interaction couplings $\lambda_{1,2\pm}$, between the non-metric degrees of freedom and the length-metric fluctuation, generically remain relevant and grow under the RG flow towards the IR even faster than their canonical scaling dimension implies. The dimensionful couplings at a scale $k_{\rm IR}$, above the scale of a possible Landau pole, can be expected to be at least of the order of $\Lambda_{\rm UV}$ in the absence of fine-tuning of the initial conditions in the UV. In principle, initial conditions for the flow can be arranged such that the vertex couplings reach an IR Landau pole only after the non-metric degrees of freedom have decoupled, as a result of their heavy mass.

In summary, on the one hand we find evidence that decoupling in area-metric gravity, as a result of large masses of the non-metric degrees of freedom, is a realistic and robust scenario. Such large masses would freeze the dynamics of the non-metric degrees of freedom in the IR. However, on the other hand, we observe that the interaction couplings between these fields and the length-metric fluctuation also stay large in units of $\Lambda_{\rm UV}$. This may result in a residual effect of the non-metric degrees of freedom in the low-energy limit of the theory. In particular, large values of these couplings may generate large Wilson coefficients in the effective field theory for the length-metric degrees of freedom. These may be reflected in dominant higher-curvature terms in the action.

4.1.4 Parity-violating directions in theory space

In this subsection, we consider the RG flow of parity-violating couplings in area-metric gravity. Parity violation in length-metric gravitational systems is observationally tightly constrained, see e.g. [131, 132]. Thus, for area-metric gravity to be phenomenologically viable, the low-energy effective action for the length-metric degrees of freedom must respect parity symmetry to a good approximation. This can be expected, if the parity-violating couplings in the area-metric action are dynamically driven to small values in the low-energy regime.

On these grounds, we will analyse if a departure from the parity-symmetric subspace of the area-metric theory space in the UV can be reversed by the RG flow towards the IR. If this is the case, parity invariance of the low-energy theory can be regarded as an emergent symmetry. However, if this is not the case, parity-violating couplings in the low-energy regime of area-metric gravity may induce parity-violating interactions in the effective action for the length-metric degrees of freedom, irrespective of the decoupling of the non-metric degrees of freedom of the area metric.

A parity transformation leaves the length-metric degrees of freedom of the area metric invariant, and interchanges the selfdual and anti-selfdual Weyl components of the area metric corresponding to the non-metric degrees of freedom,

$$h_{\mu\nu} \rightarrow h_{\mu\nu}\,, \tag{4.41}$$

$$\omega^{\pm}_{\mu\nu\rho\sigma} \rightarrow \omega^{\mp}_{\mu\nu\rho\sigma}\,. \tag{4.42}$$

This field transformation is a global symmetry of Γ_k, if

$$c_+ = c_- \equiv c \qquad \forall \text{ couplings } c_\pm \in \Gamma_k\,. \tag{4.43}$$

Concretely, for the truncation ansatz (4.18), this condition concerns the couplings $c_\pm \in \{m^2_\pm, \alpha_\pm, \lambda_{1,2\pm}\}$. Within the parity-symmetric subspace of the full theory space, defined by (4.43), the beta functions for all couplings $c_\pm$ in Γ_k are identical ,such that $\beta_c = \beta_{c_\pm}$.

To analyse directions away from the parity-symmetric subspace, we consider a transformation from couplings $\{c_+, c_-\}$ to sums and differences $\{\delta_c, \sigma_c\}$ by writing

$$\delta_c = c_+ - c_-\,, \tag{4.44}$$

$$\sigma_c = c_+ + c_-\,. \tag{4.45}$$

The differences δ_c represent the parity-violating couplings in the effective action.

To determine if parity invariance can be an emergent symmetry in the IR, we consider the beta functions for the parity-violating couplings, expanded in a regime of large mass-square sum σ_{m^2} and small differences δ_c, for all $c \in \{m^2, \alpha, \lambda_{1,2}\}$. These can be expressed as

$$\beta_{\delta_{m^2}} = \left(-2 - \frac{3\sigma_{\lambda_2}^2}{512\pi^2}\right)\delta_{m^2} + \frac{28\sigma_{\lambda_2} - 9\sigma_{m^2}\sigma_{\lambda_2} - 2\sigma_\alpha^2\sigma_{\lambda_2}}{768\pi^2}\delta_{\lambda_2} + \mathcal{O}\big(\sigma_{m^2}^{-1}, \delta_c^2, \delta_c\delta_{c'}\big)\,, \tag{4.46}$$

$$\beta_{\delta_\alpha} = \frac{3\sigma_{\lambda_2}^2}{1024\pi^2}\delta_\alpha - \frac{162\sigma_\alpha\sigma_{\lambda_2} + 15872\sqrt{\pi g}}{27648\pi^2}\delta_{\lambda_2} + \mathcal{O}\big(\sigma_{m^2}^{-1}, \delta_c^2, \delta_c\delta_{c'}\big)\,, \tag{4.47}$$

$$\beta_{\delta_{\lambda_1}} = \left(-1 - \frac{3\sigma_{\lambda_2}^2}{512\pi^2}\right)\delta_{\lambda_1} + \frac{9\sigma_{\lambda_1}\sigma_{\lambda_2} - 56\sigma_{\lambda_2}\sqrt{\pi g}}{768\pi^2}\delta_{\lambda_2} + \mathcal{O}\big(\sigma_{m^2}^{-1}, \delta_c^2, \delta_c\delta_{c'}\big)\,, \tag{4.48}$$

$$\beta_{\delta_{\lambda_2}} = \left(-1 - \frac{9\sigma_{\lambda_2}^2}{1024\pi^2}\right)\delta_{\lambda_2} + \mathcal{O}\big(\sigma_{m^2}^{-1}, \delta_c^2, \delta_c\delta_{c'}\big)\,. \tag{4.49}$$

The stability matrix, derived from the linearised beta functions in the subspace spanned by the couplings $\{\delta_{m^2}, \delta_\alpha, \delta_{\lambda_{1,2}}\}$, has an upper-triangular form. The eigenvalues, corresponding to the negative critical exponents, can be read off directly from the factors multiplying a given coupling δ_c in the expansion of its beta function β_{δ_c} above. Since the critical exponents determine if an eigendirection is relevant or irrelevant, we focus only on these terms.

The beta function for δ_α has no canonical term, as this coupling is canonically marginal. Considering the first term in (4.47), this parity-violating coupling can be suppressed dynamically, because for positive δ_α this term is positive. On the other hand, the beta functions for the other parity-violating couplings, δ_{m^2} and $\delta_{\lambda_{1,2}}$, contain canonical terms which indicate that these couplings are relevant. Moreover, the sign of the non-canonical contribution in the first terms of (4.47), (4.48) and (4.49), for non-zero σ_{λ_2}, is identical to the sign of the canonical term. As a result, taking into account only the first terms in the beta functions for δ_{m^2} and $\delta_{\lambda_{1,2}}$, these parity-violating couplings remain relevant and grow towards the IR even faster than canonically. This holds generically for the coupling δ_{λ_2}, whose beta function, to the considered order, consists only of one term proportional to δ_{λ_2}. A non-zero coupling δ_{λ_2}, in turn, induces additional terms in the beta functions for the couplings δ_{m^2}, δ_α, and δ_{λ_1}. Thus, if δ_{λ_2} is already non-zero, this coupling may in principle limit the further growth of some other couplings in parts of the parameter space.

Altogether, however, any departure from the parity-symmetric subspace of the area-metric theory in the UV cannot be reversed by the flow towards the IR. Instead, such a parity violation will be dynamically enhanced. This observation indicates that parity invariance cannot be an emergent symmetry in the low-energy regime of area-metric gravity. The only possibility for a parity-invariant low-energy theory are exact parity-symmetric initial conditions in the UV.

4.1.5 Flow of the Immirzi parameter

In this subsection, we analyse the flow of the Immirzi parameter γ [80, 81], whose inverse is proportional to the coupling in front of the Holst action [126]. In section 3.2, we have identified γ as a parity-violating coupling in area-metric actions derived from modified non-chiral Plebanski theories. If the mass parameters of the non-metric degrees of freedom $\omega^{\pm}$ are identical, the effective action, obtained by integrating out these fields from the linearised theory, is ghostfree. These results were established in subsections 3.1.3 and 3.2.3. The associated subclass of linearised area-metric actions with shift-symmetric kinetic term is the starting point for the analysis in this subsection.

According to (3.160), the non-gauge-fixed quadratic part of the classical action can be expressed as

$$\begin{aligned} S^{(2)} \quad \equiv \quad & S^{(2)}_{\mathrm{EH}}[h_{\mu\nu}] + \frac{1}{2}\sum_{\pm}\Big[\bar{\gamma}_{\pm}\partial^{\mu}h^{\rho\sigma}\partial^{\nu}\omega^{\pm}_{\mu\rho\nu\sigma} + \frac{1}{8}\bar{\gamma}_{\pm}\partial_{\alpha}\omega^{\pm}_{\mu\nu\rho\sigma}\partial^{\alpha}\omega^{\pm\mu\nu\rho\sigma} \\ & + \frac{1}{8}\bar{\gamma}_{\pm}\bar{m}^2_{\pm}\omega^{\pm}_{\mu\nu\rho\sigma}\omega^{\pm\mu\nu\rho\sigma}\Big]\,. \end{aligned} \tag{4.50}$$

Here, we have translated between the fields $\chi^{\pm}$ and $\omega^{\pm}$, using (3.158) and (3.159), and redefined their mass parameters by replacing $m^2_{\pm} \to \bar{\gamma}_{\pm}\bar{m}^2_{\pm}$ in (3.160). [1]

The dimensionless Immirzi parameter γ enters in (4.50) through the dimensionful couplings

$$\bar{\gamma}_{\pm} = \frac{1}{8\pi G}(1 \pm \hat{\gamma}) \qquad \text{where} \qquad \hat{\gamma} = \frac{1}{\gamma}\,. \tag{4.52}$$

The limit $\hat{\gamma} \to 0$ is associated originally with a suppression of the Holst term in the modified Plebanski-Holst action. In this limit, $S^{(2)}$ in (4.50) is parity-invariant, if in addition $\bar{m}^2_{+} = \bar{m}^2_{-}$ holds. Parity violation in the interaction term, and in the kinetic term for the non-metric fields, is present for any non-zero value of $\hat{\gamma}$, and maximal for $\hat{\gamma} \to \pm\infty$.

1. We remind that, in section 3.1, the field $h_{\mu\nu}$ was already canonically normalised, and related to the dimensionless perturbation of the length metric by $\delta g_{\mu\nu} = 16\pi G h_{\mu\nu}$. To match with the order of steps performed in subsection 4.1.2, we consider this rescaling to not have taken place yet in (4.50). As a result, the couplings $\bar{\gamma}_{\pm}$ in (4.50) are dimensionful, as they would have been in section 3.2, had the factor $\frac{1}{2\kappa^2} = \frac{1}{16\pi G}$ in front of the (modified) Plebanski-Holst action not been set to one there. Additionally, $S^{(2)}_{\mathrm{EH}}[h_{\mu\nu}]$ in (4.50) denotes the term at second order in $h_{\mu\nu}$ in the Taylor expansion of $S_{\mathrm{EH}}[g_{\mu\nu}]$, with $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$. This term differs from $\int \mathrm{d}^4x\,\mathcal{L}_{\mathrm{EH}}(h_{\mu\nu})$ in chapter 3, by a factor of $\frac{1}{2!}$. Taking this into account, leads to the identity (4.52) and the avatar

$$\bar{\gamma}_{+} + \bar{\gamma}_{-} = \frac{1}{4\pi G} \tag{4.51}$$

of the relation (3.161). This relation reflects the condition (3.69) for a 5-parameter shift symmetry in the kinetic term.

Comparing the classical action (4.50) to the quadratic part of the effective average action Γ_k in (4.18), we can identify the couplings in front of the $h\omega^{\pm}$ terms in Γ_k as $\bar{\alpha}'_{\pm} = \frac{1}{2}\bar{\gamma}_{\pm}$. Following the analogous steps as in subsection 4.1.2, we normalise the field h canonically by rescaling

$$h_{\mu\nu} \to \sqrt{16\pi G Z_h} h_{\mu\nu} \,. \tag{4.53}$$

Additionally, the fields $\omega^{\pm}$ have to be normalised canonically by rescaling

$$\omega^{\pm}_{\mu\nu\rho\sigma} \to \sqrt{\frac{8 Z_{\omega^{\pm}}}{\bar{\gamma}_{\pm}}} \omega^{\pm}_{\mu\nu\rho\sigma} \,. \tag{4.54}$$

As a consequence of the rescalings (4.53) and (4.54), in the same way as before, we redefine the couplings in Γ_k so as to absorb all factors, except possible wave-function renormalisations appearing in front of a given term. In particular, as a result of this redefinition, the dimensionless couplings $\alpha_{\pm}$, in front of the $h\omega^{\pm}$ interaction, terms are given by $\alpha_{\pm} = \frac{1}{2}\sqrt{16\pi G \cdot 8\bar{\gamma}_{\pm}}$, and satisfy

$$\alpha_+^2 + \alpha_-^2 = 8 \,, \tag{4.55}$$
$$\alpha_+^2 - \alpha_-^2 = 8\,\hat{\gamma} \,. \tag{4.56}$$

However, it is important to note that the identity (4.55), which indicates a 5-parameter shift symmetry as analysed in subsection 3.1.4, holds only for the classical action. Under the RG flow, this sum of couplings evolves dynamically, such that the relation (4.55) is generically not preseved. To take this into account, we introduce a dimensionless coupling σ_{α^2} proportional to the sum of α_+^2 and α_-^2 in the effective action. Concretely, we replace (4.55) and (4.56) by

$$\alpha_+^2 + \alpha_-^2 = 8\,\sigma_{\alpha^2} \,, \tag{4.57}$$
$$\alpha_+^2 - \alpha_-^2 = 8\,\hat{\gamma} \,. \tag{4.58}$$

On these grounds, we consider a transformation from the couplings $\{\alpha_+, \alpha_-\}$ to the couplings $\{\sigma_{\alpha^2}, \hat{\gamma}\}$. [1] With

$$\alpha_{\pm} = 2\sqrt{\sigma_{\alpha^2} \pm \hat{\gamma}} \,, \tag{4.60}$$

1. To this end, equations (4.57) and (4.56) must be solved for $\alpha_{\pm}$ in terms of σ_{α^2} and $\hat{\gamma}$. There are four sets of solutions,

$$\alpha_+ = s_+ 2\sqrt{\sigma_{\alpha^2} + \hat{\gamma}} \quad \text{and} \quad \alpha_- = s_- 2\sqrt{\sigma_{\alpha^2} - \hat{\gamma}} \,, \tag{4.59}$$

labelled by all possible combinations (s_+, s_-) of signs $s_+, s_- = \pm$. Solutions with mixed signs are excluded by the requirement that the condition $\alpha_+ = \alpha_-$, for parity symmetry, is respected in the limit $\hat{\gamma} \to 0$. A choice among the remaining two solutions amounts to a global choice of sign in front of the $h\omega^{\pm}$ terms in Γ_k. Thus, without loss of generality, we focus on the solution $\alpha_{\pm} = 2\sqrt{1 \pm \hat{\gamma}}$. The opposite choice $-\alpha_{\pm}$ is equivalent, at the level of Γ_k, after a field redefinition $\omega^{\pm} \to -\omega^{\pm}$ and a redefinition of the couplings $\lambda_{2\pm} \to -\lambda_{2\pm}$.

the beta functions of σ_{α^2} and $\hat{\gamma}$ can be expressed as

$$\begin{aligned} \beta_{\sigma_{\alpha^2}} &= \frac{1}{4}\big(\alpha_+\beta_{\alpha+} + \alpha_-\beta_{\alpha-}\big)\,, & (4.61)\\ \beta_{\hat{\gamma}} &= \frac{1}{4}\big(\alpha_+\beta_{\alpha+} - \alpha_-\beta_{\alpha-}\big)\,. & (4.62)\end{aligned}$$

In order to analyse the RG flow of the couplings σ_{α^2} and $\hat{\gamma}$, we expand their beta functions for large masses $m_\pm^2$ of the non-metric degrees of freedom,

$$\begin{aligned} \beta_{\sigma_{\alpha^2}} &= -\sum_\pm \left[(\sigma_{\alpha^2} \pm \hat{\gamma})\,\frac{3\lambda_{2\pm}^2}{256\pi^2} + \sqrt{\sigma_{\alpha^2} \pm \hat{\gamma}}\,\frac{31\sqrt{\pi g}\lambda_{2\pm}}{108\pi^2}\right] + \mathcal{O}\big(m_\pm^{-2}\big)\,, & (4.63)\\ \beta_{\hat{\gamma}} &= -\sum_\pm \left[\pm(\sigma_{\alpha^2} \pm \hat{\gamma})\,\frac{3\lambda_{2\pm}^2}{256\pi^2} \pm \sqrt{\sigma_{\alpha^2} \pm \hat{\gamma}}\,\frac{31\sqrt{\pi g}\lambda_{2\pm}}{108\pi^2}\right] + \mathcal{O}\big(m_\pm^{-2}\big)\,. & (4.64)\end{aligned}$$

The leading terms in $\beta_{\hat{\gamma}}$ and $\beta_{\sigma_{\alpha^2}}$ are independent of $\lambda_{1\pm}$.

From (4.63) and (4.64), we see that the flow of σ_{α^2} and $\hat{\gamma}$ freezes out in the limit $m_\pm^2 \to \infty$ and $\lambda_{2\pm} \to 0$, in which the non-length metric degrees of freedom decouple. Therefore, length-metric fluctuations alone do not induce a flow for these couplings.

Moreover, the beta functions for σ_{α^2} and $\hat{\gamma}$ vanish in the limit $\{\sigma_{\alpha^2}, \hat{\gamma}\} \to 0$, when the interaction couplings $\alpha_\pm$ in front of the $h\omega^\pm$ terms in the effective action are switched off. At this point in the parameter space, the matrix of derivatives

$$\begin{pmatrix} \partial_{\sigma_{\alpha^2}}\beta_{\sigma_{\alpha^2}} & \partial_{\hat{\gamma}}\beta_{\sigma_{\alpha^2}} \\ \partial_{\sigma_{\alpha^2}}\beta_{\hat{\gamma}} & \partial_{\hat{\gamma}}\beta_{\hat{\gamma}} \end{pmatrix} = -\sum_\pm \left[\frac{3}{256\pi^2}\begin{pmatrix} \lambda_{2\pm}^2 & \pm\lambda_{2\pm}^2 \\ \pm\lambda_{2\pm}^2 & \lambda_{2\pm}^2 \end{pmatrix} + \frac{31\sqrt{\pi g}}{216\pi^2}\,\frac{1}{\sqrt{\sigma_{\alpha^2} \pm \hat{\gamma}}}\begin{pmatrix} \lambda_{2\pm} & \pm\lambda_{2\pm} \\ \pm\lambda_{2\pm} & \lambda_{2\pm} \end{pmatrix}\right]$$

diverges, such that no well-defined critical exponents can be assigned. The sign of $\partial_{\hat{\gamma}}\beta_{\hat{\gamma}}$, when approaching the limit $\hat{\gamma} \to 0$, followed by $\sigma_{\alpha^2} \to 0$, depends on the combination $-\sum_\pm \lambda_{2\pm}$. The flow is driven towards $\hat{\gamma} = 0$ for $\lambda_{2+} + \lambda_{2-} < 0$, and away from $\hat{\gamma} = 0$ for $\lambda_{2+} + \lambda_{2-} > 0$.

More generally, any configurations of the form $\{\sigma_{\alpha^2} = +\hat{\gamma}, \lambda_{2+} = 0\}$ and $\{\sigma_{\alpha^2} = -\hat{\gamma}, \lambda_{2-} = 0\}$ are zeros of $\beta_{\sigma_{\alpha^2}}$ and $\beta_{\hat{\gamma}}$. In particular, the condition for a shift-symmetric kinetic term, satisfied for the classical action, can be realised at the fixed point $(\sigma_{\alpha^2}, \hat{\gamma}, \lambda_{2+}) = (1, 1, 0)$ or the fixed point $(\sigma_{\alpha^2}, \hat{\gamma}, \lambda_{2-}) = (1, -1, 0)$. Additionally, there are other zeros of the beta functions for σ_{α^2} and $\hat{\gamma}$, which we consider to be truncation artifacts and do not analyse further.

Finally, we consider the point associated with maximal parity violation in the $h\omega^\pm$ interaction terms. This point is characterised by vanishing Immirzi parameter γ. The beta function for γ can be obtained from the beta function for $\hat{\gamma}$, via

$$\beta_\gamma = -\gamma^2\beta_{\hat{\gamma}}\,. \tag{4.65}$$

Inserting (4.64) and expanding the result for small γ, leads to

$$\beta_\gamma = \frac{3}{256\pi^2} \sum_{\pm} \lambda_{2\pm}^2 \gamma + \mathcal{O}\big(\gamma^{3/2}, m_\pm^{-2}\big) \,. \tag{4.66}$$

The flow of γ has a fixed point at $\gamma = 0$ with a critical exponent

$$\theta_\gamma = \frac{\partial \beta_\gamma}{\partial \gamma}\bigg|_{\gamma=0} = -\frac{3}{256\pi^2}\big(\lambda_{2+}^2 + \lambda_{2-}^2\big) < 0 \,, \tag{4.67}$$

which is always negative if at least one of the couplings $\lambda_{2\pm}$ is non-zero. Therefrom we conclude that the fixed point $\gamma = 0$ is IR attractive.

Previous works have derived the RG flow for γ in the framework of Einstein-Cartan gravity, starting from the first-order Hilbert-Palatini action with a Holst term [173–177]. These analyses involve different degrees of freedom and evaluation schemes, such that we can only qualitatively compare results. Nevertheless, these works have identified a fixed point at $\gamma = 0$, at which γ is marginally irrelevant, and another one at $\gamma \to \pm\infty$, at which γ is marginally relevant. This suggests an RG trajectory starting from zero parity violation in the UV, and ending with maximal parity violation in the IR. Notably, we find that the values $\gamma = 0$ and $\hat{\gamma} = 0$ are also fixed points in the framework of area-metric gravity. Moreover, at the former, γ is also always marginally irrelevant, whereas whether γ at $\hat{\gamma} = 0$ is marginally relevant or irrelevant, depends on the sign of the sum of couplings $\sum_\pm \lambda_{2\pm}$. We can tune this sign to achieve an analogous type of RG trajectory, starting from zero parity violation in the UV and ending with maximal parity violation in the IR.

This concludes our analysis of the phenomenological viability of area-metric gravity as a local quantum effective field theory. In the next section, we consider phenomenological aspects of classical effective field theories for the metric degrees of freedom in area-metric gravity.

4.2 Non-linear effective length-metric actions in a symmetry-reduced framework

In this section, we will consider non-linear classical effective field theories for the metric degrees of freedom in area-metric gravity. Previously, we have seen that linearised area-metric gravity provides a distinct framework for a particular type of modifications to the linearised Einstein-Hilbert action for general relativity, which are quasi-local and quadratic in the Weyl curvature. On these grounds, we will focus on a non-linear completion of these actions in the form of a quadratic-curvature quasi-local Einstein-Weyl action for the metric tensor. Quadratic-curvature actions with infinite derivatives haven been analysed extensively as classical and quantum field theories which provide mechanisms for the regularisation of singularities and can lead to modifications of the gravitational field in the weak-field regime [122, 178–185].

To address analogous questions for the effective metric theories suggested by area-metric gravity, in subsection 4.2.1 we first define the quasi-local Einstein-Weyl action which forms the starting point of our analysis. In subsection 4.2.2, we derive the covariant equations of motion after localisation by means of an additional tensor field. Subsequently, we consider a symmetry-reduced static spherically symmetric ansatz constructed in subsection 4.2.3. Solutions to the equations of motion in the weak-field regime are analysed in subsection 4.2.4. Additionally, we derive a regular Frobenius solution at the radial center in subsection 4.2.5.

4.2.1 Quasi-local Einstein-Weyl action

We will consider a non-linear completion of a modified version of the linearised effective action for the length metric (3.64). More concretely, as a modification we allow a generic dimensionless coupling μ in front of the quasi-local Weyl-squared correction to the Einstein action, and additionally introduce a dimensionless parameter η in front of the covariant d'Alembert operator, $\Box = g_{\mu\nu}\nabla^\mu\nabla^\nu$. The latter appears, together with the mass parameter m^2 of the non-metric degrees of freedom, as an inverse derivative operator acting on a Weyl curvature. Altogether, we consider the higher-derivative quasi-local Einstein-Weyl action [1]

$$S[g] = \frac{m_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\left[R + \mu C_{\mu\nu\rho\sigma}\frac{1}{\eta\Box - m^2}C^{\mu\nu\rho\sigma}\right]. \tag{4.68}$$

The dimensionless parameter η quantifies the amount of non-locality, whereas the parameter $m^2 > 0$ with dimension mass squared determines the mass scale of the non-metric degrees of freedom in the context of area-metric gravity. For generic choices of the coupling μ and the parameter η, the action (4.68) contains modes which can lead to a classical instability. For instance, in an expansion around flat Minkowski background a ghostfree propagator should only be expected for a special choice of parameters which reproduces the linearised effective action (3.64). In the following, however, we will assume μ and η to be generic.

The overparametrisation in the second term of (4.68) allows us to distinguish different limits of this action. For $\mu \to 0$, the second term is absent and the action reduces to the classical Einstein action for general relativity. For $\eta \to 0$, we obtain local Einstein-Weyl gravity as a subclass of quadratic gravity [123]. For $m^2 \to 0$, the action becomes a genuinely non-local Einstein-Weyl action as a subclass of generalised non-local quadratic-curvature gravity [122]. Keeping both η and m^2 generic, for finite μ, allows us analyse the separate effects stemming from the covariant d'Alembert operator in combination with a mass term, in the form of a quasi-local inverse operator between two Weyl curvatures.

1. Here, $m_{\mathrm{Pl}}^2 = \frac{1}{8\pi G}$ is the squared reduced Planck mass in Planck units, $c = \hbar = 1$. In the following, we set $m_{\mathrm{Pl}}^2 = 1$.

4.2.2 Equations of motion after localisation

To derive the equations of motion from (4.68), we first localise the action by introducing an auxiliary tensor field $\psi_{\mu\nu\rho\sigma}$ with the same symmetries as the Weyl tensor and defined by

$$\psi_{\mu\nu\rho\sigma} = -\big(\eta\Box - m^2\big)^{-1} C_{\mu\nu\rho\sigma}\,. \tag{4.69}$$

This definition can be implemented in the action through a Lagrange multiplier $\lambda_{\mu\nu\rho\sigma}$. Let us consider the action

$$S[g,\psi,\lambda] = \frac{m_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\big[R - \mu C^{\mu\nu\rho\sigma}\psi_{\mu\nu\rho\sigma} + \lambda^{\mu\nu\rho\sigma}\big(\big(\eta\Box - m^2\big)\psi_{\mu\nu\rho\sigma} + C_{\mu\nu\rho\sigma}\big)\big]\,, \tag{4.70}$$

whose variation with respect to λ returns the local version of the defining relation (4.69),

$$\big(\eta\Box - m^2\big)\psi_{\mu\nu\rho\sigma} = -C_{\mu\nu\rho\sigma}\,. \tag{4.71}$$

We assume boundary conditions on ψ such that solutions to the homogeneous equation vanish. This amounts to requiring that the operator acting on the left hand side in (4.71) has a trivial kernel, which in turn defines the inverse operator in (4.69).

The Lagrange multiplier can be integrated out from (4.70), in order to arrive at a reduced action which depends only on the metric and the field ψ. The resulting local action is the starting point of our analysis,

$$S[g,\psi] = \frac{m_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\big[R - \mu\big(2C^{\mu\nu\rho\sigma}\psi_{\mu\nu\rho\sigma} + \psi^{\mu\nu\rho\sigma}\big(\eta\Box - m^2\big)\psi_{\mu\nu\rho\sigma}\big)\big]\,. \tag{4.72}$$

Varying this action with respect to the field ψ, we obtain its equations of motion

$$\mathcal{E}_{\mu\nu\rho\sigma}(g,\psi) \equiv \mu\big[\big(\eta\Box - m^2\big)\psi_{\mu\nu\rho\sigma} + C_{\mu\nu\rho\sigma}\big] = 0\,. \tag{4.73}$$

Therefore, onshell for ψ the action (4.72) reduces to (4.68). Varying (4.72) with respect to the metric, and substituting the Ricci-Weyl decomposition for the Riemann tensor, we arrive at the equations of motion for the metric,

$$\mathcal{E}_{\mu\nu}(g,\psi) \equiv G_{\mu\nu} - \mathcal{T}_{\mu\nu}(g,\psi) = 0\,. \tag{4.74}$$

Here, $G_{\mu\nu} = R_{\mu\nu} - 1/2 g_{\mu\nu}R$ is the Einstein tensor, whereas the effective energy-momentum tensor is given by

$$\mathcal{T}_{\mu\nu} = \mu\big(\mathcal{T}^{(0)}_{\mu\nu} + \eta\mathcal{T}^{(\eta)}_{\mu\nu}\big)\,, \tag{4.75}$$

where

$$\begin{aligned}\mathcal{T}^{(0)}_{\mu\nu} &= 2\nabla_{(\alpha}\nabla_{\beta)}\psi_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta} + \frac{1}{2}m^2 g_{\mu\nu}\psi^{\alpha\beta\gamma\delta}\psi_{\alpha\beta\gamma\delta} - 4m^2\psi_{\mu}{}^{\alpha\beta\gamma}\psi_{\nu\alpha\beta\gamma} \\ &\quad - g_{\mu\nu}C^{\alpha\beta\gamma\delta}\psi_{\alpha\beta\gamma\delta} + 6C_{(\mu|}{}^{\alpha\beta\gamma}\psi_{|\nu)\alpha\beta\gamma} + 2R^{\alpha\beta}\psi_{\mu\alpha\nu\beta}\,, \end{aligned} \tag{4.76}$$

and

$$\begin{aligned}\mathcal{T}^{(\eta)}_{\mu\nu} &= 4\psi^{\alpha\beta\gamma\delta}\nabla_{\beta}\nabla_{(\mu}\psi_{\nu)\alpha\gamma\delta} - 4\psi_{(\mu|\alpha\beta\gamma}\nabla_{\delta}\nabla_{|\nu)}\psi^{\alpha\delta\beta\gamma} \\ &\quad + 4\nabla_{(\mu|}\psi_{|\nu)\alpha\beta\gamma}\nabla_{\delta}\psi^{\alpha\delta\beta\gamma} - 4\nabla_{\delta}\psi_{(\mu|\alpha\beta\gamma}\nabla_{|\nu)}\psi^{\alpha\delta\beta\gamma} \\ &\quad + 4\psi_{(\mu|\alpha\beta\gamma}\Box\psi_{|\nu)}{}^{\alpha\beta\gamma} - \nabla_{\mu}\psi_{\alpha\beta\gamma\delta}\nabla_{\nu}\psi^{\alpha\beta\gamma\delta} + \frac{1}{2}g_{\mu\nu}\nabla_{\rho}\psi_{\alpha\beta\gamma\delta}\nabla^{\rho}\psi^{\alpha\beta\gamma\delta}\,. \end{aligned} \tag{4.77}$$

Equations (4.73) and (4.74) provide the full set of covariant equations of motion for the action (4.72).

4.2.3 Static spherically symmetric ansatz

In this subsection, we consider the equations of motion (4.73) and (4.74) in static spherical symmetry. The most general static spherically symmetric line element in spherical coordinates (t, r, θ, ϕ) can be parametrised by two free functions $f(r)$ and $h(r)$, in the form

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + h(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\ , \tag{4.78}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\phi^2$ is the area element on the unit two-sphere.

To determine the form of the tensor field $\psi_{\mu\nu\rho\sigma}$ in static spherical symmetry, we use that this field has the same symmetries as the Weyl tensor. In particular, this field satisfies

$$\psi_{\mu\nu\rho\sigma} = -\psi_{\nu\mu\rho\sigma} = \psi_{\rho\sigma\mu\nu} \quad \text{and} \quad \psi_{\mu\alpha\beta\gamma} + \psi_{\mu\beta\gamma\alpha} + \psi_{\mu\gamma\alpha\beta} = 0\,, \tag{4.79}$$

and is traceless upon contracting any two of its four indices, as is the Weyl tensor. The latter can be written in Petrov notation as a 6×6 symmetric matrix C_{AB}, where $A, B, \ldots = [\mu\nu] \in \{01, 02, 03, 12, 13, 23\}$ label antisymmetric index pairs. When evaluated for the line element (4.78), this matrix can be expressed as

$$C_{AB} = \mathcal{F} \cdot \mathcal{M}_{AB}\,, \tag{4.80}$$

with

$$\mathcal{F}(f, f', f'', h, h'; r) = \frac{1}{12}\left[2f'' - \frac{f'^2}{f} - \frac{f'h'}{h} - \frac{2f'}{r} + \frac{2gh'}{r} + \frac{4f}{r^2} - \frac{4fh}{r^2}\right], \tag{4.81}$$

where a prime denotes the derivative with respect to r. The matrix $\mathcal{M}_{AB}$ in (4.80) has components

$$\mathcal{M}_{AB} = \mathrm{diag}\left\{1,\, -r^2\frac{1}{2h},\, -r^2\sin^2(\theta)\frac{1}{2h},\, r^2\frac{1}{2f},\, r^2\sin^2(\theta)\frac{1}{2f},\, -r^4\sin^2(\theta)\frac{1}{fh}\right\}. \tag{4.82}$$

Thus, all non-zero components of the Weyl tensor are encoded in the diagonal entries of $\mathcal{M}_{AB}$. There is only one independent component of the Weyl tensor, which can be taken to be $C_{trtr} = \mathcal{F}$. Therefrom, the others can be inferred by making use of the tracelessness condition $g^{\mu\rho}C_{\mu\nu\rho\sigma} = 0$. According to the defining equation for $\psi_{\mu\nu\rho\sigma}$ in (4.73), and the structure of the Weyl tensor (4.80), we can parametrise this field analogously in the form

$$\psi_{AB} \quad = \quad \psi \cdot \mathcal{M}_{AB}\,, \tag{4.83}$$

where $\psi(r)$ is a scalar function which depends only on the radial coordinate. With this ansatz, $\psi_{\mu\nu\rho\sigma}$ is automatically traceless in each index pair and satisfies the cyclicity condition given by the second equation in (4.79).

Inserting the ansatz for the metric in (4.78) and the ansatz for the field $\psi_{\mu\nu\rho\sigma}$ in (4.83) into the equations of motion (4.74) and (4.73), we see that only the diagonal components of $\mathcal{E}_{\mu\nu}$ and $\mathcal{E}_{\mu\nu\rho\sigma}$, viewed as a 6×6 matrix $\mathcal{E}_{AB}$, are non-zero. Concretely, these matrices take the form

$$\mathcal{E}_{\mu\nu} \quad = \quad \mathrm{diag}\left\{\mathcal{E}_{tt},\, \mathcal{E}_{rr},\, \mathcal{E}_{\theta\theta},\, \sin^2(\theta)\mathcal{E}_{\theta\theta}\right\}, \tag{4.84}$$

and

$$\mathcal{E}_{AB} = \mathcal{E}_{trtr} \cdot \mathcal{M}_{AB}\,, \tag{4.85}$$

with $\mathcal{M}_{AB}$ given in (4.82). There is only one independent equation of motion encoded in $\mathcal{E}_{\mu\nu\rho\sigma}$, which we will take to be the component $\mathcal{E}_{trtr} \equiv \mathcal{E}_\psi$.

The non-zero components of the field-equation matrix (4.84) depend on the functions $\{f, h, \psi\}$ and their first and second derivatives, as well as on the coordinate r explicitly. Onshell on the equation of motion for ψ, the tensor $\mathcal{E}_{\mu\nu}$ is covariantly conserved,

$$\nabla^\mu \mathcal{E}_{\mu\nu} = 0\,. \tag{4.86}$$

For a general tensor $\mathcal{E}_{\mu\nu}$ of the form as in (4.84), the only a priori non-vanishing component on the left hand side in (4.86) is the r component,

$$\nabla^\mu \mathcal{E}_{\mu r} = \frac{f'}{2f^2}\mathcal{E}_{tt} + \frac{f'}{2fh}\mathcal{E}_{rr} + \frac{2}{hr}\mathcal{E}_{rr} + \left(\frac{\mathcal{E}_{rr}}{h}\right)' - \frac{2}{r^3}\mathcal{E}_{\theta\theta}\,. \tag{4.87}$$

The term $\mathcal{E}'_{rr}$ introduces derivatives of the functions $\{f, h, \psi\}$ up to third order. The third derivative ψ''' can be eliminated by taking the radial derivative of the equation of motion $\mathcal{E}_\psi = 0$ and substituting the result back into (4.87). The resulting expression involves derivatives of $\{f, h, \psi\}$ only up to second order. Using therein the equation of motion $\mathcal{E}_\psi = 0$ itself to substitute ψ'', leads to $\nabla^\mu \mathcal{E}_{\mu r} = 0$. In summary, we have verified explicitly that $\mathcal{E}_{\mu\nu}$ is covariantly conserved after using both the equation of motion for the field $\psi_{\mu\nu\rho\sigma}$ and its derivative.

The conservation equation (4.86) implies that there are only two algebraically independent equations of motion encoded in $\mathcal{E}_{\mu\nu} = 0$. We will take these to be $\mathcal{E}_{tt} = 0$ and $\mathcal{E}_{rr} = 0$. Altogether, we obtain a set of three algebraically independent coupled second-order non-linear differential equations for the functions $\{f, h, \psi\}$, which are denoted by

$$\mathcal{E} = \{\mathcal{E}_f, \mathcal{E}_h, \mathcal{E}_\psi\}\,. \tag{4.88}$$

These equations of motion are given explicitly in appendix A.5. They can be split into one part denoted by GR, which remains present after taking the limit $\mu \to 0$, in which case the original action reduces to the Einstein action of general relativity. The other part of the equations of motion is proportional to the parameter μ and decomposes further into a contribution proportional to the non-locality parameter η and denoted by a supersript (η), as well as a contribution which remains in the limit $\eta \to 0$ and is denoted by a superscript (0). With this notation, the equations of motion (4.88) can be expressed as

$$\mathcal{E}_f \equiv \mathcal{E}_f^{\mathrm{GR}} + \mu\Big(\mathcal{E}_f^{(0)} + \eta\mathcal{E}_f^{(\eta)}\Big) = 0\,, \tag{4.89}$$

$$\mathcal{E}_h \equiv \mathcal{E}_h^{\mathrm{GR}} + \mu\Big(\mathcal{E}_h^{(0)} + \eta\mathcal{E}_h^{(\eta)}\Big) = 0\,, \tag{4.90}$$

$$\mathcal{E}_\psi \equiv \mu\Big(\mathcal{E}_\psi^{(0)} + \eta\mathcal{E}_\psi^{(\eta)}\Big) = 0\,. \tag{4.91}$$

The equation $\mathcal{E}_\psi$ has no GR contibution, as ψ does not appear in the action for general relativity.

The three equations $\{\mathcal{E}_f, \mathcal{E}_h, \mathcal{E}_\psi\} \equiv 0$ are equivalent to the equations of motion obtained by first inserting the static spherically symmetric ansatz (4.78) and (4.83) into the action (4.72), and subsequently taking the variation with respect to f, h, and ψ. According to the principle of symmetric criticality [186, 187], this reflects consistency of the truncation to the invariant sector under a group action, which in this case is the group of spatial rotations.

4.2.4 Weak-field regime

In this subsection, we will solve the equations of motion (4.88) to linear order in an expansion around flat space. Such an approximation is appropriate to describe the regime of a weak gravitational field at asymptotically large distances. Expanding the metric functions f and h in (4.78) and the field ψ in (4.83) around their flat-space configuration, we write

$$f(r) = 1 + \delta a(r)\,, \tag{4.92}$$

$$h(r) = 1 + \delta b(r)\,, \tag{4.93}$$

$$\psi(r) = \delta c(r)\,. \tag{4.94}$$

The field ψ must be zero to leading order, when the spacetime is described by the Minkowski metric with vanishing Weyl tensor. This is consistent with the assumption that solutions to the homogenous equation in (4.71) vanish.

Inserting the ansatz (4.92)–(4.94) into the field equations (4.88), and expanding the result to first order in the perturbation parameter δ, leads to

$$0 = \big(b + rb' + \mu\big(12c + 20rc' + 4r^2c''\big)\big)\delta + \mathcal{O}\big(\delta^2\big)\,, \tag{4.95}$$

$$0 = (b - ra' - \mu(12c + 4rc'))\delta + \mathcal{O}\big(\delta^2\big)\,, \tag{4.96}$$

$$0 = \mu\big(4b + 12m^2r^2c + 2ra' - 2rb' - 2r^2a'' + \eta\big(72c - 24rc' - 12r^2c''\big)\big)\delta + \mathcal{O}\big(\delta^2\big)\,. \tag{4.97}$$

We can eliminate b from the second equation and insert the result into the first and third equations. Thereby we obtain two second-order linear coupled differential equations for the functions a and c. These can be solved exactly in terms of four integration constants C, C_0, and $C_\pm$. The general solution to the linearised equations for the two metric functions f and h in the weak-field limit is given by

$$f(r) = 1 + C + \frac{C_0}{r} + C_+\frac{e^{\hat{m}r}}{r} + C_-\frac{e^{-\hat{m}r}}{r}\,, \tag{4.98}$$

$$h(r) = 1 - \frac{C_0}{r} - C_+\frac{e^{\hat{m}r}}{2r}(1 - \hat{m}r) - C_-\frac{e^{-\hat{m}r}}{2r}(1 + \hat{m}r)\,, \tag{4.99}$$

whereas the solution for the field ψ is

$$\psi(r) = \frac{1}{m^2}\frac{C_0}{r^3} + C_+\frac{e^{\hat{m}r}}{8\mu\hat{m}^2r^3}\big(3 - 3\hat{m}r + \hat{m}^2r^2\big) + C_-\frac{e^{-\hat{m}r}}{8\mu\hat{m}^2r^3}\big(3 + 3\hat{m}r + \hat{m}^2r^2\big)\,. \tag{4.100}$$

In the previous expressions we have introduced the effective mass parameter

$$\hat{m} \equiv \sqrt{\frac{m^2}{2\mu + \eta}}\,. \tag{4.101}$$

If $2\mu < -\eta$, this effective mass becomes imaginary and produces an asymptotically non-flat and spatially oscillating profile of the metric functions f and h at large r, unless the coefficients $C_\pm$ are set to zero. In the following, we will assume $2\mu > -\eta$ such that the radicand in (4.101) is positive, and $\hat{m}$ is real and positive.

An asymptotic falloff of the solutions (4.98)–(4.100) at large distances requires setting $C_+ = 0$. In addition, we may fix the time parametrisation at infinity by setting $C = 1$, such that $f(r) \to 1$ for $r \to \infty$. Thereby the number of free parameters is reduced to two. These are $C_0 = -2M$, where M represents the Arnowitt-Deser-Misner (ADM) mass [188, 189], and C_-, which plays the role of a charge that mediates a Yukawa interaction. This interaction can be attractive or repulsive, depending on the sign of C_-. The general form of asymptotically flat solutions to the weak-field equations is therefore

$$
\begin{aligned}
f(r) &= 1 - \frac{2M}{r} + C_- \frac{e^{-\hat{m}r}}{r}\,, && (4.102)\\
h(r) &= 1 + \frac{2M}{r} - C_- \frac{e^{-\hat{m}r}}{2r}(1+\hat{m}r)\,, && (4.103)
\end{aligned}
$$

as well as

$$
\psi(r) = -\frac{1}{m^2}\frac{2M}{r^3} + C_- \frac{e^{-\hat{m}r}}{8\mu\hat{m}^2 r^3}\left(3 + 3\hat{m}r + \hat{m}^2 r^2\right). \tag{4.104}
$$

Figure 4.4 shows the metric function f in (4.98), for fixed M and C_- and different values of the effective mass parameter $\hat{m} > 0$. In the limit $\hat{m} \to \infty$, this function asymptotes to the lapse function of the Schwarzschild spacetime,

$$
\lim_{\hat{m}\to\infty} f(r) = 1 - \frac{2M}{r}\,. \tag{4.105}
$$

The latter is an exact solution to vacuum general relativity and local Einstein-Weyl gravity, but is not an exact solution to the field equations (4.88) for the quasi-local Einstein-Weyl system.

In view of (4.101), the limit $\hat{m} \to \infty$ can be achieved by taking the limit $m^2 \to \infty$. In the context of area-metric gravity, this corresponds to a limit in which the non-metric degrees of freedom of the area metric become infinitely heavy, and thereby decouple. In this limit, the quadratic term in the Weyl tensor in the effective action for the length metric (4.68) is effectively suppressed.

Notably, the limit $\hat{m} \to \infty$ can formally also be achieved by allowing $\mu \to -\frac{1}{2}\eta$. This special choice of couplings in the action (4.68) leads to a ghostfree propagator for the spin-2 mode in an expansion around flat Minkowski background, which does not exhibit any additional poles beyond the massless graviton pole. Thus, one may view the absence of corrections to the weak-field regime of general relativity as a manifestation of the ghostfree nature of the action in this regime. Nevertheless, as the limit $\mu \to -\frac{1}{2}\eta$ is not well-defined in (4.101), the equations of motion for the non-linear theory described by this particular choice of parameters should be analysed separately.

We close this subsection by noticing that the expressions for f and h in (4.102) and (4.103) resemble expressions of asymptotically flat solutions in the weak-field regime of local Einstein-Weyl gravity, associated with the limit $\eta \to 0$ in the original action (4.68), see e.g. [123, 190–

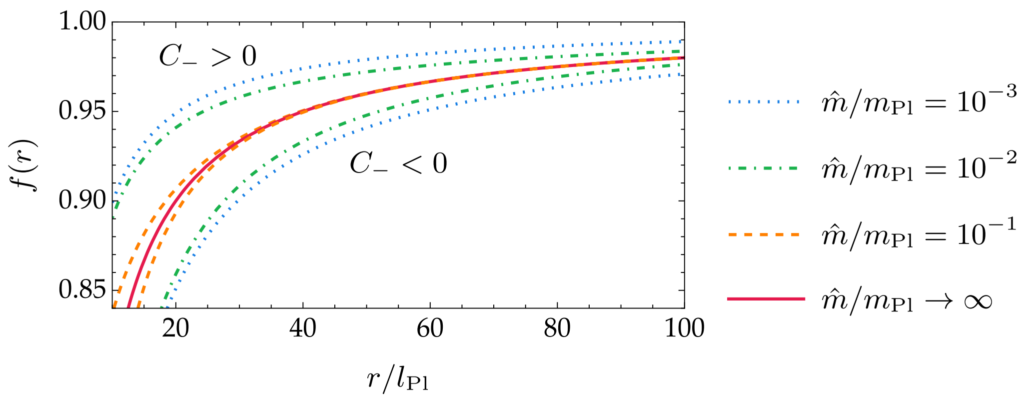

Figure 4.4: Metric function f in (4.102) for asymptotically flat solutions to the equations of motion in the weak-field regime, for different values of the effective mass parameter $\hat{m}$ defined in (4.101). The integration constants are set to $M/m_{\mathrm{Pl}} = 1$ and $C_-/m_{\mathrm{Pl}} = \pm 1$. In the limit $\hat{m}/m_{\mathrm{Pl}} \to \infty$, the function f asymptotes to the red line which coincides with the lapse function of the Schwarzschild spacetime.

194]. [1] In this limit,

$$\hat{m}^2 \;\to\; m_{\mathrm{EW}}^2 \equiv \frac{m^2}{2\mu} \tag{4.106}$$

represents the squared mass of the massive spin-2 particle appearing in the spectrum [123]. Therefore, the weak-field solutions to quasi-local Einstein-Weyl gravity can be obtained from the weak-field solutions to local Einstein-Weyl gravity, through the replacement $m_{\mathrm{EW}} \to \hat{m}$.

4.2.5 Regular Frobenius solution at $r = 0$

In this subsection, we will analyse the asymptotic behavior of solutions to the equations of motion (4.88) which can be expanded in a Frobenius series around $r = 0$. To that end, we expand the metric functions f and h, and the field ψ in the form

$$f(r) \;=\; a_\alpha\left[r^\alpha + a_{\alpha+1}r^{\alpha+1} + a_{\alpha+2}r^{\alpha+2} + \cdots\right], \tag{4.107}$$
$$h(r) \;=\; b_\beta r^\beta + b_{\beta+1}r^{\beta+1} + b_{\beta+2}r^{\beta+2} + \cdots, \tag{4.108}$$
$$\psi(r) \;=\; c_\gamma r^\gamma + c_{\gamma+1}r^{\gamma+1} + c_{\gamma+2}r^{\gamma+2} + \cdots, \tag{4.109}$$

where a_α, b_β and c_γ are the first non-zero coefficients in the expansion. By a global rescaling of the time coordinate we can fix $a_\alpha = 1$. The exponents α, β, and γ of the leading terms in the

1. In the notation of literature on spherically symmetric solutions in quadratic gravity, such as [190–198], the couplings γ and $-\alpha$, in front of the Ricci scalar and squared Weyl tensor, correspond in our notation to $(\gamma, -\alpha) \leftrightarrow \left(\frac{1}{2}, -\frac{\mu}{2m^2}\right)$. The squared mass of the massive spin-2 mode in Einstein-Weyl gravity is given by $m_{\mathrm{EW}}^2 = \frac{\gamma}{2\alpha} = \frac{m^2}{2\mu}$.

expansions are a priori unknown. They have to be determined by inserting the ansatz (4.107)–(4.109) into the field equations $\{\mathcal{E}_f, \mathcal{E}_h, \mathcal{E}_\psi\} \equiv 0$ in (4.88) and equating the coefficients in front of the lowest-order terms to zero. The corresponding indicial equations are given by

$$\begin{aligned}
0 &= -(1+\beta)b_\beta^2 r^{2\alpha+2\beta} + b_\beta^3 r^{2\alpha+3\beta} - \mu\big[f_1(\alpha,\beta,\gamma)b_\beta c_\gamma r^{\alpha+\beta+\gamma} + 6m^2 b_\beta c_\gamma^2 r^{\beta+2\gamma+2} \\
&+ \eta f_2(\alpha,\beta,\gamma)c_\gamma^2 r^{2\gamma}\big]\,, \qquad (4.110)\\
0 &= -(1+\alpha)b_\beta^2 r^{2\alpha+2\beta} + b_\beta^3 r^{2\alpha+3\beta} - \mu\big[h_1(\alpha,\beta,\gamma)b_\beta c_\gamma r^{\alpha+\beta+\gamma} + 6m^2 b_\beta c_\gamma^2 r^{\beta+2\gamma+2} \\
&+ \eta h_2(\alpha,\beta,\gamma)c_\gamma^2 r^{2\gamma}\big]\,, \qquad (4.111)\\
0 &= \mu\big[4b_\beta^2 r^{\alpha+2\beta} + \psi_1(\alpha,\beta,\gamma)b_\beta r^{\alpha+\beta} + 12m^2 b_\beta c_\gamma r^{\beta+\gamma+2} + \eta\psi_2(\alpha,\beta,\gamma)c_\gamma r^\gamma\big]\,, \qquad (4.112)
\end{aligned}$$

where $\{f_1, f_2, h_1, h_2, \psi_1, \psi_2\}$ are polynomials in α, β, and γ defined by

$$\begin{aligned}
f_1(\alpha,\beta,\gamma) &= 3\alpha^2 + 6\beta^2 + 4\gamma^2 + 11\alpha\beta - 8\alpha\gamma - 10\beta\gamma - 12\alpha - 22\beta + 16\gamma + 12\,, &(4.113)\\
f_2(\alpha,\beta,\gamma) &= -12\alpha^2 - 24\beta^2 - 18\gamma^2 - 36\alpha\beta + 30\alpha\gamma + 42\beta\gamma + 12\alpha + 12\beta - 12\gamma + 36\,, &(4.114)\\
h_1(\alpha,\beta,\gamma) &= -3\alpha^2 - \alpha\beta + 2\alpha\gamma + 12\alpha + 2\beta - 4\gamma - 12\,, &(4.115)\\
h_2(\alpha,\beta,\gamma) &= -12\beta^2 - 6\gamma^2 - 12\alpha\beta + 6\alpha\gamma + 18\beta\gamma + 12\alpha + 12\beta - 12\gamma + 108\,, &(4.116)\\
\psi_1(\alpha,\beta,\gamma) &= -\alpha^2 + \alpha\beta + 4\alpha - 2\beta - 4\,, &(4.117)\\
\psi_2(\alpha,\beta,\gamma) &= -6\alpha^2 - 18\beta^2 - 12\gamma^2 - 24\alpha\beta + 18\alpha\gamma + 30\beta\gamma + 12\alpha + 12\beta - 12\gamma + 72\,. &(4.118)
\end{aligned}$$

The expressions in (4.110)–(4.112) are grouped into two parts, similarly as the full equations of motion (4.89)–(4.91). The first terms are the terms which remain in the limit $\mu \to 0$, when the equations of motion reduce to the equations for vacuum general relativity. The other terms proportional to μ decompose into one part that remains in the limit $\eta \to 0$, and another part which is proportional to η.

In the following, we focus exclusively on the existence of a possible regular solution family. The Kretschmann scalar for a metric with line element (4.78), where $f(r) = r^\alpha$ and $h(r) = b_0 r^\beta$, is given by

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 4r^{-4} - \frac{1}{b_0}8r^{-4-\beta} + \frac{1}{4b_0^2}p(\alpha,\beta)r^{-4-2\beta}\,, \qquad (4.119)$$

where

$$p(\alpha,\beta) = \alpha^4 - 2\alpha^3\beta + \alpha^2\beta^2 - 4\alpha^3 + 4\alpha^2\beta + 12\alpha^2 + 8\beta^2 + 16\,. \qquad (4.120)$$

Thus $(\alpha,\beta) = (0,0)$ and additionally $b_0 = 1$ in the series ansatz (4.107)–(4.108) are necessary conditions for regularity at $r = 0$. [1]

None of the coefficients in (4.110)–(4.112) is annihilated by specifying the values $(\alpha,\beta) =$

1. These conditions are however not sufficient for regularity at $r = 0$. Regularity requires that the metric functions expand around $r = 0$ as $f(r) = 1 + \mathcal{O}(r^2)$ and $h(r) = 1 + \mathcal{O}(r^2)$.

$(0,0)$. With these values for α and β, the equations simplify to

$$
\begin{aligned}
0 &= -b_0^2 + b_0^3 - \mu\big[f_1(\gamma)b_0c_\gamma r^\gamma + 6m^2 b_0 c_\gamma^2 r^{2\gamma+2} + \eta f_2(\gamma)c_\gamma^2 r^{2\gamma}\big]\,, && (4.121)\\
0 &= -b_0^2 + b_0^3 - \mu\big[h_1(\gamma)b_0c_\gamma r^\gamma + 6m^2 b_0 c_\gamma^2 r^{2\gamma+2} + \eta h_2(\gamma)c_\gamma^2 r^{2\gamma}\big]\,, && (4.122)\\
0 &= \mu\big[4b_0 + \psi_1(\gamma)b_0 + 12m^2 b_0 c_\gamma r^{\gamma+2} + \eta\psi_2(\gamma)c_\gamma r^\gamma\big]\,, && (4.123)
\end{aligned}
$$

where we have denoted

$$
\begin{aligned}
f_1(\gamma) &= f_1(0,0,\gamma) = 4\gamma^2 + 16\gamma + 12\,, && (4.124)\\
f_2(\gamma) &= f_2(0,0,\gamma) = -18\gamma^2 - 12\gamma + 36\,, && (4.125)\\
h_1(\gamma) &= h_1(0,0,\gamma) = -4\gamma - 12\,, && (4.126)\\
h_2(\gamma) &= h_2(0,0,\gamma) = -6\gamma^2 - 12\gamma + 108\,, && (4.127)\\
\psi_1(\gamma) &= \psi_1(0,0,\gamma) = -4\,, && (4.128)\\
\psi_2(\gamma) &= \psi_2(0,0,\gamma) = -12\gamma^2 - 12\gamma + 72\,. && (4.129)
\end{aligned}
$$

For $\gamma < 0$, the leading terms in (4.121)–(4.123) would be the ones proportional to $r^{2\gamma}$. However, setting both coefficients $f_2(\gamma)$ and $h_2(\gamma)$ in front of these terms to zero does not yield a solution for γ. Here, we use that b_0 and c_γ are assumed to be non-zero.

For $\gamma = 0$, the leading terms would be the ones proportional to $r^{2\gamma} = r^\gamma = r^0$. These are all constant terms. However, requiring these terms to vanish does not lead to a solution under the assumption that b_0 and c_γ are non-zero.

For $\gamma > 0$, the leading terms are the two constant terms in the beginning of each equation. These depend only on the parameter b_0. Setting these terms to zero fixes the coefficient $b_0 = 1$ in the expansion of h in (4.108).

The previous observations indicate that a possible regular solution family must be of the form $(\alpha, \beta, \gamma) = (0, 0, \gamma > 0)$ with $b_0 = 1$. At this stage, the exponent γ has not been fixed yet. This exponent can be fixed once the r^γ terms become the leading terms. For this to be the case, r^γ must be of the same order as the μ-independent terms in the equations of motion $\{\mathcal{E}_f, \mathcal{E}_h\} \equiv 0$ at higher orders in the expansion.

Continuing with the equations (4.88) to the next-to-leading order in the expansion produces μ-independent terms of order r^1 and higher. However, setting $\gamma = 1$ we cannot cancel the coefficient in front of these terms under the assumption that c_γ is non-zero. Therefore, we must assume $\gamma > 1$ and in this case requiring the coefficient in front of the r^1 terms to vanish fixes the parameters $a_1 = b_1 = 0$. This in turn ensures that curvature invariants of the metric are indeed regular at $r = 0$. [1]

Evaluating the equations (4.88) at the next order produces μ-independent terms of order r^2

1. Generically, whenever the μ-independent terms in the equations $\{\mathcal{E}_f, \mathcal{E}_h\} \equiv 0$ are leading to a given order $n > 0$ in the expansion, and their order does not coincide with the order of r^γ, then the corresponding metric coefficients a_n and b_n must be set to zero. When this is the case, order by order, the metric reduces to the flat Minkowski metric. This reflects the fact that the only regular static spherically symmetric solution to the equations of motion for vacuum general relativity, obtained in the limit $\mu \to 0$, is the flat Minkowski metric.

and higher. In this case, setting $\gamma = 2$ allows us to cancel the coefficient in front of the leading terms $r^2 = r^\gamma$ by fixing $a_2 = b_2$ and $c_2 = \frac{b_2}{20\mu}$ in terms of one free parameter b_2. Thereby we conclude that $(\alpha, \beta, \gamma) = (0, 0, 2)$ defines a regular solution family around $r = 0$.

Investigating this regular Frobenius solution further by solving the equations at higher orders, we find that the metric functions f and h, as well as the field ψ are characterised by only one free parameter b_2 in addition to the global time-rescaling parameter $a_0 = 1$. Explicitly, they are given by

$$f(r) = 1 + b_2 r^2 + \frac{(240\mu^2 + 76\mu\eta - 9\eta^2)b_2 + 10\mu m^2}{200\mu(2\mu + \eta)} b_2 r^4 + \mathcal{O}\big(r^6\big)\,, \tag{4.130}$$

$$h(r) = 1 + b_2 r^2 + \frac{(120\mu^2 + 28\mu\eta - 3\eta^2)b_2 + 10\mu m^2}{100\mu(2\mu + \eta)} b_2 r^4 + \mathcal{O}\big(r^6\big)\,, \tag{4.131}$$

$$\psi(r) = \frac{1}{20\mu} b_2 r^2 - \frac{(320\mu + 147\eta)b_2 + 5m^2}{1400\mu(2\mu + \eta)} b_2 r^4 + \mathcal{O}\big(r^6\big)\,. \tag{4.132}$$

As observed previously for solutions to the equations in the weak-field regime, we see that the limit $\mu \to -\frac{1}{2}\eta$ is not well-defined. This again indicates that the theory described by the original action (4.68) with this particular choice of parameters should be analysed separately. It should also be noted that, locally near $r = 0$, the function f can be inverted in the form $f^{-1}(r) = 1 - b_2 r^2 \neq h(r)$ for non-vanishing parameter b_2. Thus, regular solutions do not exhibit a de Sitter core such as standard regular black holes [199–201], for which $f^{-1} = h$ holds and $b_2 \propto -Ml^{-3}$, where M is the ADM mass and l a regularisation length parameter playing the role of an effective cosmological constant $\Lambda \equiv Ml^{-3}$.

We close this subsection with the remark that the expressions for f and h in (4.130) and (4.131) reproduce the regular solution family $(\alpha, \beta) = (0, 0)$ of local Einstein-Weyl gravity [123, 190, 191, 202], which is obtained in the limit $\eta \to 0$ of the original action (4.68). In this case, the field ψ is proportional to the Weyl tensor, as can be seen from the equation of motion (4.73).

Generically, in local Einstein-Weyl gravity there are two other Frobenius solution families (α, β, γ) around $r = 0$ in Schwarzschild spherical coordinates [123, 190, 191, 202]. These are the $(-1, 1, -3)$ family, which among others locally contains the Schwarzschild solution, and the $(2, 2, -2)$ family, which has been used to construct so-called 2–2 holes, see e.g. [203, 204]. We do not find such singular solution families in the quasi-local Einstein-Weyl theory for generic values of the parameters η and m^2. Future analyses using a Kundt-conformal metric ansatz, as e.g. in [196–198, 205], will complete this picture and allow for a classification of Frobenius solutions around an arbitrary expansion point.

5 Outlook

Area-metric gravity is a candidate effective field theory for the continuum limit of loop quantum gravity and spin foams. As such, this theory is currently at the initial stage of development. Several challenges and open questions have to be addressed in order to advance the theory and phenomenology of area-metric gravity.

A major mathematical challenge is the absence of an established framework for differential geometry on area-metric backgrounds. A limited number of works have constructed area connections [50, 53, 55]. However, these constructions rely on a Gilkey decomposition of the area metric into a sum of area metrics induced by length metrics. Such a decomposition is non-unique. Consequently, there are many choices of area connections which satisfy the conditions of area-metricity and torsion-freeness. This is in stark contrast to the unique metric and torsion-free Levi-Civita connection in Riemannian geometry.

A natural framework for area-metric gravity are modified non-chiral Plebanski theories. However, the area metric therein is defined in terms of the length metric and the two unimodular internal metrics which parametrise the degrees of freedom of the bivector field after the reduction by a subset of the simplicity constraints. Therefore, non-linear area-metric actions derived in this framework depend on the area metric only implicitly. Inverting the defining relation for the area metric explicitly would require solving twenty polynomial equations. The desired result can be viewed as an analogue of the Urbantke formula, which reconstructs the conformal class of the metric from a basis of selfdual or anti-selfdual 2-forms [125].

In view of these complications, we have taken a bottom-up approach and constructed area-metric gravity perturbatively guided by the principle of general covariance. The procedure of deriving relations between couplings imposed by diffeomorphism invariance can be continued to higher orders in area-metric perturbations. It remains to see if the number of free parameters at higher orders reproduces tentative countings in constructive gravity [117–121].

Taking into account higher-order interaction terms in the action for area-metric perturbations, we can reconsider the question of classical stability of the theory. The indefiniteness of the area-metric Lagrangian at quadratic order is sourced by the selfduality and anti-selfduality relations of the right-handed and left-handed non-metric degrees of freedom of the area-metric perturbation in Lorentzian signature. Nevertheless, the two-parameter subclass of quadratic area-metric theories which results in a ghostfree effective action for the metric is classically stable, with a physical spectum consisting of two massless positive-energy modes and five massive negative-energy modes. Whereas these modes are decoupled at quadratic order, it is clear that higher-order interaction potentials will generically couple these two types of modes. Therefore, in the absence of fine-tuning of couplings, one may expect that the theory suffers from ghost instabilities. However, it should be emphasised that classical dynamical systems containing ghosts do not need to be instable [206, 207]. In particular, a second-order classical field theory in which a positive-energy mode interacts with a ghost mode through a non-derivative potential can exhibit a long-lived stable dynamics, provided the mass of the ghost field is sufficiently heavy [208]. Such an assumption is justified for the masses of the non-metric degrees of freedom in area-metric gravity. The implications of these recent results for the validity of standard arguments for troubles with ghosts in quantum field theory [209] are at present not understood. On these grounds, it should be cautioned against prematurely discarding the theory of area metrics based on the presence of negative-energy modes. Instead, this circumstance may be viewed as a strong motivation to pursue the direction of classical and quantum stability of effective field theories with ghosts.

Implementing diffeomorphism invariance at higher orders and extending the truncation ansatz for the area-metric effective average action will allow for a systematic computation of RG flows in area-metric gravity. In this context, it is essential to understand the impact of large interaction couplings between the metric and non-metric degrees of freedom below the mass-decoupling scale, both within and beyond the parity-symmetric subspace of area-metric gravity. Such large interaction couplings may result in dominant contributions from offshell non-metric configurations in the effective action for the metric, and thereby pose a challenge for the phenomenological viability of a theory of area metrics.

An immediate extension of the RG analysis is to evaluate the flow of the Newton coupling and the cosmological constant, in order to investigate if an asymptotically safe fixed point for quantum Einstein gravity [165, 210, 211] persists under the impact of the non-metric degrees of freedom of the area metric. The answer will complemement the current understanding of asymptotic safety in length-metric gravity-matter systems [212–217], and in frameworks which employ variables different from the length metric [173, 174, 177, 218–220]. Related to this question, one may reconsider expectations formulated on properties of the RG flow of modified Plebanski theories [94], as a step towards establishing a connection between features of the functional RG flow of continuous area-metric gravity and coarse-graining renormalisation flows derived for loop quantum gravity and spin foams [221–225].

More generally, a thorough investigation of possible fixed points in the interacting system of length-metric and non-metric degrees of freedom in area-metric gravity is a key future direction to pursue. This will in particular allow for explorations of the idea of effective asymptotic safety, according to which a fundamental UV theory features an effective field theory regime with an IR attractive interacting fixed point [226]. Different from fundamental asymptotic safety, this leads to a predictivity of RG fixed points over a finite range of scales. In particular, multiple initial conditions for the RG flow in the UV are mapped onto small intervals for the values of couplings in the IR, such that the low-energy theory is to a good approximation insensitive to the free parameters of the UV completion [227]. The scenario of effective asymptotic safety has been considered in the context of string theory [228, 229], but has not yet been considered in the context of spin foams with area-metric gravity as the intermediate effective field theory between the fundamental discreteness scale of loop quantum gravity and the scale at which the non-metric degrees of freedom decouple.

Finally, area-metric gravity suggests a particular class of corrections to the Einstein-Hilbert action for the metric in general relativity, which are quasi-local and quadratic in the Weyl curvature. Our analysis of static spherically symmetric solutions to the localised action indicates that the two singular Frobenius solution families of local Einstein-Weyl gravity around the radial center [123, 190, 191, 202] are not present in quasi-local Einstein-Weyl gravity. The regular Frobenious solution family of local Einstein-Weyl gravity attains corrections as a result of the non-locality in the action. Understanding a possible mechanism by which certain types of quasi-local quadratic-curvature operators may single out regular solutions, is a central step towards understanding conditions for black-hole singularity resolution in classical gravity. This understanding has to be supplemented by a complete classification of Frobenius solutions around an arbitrary expansion point, as has been performed extensively for quadratic gravity [196–198, 205], and initiated only recently for six-derivative gravity [230, 231]. Such a classification will set the stage for the construction of numerical solutions connecting the asymptotic weak-field regime to the radial core. Altogether, these future directions will present a central contribution towards a non-singular paradigm for black holes and mimickers [232, 233] inspired by area-metric gravity.

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A Appendix

A.1 Basis for second-order quadratic actions for area-metric perturbations

In this appendix, we describe two dictinct ways of deriving a basis for the most general local kinetic action for area-metric perturbations $a_{\mu\nu\rho\sigma}$ around a background configuration induced by a length metric in four spacetime dimensions. For concreteness, in the following the inducing metric is considered to be the flat Euclidean metric $\delta_{\mu\nu}$.

We remind the reader that the tensor of area-metric perturbations $a_{\mu\nu\rho\sigma}$ satisfies the index exchange symmetries and the cyclicity condition,

$$a_{\mu\nu\rho\sigma} = a_{\rho\sigma\mu\nu} = -a_{\rho\sigma\nu\mu} \quad \text{and} \quad a_{\mu[\nu\rho\sigma]} = 0\,. \tag{A.1}$$

In four dimensions the last condition can be replaced by $a_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = 0$, where $\epsilon_{\mu\nu\rho\sigma}$ denotes the totally antisymmetric Levi-Civita density. If a rank-4 tensor a satisfies the first set of symmetries in (A.1), but not the second, we can obtain a cyclic area metric a_c by explicitly subracting from a the totally antisymmetric part,

$$(a_c)_{\mu\nu\rho\sigma} = a_{\mu\nu\rho\sigma} - \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}a_{\alpha\beta\gamma\delta}\epsilon^{\alpha\beta\gamma\delta}\,. \tag{A.2}$$

In subsection A.1.1, we illustrate how the package `xTras` of the tensor computer algebra system `xAct` for Mathematica [234] can be used to find all possible area-metric kinetic terms. These must subsequently be reduced to a minimal set of algebraically independent tensorial structures for a kinetic Lagrangian.

Finally, we will use that in Euclidean signature the tensor $a_{\mu\nu\rho\sigma}$ can be decomposed into its irreducible representations under the action of the special orthogonal group SO(4). This decomposition contains a scalar h, a symmetric and traceless tensor $\hat{h}_{\mu\nu}$, and two respectively selfdual and anti-selfdual traceless tensors $\omega^{\pm}_{\mu\nu\rho\sigma}$ with the symmetries (A.1). In terms of these fields the area-metric perturbation can be parametrized in the form

$$a_{\mu\nu\rho\sigma} = \delta_{\mu[\rho}\delta_{\sigma]\nu}h + 2\Big(\delta_{\mu[\rho}\hat{h}_{\sigma]\nu} - \delta_{\nu[\rho}\hat{h}_{\sigma]\mu}\Big) + \omega^{+}_{\mu\nu\rho\sigma} + \omega^{-}_{\mu\nu\rho\sigma}\,. \tag{A.3}$$

Subsection A.1.2 provides an alternative derivation of an area-metric kinetic basis by making use of the representation coupling theory of the special orthogonal group.

A.1.1 Construction via tensor-algebra package **xTras**

Kinetic terms for the field $a_{\mu\nu\rho\sigma}$ are terms of the form [1]

$$a_{\alpha\beta\gamma\delta}{K_I}^{\alpha\beta\gamma\delta,\mu\nu\rho\sigma}\big(p^2\big)a_{\mu\nu\rho\sigma}\,, \tag{A.5}$$

where the tensors K_I are quadratic in the four-momentum p^μ and obey the same symmetries as the field $a_{\mu\nu\rho\sigma}$ in their first and last index quadruples. In addition, they are symmetric under the exchange of these. The momenta in K_I can be contracted either among themselves to form the covariant Laplacian p^2, or they can appear with two different indices in the form $p^\mu p^\nu$ which are not contracted. For each of these two cases, the admissible tensors can be subdivided into a subset involving only products of background metrics $\delta_{\mu\nu}$, and a subset involving products of background metrics $\delta_{\mu\nu}$ and one Levi-Civita density $\epsilon_{\mu\nu\rho\sigma}$. The possibility for additional independent contractions with two or more Levi-Civita densities is excluded, as a product of these can be rewritten into a combination of Kronecker deltas. Concretely, we can distinguish between the following tensorial structures k_I for the tensors K_I,

$$k_I = \delta\delta\delta\delta p^2\,,\; \delta\delta\epsilon p^2\,,\; \delta\delta\delta pp\,,\; \delta\delta\epsilon pp\,, \tag{A.6}$$

where pp stands for momenta which are not contracted among themselves. The last combination contains a pair of contracted indices.

To find a basis of tensors $\{K_I\}$, we make use of the package `xTras` of the tensor computer algebra system `xAct` for Mathematica [234]. We assume familiarity of the reader with the `xAct` package and only sketch the key steps performed in deriving the area-metric kinetic basis.

Given a four-dimensional spacetime manifold `M` with a metric `g`, we define a momentum vector `p` and a rank-4 tensor `a` obeying the first set of index exchange symmetries in (A.1) by assigning to its four indices the property `RiemannSymmetric`. This assignment implements the index exchange symmetries, but does not implement the algebraic Bianchi identity given by the second condition in (A.1). Consequently, the tensor `a` defined by `xTras` is an acyclic area metric. We can obtain a cyclic area metric by explictly subtracting the totally antisymmetric part from `a`, as in equation (A.2), and refer to this tensor as $\mathtt{a}_\mathtt{c}$.

With these definitions, in a first step, the function `AllContractions` can be used to generate all possible full contractions between two factors of the tensor `a` and each of the tensorial structures appearing in (A.6), where each δ stands for a factor of `g`. This results in a high number

1. Our notation for kinetic terms in Fourier space is

$$\phi_{\mu\dots}K\big(p^2\big)^{\mu\dots,\nu\dots}\psi_{\nu\dots} \equiv \frac{1}{2}\phi_{\mu\dots}(p)K\big(p^2\big)^{\mu\dots,\nu\dots}\psi_{\nu\dots}(-p) + \frac{1}{2}\phi_{\mu\dots}(-p)K\big(p^2\big)^{\mu\dots,\nu\dots}\psi_{\nu\dots}(p)\,, \tag{A.4}$$

where $K(p^2)^{\mu\dots,\nu\dots}$ is a quadratic polynomial in the momentum p^μ.

of contractions, out of which some are duplicate as a consequence of the Riemann symmetries assigned to `a`, and yet others are algebraically dependent for a cyclic area metric. We remove duplicates using the function `DeleteDuplicates` and replace `a` $\to$ $\mathtt{a}_\mathtt{c}$ to identify contractions which are algebraically dependent for a cyclic area metric. Alternatively, one may replace `a` $\to$ `RiemannCD`, where the latter denotes the Riemann tensor built from the covariant derivative `CD` of the metric `g`. Subsequently, we can apply the function `RiemannYoungProject` which projects all occurances of Riemann tensors onto their Young tableaux, and thereby implicitly implements the algebraic Bianchi identity.

To single out a minimal set of linearly independent tensors for a kinetic basis, we consider the representation of the remaining contractions as 20×20 matrices (K_I) extracted from the contractions $a(K_I)a$, where rows and columns label all independent components of the area metric: $a_{0101}, a_{0102}, \ldots$. As a result of reducing by linearly dependent matrices, we arrive at an eight-dimensional basis for the most general kinetic Lagrangian for the field $a_{\mu\nu\rho\sigma}$ representing (cyclic) area-metric perturbations. This result is in agreement with the approach of the constructive-gravity programme [117–121]. The independent tensorial structures are given by

- ${k_0}^{\alpha\beta\gamma\delta,\mu\nu\rho\sigma} = \delta^{\alpha\mu}\delta^{\beta\nu}\delta^{\gamma\rho}\delta^{\delta\sigma}p^2\,,$
- ${k_1}^{\alpha\beta\gamma\delta,\mu\nu\rho\sigma} = \delta^{\alpha\gamma}\delta^{\mu\rho}\delta^{\beta\nu}\delta^{\delta\sigma}p^2\,,$
- $k_2^{\alpha\beta\gamma\delta,\mu\nu\rho\sigma} = \delta^{\alpha\gamma}\delta^{\beta\delta}\delta^{\mu\rho}\delta^{\nu\sigma}p^2\,,$
- ${k_3}^{\alpha\beta\gamma\delta,\mu\nu\rho\sigma} = \epsilon^{\alpha\beta\mu\nu}\delta^{\gamma\rho}\delta^{\delta\sigma}p^2\,,$
- ${k_4}^{\alpha\beta\gamma\delta,\mu\nu\rho\sigma} = \delta^{\alpha\mu}\delta^{\gamma\rho}\delta^{\nu\sigma}p^\beta p^\delta\,,$
- ${k_5}^{\alpha\beta\gamma\delta,\mu\nu\rho\sigma} = \delta^{\alpha\gamma}\delta^{\mu\rho}\delta^{\nu\sigma}p^\beta p^\delta\,,$
- $k_6^{\alpha\beta\gamma\delta,\mu\nu\rho\sigma} = \delta^{\alpha\gamma}\delta^{\beta\nu}\delta^{\mu\rho}p^\delta p^\sigma\,,$
- ${k_7}^{\alpha\beta\gamma\delta,\mu\nu\rho\sigma} = \epsilon^{\alpha\beta\mu}{}_{\lambda}\delta^{\gamma\rho}\delta^{\nu\sigma}p^\lambda p^\delta\,.$

The tensors K_I, symmetrised according to the area-metric symmetries (A.1), are obtained from k_I by projecting the latter onto their part which is

- anti-symmetric in the index pairs $(\alpha\beta), (\gamma\delta), (\mu\nu), (\rho\sigma)$,
- symmetric under the exchange of the index pairs $(\alpha\beta) \leftrightarrow (\gamma\delta)$ and $(\mu\nu) \leftrightarrow (\rho\sigma)$,
- symmetric under the exchange of the index quadruples $(\alpha\beta\gamma\delta) \leftrightarrow (\mu\nu\rho\delta)$,
- cyclic in the index quadruples $(\alpha\beta\gamma\delta)$ and $(\mu\nu\rho\delta)$.

This is established by the following symmetrisation procedure,

$$\begin{aligned}
\hat{K}_I^{\alpha\beta\gamma\delta,\mu\nu\rho\sigma} \equiv\ & \frac{1}{2^7}\Bigg\{\Bigg\{\bigg\{\big\{k_I^{\alpha\beta\gamma\delta,\mu\nu\rho\sigma} - [\alpha\leftrightarrow\beta] - [\gamma\leftrightarrow\delta] + [\alpha\leftrightarrow\beta,\gamma\leftrightarrow\delta]\big\} \\
& \qquad - [\mu\leftrightarrow\nu] - [\rho\leftrightarrow\sigma] + [\mu\leftrightarrow\nu,\rho\leftrightarrow\sigma]\bigg\} \\
& \qquad + [\alpha\beta\leftrightarrow\gamma\delta] + [\mu\nu\leftrightarrow\rho\sigma] + [\alpha\beta\leftrightarrow\gamma\delta,\ \mu\nu\leftrightarrow\rho\sigma]\Bigg\} \\
& \quad + [\alpha\beta\gamma\delta\leftrightarrow\mu\nu\rho\sigma]\Bigg\}, \qquad \text{(A.7)}
\end{aligned}$$

$$\begin{aligned}
K_I^{\alpha\beta\gamma\delta,\mu\nu\rho\sigma} \equiv\ & \hat{K}_I^{\alpha\beta\gamma\delta,\mu\nu\rho\sigma} - \frac{1}{4!}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha'\beta'\gamma'\delta'}\hat{K}^{\alpha'\beta'\gamma'\delta',\mu\nu\rho\sigma} - \frac{1}{4!}\hat{K}_I^{\alpha\beta\gamma\delta,\mu'\nu'\rho'\sigma'}\epsilon_{\mu'\nu'\rho'\sigma'}\epsilon^{\mu\nu\rho\sigma} \\
& + \frac{1}{4!}\frac{1}{4!}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha'\beta'\gamma'\delta'}\hat{K}_I^{\alpha'\beta'\gamma'\delta',\mu'\nu'\rho'\sigma'}\epsilon_{\mu'\nu'\rho'\sigma'}\epsilon^{\mu\nu\rho\sigma}\,. \qquad \text{(A.8)}
\end{aligned}$$

Contracting the tensors k_I or K_I with the cyclic area-metric perturbation tensor results in the following set of kinetic terms,

$$\begin{aligned}
&\big\{a_{\mu\nu\rho\sigma}a^{\mu\nu\rho\sigma}p^2, a_{\mu\nu}{}^{\mu\rho}a_{\sigma\rho}{}^{\sigma\nu}p^2, a_{\mu\nu}{}^{\mu\nu}a_{\rho\sigma}{}^{\rho\sigma}p^2, a_{\mu\nu}{}^{\rho\sigma}a^{\mu\nu\lambda\tau}\epsilon_{\lambda\tau\rho\sigma}p^2, \\
&a_{\mu\rho\nu\sigma}a^{\rho\lambda\sigma}{}_{\lambda}p^\mu p^\nu, a_{\mu}{}^{\rho}{}_{\nu\rho}a^{\sigma\lambda}{}_{\sigma\lambda}p^\mu p^\nu, a_{\mu}{}^{\rho}{}_{\rho\sigma}a_{\lambda\nu}{}^{\sigma\lambda}p^\mu p^\nu, a_{\mu}{}^{\rho\sigma\lambda}a_{\rho}{}^{\tau}{}_{\tau}{}^{\kappa}\epsilon_{\nu\sigma\kappa\lambda}p^\mu p^\nu\big\}\,, \qquad \text{(A.9)}
\end{aligned}$$

which define the Lagrangian ansatz (3.14).

A.1.2 Derivation based on SO(4) representation theory

In this subsection, we provide an alternative derivation of the basis for the most general kinetic action for area-metric perturbations $a_{\mu\nu\rho\sigma}$ in Euclidean signature, using its decomposition (A.12) into irreducible representations of the special orthogonal group SO(4). The result extends immediately to Lorentzian signature, with the only difference manifested in the translation between $a_{\mu\nu\rho\sigma}$ and its irreducible components h, $\hat{h}_{\mu\nu}$ and $\omega^{\pm}_{\mu\nu\rho\sigma}$ as a consequence of the replacement $\delta_{\mu\nu} \to \eta_{\mu\nu}$, which in particular modifies the selfduality and anti-selfduality conditions (3.141).

A general irreducible representation of SO(4) is labeled by a tuple (j_1, j_2), where $j_{1,2}$ are both integers or both half-integers. The dimension of (j_1, j_2) is $\dim(j_1, j_2) = (2j_1+1)(2j_2+1)$. The tensor product of two irreducible representations can be decomposed by applying the Clebsch-Gordan series,

$$(j_1, j_2)\otimes(j_1', j_2') = (j_1\otimes j_1', j_2\otimes j_2') \qquad \text{where} \qquad j_i\otimes j_i' = \bigoplus_{|j_i-j_i'|\le J_i\le|j_i+j_i'|} J_i \qquad \text{(A.10)}$$

and J_i runs from $|j_i - j_i'|$ to $|j_i + j_i'|$ in integer steps. The only possibility for a tensor product of two irreducible representations to contain a singlet (0,0) is if they are identical. Concretely, according to (A.10), the tensor product of an irreducible representation with itself contains the singlet with multiplicity one,

$$(j_1, j_2) \otimes (j_1, j_2) = (0,0) \oplus \cdots \, . \tag{A.11}$$

The irreducible representations appearing in the decomposition of the area metric (A.3) are

$$a \;\; \leftrightarrow \;\; \left(h, \hat{h}, \omega^+, \omega^-\right) \;\; \in \;\; (0,0) \oplus (1,1) \oplus (2,0) \oplus (0,2) \, . \tag{A.12}$$

Kinetic terms for an area-metric Lagrangian are terms in which two copies of fields appearing in (A.12) are coupled with two momenta to form a singlet. Each momentum p^μ is in the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation of SO(4). Therefore, $p^\mu p^\nu$ lies in the tensor product

$$p \otimes p \;\; \in \;\; \left(\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\frac{1}{2}, \frac{1}{2}\right)\right)_S = (0,0) \oplus (1,1) \;\; \ni \;\; p^2 \oplus \widehat{(p \otimes p)} \, , \tag{A.13}$$

where the subscript S denotes the symmetric part. The invariant p^2 stands for the square of the momentum vector, whereas $\widehat{pp}$ denotes the traceless part of $p^\mu p^\nu$.

In the first step, we consider all possible kinetic terms which contain p^2. The tensor product with two copies of irreducible representations appearing in (A.12) must include a singlet. The only possibility to obtain a singlet is to consider the square of each irreducible component. Therefore, there are four kinetic terms which contain p^2. These are

- $h^2 p^2$,
- $\omega^+_{\mu\nu\rho\sigma} \omega^{+\mu\nu\rho\sigma} p^2$,
- $\hat{h}_{\mu\nu} \hat{h}^{\mu\nu} p^2$,
- $\omega^-_{\mu\nu\rho\sigma} \omega^{-\mu\nu\rho\sigma} p^2$.

Now we consider the second possible type of kinetic terms. These contain the traceless part $\widehat{pp}$ of $p^\mu p^\nu$, which lies in the $(1,1)$ representation. Consequently, we have to determine all possible singlets that can be constructed from this representation and two copies of irreducible representations appearing in (A.12). To that end, we proceed by first tensoring $(1,1)$ with a given one of the representations in (A.12) and decomposing this product into its irreducible representations. Subsequently, we consider the tensor product of this decomposition separately with each of the representations appearing in (A.12), to see in which cases a singlet and thereby a kinetic term can be produced.

As a starting point, the product between $(1,1)$ and the singlet representation $(0,0)$ associated with h is given by

$$h \otimes \widehat{(p \otimes p)} \;\; \in \;\; (0,0) \otimes (1,1) = (1,1) \, . \tag{A.14}$$

The only singlet involving one more factor of a field that can be constructed from this tensor product arises by forming the tensor product with the representation $(1,1)$, associated with $\hat{h}$.

This leads to the kinetic term

- $\hat{h}_{\mu\nu} h p^\mu p^\nu$.

Next, we consider terms involving $\widehat{pp}$ and $\hat{h}$ by building the tensor product of two copies of the $(1,1)$ representation. This product decomposes into

$$\begin{aligned} \hat{h} \otimes \widehat{(p \otimes p)} \;\in\; (1,1)\otimes(1,1) \;&=\; (0,0)\oplus(0,1)\oplus(0,2) \\ &\oplus\; (1,0)\oplus(1,1)\oplus(1,2) \\ &\oplus\; (2,0)\oplus(2,1)\oplus(2,2)\,. \end{aligned} \tag{A.15}$$

All of the representations appearing in (A.12) are contained on the right hand side. Therefore, we obtain four kinetic terms by tensoring the expression in (A.14) with each of these representations such that the result contains a singlet, according to (A.11). These four kinetic terms are of the form $h\hat{h}\widehat{pp}$, $\hat{h}\hat{h}\widehat{pp}$, and $\omega^\pm\hat{h}\widehat{pp}$. The first term was already stated above. The second term is given by the contraction

- $\hat{h}_{\mu\rho}\hat{h}^\rho_\nu p^\mu p^\nu$.

Finally, the other two kinetic terms are given by

- $\hat{h}_{\rho\sigma}\omega^+_{\mu\rho\nu\sigma} p^\mu p^\nu$,
- $\hat{h}_{\rho\sigma}\omega^-_{\mu\rho\nu\sigma} p^\mu p^\nu$.

There are no further new kinetic terms that can be constructed. Such terms would have to be built from $\widehat{pp}$ and two powers of fields $\omega^\pm$, in the form $\omega^+\omega^+\widehat{pp}$, $\omega^-\omega^-\widehat{pp}$, or $\omega^-\omega^+\widehat{pp}$. The relevant tensor products decompose as

$$\omega^+ \otimes \widehat{(p\otimes p)} \quad \in \quad (2,0)\otimes(1,1) = (1,1)\oplus(2,1)\oplus(3,1)\,, \tag{A.16}$$

$$\omega^- \otimes \widehat{(p\otimes p)} \quad \in \quad (0,2)\otimes(1,1) = (1,1)\oplus(1,2)\oplus(1,3)\,. \tag{A.17}$$

The right hand sides do not contain any of the representations $(2,0)$ or $(0,2)$. Therefore it is not possible to couple $\omega^\pm$ to $\omega^+\widehat{pp}$ or $\omega^-\widehat{pp}$ to produce a singlet. The only singlets that can be constructed from $\omega^\pm\widehat{pp}$, with one additional field, arise by forming the tensor product with the representation $(1,1)$, associated with $\hat{h}$. The resulting kinetic terms $\hat{h}\omega^\pm\widehat{pp}$ were already found above.

This concludes the derivation of all possible local kinetic terms for an area-metric Lagrangian, using the decomposition of $a_{\mu\nu\rho\sigma}$ into its irreducible components under SO(4). The eight-dimensional basis is given by

$$\left\{ h^2p^2\,,\, \hat{h}_{\mu\nu}\hat{h}^{\mu\nu}p^2\,,\, h\hat{h}_{\mu\nu}p^\mu p^\nu\,,\, \hat{h}_{\mu\rho}\hat{h}^\rho_\nu p^\mu p^\nu\,,\, \hat{h}^{\rho\sigma}\omega^\pm_{\mu\rho\nu\sigma}p^\mu p^\nu\,,\, \omega^\pm_{\mu\nu\rho\sigma}\omega^{\pm\mu\nu\rho\sigma}p^2 \right\}. \tag{A.18}$$

The selfdual and anti-selfdual components $\omega^{\pm}$ of the area metric couple only to the traceless tensor $\hat{h}$. The latter, together with the scalar h, can be combined into the symmetric tensor

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{4}\delta_{\mu\nu} h \,. \tag{A.19}$$

With this definition, h and $\hat{h}_{\mu\nu}$ represent the trace and traceless part of $h_{\mu\nu}$. The independent contractions appearing in the kinetic basis (A.18) are

$$\left\{h^2 p^2, h_{\mu\nu}h^{\mu\nu}p^2, h h_{\mu\nu}p^\mu p^\nu, h_{\mu\rho}h^{\rho}_{\nu}p^\mu p^\nu, h^{\rho\sigma}\omega^{\pm}_{\mu\rho\nu\sigma}p^\mu p^\nu, \omega^{\pm}_{\mu\nu\rho\sigma}\omega^{\pm\mu\nu\rho\sigma}p^2\right\} . \tag{A.20}$$

The first four terms, quadratic in $h_{\mu\nu}$, are identical to the terms which define the most general kinetic Lagrangian for a symmetric rank-2 tensor in (3.2).

Finally, for completeness, we note that the transformation from the kinetic basis (A.20) expressed in terms of the tensors h and $\omega^{\pm}$, into the kinetic basis (A.9) expressed explicitly in terms of the tensor a, in Euclidean signature is defined by the equations (3.142)–(3.143) and (3.151)–(3.152), making use of the projectors Π^{L} and $\Pi^{\pm}$ in (3.145) and (3.146). The inverse transformation is defined by substituting for a in (A.9) the decomposition (A.3), with $\hat{h}$ eliminated from (A.19), and making use of the selfduality and anti-selfduality relations for $\omega^{\pm}$ in Euclidean signature (3.141). The result is

$$\begin{aligned}
a_{\mu\nu\rho\sigma}a^{\mu\nu\rho\sigma}p^2 &= 4h^2p^2 + 8h_{\mu\nu}h^{\mu\nu}p^2 + \omega^{+}_{\mu\nu\rho\sigma}\omega^{+\mu\nu\rho\sigma}p^2 + \omega^{-}_{\mu\nu\rho\sigma}\omega^{-\mu\nu\rho\sigma}p^2 \,, && \text{(A.21)}\\
a_{\mu\nu}{}^{\mu\rho}a_{\sigma\rho}{}^{\sigma\nu}p^2 &= 8h^2p^2 + 4h_{\mu\nu}h^{\mu\nu}p^2 \,, && \text{(A.22)}\\
a_{\mu\nu}{}^{\mu\nu}a_{\rho\sigma}{}^{\rho\sigma}p^2 &= 36h^2p^2 \,, && \text{(A.23)}\\
a_{\mu\nu}{}^{\rho\sigma}a^{\mu\nu\lambda\tau}\epsilon_{\lambda\tau\rho\sigma}p^2 &= 2\omega^{+}_{\mu\nu\rho\sigma}\omega^{+\mu\nu\rho\sigma}p^2 - 2\omega^{-}_{\mu\nu\rho\sigma}\omega^{-\mu\nu\rho\sigma}p^2 \,, && \text{(A.24)}\\
a_{\mu\rho\nu\sigma}a^{\rho\lambda\sigma}{}_{\lambda}p^\mu p^\nu &= h^2p^2 + 2h_{\mu\nu}h^{\mu\nu}p^2 + 4hh_{\mu\nu}p^\mu p^\nu - 4h_{\mu\rho}h^{\rho}_{\nu}p^\mu p^\nu \\
&+ 2h^{\rho\sigma}\omega^{+}_{\mu\rho\nu\sigma}p^\mu p^\nu + 2h^{\rho\sigma}\omega^{-}_{\mu\rho\nu\sigma}p^\mu p^\nu \,, && \text{(A.25)}\\
a_{\mu}{}^{\rho}{}_{\nu\rho}a^{\sigma\lambda}{}_{\sigma\lambda}p^\mu p^\nu &= 6h^2p^2 + 12hh_{\mu\nu}p^\mu p^\nu \,, && \text{(A.26)}\\
a_{\mu}{}^{\rho}{}_{\rho\sigma}a_{\lambda\nu}{}^{\sigma\lambda}p^\mu p^\nu &= h^2p^2 + 4hh_{\mu\nu}p^\mu p^\nu + 4h_{\mu\rho}h^{\rho}_{\nu}p^\mu p^\nu \,, && \text{(A.27)}\\
a_{\mu}{}^{\rho\sigma\lambda}a_{\rho}{}^{\tau}{}_{\tau}{}^{\kappa}\epsilon_{\nu\sigma\kappa\lambda}p^\mu p^\nu &= 4h^{\rho\sigma}\omega^{+}_{\mu\rho\nu\sigma}p^\mu p^\nu - 4h^{\rho\sigma}\omega^{-}_{\mu\rho\nu\sigma}p^\mu p^\nu \,. && \text{(A.28)}
\end{aligned}$$

In Lorentzian signature with inducing background Minkowski metric $\eta_{\mu\nu}$, the right hand sides of (A.24) and (A.28) have to be multiplied by a factor of i as a result of the selfduality and anti-selfduality relations (3.19) for $\omega^{\pm}$ in Lorentzian signature.

A.2 Hamiltonian formulation of linearised shift-symmetric area-metric actions

In subsection 3.1.4, we have identified a two-parameter subclass of linearised area-metric Lagrangians with shift-symmetric kinetic term and identical effective mass parameters for the non-metric fields, for which the effective action for the metric degrees of freedom derived in subsection 3.1.3 is ghostfree. An analysis of the covariant equations of motion for $h_{\mu\nu}$ and $\chi^{\pm}_{\mu\nu}$, derived from such a type of Lagrangian, tentatively indicates that the physical spectrum of this subclass of area-metric actions consists of two massless modes, associated with a massless spin-2 particle, and five additional massive modes. This conclusion is, however, based on the parametrisation of the Lagrangian in terms of $h_{\mu\nu}$ and the fields $\chi^{\pm}_{\mu\nu}$, which arise from the original fields $\omega^{\pm}_{\mu\nu\rho\sigma}$, or equivalently from $\chi^{\pm}_{ab}$, through the non-local field redefinition (3.38). The main goal of this appendix is to confirm the expected form of the physical spectrum explicitly based on a Hamiltonian analysis, starting from the Lagrangian parametrised in terms of the local fields $h_{\mu\nu}$ and $\chi^{\pm}_{ab}$.

The Lagrangian (3.42) in position space can be expressed as

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\mathrm{EH}}(h_{\mu\nu}) + \frac{1}{2}\sum_{\pm}\left[\alpha_{\pm} h_{\mu\nu}\partial_\alpha\partial^\alpha\chi^{\pm\mu\nu} - \frac{1}{2}\partial_\alpha\chi^{\pm}_{\mu\nu}\partial^\alpha\chi^{\pm\mu\nu} - \frac{1}{2}m_{\pm}^2\chi^{\pm}_{\mu\nu}\chi^{\pm\mu\nu}\right] \\ &\equiv \mathcal{L}_{\mathrm{EH}}(h_{\mu\nu}) + \mathcal{L}_{h\chi} + \mathcal{L}_{\chi\chi}\,, \end{aligned} \tag{A.29}$$

where the linearised Einstein-Hilbert Lagrangian is given in (3.4). Without loss of generality we have fixed the global rescaling of the Lagrangian by setting $A \equiv 1$. Additionally, we have rescaled the fields $\chi^{\pm}_{\mu\nu}$ such that $\beta_{\pm} = 1$. Using the isometricity relation (3.40), we may directly replace $\chi^{\pm}_{\mu\nu} \to \chi^{\pm}_{ab}$ in the Lagrangian term $\mathcal{L}_{\chi\chi}$ quadratic in the non-metric fields,

$$\mathcal{L}_{\chi\chi} = -\frac{1}{4}\sum_{\pm}\left[\partial_\alpha\chi^{\pm}_{ab}\partial^\alpha\chi^{\pm ab} + m_{\pm}^2\chi^{\pm}_{ab}\chi^{\pm ab}\right]. \tag{A.30}$$

The interaction term $\mathcal{L}_{h\chi}$ is transformed from $\chi^{\pm}_{\mu\nu}$ to the local fields $\chi^{\pm}_{ab}$ by inserting the definition (3.38),

$$\mathcal{L}_{h\chi} = \frac{1}{2}\sum_{\pm}\alpha_{\pm}h_{\mu\nu}\partial_\alpha\partial^\alpha\chi^{\pm\mu\nu} = h_{\mu\nu}\partial_\alpha\partial^\alpha\mathrm{Re}\big[\alpha_+\chi^+_{\mu\nu}\big] = h_{\mu\nu}\mathrm{Re}\big[\alpha_+\Sigma^{+a}_{\mu\rho}\Sigma^{+b}_{\nu\sigma}\partial^\rho\partial^\sigma\chi^+_{ab}\big]\,, \tag{A.31}$$

where the selfdual and anti-selfdual Plebanski 2-forms $\Sigma^{\pm a}_{\mu\nu}$ constructed from the flat Minkowski tetrad η^I_μ are given in (3.21). In particular, we can identify internal indices $a, b, ... = 1, 2, 3$ with spatial indices $i, j, ... = 1, 2, 3$ via the background spatial triads $\eta^a_i = \delta^a_i$. Thus, from now on we denote the scalar fields by $\chi^{\pm}_{ij}$ and remind the reader that these are traceless.

The previous formulae define the general gauge-invariant linearised area-metric Lagrangian expressed in terms of the fields $h_{\mu\nu}$ and $\chi^{\pm}_{ab}$ parametrising the area-metric perturbations. The two-parameter subclass of area-metric Lagrangians analysed in subsection 3.1.4 and in this appendix is characterised by identical mass parameters $m^2_{\pm} = m^2$ for the non-metric fields, as well as by the shift-symmetry condition (3.69),

$$\alpha_+^2 + \alpha_-^2 = 2\,. \tag{A.32}$$

A.2.1 Hamiltonian formulation of linearised general relativity

In this subsection, we first review the Hamiltonian formulation of linearised general relativity starting from the linearised Einstein-Hilbert action (3.4),

$$S = \int \mathrm{d}^4x \left(-\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \partial^\mu h_{\mu\rho}\partial_\nu h^{\nu\rho} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial^\mu h \partial_\mu h \right). \tag{A.33}$$

A $3+1$ decomposition of the Lagrangian, after integration by parts, leads to

$$\begin{aligned}\mathcal{L}_{\mathrm{EH}} \;=\;& \frac{1}{2}\dot{h}_{ij}^2 - \frac{1}{2}\dot{h}_{ii}^2 - 2\partial_i h_{j0}\dot{h}_{ij} + 2\partial_i h_{i0}\dot{h}_{jj} - h_{00}\partial_i\partial_j(h_{ij} - \delta_{ij}h_{kk}) \\ & -(\partial_i h_{i0})^2 + (\partial_i h_{j0})^2 - \frac{1}{2}(\partial_i h_{jk})^2 + (\partial_j h_{ij})^2 - \partial_j h_{ii}\partial_k h_{jk} + \frac{1}{2}(\partial_i h_{jj})^2\,. \end{aligned} \tag{A.34}$$

Here, Latin letters $i, j, \ldots = 1, 2, 3$ label spatial directions and are contracted with the flat background metric δ_{ij}. To simplify the notation, we disregard the upper versus lower positioning of indices and adopt the convention that any repeated indices are summed over. The components h_{00} and h_{i0} do not appear with time derivatives in (A.34). Therefore, they have vanishing canonical momenta. The only non-vanishing canonical momentum is the one conjugated to the spatial components of the metric h_{ij} and given by

$$\pi_{ij} \equiv \frac{\partial \mathcal{L}_{\mathrm{EH}}}{\partial \dot{h}_{ij}} = \dot{h}_{ij} - \dot{h}_{kk}\delta_{ij} - 2\partial_{(i}h_{j)0} + 2\partial_k h_{k0}\delta_{ij}\,. \tag{A.35}$$

Taking the trace of this expression allows us to solve for the velocities $\dot{h}_{ij}$ as functions of the momenta,

$$\dot{h}_{ij} = \pi_{ij} - \frac{1}{2}\pi_{kk}\delta_{ij} + 2\partial_{(i}h_{j)0}\,. \tag{A.36}$$

After additional integrations by parts to remove all derivatives from the field h_{i0}, the canonical Hamiltonian for linearised general relativity,

$$\mathcal{H}_{\mathrm{EH}} \equiv \pi_{ij}\dot{h}_{ij} - \mathcal{L}_{\mathrm{EH}}\,, \tag{A.37}$$

explicitly takes the form

$$\mathcal{H}_{\rm EH} = \frac{1}{2}\pi_{ij}^2 - \frac{1}{4}\pi_{ii}^2 + \frac{1}{2}(\partial_i h_{jk})^2 - (\partial_j h_{ij})^2 + \partial_j h_{ii}\partial_k h_{jk} - \frac{1}{2}(\partial_i h_{jj})^2 + h_{00}\mathcal{C} - 2h_{i0}\mathcal{C}_i\,. \qquad \text{(A.38)}$$

The fields h_{00} and h_{i0} are recognised as Lagrange multipliers, and impose four primary constraints known as the Hamiltonian and diffeomorphism constraints

$$\mathcal{C} \equiv \partial_i\partial_j(h_{ij} - h_{kk}\delta_{ij}) \qquad \text{and} \qquad \mathcal{C}_i \equiv \partial_j\pi_{ij}\,. \qquad \text{(A.39)}$$

These constraints commute and are therefore first-class. Computing their Poisson bracket with the Hamiltonian,

$$\{\mathcal{C}, \mathcal{H}_{\rm EH}\} = \partial_i\mathcal{C}_i \qquad \text{and} \qquad \{\mathcal{C}_i, \mathcal{H}_{\rm EH}\} = 0\,, \qquad \text{(A.40)}$$

does not generate any secondary constraints. [1]

The physical phase space $\Gamma_{\rm phys}$ is obtained from the kinematical phase space $\Gamma_{\rm kin} = \{(h_{ij}, \pi_{ij})\}$ by imposing the four first-class constraints (A.39). Each of these removes two degrees of freedom. This leads to a

$$\dim(\Gamma_{\rm phys}) = \dim(\Gamma_{\rm kin}) - 2n_{\text{first-class}} - n_{\text{second-class}} = 2\cdot 6 - 2\cdot 4 - 0 = 4 \qquad \text{(A.42)}$$

dimensional physical phase space describing two propagating degrees of freedom. These are associated with the two polarisations of the massless spin-2 graviton in general relativity. The transformations generated by the Hamiltonian and diffeomorphism constraints (A.39) can be gauge-fixed by imposing

$$\pi_{ii} = 0 \qquad \text{and} \qquad \partial_i h_{ij} = 0\,. \qquad \text{(A.43)}$$

When the last condition is satisfied, the Hamiltonian constraint enforces further that the spatial metric h_{ij} be traceless. Altogether, after gauge-fixing the only physical degrees of freedom left are two transverse-traceless (tt) fields $\left(h_{ij}^{tt}, \pi_{ij}^{tt}\right)$ with manifestly positive definite physical Hamiltonian given by

$$\mathcal{H}_{\rm phys} = \frac{1}{2}\left(\pi_{ij}^{tt}\right)^2 + \frac{1}{2}\left(\partial_i h_{jk}^{tt}\right)^2. \qquad \text{(A.44)}$$

1. The time evolution of a phase-space function is obtained by evaluating its Poisson bracket with the Hamiltonian H, which is defined from the Hamiltonian density $\mathcal{H}(x)$ by a spatial integral,

$$H = \int \mathrm{d}^3x\,\mathcal{H}(x)\,. \qquad \text{(A.41)}$$

We omit this spatial integral in expressions such as (A.40) for notational convenience, but remind the reader that the second argument actually involves the Hamiltonian.

A.2.2 3+1 decomposition of the area-metric Lagrangian and Hamiltonian

In this subsection, we perform a 3+1 decomposition of the two-parameter subclass of area-metric Lagrangians (A.29) characterised by identical mass parameters $m_\pm^2 = m^2$ for the non-metric degrees of freedom, and by the shift-symmetry condition (A.32). The $3+1$ decomposition of the Einstein-Hilbert Lagrangian $\mathcal{L}_{\mathrm{EH}}$ was derived in the previous subsection and is given by the expression in equation (A.34). For the other terms in the Lagrangian it is useful to decompose the fields $\chi^\pm_{ij}$ and couplings $\alpha_\pm$ into their real and imaginary parts,

$$\chi^\pm_{ij} = \chi^1_{ij} \pm i\chi^2_{ij} \qquad \text{and} \qquad \alpha_\pm = \alpha_1 \pm i\alpha_2 \, . \tag{A.45}$$

The condition (A.32) implies that we can parametrise the real couplings $\alpha_{1,2}$ in terms of just one real parameter ξ, by setting

$$\alpha_1 = \cosh(\xi) \qquad \text{and} \qquad \alpha_2 = \sinh(\xi) \, . \tag{A.46}$$

Let us introduce the two real symmetric and traceless fields ψ_{ij} and ϕ_{ij} defined by

$$\psi_{ij} = \alpha_1\chi^1_{ij} - \alpha_2\chi^2_{ij} \, , \tag{A.47}$$

$$\phi_{ij} = \alpha_2\chi^1_{ij} - \alpha_1\chi^2_{ij} \, , \tag{A.48}$$

such that

$$\alpha_1\chi^2_{ij} + \alpha_2\chi^1_{ij} = s_\xi\psi_{ij} - c_\xi\phi_{ij} \, , \tag{A.49}$$

where the parameters s_ξ and c_ξ are defined by

$$s_\xi = \sinh(2\xi) \qquad \text{and} \qquad c_\xi = \cosh(2\xi) \, . \tag{A.50}$$

With the previous definitions, the interaction term $\mathcal{L}_{h\chi}$ in (A.31) can be written, after integration by parts, as

$$\mathcal{L}_{h\chi} = -h_{00}\partial_i\partial_j\psi_{ij} + \pi_{ij}\dot{\psi}_{ij} + \epsilon_{ikm}\epsilon_{jln}h_{kl}\partial_m\partial_n\psi_{ij} + 2\pi_{ij}\widehat{\partial}(s_\xi\psi - c_\xi\phi)_{ij} \, , \tag{A.51}$$

with π_{ij} defined by the expression in (A.35). Thus, π_{ij} corresponds to the momentum conjugated to the spatial metric h_{ij} in linearised general relativity. In (A.51) we have introduced a spatial derivative operator $\widehat{\partial}$ acting on symmetric tensors τ_{ij} as

$$\widehat{\partial}\tau_{ij} \equiv \epsilon_{(i|kl}\partial_k\tau_{l|j)} \, . \tag{A.52}$$

If τ_{ij} is traceless, then $\widehat{\partial}\tau_{ij}$ is also traceless.

Finally, we are left with Lagrangian term $\mathcal{L}_{\chi\chi}$ in (A.30) which is quadratic in ψ and ϕ. The

$3+1$ decomposition of this term is easily computed as

$$\mathcal{L}_{XX} = \frac{1}{2}\Big(\dot{\psi}_{ij}^2 - \dot{\phi}_{ij}^2 - (\partial_i\psi_{jk})^2 + (\partial_i\phi_{jk})^2 - m^2\psi_{ij}^2 + m^2\phi_{ij}^2\Big)\,. \tag{A.53}$$

Notably, the field ϕ_{ij} has a wrong sign in front of its kinetic and mass term.

We now collect all pieces together, using (A.34) for the linearised Einstein-Hilbert action for $h_{\mu\nu}$, and (A.35) to express π_{ij}. Moreover, we use that ψ_{ij} is traceless, i.e., $\psi_{ii} = 0$. This allows us in a first step to write the area-metric Lagrangian in a $3+1$ decomposition as

$$\begin{aligned}
\mathcal{L} \;=\;& \frac{1}{2}\Big(\pi_{ij} + \dot{\psi}_{ij}\Big)^2 - \frac{1}{4}\Big(\pi_{ii} + \dot{\psi}_{ii}\Big)^2 - h_{00}\partial_i\partial_j(h_{ij} + \psi_{ij} - \delta_{ij}(h_{kk} + \psi_{kk}))\\
-\;& \frac{1}{2}(\partial_i h_{jk})^2 + (\partial_j h_{ij})^2 + h_{ii}\partial_j\partial_k h_{jk} + \frac{1}{2}(\partial_i h_{jj})^2\\
+\;& \epsilon_{ikm}\epsilon_{jln}h_{kl}\partial_m\partial_n\psi_{ij} + 2\pi_{ij}\widehat{\partial}(s_\xi\psi - c_\xi\phi)_{ij}\\
+\;& \frac{1}{2}\Big(-\dot{\phi}_{ij}^2 - (\partial_i\psi_{jk})^2 + (\partial_i\phi_{jk})^2 - m^2\psi_{ij}^2 + m^2\phi_{ij}^2\Big)\,.
\end{aligned} \tag{A.54}$$

It should be recognised that terms of the form π^2, $\pi\dot{\psi}$, and $\dot{\psi}^2$ have combined into a square. The only other terms involving ψ are the ones in the third line and do not contain time derivatives. These terms can be absorbed by introducing the combinations

$$\Pi_{ij} \;\equiv\; \pi_{ij} + \dot{\psi}_{ij} + 2\widehat{\partial}(s_\xi\psi - c_\xi\phi)_{ij}\,, \tag{A.55}$$

$$\Pi^\phi_{ij} \;\equiv\; -\dot{\phi}_{ij} - 2c_\xi\hat{\partial}\psi_{ab}\,, \tag{A.56}$$

and rewriting (A.54) in terms of these. The thereby generated new term $\psi\dot{\psi}$ vanishes after repeatingly integrating by parts and discarding boundary terms. The other new term $\dot{\psi}\phi$ can be rewritten after partial integration such that the time derivative acts on ϕ. Thereby, the Lagrangian can be expressed in terms of the shifted spatial metric

$$H_{ij} \equiv h_{ij} + \psi_{ij} \tag{A.57}$$

as

$$\begin{aligned}
\mathcal{L} \;=\;& \frac{1}{2}\Pi_{ij}^2 - \frac{1}{4}\Pi_{ii}^2 - h_{00}\partial_i\partial_j(H_{ij} - \delta_{ij}H_{kk})\\
-\;& \frac{1}{2}(\partial_i H_{jk})^2 + (\partial_j H_{ij})^2 + H_{ii}\partial_j\partial_k H_{jk} + \frac{1}{2}(\partial_i H_{jj})^2\\
+\;& \epsilon_{ikm}\epsilon_{jln}h_{kl}\partial_m\partial_n\psi_{ij} + 4c_\xi s_\xi\widehat{\partial}\psi_{ij}\widehat{\partial}\phi_{ij} - \frac{1}{2}m^2\psi_{ij}^2\\
-\;& \frac{1}{2}{\Pi^\phi_{ij}}^2 + \frac{1}{2}(\partial_i\phi_{jk})^2 - 2c_\xi^2\Big(\widehat{\partial}\phi_{ij}\Big)^2 + \frac{1}{2}m^2\phi_{ij}^2\,.
\end{aligned} \tag{A.58}$$

To state the Hamiltonian, we have to write the Lagrangian in terms of the fields H_{ij}, ψ_{ij}, and

ϕ_{ij}, as well as their time derivatives. This can be done by expressing Π_{ij} and Π^{ϕ}_{ij} via (A.55) and (A.56), with π_{ij} defined in (A.35). This leads to

$$\begin{aligned}\Pi_{ij} &= \dot{H}_{ij} - \delta_{ij}\dot{H}_{kk} - 2\partial_{(i}h_{j)0} + 2\delta_{ij}\partial_i h_{j0} + 2s_\xi\widehat{\partial}\psi_{ij} - 2c_\xi\widehat{\partial}\phi_{ij}\,, && \text{(A.59)}\\ \Pi^{\phi}_{ij} &= -\dot{\phi}_{ij} - 2c_\xi\widehat{\partial}\phi_{ij}\,. && \text{(A.60)}\end{aligned}$$

Inserting these expressions into (A.58), we arrive at the Lagrangian as a function

$$\mathcal{L} = \mathcal{L}(H_{ij}, \psi_{ij}, \phi_{ij}, \dot{H}_{ij}, \dot{\psi}_{ij}, \dot{\phi}_{ij})\,. \tag{A.61}$$

Therefrom it is straightforward to see that the momenta conjugated to H_{ij} and ϕ_{ij} are identical to Π_{ij} and Π^{ϕ}_{ij}, i.e.,

$$\Pi_{ij} = \frac{\partial\mathcal{L}}{\partial\dot{H}_{ij}} \qquad \text{and} \qquad \Pi^{\phi}_{ij} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_{ij}}\,. \tag{A.62}$$

The momentum conjugated to ψ_{ij} vanishes,

$$\Pi^{\psi}_{ij} \equiv \frac{\partial\mathcal{L}}{\partial\dot{\psi}_{ij}} = 0\,, \tag{A.63}$$

which represents a primary constraint.

Finally, after inverting the relations (A.59) and (A.60) for $\dot{H}_{ij}$ and $\dot{\phi}_{ij}$, and integrating by parts, we arrive at the Hamiltonian density

$$\begin{aligned}\mathcal{H} &\equiv \Pi_{ij}\dot{H}_{ij} + \Pi^{\phi}_{ij}\dot{\phi}_{ij} - \mathcal{L}\\ &= \frac{1}{2}\Pi^2_{ij} - \frac{1}{4}\Pi^2_{ii} + \frac{1}{2}(\partial_i H_{jk})^2 - (\partial_j H_{ij})^2 + \partial_j H_{ii}\partial_k H_{jk} - \frac{1}{2}(\partial_i H_{jj})^2\\ &+ h_{00}\mathcal{C} - 2h_{i0}\mathcal{C}_i\\ &- 2\Pi_{ij}\left(s_\xi\widehat{\partial}\psi_{ij} - c_\xi\widehat{\partial}\phi_{ij}\right) - 2c_\xi\Pi^{\phi}_{ij}\widehat{\partial}\phi_{ij} - 2\chi_{ij}\widehat{\partial\partial}H_{ij} - 4c_\xi s_\xi\widehat{\partial}\psi_{ij}\widehat{\partial}\phi_{ij} + \frac{1}{2}m^2\psi^2_{ij}\\ &- \frac{1}{2}\Pi^{\phi\,2}_{ij} - \frac{1}{2}(\partial_i\phi_{jk})^2 + 2c_\xi^2\left(\widehat{\partial}\phi_{ij}\right)^2 - \frac{1}{2}m^2\phi^2_{ij}\,. \end{aligned} \tag{A.64}$$

Here, in addition to the spatial derivative operator $\widehat{\partial}$, we have introduced the double spatial derivative operator $\widehat{\partial\partial}$ acting on symmetric tensors τ_{ij} as

$$\widehat{\partial\partial}\tau_{ij} \equiv \epsilon_{ikm}\epsilon_{jln}\partial_k\partial_l\tau_{mn}\,. \tag{A.65}$$

A.2.3 Constraints and reduced Hamiltonian

Comparing the Hamiltonian density (A.64) with (A.38) and (A.39), we recognise the first two lines as the linearised Einstein-Hilbert Hamiltonian density for the shifted spatial metric H_{ij} and its conjugate momentum Π_{ij}, with Lagrange multipliers h_{00} and h_{i0} imposing the Hamiltonian and diffeomorphism constraints,

$$\mathcal{C} \equiv \partial_i \partial_j (H_{ij} - \delta_{ij} H_{kk}) \qquad \text{and} \qquad \mathcal{C}_i \equiv \partial_j \Pi_{ij} \,. \tag{A.66}$$

These constraints commute with the parts of the Hamiltonian involving the fields ψ_{ij} and ϕ_{ij}. In the same way as in linearised general relativity, evaluating their Poisson bracket with the Hamiltonian does not generate any secondary constraints,

$$\{\mathcal{C}, \mathcal{H}\} = \partial_i \mathcal{C}_i \qquad \text{and} \qquad \{\mathcal{C}_i, \mathcal{H}\} = 0 \,. \tag{A.67}$$

However, there is in addition the primary constraint in (A.63) imposing that the momentum Π^{ψ}_{ij} conjugated to ψ_{ij} vanishes. Its Poisson bracket with the Hamiltonian can be evaluated using the relation

$$\tau_{ij} \widehat{\partial} \sigma_{ij} = \sigma_{ij} \widehat{\partial} \tau_{ij} \,, \tag{A.68}$$

which holds for any two symmetric tensors τ_{ij} and σ_{ij} after integration by parts, up to a boundary term. This leads to a secondary constraint

$$\mathcal{C}^{\psi}_{ij} \equiv \left\{ \Pi^{\psi}_{ij}, \mathcal{H} \right\} = -m^2 \psi_{ij} + \mathcal{F}_{ij} \,, \tag{A.69}$$

where $\mathcal{F}_{ij}$ is given by

$$\mathcal{F}_{ij}\Big(H_{kl}, \Pi_{kl}, \phi_{kl}, \Pi^{\phi}_{kl}\Big) = 2 s_\xi \widehat{\partial} \Pi_{ij} + 2 c_\xi \widehat{\partial} \Pi^{\phi}_{ij} + 2 \widehat{\partial\partial} H_{ij} + 4 c_\xi s_\xi \widehat{\partial}^2 \phi_{ij} \,. \tag{A.70}$$

The commutator between $\mathcal{F}_{ij}$ and $\mathcal{F}_{kl}$ vanishes such that $\mathcal{C}^{\psi}_{ij}$ commutes with $\mathcal{C}^{\psi}_{kl}$. Furthermore, $\mathcal{C}^{\psi}_{ij}$ commutes with the Hamiltonian and diffeomorphism constraints $\mathcal{C}$ and $\mathcal{C}_k$. However, due to the mass term for ψ_{ij}, the Poisson bracket between the primary and secondary constraints Π^{ψ}_{ij} and $\mathcal{C}^{\psi}_{kl}$ does not vanish and evaluates to

$$\left\{ \Pi^{\psi}_{ij}, \mathcal{C}^{\psi}_{kl} \right\} = m^2 \hat{\delta}_{ijkl} \,, \tag{A.71}$$

where $\hat{\delta}_{ijkl}$ denotes the identity on the space of symmetric and traceless rank-2 tensors τ_{ij}. Therefore, different from linearised general relativity, the constraint algebra is second-class. According to the Dirac stabilisation procedure, both the primary and secondary constraints have to be added with independent Lagrange multipliers to the Hamiltonian. Requiring these constraints to be preserved under time evolution fixes the Lagrange multipliers and does not generate any tertiary constraints. Altogether, the physical phase space Γ_{phys} is obtained from the kinematical phase space $\Gamma_{\text{kin}} = \{(H_{ij}, \psi_{ij}, \phi_{ij}, \Pi_{ij}, \Pi^{\psi}_{ij}, \Pi^{\phi}_{ij})\}$ by imposing the 4 first-class Hamiltonian and diffeomorphism constraints $\mathcal{C}$ and $\mathcal{C}_i$, as well as the 5+5 second-class constraints Π^{ψ}_{ij} and $\mathcal{C}^{\psi}_{ij}$. This leads to a

$$\begin{aligned} \dim(\Gamma_{\text{phys}}) &= \dim(\Gamma_{\text{kin}}) - 2n_{\text{first-class}} - n_{\text{second-class}} \\ &= (2\cdot 6 + 2\cdot 5 + 2\cdot 5) - 2\cdot 4 - (5+5) = 14 \end{aligned} \tag{A.72}$$

dimensional physical phase space. The last line in the expression of the Hamiltonian (A.64) indicates that there are 5 physical degrees of freedom with negative energy. Accounting for these would leave 2 physical degrees of freedom sufficient to describe a massless spin-2 particle such as the graviton in general relativity. In the next subsection we will solve the linearised Hamiltonian dynamics to confirm this expectation.

To that end, we first eliminate the constraints Π^{ψ}_{ij} and $\mathcal{C}^{\psi}_{ij}$ from the system by solving (A.69) for

$$\psi_{ij} = \frac{1}{m^2}\Big(\mathcal{F}_{ij} - C^{\psi}_{ij}\Big) \approx \frac{1}{m^2}\mathcal{F}_{ij} \tag{A.73}$$

on the constraint surface, and insert this expression into the Hamiltonian density (A.64). As Π^{ψ}_{ij} and $\mathcal{C}^{\psi}_{ij}$ are conjugated to each other, the Dirac brackets amount to the usual Poisson brackets when restricting to the phase space $\{(H_{ij}, \phi_{ij}, \Pi_{ij}, \Pi^{\phi}_{ij})\}$. Thereby we arrive at a reduced Hamiltonian density

$$\begin{aligned} \mathcal{H}' &= \frac{1}{2}\Big(\Pi_{ij} + 2c_\xi \widehat{\partial}\phi_{ij}\Big)^2 - \frac{1}{4}\Pi^2_{ii} + \frac{1}{2}(\partial_i H_{jk})^2 - (\partial_j H_{ij})^2 + \partial_j H_{ii}\partial_k H_{jk} - \frac{1}{2}(\partial_i H_{jj})^2 \\ &+ h_{00}\mathcal{C} - 2h_{i0}\mathcal{C}_i \\ &- \frac{1}{2}{\Pi^{\phi}_{ij}}^2 - \frac{1}{2}(\partial_i\phi_{jk})^2 - \frac{1}{2m^2}\mathcal{F}^2_{ij} - \frac{1}{2}m^2\phi^2_{ij}\,. \end{aligned} \tag{A.74}$$

It should be noted that all terms in the last line are negative definite and, moreover, ϕ_{ij} couples to the conjugated momentum of the spatial metric H_{ij}. We now proceed to analyse the dynamics generated by the reduced Hamiltonian (A.74).

A.2.4 Hamiltonian equations of motion and mode decomposition

The Hamiltonian equations of motion are given by

$$\dot{H}_{ij} = \{H_{ij}, \mathcal{H}'\} = \Pi_{ij} + 2c_\xi \widehat{\partial}\phi_{ij} - \frac{1}{2}\delta_{ij}\Pi_{kk} + 2\delta_{k(i}\delta_{j)l}\partial_k h_{l0} - \frac{2s_\xi}{m^2}\widehat{\partial}\mathcal{F}_{ij}\,, \tag{A.75}$$

$$\begin{aligned}\dot{\Pi}_{ij} = \{\Pi_{ij}, \mathcal{H}'\} &= \partial_k^2 H_{ij} - \partial_i\partial_j h_{00} + \delta_{ij}\partial_k^2 h_{00} - 2\partial_k\partial_{(i}H_{j)k} + \delta_{ij}\partial_k\partial_l H_{kl} \\ &\quad + \partial_i\partial_j H_{kk} - \delta_{ij}\partial_k^2 H_{ll} + \frac{2}{m^2}\widehat{\partial\partial}\mathcal{F}_{ij}\,,\end{aligned} \tag{A.76}$$

$$\dot{\phi}_{ij} = \{\phi_{ij}, \mathcal{H}'\} = -\Pi^\phi_{ij} - \frac{2}{m^2}c_\xi\widehat{\partial}\mathcal{F}_{ij}\,, \tag{A.77}$$

$$\dot{\Pi}^\phi_{ij} = \left\{\Pi^\phi_{ij}, \mathcal{H}'\right\} = \left(-\partial_k^2 - 4c_\xi^2\widehat{\partial}^2 + m^2\right)\phi_{ij} - 2c_\xi\widehat{\partial}\Pi_{ij} + \frac{4}{m^2}c_\xi s_\xi\widehat{\partial}^2\mathcal{F}_{ij}\,. \tag{A.78}$$

To solve these differential equations, we consider two types of mode decompositions for symmetric tensors τ_{ij}. These correspond to different choices of bases discussed in the context of gravitational-wave polarisations, see e.g. [235]. To that end, we first apply a Fourier transform in the spatial coordinates. This amounts to replacing $\partial_j \to ik_j$, where $\vec{k}$ is the spatial three-momentum wave vector with magnitude denoted by $k = |\vec{k}|$. The spatial Laplacian is $\partial_i^2 = -k^2$. Consider a Cartesian right-handed orthonormal coordinate frame such that the z-axis aligns with the direction of wave propagation, i.e.,

$$\left(\hat{x}, \hat{y}, \hat{z} = \frac{\vec{k}}{k}\right), \tag{A.79}$$

with

$$\begin{aligned}\hat{x}_i\hat{x}_i &= \hat{y}_i\hat{y}_i = \hat{z}_i\hat{z}_i = 1\,, \\ \hat{x}_i\hat{y}_i &= \hat{x}_i\hat{z}_i = \hat{y}_i\hat{z}_i = 0\,, \\ \epsilon_{ijk}\hat{x}_i\hat{y}_j\hat{z}_k &= +1\,.\end{aligned} \tag{A.80}$$

A generic symmetric tensor τ_{ij} can be decomposed into transverse-traceless (tt) and longitudinal-traceless (l) modes, plus a trace (tr) mode. This allows us to write τ_{ij} in the form

$$\tau_{ij} = \tau_{ij}^{tt} + \tau_{ij}^{l} + \tau_{ij}^{tr}\,. \tag{A.81}$$

In the following, we introduce orthonormal bases for each of the three components appearing in (A.81).

Bases for the transverse-traceless modes

For the transverse-traceless modes τ_{ij}^{tt}, we consider two different bases.

On the one hand, we consider a real orthonormal basis $\left(e_{ij}^{+}, e_{ij}^{\times}\right)$, referred to as the linear polarisation basis, and write

$$\tau_{ij}^{tt} = \tau^{+} e_{ij}^{+} + \tau^{\times} e_{ij}^{\times}\,, \tag{A.82}$$

where

$$e_{ij}^{+} \;=\; \frac{1}{\sqrt{2}}(\hat{x}_i\hat{x}_j - \hat{y}_i\hat{y}_j)\,, \tag{A.83}$$
$$e_{ij}^{\times} \;=\; \frac{1}{\sqrt{2}}(\hat{x}_i\hat{y}_j + \hat{y}_i\hat{x}_j)\,. \tag{A.84}$$

Orthonormality can be stated as $e_{ij}^{p} e_{ij}^{p'} = \delta^{pp'}$ for $p, p' \in \{+, \times\}$, and similarly for other othonormal basis sets introduced below.

Alternatively, we consider a complex orthonormal basis $\left(e_{ij}^{R}, e_{ij}^{L}\right)$, referred to as the circular polarisation basis, and write

$$\tau_{ij}^{tt} = \tau^{R} e_{ij}^{R} + \tau^{L} e_{ij}^{L}\,, \tag{A.85}$$

where

$$e_{ij}^{R} \;=\; \frac{1}{\sqrt{2}}\left(e_{ij}^{+} + i e_{ij}^{\times}\right), \tag{A.86}$$
$$e_{ij}^{L} \;=\; \frac{1}{\sqrt{2}}\left(e_{ij}^{+} - i e_{ij}^{\times}\right). \tag{A.87}$$

Bases for the longitudinal-traceless modes

For the longitudinal-traceless modes τ_{ij}^{l}, we can similarly consider two types of bases.

First, we consider a real othonormal basis $\left(e_{ij}^{lx}, e_{ij}^{ly}, e_{ij}^{ll}\right)$ and write

$$\tau_{ij}^{l} = \tau^{lx} e_{ij}^{lx} + \tau^{ly} e_{ij}^{ly} + \tau^{ll} e_{ij}^{ll}\,, \tag{A.88}$$

where

$$e_{ij}^{lx} = \frac{1}{\sqrt{2k^2}}(k_i\hat{x}_j + \hat{x}_i k_j)\,, \tag{A.89}$$

$$e_{ij}^{ly} = \frac{1}{\sqrt{2k^2}}(k_i\hat{y}_j + \hat{y}_i k_j)\,, \tag{A.90}$$

$$e_{ij}^{ll} = \sqrt{\frac{3}{2k^4}}\left(k_i k_j - \frac{1}{3}k^2\delta_{ij}\right). \tag{A.91}$$

These basis tensors are orthogonal to the basis tensors for transverse-traceless modes.

From the definitions in (A.89)–(A.91), we can alternatively construct a complex basis $\left(e_{ij}^{l+}, e_{ij}^{l-}, e_{ij}^{ll}\right)$ and write

$$\tau_{ij}^{l} = \tau^{l+}e_{ij}^{l+} + \tau^{l-}e_{ij}^{l-} + \tau^{ll}e_{ij}^{ll}\,, \tag{A.92}$$

where

$$e_{ij}^{l\pm} = \frac{1}{\sqrt{2}}\left(e_{ij}^{lx} \pm i e_{ij}^{ly}\right), \tag{A.93}$$

$$e_{ij}^{ll} = \sqrt{\frac{3}{2k^4}}\left(k_i k_j - \frac{1}{3}k^2\delta_{ij}\right). \tag{A.94}$$

Basis for the trace mode

For the trace mode τ_{ij}^{tr}, we choose an orthonormal basis consisting of a single basis tensor $\left(e_{ij}^{tr}\right)$, and write

$$\tau_{ij}^{tr} = \tau^{tr}e_{ij}^{tr}\,, \tag{A.95}$$

where

$$e_{ij}^{tr} = \frac{1}{\sqrt{3}}\delta_{ij}\,. \tag{A.96}$$

This basis tensor is orthogonal to all the basis tensors for transverse-traceless and longitudinal-traceless modes.

Action of $\widehat{\partial}$ and $\widehat{\partial\partial}$ on the various modes

To evaluate the Hamiltonian equations of motion (A.75)–(A.78), we have to consider the action of the spatial derivative operator $\widehat{\partial}$ defined in (A.52), on the various modes of a symmetric tensor τ_{ij}. This action is given for the transverse-traceless modes by

$$\begin{aligned} \widehat{\partial} e_{ij}^{+,\times} &= \pm ik e_{ij}^{\times,+}\,, && \text{(A.97)} \\ \widehat{\partial} e_{ij}^{R,L} &= \pm k e_{ij}^{R,L}\,. && \text{(A.98)} \end{aligned}$$

Thus, the spatial derivative operator $\widehat{\partial}$ mixes the two linear polarisation modes τ_{ij}^{+} and $\tau_{ij}^{\times}$, but acts diagonally on the two circular polarisation modes τ_{ij}^{R} and τ_{ij}^{L}. Moreover, $\widehat{\partial}^2$ acts on the transverse-traceless tensor τ_{ij}^{tt} as the negative spatial Laplacian,

$$\widehat{\partial}^2 \tau_{ij}^{tt} = -\partial_k^2 \tau_{ij}^{tt} = k^2 \tau_{ij}^{tt}\,. \tag{A.99}$$

The action of $\widehat{\partial}$ on the longitudinal-traceless modes is given by

$$\begin{aligned} \widehat{\partial} e_{ij}^{lx,y} &= \pm \frac{ik}{2} e_{ij}^{ly,x}\,, && \text{(A.100)} \\ \widehat{\partial} e_{ij}^{l\pm} &= \pm \frac{k}{2} e_{ij}^{l\pm}\,, && \text{(A.101)} \\ \widehat{\partial} e_{ij}^{ll} &= 0\,. && \text{(A.102)} \end{aligned}$$

Thus, $\widehat{\partial}$ mixes the two modes τ_{ij}^{lx} and τ_{ij}^{ly}, but acts diagonally on the two modes τ_{ij}^{l+} and τ_{ij}^{l-}. Finally, we remind the reader that $\widehat{\partial}$ annihilates the trace mode,

$$e_{ij}^{tr} = 0\,. \tag{A.103}$$

The Hamiltonian equations of motion (A.75)–(A.78) involve furthermore the double spatial derivative operator $\widehat{\partial\partial}$ defined in (A.65). This operator acts on the transverse-traceless modes as

$$\widehat{\partial\partial} e_{ij}^{+,\times,R,L} = k^2 e_{ij}^{+,\times,R,L}\,. \tag{A.104}$$

Thus, the action of $\widehat{\partial\partial}$ on the transverse-traceless tensor τ_{ij}^{tt} is identical to the one of $\widehat{\partial}^2$ and given by the negative spatial Laplacian,

$$\widehat{\partial\partial} \tau_{ij}^{tt} = \widehat{\partial}^2 \tau_{ij}^{tt} = k^2 \tau_{ij}^{tt}\,. \tag{A.105}$$

Proceeding with the longitudinal-traceless modes, we compute

$$\begin{aligned} \widehat{\partial\partial} e_{ij}^{lx,ly,l\pm} &= 0\,, && \text{(A.106)} \\ \widehat{\partial\partial} \tau_{ij}^{ll} &= -\frac{k^2}{3} \tau_{ij}^{ll} + \frac{\sqrt{2}k^2}{3} \tau_{ij}^{tr}\,, && \text{(A.107)} \end{aligned}$$

whereas for the trace mode we compute

$$\widehat{\partial\partial} e^{tr}_{ij} = \frac{\sqrt{2}k^2}{3} e^{ll}_{ij} - \frac{2}{3} k^2 e^{tr}_{ij}\,. \tag{A.108}$$

Thus, $\widehat{\partial\partial}$ mixes the double-longitudinal mode τ^{ll}_{ij} and the trace mode τ^{tr}_{ij}.

The previous identities allow us to solve the Hamiltonian equations of motion (A.75)–(A.78) for the dynamical variables $\left(H_{ij}, \Pi_{ij}, \phi_{ij}, \Pi^{\phi}_{ij}\right)$.

In subsection A.2.5, we analyse the Hamiltonian dynamics in the complex circular polarisation basis

$$\left(e^R_{ij}, e^L_{ij}, e^{l+}_{ij}, e^{l-}_{ij}, e^{ll}_{ij}, e^{tr}_{ij}\right)\,. \tag{A.109}$$

In subsection A.2.6, we analyse additionally the Hamiltonian dynamics of the transverse-traceless modes in the real linear polarisation basis

$$\left(e^{+}_{ij}, e^{\times}_{ij}, e^{l\theta}_{ij}, e^{l\varphi}_{ij}, e^{ll}_{ij}, e^{tr}_{ij}\right)\,. \tag{A.110}$$

To state the solution to the Hamiltonian equations, we will use that the general solution to the differential equation

$$\dot{\vec{X}}(\tau) = A \cdot \vec{X}(\tau) \tag{A.111}$$

with initial conditions $\vec{X}(0) = \vec{X}_0$, for a τ-independent matrix A, is given by

$$\vec{X} = \exp(A\tau) \cdot \vec{X}_0\,. \tag{A.112}$$

Moreover, let us recall that the Hamiltonian for a non-relativistic particle in a quadratic potential can be written as

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right)\,, \tag{A.113}$$

where q and p denote the position and momentum of the particle, and $\omega \neq 0$. The Hamiltonian equations of motion are

$$\dot{\vec{X}} = A \cdot \vec{X} \qquad \text{where} \qquad \vec{X} = (q,p)^T \qquad \text{and} \qquad A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}. \tag{A.114}$$

The matrix A has eigenvalues $\pm i\omega$ with corresponding eigenvectors $\vec{X}_E$ given by

$$\vec{X}_E(\pm i\omega) = \left(\mp\frac{i}{\omega},\, 1\right)^T . \tag{A.115}$$

According to equation (A.112), the classical motion in the real variables $q(\tau)$ and $p(\tau)$ is bounded for generic initial data $\vec{X}_0 = (q_0, p_0)^T$ provided ω is real, such that $\omega^2 > 0$. In this case, the system describes a simple harmonic oscillator with frequency ω. If ω is imaginary and therefore $\omega^2 < 0$, the potential term in the Hamiltonian (A.113) has the wrong sign and the motion in phase space is unbounded.

A.2.5 Evolution in the circular polarisation basis

In the circular polarisation basis, we decompose the dynamical variables in the form

$$\begin{aligned}
H_{ij} &= H^R e^R_{ij} + H^L e^L_{ij} + H^{l+} e^{l+}_{ij} + H^{l-} e^{l-}_{ij} + H^{ll} e^{ll}_{ij} + H^{tr} e^{tr}_{ij}\,, && \text{(A.116)}\\
\Pi_{ij} &= \Pi^R e^R_{ij} + \Pi^L e^L_{ij} + \Pi^{l+} e^{l+}_{ij} + \Pi^{l-} e^{l-}_{ij} + \Pi^{ll} e^{ll}_{ij} + \Pi^{tr} e^{tr}_{ij}\,, && \text{(A.117)}\\
\phi_{ij} &= \phi^R e^R_{ij} + \phi^L e^L_{ij} + \phi^{l+} e^{l+}_{ij} + \phi^{l-} e^{l-}_{ij} + \phi^{ll} e^{ll}_{ij}\,, && \text{(A.118)}\\
\Pi^{\phi}_{ij} &= \Pi^{\phi R} e^R_{ij} + \Pi^{\phi L} e^L_{ij} + \Pi^{\phi l+} e^{l+}_{ij} + \Pi^{\phi l-} e^{l-}_{ij} + \Pi^{\phi ll} e^{ll}_{ij}\,, && \text{(A.119)}
\end{aligned}$$

where we have used that ϕ and Π^ϕ are traceless. The derivative operators $\widehat{\partial}$ and $\widehat{\partial\partial}$ act diagonally on each of the modes labelled by R, L, $l+$, and $l-$. Moreover, $\widehat{\partial}$ annihilates the modes labelled by ll and tr, whereas $\widehat{\partial\partial}$ mixes these two. We may therefore analyse the dynamics of the R and L modes, and of the $l+$ and $l-$ modes as four closed subsystems independent from the dynamics of remaining two modes. The latter form another closed subsystem (ll, tr).

Evolution of the transverse-traceless modes R and L

For the transverse-traceless R and L modes, we can write the Hamiltonian equations (A.75)–(A.78), taking into account the previously determined action of the derivative operators $\widehat{\partial}$ and $\widehat{\partial\partial}$, in the form

$$\dot{\vec{X}}^{R,L} = A^{R,L} \cdot \vec{X}^{R,L} \qquad \text{where} \qquad \vec{X}^{R,L} = \left(H^{R,L},\, \Pi^{R,L},\, \phi^{R,L},\, \Pi^{\phi R,L}\right)^T \tag{A.120}$$

and

$$A^{R,L} = \begin{pmatrix}
\mp\frac{4s_\xi k^3}{m^2} & 1 - \frac{4s_\xi^2 k^2}{m^2} & \pm 2c_\xi k \mp \frac{8c_\xi s_\xi^2 k^3}{m^2} & -\frac{4c_\xi s_\xi k^2}{m^2} \\
-k^2 + \frac{4k^4}{m^2} & \pm\frac{4s_\xi k^3}{m^2} & \frac{8c_\xi s_\xi k^4}{m^2} & \pm\frac{4c_\xi k^3}{m^2} \\
\mp\frac{4c_\xi k^3}{m^2} & -\frac{4c_\xi s_\xi k^2}{m^2} & \mp\frac{8c_\xi^2 s_\xi k^3}{m^2} & -1 - \frac{4c_\xi^2 k^2}{m^2} \\
\frac{8c_\xi s_\xi k^4}{m^2} & \mp 2c_\xi k \pm \frac{8c_\xi s_\xi^2 k^3}{m^2} & \left(1 - 4c_\xi^2\right)k^2 + \frac{16c_\xi^2 s_\xi^2 k^4}{m^2} & \pm\frac{8c_\xi^2 s_\xi k^3}{m^2}
\end{pmatrix} .$$

Here, and in following expressions, $c_\xi(s_\xi) = \sqrt{1+s_\xi^2}$. The matrices $A^{R,L}$ are diagonalisable and have identical eigenvalues given by

$$\left\{+ik,\, -ik,\, +i\sqrt{k^2+m^2},\, -i\sqrt{k^2+m^2}\right\}. \tag{A.121}$$

We can explicitly compute the eigenvectors and consider their expansion for large values of the mass. A dimensionless expansion parameter is $\frac{k^2}{m^2} \ll 1$.

The eigenvectors $\vec{X}_E^{R,L}$ of $A^{R,L}$ associated with the first pair of eigenvalues are given by

$$\begin{aligned} \vec{X}_E^R(\pm ik) &= (1,\, \pm ik,\, 0,\, 0)^T + \frac{k^2}{m^2}(0,\, +4(s_\xi \mp i)k,\, \pm 2ic_\xi,\, \mp 2c_\xi(2is_\xi \pm 1)k)^T \\ &+ \mathcal{O}\left(\frac{k^3}{m^3}\right), \end{aligned} \tag{A.122}$$

$$\begin{aligned} \vec{X}_E^L(\pm ik) &= (1,\, \pm ik,\, 0,\, 0)^T + \frac{k^2}{m^2}(0,\, -4(s_\xi \pm i)k,\, \mp 2ic_\xi,\, \mp 2c_\xi(2is_\xi \mp 1)k)^T \\ &+ \mathcal{O}\left(\frac{k^3}{m^3}\right). \end{aligned} \tag{A.123}$$

In the limit of infinite mass, the dynamics of the two massless transverse-traceless modes approaches the one of the pure-gravitational system associated with the variables $H^{R,L}$ and $\Pi^{R,L}$. We remind the reader that H_{ij} is the shifted spatial metric defined in (A.57) and Π_{ij} its conjugate momentum given in (A.59). To higher orders, however, the eigenvectors of the massless transverse-traceless modes for generic values of the coupling ξ receive a correction term in the $\Pi^{R,L}$ component, and also non-vanishing $\phi^{R,L}$ and $\Pi^{\phi R,L}$ components.

Similarly, we can expand the eigenvectors $\vec{X}_E^{R,L}$ of $A^{R,L}$ associated with the second pair of eigenvalues as

$$\vec{X}_E^R\left(\pm i\sqrt{k^2+m^2}\right) = \left(0,\, 0,\, \pm\frac{i}{m},\, 1\right)^T + \frac{k^2}{m^2}\left(\frac{2c_\xi}{k},\, 0,\, 0,\, 0\right)^T + \mathcal{O}\left(\frac{k^3}{m^3}\right), \tag{A.124}$$

$$\vec{X}_E^L\left(\pm i\sqrt{k^2+m^2}\right) = \left(0,\, 0,\, \pm\frac{i}{m},\, 1\right)^T + \frac{k^2}{m^2}\left(-\frac{2c_\xi}{k},\, 0,\, 0,\, 0\right)^T + \mathcal{O}\left(\frac{k^3}{m^3}\right). \tag{A.125}$$

In the limit of infinite mass, the massive transverse-traceless modes describe simple harmonic oscillators. Their dynamics approaches the one of the non-metric degrees of freedom $\phi^{R,L}$ and their conjugate momenta $\Pi^{\phi R,L}$. To subleading order, however, the eigenvectors of the massive transverse-traceless modes receive a non-vanishing $H^{R,L}$ component.

Altogether, the spectrum of transverse-traceless modes consists of two massless propagating degrees of freedom and two massive propagating degres of freedom. To lowest order in k^2/m^2, these can be identified with the transverse-traceless right-left handed components of the shifted spatial metric H_{ij} and the non-metric degrees of freedom ϕ_{ij}. The eigenvalues of the Hamiltonian evolution matrices are independent of the coupling parameter ξ and the classical motion in phase space is stable. The parameter ξ also does not lead to modified dispersion relations.

Evolution of the longitudinal-traceless modes $l+$ and $l-$

In the next step, we consider the Hamiltonian dynamics of the longitudinal-traceless modes $l+$ and l_-. From the equation of motion for H_{ij} in (A.75) and the action of the derivative operators $\widehat{\partial}$ and $\widehat{\partial\partial}$, it follows that these modes form two closed subsystems, up to a term involving the parameters h_{i0}. These parameters were previously identified as Lagrange multipliers imposing the diffeomorphism constraint $\mathcal{C}_i$ in (A.66). Using the orthogonality of the spatial basis vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$, as well as the definitions of the various basis tensors e_{ij} in the decomposition of Π_{ij} as in (A.117), this constraint can be rewritten as

$$\begin{aligned}
\mathcal{C}_i &= ik_j\Pi_{ij} = ik_j\big(\Pi^{l+}e^{l+}_{ij} + \Pi^{l-}e^{l-}_{ij} + \Pi^{ll}e^{ll}_{ij} + \Pi^{tr}e^{tr}_{ij}\big) \\
&= \Pi^{l+}\frac{ik}{2}(\hat{x}_i + i\hat{y}_i) + \Pi^{l-}\frac{ik}{2}(\hat{x}_i - i\hat{y}_i) + \Pi^{ll}\sqrt{\frac{2}{3}}ik_i + \Pi^{tr}\frac{1}{\sqrt{3}}ik_i \\
&= \frac{ik}{2}\big(\Pi^{l+} + \Pi^{l-}\big)\hat{x}_i - \frac{k}{2}\big(\Pi^{l+} - \Pi^{l-}\big)\hat{y}_i + \frac{i}{\sqrt{3}}\Big(\sqrt{2}\Pi^{ll} + \Pi^{tr}\Big)k_i\,. \qquad \text{(A.126)}
\end{aligned}$$

The linear independence of the basis for 3-vectors implies that each coefficient in front of $\hat{x}_i$, $\hat{y}_i$, and k_i has to vanish separately in order to guarantee $\mathcal{C}_i = 0$. Thus, we conclude that

$$\Pi^{l\pm} = 0\,. \qquad \text{(A.127)}$$

Similarly, we can apply the mode expansion to the equation of motion for $\dot{P}_{ij}$ in (A.76) to conclude that

$$\dot{\Pi}^{l\pm} = 0\,. \qquad \text{(A.128)}$$

Alternatively, this result follows from equation (A.67), which states that the diffeomorphism constraint is preseved during time evolution. The only $l+$ and $l-$ modes left are $\phi^{l\pm}$ and $\Pi^{\phi l\pm}$, as well as $H^{l\pm}$. Using the action of $\widehat{\partial}$ and $\widehat{\partial\partial}$, the equations of motion (A.77) and (A.78) for the first two variables can be written as

$$\dot{\vec{X}}^{l\pm} = A^{l\pm}\cdot\vec{X}^{l\pm} \qquad \text{where} \qquad \vec{X}^{l\pm} = \big(\phi^{l\pm},\,\Pi^{\phi l\pm}\big)^T \qquad \text{(A.129)}$$

and

$$A^{l\pm} = \begin{pmatrix} \mp\frac{c_\xi^2 s_\xi k^3}{m^2} & -1 - \frac{c_\xi^2 k^2}{m^2} \\ -s_\xi^2 k^2 + m^2 + \frac{c_\xi s_\xi^2 k^4}{m^2} & \pm\frac{c_\xi^2 s_\xi k^3}{m^2} \end{pmatrix}.$$

The matrices $A^{l\pm}$ are diagonalisable and have identical eigenvalues given by

$$\left\{+i\sqrt{k^2+m^2},\ -i\sqrt{k^2+m^2}\right\}. \tag{A.130}$$

Computing the eigenvectors $\vec{X}_E^{l\pm}$ of $A^{l\pm}$ and these expanding in inverse powers of the mass leads to

$$\vec{X}_E^{l+}\left(\pm i\sqrt{k^2+m^2}\right) = \vec{X}_E^{l-}\left(\pm i\sqrt{k^2+m^2}\right) = \left(\pm\frac{i}{m},\, 1\right)^T + \mathcal{O}\left(\frac{k^3}{m^3}\right). \tag{A.131}$$

Altogether, the subsystems of longitudinal $l+$ and $l-$ modes associated with the variables $\phi^{l\pm}$ and their conjugate momenta $\Pi^{\phi l\pm}$ describe two massive propagating degrees of freedom, whose dynamics is stable.

Finally, there is the evolution of the modes $H^{l\pm}$. According to the equation of motion (A.75), the oscillations of the $l+$ and $l-$ modes $\phi^{l\pm}$ and $\Pi^{\phi l\pm}$ induce a time evolution for the variables $H^{l\pm}$ given by

$$\dot{H}^{l\pm} = \pm\left(c_\xi k - \frac{c_\xi s_\xi k^3}{m^2}\right)\phi^{l\pm} - \frac{c_\xi s_\xi k^2}{m^2}\Pi^{\phi l\pm} + 2ik_{(i}h_{j)0}e_{ij}^{l\pm}\,. \tag{A.132}$$

This indicates that by a suitable choice of the gauge parameters h_{i0} and initial data, we can achieve

$$\dot{H}^{l\pm} = 0 \qquad \text{and} \qquad H^{l\pm} = 0\,. \tag{A.133}$$

Evolution of the longitudinal-traceless and trace modes ll and tr

It remains to discuss the longitudinal-traceless mode ll and trace mode tr. To that end, we make use of the Hamiltonian constraint $\mathcal{C}$ in (A.66). Using the orthogonality of the spatial basis vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$, the definitions of the various basis tensors e_{ij} in the decomposition of H_{ij} as in (A.116), and $H^{l\pm}=0$, this constraint can be rewritten as

$$\begin{aligned}\mathcal{C} &= -k_ik_j(H_{ij}-\delta_{ij}H_{kk}) = -k_ik_j\left(H^{ll}e_{ij}^{ll} + H^{tr}e_{ij}^{tr} - \sqrt{3}H^{tr}\delta_{ij}\right)\\ &= -\sqrt{\frac{2}{3}}k^2\left(H^{ll}-\sqrt{2}H^{tr}\right).\end{aligned} \tag{A.134}$$

In addition, there is the condition that the expression in brackets in the last term of equation (A.126) vanishes. Therefore, the Hamiltonian and diffeomorphism constraints imply

$$H^{ll} - \sqrt{2}H^{tr} = 0 \qquad \text{and} \qquad \sqrt{2}\Pi^{ll} + \Pi^{tr} = 0\,. \tag{A.135}$$

The Poisson brackets of these constraint with the Hamiltonian (A.67) state that the time derivative of these constraints vanishes. In other words, knowing the time evolution of H^{ll} and Π^{ll} we can infer the time evolution of H^{tr} and Π^{tr}. It is therefore sufficient to focus on the dynamical equations for the H^{ll} and P^{ll} modes. Moreover, is is sufficient to consider the equations of motion for the ϕ^{ll} and $\Pi^{\phi ll}$ modes of the field ϕ_{ij} and its conjugate momentum Π^{ϕ}_{ij}, as the trace mode of these tensors vanishes.

The time evolution of the double-longitudinal modes H^{ll} and Π^{ll} is decoupled from the time evolution of ϕ^{ll} and $\Pi^{\phi ll}$. According to (A.75) and (A.76), the equations of motion for the first two variables are

$$\dot{H}^{ll} = \Pi^{ll} + 2i\sqrt{\frac{2}{3}}k_i h_{i0} \qquad \text{and} \qquad \dot{\Pi}^{ll} = \sqrt{\frac{2}{3}}k^2 h_{00}\,. \tag{A.136}$$

Thus, by a suitable choice of the gauge parameters h_{00} and h_{i0} and initial data, we can achieve

$$\dot{H}^{ll} = H^{ll} \;=\; 0 \qquad \text{and} \qquad \dot{\Pi}^{ll} = \Pi^{ll} = 0\,, \tag{A.137}$$

$$\dot{H}^{tr} = H^{tr} \;=\; 0 \qquad \text{and} \qquad \dot{\Pi}^{tr} = \Pi^{tr} = 0\,. \tag{A.138}$$

Finally, the equations of motion (A.77) and (A.78) for ϕ^{ll} and $\Pi^{\phi ll}$ can be written as

$$\dot{\vec{X}}^{ll} = A^{ll}\cdot\vec{X}^{ll},\,, \qquad \text{where} \qquad \vec{X}^{ll} = \left(\phi^{ll},\,\Pi^{\phi ll}\right)^T \tag{A.139}$$

and

$$A^{ll} = \begin{pmatrix} 0 & -1 \\ k^2+m^2 & 0 \end{pmatrix}.$$

The matrix A^{ll} is diagonalisable with eigenvalues given by

$$\left\{+i\sqrt{k^2+m^2},\,-i\sqrt{k^2+m^2}\right\}. \tag{A.140}$$

The eigenvectors $\vec{X}^{ll}_E$ of the matrix A^{ll} to leading order in inverse powers of the mass are given by

$$\vec{X}^{ll}_E\left(\pm i\sqrt{k^2+m^2}\right) = \left(\pm\frac{i}{m},\,1\right)^T + \mathcal{O}\left(\frac{k^3}{m^3}\right). \tag{A.141}$$

Altogether, the subsystem of double-longitudinal ll modes associated with the variable ϕ^{ll} and its conjugate momentum $\Pi^{\phi ll}$ describes one additional massive propagating degree of freedom, whose dynamics is stable.

A.2.6 Evolution in the linear polarisation basis

In the linear polarisation basis, we decompose the dynamical variables in the form

$$
\begin{aligned}
H_{ij} &= H^{+}e^{+}_{ij} + H^{\times}e^{\times}_{ij} + H^{lx}e^{lx}_{ij} + H^{ly}e^{ly}_{ij} + H^{ll}e^{ll}_{ij} + H^{tr}e^{tr}_{ij}\,, && \text{(A.142)}\\
\Pi_{ij} &= \Pi^{+}e^{+}_{ij} + \Pi^{\times}e^{\times}_{ij} + \Pi^{lx}e^{lx}_{ij} + \Pi^{ly}e^{ly}_{ij} + \Pi^{ll}e^{ll}_{ij} + \Pi^{tr}e^{tr}_{ij}\,, && \text{(A.143)}\\
\phi_{ij} &= \phi^{+}e^{+}_{ij} + \phi^{\times}e^{\times}_{ij} + \phi^{lx}e^{lx}_{ij} + \phi^{ly}e^{ly}_{ij} + \phi^{ll}e^{ll}_{ij}\,, && \text{(A.144)}\\
\Pi^{\phi}_{ij} &= \Pi^{\phi+}e^{+}_{ij} + \Pi^{\phi\times}e^{\times}_{ij} + \Pi^{\phi lx}e^{lx}_{ij} + \Pi^{\phi ly}e^{ly}_{ij} + \Pi^{\phi ll}e^{ll}_{ij}\,, && \text{(A.145)}
\end{aligned}
$$

where we have used that ϕ_{ij} and Π^{ϕ}_{ij} are traceless. The derivative operator $\widehat{\partial}$ mixes the pairs of modes labelled by $(+,\times)$ and (lx, ly), and annihilates the double-longitudinal mode (ll) and the trace mode (tr). The derivative operator $\widehat{\partial\partial}$ acts diagonally on the pairs $(+,\times)$ and (lx, ly), and mixes the pair (ll, tr). We can therefore analyse the dynamics by focusing separately on the three closed subsystems $(+,\times)$, $(l\theta, l\varphi)$, and (ll, tr). The analysis of the longitudinal and trace sectors can be done analogously as in the helicity basis. In the following, we consider only the evolution of the transverse-traceless modes in the duality basis.

Evolution of the transverse-traceless modes + and ×

For the transverse-traceless $+$ and $\times$ modes, we can write the Hamiltonian equations (A.75)–(A.78), taking into account the previously determined actions of the derivative operators $\hat{\partial}$ and $\widehat{\partial\partial}$, in the form

$$
\dot{\vec{X}}^{+\times} = A^{+\times}\cdot\vec{X}^{+\times} \qquad \text{where} \qquad \vec{X}^{+\times} = \left(H^{+},\,\Pi^{+},\,H^{\times},\,\Pi^{\times},\,\phi^{+},\,\Pi^{\phi+},\,\phi^{\times},\,\Pi^{\phi\times}\right)^{T} \tag{A.146}
$$

and $A^{+\times} = \left(A^{+\times}_1\;\; A^{+\times}_2\right)$, with

$$
A^{+\times}_1 = \begin{pmatrix}
0 & 1-\frac{4s_\xi^2k^2}{m^2} & \frac{4is_\xi k^3}{m^2} & 0\\
-k^2+\frac{4k^4}{m^2} & 0 & 0 & -\frac{4is_\xi k^3}{m^2}\\
-\frac{4is_\xi k^3}{m^2} & 0 & 0 & 1-\frac{4s_\xi^2k^2}{m^2}\\
0 & \frac{4is_\xi k^3}{m^2} & -k^2+\frac{4k^4}{m^2} & 0\\
0 & -\frac{4c_\xi s_\xi k^2}{m^2} & \frac{4ic_\xi k^3}{m^2} & 0\\
\frac{8c_\xi s_\xi k^4}{m^2} & 0 & 0 & 2ic_\xi k-\frac{8ic_\xi s_\xi^2k^3}{m^2}\\
-\frac{4ic_\xi k^3}{m^2} & 0 & 0 & -\frac{4c_\xi s_\xi k^2}{m^2}\\
0 & -2ic_\xi k+\frac{8ic_\xi s_\xi^2k^3}{m^2} & \frac{8c_\xi s_\xi k^4}{m^2} & 0
\end{pmatrix}
$$

and

$$
A_2^{+\times} = \begin{pmatrix}
0 & -\frac{4c_\xi s_\xi k^2}{m^2} & -2ic_\xi k + \frac{8ic_\xi s_\xi^2 k^3}{m^2} & 0 \\
\frac{8c_\xi s_\xi k^4}{m^2} & 0 & 0 & -\frac{4ic_\xi k^3}{m^2} \\
2ic_\xi k - \frac{8ic_\xi s_\xi^2 k^3}{m^2} & 0 & 0 & -\frac{4c_\xi s_\xi k^2}{m^2} \\
0 & \frac{4is_\xi k^3}{m^2} & \frac{8c_\xi s_\xi k^4}{m^2} & 0 \\
0 & -1 - \frac{4c_\xi^2 k^2}{m^2} & \frac{8is_\xi c_\xi^2 k^3}{m^2} & 0 \\
\left(1-4c_\xi^2\right)k^2 + m^2 + \frac{16c_\xi^2 s_\xi^2 k^4}{m^2} & 0 & 0 & -\frac{8ic_\xi^2 s_\xi k^3}{m^2} \\
-\frac{8ic_\xi^2 s_\xi k^3}{m^2} & 0 & 0 & -1 - \frac{4c_\xi^2 k^2}{m^2} \\
0 & \frac{8ic_\xi^2 s_\xi k^3}{m^2} & \left(1-4c_\xi^2\right)k^2 + m^2 + \frac{16c_\xi^2 s_\xi^2 k^4}{m^2} & 0
\end{pmatrix}.
$$

The matrix $A^{+\times}$ is diagonalisable with eigenvalues identical to the ones of the combined (R, L) system,

$$
\left\{+ik,\ -ik,\ +ik,\ -ik,\ +i\sqrt{k^2+m^2},\ -i\sqrt{k^2+m^2},\ +i\sqrt{k^2+m^2},\ -i\sqrt{k^2+m^2}\right\}. \tag{A.147}
$$

The eigenvectors $\vec{X}_E^{+\times}$ associated with the first and second pair of eigenvalues can be expanded in inverse powers of the mass as

$$
\begin{aligned}
\vec{X}_{E_1}^{+\times}(\pm ik) &= (1,\ \pm ik,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)^T \\
&+ \frac{k^2}{m^2}(0,\ \mp 4ik,\ 0,\ 4is_\xi k,\ 0,\ \mp 4is_\xi c_\xi k,\ \mp 2c_\xi,\ -2ic_\xi k)^T + \mathcal{O}\left(\frac{k^3}{m^3}\right), && \text{(A.148)} \\
\vec{X}_{E_2}^{+\times}(\pm ik) &= (0,\ 0,\ 1,\ \pm ik,\ 0,\ 0,\ 0,\ 0)^T \\
&+ \frac{k^2}{m^2}(0,\ \mp 4is_\xi k\,,0,\ \mp 4ik,\ \pm 2c_\xi,\ 2ic_\xi k,\ 0,\ \mp 4is_\xi c_\xi k)^T + \mathcal{O}\left(\frac{k^3}{m^3}\right). && \text{(A.149)}
\end{aligned}
$$

Similarly as in the circular polarisation basis, in the limit of infinite mass the dynamics of two massless transverse-traceless modes approaches the one of the pure-gravitational system associated with the variables $H^{+,\times}$ and $\Pi^{+,\times}$. At the next order, the eigenvectors associated with the massless transverse-traceless modes receive a correction term in the $\Pi^{+,\times}$ entries, respectively, which is independent of the coupling ξ.

Morover, we observe that at subleading order there is a mixing between the + and × polarisations of the shifted spatial metric H_{ij}. Using (A.46) and (A.45), this mixing is quantified by the parameter

$$
s_\xi = \sinh(2\xi) = 2\alpha_1\alpha_2 = 2\mathrm{Re}[\alpha_+]\mathrm{Im}[\alpha_+]\,, \tag{A.150}
$$

which is non-zero only if both the real and imaginary parts of the coupling α_+ are non-zero. Therefore, this parameter is a manifestation of parity-violation in the original area-metric Lagrangian. It should be emphasised that this mixing effect is established here for the $(+,\times)$

modes of the shifted spatial metric H_{ij} defined in (A.57).

For completeness, we state the eigenvectors associated with the third and fourth pair of eigenvalues of the matrix $A^{+\times}$,

$$\begin{aligned}\vec{X}^{+\times}_{E_3}(\pm i\sqrt{k^2+m^2}) &= \left(0,\,0,\,0,\,0,\,\pm\frac{i}{m},\,1,\,0,\,0\right)^T \\ &+ \frac{k^2}{m^2}\left(0,\,0,\,\pm\frac{2ic_\xi}{k},\,0,\,0,\,\mp\frac{1}{2}\left(8s_\xi^2+7\right),\,0,\,0\right)^T + \mathcal{O}\left(\frac{k^3}{m^3}\right), \qquad \text{(A.151)}\\ \vec{X}^{+\times}_{E_4}(\pm i\sqrt{k^2+m^2}) &= \left(0,\,0,\,0,\,0,\,0,\,0,\,\pm\frac{i}{m},\,1\right)^T \\ &+ \frac{k^2}{m^2}\left(\mp\frac{2ic_\xi}{k},\,0,\,0,\,0,\,0,\,0,\,0,\,\mp\frac{1}{2}\left(8s_\xi^2+7\right)\right)^T + \mathcal{O}\left(\frac{k^3}{m^3}\right). \qquad \text{(A.152)}\end{aligned}$$

To leading order, these eigenvectors correspond to two massive degrees of freedom associated with the $+$ and $\times$ polarisations of the field ϕ_{ij}. At subleading order, the $\phi^{+,\times}$-dominated eigenmodes receive a non-vanishing $H^{\times,+}$ component.

In summary, the physical spectrum of the shift-symmetric area-metric actions with identical masses for the Weyl components consist of two massless and two massive transverse-traceless propagating degrees of freedom, as well as three massive longitudinal-traceless propagating degrees of freedom. The energy eigenvalues are determined by the usual dispersion relations for massless and massive particles, and are independent of the coupling between the trace components and the Weyl components of the area metric. Despite the negative-sign kinetic and mass terms for a subset of the non-metric degrees of freedom of the area metric, the classical motion in phase space is bounded.

A.3 Third-order momentum-independent area-metric contractions

In this section, we derive all scalar invariants at third order in the perturbations of a cyclic area metric. To that end, we make use of the decomposition (A.3) of the tensor $a_{\mu\nu\rho\sigma}$ into the scalar h and the tensors $\hat{h}_{\mu\nu}$ and $\omega^{\pm}_{\mu\nu\rho\sigma}$ associated with the SO(4) irreducible representations appearing in (A.12). For convenience, we remind the reader of these explicitly,

$$a \;\leftrightarrow\; \left(h, \hat{h}, \omega^+, \omega^-\right) \;\in\; (0,0)\oplus(1,1)\oplus(2,0)\oplus(0,2)\,. \qquad \text{(A.153)}$$

Third-order scalar invariants require the irreducible decomposition of a tensor product of three copies of representations appearing in (A.153) to contain a singlet $(0,0)$.

First, we consider tensor products involving the scalar h, which is in the $(0,0)$ representation. In this case, the tensor product between the remaining two factors of fields must include the singlet representation. This is only possible if these two fields are in the same representation. Therefrom, we obtain four invariants in which h couples to the square of each field,

- h^3 ,
- $\hat{h}_{\mu\nu}\hat{h}^{\mu\nu}h$,
- $\omega^+_{\mu\nu\rho\sigma}\omega^{+\mu\nu\rho\sigma}h$,
- $\omega^-_{\mu\nu\rho\sigma}\omega^{-\mu\nu\rho\sigma}h$.

These all the terms in which the scalar h can appear.

Next, we consider combinations involving the tensor $\hat{h}$. The scalar invariant in which this tensor couples to the scalar h is already captured by one of the previous terms. Thus, we proceed and consider terms with two factors of $\hat{h}$. The corresponding tensor product between two $(1,1)$ representations decomposes in the same way as in (A.15) into

$$\begin{aligned} \hat{h}\otimes\hat{h} \;\in\; (1,1)\otimes(1,1) \;&=\; (0,0)\oplus(0,1)\oplus(0,2) \\ &\oplus\; (1,0)\oplus(1,1)\oplus(1,2) \\ &\oplus\; (2,0)\oplus(2,1)\oplus(2,2)\,. \end{aligned} \tag{A.154}$$

All of the representations in (A.153) appear in this decomposition. Accordingly, we obtain four singlets of the form $h\hat{h}\hat{h}$, $\hat{h}\hat{h}\hat{h}$, and $\omega^\pm\hat{h}\hat{h}$, whereby the first term was already found previously. The other three scalar invariants are given by

- $\hat{h}^{\nu}_{\mu}\hat{h}^{\rho}_{\nu}\hat{h}^{\mu}_{\rho}$,
- $\omega^+_{\mu\rho\nu\sigma}\hat{h}^{\mu\nu}\hat{h}^{\rho\sigma}$,
- $\omega^-_{\mu\rho\nu\sigma}\hat{h}^{\mu\nu}\hat{h}^{\rho\sigma}$.

Finally, it remains to consider terms with at least two factors of $\omega^\pm$. The tensor products between two copies of representations $(2,0)$ or $(0,2)$ decompose into

$$\omega^+\otimes\omega^+ \;\in\; (2,0)\otimes(2,0) = (0,0)\otimes(1,0)\otimes(2,0)\otimes(3,0)\otimes(4,0)\,, \tag{A.155}$$
$$\omega^-\otimes\omega^- \;\in\; (0,2)\otimes(0,2) = (0,0)\otimes(0,1)\otimes(0,2)\otimes(0,3)\otimes(0,4)\,. \tag{A.156}$$

The only representations which appear in these decompositions and are also contained in (A.153), are $(0,0)$ and the $(2,0)$ or $(0,2)$ representations, respectively. Therefore, we obtain four singlets by tensoring (A.155) and (A.156) with the $(0,0)$ representation associated with h or with the $(2,0)$ or $(0,2)$ representations, respectively, associated with ω^+ and ω^-. The resulting invariants of the first type, $h\omega^\pm\omega^\pm$, were already identified above. The second type of scalar invariants are of the form $\omega^\pm\omega^\pm\omega^\pm$, and are explicitly given by

- $w^{+\;\;\alpha\beta}_{\mu\nu}w^{+\;\;\gamma\delta}_{\alpha\beta}w^{+\;\;\mu\nu}_{\gamma\delta}$,
- $w^{-\;\;\alpha\beta}_{\mu\nu}w^{o\;\;\gamma\delta}_{\alpha\beta}w^{-\;\;\mu\nu}_{\gamma\delta}$.

Lastly, the mixed tensor product between the $(2,0)$ and $(0,2)$ representations is given by

$$\omega^+ \otimes \omega^- \in (2,0)\otimes(0,2) = (2,2)\,, \tag{A.157}$$

and contains none of the representations appearing in (A.153). Therefore, it is not possible to obtain a third-order scalar invariant in which ω^+ and ω^- couple to each other.

Altogether, there are nine invariants at third order in cyclic area-metric perturbations without derivatives. This result is in agreement with the approach of the constructive-gravity programme [117–121]. These nine invariants can be expressed as

$$\left\{h^3, \hat{h}_{\mu\nu}\hat{h}^{\mu\nu}h, \hat{h}^{\nu}_{\mu}\hat{h}^{\rho}_{\nu}\hat{h}^{\mu}_{\rho}, \omega^{\pm}_{\mu\rho\nu\sigma}\hat{h}^{\mu\nu}\hat{h}^{\rho\sigma}, \omega^{\pm}_{\mu\nu\rho\sigma}\omega^{\pm\mu\nu\rho\sigma}h, w^{\pm}_{\mu\nu}{}^{\alpha\beta}w^{\pm}_{\alpha\beta}{}^{\gamma\delta}w^{\pm}_{\gamma\delta}{}^{\mu\nu}\right\}. \tag{A.158}$$

Combining h and $\hat{h}_{\mu\nu}$ into the symmetric tensor $h_{\mu\nu}$ defined in (A.19), we can write the four independent invariants involving both $h_{\mu\nu}$ and $\omega^{\pm}_{\mu\nu\rho\sigma}$ as

$$h\omega^{\pm}_{\mu\nu\rho\sigma}\omega^{\pm\mu\nu\rho\sigma} \qquad \text{and} \qquad h^{\mu\nu}h^{\rho\sigma}\omega^{\pm}_{\mu\rho\nu\sigma}\,. \tag{A.159}$$

In the last term, $\omega^{\pm}$ projects onto the traceless parts of the tensor $h_{\mu\nu}$. The invariants (A.159) provide the interaction vertices for the evaluation of the length-metric induced RG flow of the masses of the non-metric degrees of freedom in area-metric gravity, analysed in section 4.1.

A.4 Area-metric propagator

In this section, we illustrate the structural form of the regularised propagator $\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}$ which enters the RG flow equation (4.1) for the effective action Γ_k defined in (4.18). Here, $\Gamma_k^{(2)}$ denotes the Hessian of second functional derivatives of Γ_k with respect to the fields $\Phi \equiv (h, \omega^+, \omega^-)$ evaluated at fixed background. In matrix notation, this Hessian takes the form

$$\Gamma_k^{(2)} \equiv \frac{\delta^2\Gamma_k}{\delta\Phi_i\delta\Phi_j}\bigg|_{\Phi=0} = \begin{pmatrix} \Gamma^{(2)}_{k\ \ hh} & \Gamma^{(2)}_{k\ \ h\omega^+} & \Gamma^{(2)}_{k\ \ h\omega^-} \\ \left[\Gamma^{(2)}_{k\ \ h\omega^+}\right]^T & \Gamma^{(2)}_{k\ \ \omega^+\omega^+} & 0 \\ \left[\Gamma^{(2)}_{k\ \ h\omega^-}\right]^T & 0 & \Gamma^{(2)}_{k\ \ \omega^-\omega^-} \end{pmatrix}. \tag{A.160}$$

To express the separate entries of $\Gamma_k^{(2)}$, it is useful to rewrite the parts involving $\omega^{\pm}$ in Γ_k in terms of the projectors $\Pi^{\pm}$ which project the area-metric fluctuation a onto $\omega^{\pm}$, respectively. $\Pi^{\pm}$ acts as the identity on $\omega^{\pm}$. The relevant definitions and identities were introduced in subsection 3.2.3. For completeness, we remind the reader of the projector and orthogonality relations

$$\Pi^{\pm}\cdot\Pi^{\pm} = \Pi^{\pm}\,, \tag{A.161}$$
$$\Pi^{\pm}\cdot\Pi^{\mp} = 0\,. \tag{A.162}$$

With these projectors, the quadratic part of Γ_k in (4.18) can be expressed in momentum space as

$$\begin{aligned}
\Gamma_k \;&\supset\; h_{\mu\nu}\mathcal{E}^{\mu\nu\rho\sigma}_{\mathrm{EH+gf}}h_{\rho\sigma} + \sum_{\pm} \bar{\alpha}'_{\pm} p^{\nu} p^{\sigma} h^{\mu\rho}\omega^{\pm}_{\mu\nu\rho\sigma} + \frac{1}{2}\left(p^2 + \bar{m}^2_{\pm}\right)\omega^{\pm}_{\mu\nu\rho\sigma}\omega^{\pm\,\mu\nu\rho\sigma} \\
&= h_{\mu\nu}\mathcal{E}^{\mu\nu\rho\sigma}_{\mathrm{EH+gf}}h_{\rho\sigma} + \sum_{\pm} h_{\mu\nu}\left[\bar{\alpha}'_{\pm}\mathbb{I}^{\mu\nu\mu'\nu'} p^{\rho'} p^{\sigma'} \Pi^{\pm}{}_{\mu'\rho'\nu'\sigma'}{}^{\alpha\beta\gamma\delta}\right]\omega^{\pm}_{\alpha\beta\gamma\delta} \\
&+ \omega^{\pm}_{\mu\nu\rho\sigma}\left[\frac{1}{2}\left(p^2 + \bar{m}^2_{\pm}\right)\Pi^{\pm\,\mu\nu\rho\sigma,\alpha\beta\gamma\delta}\right]\omega^{\pm}_{\alpha\beta\gamma\delta}\,, \qquad \text{(A.163)}
\end{aligned}$$

where $\mathbb{I}_{\mu\nu}{}^{\mu'\nu'}$ is the identity on the space of symmetric tensors $h_{\mu\nu}$. The part quadratic in h is denoted formally by $\mathcal{E}_{\mathrm{EH+gf}}$ and consists of the Fierz-Pauli operator, which arises by expanding the Einstein-Hilbert action at second order in h, together with the gauge-fixing contribution in equation (4.20). The other matrix entries of $\Gamma^{(2)}_k$ are

$$\Gamma^{(2)}_{k}{}_{h\omega^{\pm}}{}^{\alpha\beta\gamma\delta}{}_{\mu\nu} \equiv \left.\frac{\delta^2\Gamma_k}{\delta\omega^{\pm}_{\alpha\beta\gamma\delta}\delta h^{\mu\nu}}\right|_{\Phi=0} = \bar{\alpha}'_{\pm}\mathbb{I}_{\mu\nu}{}^{\mu'\nu'} p^{\rho'} p^{\sigma'}\Pi^{\pm}{}_{\mu'\rho'\nu'\sigma'}{}^{\alpha\beta\gamma\delta}\,, \qquad \text{(A.164)}$$

$$\Gamma^{(2)}_{k}{}_{\omega^{\pm}\omega^{\pm}}{}^{\alpha\beta\gamma\delta}{}_{\mu\nu\rho\sigma} \equiv \left.\frac{\delta^2\Gamma_k}{\delta\omega^{\pm}_{\alpha\beta\gamma\delta}\delta\omega^{\pm\mu\nu\rho\sigma}}\right|_{\Phi=0} = \left(p^2 + \bar{m}^2_{\pm}\right)\Pi^{\pm}{}_{\mu\nu\rho\sigma}{}^{\alpha\beta\gamma\delta}\,. \qquad \text{(A.165)}$$

The regularised propagator G is defined by

$$\begin{aligned}
\left(\Gamma^{(2)}_k + \mathcal{R}_k\right)\cdot G \;&\equiv\; \left(\Gamma^{(2)}_k + \mathcal{R}_k\right)\cdot \begin{pmatrix} G_{hh} & G_{h\omega^+} & G_{h\omega^-} \\ [G_{h\omega^+}]^T & G_{\omega^+\omega^+} & G_{\omega^+\omega^-} \\ [G_{h\omega^-}]^T & [G_{\omega^+\omega^-}]^T & G_{\omega^-\omega^-} \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{I}_{hh} & 0 & 0 \\ 0 & \mathbb{I}_{\omega^+\omega^+} & 0 \\ 0 & 0 & \mathbb{I}_{\omega^-\omega^-} \end{pmatrix}\,, \qquad \text{(A.166)}
\end{aligned}$$

where $\mathbb{I}_{hh}$ and $\mathbb{I}_{\omega^{\pm}\omega^{\pm}}$ are the tensor identities on the respective field subspaces. Using the projectors onto each subspace, the matrix entries of G can be computed explicitly. They take the form

$$\begin{aligned}
G_{hh}{}^{\mu\nu}{}_{\rho\sigma} &= f_{(2)}\Pi^{(2)\mu\nu}{}_{\rho\sigma} + f_{(0)}\Pi^{(0)\mu\nu}{}_{\rho\sigma}\,, && \text{(A.167)} \\
G_{h\omega^{\pm}}{}^{\mu\nu}{}_{\alpha\beta\gamma\delta} &= f_{h\omega^{\pm}} p_{\rho'} p_{\sigma'}\Pi^{\pm\mu\rho'\nu\sigma'}{}_{\alpha\beta\gamma\delta}\,, && \text{(A.168)} \\
G_{\omega^{\pm}\omega^{\pm}}{}^{\mu\nu\rho\sigma}{}_{\alpha\beta\gamma\delta} &= f_{\omega^{\pm}\omega^{\pm}}\Pi^{\pm\mu\nu\rho\sigma}{}_{\alpha\beta\gamma\delta}\,, && \text{(A.169)} \\
G_{\omega^{\pm}\omega^{\mp}}{}^{\mu\nu\rho\sigma}{}_{\alpha\beta\gamma\delta} &= f_{\omega^{\pm}\omega^{\mp}}\frac{p^{\mu'}p^{\nu'}p_{\rho'}p_{\sigma'}}{p^4}\Pi^{\pm\mu\nu\rho\sigma}{}_{\mu'\alpha'\nu'\beta'}\,\Pi^{\mp\rho'\alpha'\sigma'\beta'}{}_{\alpha\beta\gamma\delta}\,. && \text{(A.170)}
\end{aligned}$$

The coefficients f_i are functions of the momentum p^2 and the shape function r_k which determines the regulator $\mathcal{R}_k$. The beta functions were computed with the choice of a spectrally adjusted regulator

$$\mathcal{R}_k(p^2) = \frac{k^2}{p^2} r_k\left(\frac{k^2}{p^2}\right)\Gamma_k^{(2)} \qquad \text{where} \qquad r_k(x) = (1-x)\Theta(1-x) \tag{A.171}$$

is the Litim-type shape function [236]. The entries of the regularised propagator depend furthermore on the couplings $\bar{\alpha}'_\pm$ and on the mass parameters $\bar{m}_\pm^2$. $\Pi^{(2)} = P^{(2)}$ in (A.167) is the projector onto the transverse-traceless component of the length-metric fluctuation as defined in (3.8). Explicitly, this projector in $d > 2$ dimensions with background metric $\delta_{\mu\nu}$ can be expressed as

$$\begin{aligned} {\Pi^{(2)}}_{\mu\nu}{}^{\rho\sigma} &= \delta^\rho_{(\mu}\delta^\sigma_{\nu)} - \frac{1}{d-1}\delta_{\mu\nu}\delta^{\rho\sigma} - \frac{2}{p^2}\delta^{(\rho}_{(\mu}p_{\nu)}p^{\sigma)} \\ &+ \frac{1}{d-1}\frac{1}{p^2}(\delta_{\mu\nu}p^\rho p^\sigma + p_\mu p_\nu \delta^{\rho\sigma}) + \frac{d-2}{d-1}\frac{1}{p^4}p_\mu p_\nu p^\rho p^\sigma\,, \end{aligned} \tag{A.172}$$

and is manifestly gauge-independent. By contrast, $\Pi^{(0)}$ in (A.167) is the projector onto the scalar component of the length-metric fluctuation, which can be expressed as

$${\Pi^{(0)}}_{\mu\nu}{}^{\rho\sigma} = \frac{B^2}{C}\left(\delta_{\mu\nu} + \frac{A}{B}\frac{p_\mu p_\nu}{p^2}\right)\left(\delta^{\rho\sigma} + \frac{A}{B}\frac{p^\rho p^\sigma}{p^2}\right), \tag{A.173}$$

where

$$A = d\beta_h \tag{A.174}$$
$$B = (d-1-\beta_h) \tag{A.175}$$
$$C = (d-1)\big(d^2 - d\big(1-\beta_h^2\big)\big)\,. \tag{A.176}$$

The explicit dependence of $\Pi^{(0)}$ on the gauge parameter β_h in (4.20) ensures that the propagating scalar mode is projected on for any choice of β_h [237, 238]. In the limit $\beta_h \to 0$, the projector $\Pi^{(0)}$ reduces to the projector onto the trace mode of $h_{\mu\nu}$.

The regularised propagator G contains a mixing term $G_{\omega^\pm\omega^\mp}$ given by (A.170), which originates from the inversion of the kinetic matrix in field space. We do not take into account such a mixing term in the truncation ansatz for Γ_k, as this would introduce a coupling of negative mass dimension which we would be of higher order in the expansion scheme underlying our truncation.

A.5 Equations of motion in static spherical symmetry

In this appendix, we state the equations of motion (4.88) derived from the localised action (4.72) for the static spherically symmetric ansatz detailed in subsection 4.2.3. These can be split into one part which remains present after taking the limit $\mu \to 0$, in which case the action reduces to the Einstein action of general relativity. The other part of the equations of motion is proportional to the parameter μ and decomposes further into a contribution proportional to the non-locality parameter η, and denoted by a superscript (η), as well as a contribution which remains in the limit $\eta \to 0$, and is denoted by a superscript (0). With this notation, the equations of motion (4.88) are explicitly given by

$$\mathcal{E}_f \equiv \mathcal{E}_f^{\rm GR} + \mu\Big(\mathcal{E}_f^{(0)} + \eta\mathcal{E}_f^{(\eta)}\Big) = 0\,, \tag{A.177}$$

$$\mathcal{E}_h \equiv \mathcal{E}_h^{\rm GR} + \mu\Big(\mathcal{E}_h^{(0)} + \eta\mathcal{E}_h^{(\eta)}\Big) = 0\,, \tag{A.178}$$

$$\mathcal{E}_\psi \equiv \mu\Big(\mathcal{E}_\psi^{(0)} + \eta\mathcal{E}_\psi^{(\eta)}\Big) = 0\,, \tag{A.179}$$

where

$$\mathcal{E}_f^{\rm GR} = f^4h^3\big(rh' + h^2 - h\big)\,, \tag{A.180}$$

$$\begin{aligned}\mathcal{E}_f^{(0)} =& -fh\Big(rfh\big(11r\psi f'h' - 2h\big(-3m^2r\psi^2 + 4rf'\psi' + 3\psi(rf'' + 3f')\big)\big) + 9r^2h^2\psi f'^2 \\ & +2f^2\big(5r^2\psi h'^2 - rh(5rh'\psi' + \psi(2rh'' + 13h')) + 2h^2(r(r\psi'' + 5\psi') + 3\psi)\big)\Big)\,,\end{aligned} \tag{A.181}$$

$$\begin{aligned}\mathcal{E}_f^{(\eta)} =& \, 24r^2h^2\psi^2f'^2 + 6f^2\Big(6r^2\psi^2h'^2 + h^2\big(r^2\psi'^2 + 2r\psi(r\psi'' + 2\psi') - 6\psi^2\big) - \\ & rh\psi(7rh'\psi' + 2\psi(rh'' + 2h'))\Big) - 6rfh\psi\big(h(2r\psi f'' + 5rf'\psi' + 4\psi f') \\ & -6r\psi f'h'\big)\,,\end{aligned} \tag{A.182}$$

and

$$\mathcal{E}_h^{\rm GR} = f^4(h-1)h^4 - rf^3h^4f'\,, \tag{A.183}$$

$$\begin{aligned}\mathcal{E}_h^{(0)} =& \, fh^2\Big(2rfh\big(-3m^2r\psi^2 + r\psi f'' - rf'\psi' - 5\psi f'\big) + r^2\psi f'h' + r^2h\psi f'^2 \\ & +2f^2(2h(r\psi' + 3\psi) - r\psi h')\Big)\,,\end{aligned} \tag{A.184}$$

$$\begin{aligned}\mathcal{E}_h^{(\eta)} =& \, 12r^2h^2\psi^2f'^2 + 6f^2\Big(4r^2\psi^2h'^2 - h^2\big(r^2\psi'^2 - 2r\psi(r\psi'' + 2\psi') + 18\psi^2\big) \\ & -rh\psi(3rh'\psi' + 2\psi(rh'' + 2h'))\Big) - 6rfh\psi\big(h(2r\psi f'' + rf'\psi' + 4\psi f') \\ & -2r\psi f'h'\big)\,,\end{aligned} \tag{A.185}$$

and

$$\mathcal{E}_\psi^{(0)} = fh^2\big(rf\big(rf'h' - 2h\big(-6m^2r\psi + rf'' - f'\big)\big) + r^2hf'^2 + f^2\big(-2rh' + 4h^2 - 4h\big)\big)\,, \tag{A.186}$$

$$\begin{aligned}\mathcal{E}_\psi^{(\eta)} =& -18r^2h^2\psi f'^2 - 6f^2\Big(5r^2\psi h'^2 - rh(5rh'\psi' + 2\psi(rh'' + 2h')) \\ & -2h^2(6\psi - r(r\psi'' + 2\psi'))\Big) + 6rfh(h(2r\psi f'' + 3rf'\psi' + 4\psi f') - 4r\psi f'h')\,.\end{aligned} \tag{A.187}$$

The equation of motion $\mathcal{E}_\psi$ has no GR contibution, as ψ does not appear in the action for general relativity.