DUALITY AND STRONGLY CORRELATED SYSTEMS IN TWO DIMENSIONS

BY

HART GOLDMAN

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Physics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2020

Urbana, Illinois

Doctoral Committee:

Assistant Professor Thomas Faulkner, Chair
Professor Eduardo Fradkin, Director of Research
Professor Nadya Mason
Professor Michael Stone

Abstract

Many of the most vexing phenomena observed in condensed matter physics involve strongly correlated systems near quantum critical points in two spatial dimensions. Examples range from the sharing of critical exponents (superuniversality) among quantum Hall plateau transitions to the emergence of a charge-vortex symmetry near the field-tuned superconductor-insulator transition in thin films to the appearance of "anomalous" metallic states which evade localization. These problems elude theoretical understanding because they lack small parameters; standard perturbative approaches constitute poor, often uncontrolled approximations.

This thesis concerns itself with the development of techniques reaching beyond the perturbative paradigm, focusing especially on duality, the idea that two seemingly different theories are actually one and the same. While a duality is most useful when it relates a strongly interacting theory to a weakly interacting one, even dualities between strongly coupled theories can reveal emergent symmetries and exotic gapped phases that may be obscure in one theory but not its dual. Recently, starting from a relativistic analogue of flux attachment, an entire web of quantum field theory dualities was proposed connecting a wide variety of quantum critical states in two spatial dimensions. This web of dualities has led to a surge of progress on a wide variety of condensed matter problems, several examples of which are presented in this thesis.

One of the major challenges that comes with the proposal of new dualities is to "derive" them. In Chapter 2, we construct simple, explicit derivations of many members of the web of dualities in (particle-hole symmetric) models of relativistic current loops. In relativistic theories, flux attachment necessarily involves transmutation of both statistics and spin, an operation which can be made precise in the context of loop models. We further show that while non-relativistic theories are invariant under attachment of even numbers of flux quanta due to the periodicity of statistics, this symmetry is completely lost in relativistic theories due to the presence of spin, clarifying the interpretation of earlier loop models in which this symmetry appeared to be present.

Motivated in part by this lack of statistical periodicity in relativistic flux attachment, in Chapter 3 we turn our attention to the metallic, "composite Fermi liquid" states occuring in quantum Hall systems at filling fraction $\nu = 1/2n$. Famously, the state at $\nu = 1/2$ is known to display particle-hole symmetry, which has

recently led to the proposal of a manifestly particle-hole symmetric theory of relativistic, or Dirac, composite fermions that features prominently in the web of dualities. Surprisingly, however, an analogous "reflection symmetry" has also been observed in transport experiments about $\nu = 1/4$. To explain this symmetry, we propose a series of relativistic composite fermion theories for the compressible states at $\nu = 1/2n$, in which the reflection symmetry is incorporated as a mean field time-reversal symmetry of the composite fermions. These theories consist of electrically neutral Dirac fermions attached to $2n$ flux quanta via an emergent Chern-Simons gauge field. While not possessing an explicit particle-hole symmetry, these theories reproduce the known Jain sequence states proximate to $\nu = 1/2n$, and we show that such states can be related by the observed reflection symmetry, at least at mean field level.

In Chapter 4, we describe how duality can be used to access exotic gapped phases, in particular non-Abelian quantum Hall states. Using proposed non-Abelian bosonization dualities in two spatial dimensions, which morally relate $U(N)_k$ and $SU(k)_{-N}$ Chern-Simons-matter theories, we present pairing scenarios for which non-Abelian quantum Hall states can be obtained starting from theories of Abelian composite particles. The advantage of these dualities is that regions of the phase diagram which may be obscure on one side of the duality can be accessed by condensing local operators on the other side. Starting from parent Abelian states, we use this approach to construct Landau-Ginzburg theories of non-Abelian states through a pairing mechanism. In particular, we obtain the bosonic Read-Rezayi sequence at fillings $\nu = k/(kM+2)$ by starting from k layers of bosons at $\nu = 1/2$ with M Abelian fluxes attached and then condensing k-clusters of the dual non-Abelian bosons. We further extend this constructions to obtain generalizations of the Read-Rezayi states with emergent global symmetries.

In Chapter 5, we describe a context in which non-perturbative ideas from duality and perturbative results in the language of the renormalization group (RG) can inform one another: the interplay of quenched disorder and strong interaction effects near quantum critical points. In particular, we focus on the problem of quenched disorder at the superfluid-insulator transition of the $O(N)$ model in the large-N limit. While a random mass is strongly relevant at the free fixed point, its effect is screened by the strong interactions of the Wilson-Fisher fixed point in this model, enabling a perturbative RG study of the interplay of disorder and interactions about this fixed point. In contrast to the spiralling flows obtained in earlier double-ϵ expansions, we show that the theory flows directly to a quantum critical point characterized by finite disorder and interactions, with critical exponents in remarkable agreement with numerical studies of the superfluid-Mott glass transition. With these results in hand, we apply duality to discuss the possible implications of this result for the dual Abelian Higgs and Chern-Simons-Dirac fermion theories when $N = 1$.

In loving memory of my grandparents,

Ann and George Hart, and Norma and Ozzy Goldman.

Acknowledgements

My time at the University of Illinois has been one of immense intellectual and personal growth. Primary responsibility for this growth falls on my advisor, Eduardo Fradkin, whose warmth, support, humility, and dedication to mentorship gave me the confidence to craft my identity as a physicist. I will never cease to be amazed by the breadth and profundity of Eduardo's understanding of theoretical physics, which always pushed me to think in new ways about ideas I mistakenly thought I understood completely. One of the most inspiring things about Eduardo is his tenacity, whether in pursuit of a new idea or commitment to a joke. The work comprising this thesis would not exist were it not for that tenacity, which drove me forward even when a challenging question appeared beyond answering or an approach looked fruitless.

Throughout my PhD, I have had the privilege of working with a wide variety of collaborators, who welcomed me into what may best be described as a scientific family. I especially want to thank Sri Raghu, who (perhaps unwittingly) taught me that the hardest problems can be more beautiful than the simplest solutions, and whose encouragement brought me into condensed matter physics in the first place. I also want to thank Mike Mulligan, whose mentorship, deep intuition for condensed matter physics and quantum field theory, and willingness to provide detailed feedback on every last one of my papers have played singular roles in my development as a physicist. Further still, I want to thank Zhen Bi, Prashant Kumar, Laimei Nie, Alex Rasmussen, Jonathan San Miguel, Raman Sohal, Alex Thomson, Gonzalo Torroba, and Max Zimet for their collaboration and friendship, as well as for patiently reading the walls of text I send in emails and messaging apps. I also want to thank Jason Alicea, Meng Cheng, Liang Fu, Sean Hartnoll, Yin-Chen He, Steve Kivelson, Sung-Sik Lee, Lesik Motrunich, Shinsei Ryu, T. Senthil, Dam Son, and Chong Wang for their warmth and openness in discussing physics, as well as for welcoming me as a visitor in their groups at various points in my PhD. Finally, I want to thank the members of my dissertation committee: Tom Faulkner, Nadya Mason, and Mike Stone for subjecting themselves to a virtual defense during a pandemic.

Lastly, I want to thank my family and friends both in physics and beyond. Foremost of whom is my partner, Kate Mayer, who against common sense agreed to follow me to Urbana from San Francisco. In addition to those already mentioned above, I also want to thank Luis Aguilar, Ginny Bailey, Keegan Boyar, John Bowers, Alfred Cheung, Sophie Christel, Luca Delacretaz, John Dodaro, Oleg Dubinkin, Ilya Esterlis, Ari Esters, Arturo Ferrari, Armelle Goreux, Davneet Kaur, Lazar Kish, Howon Lee, Ben Lehmann, Ryan Levy, Julian May-Mann, Mayisha Nakib, V. Nguyen, Pranav Rao, Jun Ho Son, Xiao-Qi Sun, Melis Tekant, Natalie Telis, Ian Tenney, Saavanth Velury, Chao Wang, and Penghao Zhu for their enduring friendship and commitment to preserving my sanity while in graduate school. I also want to thank my parents, Steve and Melissa Goldman, whose support has been the backbone of my development as a person and as a physicist.

Funding acknowledgements: The work represented in this thesis was supported by the National Science Foundation (NSF) Graduate Research Fellowship Program under Grant No. DGE-1144245 as well as NSF grants DMR-1408713 and DMR-1725401 (awarded to Eduardo Fradkin).

Table of Contents

Chapter 1

Introduction

1.1 From Strongly Interacting Quantum Phase Transitions to Anomalous Metals

Many of the most mysterious phenomena in condensed matter physics involve systems near phase transitions at zero temperature. Unlike classical phase transitions, which are driven by thermal fluctuations, these phase transitions are driven by quantum fluctuations stemming from the uncertainty principle, thereby earning the name of quantum phase transitions. Frequently, the gapless degrees of freedom at a quantum phase transition are very strongly interacting at all energy scales, placing them well beyond the realm of single particle physics. Quantum phase transitions are therefore natural places to expect exotic, strongly correlated quantum phenomena, an expectation borne out by a wide breadth of unexplained experimental observations in everything from quantum Hall systems to superconducting thin films to correlated magnets to topological insulators.

Among the most prominent quantum phase transitions are those occuring between quantum Hall plateaus, which arise in 2d systems of electrons in a strong perpendicular magnetic field. The presence of the magnetic field causes the electrons to fill out highly degenerate Landau levels (LLs). Famously, at both integer and fractional LL fillings, these systems organize themselves into a rich variety of incompressible phases – the titular plateaus – with quantized Hall response. Although an understanding of these incompressible phases was a major achievement of twentieth century physics, the transitions between quantum Hall plateaus exhibit a range of phenomena that continue to elude explanation. For example, all appear to share the same critical exponents, $\nu \approx 7/3$ and $z \approx 1$, irrespective of the particular quantum Hall phases on either side[1] [1–9], a phenomenon known as *superuniversality* [10, 11]. While the concept of universality is a bedrock organizing principle of modern condensed matter physics, plateau transitions are the only known family of superuniversal phase transitions in nature. Clearly, the observation of superuniversality hints at an intimate

[1]While it is often an operating assumption that this is also true for fractional quantum Hall transitions, only the product νz has been measured at those transitions. Their individual values of ν and z have not been measured.

connection – perhaps a symmetry – between these ostensibly distinct quantum phase transitions, yet capturing this connection theoretically has proven exceptionally difficult due to the strongly interacting nature of the problem. Indeed, the known theoretical examples of superuniversality have proven rare and idiosyncratic [12, 13]. Moreover, they fail to explain the values of the exponents themselves, which appear impossible to obtain without incorporating both strong interactions and quenched disorder: $\nu \approx 7/3$ and $z = 2$ can be obtained in non-interacting descriptions of integer quantum Hall transitions with quenched disorder [14–16], but the presence of interactions is believed to play a crucial role in producing $z \approx 1$ [17]. The problem of superuniversality is therefore inherently beyond the usual tools of condensed matter physics and quantum field theory, which rely on *something* being small enough for perturbation theory to be reliable.

In addition to the sharing of exotic critical indices, plateau transitions are also observed to host poorly understood emergent symmetries. About the $\nu = 1$ integer quantum Hall (IQH) to insulator transition[2], located at $\nu = 1/2$, the current-voltage curves appear to satisfy a *reflection symmetry* (**R**) across the transition, in which current and voltage exchange roles [18–20]. More precisely, for each filling fraction ν on one side of the transition, there is a filling fraction ν' on the other side such that

$$\rho_{xx}(\nu) = \frac{1}{\rho_{xx}(\nu')} \left(\frac{h}{e^2}\right)^2 , \; \rho_{xy}(\nu) = \rho_{xy}(\nu') = \frac{h}{e^2} \, , \tag{1.1}$$

where ρ_{ij} is the resistivity tensor. Since this symmetry amounts to exchange of current and voltage, it can be interpreted as a symmetry relating flux and charge. It can also be understood as a particle-hole symmetry (**PH**), since the relationship between ν and ν' is $\nu' = 1 - \nu$, which is the exchange of empty and filled states. Such a **PH** symmetry is expected to appear at $\nu = 1/2$ because it is an exact symmetry of the lowest Landau level (LLL) Hamiltonian, yet it has been notoriously difficult to capture theoretically. The classic Halperin-Lee-Read (HLR) description of the $\nu = 1/2$ state in terms of composite fermions interacting with a fluctuating Chern-Simons gauge field [21] fails to manifest this symmetry at mean field level[3] as long as ρ_{xx} is finite, and attempts to incorporate gauge field fluctuations have not altered this conclusion [26, 27]. In addition, **R** symmetry is also observed at the $\nu = 1/3$ fractional quantum Hall (FQH) to insulator transition, which occurs at $\nu = 1/4$. The only differences in this case are that $\rho_{xy} = 3h/e^2$, and the relationship between ν and ν' takes a form which maps filling fractions on one side of $\nu = 1/4$ to the other. Unlike the case of $\nu = 1/2$, there is no reason to believe *a priori* that **R** symmetry should arise at $\nu = 1/4$, rendering the problem of explaining it even more challenging.

[2]Here $\nu = 2\pi\rho_e/B$ denotes the filling fraction in units $\hbar = c = 1$, with ρ_e being the electron density and B the external magnetic field. Whether we are referring to filling fraction or correlation length exponent will always be apparent from context.

[3]This conclusion has been recently challenged in a number of works [22–25], which find that a proper incorporation of the flux attachment constraint in HLR can lead to **PH**-symmetric DC response in mean field theory. This, however, is not the same as demonstrating that **PH** is a genuine global symmetry.

A more common way of thinking about quantum Hall systems at $\nu = 1/2$ and $\nu = 1/4$ is not as quantum phase transitions, but as metallic phases that persist for a range of magnetic field strengths [28]. This is because, in clean enough samples, the corresponding plateau transitions broaden out into what appear to be stable compressible phases. Within the HLR language, these states are described in terms of Fermi surfaces of composite fermions coupled to gauge fields. Indeed, the composite Fermi surface seems to reveal itself in quantum oscillations measured near $\nu = 1/2$ [29–35]. However, the existence of such a stable metallic phase contradicts the widespread belief, based on the scaling theory of localization [36], that there are no stable metallic ground states in two dimensions in the presence of weak disorder[4]. States of this kind are thus referred to as *anomalous metals*. Although the scaling theory of localization is only valid for systems of non-interacting electrons, attempts to construct stable (finite density) metallic theories with disorder and interactions have been largely unsuccessful. Even in the clean limit, the problem of a Fermi surface strongly coupled to a gapless boson (either a gauge field or a scalar order parameter) in 2d is plagued with infrared divergences that invalidate standard perturbative approaches such as the random phase approximation (RPA) [37], although some controlled expansions have been developed [38–42].

Surprisingly, much of this phenomenology is not limited to quantum Hall systems. For example, the same superuniversal exponents have been observed at the magnetic field-tuned superconductor-insulator transition (SIT) in thin films, as well as a similar charge-flux symmetry known as self-duality [43–51]. At the transition, self-duality is the statement that Cooper pairs and vortices have the same transport properties, but since conductivity of charge is resistivity of vortices (current and voltage again exchange roles), this implies

$$(\rho_{xx})^2 + (\rho_{xy})^2 = \left(\frac{h}{4e^2}\right)^2 . \tag{1.2}$$

Here $h/4e^2$ is the quantum of Cooper pair resistance. Experimentally, ρ_{xy} has been observed to be vanishingly small, meaning $\rho_{xx} \approx h/4e^2$ [45]. In further analogy to the quantum Hall transitions, the field-tuned SIT also has been observed to broaden into an anomalous metal at weak disorder [52–56]. Such parallels are are shocking: the field-tuned SIT and the plateau transition problems appear fundamentally distinct, as the degrees of freedom governing the SIT are bosons, but those governing plateau transitions are fermions.

Like the quantum Hall transitions, the field-tuned SIT lies beyond the scope of perturbative techniques, as both strong interaction and disorder effects appear to play essential roles. The apparent connection between these two very different systems, however, indicates that a deep set of principles may be at work which is inherently non-perturbative. Indeed, the most natural inference one can make is that these two types of quantum critical points are one and the same, with the different apparent degrees of freedom at each simply

[4] This conclusion assumes the absence of spin-orbit coupling.

constituting different choices of variables. In more precise terms, this would mean that the quantum field theories (QFTs) describing these transitions are *dual* to one another. Duality is most useful when it relates strongly and weakly interacting theories, or theories to themselves. In such cases, the duality represents a complete, non-perturbative solution to a problem. However, even in cases where duality connects two strongly interacting theories, it can be leveraged to make a variety of useful, non-perturbative statements, particularly regarding emergent symmetries and the accessibility of exotic gapped phases.

This thesis focuses on the development and application of QFT dualities to better understand quantum phase transitions in two spatial dimensions. Along the way, we will engage with each of the problems described above. In some cases, like that of reflection symmetry at $\nu = 1/4$ or the study of the interplay of strong interactions and quenched disorder, we will demonstrate significant progress and motivate new, non-perturbative principles; in others, particularly that of superuniversality, we will show how duality can lead to a more refined understanding of the underlying problem, helping to set the stage for an ultimate solution. The remainder of this Chapter will consist of a brief historical overview of duality, followed by a discussion of the different ways that duality can be used to generate new, non-perturbative results and an outline of the remainder of the thesis.

1.2 A Brief History of Duality

1.2.1 Electromagnetic Duality and the Dirac Monopole

The idea of duality has a long history in condensed matter, high energy, and statistical physics. Perhaps the earliest example of duality is electromagnetic (EM) duality, which is simply the statement that Maxwell's equations in a vacuum,

$$\nabla \cdot \mathbf{E} = 0\,, \qquad \nabla \times \mathbf{E} = -\partial_t \mathbf{B}\,, \tag{1.3}$$

$$\nabla \cdot \mathbf{B} = 0\,, \qquad \nabla \times \mathbf{B} = \partial_t \mathbf{E}\,, \tag{1.4}$$

are invariant under the exchange of electric and magnetic fields, $\mathbf{E} \leftrightarrow -\mathbf{B}$. This means that Maxwell's electromagnetism in 3+1D is invariant under the exchange of electric charges and magnetic monopoles, making it *self-dual*.

What are the monopoles that participate in EM duality? Although monopoles have not been observed in nature, the possibility of their existence has important consequences for gauge theories. The magnetic field of a monopole is given by

$$\mathbf{B} = h\frac{\hat{\mathbf{r}}}{|\mathbf{r}|^2}\,. \tag{1.5}$$

where h is the magnetic charge.

The challenge of defining monopoles is that the field of a monopole is in conflict with the use of a vector potential, $\mathbf{A}$, such that $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. The use of a vector potential is essential to describing magnetism for a variety of reasons. For example, we know from quantum mechanics that vector potentials can contain physical information, leading, for example, to the Aharanov-Bohm effect.

The existence of monopoles can be made consistent with the use of a vector potential if said vector potential is allowed to be singular. Pursuing this line of thinking, Dirac imagined that a monopole could be defined as the end of an infinitely long, infinitely thin solenoid, now known as a Dirac string [57]. Along the Dirac string, $\mathbf{A}$ is singular, but this does not matter so long as the presence of the string itself cannot be detected in an experiment. While such an experiment cannot be constructed using classical ideas, in quantum mechanics it is possible to detect the Dirac string by bringing an electric charge e around it and measuring the Aharonov-Bohm phase the string imparts,

$$\Delta\Phi = e\oint d\boldsymbol{\ell}\cdot\mathbf{A} = e\,h\,. \tag{1.6}$$

This means that the string cannot be detected if the magnetic charge is quantized in inverse units of the (minimum) electric charge,

$$h = \frac{2\pi n}{e}\,, n\in\mathbb{Z}\,. \tag{1.7}$$

This statement is known as the Dirac quantization condition, and it holds for any compact gauge theory[5]. We will encounter versions of it frequently in this thesis. The fact that the fundamental electric and magnetic charges are inverses of each other indicates that EM duality is not only a self-duality, but also a strong-weak duality: when electric charges are strongly coupled, magnetic monopoles are weakly coupled and vice versa.

A common feature of the dualities we consider in this thesis is that they involve exchange of electric and magnetic degrees of freedom. While the use of a Dirac string is not essential in defining a monopole [58], the nonlocal character of the Dirac string nevertheless provides useful intuition, as we will see below.

[5]The above discussion relies on taking electromagnetism to be a compact, $U(1)$ gauge theory rather than a $\mathbb{R}$ gauge theory, which is not compact. Throughout this thesis, we will always use this 'high energy' definition of a compact gauge theory: namely, a gauge theory which can support monopoles. Often, in condensed matter contexts, a different definition of compactness is used in which monopoles are required to have finite energy, and flux is not conserved.

1.2.2 Order and Disorder, Charge and Flux

Kramers-Wannier Duality

One of the most beautiful insights in statistical physics is that the intuition of EM duality can be applied to the 2d (classical[6]) Ising model,

$$Z[K] = \sum_{\{\sigma_i = \pm 1\}} \prod_{\langle s,s' \rangle} \exp\left(-K\sigma_s\sigma_{s'}\right) \,, \tag{1.8}$$

where $\langle s, s' \rangle$ denotes nearest-neighbor sites s and s', and $K = J/T$, with J being the nearest-neighbor coupling and T the temperature. Kramers and Wannier [59] showed that the high temperature (disordered, small K) and low temperature (ordered, large K) phases are dual to one another, i.e.

$$Z[K] = Z\left[\tilde{K}\right], \qquad \tanh K = e^{-2\tilde{K}} \,. \tag{1.9}$$

Here the Ising variables of the dual theory are viewed as living on the sites located at the centers of each plaquette, which form what is known as the dual lattice. This duality enabled Kramers and Wannier to solve exactly for the critical temperature, where the partition function is exactly self-dual and $K = \tilde{K}$.

Already, the existence of a self-duality mapping strong and weak coupling is reminiscent of EM duality, and, indeed, Kramers-Wannier duality can be viewed as an exchange of "electric" and "magnetic" variables. This was first understood by Kadanoff and Ceva [60], who considered an operator $\mu_{\tilde{s}}(\Gamma)$ which creates an half-infinitely long domain wall along a path Γ on the dual lattice which ends at the site $\tilde{s}$, i.e. $\mu_{\tilde{s}}(\Gamma)$ breaks every bond crossed by Γ. This operator is called a disorder operator, meaning that it acquires a vacuum expectation value in the *disordered* phase of the original theory. This can be observed from the fact that, in the disordered phase, the free energy cost ΔF of a fractional domain wall is essentially constant in its length (domain walls have proliferated, i.e. $\langle \mu \rangle \neq 0$), while in the ordered phase ΔF grows linearly with the length of the domain wall (domain walls are rare, $\langle \mu \rangle = 0$).

Equivalently we can view the disorder parameter as $\mu_{\tilde{s}}$ is the dual of the order parameter, meaning that duality is the exchange of order and disorder operators. The notion of disorder operators therefore generalizes the role of the monopole of EM duality in a manner that can be applied to any duality. Indeed, the definition of the disorder operators of the 2d Ising model as a half-infinite domain wall recalls that of the Dirac string as a half-infinite solenoid.

[6] The conclusions of this section are also valid for the quantum problem in 1+1 dimensions, which is related to the classical problem via the transfer matrix formalism.

The Ising Gauge Theory

Kramers-Wannier duality can be generalized to higher dimensions, although it is no longer a self-duality. For example, in 3d, domains in the Ising model are no longer bounded by lines, but by surfaces, implying that the disorder variables will live on links rather than sites of the dual lattice. Theories of this kind are lattice gauge theories, and the Kramers-Wannier dual of the 3d Ising model is the so-called Ising gauge theory [61],

$$Z_{\text{Ising}}[K] = Z_{\text{IGT}}\left[\tilde{K}\right] = \sum_{\{\sigma_\ell=\pm1\}} \prod_p \exp\left(-\tilde{K}\prod_{\ell\in p}\sigma_\ell\right)\,,\ \tanh K = e^{-2\tilde{K}}\,, \tag{1.10}$$

where σ_ℓ is an Ising variable on a link ℓ on the dual lattice, and p denotes a plaquette (which contains four links). It can be physically interpreted as a discrete magnetic flux variable. Note that we have dropped the tildes on dual lattice coordinates. The Ising gauge theory has a $\mathbb{Z}_2$ gauge symmetry, $\sigma_\ell \mapsto V_s\sigma_{s,s+e_i}V_{s+e_i}$, $V_s \in \mathbb{Z}_2$, if $\ell = (s, s+e_i)$, where e_i is the unit lattice vector in the i-direction.

Unlike the 3d Ising model, the Ising gauge theory does not have the usual kind of ordered and disordered phases. Instead, its phases are characterized by confinement and deconfinement, which are not associated with a local order parameter. Confinement is diagnosed from the behavior of the (non-local) Wilson loop operator [62],

$$W_\Gamma = \prod_{\ell\in\Gamma}\sigma_\ell\,, \tag{1.11}$$

where $\Gamma = \partial\Sigma$ is a closed path on the dual lattice. Whether the theory is in the confined (high temperature, $\tilde{K} < \tilde{K}_c$) or deconfined (low temperature, $\tilde{K} > \tilde{K}_c$) phase depends on whether the Wilson loop satisfies an area or a perimeter law respectively,

$$\text{confined } (\tilde{K} < \tilde{K}_c) : \langle W_\Gamma\rangle \sim \exp\left[-f(\tilde{K})\ \text{Area}(\Sigma)\right]\,, \tag{1.12}$$

$$\text{deconfined } (\tilde{K} > \tilde{K}_c) : \langle W_\Gamma\rangle \sim \exp\left[-\bar{f}(\tilde{K})\ \text{Perimeter}(\Gamma)\right]\,. \tag{1.13}$$

The statement of Kramers-Wannier duality for the Ising gauge theory and the 3d Ising model is that the high and low temperature phases map to one another. Confinement therefore corresponds to the ordered phase of the Ising model, while the deconfined phase corresponds to the disordered phase.

The physical meaning of confinement is apparent if we consider the potential between two static point particles charged under the $\mathbb{Z}_2$ gauge field. Say that the are a distance R apart. Their world lines simply wrap the time axis (which has circumference $\tilde{\beta}$), and so the potential between the two point charges is none other than the free energy cost of the Wilson loop formed by the two particles' world lines. In the confining phase, one obtains

$$\tilde{\beta}V(R) \sim \text{Area}(\Sigma) = \tilde{\beta}R \Rightarrow V(R) \sim R\,. \tag{1.14}$$

This is a confining potential: it grows linearly with the separation of the two charges, like a string tension. In contrast, in the deconfined phase,

$$\tilde{\beta}V(R) \sim \text{Perimeter}(\Gamma) \sim \tilde{\beta} + R \Rightarrow V(R) \sim \text{constant as } \tilde{\beta} \to \infty. \tag{1.15}$$

The potential is independent of the separation as the temperature goes to infinity, and so the charges are free to live apart. They are deconfined. This phase takes on special importance in modern condensed matter physics: it is the simplest example of a phase hosting topological order, manifest in a topological ground state degeneracy of the theory on a torus (for a review, see Ref. [63]). However, this is not captured by the duality with the 3d Ising model, which exchanges open and periodic boundary conditions. In other words, Kramers-Wannier duality successfully captures local but not global properties of the two dual theories in the thermodynamic limit.

A deep consequence of the duality between the 3d Ising model and Ising gauge theory is that confinement can be interpreted as the proliferation of monopoles. While we introduced the Ising gauge theory as the theory of disorder operators of the 3d Ising model, we can equivalently view the variables, σ_s, of the 3d Ising model as "Dirac strings" in the Ising gauge theory. From the point of view of the Ising gauge theory, the operator σ_s creates a flux tube which excites $\mathbb{Z}_2$ flux around each of the plaquettes it pierces, ending on site s. In the confining phase, these Ising variables order, meaning that there is a condensate of "magnetic" charges. The principle that confinement corresponds to the condensation of monopoles is not limited to this example and has broad implications in the study of gauge theories.

1.2.3 Boson-Vortex Duality

One of the dualities that will feature prominently in this thesis is boson-vortex duality [64–66]. Consider the 3d (classical) XY model,

$$Z_{\text{XY}}[K] = \int \mathcal{D}\theta \prod_{s,\mu} \exp\left[-K\cos(\Delta_\mu\theta_s)\right]\,, \tag{1.16}$$

where s again is a site index, $\theta_s \in [0, 2\pi)$, $\mu = 0, 1, 2$ is a lattice vector index, and $\Delta_\mu\theta_s = \theta_{s+e_\mu} - \theta_s$ is a lattice derivative. Physically, this theory can be thought of as describing an array of coupled Josephson junctions, where θ_s is the phase of the order parameter. As K is tuned, the theory passes from an ordered, superfluid phase phase ($K > K_c$), in which $\langle e^{i\theta_s}\rangle \neq 0$ and there is a gapless Goldstone mode, to a disordered, insulating phase ($K < K_c$) in which $\langle e^{i\theta_s}\rangle = 0$ and all excitations are gapped. The transition between these two phases is known to be second order, and QFT describing it the Wilson-Fisher fixed point,

$$\mathcal{L}_{\rm WF} = |\partial_\mu \phi|^2 - |\phi|^4 \,. \tag{1.17}$$

Here the complex scalar field ϕ is the order parameter of the transition, and we use the notation $-|\phi|^4$ to denote tuning to the fixed point, which has $\mathcal{O}(1)$ self-interactions. The transition is tuned by introducing the mass operator $-\delta r|\phi|^2$, with the ordered phase corresponding to $\delta r < 0$ and the disordered phase corresponding to $\delta r > 0$.

Boson-vortex duality is the statement that, up to irrelevant operators, the partition function of the 3d XY model is equal to that of the Abelian Higgs model, which is a theory of bosonic "vortices" interacting logarithmically via a fluctuating $U(1)$ gauge field $a_{\mu,s}$,

$$Z_{\text{Abelian Higgs}}[\tilde{K}, g] = \int \mathcal{D}\tilde{\theta}\, \mathcal{D}a_\mu \prod_{s,\mu} \exp\left[\tilde{K}\cos(\Delta_\mu \tilde{\theta}_s - a_{\mu,s}) - \frac{1}{2g^2}(\Delta \times a)^2\right] , \tag{1.18}$$

$$Z_{\rm XY}[K] = Z_{\text{Abelian Higgs}}\left[\tilde{K} \to \infty, g^2 = (2\pi)^2 K\right] . \tag{1.19}$$

The Abelian Higgs model has two phases. The first is a superconductor ($g < g_c$), in which the vortices condense, a_μ is Higgsed, and all excitations are gapped. This phase is dual to the insulating phase of the 3d XY model. The second phase is an insulator ($g > g_c$), in which the spectrum contains a gapless photon. This phase is dual to the superfluid phase of the 3d XY model because since in three spacetime dimensions the photon only has one physical polarization, and so it is equivalent to a compact scalar field. The phase transition of the Abelian Higgs model is governed by a *gauged* Wilson-Fisher theory,

$$\mathcal{L}_{\text{Abelian Higgs}} = |D_a\tilde{\phi}|^2 - |\tilde{\phi}|^4 - \frac{1}{4g_{\rm M}^2} f_{\mu\nu}^2 \,, \tag{1.20}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. Here the Maxwell term only appears as a regulator: the fixed point corresponds to the infrared (IR) limit $g_{\rm M}^2 \to \infty$. While the derivation of the duality on the lattice does not necessarily imply the equivalence of the continuum field theories (1.17) and (1.20), it is highly suggestive of it. Indeed, if the duality holds at the critical point, then it explains the experimental observation that superconductor-insulator transitions and nematic-smectic transitions in liquid crystals are second order, in contrast to early conclusions from the ϵ-expansion [67].

The important conceptual content of boson-vortex duality is that charge in the XY model is dual to magnetic vortices in the Abelian Higgs model and vice-versa, i.e. if $J_\mu \sim \Delta_\mu\theta$ is the charge current in the

XY model, then

$$J_\mu = \frac{1}{2\pi}\, \varepsilon_{\mu\nu\lambda} \Delta_\nu a_\lambda \,. \tag{1.21}$$

This equivalence follows from the fact that a conserved current in three spacetime dimensions can always be expressed as the curl of a vector potential. Similarly, if we were to introduce a background EM gauge field A_μ under which the matter of the 3d XY model is charged, then integrating out a_μ can be shown to set

$$\langle \tilde{J}_\mu \rangle = \frac{1}{2\pi}\, \varepsilon_{\mu\nu\lambda} \Delta_\nu A_\lambda \,, \tag{1.22}$$

where $\tilde{J}_\mu \sim \Delta_\mu \tilde{\theta} - a_\mu$ is the gauge current in the Abelian Higgs model. Additionally, an immediate consequence of these relations is that the conductivity tensor of the charge of the XY model, σ_{ij}, is related to the resistivity of the vortices of the Abelian Higgs model, $\tilde{\rho}_{ij} = (\tilde{\sigma}_{ij})^{-1}$,

$$\sigma_{ij} = \frac{1}{(2\pi)^2}\, \varepsilon^{ik}\, \tilde{\rho}_{kl}\, \varepsilon^{jl} \,, \tag{1.23}$$

in units $e = \hbar = 1$. The exchange of conductivity with resistivity is a general feature of particle-vortex dualities, and it will play an important role at various points in this thesis.

We finally comment on the statement we made earlier that boson-vortex duality is only valid up to the addition of irrelevant operators. In the process of deriving the duality, it is necessary to introduce terms to the energy functional that flow to zero at long distances. This is actually the reason for the $\tilde{K} \to \infty$ limit in Eq. (1.19), which can be viewed as an IR limit. A consequence of this is that the duality is an *IR duality*, i.e. only an identification of ground states, not of the entire spectrum. The dualities we will make use of in this thesis will all be IR dualities.

1.2.4 Flux Attachment

Another duality that will play an important role in this thesis is flux attachment [68]. In theories of non-relativistic particles (fermions or bosons) in 2+1d, m flux quanta can be adiabatically attached to each particle via an emergent Chern-Simons gauge field [69]. This leads to an equivalent theory of composite particles which have statistics that have been shifted by $m\pi$, the Aharanov-Bohm phase associated with each flux quantum, and which experience a screened magnetic field. This duality holds in general for non-relativistic theories because such theories do not respect the spin-statistics theorem, so statistics can be transmuted without affecting spin.

Explicitly, if the original electric charges are described by a field ψ and the composite particles by a field

$\tilde{\psi}$, then flux attachment is the duality

$$\mathcal{L}[\psi, A] \longleftrightarrow \mathcal{L}[\tilde{\psi}, a] + \frac{1}{4\pi m}(a - A)d(a - A)\,, \tag{1.24}$$

where A_μ is the background EM field, a_μ is the statistical $U(1)$ gauge field, we use the notation $AdB = \varepsilon_{\mu\nu\lambda}A_\mu\partial_\nu B_\lambda$, and $\longleftrightarrow$ denotes duality. If ψ is a fermion, then $\tilde{\psi}$ is a composite fermion (boson) for m even (odd). From the equation of motion for a_0, it can be observed that the magnetic field felt by $\tilde{\psi}$ screened. If $b_* = \varepsilon_{ij}\partial_i a_j$ is the magnetic field felt by $\tilde{\psi}$, $B = \varepsilon_{ij}\partial_i A_j$, and $\rho = \langle\psi^\dagger\psi\rangle = \langle\tilde{\psi}^\dagger\tilde{\psi}\rangle$ is the electric charge density, then the equation of motion of a_0 can be written as

$$b_* = B - 2\pi m\rho\,. \tag{1.25}$$

The effective magnetic field is therefore reduced by an amount proportional to the number of flux quanta attached.

Flux attachment has played a central role in our understanding of the fractional quantum Hall (FQH) effect. By invoking the notion of composite fermions (electrons with $2n$ flux quanta attached), the observed Jain sequences of Abelian FQH states,

$$\nu_{\text{Jain}} = 2\pi\frac{\rho}{B} = \frac{p}{2np+1}\,,\ n, p \in \mathbb{Z}\,, \tag{1.26}$$

can be explained as $\nu = p$ IQH states of composite fermions [70, 71]. In addition, at $\nu = 1/2n$ (the $p \to \infty$ limit of ν_{Jain}), b_* vanishes. In the absence of a magnetic field, the composite fermions have no choice but to form a Fermi surface strongly coupled to the emergent gauge field a_μ. This is the HLR theory discussed in the introduction [72]. Furthermore, FQH states can also be understood as condensed phases of composite bosons, a picture which has proven essential in understanding the topological orders of FQH phases [73, 74].

Before proceeding, it is worth noting that Eq. (1.24) only makes sense locally. As written, the Chern-Simons term is not invariant under large gauge transformations because it does not have integer level. This issue can be resolved by introducing auxiliary gauge fields with properly quantized Chern-Simons levels (see e.g. Refs. [63, 75]),

$$\mathcal{L}[\psi, A] \longleftrightarrow \mathcal{L}[\tilde{\psi}, a] + \frac{1}{2\pi}cd(a - A) - \frac{m}{4\pi}cdc\,, \tag{1.27}$$

where c_μ is a $U(1)$ gauge field.

1.2.5 A Web of Quantum Critical Dualities

We now arrive at the dualities we will primarily consider in this thesis, which comprise a recently proposed "web" of QFT dualities connecting a wide range of quantum phase transitions in two spatial dimensions. This web of dualities was motivated the discovery of an equivalence between the 't Hooft limits of $U(N)_k$ Chern-Simons theories coupled to gapless complex scalar bosons and $SU(k)_{-N}$ Chern-Simons theories coupled to gapless Dirac fermions (here the subscript refers to the level of the Chern-Simons term) [76, 77]. This result led Aharony to conjecture the non-Abelian dualities [78],

$$N_f \text{ scalars} + U(N)_{k,k} \quad \longleftrightarrow \quad N_f \text{ fermions} + SU(k)_{-N+N_f/2}\,, \tag{1.28}$$

$$N_f \text{ scalars} + SU(N)_{k} \quad \longleftrightarrow \quad N_f \text{ fermions} + U(k)_{-N+N_f/2,-N+N_f/2}\,, \tag{1.29}$$

$$N_f \text{ scalars} + U(N)_{k,k+N} \quad \longleftrightarrow \quad N_f \text{ fermions} + U(k)_{-N+N_f/2,-N-k+N_f/2}\,. \tag{1.30}$$

where we use the notation $U(N)_{k,k'} = [SU(N)_k \times U(1)_{Nk'}]/\mathbb{Z}_N$ to refer to theories with differing Abelian and non-Abelian Chern-Simons levels.

The non-Abelian dualities presented by Aharony will play a central role in Chapter 4. For the most part, however, our interest will be in the dualities that spring forth when $N = k = 1$. In that case, the duality (1.28) becomes a "seed duality," from which an entire web of other 2+1d Abelian dualities can be derived [79, 80]. It relates a SIT of a Wilson-Fisher boson coupled to a fluctuating Chern-Simons gauge field to an IQH plateau transition of a free Dirac fermion[7],

$$\mathcal{L}_\phi = |D_a\phi|^2 - |\phi|^4 - \frac{1}{4\pi}ada + \frac{1}{2\pi}adA \longleftrightarrow \mathcal{L}_\Psi = i\bar{\Psi}\not{D}_A\Psi + \frac{1}{8\pi}AdA\,. \tag{1.31}$$

Here a_μ is a fluctuating $U(1)$ gauge field, while A_μ represents the background electromagnetic field. We use the notation $(D_B)_\mu = \partial_\mu - iB_\mu$, $\not{v} = v_\mu\gamma^\mu$. It can be immediately seen that the phase diagrams of these two theories match: when ϕ is gapped, integrating out a_μ leads to an integer Hall effect, $\sigma_{xy} = \frac{1}{2\pi}$, while when ϕ condenses, a_μ is Higgsed, and the resulting theory is a trivial insulator. Similarly, integrating out Ψ with a positive mass yields $+AdA/8\pi$, leading to an IQH state, while a negative mass yields $-AdA/8\pi$, leading to the trivial insulator. This means that the boson and fermion mass operators are dual to one another, $|\phi|^2 \leftrightarrow \bar{\Psi}\Psi$.

The duality, Eq. (1.31), is a relativistic version of flux attachment, since the Dirac fermion carries spin. From the point of view of the bosonic theory, the Dirac fermion is understood as a composite of a boson

[7] Throughout this thesis, we approximate the Atiyah-Patodi-Singer η-invariant by a level-1/2 Chern-Simons term and include it in the action. Also throughout this thesis: footnotes reminding the reader of this.

with a monopole of a_μ. Furthermore, one of the immediate consequences of this duality is that it formalizes one of experimental observations discussed at the beginning of the Introduction. The statement of duality implies that the **PH** symmetry of the Dirac fermion must also appear in the bosonic theory. It turns out that the action of this symmetry in the bosonic theory yields precisely its dual under boson-vortex duality, meaning that it exchanges bosons with vortices. This means that the bosons display self-duality!

This web of dualities represents a unifying, non-perturbative framework in which to study a wide range of strongly correlated quantum critical points and phases [13, 81–90], and it even includes the boson-vortex duality discussed above. One particularly important member of the web of dualities is a *fermion-vortex* duality relating a free Dirac fermion Ψ to a theory of Dirac composite fermions ψ interacting with a fluctuating gauge field (QED_3),

$$\mathcal{L}_\Psi = i\bar{\Psi}\not{D}_A\Psi + \frac{1}{8\pi}AdA \longleftrightarrow \mathcal{L}_\psi = i\bar{\psi}\not{D}_a\psi - \frac{1}{4\pi}adA + \frac{1}{8\pi}AdA\,. \tag{1.32}$$

This duality was initially proposed by Son [81] as a means of explaining **PH** symmetry at $\nu = 1/2$. Son noticed that the $\nu = 1/2$ problem of non-relativistic electrons with $g = 2$ could be mapped to that of free Dirac fermions, and his insight was to construct a composite fermion theory starting with the Dirac fermion description. This led him to conjecture the duality in Eq. (1.32), in which the magnetic field experienced by the free Dirac "electrons" sets the density of the Dirac composite fermions, which are thus interpreted as magnetic vortices (the equation of motion of a_0 sets $\rho_\psi = \langle\psi^\dagger\psi\rangle = B/4\pi$). Consequently, the **PH** symmetry of the free Dirac fermions (the physical electric charges) half filling the lowest LL is encoded as a time-reversal (**T**) symmetry of the composite fermions. However, despite providing an explanation for the **PH** symmetry at $\nu = 1/2$, it is important to emphasize that, on its own, Son's Dirac composite fermion theory does not explain **R** symmetry at more general compressible filling fractions like $\nu = 1/4$. Explaining **R** symmetry in the language of composite fermions is the topic of Chapter 3.

One of the major challenges when new dualities are proposed is to "derive" them, although it should be emphasized that any such derivation is necessarily done away from the critical point, meaning that it does not constitute a proof. Since the web of dualities was proposed, its members have been derived in the context of wire constructions [91, 92], Euclidean lattice models [93, 94], and from supersymmetric dualities [95–97]. However, none of these constructions explicitly involve the physics of flux attachment. In Chapter 2, we will show how the concept of relativistic flux attachment can be made precise in the context of loop models, leading to simple derivations of many of the members of the duality web.

1.3 Outline of the Thesis: The Uses of Duality

The main work of this thesis begins in Chapter 2 with the development of simple derivations of many of the members of the "web of dualities" using flux attachment in relativistic models of interacting current loops with particle-hole symmetry. Any gapped quantum field theory can be written as a loop model by passing to a world line path integral, providing a useful representation of the theories theories valid near (but not at) criticality. In particular, we will describe how any consistent choice of regulator implies that attaching flux to these world lines imparts not only braiding statistics, but also spin, in the form of a so-called fractional spin contribution to the action. Unlike statistics, fractional spin is not periodic under attachment of even numbers of flux quanta, implying that relativistic theories do not share the periodicity of their non-relativistic counterparts. Deriving the members of the web of dualities then amounts to showing that dual theories all correspond to the same loop model, with the same fractional spin term.

The remainder of this thesis is devoted to the application of QFT dualities, particularly the constituents of the duality web, to generate non-perturbative results related to strongly correlated quantum critical points and phases in two spatial dimensions. Such applications fall into three major categories

(i) Explaining emergent symmetries (Chapter 3)

If two theories are dual, then they must have the same global symmetries. Frequently, a symmetry that is manifest on one side of a duality is not manifest on the other, meaning that it must be *emergent* on that side. Sometimes, comparing the symmetries on either side of a duality can imply that the global symmetry is actually larger than is manifest on either side, as is the case in some theories with multiple matter species that have appeared in the study of phase transitions beyond the Landau paradigm [86, 98]. Duality can also be used to understand how symmetries are encoded across dual theories and how they manifest in different transport properties. An example of this is the connection between **PH** symmetry and self-duality in boson-fermion dualities.

The goal of Chapter 3 is to present a possible explanation of a particularly confounding emergent symmetry, the reflection symmetry observed at $\nu = 1/4$ in quantum Hall systems. Namely, we propose a series of relativistic composite fermion theories for general compressible states at filling $\nu = 1/2n$. These theories consist of electrically neutral Dirac fermions attached to $2n$ flux quanta via an emergent Chern-Simons gauge field. While not possessing an explicit particle-hole symmetry, these theories reproduce the known Jain sequence states proximate to $\nu = 1/2n$, and we show that such states can be related by the observed reflection symmetry, at least at mean field level. In particular, the reflection symmetry manifests itself as a $\mathbf{T}$ symmetry of the composite fermions, in analogy with Son's theory of $\nu = 1/2$. We further argue that the LLL limit requires that the mass of the Dirac fermions be tuned to zero, whether or not this symmetry extends to the compressible states themselves, and we describe how reflection symmetry is connected to self-duality of dual theories of composite bosons.

(ii) Uncovering exotic gapped phases (Chapter 4)

Another implication of duality is that dual quantum critical points should share a phase diagram. Indeed, this requirement is often used to check dualities. However, not all relevant perturbations map locally across dualities. Consequently, a phase accessible by condensing a local operator in one theory may be obscure in its dual. This means that dualities can be used to predict the presence of exotic phases. In one major example, s-wave pairing in Son's Dirac composite fermion theory (1.32) leads to a non-Abelian topological phase, the **PH**-Pfaffian superconductor [82, 83], but the superconducting order parameter is not dual to a local operator in the free Dirac theory.

In Chapter 4, we use this strategy to explore the landscape of non-Abelian fractional quantum Hall phases which can be obtained starting from physically motivated theories of Abelian composite particles. To do this, we use recently proposed non-Abelian bosonization dualities in 2+1 dimensions, which morally relate $U(N)_k$ and $SU(k)_{-N}$ Chern-Simons-matter theories. The advantage of these dualities is that regions of the phase diagram which may be obscure on one side of the duality can be accessed by condensing local operators on the other side. Starting from parent Abelian states, we use this approach to construct Landau-Ginzburg theories of non-Abelian states through a pairing mechanism. In particular, we obtain the bosonic Read-Rezayi sequence at fillings $\nu = k/(kM+2)$ by starting from k layers of bosons at $\nu = 1/2$ with M Abelian fluxes attached. The Read-Rezayi states arise when k-clusters of the dual non-Abelian bosons condense. We extend this construction by showing that N_f-component generalizations of the Halperin $(2,2,1)$ bosonic states have dual descriptions in terms of $SU(N_f+1)_1$ Chern-Simons-matter theories, revealing an emergent global symmetry in the process. Clustering k layers of these theories yields a non-Abelian $SU(N_f)$-singlet state at filling $\nu = kN_f/(N_f + 1 + kMN_f)$.

(iii) "Solving" a theory (Chapter 5)

Sometimes, one gets very lucky by having a strong-weak duality or a self-duality, which allow for one to non-perturbatively solve for many properties of the strongly coupled theory. Such dualities also enable one to study the effects of perturbations on the strongly interacting theory by considering its dual. In one such example (not featured in this thesis), we used Son's duality to study QED_3 with a single species of fermion in the presence of quenched disorder [85]. Because the problem of free Dirac fermions with quenched disorder is very well understood, this led to a variety of new non-perturbative results, including universal transport properties.

Chapter 5, in part, extends this approach to make new conjectures for the fate of Dirac fermions coupled to a (fluctuating) level-1/2 Chern-Simons gauge field in the presence of disorder, by studying its dual: a Wilson-Fisher boson. While in the clean limit the Wilson-Fisher fixed point is among the most well understood interacting QFTs, the physics of a Wilson-Fisher boson with a random mass has been debated for many years. Using a large-N limit of the $O(2N)$ model, we find that this problem is not only tractable, but that the clean Wilson-Fisher fixed point directly gives way to a quantum critical point with finite disorder and interaction strengths, as well as critical exponents in remarkable agreement with numerical studies of the superfluid-Mott glass transition when N is extrapolated to 1. This is in contrast to earlier results using a double-ϵ expansion, which yields RG flows that are spirals, suggesting violation of unitarity. When used to make conjectures about the dual Dirac fermion theory, the analysis of this Chapter demonstrates how progress in developing perturbative and non-perturbative techniques can inform one another.

Chapter 2

Deriving Duality: Relativistic Flux Attachment in Loop Models

*This Chapter is adapted from Hart Goldman and Eduardo Fradkin, Phys. Rev. B **97**, 195112 (2018).* *This paper is also cited as Ref. [99] in the References section of the thesis.*

2.1 Introduction

In theories of non-relativistic particles in 2+1 dimensional flat spacetime, it is an established fact that attachment of even numbers of flux quanta to each particle does not change their statistics, provided the world lines of the particles do not intersect [68]. This mapping from the original system of interacting particles to an equivalent system of (also interacting) "composite particles" (fermions or bosons) coupled to a dynamical Abelian Chern-Simons gauge field is an identity at the level of their partition functions (see Ref. [63] for a review). These mappings have played a key role in the theory of the fractional quantum Hall fluids [70–73], in particular in elucidating their topological nature [100–103], and showing that they are described by a Chern-Simons gauge theory at low energies [69]. With subtle but important differences, analogous mappings for relativistic quantum field theories in 2+1 dimensions between massive scalar fields and Dirac fermions were argued by Polyakov [104]. Because this duality involves transmutation of both statistics and spin, it does not accommodate the exact invariance under flux attachment seen in its non-relativistic counterpart.

Recently, a similar duality to Polyakov's was conjectured to hold, relating a Wilson-Fisher boson coupled to a Chern-Simons gauge field to one of a free Dirac fermion. From this "3D bosonization" duality, it was shown that one can derive a web of new dualities[1] between relativistic quantum field theories in 2+1 dimensions [79, 80]. These conjectures were motivated, in part, by the remarkable duality found between non-Abelian Chern-Simons gauge theories coupled to matter in the 't Hooft large-N limit [76–78] and by Son's proposal to map the problem of the half-filled Landau level [72] to a theory of massless Dirac fermions

[1]See, also, the early approaches to duality in 2+1 dimensions in Refs. [105–107].

in 2+1 dimensions [81], as well as the work connecting this problem to the theory of topological insulators in 3+1 dimensions [82, 83]. The evidence for these dualities has been steadily mounting, with derivations from Euclidean lattice models [93], wire constructions [91, 92], and deformations of supersymmetric dualities [95–97]. However, it has remained an open problem to construct derivations of these dualities in which relativistic flux attachment is implemented in a simple and transparent way explicitly using the Chern-Simons term.

In this Chapter, we show that such derivations can be constructed using relativistic models of current loops in 2+1 dimensions coupled to Chern-Simons gauge fields[2]. Such models can capture the physics of the theories of interest near criticality. They are analogues of models originally studied by Kivelson and one of us [12], which take the schematic form

$$\frac{1}{2}\left[g^2 J^\mu \frac{1}{\sqrt{\partial^2}} J_\mu + 2i\theta\epsilon^{\mu\nu\rho} J_\mu \frac{\partial_\nu}{\partial^2} J_\rho\right], \tag{2.1}$$

where J^μ is a configuration of closed bosonic world lines satisfying $\partial_\mu J^\mu = 0$. Here the first term is a long-ranged interaction of strength g^2, and the second term is a linking number which endows the matter with statistical angle θ. The model of Ref. [12] displays self-duality under the modular group[3] $\mathrm{PSL}(2,\mathbb{Z})$ generated by particle-vortex duality, which maps a theory of matter to one of vortices interacting with an emergent gauge field [65, 66], and flux attachment, which shifts θ by π. Similar $\mathrm{PSL}(2,\mathbb{Z})$ structures arise in the study of the phase diagram of the quantum Hall effect [10, 111–114] as well as in lattice models exhibiting oblique confinement [115–120]. Another appears as electric-magnetic duality and Θ-angle periodicity in 3+1 dimensions, which can be extended to correlation functions in 2+1 dimensional conformal field theories (CFTs) [121, 122]. More recently, this modular group has appeared as a way of organizing the above mentioned web of 2+1 dimensional field theory dualities [79]. It is important to note, however, that the PSL(2,$\mathbb{Z}$) of the duality web is not a group of dualities. Rather, it generates new dualities from known ones. On the other hand, the PSL(2,$\mathbb{Z}$) of the loop models we discuss here is to be taken as a group of dualities.

The invariance under flux attachment appearing in Ref. [12] is surprising given the apparent absence of such a symmetry in relativistic theories mentioned above. However, we will see that this is a consequence of a choice of regularization which is impossible to apply to continuum Chern-Simons gauge theories coupled to matter. This choice of regularization dispenses with the "fractional spin" which massive particles are endowed with due to their interaction with the Chern-Simons gauge field. This fractional spin was at the center of Polyakov's original argument for boson-fermion duality, and it is responsible for the complete breaking of

[2] Loop models have also been used by constructive field theorists to represent quantum field theories in 3+1 dimensional Euclidean spacetime [108–110].

[3] PSL(2,$\mathbb{Z}$) is the group of all 2×2 matrices with integer entries and unit determinant, defined up to an overall sign.

statistical periodicity[4]. Therefore, the inclusion of fractional spin allows contact with the Chern-Simons-matter theories comprising the web of dualities, enabling us to show that theories related by boson-fermion duality correspond to the same loop model. We are thus able to derive a duality web of loop models which parallels that of Refs. [79, 80].

We proceed as follows. In Section 2.2, we review the model of Ref. [12], discuss the inconsistency of statistical periodicity with the web of field theory dualities of Refs. [79, 80], and review the appearance of PSL$(2,\mathbb{Z})$ in both contexts. We then introduce the notion of fractional spin in Section 2.3, and we describe how it breaks statistical periodicity and is generic if our goal is to realize theories of relativistic matter coupled to Chern-Simons gauge fields at criticality. In Section 2.4, we show that the inclusion of fractional spin leads to consistency with the duality web and derive a parallel duality web of loop models. We conclude in Section 2.5.

2.2 Flux Attachment in a Self-Dual Loop Model

2.2.1 Model

Motivated by the fact that all quantum Hall plateau transitions appear to have essentially the same critical exponents [10, 11, 123], a phenomenon referred to as superuniversality, Kivelson and one of us wrote down a model of current loops with long-ranged ($1/r^2$, where r is the distance in 2+1 dimensional Euclidean spacetime) and statistical (linking number) interactions on a 3D Euclidean lattice displaying invariance under flux attachment ($\mathcal{T}$) and self-duality under particle-vortex duality ($\mathcal{S}$) [12]. This model therefore describes superuniversal families of fixed points related by elements of the modular group generated by $\mathcal{S}$ and $\mathcal{T}$, PSL$(2,\mathbb{Z})$. These fixed points have the surprising property that they not only share critical exponents, but also conductivities and other transport properties.

The loop model of Ref. [12] consists of integer-valued current loop variables J_μ representing the world lines of bosons on a 2+1 dimensional Euclidean cubic lattice, with marginally long-ranged and statistical interactions. The partition function is

$$Z = \sum_{\{J_\mu\}} \delta(\Delta_\mu J^\mu) e^{-S}\,, \tag{2.2}$$

where the delta function enforces the condition that the currents J_μ are conserved, or that the world lines

[4] These issues with modular invariance do not arise in models consisting of two species of loops which can only statistically interact with one another. Such models display modular invariance and may be related to BF theories in the continuum limit [118–120].

form closed loops. We require that the world lines are *non intersecting*, meaning that the bosons have a strong short-ranged repulsive interaction ("hard-core"). The action S is defined to be

$$
\begin{aligned}
S =& \frac{1}{2}\sum_{r,r'} J^{\mu}(r)G_{\mu\nu}(r-r')J^{\nu}(r') + \frac{i}{2}\sum_{r,R} J^{\mu}(r)K_{\mu\nu}(r,R)J^{\nu}(R) \\
&+ i\sum_{r,r'} e(r-r')J^{\mu}(r)A_{\mu}(r') + \sum_{R,R'} h(R-R')\epsilon^{\mu\nu\rho}J_{\mu}(R)\Delta_{\nu}A_{\rho}(R') \\
&+ \frac{1}{2}\sum_{r,r'} A_{\mu}(r)\Pi^{\mu\nu}(r,r')A_{\nu}(r')\,,
\end{aligned}
\tag{2.3}
$$

where r, r' are sites on the direct lattice, R are sites on the dual lattice, Δ_{μ} is a right lattice derivative, and A_{μ} is a background probe electromagnetic field. Importantly, in this model we regard the loops as matter and flux world lines which follow each other, being separated by a rigid translation, so $R = r+(1/2, 1/2, 1/2)$.

The symmetric tensor $G_{\mu\nu}$ and the antisymmetric tensor $K_{\mu\nu}$ are assumed to behave at long distances such that in momentum space they take the form

$$G_{\mu\nu}(p) = \frac{g^2}{|p|}\left(\delta_{\mu\nu} - p_{\mu}p_{\nu}/p^2\right)\,, \tag{2.4}$$

$$K_{\mu\nu}(p) = 2i\theta\,\epsilon_{\mu\nu\rho}\frac{p^{\rho}}{p^2}\,. \tag{2.5}$$

Here $G_{\mu\nu}$ represents long-ranged interactions (i.e. a $1/r^2$ interaction, where r is the Euclidean distance in 2+1 spacetime dimensions), and $K_{\mu\nu}$ represents the statistical interaction between matter fields (direct lattice) and flux (dual lattice) currents. In Eq. (2.4), the parameter g is the coupling constant. The parameter θ of Eq. (2.5) is the statistical angle of the world lines, and so θ/π is the number of flux quanta attached to each matter particle.

Because we have defined matter and flux world lines to follow one another, conventional self-linking processes are absent. As a result, the action for the statistical interaction of a given closed loop configuration J_{μ} is

$$\theta \times \Phi[J], \quad \text{where } \Phi[J] \in 2\mathbb{Z}\,, \tag{2.6}$$

where $\Phi[J]$ is *twice* the linking number of the loop configuration. In the continuum limit, $\Phi[J]$ is given by

$$
\begin{aligned}
\Phi[J] &= \frac{1}{2\theta}\int \frac{d^3p}{(2\pi)^3}\,J_{\mu}(-p)\,K^{\mu\nu}(p)\,J_{\nu}(p) \\
&= \frac{1}{4\pi}\int d^3x \int d^3y\,\epsilon^{\mu\nu\rho}\,J_{\mu}(x)\,\frac{(x_{\nu}-y_{\nu})}{|x-y|^3}\,J_{\rho}(y)
\end{aligned}
\tag{2.7}
$$

and is an even integer, so long as J_{μ} does not include any self-linking processes. This is twice the linking number since it counts each link twice (or each particle exchange once). The phase $\theta\Phi[J]$ should be regarded as the Berry phase of a configuration of closed loops labeled by the currents J_μ.

In terms of Φ, the partition function for long, closed loops can be written as

$$Z = \sum_{\{J_\mu\}} \delta(\Delta_\mu J^\mu)\, e^{i\theta\Phi[J]}\, e^{-\frac{1}{2}\sum_{r,r'} J^\mu(r) G_{\mu\nu}(r-r') J^\nu(r')} \,. \tag{2.8}$$

where we have suppressed source terms. Because $\Phi[J]$ is an even integer, the partition function is invariant under

$$\mathcal{T} : \theta \mapsto \theta + \pi \,. \tag{2.9}$$

In other words, this theory is invariant under attachment of any number of flux quanta. Physically, this is due to our neglect of self-linking, which would correspond to single exchange processes, and so the exchange processes allowed in the theory come in pairs.[5] Allowed exchange processes involving fermions therefore have the same amplitudes as their bosonic counterparts.

The reader may worry about the fact that we seem to allow θ to take fractional values. If we were to think of the statistical interaction as being obtained by integrating out a Chern-Simons gauge field, this would be inconsistent with gauge invariance in a purely 2+1 dimensional theory. Additionally, because $\theta \sim 1/k$, where k is the Chern-Simons level, $\mathcal{T}$ transformations do not map integer levels to integer levels. This can be resolved by the introduction of auxiliary gauge fields so that no gauge field in the theory has a fractional level, as has been done in the study of the fractional quantum Hall effect (see e.g. Refs. [63, 124]). We will nevertheless proceed with fractional values of θ and k for now since they should not affect local properties of the theory, and they do not run afoul of gauge invariance if the theory is defined on the boundary of a 3+1 dimensional system.

In addition to invariance under shifts of the statistical angle of Eq. (2.9), it can easily be seen that the model in Eq. (2.3) is also *self-dual* (in the absence of background fields, which break self-duality explicitly) under bosonic particle-vortex duality [65, 66]. This duality is a consequence of the fact that in 2+1 dimensions, a conserved current J_μ can be related to the field strength of an emergent gauge field a_μ

$$J_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\Delta^\nu a^\lambda \,. \tag{2.10}$$

In the case of the 3D XY model, this allows one to rewrite the partition function as one of bosonic vortex

[5] In Ref. [12], the statistical angle is defined (in the current notation) as 2θ. This was convenient in that work since self-linking processes were not allowed. Since we will relax this constraint soon, we have chosen to use the more conventional definition of the statistical angle as θ.

variables strongly interacting via a logarithmic potential mediated by a_μ. This dual theory is known as the Abelian Higgs model. In general, bosonic particle-vortex duality relates the symmetric, or insulating, phase of the matter variables to the broken symmetry, or superfluid, phase of the vortex variables: matter loops are scarce when vortex loops condense and vice versa. For the models described by Eq. (2.3), particle-vortex duality is the map (see Appendix A.1),

$$\mathcal{S} : \tau \mapsto -\frac{1}{\tau}\,, \tag{2.11}$$

where we have defined the modular parameter

$$\tau = \frac{\theta}{\pi} + i\frac{g^2}{2\pi}\,. \tag{2.12}$$

Together, $\mathcal{S}$ and $\mathcal{T}$ generate the modular group $\mathrm{PSL}(2,\mathbb{Z})$, which is the group of transformations

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \tag{2.13}$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

Invariance of the partition function under $\mathrm{PSL}(2,\mathbb{Z})$ enabled the authors of Ref. [12] to make predictions for the DC conductivities of this theory at the so-called modular fixed point values of τ. These are the points which are invariant under a particular modular transformation. We briefly review these results in the following subsection. In particular, since at the fixed points of $\mathrm{PSL}(2,\mathbb{Z})$ the longitudinal conductivity σ_{xx} is *finite*, the theory at such fixed points must also be at a fixed point in the sense of the renormalization group. One of the goals of the present work is to understand the nature of the conformal field theories describing these fixed points.

2.2.2 Modular Fixed Points and Superuniversal Transport

If we consider the partition function to be invariant under modular transformations in $\mathrm{PSL}(2,\mathbb{Z})$, then we can fully constrain transport properties at the modular fixed points. Each fixed point can be related to one of $\tau = i$ (invariant under $\mathcal{S}$), $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ (invariant under $\mathcal{TS}$), $\tau = i\infty$, or $\tau = \infty$.

Under a modular transformation which leaves a fixed point invariant, we expect invariance of the loop-loop correlation function

$$D_{\mu\nu}(p;\tau) = \langle J_\mu(p) J_\nu(p)\rangle = D_{\mathrm{even}}(p;\tau)(\delta_{\mu\nu} - p_\mu p_\nu/p^2) + D_{\mathrm{odd}}(p;\tau)\epsilon_{\mu\nu\lambda}\frac{p^\lambda}{|p|}\,. \tag{2.14}$$

Calculating $D_{\mu\nu}$ then amounts to writing down how it transforms under the modular transformation which leaves the fixed point invariant, equating that result to $D_{\mu\nu}$, and then solving. It is convenient to define

$$D(\tau) = \frac{2\pi}{|p|}(D_{\text{odd}}(\tau) - iD_{\text{even}}(\tau)) \,. \tag{2.15}$$

One can derive the transformation law for $D(\tau)$ under $\mathcal{S}$ by exploiting the invariance of the current-current correlation function

$$\mathcal{K}_{\mu\nu} = -\frac{\delta}{\delta A_\mu}\frac{\delta}{\delta A_\nu}\log Z[A]\big|_{A=0} \,, \tag{2.16}$$

which is invariant under *any* duality transformation and tracks how the source terms transform. It is *not* the same as the loop-loop correlation function, although they are related. Some algebra [12] shows that the invariance of $\mathcal{K}_{\mu\nu}$ implies

$$D\left(-\frac{1}{\tau}\right) = \tau^2 D(\tau) + \tau \,. \tag{2.17}$$

This equation implies that $D(\tau)$ is not invariant under $\mathcal{S}$, instead transforming almost as a rank 2 modular form [12]. We say almost because of the the last term in Eq. (2.17), which is known as the modular anomaly.

The conductivity,[6] in units of $e^2/\hbar$, is defined in terms of the loop-loop correlation function as

$$\sigma_{xx}(\tau) = \frac{1}{2\pi}\,\text{Im}[D(\tau)] \,, \qquad \sigma_{xy}(\tau) = \frac{1}{2\pi}\,\text{Re}[D(\tau)] \,. \tag{2.18}$$

This result enables us to immediately calculate the conductivity at the fixed point $\tau = i$, which is invariant under $\mathcal{S}$ transformations

$$D(i) = -D(i) + i \Rightarrow D(i) = \frac{i}{2} \,, \tag{2.19}$$

so the conductivity at $\tau = i$ is

$$\sigma_{xx}(i) = \frac{1}{2\pi}\,\text{Im}[D(i)] = \frac{1}{4\pi}, \qquad \sigma_{xy}(i) = 0 \,. \tag{2.20}$$

This gives a consistent result with continuum particle-vortex duality because it only requires self-duality under $\mathcal{S}$. The transport properties of this fixed point with Dirac fermion matter have been explored in detail in Ref. [126].

Before moving on to the other fixed points, a general result valid for all fixed points can be derived if we

[6] In general, the conductivity is a function of the ratio of frequency and temperature [125]. In this Chapter, we will exclusively consider optical conductivities, or the limit $T/\omega \to 0$.

consider $D(\tau)$ to be invariant under $\mathcal{T}$ transformations

$$D(\tau+1) = D(\tau)\,. \tag{2.21}$$

This can be thought of as a statement of superuniversality, as it equates conductivities at different values of $\theta = \pi\,\mathrm{Re}[\tau]$. It implies that the general transformation law for $D(\tau)$ is

$$D\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 D(\tau) + c(c\tau+d)\,, \tag{2.22}$$

This enables us to solve for $D(\tau)$ at an arbitrary fixed point. In particular, it enables us to uniquely determine $D(\tau)$ at the fixed points

$$D(\tau) = \frac{i}{2\,\mathrm{Im}[\tau]}\,. \tag{2.23}$$

Notice that this implies that the Hall conductivity is fixed at zero. The only ingredient required to obtain this result is modular invariance, manifested in duality, Eq. (2.17), and periodicity, Eq. (2.21). In Ref. [12], this result was interpreted as implying that when the loop model is at a modular fixed point where σ_{xx} is finite (e.g. $\tau = i$), it is at a critical point, where the loops become arbitrarily large and proliferate. In other words, at these modular fixed points the loop model must at a renormalization group fixed point represented by a scale-invariant (and, presumably, conformally invariant) quantum field theory.[7]

We can use Eq. (2.23) to derive conductivities for the other fixed points. The point at $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, referred to as the self-dual fermion point in Ref. [12] despite having $\theta = \pi/2$, has conductivity $\sigma_{xx}(1/2 + i\sqrt{3}/2) = \frac{1}{2\pi}\frac{\sqrt{3}}{3}$. Additionally, in Ref. [12], it was noted that there are modular fixed points on the real axis, which formally have $\sigma_{xx}(\infty) \to \infty$. However, in this limit, where the parity-even long-ranged interactions vanish, the short-ranged interactions can no longer be neglected and, in a sense, become dominant. In the next subsection, we will see that these "pathological" fixed points of the modular symmetry are in conflict with results derived from the duality web. We will later see in Section 2.3 that the correct definition of the loop models at short distances necessarily implies fractional spin, which spoils the periodicity symmetry (and hence modular invariance).

[7]Up to subtle orders of limits, a 2+1-dimensional theory of charged fields at criticality, i.e. a CFT, can have a finite longitudinal conductivity in the thermodynamic limit. This finite dimensionless quantity is a universal property of the CFT.

2.2.3 Modular Invariance and the Web of Dualities

An Attempt at a Field Theory Description

It is natural to ask whether at criticality the loop models in Eq. (2.3) approach relativistically invariant CFTs which inherit modular invariance and what the interpretation of this might be in the context of the duality web of Refs. [79, 80] and its own $\mathrm{PSL}(2,\mathbb{Z})$ structure. These theories would display superuniversality in both critical exponents *and transport*. The only obvious local candidates for such theories would consist of matter fields on a 2+1 dimensional surface in a bulk 3+1 dimensional spacetime interacting via an emergent, dynamical gauge field that propagates in the bulk with Maxwell and Θ terms. Such theories are analogous to models of fractional topological insulators [127–135], which in the bulk have fractional Θ-angles and support gapless matter on their boundaries. The connection to the loop model Eq. (2.3) is immediate: the Maxwell term would then integrate to the surface as long-ranged $1/r^2$ interactions between the matter particles, and the Θ term would become a Chern-Simons term of level $k = \Theta/2\pi$ which endows the matter with fractional statistics. One may also consider the surface theory on its own without a bulk, but this theory would be non-local. Without referencing a bulk, the Lagrangian for these theories takes the form

$$\mathcal{L}_{\mathrm{CFT}} = \mathcal{L}_{\mathrm{matter}}[a] - \frac{1}{4e^2} f^{\mu\nu} \frac{i}{\sqrt{\partial^2}} f_{\mu\nu} + \frac{k}{4\pi} a da \,, \tag{2.24}$$

where a_μ is a dynamical gauge field, $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, we use the notation $AdB = \epsilon^{\mu\nu\rho} A_\mu \partial_\nu B_\rho$, and we have again suppressed background terms. $\mathcal{L}_{\mathrm{matter}}[a]$ can be taken to be the Lagrangian either for a single species of Dirac fermion or Wilson-Fisher boson coupled to a_μ. In the case of bosonic matter, a natural modular parameter for this theory is $k + i\frac{2\pi}{e^2} = \frac{\Theta}{2\pi} + i\frac{2\pi}{e^2}$, which would correspond to $-\frac{1}{\tau}$ in the loop model language.

Models of the form Eq. (2.24) are self-dual under $\mathcal{S}$, which can be taken to be (fermionic or bosonic) particle-vortex duality [65, 66, 82, 83]. Recently, inspired by the web of field theory dualities, this self-duality has been explored anew [92, 126], building on the earlier analytic work on bosonic loop models in Ref. [12] and on numerical work at $\theta = 0$ [136]. However, invariance under $\mathcal{T}$ is far from manifest in these theories. It is a deformation of the Chern-Simons level, which does not preserve the phase diagram of the theory, affecting both local (e.g. Hall conductivities) and global (e.g. ground state degeneracy on a torus) properties of the gapped phases. Moreover, invariance of transport properties under $\mathcal{T}$ leads to predictions which are inconsistent with those of the duality web, which does not accommodate sharing of transport properties amongst theories with general values of k, as we will see below in the next subsection. Theories of the form of Eq. (2.24) related by $\mathcal{T}$ therefore cannot be dual. In Section 2.3, we will see that this apparent tension is resolved upon the introduction of fractional spin, which breaks periodicity in the loop models completely.

Inconsistency of Modular Invariance with the Duality Web

We can check for consistency of the transport predictions one obtains from modular invariance with those from the duality web of Refs. [79, 80]. While the predictions for the modular fixed point at $\tau = i$ are consistent whether the matter content is fermionic or bosonic [126, 136], this is not the case for the fixed points on the real τ line. We can see this by studying the (conjectured) duality between a free Dirac fermion[8] and a gauged Wilson-Fisher fixed point

$$i\bar{\psi}\not{D}_A\psi - \frac{1}{8\pi}AdA \longleftrightarrow |D_a\phi|^2 - |\phi|^4 + \frac{1}{4\pi}ada + \frac{1}{2\pi}adA \longleftrightarrow |D_{b-A}\tilde{\phi}|^2 - |\tilde{\phi}|^4 - \frac{1}{4\pi}bdb\,, \tag{2.25}$$

where A is a background gauge field, $D^\mu_\alpha = \partial^\mu - i\alpha^\mu$, and we use the notation $\not{\alpha} = \alpha^\mu\gamma_\mu$, where the γ_μ's are the Dirac gamma matrices. Throughout this Chapter, we will use $\longleftrightarrow$ to indicate duality. The duality between the bosonic theories is a particle-vortex duality. We expect the bosonic theories to correspond to the loop model fixed points at $\tau = \mp 1$, $k = \pm 1$ (invariant under $\mathcal{T}^2\mathcal{S}$), where k is the level of the Chern-Simons gauge field.

Because the Dirac fermion on the left hand side of Eq. (2.25) is free, we can use this duality to calculate the optical conductivities of the strongly coupled bosonic theories. As with the loop models, duality implies that the correlation function,

$$\mathcal{K}_{\mu\nu} = -\frac{\delta}{\delta A_\mu}\frac{\delta}{\delta A_\nu}\log Z[A]\big|_{A=0}\,, \tag{2.26}$$

should be the same for each of these theories. From the free fermion theory, it is easy to calculate the conductivity (again in units of $e^2/\hbar$)

$$\frac{1}{i\omega}\mathcal{K}_{xx} = \frac{1}{16},\ \frac{1}{i\omega}\mathcal{K}_{xy} = -\frac{1}{4\pi}\,. \tag{2.27}$$

From the particle-vortex duality in Eq. (2.25), we see that the current-current correlation functions for the ϕ and $\tilde{\phi}$ gauge currents, J_μ and $\tilde{J}_\mu$ respectively, differ only in the Hall conductivity

$$\begin{aligned}\frac{1}{i\omega}\mathcal{K}_{ij}(\omega) &= \frac{1}{i\omega}\langle\tilde{J}_i(-\omega)\tilde{J}_j(\omega)\rangle \\ &= \frac{1}{i\omega}\langle J_i(-\omega)J_j(\omega)\rangle - \frac{1}{2\pi}\epsilon_{ij}.\end{aligned} \tag{2.28}$$

[8] In this Chapter, we approximate the η-invariant by $\frac{1}{8\pi}AdA$ and include it in the action.

Notice that this matches Eq. (2.17) for the case $\tau = -1$. Upon denoting[9]

$$\sigma^{\phi}_{ij} = \frac{1}{i\omega}\langle J_i(-\omega)J_j(\omega)\rangle\,, \tag{2.29}$$

$$\sigma^{\tilde{\phi}}_{ij} = \frac{1}{i\omega}\langle \tilde{J}_i(-\omega)\tilde{J}_j(\omega)\rangle\,, \tag{2.30}$$

we obtain

$$\sigma^{\phi}_{xx} = \sigma^{\tilde{\phi}}_{xx} = \frac{1}{16}\,, \qquad \sigma^{\phi}_{xy} = -\sigma^{\tilde{\phi}}_{xy} = \frac{1}{4\pi}\,. \tag{2.31}$$

This result disagrees with the prediction of Eq. (2.23) of modular invariance! Not only is the Hall conductivity nonvanishing, but the transverse conductivity is finite. Hence, if the bosonization duality is to be trusted, periodicity cannot extend to transport in theories of gapless matter coupled to a Chern-Simons gauge field, i.e. a version of Eq. (2.21) cannot hold. The conclusion one is driven toward is that any loop model description of these theories cannot be periodic either.

A Multitude of Modular Groups

Before concluding this section, we note the appearance of $\mathrm{PSL}(2,\mathbb{Z})$ in the context of the duality web of Ref. [79] in order to distinguish it from the modular group we are primarily concerned with. In that work, new dualities are obtained from old ones by the application of modular transformations to the conformal field theories on either side of a duality. If Φ denotes a set of dynamical fields and A is a background gauge field, these transformations act on a Lagrangian $\mathcal{L}[\Phi, A]$ as [121]

$$\tilde{\mathcal{S}} \quad : \quad \mathcal{L}[\Phi, A] \mapsto \mathcal{L}[\Phi, a] + \frac{1}{2\pi} A da\,, \tag{2.32}$$

$$\tilde{\mathcal{T}} \quad : \quad \mathcal{L}[\Phi, A] \mapsto \mathcal{L}[\Phi, A] + \frac{1}{4\pi} A dA\,, \tag{2.33}$$

where a is a dynamical gauge field. Here $\tilde{\mathcal{S}}$ involves gauging $A \to a$ and adding a BF term coupling a to a new background gauge field (also denoted A), and $\tilde{\mathcal{T}}$ is simply the addition of a background Chern-Simons term. If A is allowed to exist in a bulk 3+1 dimensional spacetime for which $\mathcal{L}[\Phi, A]$ is the boundary Lagrangian, $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{T}}$ correspond respectively to electromagnetic duality and Θ-angle periodicity of the bulk theory. The modular group generated by $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{T}}$ also organizes the global phase diagram of the fractional quantum Hall effect, where it has a natural action on the conductivities of the incompressible phases [10, 111–113].

[9] Usually, in gauge theories one is interested in the 1PI conductivity, which is defined using the polarization tensor $\Pi_{ij}(\omega)$ as $\sigma^{\mathrm{1PI}}_{ij} = \Pi_{ij}(\omega)/i\omega$ and characterizes the current response to the sum of the probe and emergent electric fields. In general, these conductivities map to resistivities under particle-vortex duality [85, 118, 119, 126, 137]. However, for correct comparison to the constraints obtained in the loop model of Ref. [12], we use the conductivities associated with the full current-current correlation functions.

It has also provided insight into the problem of superuniversality, relating theories which appear to share correlation length exponents despite having distinct transport properties [114].

In general, the modular transformations $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{T}}$ *are not* duality transformations themselves: they do not always leave the partition function of a particular theory invariant. This is obvious for $\tilde{\mathcal{T}}$, which shifts the Hall conductivity[10]. $\tilde{\mathcal{S}}$, on the other hand, is only occasionally a duality transformation, e.g. in the case of the duality between the Abelian Higgs model and a boson at its Wilson-Fisher fixed point. In contrast, the $\mathrm{PSL}(2,\mathbb{Z})$ associated with the self-dual loop model of Ref. [12] is a group of dualities: there $\mathcal{S}$ is identified with particle-vortex duality, and $\mathcal{T}$ is periodicity. $\mathcal{S}$ can also be related to bulk electromagnetic duality in the case where A is a dynamical field, albeit in a way slightly different from $\tilde{\mathcal{S}}$ [126].

2.3 Fractional Spin and the Fate of Periodicity

2.3.1 Fractional Spin and the Framing Anomaly

The full $\mathrm{PSL}(2,\mathbb{Z})$ invariance of the loop model of Ref. [12] above relies on the absence of self-linking and thus of fractional spin. This corresponds to a convenient choice of regularization, but we will find that such regularization is not available for CFTs of the form of Eq. (2.24). To see this, we must carefully include self-linking in the loop models reviewed above, as such processes generically appear in continuum field theories. Moreover, whenever one considers self-linking processes in a Chern-Simons theory, they are confronted with the framing anomaly, with which fractional spin is associated. We will find that (1) the inclusion of self-linking processes while neglecting fractional spin breaks the $\mathcal{T}$-invariance of the loop model not only down to invariance under $\mathcal{T}^2$ (the usual statistical periodicity $\theta \sim \theta + 2\pi$), and that (2) fractional spin breaks $\mathcal{T}$-invariance entirely.

Consider the loop model of Eq. (2.3) with the inclusion of self-linking processes. For convenience, now and in the remainder of this Chapter we will use a continuum description, replacing lattice sums with integrals. The reader may be concerned that this passage to the continuum is too cavalier. However, starting from a continuum, gapped field theory, we can always rewrite the partition function as a world line path integral without referencing a lattice. See, for example, Refs. [63, 139–142].

[10]Despite the fact that it is not a duality transformation, $\tilde{\mathcal{T}}$ does not change the fractional part of the Hall conductivity, which an universal obsevable in a topological phase [10, 102] and in a CFT [138]. This statement can be thought of as the 2+1 dimensional analogue of Θ-angle periodicity in 3+1 dimensions, although it is important to emphasize that this is a statement about Chern-Simons terms of *background* gauge fields.

The linking number term in the action is

$$\theta\,\Phi[J]=\theta\int d^3x J d^{-1}J=\frac{\theta}{4\pi}\int d^3x\int d^3y\,\epsilon^{\mu\nu\rho}J_\mu(x)\frac{(x_\nu-y_\nu)}{|x-y|^3}J_\rho(y)\,. \tag{2.34}$$

Note that to properly define this term, we must assume that the configuration J does not involve any loops which cross. This constraint can be implemented through additional short-ranged interactions like those which characterize the Wilson-Fisher fixed point. Now consider a configuration of two loops, $J(x)=\ell^{(1)}(x)+\ell^{(2)}(x)$, where each $\ell^{(i)}$ represents a single closed loop with unit charge. The action of this configuration is

$$\theta\,\Phi[J]=\theta\left(2\varphi[\ell^{(1)},\ell^{(2)}]+\varphi[\ell^{(1)},\ell^{(1)}]+\varphi[\ell^{(2)},\ell^{(2)}]\right)\,, \tag{2.35}$$

where

$$\varphi[\ell^{(i)},\ell^{(j)}]=\frac{1}{4\pi}\int d^3xd^3y\,\epsilon^{\mu\nu\rho}\ell^{(i)}_\mu(x)\frac{(x_\nu-y_\nu)}{|x-y|^3}\ell^{(j)}_\rho(y)\,. \tag{2.36}$$

The first term in Eq. (2.35) is twice the linking number of the two loops and is an integer-valued topological invariant: it simply counts the number of times the two loops link. This is the only term which appears in the model discussed in Section 2.2. The last two terms are referred to as the *writhes* of $\ell^{(1)}$ and $\ell^{(2)}$ respectively, denoted below as $W[\ell^{(i)}]=\varphi[\ell^{(i)},\ell^{(i)}]$.

The writhe contains a "fractional spin" term, which Polyakov showed can transmute massive scalar bosons to massive Dirac fermions in 2+1-dimensions [104]. Unlike the linking number, the writhe is not a topological invariant: it depends on the metric. It also generically breaks invariance under orientation-reversal of the loops, which can be thought of as particle-hole (or charge conjugation) symmetry (**PH**), in addition to time-reversal (**T**) and parity[11] (**P**). This metric dependence can in principle be eliminated by calculating self-linking numbers using a point-splitting regularization following Witten [69], in which the loops are broadened into ribbons with a framing vector $a_{\mathrm{f}}\hat{n}$, the edges of which having a well defined linking number SL,

$$SL[\ell]=\lim_{a_{\mathrm{f}}\to 0}\frac{1}{4\pi}\oint_\ell dx^\mu\oint_\ell dy^\nu\epsilon_{\mu\nu\rho}\frac{(x^\rho-y^\rho+a_{\mathrm{f}}\hat{n}^\rho)}{|x-y+a_{\mathrm{f}}\hat{n}|^3}\,, \tag{2.37}$$

However, this is at the cost of introducing a framing ambiguity in the calculation of this linking number: there is in general no canonical way to convert a loop into a ribbon. On the other hand, we can break the topological character of the theory along with **PH**, **T**, and **P** by including the fractional spin, eliminating the framing ambiguity. This choice is the manifestation of the framing anomaly [69, 143] in the language of

[11] Reflection about one of the spatial axes.

loop models.

Self-Linking Without Fractional Spin: Point Splitting

Let us consider what happens if we choose Witten's point-splitting procedure, which looks appealing because we may replace the writhe with a topological invariant. If we replace the writhe with the self-linking number and plug this into the action for the linking of two loops in Eq. (2.35), we obtain the action,

$$S = \theta(SL[\ell^{(1)}] + SL[\ell^{(2)}]) + 2\theta\varphi[\ell^{(1)}, \ell^{(2)}]\,. \tag{2.38}$$

The self-linking number SL can take any integral value, so here S is only invariant mod 2π under

$$\mathcal{T}^2 : \theta \mapsto \theta + 2\pi\,. \tag{2.39}$$

Here $e^{i\theta}$ is the phase the wave function picks up upon a single exchange process of two particles, as discussed in the previous section. It is worth noting that exchange processes which form closed loops are only possible in relativistic theories, where we have particles and antiparticles available for braiding. In non-relativistic systems, to obtain this type of process, one must compactify time and wrap the particle world lines around the time direction.

We have now found that the $\mathcal{T}$ invariance of a model without self-linking is broken down to the usual periodicity of the statistical angle when self-linking, but not fractional spin, is included. This means that the PSL$(2,\mathbb{Z})$ modular invariance of the model of Ref. [12] is broken down to a subgroup generated by particle-vortex duality ($\mathcal{S}$) and $\mathcal{T}^2$. We may therefore be inclined to accept the framing ambiguity and proceed by calculating the partition function with a point-splitting regularization of the linking integral. However, we will soon see that not even this symmetry can be accommodated by the continuum CFTs we might hope to describe.

Introducing Fractional Spin

Now consider the regularization in which the full writhe remains in the action without adopting a point-splitting regularization, following Polyakov. In this case, the action is frame independent, but this comes at the cost of reintroducing the metric. There is a general relation in knot theory relating $W[\ell]$ and $SL[\ell]$ [144],

$$W[\ell] = SL[\ell] - T[\ell]\,, \tag{2.40}$$

where $T[\ell]$ is referred to as the twist of the world line ℓ. It is Polyakov's fractional spin term, and it can be written as [104]

$$T[\ell] = \frac{1}{2\pi}\oint_\ell \omega' = \frac{1}{2\pi}\oint_\ell ds\,\hat{e}\cdot(\hat{n}\times\partial_s\hat{n})\,, \tag{2.41}$$

where ℓ is parameterized by the variable $s \in [0, L]$, $\hat{e}$ is the unit tangent vector to ℓ, and $\hat{n}$ is again a chosen frame vector normal to ℓ. This integral clearly depends on the metric, and it measures the angular rotation of $\hat{n}$ about $\hat{e}$. ω' is the angular velocity of $\hat{n}$. It can be thought of as a spin connection restricted to ℓ. However, this integral need not vanish on a flat manifold, and it can take non-integer values because it depends on the embedding of ℓ in spacetime.[12]

Up to addition by an integer, the integral of Eq. (2.41) can be written as a Berry phase by extending $\hat{e}$ to a disk: $\hat{e}(s) \to \hat{e}(s,u)$, where $u \in [0,1]$ and $\hat{e}(s,u=1) = \hat{e}(s), \hat{e}(s,u=0) = \hat{e}_0 \equiv$ constant,

$$T[\ell] = \frac{1}{2\pi}\int_0^L ds\int_0^1 du\,\hat{e}\cdot(\partial_s\hat{e}\times\partial_u\hat{e}) + n,\; n\in\mathbb{Z}\,. \tag{2.42}$$

This Berry phase form is what earns this term the name of fractional spin.

For $\theta = \pm\pi$, Polyakov argued that the loop model partition function for particles with this Berry phase (massive charged scalar bosons coupled to a Chern-Simons gauge field) is that of a single massive Dirac fermion of mass M, [13]

$$Z_{\text{fermion}} = \det[i\not{\partial} - M] = \int \mathcal{D}J\,\delta(\partial_\mu J^\mu)\,e^{-|m|L[J] - i\,\text{sgn}(M)\pi\Phi[J]}\,. \tag{2.43}$$

This relation[14] requires some unpacking, especially since we will encounter several more like it in Section 2.4. Here we have fully passed to a continuum picture where J^μ is a current density and so is not restricted to be an integer (although $\int_S d\Sigma_\mu J^\mu \in \mathbb{Z}$ for any closed surface S), thus the use of $\int \mathcal{D}J$ rather than $\sum_{\{J\}}$. $L[J]$ is the sum of the lengths of the loops in the configuration J, and the term $-|m|L[J]$ represents tuning away from criticality into a phase of small loops so that the partition function converges. It is generic in loop models, despite the fact that we have suppressed it thus far. $\Phi[J]$ is the linking number (2.34) of J (without appeal to point splitting), which contains the full writhes and, therefore, the fractional spin factor of each loop. The sign of the linking number term matches the sign of the fermion mass M, which is proportional to m. Note that J is not coupled to a gauge field here: this leads to the appearance of the parity anomaly, which breaks $\mathbf{P}, \mathbf{T}$, and $\mathbf{PH}$ even in the approach to criticality $m \to 0$. We will discuss how to couple J to

[12] For an explicit example, see Ref. [141]

[13] Here, and in several places below, we do not make explicit the fact that loops have strong short range repulsive interactions, without which these expressions involving linking numbers do not make sense.

[14] An unpublished work by Ferreirós and one of us [145] discusses an extension of Polyakov's duality in curved spacetimes.

a gauge field in detail in Section 2.4. For a review of Polyakov's original argument, see Appendix A.2. See also later work fleshing out some of the details, Refs. [141, 142, 146–148].

For general statistical angle θ, we have the loop model partition function

$$Z = \int \mathcal{D}J\, \delta(\partial_\mu J^\mu)\, e^{-|m|L[J]+i\theta\Phi[J]}\,. \tag{2.44}$$

Unless $\theta = 0, \pm\pi$, **T**, **P**, and **PH** are broken explicitly even in the $m \to 0$ limit. Since the value of the twist $T[\ell]$ is not restricted to the integers, $\Phi[J]$ is not restricted to the integers either. This means that fractional spin eliminates even the $\mathcal{T}^2$ symmetry we found via point-splitting. This has consequences for universal physics: for example, in the case $\theta = \pi$, the theory of a free Dirac fermion one obtains with fractional spin has a different correlation length exponent from the theory of spinless fermions one would have obtained neglecting fractional spin.

If periodicity is broken in the presence of fractional spin, how should one interpret shifts of the statistical angle? Say that we start with $\theta = \pi$, or the free Dirac fermion. When θ is shifted, the fractional spin is in turn shifted, and we can no longer make the mapping to a free Dirac fermion. It cannot be a theory of a higher spin particle either, since there are no non-trivial higher spin particles in 2+1-dimensions.[15] However, in Section 2.4, we will argue that the theory which can reproduce the same spin factor is a theory of Dirac fermions strongly coupled to Chern-Simons gauge fields.

2.3.2 Fractional Spin is Generic

Having established that the introduction of fractional spin breaks periodicity of the statistical phase completely, we now describe how fractional spin is a generic feature of loop models of the form of Eq. (2.3). It is known that, in the presence of ultraviolet (UV) scales, Witten's point-splitting regularization described above does not generally eliminate fractional spin. If we decouple the J variables by introducing an emergent gauge field a, this can be seen if we turn on arbitrarily weak short-ranged interactions in the form of a Maxwell term[16]

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4g_{\text{M}}^2} f^2\,. \tag{2.45}$$

Due to the existence of this term point-splitting no longer has the desired effect: one continues to obtain the metric-dependent $W[\ell]$ rather than the topological invariant $SL[\ell]$ [149]. This is because the Maxwell

[15] The reader can convince themselves of this by writing down the action for a Rarita-Schwinger (spin-3/2) field in 2+1 dimensions and considering its equations of motion. They will find that such a field does not propagate.

[16] The same thing happens if we UV complete the Chern-Simons terms to lattice fermions coupled to dynamical gauge fields, as was done in Ref. [93] to construct a lattice proof of the duality between a gauged Wilson-Fisher boson and a free Dirac fermion.

term introduces a short-distance cutoff $a_{\rm M} = 2\pi(g_{\rm M}^2 k)^{-1}$, where k is again the level of the Chern-Simons term, and the different result obtained by point-splitting is a consequence of the short-distance singularity of the Chern-Simons propagator in the absence of a natural cutoff. More physically, with the Maxwell term, flux is no longer localized on the matter world lines, but is smeared around the world line out to lengths of order $a_{\rm M}$. When this singularity is smoothed out, the self-linking number becomes metric-dependent but frame independent, leading to the full writhe. The existence of a Maxwell term therefore removes the UV ambiguities that exist in pure Chern-Simons theory and renders fractional spin unavoidable: the Maxwell term is dangerously irrelevant.

The above argument assumes a particular order of limits. When we consider Witten's point-splitting regularization, there is also the length scale $a_{\rm f}$ associated with the point-splitting, as in Eq. (2.37). If this scale is kept longer than $a_{\rm M}$ as we take the infrared (IR) limit $a_{\rm M} \to 0$, then we would obtain $SL[\ell]$ rather than $W[\ell]$. In other words, in this order of limits, it is as if the Maxwell term was never introduced. In the presence of long-ranged interactions, we might think that such an order of limits would be allowed since the Maxwell term is not required to suppress fluctuations of the emergent gauge field (without long-ranged interactions, the Maxwell term must be included for this purpose). However, to obtain correctly propagating matter at criticality, the particle-vortex duals of these theories *must* have nonvanishing Maxwell terms [137]. To see this, notice that the core energy term,

$$\mathcal{L}_{\rm core} = \frac{\varepsilon_c}{2} J_\mu^2(r)\,, \tag{2.46}$$

is a Maxwell term for the emergent gauge field b in the dual theory since particle-vortex duality relates $J_\mu = \epsilon^{\mu\nu\rho}\Delta_\nu b_\rho/2\pi$. Core energy terms can be rewritten as the kinetic terms for the phase fluctuations of the matter fields, and so are crucial for giving rise to the right kinetic terms for the matter fields as we approach criticality. Thus, it is not possible to simultaneously eliminate the Maxwell term in both a theory and its particle-vortex dual, and so it is inconsistent to take $a_{\rm M} \to 0$ before $a_{\rm f} \to 0$.

2.3.3 Fractional Spin and Conformal Field Theories

The arguments we have presented in this section are well defined in the UV with a specific regulator assumed. Such an analysis amounts to defining a continuum field theory for the loop model. From the discussion above, it is clear that this limit must be subtle given that Witten's and Polyakov's regularizations are not equivalent. Furthermore, as the non-trivial fixed point is approached, the relevant loop configurations become large and fractal-like (reflecting the anomalous dimensions at the fixed point), hence reaching all the way from the UV to the IR. An understanding of these limits is essentially what is needed for a "derivation" of the conjectured web of field theory dualities of Refs. [79, 80].

The arguments of the previous subsection immediately imply that Polyakov's regularization, in which fractional spin appears, is significantly more natural than Witten's point splitting procedure. We therefore conclude that any loop model description of CFTs of the general form,

$$\mathcal{L}_{\mathrm{CFT}} = \mathcal{L}_{\mathrm{matter}}[a] - \frac{1}{4e^2} f^{\mu\nu} \frac{i}{\sqrt{\partial^2}} f_{\mu\nu} + \frac{k}{4\pi} ada \,, \tag{2.47}$$

should include fractional spin. This is because we can always build a loop model by deforming these theories into a phase, and this loop model will generically include fractional spin. Unlike the loop model of Ref. [12], loop models with fractional spin do not display invariance under periodicity $\mathcal{T}$, so we no longer encounter the issue that periodicity relates theories with different phase diagrams, which should not display duality. Moreover, loop models with fractional spin should yield transport predictions consistent with those of the duality web in Section 2.2.3. This is because Polyakov's duality, Eq. (2.43), uses fractional spin to relate a free massive Dirac fermion to a massive boson with strong short-ranged interactions coupled to a Chern-Simons gauge field at level ± 1. Extrapolated to criticality, this duality would simply be the one featured in Eq. (2.25), so the transport predictions of this duality would match those of Section 2.2.3 (we will explain how to couple Polyakov's duality to background fields in Section 2.4.1).

A more subtle question is whether periodicity somehow survives in any of the correlation functions or critical exponents of the theories of Eq. (2.47), even though it does not appear in general. This is one way of phrasing the problem of superuniversality of quantum Hall plateau transitions. On general grounds, because we lack a duality relation between theories related by periodicity, there is no reason to expect the theories of Eq. (2.47) to have observables which are invariant under periodicity. For example, by the arguments of Section 2.2.3, we do not expect that DC transport in these theories has a simple transformation law under periodicity. However, there is some reason to be optimistic about critical exponents: recently it has been argued using non-Abelian bosonization dualities that certain theories related by periodicity share correlation length exponents [114].

We now return to the question of whether there is any CFT for which the model of Ref. [12], with full or partial modular invariance, is a good lattice regularization. The answer seems to be negative. As argued above, fractional spin is quite generic, and it prevents us from using generic theories of Chern-Simons gauge fields coupled to gapless matter. However, perhaps there exists an exotic CFT (either local or nonlocal) which can realize periodicity as a symmetry of the partition function along with self-duality. This is an open question.

2.4 Fractional Spin and a Duality Web of Loop Models

Having argued that any loop model with hope of describing Chern-Simons theories coupled to matter should include fractional spin, we can ask whether such loop models satisfy the dualities of Refs. [79] and [80]. Our strategy will be to use Polyakov's duality, Eq. (2.43), which expresses the partition function of a massive fermion as a bosonic loop model with fractional spin, to derive new dualities. This parallels the philosophy of Refs. [79] and [80], which derives the duality web of field theories starting from the assumption of the duality between a gauged Wilson-Fisher boson and a free Dirac fermion. An advantage of working with bosonic loop models is that we never have to work with fermionic matter explicitly. Instead, we derive dualities of the corresponding bosonic loop models. As a result, it is inconvenient to derive boson-boson dualities starting from the seed bosonization duality of Eq. (2.65). Such dualities are better thought of as following from the Peskin-Halperin-Dasgupta procuedure [65, 66] for deriving the particle-vortex duality of lattice loop models. This duality is exact in these models assuming that the statistical interactions between the loops (including fractional spin) can be suitably defined on a lattice [12].

2.4.1 Coupling Polyakov's Duality to Gauge Fields

In order to obtain new loop model dualities from Polyakov's duality, Eq. (2.43), we must couple the loop variables to a gauge field A, which here we will take to be a background field satisfying the Dirac quantization condition,

$$\int_{S^2} \frac{dA}{2\pi} \in \mathbb{Z}\,, \tag{2.48}$$

for any S^2 submanifold of the spacetime. In theories of a single Dirac fermion, coupling to gauge fields leads to the parity anomaly, so we should expect the loop model partition function Eq. (2.43) to also exhibit the parity anomaly. To see how this works, we start with a theory of massive scalar bosons coupled to a Chern-Simons gauge field at level $+1$,[17]

$$|D_a\phi|^2 - m_0^2|\phi|^2 - |\phi|^4 + \frac{1}{4\pi} a da\,. \tag{2.49}$$

This is the bosonic theory in Polyakov's duality, and its partition function can be rewritten as the loop model on the right hand side of Eq. (2.43). m_0 is related to the mass m in that equation, but it is not exactly equal to it [142]. Notice that we work in the symmetric (insulating) phase of the theory where the

[17] Throughout this section, we use a metric with the Minkowski signature.

global $U(1)$ symmetry is unbroken, so that a is not Higgsed. We couple this theory to A as follows,

$$|D_a\phi|^2 - m_0^2|\phi|^2 - |\phi|^4 + \frac{1}{4\pi}ada + \frac{1}{2\pi}adA\,. \tag{2.50}$$

Coupling this theory to a gauge field should be the same as coupling the Dirac fermion to a gauge field. The theory in Eq. (2.50) can be rewritten in a more useful form by shifting $a \to a + A$,

$$|D_{a-A}\phi|^2 - m_0^2|\phi|^2 - |\phi|^4 + \frac{1}{4\pi}ada - \frac{1}{4\pi}AdA\,. \tag{2.51}$$

This theory is anomaly free and gauge invariant by construction, so the Dirac fermion it describes should have the right parity anomaly term to enforce gauge invarance. The loop model partition function for this theory has the same form as that with $A = 0$, except now J couples to A, and we have the background Chern-Simons term

$$Z[A] = \int \mathcal{D}J\mathcal{D}a\,\delta(\partial_\mu J^\mu)\,e^{-|m|L[J]+iS[J,a,A]}\,, \tag{2.52}$$

where

$$S[J,a,A] = \int d^3x\left[J(a-A) + \frac{1}{4\pi}ada - \frac{1}{4\pi}AdA\right]\,, \tag{2.53}$$

and we suppress all contractions of spacetime indices in the action. Please note that in Eq. (2.52), as in previous sections, short-ranged interactions are not made explicit. The manipulations that follow in the context of Chern-Simons theory are only consistent if the bosons have (strong) short-ranged repulsive interactions. This is also natural since for $D < 4$ spacetime dimensions the free massless scalar field fixed point is essentially inaccessible.

As we know well now, integrating out a results in a nonlocal linking number term for J

$$-\pi\,\Phi[J] + \int d^3x\left[JA - \frac{1}{4\pi}AdA\right]\,, \tag{2.54}$$

where we have changed variables $J \to -J$ since the partition function does not depend on the overall sign of J. For $A = 0$, we recover Eq. (2.43) with $\theta = -\pi$. Since the J variables are gapped and bosonic, the response of this theory is determined solely by the background Chern-Simons term, which gives a Hall conductivity $\sigma_{xy} = -\frac{1}{4\pi}$, precisely what we would expect from a massive, properly regulated Dirac fermion, which has a parity anomaly term[18]

$$\bar{\Psi}(i\not{D}_A - M)\Psi - \frac{1}{8\pi}AdA\,, \tag{2.55}$$

[18] For a derivation using ζ function regularization see Ref. [150].

with $M < 0$. Indeed, this is what one finds by following Polyakov's logic starting with Eq. (2.54). This identification already suggests that the sign of the linking number term is identified with the sign of the mass of the fermion in the phase. See Appendix A.2 for a more explicit justification of this statement. Again, M is related, but not equal, to m.

Thus, we find that the properly regulated loop model partition function for a Dirac fermion in its **T**-broken (integer quantum Hall) phase is

$$Z_{\text{fermion}}[A; M < 0]\, e^{-i\,\text{CS}[A]/2} = \int \mathcal{D}J\, \delta(\partial_\mu J^\mu)\, e^{-|m|L[J] + iS_{\text{fermion}}[J,A;M<0]} e^{-i\,\text{CS}[A]/2}\,, \tag{2.56}$$

where

$$S_{\text{fermion}}[J, A; M < 0] = \int d^3x\, JA - \left(\pi\Phi[J] + \frac{1}{2}\,\text{CS}[A]\right)\,, \tag{2.57}$$

and we define

$$\text{CS}[A] = \int d^3x\, \frac{1}{4\pi} AdA\,. \tag{2.58}$$

Our reason for factoring out a $\text{CS}[A]/2$ term in Eq. (2.56) is to isolate the effect of the parity anomaly, which can be thought of as arising from a heavy fermion doubler (or regulator). In practice, Eq. (2.56) tells us how to write the loop model partition function of a Dirac fermion in its **T**-broken phase, where the low energy effective action is $-\,\text{CS}[A]$.

Having completed our analysis for the unbroken phase of Eq. (2.50), how do we write the loop model partition function in the broken symmetry (superfluid) phase of Eq. (2.50)? In the Dirac fermion picture, this should be the **T**-symmetric (trivial insulator) phase, obtained from Eq. (2.55) with $M > 0$. Instead of modeling the broken symmetry phase of the ϕ variables directly, we use bosonic particle-vortex duality [65, 66] to exactly write the loop model partition function in this phase as one describing the symmetric (insulator) phase of vortex variables $\tilde{\phi}$ (with corresponding loop variables $\tilde{J}$). Recall that this changes the dependence on the background fields and inverts the sign of the linking number term, as we saw with the loop model in Section 2.2 and demonstrate in Appendix A.1. This leads to a loop model with statistical angle $\theta = +\pi$. The loop model partition function in the **T**-symmetric phase is therefore

$$\tilde{Z}[A] = \int \mathcal{D}\tilde{J}\mathcal{D}b\, \delta(\partial_\mu \tilde{J}^\mu)\, e^{-|m|L[\tilde{J}] + i\tilde{S}[\tilde{J},b,A]}\,, \tag{2.59}$$

where

$$\tilde{S}[\tilde{J}, b, A] = \int d^3x \left[\tilde{J}(b - A) - \frac{1}{4\pi} bdb\right] . \tag{2.60}$$

Integrating out b and changing variables $\tilde{J} \to -\tilde{J}$ yields the action

$$+\,\pi\Phi[\tilde{J}\,] + \int d^3x\, \tilde{J}A\,, \tag{2.61}$$

consistent with a Hall conductivity $\sigma_{xy} = 0$, as we would expect from Eq. (2.55) with $M > 0$. Following through with Polyakov's argument from here allows us to write

$$Z_{\text{fermion}}[A; M > 0]\, e^{-i\,\text{CS}[A]/2} = \int \mathcal{D}J\, \delta(\partial_\mu J^\mu)\, e^{-|m|L[J] + iS_{\text{fermion}}[J,A;M>0]} e^{-i\,\text{CS}[A]/2}\,, \tag{2.62}$$

where

$$S_{\text{fermion}}[J, A; M > 0] = \int d^3x\, JA + \left(\pi\Phi[J] + \frac{1}{2}\,\text{CS}[A]\right)\,, \tag{2.63}$$

Bringing everything together, the loop model partition function of a free Dirac fermion with Lagrangian,

$$\bar{\Psi}(i\not{D}_A - M)\Psi - \frac{1}{8\pi} AdA\,, \tag{2.64}$$

having fixed the sign of the parity anomaly term, is

$$\begin{aligned} Z_{\text{fermion}}[A; M]e^{-i\,\text{CS}[A]/2} &= \det[i\not{D}_A - M]e^{-i\,\text{CS}[A]/2} \\ &= \int \mathcal{D}J\, \delta(\partial_\mu J^\mu)\, \exp\left(-|m|L[J] + iS_{\text{fermion}}[J, A; M] - \frac{i}{2}\,\text{CS}[A]\right). \end{aligned} \tag{2.65}$$

where the loop model action S_{fermion} for general M is

$$S_{\text{fermion}}[J, A; M] = \int d^3x\, JA + \,\text{sgn}(M)\left(\pi\Phi[J] + \frac{1}{2}\,\text{CS}[A]\right)\,, \tag{2.66}$$

Here, too, we have left implicit the necessary interactions between the loops. Eq. (2.65) is the main result of this subsection.

In field theory language, we have derived the duality of a massive free Dirac fermion

$$\bar{\Psi}(i\not{D}_A - M)\Psi - \frac{1}{8\pi} AdA \tag{2.67}$$

to a gauged Wilson-Fisher scalar with a mass term,

$$|D_a\phi|^2 - r|\phi|^2 - |\phi|^4 + \frac{1}{4\pi}ada + \frac{1}{2\pi}adA\,, \tag{2.68}$$

starting from the loop model representation of the scalar theory. For $r > 0$, the scalar theory is in its symmetric phase, which we showed can be related to the $\mathbf{T}$-broken, $M < 0$ phase of the fermionic theory, which has $\sigma_{xy} = -1/(4\pi)$. Conversely, the $\mathbf{T}$-symmetric, $M < 0$ phase of the fermionic theory is dual to the symmetry broken phase of the scalar theory, where $r < 0$. In this phase, $\sigma_{xy} = 0$.

We can also consider acting time reversal $\mathbf{T}$ on the duality (2.65). This flips the signs of the Chern-Simons and BF terms, as well as the fermion mass M, i.e. this duality corresponds to Eq. (2.65) with $M \to -M$ and a parity anomaly term with positive sign. In this case, the $\mathbf{T}$-broken phase now has Hall conductivity $\sigma_{xy} = +\frac{1}{4\pi}$.

2.4.2 A Duality Web of Loop Models

Fermionic Particle-Vortex Duality

Equipped with the loop model partition function for a Dirac fermion coupled to a gauge field, we now proceed to derive new loop model dualities. We start by deriving a loop model version of the duality between a free Dirac fermion and 2+1 dimensional quantum electrodynamics (QED_3) [81–83][19],

$$i\bar{\Psi}\not{D}_A\Psi - \frac{1}{8\pi}AdA \longleftrightarrow i\bar{\psi}\not{D}_a\psi - \frac{1}{4\pi}adA - \frac{1}{8\pi}AdA\,. \tag{2.69}$$

Here, it will be more convenient to consider the version of this duality with properly quantized coefficients of Chern-Simons and BF terms in the strongly interacting theory [79],

$$i\bar{\Psi}\not{D}_A\Psi - \frac{1}{8\pi}AdA \longleftrightarrow i\bar{\psi}\not{D}_a\psi + \frac{1}{8\pi}ada - \frac{1}{2\pi}adb + \frac{2}{4\pi}bdb - \frac{1}{2\pi}bdA\,, \tag{2.70}$$

where Eq. (2.69) can be recovered by integrating out the auxiliary gauge field b, which comes at the cost of violating flux quantization, Eq. (B.1). Note that the signs of the $\frac{1}{8\pi}ada$ and $\frac{1}{8\pi}AdA$ terms, which can be thought to arise from heavy fermion doublers coupled to a and A respectively, need not match across this duality.

To obtain loop models, we add a mass term $-M\bar{\Psi}\Psi$, $M < 0$, to the free theory and a mass term $-M'\bar{\psi}\psi$,

[19] As usual, we will only explicitly include the leading relevant operators and suppress irrelevant operators. In particular, Maxwell terms $-\frac{1}{4g_{\mathrm{M}}^2}f^2$ for the gauge fields are always implicitly included. Thus, the loop model dualities derived here are only meant to hold at energies $E << g_{\mathrm{M}}^2$.

$M' > 0$, to QED_3 so that both theories are in their **T**-broken phase. The partition function of the free theory is Eq. (2.65) with $M < 0$. Similarly, we can obtain a loop model analogue of QED_3 by acting **T** on Eq. (2.65), plugging in $M' > 0$, gauging $A \to a$, and adding the correct couplings to b

$$Z_{\mathrm{QED}_3}[A; M' > 0]\, e^{-i\,\mathrm{CS}[A]/2} = \int \mathcal{D}J\mathcal{D}a\mathcal{D}b\, \delta(\partial_\mu J^\mu) e^{-|m|L[J] + iS_{\mathrm{QED}_3}[J,a,b,A;M'<0]}\,, \tag{2.71}$$

where

$$S_{\mathrm{QED}_3}[J, a, b, A; M' > 0] = \pi\,\Phi[J] + \int d^3x \left[Ja + \frac{1}{4\pi}ada - \frac{1}{2\pi}adb + \frac{2}{4\pi}bdb - \frac{1}{2\pi}bdA\right]. \tag{2.72}$$

We can integrate out a without violating flux quantization to obtain

$$\begin{aligned} S_{\mathrm{eff}} =& \pi\,\Phi[J] + \int d^3x \left[-\pi\left(J - \frac{db}{2\pi}\right) d^{-1} \left(J - \frac{db}{2\pi}\right) + \frac{2}{4\pi}bdb - \frac{1}{2\pi}bdA\right] \\ =& \int d^3x \left[Jb + \frac{1}{4\pi}bdb - \frac{1}{2\pi}bdA\right], \end{aligned} \tag{2.73}$$

where we have used the fact that $\Phi[J] = \int Jd^{-1}J$. Integrating out b gives the action of Eq. (2.65) with $M < 0$, so we obtain the loop model duality

$$Z_{\mathrm{fermion}}[A; M < 0] = Z_{\mathrm{QED}_3}[A; M' > 0]\,. \tag{2.74}$$

If we instead work with the trivial insulating phase, similar manipulations lead to

$$Z_{\mathrm{fermion}}[A; M > 0] = Z_{\mathrm{QED}_3}[A; M' < 0]\,. \tag{2.75}$$

The interpretation of these loop model dualities as fermionic particle-vortex dualities is immediate. First, since the sign of the linking number term is the same as the sign of the mass of the fermion, we recover the mapping of mass operators $\bar{\Psi}\Psi \longleftrightarrow -\bar{\psi}\psi$. What's more, if we violate flux quantization by integrating out b, we recover the matter-flux mapping:

$$J_\Psi \longleftrightarrow \frac{1}{4\pi}da\,, \tag{2.76}$$

where $J^\mu_\Psi = \bar{\Psi}\gamma^\mu\Psi$ is the global $U(1)$ current of the free fermion. This may seem odd since we never actually changed variables in deriving these dualities. However, we were never working with fermionic variables to begin with, but *bosonic* ones. Thus, the loop variables above should not be interpreted as the currents of the free fermion. Instead, Polyakov's duality (2.65) makes clear that since A couples to J_Ψ in S_{fermion}, correlation functions of J_Ψ are generated by derivatives of

$$F_{J_\Psi}[A; M] = \log Z_{\text{fermion}}[A; M] \,, \tag{2.77}$$

where we have subtracted off the $\pm\frac{1}{2}\text{CS}[A]$ parity anomaly term, as it does not contribute to the correlation functions of J_Ψ. Since A couples as $\frac{1}{4\pi}Ada - \frac{1}{8\pi}AdA$ in S_{QED_3} after b is integrated out, subtracting off the same parity anomaly term in the QED_3 theory implies the mapping of Eq. (2.76).

General Abelian Bosonization Dualities

We now consider more general boson-fermion dualities. The field theory duality web [79, 80] can be used to relate a theory of a Wilson-Fisher scalar coupled to a Chern-Simons gauge field at level $k_\phi \in \mathbb{Z}$,

$$|D_a\phi|^2 - |\phi|^4 + \frac{k_\phi}{4\pi}ada + \frac{1}{2\pi}adA \,, \tag{2.78}$$

to a dual theory of Dirac fermions coupled to a Chern-Simons gauge field. To do this, we invoke the duality between a Wilson-Fisher boson and a Dirac fermion coupled to a Chern-Simons gauge field at level 1/2

$$|D_A\phi|^2 - |\phi|^4 \longleftrightarrow i\bar{\psi}\not{D}_b\psi + \frac{1}{8\pi}bdb + \frac{1}{2\pi}bdA + \frac{1}{4\pi}AdA \,. \tag{2.79}$$

Plugging this result into Eq. (2.78), we obtain a duality

$$|D_a\phi|^2 - |\phi|^4 + \frac{k_\phi}{4\pi}ada + \frac{1}{2\pi}adA \longleftrightarrow i\bar{\psi}\not{D}_b\psi + \frac{1}{8\pi}bdb + \frac{1}{2\pi}ad(b+A) + \frac{k_\phi+1}{4\pi}ada \,. \tag{2.80}$$

Integrating out the gauge field a on the fermionic side, would run into conflict with flux quantization Eq. (B.1) and gauge invariance (if the theory is defined purely in 2+1 dimensions). However, continuing in spite of this, one would obtain a duality between bosons coupled to a Chern-Simons gauge field at level k_ϕ and fermions coupled to a Chern-Simons gauge field at level [92]

$$k_\psi = \frac{1}{2}\frac{k_\phi - 1}{k_\phi + 1} \,. \tag{2.81}$$

This relation can be generalized to the self-dual theories which occupied our attention for much of this work, Eq. (2.47), which, in addition to Chern-Simons terms, have marginally long-ranged interactions. The introduction of such long-ranged interactions can be accommodated by replacing k_ϕ and k_ψ with $\tau_\phi = k_\phi + i\frac{2\pi}{e_\phi^2}$

and $\tau_\psi = 2k_\psi + i\frac{4\pi}{e_\psi^2}$ respectively,

$$\tau_\psi = \frac{\tau_\phi - 1}{\tau_\phi + 1}\,. \tag{2.82}$$

For clarity, in this section we will only explicitly consider the limit $e^2_{\phi,\psi} \to \infty$. Our results can be readily generalized away from this limit by replacing k's with τ's.

We can easily check that the theories on either side of the duality Eq. (2.80) have the same phase diagram. Adding $+m^2|\phi|^2$ to the scalar theory Higgses out the emergent gauge field a and leaves the theory in a trivial insulating phase. Similarly, adding $-M\bar{\psi}\psi$, with $M<0$, to the fermion theory and integrating out b also Higgses out a (the parity anomaly term being cancelled), leading to the same phase. Conversely, adding $-m^2|\phi|^2$ to the scalar theory leads to a topological quantum field theory of the form

$$\frac{k_\phi}{4\pi} ada + \frac{1}{2\pi} adA\,, \tag{2.83}$$

which can also be obtained on the fermionic side by adding $-M\bar{\psi}\psi$ with $M>0$. Integrating out ψ adds a parity anomaly term to the action,

$$\frac{1}{4\pi} bdb + \frac{1}{2\pi} ad(b+A) + \frac{k_\phi + 1}{4\pi} ada\,. \tag{2.84}$$

Integrating out b leads to Eq. (2.83). Our interest will be in this phase: our goal will be to show that in this phase the loop model partition function of massive fermions coupled to b is the same of that of the massive bosons coupled to a.

The world line partition function for the gapped bosons in the phase described by Eq. (2.83) is (turning off background fields)

$$Z_{\text{boson}}[k_\phi] = \int \mathcal{D}J\mathcal{D}a\,\delta(\partial_\mu J^\mu)\, e^{-|m|L[J] + iS_{\text{boson}}[J,a;k_\phi]}, \tag{2.85}$$

where we used the definition

$$S_{\text{boson}}[J,a;k_\phi] = \int d^3x \left[Ja + \frac{k_\phi}{4\pi} ada\right]\,. \tag{2.86}$$

Integrating out a yields an effective action

$$S_{\text{eff}}[J] = -\frac{\pi}{k_\phi}\Phi[J]\,. \tag{2.87}$$

The loop model partition function for the fermion can be written down by acting with $\mathbf{T}$ and gauging the background field $A \to b$ in Eq. (2.65). The world line partition function for fermions in the phase (2.84) is therefore

$$Z_{\text{fermion}}[k_\psi, M>0] = \int \mathcal{D}J\mathcal{D}a\mathcal{D}b\,\delta(\partial_\mu J^\mu)\,e^{-|m|L[J]+iS_{\text{fermion}}[J,b,a;k_\psi,M<0]}\,, \tag{2.88}$$

where k_ψ is the level of the gauge field b coupled to the fermion with the auxiliary gauge field a integrated out[20], Eq. (2.81), and we define

$$S_{\text{fermion}}[J,b,a;k_\psi,M>0] = \pi\,\Phi[J] + \int d^3x\left[Jb + \frac{1}{4\pi}bdb + \frac{1}{2\pi}adb + \frac{k_\phi+1}{4\pi}ada\right]. \tag{2.89}$$

We now have two options. The first is to integrate out a, but that would violate flux quantization, Eq. (B.1). Instead, we integrate out b first. Its equation of motion is

$$J + \frac{da}{2\pi} = -\frac{db}{2\pi}\,. \tag{2.90}$$

Charge quantization of the bosonic J variables implies that this equation is consistent with flux quantization. Integrating out b therefore gives

$$\begin{aligned} S_{\text{eff}}[J,a] &= \pi\,\Phi[J] + \int d^3x\left[-\pi\left(J+\frac{da}{2\pi}\right)d^{-1}\left(J+\frac{da}{2\pi}\right) + \frac{k_\phi+1}{4\pi}ada\right] \\ &= \pi\,\Phi[J] - \pi\,\Phi[J] + \int d^3x\left[-Ja - \frac{1}{4\pi}ada + \frac{k_\phi+1}{4\pi}ada\right] \\ &= \int d^3x\left[-Ja + \frac{k_\phi}{4\pi}ada\right], \end{aligned} \tag{2.91}$$

where we have used the definition of $\Phi[J]$ in passing to the second line. The path integral does not depend on the sign of J here, so integrating out a yields the same answer as in the bosonic case, Eq. (2.87).

We therefore find an equality of the loop model partition functions

$$Z_{\text{boson}}[k_\phi] = Z_{\text{fermion}}[k_\psi, M>0]\,. \tag{2.92}$$

This is the general three dimensional bosonization identity for loop models. The same analysis can be carried out in the superfluid phase of the theory in Eq. (2.78), which is the insulating phase of its particle-vortex dual, in which the Chern-Simons gauge field has "level" $-1/k_\phi$ (again a statement about the theory obtained after violating flux quantization and integrating out auxiliary gauge fields). Denoting the partition function

[20] Note that we defined k_ψ as the level of b expected at the critical point, so it does not include the extra parity anomaly term acquired by gapping ψ.

of this theory as $Z_{\text{boson}}[-1/k_\phi]$, we indeed find

$$Z_{\text{boson}}[-1/k_\phi] = Z_{\text{fermion}}[k_\psi, M < 0]\,. \tag{2.93}$$

Starting from Polyakov's loop model duality (2.65), we have thus derived a loop model version of the general CFT duality of Eq. (2.80) by matching fractional spin factors!

This derivation also provides an answer to the question of what it means to have fractional spin different from 0 or 1/2, a vexing issue since Polyakov's argument first appeared. From the perspective of the duality of Eq. (2.80), it is incorrect to think of theories of world lines with general fractional spin as theories of free particles with a strange spin. Rather, one should think of the theories on either side of the duality as strongly interacting Chern-Simons theories coupled to matter. We further note that these theories are also thought to be dual to non-Abelian Chern-Simons-matter theories [78], but it is not clear to us how to construct an explicit loop model description of these theories. This may be an interesting direction for future work.

2.5 Discussion

In this Chapter we have shown that, upon introducing fractional spin, 2+1 dimensional loop models with statistical and long-ranged interactions of Eq. (2.3) are not invariant under shifts of the statistical angle, despite remaining self-dual under particle-vortex duality. This means that, while $\mathrm{PSL}(2,\mathbb{Z})$ still has a natural action on these theories, only the $\mathcal{S}$ transformation should be taken as a good duality transformation. It also means that the superuniversal transport properties of the loop models in Ref. [12] do not have an analogue in Chern-Simons theories coupled to gapless matter, which we argued must include fractional spin.

By introducing fractional spin into the loop models of Ref. [12], we were led to develop simple loop model versions of various members of the web of 2+1 dimensional field theory dualities [79, 80], starting from a seed duality relating the partition function of a massive Dirac fermion to a bosonic loop model with a fractional spin term. This makes clear the consistency of relativistic loop model dualities with the duality web of conformal field theories. It also should be considered a nontrivial check of these dualities.

We emphasize that the duality of the loop models suggests that these theories may have a critical point, which should presumably be a relativistic CFT. Proving this statement requires solving the loop model and finding its continuum limit at the phase transition. This has not been done. Our purpose here was to inquire to what extent the critical points of the loop models can be described by a CFT on the duality web. A successful construction of this continuum limit would in fact be a derivation of the duality web.

Because of the simplicity of the loop model dualities presented here, it would be of interest to use loop models to motivate new field theory dualities or derive already proposed dualities which live outside the duality web. However, some difficulties persist. It still remains to carefully implement the short-ranged interactions in the loop model dualities presented here. This is a necessary requirement to develop loop model derivations of dualities with multiple flavors of matter fields, which can have different global symmetries depending on the form of the short-ranged interactions. Such dualities have been of interest in the study of deconfined quantum critical points [151]. It also remains to construct a precise lattice formulation of the loop models presented here, which we defined based on their long distance properties due to the subtleties surrounding placing Chern-Simons theories on a lattice.

Chapter 3

Duality and Emergent Symmetry: Reflection Symmetry of Composite Fermi Liquids

This Chapter is reproduced from Hart Goldman and Eduardo Fradkin, Phys. Rev. B ***98****, 165137 (2018).* *This paper is also cited as Ref. [88] in the References section of the thesis.*

3.1 Introduction

A paradigmatic example of a strongly interacting metallic state arises in the context of 2d systems of electrons in a strong magnetic field when the lowest Landau level (LLL) is at filling $\nu = 1/2$. However, despite many years of effort, concrete theoretical understanding of this state remains elusive. Historically, the most successful approach to this problem has been that of Halperin, Lee, and Read (HLR) [21], which utilizes the notion of flux attachment, in which a theory of non-relativistic particles is exactly mapped to a theory of "composite particles" (fermions or bosons) coupled to an Abelian Chern-Simons gauge field [68]. Such mappings have been foundational in the theory of the fractional quantum Hall (FQH) effect [70, 71, 73], explaining the observed Jain sequence FQH states as integer quantum Hall (IQH) states of composite fermions. In the HLR approach, two flux quanta are attached to each electron, completely screening the magnetic field and yielding a theory of a Fermi surface of non-relativistic composite fermions f, strongly coupled to a Chern-Simons gauge field $a_\mu = (a_t, a_x, a_y)$,

$$\mathcal{L}_{\mathrm{HLR}} = f^\dagger(i\partial_t + \mu + a_t)f - \frac{1}{2m}|(i\partial_i + a_i + A_i)f|^2 + \frac{1}{4\pi}\frac{1}{2}ada + \cdots , \tag{3.1}$$

where $A_i = \frac{B}{2}(x\hat{y} - y\hat{x})$ is the background vector potential, and we use the notation $AdB = \varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu B_\lambda$. Here we require that the emergent gauge field cancels the external magnetic field, i.e. $\langle\varepsilon^{ij}\partial_i a_j\rangle = -\varepsilon^{ij}\partial_i A_j$. The HLR theory has seen great phenomenological success: it explains the existence of the observed metallic state [28], and the large cyclotron radii of the composite fermions near $\nu = 1/2$ lead to quantum oscillations which have been observed experimentally [29–33].

Nevertheless, the HLR theory suffers from several well known problems. First, as a theory of a Fermi surface strongly coupled to a gauge field, it is plagued by infrared (IR) divergences, and the random phase approximation (RPA) is uncontrolled. Additionally, the HLR theory is not a proper LLL theory, since a theory composite fermions which are charged under electromagnetism will not have holomorphic wave functions [152]. Finally, while the LLL Hamiltonian at $\nu = 1/2$ is particle-hole (**PH**) symmetric [153], and **PH** symmetric response has been observed experimentally [18, 19, 154], the HLR theory does not seem to possess this symmetry: flux is attached to electrons, rather than holes. This issue has also found new relevance in recent quantum oscillation experiments [34, 35].

A great deal of progress on the latter two problems was made recently, when Son proposed a Dirac composite fermion theory of the $\nu = 1/2$ state [81]. This theory is based on the fact that the LLL limit of a system of electrons with gyromagnetic ratio $g = 2$ can be identified with the massless limit of a Dirac "electron" in a magnetic field

$$\mathcal{L}_e = i\bar{\Psi}\not{D}_A\Psi + \frac{1}{8\pi}AdA\,, \tag{3.2}$$

where we have introduced the notation $D^\mu_A = \partial^\mu - iA^\mu$ and $\not{D} = D^\mu\gamma_\mu$, and γ_μ are the Dirac gamma matrices. The term $AdA/8\pi$ can be thought of as coming about due to the presence of a heavy fermion doubler[1]. Since Dirac fermions have Landau levels with both positive and negative energies, with one sitting at zero energy, the zeroth Landau level is half filled when the chemical potential is zero. Such a state is automatically symmetric under **PH**, which is just the exchange of empty and filled states. This led Son to conjecture that this theory is dual to one of Dirac composite fermion vortices ψ at finite density, strongly coupled to an emergent gauge field a_μ without a Chern-Simons term (QED_3),

$$\mathcal{L}_{\text{Son}} = i\bar{\psi}\not{D}_a\psi + \frac{1}{4\pi}adA + \frac{1}{8\pi}AdA + \cdots\,. \tag{3.3}$$

where the **PH** symmetry of the Dirac electron problem now manifests as a time reversal (**T**) symmetry of the composite fermions. The $\cdots$ denote irrelevant operators, such as the Maxwell term for a_μ. This duality between a free Dirac fermion and QED_3 was quickly shown to be a part of a "web of dualities," at the center of which is a relativistic flux attachment duality relating a free Dirac fermion to a Wilson-Fisher boson coupled to a Chern-Simons gauge field [79, 80]. This fermion-vortex duality has also led to progress in other areas of condensed matter physics [82, 83, 85, 155]. Despite its success in incorporating **PH** symmetry, it still remains to understand how Son's theory might emerge from microscopics and the extent to which it can be experimentally distinguished from the HLR theory, although very interesting arguments have been put forward suggesting that Son's theory may emerge from the HLR theory upon incorporating the effect of

[1]Throughout this Chapter, we approximate the Atiyah-Patodi-Singer η-invariant as a level-1/2 Chern-Simons term and include it in the action.

quenched disorder [22, 23, 151] or as a percolation transition between the HLR theory and its **PH** conjugate [156]. Encouragingly, evidence for the Dirac composite fermion theory has been found in numerical studies [157, 158].

A major open question has been whether Son's proposal can be extended to describe the compressible states appearing at other even denominator filling fractions $\nu = 1/2n$. In the HLR theory, descriptions of these states arise trivially: one can simply attach an even number of flux quanta so that the external field is again completely screened. For non-relativistic particles, this transformation should be an identity at the level of the partition function, implying that all of these theories lie in the same universality class. On the other hand, in relativistic theories, flux attachment influences both statistics and spin, and so this transformation is no longer innocuous. More saliently, while the LLL Hamiltonian for these states is not **PH** symmetric, transport experiments have observed an analogous "reflection symmetry" in the $I - V$ curves about the $\nu = 1/3$ FQH to insulator transition, which occurs at $\nu = 1/4$ [18–20], suggesting that this state might host its own kind of **PH** symmetry, or at least that there is a symmetry relating the Jain sequence FQH states proximate to it. More precisely, the observed symmetry maps a longitudinal $I - V_{xx}$ curve at a filling fraction $\nu < 1/2n$ to a $I - V_{xx}$ curve at a dual filling fraction $\nu' > 1/2n$ in which the roles of current and voltage are exchanged,

$$(E_i(\nu), J_i(\nu)) = \left(\frac{h}{e^2} J_i(\nu'), \frac{e^2}{h} E_i(\nu')\right) . \tag{3.4}$$

and the transverse $I - V_{xy}$ curves in the observed region are all linear with slope $3\frac{h}{e^2}$. Surprisingly, the observed longitudinal $I - V_{xx}$ curves do not appear to be linear except very close to $\nu = 1/4$, meaning that the observed symmetry extends to *nonlinear* response.

The presence of this symmetry makes sense if we view the state at $\nu = 1/2n$ as a limit of the Jain states $\nu = p/(2np + 1)$, where n and p are integers. Such states have reflection conjugates on either side of the point at $\nu = 1/2n$, and we thus might expect the composite fermions in these states to experience the same physics. Indeed, to impressively high precision, the reflection symmetry observed in experiment appears identical to the one which relates conjugate Jain states. However, the HLR theory is incompatible with this symmetry, since the conjugate states in question correspond to *different* IQH states of composite fermions.

In this Chapter, we propose a series of Dirac composite fermion theories to describe the compressible states at $\nu = 1/2n$. These theories are obtained by attaching an even number of fluxes to the composite fermions of Son's theory (3.3). They therefore can be thought of as existing in a unified framework with Son's theory of $\nu = 1/2$. Although they lack an explicit analogue of **PH** symmetry, we argue that they can explain the reflection symmetry observed in experiments. In particular, we show that in reflection conjugate Jain states, the composite fermions fill the *same* number of Landau levels, in contrast to HLR. Moreover, our theories are consistent with the LLL limit: as in Son's theory, our Dirac composite fermions are electrically neutral. In fact, we show that the LLL limit ensures that the Dirac composite fermion is massless, whether or not a mass is allowed by symmetry. If a mass is indeed allowed by symmetry, that would suggest that the states at $\nu = 1/2n$ for $n > 1$ are tuned to a quantum critical point, rather than constituting a genuine phase.

It is not clear to us whether the reflection symmetry of the Jain states proximate to $\nu = 1/2n$ extends to a full blown symmetry of the theories *at* $\nu = 1/2n$ for $n > 1$. The case of $\nu = 1/2$ is special in this regard, since there the reflection symmetry is identical to **PH** symmetry, which manifests itself as the **T** symmetry of the composite fermion theory. While the experiments do strongly hint that the compressible states at $\nu = 1/2n$ have this reflection symmetry, they do not necessarily imply it. This is because the experiments may not have truly observed the compressible state, instead seeing the signatures of the phases asymptotically close to $\nu = 1/4$. However, it is entirely possible that our composite fermion theories flow to fixed points hosting an enhanced **T** symmetry which is the continuation of the reflection symmetry of the Jain states. The presence of such a symmetry would also ensure the masslessness of the Dirac fermions, and it would imply that our theories display "self-dual" transport at $\nu = 1/2n$ [20].

We finally note that other descriptions of the $\nu = 1/2n$ states have been proposed in Refs. [159–161] using semiclassical arguments[2]. These theories are variants of HLR involving Fermi surfaces with nonvanishing Berry phases, which are related to an "anomalous velocity" term associated with the non-commutative geometry of the LLL [162]. The effect of these Berry phases is to generate anomalous Hall conductivities that completely cancel the Chern-Simons terms of HLR. However, it is not clear whether this cancellation truly occurs beyond $\nu = 1/2$: without **PH** symmetry, the Berry phase can run. These theories also do not seem compatible with the reflection symmetry of the Jain states. Our expectation is that the same kind of anomalous velocity that is associated with the Berry phase can be equally well explained via interactions with a Chern-Simons gauge field. Evidence for this comes from the fact that our theories lead to the same set of magnetoresistance minima as the theories of Fermi surfaces with π/n Berry phases. However, it is difficult to make these connections precise because band theory intuition cannot be applied to our strongly interacting problem. In future work, we hope to elucidate the connections between these theories and the ones presented here.

We proceed as follows. In Section 3.2, we present our proposed effective field theories for the $\nu = 1/2n$ states. In Section 3.3, we describe how these theories can explain the reflection symmetry of the Jain states proximate to $\nu = 1/2n$. In Section 3.4, we argue for the Dirac composite fermions should be massless by

[2]We thank Yizhi You and Jie Wang for very enlightening discussions about these theories and their relationship to those presented in this Chapter.

viewing the state at e.g. $\nu = 1/4$ as the LLL limit of the HLR theory when the non-relativistic composite fermions are placed at half filling. We then discuss some additional observables in Section 3.5, in particular describing how to couple our theories to background geometry. We conclude in Section 3.6.

3.2 Proposed Effective Field Theories

We conjecture that the $\nu = 1/2n$ state can be described as a theory of $2n$ flux quanta attached to a free Dirac fermion. This flux attachment transformation can be implemented on the Lagrangian (3.2) by making the background gauge field dynamical, $A \to a$, and introducing a new auxiliary gauge field c at level $2n$ that couples to a and the background vector potential $A = \frac{B}{2}(x\hat{y} - y\hat{x})$ through BF terms. For a review of such flux attachment transformations, see Ref. [63],

$$i\bar{\psi}\not{D}_a\psi - \frac{1}{8\pi}ada + \frac{1}{2\pi}adc - \frac{2n}{4\pi}cdc + \frac{1}{2\pi}cdA\,. \tag{3.5}$$

We note that these transformations are the elements $\mathcal{S}\mathcal{T}^{-2n}\mathcal{S}$ of the modular group $\mathrm{PSL}(2,\mathbb{Z})$, described in condensed matter and high energy contexts by Kivelson, Lee, and Zhang [10] and Witten [121] respectively. This theory is gauge invariant with all of the gauge fields satisfying the Dirac flux quantization condition. If we loosen this requirement (an innocuous thing if our interest is in local properties) and integrate out the auxiliary gauge field c, we arrive at the theory[3]

$$\mathcal{L}_{1/2n} = i\bar{\psi}\not{D}_a\psi - \frac{1}{4\pi}\left(\frac{1}{2} - \frac{1}{2n}\right)ada + \frac{1}{2\pi}\frac{1}{2n}Ada + \frac{1}{4\pi}\frac{1}{2n}AdA\,. \tag{3.6}$$

notice that we recover Son's theory (3.3) for $n = 1$. For $n > 1$, this theory breaks **PH**, **T**, and parity (**P**) due to the presence of the nonvanishing Chern-Simons term for a. Thus, naïvely, a Dirac mass is allowed by symmetry, unless this theory harbors an enhanced symmetry which prohibits a mass. In the absence of such a symmetry, these theories are taken to be tuned to a quantum critical point.

The $\nu = 1/2n$ state corresponds to the case where the composite fermions ψ are at finite density but see a vanishing magnetic field. If we denote the physical electron density as $\rho_e = \left\langle \frac{\delta\mathcal{L}_{1/2n}}{\delta A_t} \right\rangle$ and the magnetic field seen by the composite fermions as $b_* = \langle \varepsilon^{ij}\partial_i a_j \rangle$, then

$$\nu = 2\pi\frac{\rho_e}{B} = \frac{1}{2n}\left(1 + \frac{b_*}{B}\right)\,, \tag{3.7}$$

meaning that, indeed, $\nu = 1/2n$ implies $b_* = 0$. Thus, the composite fermions form a strongly interacting

[3] In the remainder of this Chapter, we will work exclusively with theories having improperly quantized Chern-Simons levels.

metallic state.

In the sections that follow, we will see that there are several reasons to believe that this theory correctly describes the physics of the $\nu = 1/2n$ state. First, we will check that the IQH states of composite fermions reproduce the Jain sequences, $\nu = p/(2np+1)$, where p and n are integers. We will then introduce a **PH**-like reflection transformation which maps between Jain states on either side of $\nu = 1/2n$, and we will see that conjugate Jain states correspond to *the same* IQH state of composite fermions, up to a **T** transformation ($\nu \mapsto -\nu$). This transformation can be related to boson-vortex exchange upon invoking boson-fermion duality to obtain theories of composite bosons at $\nu = 1$. This goes a long way toward explaining the reflection symmetry observed in experiments.

The theories (3.6) are also consistent with the LLL limit. Not only is the composite fermion charge neutral, but we will use Son's particle-vortex duality to argue that the dual theory to Eq. (3.6), given by

$$\tilde{\mathcal{L}}_{1/2n} = i\bar{\chi}\not{D}_b\chi + \frac{1}{8\pi}bdb + \frac{1}{4\pi}\frac{1}{2(n-1)}(b+A)d(b+A)\,, \tag{3.8}$$

where b is another emergent gauge field (*note the difference with* b_*), reproduces the same LLL physics as a theory of non-relativistic electrons with $2(n-1)$ flux quanta attached, at least at mean field level. This leads to an explanation for why ψ and χ are massless, despite the fact that a mass may be allowed by symmetry. Moreover, it is easy to see that the $\nu = 1/2n$ state corresponds to a half filled zeroth Landau level of χ particles.

3.3 Reflection Symmetry of the Jain Sequences

3.3.1 Reproducing the Jain Sequences

We now show that the theories (3.6) reproduce the Jain sequences. We will see that the presence of the Chern-Simons term is crucial in making this work out. For simplicity, we will work with the version of the theory with improperly quantized Chern-Simons levels, although the computations for the properly quantized theory are similar. We start by considering the equation of motion for a_t,

$$0 = \langle\psi^\dagger\psi\rangle - \frac{1}{2\pi}\left(\frac{1}{2} - \frac{1}{2n}\right)b_* + \frac{1}{2\pi}\frac{1}{2n}B\,, \tag{3.9}$$

meaning, if we define the composite fermion filling fraction as $\nu_\psi = 2\pi\rho_\psi/b_*$, where $\rho_\psi = \langle\psi^\dagger\psi\rangle$, then

$$\nu_\psi = \frac{1}{2} - \frac{1}{2n} - \frac{1}{2n}\frac{B}{b_*}\,. \tag{3.10}$$

Notice that for $n = 1$, the first two terms on the right hand side cancel, as the density of composite fermions is proportional to the background magnetic field in that case. For $n \neq 1$, the non-cancellation of the first two terms reflects the fact that the Chern-Simons level is non-vanishing: the density of the composite fermions depends on the external magnetic field *and* the emergent magnetic field, b_*.

To produce the Jain sequences, we first fill p Landau levels of the composite fermions, leading to an incompressible integer quantum Hall state

$$\nu_\psi = p + \frac{1}{2}\,. \tag{3.11}$$

Eq. (3.10) is now

$$2np + 1 = -\frac{B}{b_*}\,. \tag{3.12}$$

We know B/b_* in terms of the physical electron filling fraction ν from Eq. (3.7). Plugging this in, we have

$$2np + 1 = -\frac{1}{2n\nu - 1}\,. \tag{3.13}$$

Solving for ν, we finally obtain

$$\nu = \frac{p}{2np + 1}\,. \tag{3.14}$$

This is the Jain sequence.

3.3.2 Reflection Symmetry

Having shown that the theories (3.6) reproduce the Jain sequences, our goal now is to determine if they can reproduce "reflection" symmetry indicated by experiments, which relates the (nonlinear) response of conjugate Jain states proximate to $\nu = 1/2n$. For us, this amounts to showing that for each Jain state proximate $\nu = 1/2n$, there is a conjugate Jain state where the composite fermions fill the same number of Landau levels but are $\mathbf{T}$ conjugated. At least at mean field level, the composite fermion response of such states should be essentially identical. Note that while this symmetry of the incompressible plateau states can give us intuition that there is an emergent reflection symmetry at the compressible state $\nu = 1/2n$, such a symmetry is not implied. It is interesting enough that the theory (3.6) explains the symmetry of the plateau states.

Each state on the Jain sequence (3.14) with filling $\nu < 1/2n$ ($p \geq 0$) has a reflection conjugate with filling $\nu' > 1/2n$ given by

$$\nu' = \frac{1 + p}{2n(1 + p) - 1}\,. \tag{3.15}$$

This can be written as the following transformation of the filling fraction ν,

$$\nu' = \frac{-(2n-1)\nu + 1}{[1-(2n-1)^2]\nu + (2n-1)} \,. \tag{3.16}$$

Notice that for $n = 1$, this relation is none other than the **PH** transformation $\nu' = 1 - \nu$.

The statement of the reflection symmetry between the conjugate Jain states is that they correspond to composite fermion IQH states with $\nu'_\psi = -\nu_\psi = -(p + 1/2)$. In other words, an IQH state of composite fermions is mapped to *the same* IQH state up to a $\mathbf{T}$ transformation (i.e. with magnetic field pointing in the opposite direction). To see that this is the case, we start by rewriting Eqs. (3.7) and (3.10) as a relation between Dirac and electron filling fractions,

$$\nu_\psi - \frac{1}{2} = \frac{\nu}{1 - 2n\nu} \,. \tag{3.17}$$

Plugging Eq. (3.15) into Eq. (3.17), the dependence on the compressible state index n cancels, and we obtain

$$\nu'_\psi = -\left(p + \frac{1}{2}\right) \,. \tag{3.18}$$

Thus, the conjugate state can be thought of as filling p Landau levels with the magnetic field pointing in the opposite direction! This is a related to the more general fact that reflection symmetry acts as $\mathbf{T}$ on the composite fermion filling fraction: Eq. (3.17) implies that mapping $\nu \mapsto \nu'$ is the same as $\mathbf{T} : \nu_\psi \mapsto -\nu_\psi$. Note that in the language of the dual theory, Eq. (3.8), this $\mathbf{T}$ symmetry can be interpreted as a particle-hole symmetry **CT** (we reserve **PH** for the electron particle-hole symmetry $\nu \mapsto 1 - \nu$).

The above results go a long way toward explaining the reflection symmetry observed experimentally. However, it is important to note that since the composite fermion theories under consideration do not appear to be $\mathbf{T}$ symmetric at $\nu = 1/2n$, the physics of the Jain state at filling factor ν might differ from that at its conjugate ν' due to the effect of fluctuations of the emergent gauge field. This being said, since these states are gapped, we expect the effect of such fluctuations to be small and essentially unobservable, and we believe that this mean field argument should suffice to explain what is observed in experiments. As the compressible state is approached, however, gauge field fluctuations will become important, and the reflection symmetry may be broken. Whether the symmetry persists to the compressible state ultimately requires an understanding of the interplay of disorder and the strong interactions with the Chern-Simons gauge field. In the next subsection, we will consider the implications of this reflection symmetry for transport in more detail and discuss what the experimental observations can tell us about whether this symmetry emerges at the compressible states at $\nu = 1/2n$.

We can develop a complementary interpretation of the transformation (3.16) in the language of composite bosons as an exchange symmetry between composite bosons and vortices, or "self-duality". A similar interpretation was also introduced in the previous, non-relativistic approaches to this problem [10, 19, 20]. It will also be a particularly useful language for writing down constraints on transport, which is the topic of the next subsection. We can obtain a composite boson theory by invoking the duality between a gauged Wilson-Fisher boson and a free Dirac fermion described in Refs. [79, 80],

$$i\bar{\Psi}\not{D}_A\Psi + \frac{1}{8\pi}AdA \longleftrightarrow |D_a\phi|^2 - |\phi|^4 - \frac{1}{4\pi}ada + \frac{1}{2\pi}adA\,, \tag{3.19}$$

where $\longleftrightarrow$ denotes duality and we use the notation "$-|\phi|^4$" to indicate tuning to the Wilson-Fisher fixed point. It is not difficult to see that the theories (3.6) have bosonic duals,

$$|D_{g-A}\phi|^2 - |\phi|^4 + \frac{1}{4\pi}\frac{1}{2n-1}gdg\,. \tag{3.20}$$

Here g is another fluctuating emergent gauge field. For $A = \frac{B}{2}(x\hat{y} - y\hat{x})$, these bosons find themselves at finite density and magnetic field. Differentiating with respect to g_t and A_t gives

$$\langle j^t_\phi\rangle = -\rho_e = -\frac{1}{2n-1}\frac{\langle\varepsilon^{ij}\partial_i g_j\rangle}{2\pi}\,, \tag{3.21}$$

where j^μ_ϕ is the *gauged* $U(1)$ current of the bosons. If we define the filling of the bosons to be $\nu_\phi = 2\pi\frac{\langle j^t_\phi\rangle}{\langle\varepsilon^{ij}\partial_i(g_j - A_j)\rangle}$, then

$$\nu_\phi = -[2n - 1 - \nu^{-1}]^{-1}\,, \tag{3.22}$$

Thus, for $\nu = \frac{1}{2n}$, we have

$$\nu_\phi = 1\,. \tag{3.23}$$

Thus, we now are facing a problem of gauged Wilson-Fisher bosons at $\nu_\phi = 1$. Interestingly, the filling of the bosons in this theory is independent of n.

The theory (3.20) can be shown to be dual to a theory of bosonic vortices [65, 66] with action[4]

$$|D_h\tilde{\phi}|^2 - |\tilde{\phi}|^4 - \frac{2n-1}{4\pi}hdh + \frac{1}{2\pi}hdA\,, \tag{3.24}$$

where g is a new emergent gauge field. Notice that, in the composite boson language, the lack of explicit

[4]For explicit derivations of the particular boson-vortex duality discussed here, see Refs. [92, 99].

$\mathbf{T}$ symmetry in the fermionic theory manifests itself as an apparent lack of symmetry between bosons and vortices: they are interacting with Chern-Simons gauge fields having different levels. Similar manipulations to those for the ϕ theory imply a relationship between the filling fractions of the dual theories,

$$\nu_\phi = -\frac{1}{\nu_{\tilde{\phi}}}\,, \tag{3.25}$$

which is the modular $\mathcal{S}$ transformation of Refs. [10, 121]. Conjugate filling fractions are those for which the roles of bosons and vortices have been exchanged, i.e. $\nu_\phi(\nu) = -\nu_{\tilde{\phi}}(\nu')$. Thus,

$$\nu^{-1} - (2n-1) = \frac{1}{\nu'^{-1} - (2n-1)}\,. \tag{3.26}$$

Solving for ν' leads to Eq. (3.18). Thus, reflection symmetry can be interpreted equivalently as a $\mathbf{T}$ (or $\mathbf{CT}$) symmetry of composite fermions and as a composite boson-vortex exchange symmetry, or "self-duality."

3.3.3 Constraints on Transport and Connections to Experiment

We now describe the implications of the reflection symmetry introduced above for transport and relate them to the experimental observations of Refs. [18, 19]. For convenience, we start by working with the composite boson description of Eqs. (3.20) and (3.24) since the experiments are most easily interpreted in that language, although we describe how to translate these results into the composite fermion language at the end of this section. Note that the arguments in this section are essentially the same as those for non-relativistic composite boson theories given in Ref. [20]. However, we emphasize that in our case we are starting with theories which naturally manifest the reflection symmetry of the Jain states proximate to $\nu = 1/2n$ (as is evident in the Dirac composite fermion language). In the previous non-relativistic work, this symmetry had to be postulated.

Composite Boson Language: Self-Duality

We start by defining the conductivity of the composite bosons of Eq. (3.20), which respond to both the background probe electric field E_i and the emergent electric field due to the Chern-Simons gauge field g, $\langle e_i(g)\rangle = \langle f_{it}(g)\rangle$,

$$\langle j_{\phi,i}\rangle = \sigma^{\mathrm{CB}}_{ij}(\langle e^j(g)\rangle - E^j)\,. \tag{3.27}$$

Note that, in this section, all conductivities (resistivities) are in units of $e^2/\hbar$ ($\hbar/e^2$).

In the composite vortex theory (3.24), the roles of charge and flux are exchanged. Consequently, $\sigma^{\mathrm{CB}}_{ij}$

is the *resistivity tensor* of the vortices. To see this, we plug the charge-flux mappings $j_\phi = dh/2\pi$ and $j_{\tilde{\phi}} = d(g-A)/2\pi$, i.e. $j^i_\phi = \varepsilon^{ij}(\partial_j h_t - \partial_t h_j)/2\pi \equiv \varepsilon^{ij}\tilde{e}_j/2\pi$ and $j^i_{\tilde{\phi}} = \varepsilon^{ij}(e_j - E_j)/2\pi$, into Eq. (3.28) to obtain the transport dictionary,

$$\sigma^{\mathrm{CB}}_{ij} = \frac{1}{(2\pi)^2}\varepsilon^{ik}\varepsilon^{jl}\,\tilde{\rho}^{\mathrm{CB}}_{kl}\,. \tag{3.28}$$

In a rotationally invariant system, this simply reduces to $\sigma^{\mathrm{CB}}_{ij} = \tilde{\rho}^{\mathrm{CB}}_{ij}/(2\pi)^2$. Because the dictionary (3.28) is a consequence of particle-vortex duality, it is valid at finite wave vector and frequency, as well as in the presence of disorder. Moreover, since we never explicitly required linear response in its derivation, we also expect the dictionary to hold beyond the linear regime. This last point is necessary if our wish is to understand the experiments of Refs. [18, 19], since the symmetry observed there was one of nonlinear response.

The equality between composite boson conductivity and vortex resistivity is the reason why the reflection symmetry described above exchanges the role of current and voltage about $\nu = 1/2n$. In the bosonic language, the reflection symmetry is the statement that composite bosons at electron filling fraction ν have identical transport to the composite vortices at conjugate electron filling fraction ν', so

$$\rho^{\mathrm{CB}}_{ij}(\nu) = \tilde{\rho}^{\mathrm{CB}}_{ji}(\nu') = (2\pi)^2\sigma^{\mathrm{CB}}_{ji}(\nu')\,. \tag{3.29}$$

From the analysis of the Dirac composite fermion theories earlier in this section, we saw that this symmetry holds for FQH states proximate to $\nu = 1/2n$, at least at mean field level.

We can connect ρ^{CB}_{ij} to the observable electron resistivity ρ_{ij} as follows. If $J^i = -j^i_\phi$ is the electron current, then we define ρ_{ij} via

$$E_i = \rho_{ij}\langle J^j\rangle\,. \tag{3.30}$$

The difference between ρ_{ij} and ρ^{CB}_{ij} comes from a shift in the Hall resistivity due to the Chern-Simons gauge field, which enforces flux attachment, $\langle e_i\rangle = 2\pi(2n-1)\varepsilon_{ij}\langle j^j_\phi\rangle$. Plugging this into the definition of $\sigma^{\mathrm{CB}}_{ij}$, (3.27), and rearranging, one finds

$$\rho_{ij} = \rho^{\mathrm{CB}}_{ij} - (2n-1)\,2\pi\varepsilon_{ij}\,. \tag{3.31}$$

Thus, the observed resistivity is just the composite boson resistivity with shifted Hall components.

If the reflection symmetry persists to $\nu = 1/2n$ (which is mapped to itself), then Eq. (3.29) implies that the composite boson resistivity must satisfy the "self-duality" condition

$$[\rho^{\mathrm{CB}}_{xx}(1/2n)]^2 + [\rho^{\mathrm{CB}}_{xy}(1/2n)]^2 = (2\pi)^2\,. \tag{3.32}$$

For $n = 1$, or $\nu = 1/2$, this constraint and the relation (3.31), implies the **PH**-symmetric Hall response $\sigma_{xy} = \frac{1}{4\pi}$ For $n \neq 1$, however, the constraint is weaker: σ_{xy} depends on the composite boson conductivity.

We are now prepared to interpret the experimental results of Refs. [18, 19], which correspond to the case of $\nu = 1/4$, or $n = 2$. Throughout the observed region of ν values, the Hall response was observed to be linear, with resistivity taking the value

$$\rho_{xy} = -3(2\pi)\,. \tag{3.33}$$

meaning that, by Eq. (3.31), the Hall resistivity of the composite bosons vanishes

$$\rho_{xy}^{\mathrm{CB}} = 0\,. \tag{3.34}$$

This constraint is surprising, since *there does not appear to be any symmetry in the problem which requires this.* Understanding the mechanism by which the composite boson Hall resistivity vanishes continues to be an open question. Plugging this into Eq. (3.29), the reflection symmetry can be expressed in terms of the electron longitudinal resistivities as

$$\rho_{xx}(\nu) = \frac{(2\pi)^2}{\rho_{xx}(\nu')}\,. \tag{3.35}$$

since $\rho_{xx}^{\mathrm{CB}} = \rho_{xx}$. This is consistent with what was observed in the longitudinal $I - V_{xx}$ curves, assuming that Eqs. (3.28) and (3.31) are valid in the nonlinear regime. It was also observed that, as ν approaches $1/4$, ρ_{xx} seems to become linear and approach the "self-dual" value $\rho_{xx} = 2\pi$. This constitutes fairly compelling evidence that reflection symmetry emerges at the compressible states at $\nu = 1/2n$, but we emphasize that this is not necessary. In the next section, we will show that the LLL limit can suffice to tune the Dirac composite fermions to criticality, whether or not the states at $\nu = 1/2n$ truly host an emergent reflection symmetry themselves.

Composite Fermion Language: T Symmetry

We close this section by considering the implications of reflection symmetry for the transport of the composite fermion theory (3.6), for which the reflection symmetry is a **T** symmetry. If we define the composite fermion conductivity $\sigma_{ij}^{\mathrm{CF}}$ via

$$\langle j_{\psi,i}\rangle = \sigma_{ij}^{\mathrm{CF}}\langle e^j(a)\rangle\,, \tag{3.36}$$

where $e_i(a) = f_{it}(a)$ and $j_\psi^\mu = \bar{\psi}\gamma^\mu\psi$, then reflection symmetry implies

$$\sigma_{ij}^{\mathrm{CF}}(\nu) = \sigma_{ji}^{\mathrm{CF}}(\nu')\,. \tag{3.37}$$

Thus, if the reflection symmetry persists to $\nu = 1/2n$, this means

$$\sigma_{xy}^{\rm CF} = 0\,. \tag{3.38}$$

Indeed, we will quickly see that the duality between the composite fermion theory (3.6) and the composite boson theory (3.20) that this $\mathbf{T}$ symmetry implies self-duality of the bosons and vice versa.

We can relate the composite fermion conductivity to the measured electron conductivity as follows. Differentiating the Lagrangian (3.6) with respect to A_j and a_j give the electron and composite fermion currents respectively

$$\langle J_i\rangle = \frac{1}{2\pi}\frac{1}{2n}\varepsilon_{ij}(\langle e^j\rangle + E^j),\ \langle j_{\psi,i}\rangle = \frac{1}{2\pi}\left(\frac{1}{2}-\frac{1}{2n}\right)\varepsilon_{ij}\langle e^j\rangle - \frac{1}{2\pi}\frac{1}{2n}\varepsilon_{ij}E^j\,, \tag{3.39}$$

plugging in the definitions of the electron and composite fermion resistivities $\rho_{ij}^{\rm CF} = (\sigma_{ij}^{\rm CF})^{-1}$ (assuming rotation invariance) and solving the system of equations, one finds that the electron and composite fermion resistivities are related by

$$\rho_{xx} = (2\pi)^2\frac{4\rho_{xx}^{\rm CF}}{(\rho_{xx}^{\rm CF})^2 + [\rho_{xy}^{\rm CF}+2(2\pi)]^2}\,, \tag{3.40}$$

$$\rho_{xy} = 2\pi\left[-2(n-1) - 8\pi\frac{2(2\pi)+\rho_{xy}^{\rm CF}}{(\rho_{xx}^{\rm CF})^2 + [\rho_{xy}^{\rm CF}+2(2\pi)]^2}\right]. \tag{3.41}$$

There are several things to note about these expressions. First, reflection symmetry (3.37) along with the observed Hall resistivity $\rho_{xy} = -3(2\pi)$ again imply the observed $I - V_{xx}$ reflection symmetry, Eq. (3.35). Moreover, plugging $\mathbf{T}$ symmetry of the composite fermions ($\rho_{xy}^{\rm CF} = 0$) into these equations and combining them with Eq. (3.31) immediately leads to the self-duality of the composite bosons, Eq. (3.32). Finally, assuming that $\mathbf{T}$ symmetry extends to the compressible state at $\nu = 1/2n$ and plugging in the observed $\rho_{xy} = -3(2\pi)$ implies

$$\sigma_{xx}^{\rm CF}(1/2n) = \frac{1}{4\pi} = \frac{1}{2}\frac{e^2}{h}\,. \tag{3.42}$$

Thus, the problem of understanding the physical origin of the observed Hall resistivity at the $\nu = 1/3$ – insulator transition may in fact be identical to the problem of understanding why $\sigma_{xx}^{\rm CF} = 1/4\pi$. Intriguingly, this value is the same as that obtained in Pruisken's two-parameter scaling theory of IQH plateau transitions [163, 164].

3.4 Massless Composite Fermions from the LLL Limit

We now argue that the LLL limit requires that our theories (3.6) be tuned so that the Dirac composite fermions are massless. This argument essentially follows the logic of Son's argument that the LLL limit of non-relativistic fermions with $g = 2$ can be identified with that of a massless Dirac fermion [81], provided that the effect of transitions between Landau levels is neglected. The difference here will be that instead of starting with a non-interacting theory of non-relativistic fermions in a magnetic field, we consider non-relativistic fermions in a magnetic field with $2(n-1)$ flux quanta attached via a Chern-Simons gauge field,

$$if^\dagger D_{a,t}f - \frac{1}{2m}|D_{a,i}f|^2 + \frac{\varepsilon^{ij}\partial_i a_j}{2m}f^\dagger f + \frac{1}{4\pi}\frac{1}{2(n-1)}(a+A)d(a+A)\,, \tag{3.43}$$

where again $A_i = \frac{B}{2}(x\hat{y} - y\hat{x})$ is the magnetic vector potential, and $A_t = \mu$. Here we work in a regime where $b = \langle \varepsilon_{ij}\partial_i a_j \rangle \neq 0$, i.e. the fermions experience a net (uniform) magnetic field, organizing themselves into Landau levels. Our ultimate interest will be in the case where the non-relativistic composite fermions are at half-filling, which corresponds to a physical electron filling fraction $\nu = 1/2n$.

The LLL limit of this theory $m \to 0$ can be understood to be finite by introducing a Hubbard-Stratonovich field c as follows,

$$if^\dagger D_{a,t}f + ic^\dagger(D_{a,x} + iD_{a,y})f - if^\dagger(D_{a,x} - iD_{a,y})c + 2mc^\dagger c + \frac{1}{4\pi}\frac{1}{2(n-1)}(a+A)d(a+A)\,, \tag{3.44}$$

Upon taking the limit $m \to 0$, we see that c becomes a Lagrange multiplier which implements the LLL constraint $(D_{a,x} + iD_{a,y})f = 0$.

We now argue that the theory (3.44) is identical to that which would be obtained by taking the LLL limit of a Dirac fermion coupled to a Chern-Simons gauge field with Lagrangian,

$$i\bar{\chi}\not{D}_a\chi + \frac{1}{8\pi}ada + \frac{1}{4\pi}\frac{1}{2(n-1)}(a+A)d(a+A)\,. \tag{3.45}$$

Notice that this theory is none other than the particle-vortex dual of our composite fermion theory for the $\nu = 1/2n$ state (3.6). Writing $\chi = (f, c)$ and choosing $\gamma^t = \sigma^z, \gamma^x = i\sigma^y, \gamma^y = -i\sigma^x$, we obtain

$$if^\dagger D_{a,t}f + ic^\dagger D_{a,t}c + ic^\dagger(D_{a,x} + iD_{a,y})f - if^\dagger(D_{a,x} - iD_{a,y})c + \frac{1}{8\pi}ada + \frac{1}{4\pi}\frac{1}{2(n-1)}(a+A)d(a+A)\,. \tag{3.46}$$

This looks almost identical to our non-relativistic Lagrangian (3.44), with two differences. The first is the presence of a time derivative term for c. However, this term is negligible upon taking the LLL limit, which for a Dirac fermion is the limit[5] of infinite fermion velocity, $v \to \infty$. The second difference is the appearance of the parity anomaly term $ada/8\pi$, which can be thought of as implementing the effect of the Dirac sea of filled negative energy states. At mean field level, where a is not dynamical, then this term would simply lead to a constant shift in the filling fraction and Hall conductivity with respect to the non-relativistic case, i.e.

$$\nu_f = \nu_\chi + \frac{1}{2}\,. \tag{3.47}$$

However, the equivalence between the LLL physics of the Dirac and non-relativistic theories may be spoiled upon taking into account fluctuations of a, although it is reasonable to expect that fluctuations of a about its mean field value do not contribute large corrections.

Thus, the LLL limits of the non-relativistic, flux attached theory (3.44) and the *massless* Dirac fermion theory (3.45) match, at least at mean field level, meaning that a proper description of the LLL requires tuning (3.45) to criticality. In other words, if we view the problem of non-relativistic electrons at filling $\nu = 1/2n$ as the $\nu = 1/2$ state of of non-relativistic electrons attached to $2(n-1)$ units of flux (which should be an exact rewriting of the original problem), then the LLL limit connects the problem to one of massless Dirac fermions coupled to a Chern-Simons gauge field with its zeroth Landau level half filled, Eq. (3.45). We then obtain our Dirac composite fermion theory (3.6) upon invoking particle-vortex duality. The beauty of this approach is that we can leverage the flux attachment invariance of the underlying non-relativistic problem to obtain a relativistic composite fermion description of the states at $\nu = 1/2n$, even though relativistic theories are not invariant under flux attachment (for an extended discussion of this point, see Ref. [99]).

The analysis of this section leads to an interesting interpretation of the theories (3.6). Unlike in HLR, where e.g. the composite fermion for the $\nu = 1/2n$ state is related to that of the $\nu = 1/2(n-1)$ state by attachment of two flux quanta, here the Dirac composite fermion of the $\nu = 1/2n$ state is the *dual vortex* of that at $\nu = 1/2(n-1)$, placed at filling $3/2$. The reason the filling is $3/2$ instead of $1/2$ is related to the fact that the composite fermion Lagrangian (3.6) and its dual (3.45) differ by a filled Landau level, $\frac{1}{4\pi}ada$. This actually makes sense from the perspective of Son's original duality, in which the state at $\nu = 1/4$ is the $\nu = 3/2$ state of the composite fermions.

[5]Here we have written the theory with Dirac fermion velocity $v = 1$ (in units of the speed of light; v is not to be confused with the Fermi velocity, $v_F = \partial_k \epsilon(k)|_{k_F}$, where $\epsilon(k)$ is the dispersion). Reintroducing v and rescaling $c \mapsto c' = vc$, one sees that the term in question becomes $ic'^\dagger D_{t,a} c'/v^2$, which vanishes as $v \to \infty$.

3.5 Further Observables

3.5.1 Shift and Hall Viscosity

Jain States

We now describe how to couple our Dirac composite fermion theories (3.6) to background geometry and show that that it is possible to reproduce the remaining universal data associated with the Jain states: the total orbital spin per particle s [74, 165], which determines the shift of the Jain states on the sphere $\mathscr{S} = 2s$, as well as the Hall viscosity [166],

$$\eta_H = \frac{s\rho_e}{2}\,. \tag{3.48}$$

The Hall viscosity measures the response to external shear deformations, and is associated with stress tensor correlation functions. In Galilean invariant systems, it also determines the leading contribution to the Hall conductivity at finite wave vector [167, 168].

In order to obtain s, we need to understand how to couple our Dirac composite fermions to the *Abelian*[6] spin connection ω_μ. The strength of this coupling is the orbital spin of the Dirac composite fermions, S_z, which is *not* restricted to be $1/2$. This is because flux attachment generally leads to a Berry phase which depends on the geometry. The presence of this Berry phase means that composite particles behave like they have an emergent "fractional spin" due to their strong interactions with the Chern-Simons gauge field[7]. The orbital spin of the composite fermions, S_z, which determines the coupling to ω_μ, can be identified with this fractional spin [169]. This can be seen explicitly by rewriting the partition function of the theory (3.6) as a path integral over composite fermion worldlines, as described in detail in Ref. [99]. For the sake of brevity, however, we instead argue for the value of S_z by analogy with the non-relativistic case, where attaching $2n$ flux quanta to a spinless fermion leads to an orbital spin $S_z = -n$ (the sign flip comes from integrating out the Chern-Simons gauge field). Since the Dirac fermion starts with spin $1/2$, we expect

$$S_z = \frac{1}{2} - n\,. \tag{3.49}$$

We thus claim that our Dirac composite fermion theories can be coupled to geometry by using the covariant

[6]The presence of an Abelian spin connection breaks Lorentz invariance, but this is not problematic here since Lorentz invariance is already broken explicitly by the external magnetic field.

[7]This can be thought of as a manifestation of the framing anomaly [69].

derivative $D^\mu(a,\omega) = \partial^\mu - ia^\mu + \frac{i}{2}\gamma^0\omega^\mu$ and shifting $A \to A + (2n-1)\omega/2$,

$$\begin{aligned}\mathcal{L}_{1/2n}[\psi, a, A, \omega] = i\,\bar{\psi}\,\gamma^\mu\left(\partial_\mu - ia_\mu + \frac{i}{2}\gamma^0\omega_\mu\right)\psi - \frac{1}{4\pi}\left(\frac{1}{2} - \frac{1}{2n}\right)ada \\ - \frac{1}{2\pi}\frac{1}{2n}\left(A + \frac{2n-1}{2}\omega\right)da + \frac{1}{4\pi}\frac{1}{2n}\left(A + \frac{2n-1}{2}\omega\right)d\left(A + \frac{2n-1}{2}\omega\right) \\ + \cdots\end{aligned} \tag{3.50}$$

where the $\cdots$ refer to additional purely gravitational contact terms which we will neglect and which are discussed in detail in Ref. [170]. Note that in this section we conjugate the sign of the BF term relative to the rest of the paper so that the Wen-Zee terms we obtain have positive sign.

Equipped with the Lagrangian (3.50), we now proceed to calculate the shift of the Jain states on the sphere, from which we can extract the orbital spin s as the coefficient of the Wen-Zee ($\frac{1}{2\pi}Ad\omega$) term when all of the dynamical fields have been integrated out. The degeneracy of the p^{th} Dirac fermion Landau level on the sphere is

$$d_p = \int d^2\mathbf{x}\,\frac{b_*}{2\pi} + 2|p| \equiv N_\phi + 2|p|\,. \tag{3.51}$$

This means that the number of composite fermions required to fill up to the p^{th} Landau level is

$$N_\psi = N_\phi\left(p + \frac{1}{2}\right) + p\,(p+1)\,. \tag{3.52}$$

The shift $\mathscr{S}$ of the electron filling fractions on the Jain sequence is defined via

$$N_e = \nu_e(N_\phi + \mathscr{S})\,. \tag{3.53}$$

To calculate $\mathscr{S}$, we start by integrating out the composite fermions. This generates new Chern-Simons and Wen-Zee terms, which are (for $b_* > 0$)

$$\frac{p+\frac{1}{2}}{4\pi}ada + \frac{p\,(p+1)}{4\pi}ad\omega\,. \tag{3.54}$$

So now we have a Lagrangian

$$\frac{1}{4\pi}\left(p + \frac{1}{2n}\right)ada + \frac{1}{4\pi}\left[p(p+1) - \frac{2n-1}{2n}\right]ad\omega - \frac{1}{2\pi}\frac{1}{2n}adA + \frac{1}{4\pi}\frac{2n-1}{2n}Ad\omega + \frac{1}{4\pi}\frac{1}{2n}AdA + \cdots \tag{3.55}$$

Notice that the contribution of the parity anomaly of the Dirac composite fermion has been cancelled, so we can already expect that this will yield *the same* answer that we would have obtained from HLR. We now integrate out a, which has the equation of motion,

$$da = \frac{1}{2np+1}\left[dA + \frac{1}{2}\left(2n-1-2np(p+1)\right)d\omega\right]. \tag{3.56}$$

Thus, suppressing purely gravitational terms, we obtain

$$\frac{1}{4\pi}\frac{p}{2np+1}Ad\left(A+(p+2n)\omega\right) = \frac{\nu}{4\pi}Ad(A+\mathscr{S}\omega)\,. \tag{3.57}$$

Thus, the shift is

$$\mathscr{S} = p + 2n\,, \tag{3.58}$$

which is precisely the known result for the Jain states [74, 169]! The orbital spin s and Hall viscosity η_H are therefore

$$s = \frac{\mathscr{S}}{2} = \frac{p}{2} + n,\ \eta_H = \frac{1}{2}\left(\frac{p}{2}+n\right)\rho_e\,, \tag{3.59}$$

again consistent with previously known results.

Some Speculation about the Compressible States

We take this opportunity to speculate about the geometric response of the theories (3.6) at the compressible filling fractions $\nu = 1/2n$. In the case of the state at $\nu = 1/2$, this has been seen as a source of disagreement between Son's Dirac composite fermion theory and HLR, essentially because the composite fermion appears to have different orbital spin in the two approaches [171]. Moreover, it is not even clear if the Hall viscosity should be viewed as universal in the HLR theory [172]. However, recent results seem to indicate consistency between the Dirac composite fermion approach and a non-relativistic "bimetric theory" of FQH states near $\nu = 1/2$ [173], and it appears likely that an approach starting from HLR with quenched disorder can match these results as well [22, 23, 151].

A naïve approach to obtaining the Hall viscosity for the compressible states might involve considering the result for the Jain states and taking the limit $p \to \infty$. Unfortunately, this clearly leads to a divergent result given Eq. (3.59). However, we have already argued that the fractional spin of the composite Dirac fermions is given by (3.49). Since the composite fermions feel a vanishing magnetic field, this should be the only contribution to the total orbital spin s. Thus, we expect at mean field level,

$$\eta_H(\nu = 1/2n) = -\frac{1}{2}S_z\rho_e = \frac{1}{2}\left(n-\frac{1}{2}\right)\rho_e\,. \tag{3.60}$$

This result should not receive large quantum corrections so long as the composite fermions remain massless, which we argued can be guaranteed both by the LLL limit as well as the reflection symmetry (assuming that it can be continued to $\nu = 1/2n$). As already mentioned above, this quantity can be measured [174] from the finite wave vector part of the Hall response [167, 168], and so it may be possible to use this to distinguish our theories from HLR as well as the HLR-like theories with π/n Berry phase discussed in the Introduction.

3.5.2 Quantum Oscillations

Classic signatures of composite fermions are the quantum oscillations in magnetoresistance which occur as the filling is tuned away from $\nu = 1/2n$, meaning that the composite fermions feel a small magnetic field. It is known that magnetoresistance minima occur along the Jain sequences $\nu = \frac{p}{2np+1}$, where the composite fermions feel a magnetic field b_* which can be obtained from Eq. (3.12),

$$\frac{1}{b_*} = -\frac{p + \frac{1}{2n}}{\frac{B}{2n}} \,. \tag{3.61}$$

Up to the overall sign (which comes from the sign of the BF term in Eq. (3.6) and is a matter of convention), this is precisely the same result that would have been obtained from a theory of a Fermi surface with π/n Berry phase, as in Refs. [159, 160]. In those references, the shift from an integer value in the numerator was seen as a consequence of the Berry phase. However, this shift can equally well be obtained by attaching flux to Dirac fermions.

3.6 Discussion

In this Chapter, we have proposed a series of Dirac composite fermion theories to describe the metallic states appearing at filling fraction $\nu = 1/2n$ in quantum Hall systems. These theories are related to Son's theory of $\nu = 1/2$ by attachment of $2n$ flux quanta. A major advantage of our theories is that they explain the **PH**-like reflection symmetry observed in transport experiments, which relates Jain sequence states on either side of $\nu = 1/2n$, since the composite fermions at conjugate filling fractions experience the same physics. No other theory presented thus far has been shown to accommodate these observations. In addition, we showed that at mean field level our theories are consistent with the LLL limit, provided that we view the state at e.g. $\nu = 1/4$ as a half filled Landau level of the (non-relativistic) composite fermions at $\nu = 1/2$.

Many open questions remain. Foremost is the question of whether the reflection symmetry emerges at the compressible states at $\nu = 1/2n$, rather than just being a property of their proximate phases. Answering this question conclusively from a theoretical point of view requires an understanding of the interplay of disorder and strong interactions in the Chern-Simons-matter theories we have presented here: even if reflection symmetry does not emerge in the clean limit, it may appear when disorder is introduced. Such problems are poorly understood at charge neutrality, let alone in the presence of a Fermi surface. Moreover, a potentially related issue is the problem of explaining the observed Hall resistivity at the $\nu = 1/3$ FQH – insulator transition (3.33) (and also the $\nu = 1$ IQH – insulator transition), which does not appear to be set by any symmetry of the problem. Progress on both of these issues can be made by studying the (uncontrolled) mean field problem with disorder [22, 23, 151], exploiting new or existing dualities [85], or by searching for perturbative approaches which can capture the effects of both disorder and interactions – a direction which has been fruitful at least in the zero density limit and which can give us hints about general principles that can extend beyond the perturbative regime [175, 176]. We intend to pursue all of these directions in the future.

It also remains to understand the precise relationship between the theories presented here and the theories of Fermi surfaces with π/n Berry phases coupled to gauge fields (with no Chern-Simons term) introduced in Refs. [159, 160]. These theories are argued to emerge as a result of the non-commutative guiding center geometry of the LLL. However, as mentioned in the Introduction, it seems likely that our theories are also consistent with the geometry of the LLL, with the Chern-Simons term playing a similar role to the Berry phase. This is borne out by the fact that observables which naïvely appear to probe the Fermi surface Berry phase are the same in our theory. For example, quantum oscillation minima for the two theories are identical, and, in a $\nu = 1/2n$ and $\nu = 1 - 1/2n$ bilayer system, we expect that the π Berry phase of our Dirac composite fermions should lead to the same suppression of $2k_F$ backscattering as seen in the π/n Berry phase theories. Of course, a major distinguishing feature of our theories from the Berry phase theories is the reflection symmetry of the Jain sequence states.

Chapter 4

Duality and the Origins of Exotic Gapped Phases: Parent Theories of Non-Abelian Quantum Hall States

*This Chapter is adapted with the permission of the coauthors from Hart Goldman, Ramanjit Sohal, and Eduardo Fradkin, Phys. Rev. B **100**, 115111 (2019). ©2019 American Physical Society. This paper is also cited as Ref. [89] in the References section of the thesis.*

4.1 Introduction

Two-dimensional charged quantum fluids in a strong magnetic field exhibit an impressive array of topologically ordered incompressible states at partial Landau level (LL) fillings ν, in what is known as the fractional quantum Hall (FQH) effect. Of these states, those exhibiting Abelian topological order are readily understood through the notion of flux attachment [68], which exactly relates fermions or (hard-core) bosons at fractional LL filling to a theory of either composite fermions [70, 71] or bosons [73] in a reduced magnetic field. After flux attachment, the Abelian FQH states may either be viewed as integer quantum Hall (IQH) states of composite fermions or as a condensate of composite bosons governed by a Landau-Ginzburg (LG) theory.

Despite the success over the past several decades in understanding the Abelian FQH states, an understanding of the dynamics which can lead to *non-Abelian* FQH states has remained elusive. Such states cannot arise directly from the application of flux attachment, which is by definition Abelian. For example, while it is believed that the observed $\nu = 5/2$ FQH plateau is a non-Abelian state arising from composite fermion pairing [177], the origin and nature of the pairing instability leading to this state continues to be debated, with seemingly contradictory results between experiment and numerics [178–182]. Nevertheless, assuming a particular pairing channel, a non-Abelian phase appears quite naturally [177, 183].

Unfortunately, this physical picture does not appear to translate simply to the other proposed non-Abelian states, such as the Read-Rezayi (RR) states [184]. Wave functions for these states can be constructed using conformal field theory (CFT) techniques [183], but it is not clear which of these states can be obtained starting from a (physically motivated) field theory of composite particles. To make matters worse, the wave functions for generic non-Abelian states are typically characterized by clustering of more than two particles [184, 185]. Naïvely, from perturbative scaling arguments, such states could not arise unless the clusters with fewer particles are disallowed by symmetry. Most theories of interest do not appear to have such a symmetry, implying that non-perturbatively strong interaction effects are required to give rise to such states. While we note that projective/parton constructions can be used to formulate effective bulk theories of non-Abelian states [186–188], in such constructions the electron operator is fractionalized by hand, and it must be taken by fiat that the fractionalized degrees of freedom are deconfined. Consequently, although the projective approach can formally generate many candidate states, it does not shed much light on their dynamical origin.

Recent progress in the study of non-Abelian Chern-Simons-matter theories in their large-N ("planar") limit [76, 77] has led to the proposal of non-Abelian Chern-Simons-matter theory dualities by Aharony [78], which take the shape of level-rank dualities. Along with the Abelian web of dualities they imply [79, 80], these dualities constitute tools with which it may be possible to make non-perturbative progress on the above problem. Such dualities can relate theories of Abelian composite particles to theories of non-Abelian monopoles, and they have led to progress on several important problems in condensed matter physics [13, 81–86, 88, 176]. Of particular importance for us, pairing deformations of a dual non-Abelian theory can lead to non-Abelian topological phases which appear inaccessible to the original Abelian theory, in which this pairing corresponds to a highly non-local product of monopole operators.

Our strategy is to use these non-Abelian dualities to begin to map the landscape of non-Abelian topological phases accessible from a "composite particle" picture, by way of "projecting down" from a multi-layer parent Abelian state. This type of approach, in which the transition to the non-Abelian phase can be physically interpreted as being driven by interlayer tunneling [177, 189–195] or pairing [196, 197], has formed the foundation of several lines of attack on the non-Abelian FQH problem. Such projections have been implemented at the formal level of the edge CFT ("ideal") wave function [198, 199] and in coupled wire constructions [200, 201]. Numerical studies of bilayer systems have also lent support to this idea [202–208]. However, a robust bulk LG description of generic non-Abelian FQH states continues to be lacking. In one major attempt to fill this gap, the authors of Ref. [196] constructed a non-Abelian LG theory for a subset of the bosonic RR states by considering layers of $\nu = \frac{1}{2}$ (bosonic) Laughlin states. Using the well-known level-rank duality of the (gapped) bulk Chern-Simons topological quantum field theory (TQFT) [209–211] (see Ref. [63] for a review), the authors motivated a description of these states involving $SU(2)$ Chern-Simons gauge fields coupled to scalar matter in the adjoint (matrix) representation, obtaining the non-Abelian QH state by pairing across the different layers. In this approach, the anyon content of the non-Abelian state is furnished by the vortices of the pairing order parameter. While this construction is conceptually appealing, it does not originate from a duality satisfied by the parent Abelian LG theory, which describes a quantum critical point, but, rather, a duality satisfied only deep in the gapped Abelian FQH phase. Moreover, in order to give the anyons electric charge in this approach, it is necessary for the external electromagnetic field to couple to the $U(1)$ subgroup of the full non-Abelian gauge group, explicitly breaking the larger gauge invariance.

Using the non-Abelian bosonization dualities, we construct LG theories of the full bosonic RR sequence at filling fractions $\nu = k/(kM+2)$, $k, M \in \mathbb{Z}$, which do not suffer from these problems. These theories are obtained by starting with k layers of $\nu = 1/2$ bosonic QH states, using the dualities to obtain a LG theory of non-Abelian composite bosons, and attaching M fluxes to the resulting theory. For example, we obtain a LG theory of the bosonic $\nu = 1$ Moore-Read state consisting of two layers of bosons ϕ_n, $n = 1, 2$, which we call the "composite vortices," each at their Wilson-Fisher fixed point and coupled in the *fundamental* representation to a $SU(2)$ gauge field a_n,

$$\mathcal{L} = \sum_{n=1}^{2} \left[|D_{a_n - A\mathbf{1}/2}\, \phi_n|^2 - |\phi_n|^4 + \frac{1}{4\pi} \operatorname{Tr}\left(a_n da_n - \frac{2i}{3} a_n^3 \right) \right] - \frac{1}{4\pi} AdA \,. \tag{4.1}$$

where $D_{a_n - A\mathbf{1}/2} = \partial - i(a_n^b t^b - A\mathbf{1}/2)$ is the covariant derivative, we use the notation $AdB = \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu B_\lambda$, $t^b = \sigma^b/2$ are the $SU(2)$ generators, and $\mathbf{1}$ is the 2×2 identity matrix. We use the notation $-|\phi|^4$ to denote tuning to the Wilson-Fisher fixed point. Although the gauge fields a_n are non-Abelian, the topological phase accessed by simply gapping out the composite vortices will only support excitations with *Abelian* statistics. For a $SU(N)$ gauge group, non-Abelian statistics require the presence of a Chern-Simons term at level greater than one. To obtain the non-Abelian FQH state, we condense clusters of the non-Abelian composite vortices across the layers (see Fig. 4.1), in this case condensing $\phi_1^\dagger \phi_2$ without condensing ϕ_1, ϕ_2 individually. This Higgses the linear combination $a_1 - a_2$ of the $SU(2)_1$ gauge fields, causing the bilayer $SU(2) \times SU(2)$ gauge group to be broken down to its diagonal $SU(2)$ subgroup. The Chern-Simons levels of the resulting gapped phase add, leading to the desired $SU(2)_2$ Chern-Simons theory at low energies (the subscript refers to the Chern-Simons level). We will show below that the composite vortices individually have the proper quantum numbers to fill out the anyon spectrum of the theory. The clarity of the topological content of the non-Abelian states is a general advantage of the bosonic LG approach. However, alternative descriptions of non-Abelian FQH states involving dual non-Abelian composite *fermions* are also possible. We plan to describe this complementary perspective in future work.

In addition to the the RR states, by considering N_f-component generalizations of the Halperin (2,2,1) spin-singlet states on each layer, we are able to generalize this approach to construct bulk LG descriptions of generalized non-Abelian $SU(N_f)$-singlet (NASS) states at fillings [201, 212],

$$\nu = \frac{kN_f}{N_f + 1 + kMN_f}\,, \qquad k, N_f, M \in \mathbb{Z}\,, \tag{4.2}$$

which are bosonic (fermionic) for M even (odd). These states generalize the clustering properties of the RR states to N_f-component systems and, as their name suggests, are singlets under $SU(N_f)$ rotations. Indeed, for $N_f = 1$, these states reduce to the RR states while for $N_f = 2$, they describe the non-Abelian spin singlet (also NASS) states of Ardonne and Schoutens [197, 213]. These generalized NASS states morally possess $SU(N_f + 1)_k$ topological order, and so support anyons obeying the fusion rules of Gepner parafermions [214], generalizations of the $\mathbb{Z}_k$ parafermions [215] found in the RR states. Although the physical relevance of an N_f-component FQH state may seem dubious for larger values of N_f, the generalized NASS states provide candidate ground states in systems of cold atoms [212, 216] and fractional Chern insulators [217]. In building LG theories of these states, we find a new duality relating (A) N_f Wilson-Fisher bosons coupled to $U(1)$ Chern-Simons gauge fields with Lagrangian given by the N_f-component generalization of the Halperin $(2, 2, 1)$ K-matrix theory to (B) a $SU(N_f + 1)_1$ Chern-Simons theory coupled to N_f Wilson-Fisher bosons in the fundamental representation. This non-Abelian dual description makes manifest the emergent $SU(N_f)$ global symmetry and reflects the fact that the edge theory of the N_f-component $(2, 2, 1)$ state supports an $SU(N_f + 1)_1$ Kac-Moody algebra.

The remainder of this Chapter is organized as follows. We begin in Section 4.2 by elaborating on the motivation for our construction both from the perspective of wave functions and that of the earlier Landau-Ginzburg approach of Ref. [196]. We then proceed to our analysis in Section 4.3 of the RR states using non-Abelian bosonization, resolving the lingering issues of the LG construction of Ref. [196]. We then extend our construction to the generalized NASS states in Section 4.4. Future directions are discussed in Section 5.6.

4.2 "Projecting Down" to Non-Abelian States

4.2.1 Perspective from the Boundary: Wave Functions and their Symmetries

If we wish to construct a LG description of non-Abelian FQH states involving pairing between Abelian states, it is first necessary to identify which Abelian states to pair. Such states can be motivated by considering "ideal" wave functions. These can be constructed from certain correlation functions, known as conformal

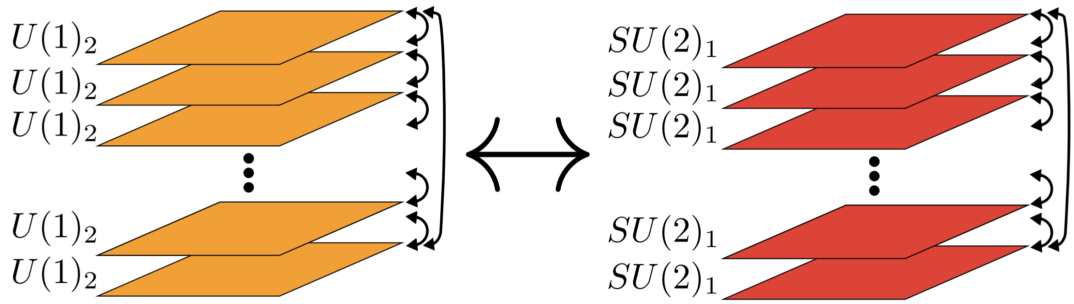

Figure 4.1: A schematic of our construction of LG theories for the RR states. k copies of the $\nu = \frac{1}{2}$ Laughlin state coupled to scalars (left) are dual to k copies of $SU(2)_1$ coupled to scalars (right). The $SU(2)_k$ Read-Rezayi states are obtained in the dual, non-Abelian language via pairing of the layers, represented by double-headed arrows. In the original Abelian theory, these correspond to non-local, monopole interactions.

blocks, of the edge CFT. In this language, the strategy of obtaining non-Abelian states from parent Abelian states through "projecting down" is well established [198].

Consider for example the bosonic RR states at $\nu = k/2$. The ideal wave functions of these states are defined as the ground states of ideal $k+1$-body Hamiltonians, which can be shown to be given by the conformal blocks of the $SU(2)_k$ Wess-Zumino-Witten (WZW) CFT [184]. This tells us that the RR wave functions describe FQH states with edges governed by $SU(2)_k$ WZW theories [183], corresponding in the bulk to a $SU(2)_k$ Chern-Simons gauge theory [69]. A natural way to obtain the ideal wave functions for the $\nu = k/2$ RR states uses the state with $k = 1$ – the $\nu = 1/2$ bosonic Laughlin state, which is Abelian – as a building block [198]. This state is described by the wave function

$$\Psi_{1/2}(\{z_i\}) = \prod_{i<j}(z_i - z_j)^2 e^{-\frac{1}{4}\sum_i |z_i|^2}, \tag{4.3}$$

where $z_j = x_j + iy_j$ denotes the complex coordinates of the j^{th} particle (a boson). The $\nu = k/2$ RR wave functions may be obtained from this one by "clustering" bosons across k copies of this state. This corresponds to taking $N = km$ bosons, dividing them into k groups, writing down a $\nu = \frac{1}{2}$ Laughlin wave function for each group, multiplying them together, and then symmetrizing over all possible assignments of bosons to groups. The resulting wave function is represented as

$$\Psi_k(\{z_i\}) = \mathcal{S}_k\left[\prod_{i=0}^{k-1}\Psi_{1/2}(z_{1+iN/k},\ldots,z_{(i+1)N/k})\right], \tag{4.4}$$

where $\mathcal{S}_k$ denotes symmetrization. It can be shown that this wave function is equivalent to that first proposed by Read and Rezayi [184] and exhibits the correct clustering properties: the wave function does not vanish unless the coordinates of $k+1$ bosons coincide. The RR wave functions for general k and M are obtained by multiplying Eq. (4.4) by a $\nu = \frac{1}{M}$ Laughlin factor.

The relation between the $k = 1$ and the $k > 1$ RR wave functions suggests that it should be possible to construct such a LG theory by considering k copies of the effective theory of the (*Abelian*) $k = 1$ state, the first attempt at which we describe in the next subsection. That a state with $SU(2)_k$ topological order can be obtained from the Abelian $\nu = \frac{1}{2}$ Laughlin state is also made plausible by the fact that the latter has an alternative description as an $SU(2)_1$ Chern-Simons theory. This is a consequence of the level-rank duality between $U(1)_2$ and $SU(2)_1$, which is reflected in the above description by the fact that the $\nu = \frac{1}{2}$ wave function can be obtained from the $SU(2)_1$ WZW CFT [196, 201, 218, 219]. We review this level-rank duality in the subsection below.

4.2.2 Perspective from the Bulk: Early LG Theories from Level-Rank Duality

To approach the problem of constructing a bulk description of the Read-Rezayi states, the authors of Ref. [196] sought to obtain a non-Abelian Landau-Ginzburg theory of the $\nu = k/2$ RR states by also considering k layers of $\nu = 1/2$ bosonic Laughlin states, or $U(1)_2$ Chern-Simons theories and recognizing that each $U(1)_2$ theory is level-rank dual to a $SU(2)_1$ theory. They therefore conjectured that an alternate LG description was possible, one involving scalar matter coupled to $SU(2)_1$ gauge fields. These scalars could then pair and lead to the symmetry breaking pattern,

$$SU(2)_1 \times \cdots \times SU(2)_1 \to SU(2)_k\,. \tag{4.5}$$

What remained was to (1) determine how the scalars transformed under $SU(2)$ and how they coupled to the physical background electromagnetic (EM) field, and (2) determine precisely how to pair these fields to obtain non-Abelian states.

For simplicity, we consider first the case of $k = 2$, a bilayer of $\nu = 1/2$ bosonic FQH liquids. This will constitute a parent state for the $\nu = 1$ bosonic Moore-Read state. To motivate the level-rank duality to a non-Abelian representation, we again consider the edge physics. The edge theory of the $U(1)_2$ state is one of a chiral boson,

$$\mathcal{L}_{\text{edge}} = \frac{1}{4\pi\nu}\,\partial_x\varphi\,(\partial_t\varphi - v\partial_x\varphi)\,, \tag{4.6}$$

where φ has compactification radius $R = 1$ and $\nu = 1/2$. The charge density is therefore $\rho = \frac{1}{2\pi}\partial_x\varphi$. The local particles (i.e. the physical bosons) of this theory are represented by the vertex operators,

$$\psi_1 = e^{i\varphi/\nu}. \tag{4.7}$$

In addition, the theory hosts anyonic quasiparticles, which are semions of charge 1/2 and correspond to the vertex operators

$$\psi_{1/2} = e^{i\varphi} \,. \tag{4.8}$$

The $\psi_1, \psi_1^\dagger$, and ρ operators all have the same scaling dimension and furnish a $SU(2)_1$ Kac-Moody algebra. This is a manifestation of the level-rank duality at the level of the edge CFT, and we can write the bulk theory on each layer as a $SU(2)_1$ gauge theory with gauge field $a_\mu = a_\mu^b t^b$, where t^b are the generators of $SU(2)$. Importantly, the ρ operator appears as the diagonal generator of $SU(2)$. Therefore, the authors guessed that in the LG theory the background EM field couples through a BF term to the Cartan component of the bulk $SU(2)$ gauge field[1],

$$\mathcal{L}_{\rm EM}[a^a, A] = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu a^3_\lambda \,. \tag{4.9}$$

This explicitly breaks gauge invariance and would indicate that the physical EM current is not conserved. We will eventually see in Section 4.3 that the new dualities will allow us to avoid this difficulty by granting us a gauge invariant way of coupling to the background electromagnetic field.

From this discussion, a natural guess for the matter variables for the bulk LG theory is a $SU(2)$ triplet on each layer consisting of boson creation and annihilation operators $B_n, B_n^\dagger$ and a boson number operator B_n^3 which essentially corresponds to the EM charge. Here $n = 1, 2$ is a layer index. If we write $B_n = B_n^1 + iB_n^2$ with $B_n^{1,2}$ real, the adjoint field B_n^a transforms like a vector under $SO(3)$. It is important to note, however, that any non-Abelian LG theory should be thought of as describing a (UV) quantum critical point proximate to the (IR) FQH state which shares universal features with the Abelian theory we started with. Since the level-rank duality is invoked deep in the FQH phase, it is a guess that these variables are the proper degrees of freedom at the UV quantum critical point (they may be alternatively understood as bound states – we will see later on that this interpretation is more accurate). Nevertheless, pairing these fields will lead to both the desired symmetry breaking pattern (4.5) as well as the existence of solitons with non-Abelian statistics.

The LG theory for the pairing of these fields can be explicitly constructed as follows. Each layer consists of a B_n^a field minimally coupled to its own $SU(2)_1$ gauge field,

$$\mathcal{L}_0[B_n, a_n] = \sum_{n=1,2}\left(|D_{a_n} B_n|^2 + \frac{1}{4\pi}\,\mathrm{Tr}\left[a_n da_n - \frac{2i}{3}a_n^3\right]\right) + \cdots \,, \tag{4.10}$$

where we have suppressed Lorentz and $SU(2)$ indices, used the notation $AdC = \varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu C_\lambda$, and defined the covariant derivative $D_{a_n}B_n \equiv \partial B_n^a - i\varepsilon^{abc}a_n^b B_n^c$. The ellipsis refers to additional contact terms, Maxwell

[1]Note that, depending on context, we use a^3 to denote both the diagonal element of a as well as $a \wedge a \wedge a$.

terms, etc. These are set up so that, taken individually, when each layer is at filling $\nu = 1/2$, the diagonal color flux $b_I^3 = \langle f_{I,xy}^3 \rangle / 2\pi$, vanishes.

Although the B_n fields are bosons, we assume that they do not condense. Rather, we consider pairing them using a method analogous to that of Jackiw and Rossi [220], who considered pairing Dirac fermions by coupling them to a scalar order parameter which mediates the pairing interaction. Let us introduce a field $\mathcal{O}^{ab}$ which transforms as an adjoint under each layer's $SU(2)$, $\mathcal{O} \mapsto G_1^{-1}\,\mathcal{O}\,G_2$, where $G_1, G_2 \in SO(3)$. Here we have used the fact that, as an adjoint field, $\mathcal{O}$ is blind to the $\mathbb{Z}_2$ centers of the two $SU(2)$ factors, and so effectively transforms under $SU(2)/\mathbb{Z}_2 \cong SO(3)$. The field $\mathcal{O}$ mediates a pairing interaction between the B_I^a fields as follows,

$$\mathcal{L}_{\text{pair}} = \lambda\, B_1^a \mathcal{O}^{ab} B_2^b\,. \tag{4.11}$$

We now require that $\mathcal{O}$ acquires a vacuum expectation value (VEV), which breaks $SU(2) \times SU(2)$ down to its diagonal subgroup $SU(2)_{\text{diag}}$, implementing the constraint $a_1 = a_2$. Any VEV equivalent to $\langle \mathcal{O} \rangle \propto \delta^{ab}$ is sufficient to achieve this. Therefore, in the final IR theory, the CS terms for a_1 and a_2 add, yielding a $SU(2)_2$ CS term, which describes precisely the $\nu = 1$ bosonic Moore-Read state. The authors of Ref. [196] then argued that, since the order parameter is valued on $[SO(3) \times SO(3)]/SO(3)$, that it can host non-trivial vortices which furnish the anyon content. This is in contrast to if we had chosen to pair fields in the fundamental representation, for which the order parameter has no non-trivial vortices. Finally, we note that because $\mathcal{O}$ is blind to the centers of the two original $SU(2)$ factors, the final gauge group is in fact $SU(2)_{\text{diag}} \times \mathbb{Z}_2$. This means that the resulting topological order is not quite that of the $\nu = 1$ bosonic Moore-Read state. We will elaborate on this point as well as the interpretation of the vortices in Section 4.3.3.

In spite of its successes, the LG theory described here has several problems. As mentioned above, the BF coupling between a_n^3 and the EM field A explicitly breaks the $SU(2)$ gauge symmetry. In addition, the theory of adjoint fields (4.10) cannot be the same as the Abelian LG theory of the original layers – the theories have different phase diagrams and so do not represent the same fixed point. Moreover, the final gauge group after pairing is not just $SU(2)$ but includes additional discrete gauge group factors. Finally, it is not entirely obvious how to generalize this approach to the rest of the Read-Rezayi states and beyond. In this Chapter, using non-Abelian boson-fermion dualities, we repair all of these problems.

4.3 LG Theories of the RR States from Non-Abelian Bosonization

4.3.1 Setup

Our setup for obtaining LG theories of the RR states is depicted in Figure 4.1. We again consider k layers of bosonic quantum Hall fluids at $\nu = 1/2$. The standard LG theory [73] of these states consists of Wilson-Fisher bosons – the Laughlin quasiparticles – on each layer, denoted Φ_n, with $n = 1, \cdots, k$ being the layer index. Each of these fields is coupled to an *Abelian* $U(1)_2$ Chern-Simons gauge field a_n as follows (the total gauge group is $[U(1)]^k$),

$$\mathcal{L}_A = \sum_n \left(|D_{a_n}\Phi_n|^2 - |\Phi_n|^4 + \frac{2}{4\pi} a_n da_n + \frac{1}{2\pi} A da_n \right) . \tag{4.12}$$

where again $-|\Phi|^4$ denotes tuning to the Wilson-Fisher fixed point and $D_{a_n} = \partial - ia_n$ is the covariant derivative. Since we wish to impose particle-hole symmetry on the bosons in the FQH state, these theories are relativistic. We take the background EM field A_μ to couple to the sum of the global $U(1)$ currents on each layer $j_{\text{top}} = \frac{1}{2\pi}\sum_n da_n$, although we could have in principle coupled background fields to each of these currents individually [74]. Notice that there is no continuous flavor symmetry manifest in $\mathcal{L}_A$ since each Φ_n couples to its own gauge field a_n. Being a theory of Laughlin quasiparticles, the Abelian quantum Hall state arises when the Φ fields are gapped, or $\rho_\Phi = \sum_{I,n}\langle i(\Phi_n^\dagger \overleftrightarrow{D}_{a_n,t}\Phi_n)\rangle = 0$. We note here that throughout this Chapter we define the filling fraction with a minus sign $\nu = -2\pi\rho_e/B$, where ρ_e is the physical EM chage and B is the background magnetic field.

We call the Abelian theory whose Lagrangian $\mathcal{L}_A$ is shown in Eq. (4.12), **Theory A**. In order to obtain a non-Abelian $SU(2)_k$ theory, our strategy is to invoke a non-Abelian duality to trade $\mathcal{L}_A$ for a theory of k bosons which are charged under emergent non-Abelian gauge fields. Since these particles are non-Abelian analogues of the Laughlin quasiparticles (they are *gapped* in the Abelian QH state), we will refer to them as non-Abelian composite *vortices*. Indeed, we will see that these theories are the k-component generalizations of the theory of Eq. (4.1). We call this non-Abelian theory **Theory B**. By pairing these fields across the layers, we will obtain the final $SU(2)_k$ theory. Thus, the non-Abelian FQH states we obtain can be interpreted as clustered states of the dual non-Abelian composite vortices, in analogy to the clustering interpretation of the wave functions. Moreover, from products of the non-Abelian vortex fields, analogues of the adjoint B_n operators of Section 4.2 can be constructed and paired, leading to a "quartetted" non-Abelian state. We now turn to a procedure for obtaining these dualities.

4.3.2 A Non-Abelian Duality: $U(1)_2$ + bosons $\longleftrightarrow$ $SU(2)_1$ + bosons

The non-Abelian dualities presented by Aharony [78] relate Chern-Simons theories coupled to complex scalar fields at their Wilson-Fisher fixed point to dual Chern-Simons theories coupled to Dirac fermions,

$$N_f \text{ scalars} + U(N)_{k,k} \quad \longleftrightarrow \quad N_f \text{ fermions} + SU(k)_{-N+N_f/2}\,, \tag{4.13}$$

$$N_f \text{ scalars} + SU(N)_{k} \quad \longleftrightarrow \quad N_f \text{ fermions} + U(k)_{-N+N_f/2,-N+N_f/2}\,, \tag{4.14}$$

$$N_f \text{ scalars} + U(N)_{k,k+N} \quad \longleftrightarrow \quad N_f \text{ fermions} + U(k)_{-N+N_f/2,-N-k+N_f/2}\,, \tag{4.15}$$

where all matter is in the *fundamental* representation of the gauge group. These take the shape of level-rank dualities, but a crucial difference is that they relate critical theories of matter coupled to Chern-Simons gauge fields rather than gapped TQFTs. Across these dualities, baryons of the $SU(k)_{-N}$ theories are mapped to monopoles of the $U(N)_k$ theories. We list our conventions for the non-Abelian Chern-Simons gauge fields in the Appendix.

Using these dualities as building blocks, it is possible to obtain new dualities relating the Abelian **Theory A** to a non-Abelian **Theory B**. The dualities obtained in this section are described in Refs. [98, 221], although we show in Section 4.4 that new, more general dualities can be obtained with an analogous strategy. To begin, let us consider the case of a single layer $k = 1$ of bosons at $\nu = 1/2$. The Landau-Ginzburg theory for this state consists of Wilson-Fisher bosons Φ coupled to a $U(1)_2$ gauge field a,

$$\mathcal{L}_A = |D_a\Phi|^2 - |\Phi|^4 + \frac{2}{4\pi}ada + \frac{1}{2\pi}Ada\,. \tag{4.16}$$

We start by invoking an Abelian boson-fermion duality, Eq. (4.14) with $N = k = 1$, which relates a Wilson-Fisher boson to a Dirac fermion with a unit of flux attached [79, 80],

$$|D_A\Phi|^2 - |\Phi|^4 \longleftrightarrow i\bar{\psi}\not{D}_b\psi - \frac{1}{2}\frac{1}{4\pi}bdb + \frac{1}{2\pi}bdA - \frac{1}{4\pi}AdA\,, \tag{4.17}$$

where b is a new dynamical $U(1)$ gauge field[2]. Applying this duality to $\mathcal{L}_A$ by treating a as a background field, one obtains **Theory C**,

$$\mathcal{L}_A \longleftrightarrow \mathcal{L}_C = i\bar{\psi}\not{D}_b\psi - \frac{1}{2}\frac{1}{4\pi}bdb + \frac{1}{4\pi}ada + \frac{1}{2\pi}ad(b+A)\,. \tag{4.18}$$

We can integrate out a without violating the Dirac quantization condition: its equation of motion is simply

[2]Throughout this Chapter, we approximate the Atiyah-Patodi-Singer η-invariant by a level-1/2 Chern-Simons term and include it in the action.

$-da = db + dA$. Thus,

$$\mathcal{L}_A \longleftrightarrow \mathcal{L}_C = i\bar{\psi} \not{D}_b \psi - \frac{3}{2}\frac{1}{4\pi} bdb - \frac{1}{2\pi} bdA - \frac{1}{4\pi} AdA \,. \tag{4.19}$$

Theory C was motivated as a description of the $\nu = 1/2$ FQH-insulator transition in Ref. [222]. The duality (4.19) is a special case of more general Abelian dualities described (and derived) in Refs. [92, 99]. However, of those dualities, it is one of the unique ones for which the Chern-Simons level is properly quantized. Notice also that this is the duality (4.15) with $N_f = N = k = 1$. The reason that we took a detour through the Abelian duality will become apparent in Section 4.4.

Applying the duality of Eq. (4.14) to **Theory C**, we obtain **Theory B**, which consists of bosons ϕ coupled to a $SU(2)_1$ gauge field u,

$$\mathcal{L}_A \longleftrightarrow \mathcal{L}_B = |D_{u-A\mathbf{1}/2}\phi|^2 - |\phi|^4 + \frac{1}{4\pi}\,\mathrm{Tr}\left[udu - \frac{2i}{3}u^3\right] - \frac{1}{2}\frac{1}{4\pi} AdA \,, \tag{4.20}$$

where $\mathbf{1}$ denotes the 2×2 identity matrix. Like its Abelian dual, Eq. (4.16), this theory describes a quantum phase transition between a $\nu = 1/2$ bosonic Laughlin state (gapped ϕ – the topological sector is decoupled) and a trivial insulator (condensed ϕ). Across this duality, the monopole current of **Theory A** is related to the baryon number current of **Theory B**,

$$\frac{\delta \mathcal{L}_A}{\delta A} = \frac{da}{2\pi} \longleftrightarrow \frac{\delta \mathcal{L}_B}{\delta A} = -\frac{i}{2}\,\phi^\dagger \overleftrightarrow{D}_{u-A\mathbf{1}/2}\,\phi - \frac{1}{2}\frac{dA}{2\pi} \tag{4.21}$$

Both of these currents correspond to the physical EM charge current J_e. We have suppressed Lorentz indices for clarity.

We can check explicitly that the $\nu = 1/2$ state has particle-hole symmetry in the composite vortex variables of **Theory B**. The physical EM charge density corresponds to the zeroth component of the currents (4.21),

$$\rho_e = \langle J_e^0 \rangle = -\frac{1}{2}\,\rho_\phi - \frac{1}{2}\frac{B}{2\pi}\,, \tag{4.22}$$

where ρ_ϕ denotes the number density of the non-Abelian composite vortices, so, when $\rho_\phi = 0$, the filling fraction is

$$\nu = -2\pi \frac{\rho_e}{B} = \frac{1}{2}\,. \tag{4.23}$$

This means that the $\nu = 1/2$ bosonic Laughlin state can be thought of as a gapped, particle-hole symmetric phase of non-Abelian composite vortices just as well as Abelian ones! By copying this duality k times, we will see in the next subsection how to obtain a non-Abelian LG theory of the RR states.

By applying the duality of Eq. (4.13) with $N = 1$ and $k = 2$ to **Theory A**, it is also possible to obtain a non-Abelian fermionic **Theory D** with gauge group $SU(2)_{-1/2}$. However, in this Chapter we focus on the non-Abelian bosonic LG theories, since in these theories the nature of the topological order and anyon content are manifest. Understanding the emergence of the RR states and other non-Abelian FQH states from the perspective of these non-Abelian composite fermion theories will be the subject of a forthcoming work. Combining all of these dualities, we see that

$$\begin{array}{c} \textbf{Theory A:}\text{ a scalar} + U(1)_2 \longleftrightarrow \textbf{Theory D:}\text{ a fermion} + SU(2)_{-1/2} \\ \updownarrow \\ \textbf{Theory C:}\text{ a fermion} + U(1)_{-3/2} \longleftrightarrow \textbf{Theory B:}\text{ a scalar} + SU(2)_1\,. \end{array} \tag{4.24}$$

It is a miracle of arithmetic that, like the boson/fermion dualities, the boson/boson and fermion/fermion dualities above also have the flavor of level-rank dualities. Indeed, it is easy to show that the topological phases of these theories are all dual to one another [98]. This can be thought of as a consequence of the fact that we were able to integrate out the gauge field a above without violating flux quantization. It is an interesting question to ask whether there are more general dualities which exhibit the same miracle. We will show that this is indeed the case in Section 4.4. We finally note that the dualities of Eq. (4.24) also have the feature of hosting an emergent $SO(3)$ global symmetry, a consequence of the fact $SU(2) \simeq USp(2)$ [223, 224]. This symmetry is manifest upon rewriting the theory in the $USp(2)$ language, which involves replacing the single complex matter field with two (pseudo)real ones [225].

4.3.3 Building Non-Abelian States from Clustering

Equipped with the duality (4.20), we now revisit the construction of Ref. [196], which we described in Section 4.2.2. We again start by considering the case where **Theory A** consists of $k = 2$ layers of $U(1)_2$ LG theories,

$$\mathcal{L}_A = \sum_n \left(|D_{a_n}\Phi_n|^2 - |\Phi_n|^4\right) + \frac{2}{4\pi} a_n da_n + \frac{1}{2\pi} Ad(a_1 + a_2)\,, I = 1, 2\,. \tag{4.25}$$

Invoking Eq. (4.20), **Theory B** is two $SU(2)_1$ theories,

$$\mathcal{L}_B = \sum_n \left(|D_{u_n - A\mathbf{1}/2}\phi_n|^2 - |\phi_n|^4\right) + \frac{1}{4\pi} \sum_n \mathrm{Tr}\left[u_n du_n - \frac{2i}{3} u_n^3\right] - \frac{1}{4\pi} AdA\,, \tag{4.26}$$

The half-filling condition here is simply $\nu = 1$. Notice that the background gauge field A couples to the "baryon number" current of the ϕ's in a *gauge invariant* way, in contrast to the theory of Ref. [196]. This also means that the physical bosons can be interpreted as baryons, or color singlet bound states of two ϕ's. However, these are monopoles from the point of view of **Theory A**.

To obtain a $SU(2)_2$ bosonic Moore-Read state at $\nu = 1$, we again seek the symmetry breaking pattern

$$SU(2)_1 \times SU(2)_1 \to SU(2)_2 \,. \tag{4.27}$$

As described in Section 4.2.2, the authors of Ref. [196] achieved this via pairing of adjoint fields so that the theory would support vortices of the order parameter with non-Abelian statistics. Instead, we will argue that singlet pairing of our *fundamental* composite vortices is sufficient to both obtain this symmetry breaking pattern and to capture the full anyon spectrum from the matter content. Nevertheless, it is still possible to obtain an analogue of the theory described in Section 4.2.2 by "quartetting" the composite vortices. In this case, the order parameter contributes non-trivial vortex excitations which possess non-Abelian statistics. These vortices arise because the order parameter sees $SO(3)$ rather than $SU(2)$ gauge fields, as in Ref. [196], and the resulting topological order again does not quite match that of the RR states. We provide a brief account of the quartetted phase at the end of this section.

Singlet Pairing

We pair the non-Abelian composite vortices by adding to **Theory B**, Eq. (4.26), an interaction with an electromagnetically neutral fluctuating scalar field $\Sigma_{mn}(x)$,

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_\Sigma + \mathcal{L}_{\text{singlet pair}} \,, \tag{4.28}$$

$$\mathcal{L}_\Sigma = \sum_{m,n} |\partial\Sigma_{mn} - iu_m\Sigma_{mn} + i\Sigma_{mn}u_n|^2 - V[\Sigma] \,, \tag{4.29}$$

$$\mathcal{L}_{\text{singlet pair}} = -\sum_{m,n} \phi_m^\dagger \Sigma_{mn} \phi_n \,, \tag{4.30}$$

where Σ_{mn} is Hermitian in the layer indices m, n, and $V[\Sigma]$ is the potential for Σ. The off-diagonal components, $\Sigma_{12} = \Sigma_{21}^\dagger$, induce interlayer pairing, while the diagonal components, Σ_{11} and Σ_{22}, induce intralayer pairing. Under a gauge transformation, Σ_{nn} (no summation intended) transforms in the adjoint representation of the $SU(2)$ gauge group on layer n, while Σ_{12} transforms as a bifundamental field under the bilayer

$SU(2) \times SU(2)$ gauge group,

$$\Sigma_{mn} \mapsto U_m \Sigma_{mn} U_n^\dagger, \quad U_m \in SU(2) \text{ on layer } m. \tag{4.31}$$

In both Eq. (4.29) and Eq. (4.31), left (right) multiplication indicates contraction with Σ's color indices in the fundamental (antifundamental) representation of $SU(2)$.

In order to achieve the symmetry breaking pattern (4.27), we choose the potential V so that Σ_{mn} condenses in such a way that $\langle \phi_1^\dagger \phi_2 \rangle \neq 0$ while $\langle \phi_1 \rangle = \langle \phi_2 \rangle = 0$. Explicitly,

$$\langle \Sigma_{nm} \rangle = M_{nm} \mathbf{1}, \qquad M_{11}, M_{22}, \det M > 0 \tag{4.32}$$

The requirement $M_{11}, M_{22}, \det M > 0$ guarantees that the resulting effective potential for $\phi_{1,2}$ is minimized only for $\langle \phi_1 \rangle = \langle \phi_2 \rangle = 0$, while the off-diagonal components $M_{12} = M_{21}^\dagger$ break the $SU(2) \times SU(2)$ gauge symmetry down to the diagonal $SU(2)$. As described in Section 4.2.2, in the low energy limit, this sets $u_1 = u_2$, and the Chern-Simons levels add to yield the correct $SU(2)_2$ Chern-Simons theory (the bosonic Moore-Read state) as the low energy TQFT.

Having obtained the $SU(2)_2$ RR state, we now show that its anyon spectrum is furnished by the non-Abelian composite vortices $\phi_{1,2}$. Both ϕ_1 and ϕ_2 carry electric charge $Q = \frac{1}{2}$ and transform in the spin-$\frac{1}{2}$ representation of the $SU(2)_2$ gauge group, endowing them with non-Abelian braiding statistics. These are precisely the properties of the minimal charge anyon in the $\nu = 1$ bosonic Moore-Read state, the half-vortex! Even though there are two bosonic fields $\phi_{1,2}$, these do not represent distinct anyons: ϕ_1 and ϕ_2 can be freely transformed into one another via the bilinear condensate $\langle \phi_1^\dagger \phi_2 \rangle$. In other words, their currents are no longer individually conserved, and the layer index is no longer a good quantum number. The remainder of the anyon spectrum is obtained by constructing composite operators of the ϕ fields or, equivalently, by fusing multiple minimal charge anyons. In the present case, the only remaining anyon is the Majorana fermion, which transforms in the spin-1 representation of $SU(2)$, and so is represented by the local bilinear $\chi_n^a = \phi_n^\dagger t^a \phi_n$ (see Table 4.1). We note that, unlike in Ref. [196], there are no non-trivial vortices in this approach, since an order parameter valued on $[SU(2) \times SU(2)]/SU(2)$ cannot host non-trivial vortices.

The reader might object to our identification of the individual particles making up the pairs with the fundamental anyons, since the energy cost to break up a pair will be on the order of the UV cutoff. However, this is not a significant shortcoming of our construction, since anyons are only well defined upon projecting into the (topologically ordered) ground state. They should therefore always be viewed as infinite energy excitations representated as Wilson lines.

Table 4.1: List of quasi-particles in the $\nu = 1$ bosonic Moore-Read state, their spin, θ, $U(1)_{EM}$ charges, Q, and the corresponding operator in our LG theory. We label the anyons by the corresponding operators in the edge CFT (see e.g. Refs. [63, 183]). Note that we do not sum over the layer index n.

	1 (vacuum)	$\sigma e^{i\varphi/2}$ (half-vortex)	χ (Majorana fermion)
θ	0	$\frac{3}{16}$	$\frac{1}{2}$
Q	0	$\frac{1}{2}$	0
Field theory	-	ϕ_n	$\phi_n^\dagger t^a \phi_n$

Quartetting and Vortices

Although singlet pairing is sufficient to obtain the RR states, it is interesting to consider an alternative mechanism for obtaining non-Abelian states that more closely resembles the construction of Ref. [196] that was discussed in Section 4.2.2. In this scenario, rather than pairing the non-Abelian bosons of **Theory B** (4.26), we imagine *quartetting* them. To do this, we define the adjoint operators,

$$B_n^a = \phi_n^\dagger t^a \phi_n\,, \tag{4.33}$$

where the repeated n index on the right hand side is not summed over. These operators are neutral under $U(1)_{\rm EM}$, and they will serve the same purpose for us here as the B_n^a fields disucussed in Section 4.2 and Ref. [196]. We thus consider a pairing interaction of the B_n^a's, or a *quartetting* interaction of the ϕ's, by introducing a scalar field $\mathcal{O}$ to mediate the pairing interaction,

$$\mathcal{L}_{\rm quartet} = \lambda\, B_1^a\, \mathcal{O}^{ab}\, B_2^b = \lambda\, (\phi_1^\dagger t^a \phi_1)\, \mathcal{O}^{ab}\, (\phi_2^\dagger t^b \phi_2)\,. \tag{4.34}$$

The quartetted phase, where $\langle \mathcal{O}^{ab}\rangle = v\delta^{ab}$ and $\langle\phi_1\rangle = \langle\phi_2\rangle = 0$, is accessed by adding a suitable potential $V[\mathcal{O}]$ and ensuring that $\phi_{1,2}$ are gapped via a mass term $-m^2\sum_n |\phi_n|^2$. Because $\mathcal{O}$ radiatively acquires a kinetic term of the form of a gauged nonlinear sigma model (NLSM), the resulting effective theory in the quartetted phase is

$$\mathcal{L}_{\rm eff} = \mathcal{L}_B + \mathcal{L}_{\rm quartet} - m^2 \sum_n |\phi_n|^2 - V[\mathcal{O}] + \kappa\ {\rm Tr}\left[\mathcal{O}^{-1} D_{u_1-u_2}\mathcal{O}\,\mathcal{O}^{-1} D_{u_1-u_2}\mathcal{O}\right] \tag{4.35}$$

where κ is a coupling constant defined so that $\mathcal{O}$ is properly normalized.

Since $\mathcal{O}$ transforms in the adjoint representation of the $SU(2)$ of each layer, it is blind to their $\mathbb{Z}_2$ centers. This means that the quartetted phase hosts not only the non-Abelian $SU(2)_2$ topological order (since $u_1 - u_2$ is again Higgsed), but also an additional Abelian $\mathbb{Z}_2$ sector. Explicitly, as noted in Section 4.2.2, the condensation of $\mathcal{O}$ yields the symmetry breaking pattern $SU(2)\times SU(2) \to SU(2)_{\rm diag}\times\mathbb{Z}_2$, where the residual $\mathbb{Z}_2$ can be chosen to act on either ϕ_1 or ϕ_2 (amounting to a choice of basis). Hence, the full topological order of the ground state is $SU(2)_2 \times \mathbb{Z}_2$. This is also true of the original construction of Ref. [196], meaning that the singlet pairing mechanism discussed above carries the significant advantage that it yields the $\nu = 1$ Moore-Read state *alone*, with no additional Abelian sector. We therefore focus on singlet pairing for the remainder of this Chapter.

How do we account for the new Abelian anyon content? As discussed in Section 4.2.2, because of the order parameter's blindness to the $\mathbb{Z}_2$ centers, the NLSM above admits vortex solutions. These vortices can carry fluxes of *both* of the residual $\mathbb{Z}_2$ and $SU(2)$ gauge groups, and so they possess non-trivial braiding statistics with respect to each other and the scalar fields. However, since the B_n^a fields here are electrically neutral, the vortices of the order parameter should not carry any electric charge either. These vortices should therefore correspond to anyon excitations which are distinct from those that can be obtained from the $\phi_{1,2}$ fields alone, as these fields carry electric charge. We leave a detailed understanding of this Abelian sector to future work.

As in the singlet pairing case, this quartetting procedure can be generalized to the case of k layers, or $\nu = k/2$, which can be easily shown to have $SU(2)_k \times \mathbb{Z}_2^{k-1}$ topological order (each factor of $\mathbb{Z}_2$ corresponds to the unbroken center of a broken $SU(2)$). In the next subsection, we describe how both the singlet pairing and quartetting constructions can be generalized to the remaining RR fillings through a flux attachment transformation.

4.3.4 Generating the Full Read-Rezayi Sequence through Flux Attachment

By attaching M fluxes to the k-layer generalization of **Theory A** (4.25) and performing the same transformation on **Theory B** (4.26), it is possible to obtain LG theories of the remaining RR states at filling fractions

$$\nu = \frac{k}{Mk+2}\,. \tag{4.36}$$

Flux attachment can be performed on **Theory A** as a modular transformation $\mathcal{S}\mathcal{T}^M\mathcal{S}$ [10, 121], where

$$\mathcal{S} : \mathcal{L}[A] \mapsto \mathcal{L}[b] + \frac{1}{2\pi} Adb\,, \qquad \mathcal{T} : \mathcal{L}[A] \mapsto \mathcal{L}[A] + \frac{1}{4\pi} AdA\,, \tag{4.37}$$

where again A is the background EM field, and b is a new dynamical $U(1)$ gauge field. Thus, attaching M fluxes to **Theory A** amounts to

$$\mathcal{S}\mathcal{T}^M\mathcal{S} : \mathcal{L}_A[A] \mapsto \mathcal{L}_A[b] + \frac{1}{2\pi} cd(b+A) + \frac{M}{4\pi} cdc\,, \tag{4.38}$$

where c is a new dynamical $U(1)$ gauge field. It is straightforward to see that this transformation is equivalent to the usual attachment of M fluxes to the composite bosons (related to the composite vortex variables – or Laughlin quasiparticles – of **Theory A** by boson-vortex duality [65, 66]). One of the insights of Refs. [79, 80] was that the modular group $\mathrm{PSL}(2,\mathbb{Z})$ generated by $\mathcal{S}$ and $\mathcal{T}$ can generate new dualities from old ones. Restricting for the moment to $k=2$ layers, the transformed **Theory A** is dual to

$$\tilde{\mathcal{L}}_B = \sum_n \left(|D_{u_n - b\mathbf{1}/2}\phi_n|^2 - |\phi_n|^4\right) + \frac{1}{4\pi}\sum_n \mathrm{Tr}\left[u_n du_n - \frac{2i}{3}u_n^3\right] - \frac{1}{4\pi}bdb + \frac{1}{2\pi}cd(b+A) + \frac{M}{4\pi}cdc\,. \quad (4.39)$$

We can repackage the $SU(2)$ gauge fields u_n as new $U(2)$ gauge fields u_n' with trace $\mathrm{Tr}[u_1'] = \mathrm{Tr}[u_2'] = b$. This gluing of the traces together can be implemented by introducing a new auxiliary gauge field α,

$$\begin{aligned}\tilde{\mathcal{L}}_B = \quad & \sum_n \left(|D_{u_n'}\phi_n|^2 - |\phi_n|^4\right) + \frac{1}{4\pi}\sum_n \mathrm{Tr}\left[u_n' du_n' - \frac{2i}{3}{u_n'}^3\right] \\ & - \frac{2}{4\pi}\mathrm{Tr}[u_1']d\,\mathrm{Tr}[u_1'] + \frac{1}{2\pi}cd(\mathrm{Tr}[u_1'] + A) + \frac{M}{4\pi}cdc + \frac{1}{2\pi}\alpha d\left(\mathrm{Tr}[u_1'] - \mathrm{Tr}[u_2']\right)\,.\end{aligned} \quad (4.40)$$

This transformation does not impact the singlet pairing nor the quartetting procedure discussed in the previous subsection, and it readily generalizes to k layers (more constraints need to be introduced in that case to glue the Abelian gauge fields together). We therefore obtain the $SU(2)_2$ Chern-Simons theory at low energies, albeit with the additional Abelian sector introduced above. For the general case of k layers, the u_n''s on each layer are set equal to one another, and the low energy TQFT is a $U(2)_{k,-2k} \times U(1)_M$ Chern-Simons-BF theory given by

$$\mathcal{L} = \frac{k}{4\pi}\mathrm{Tr}\left[u'du' - \frac{2i}{3}{u'}^3\right] - \frac{k}{4\pi}\mathrm{Tr}[u']d\,\mathrm{Tr}[u'] + \frac{1}{2\pi}cd(\mathrm{Tr}[u'] + A) + \frac{M}{4\pi}cdc\,. \quad (4.41)$$

This is indeed the proper bulk TQFT describing the RR states at filling (4.36), first described in Ref. [226]. As in the case of the $\nu = 1$ bosonic Moore-Read state discussed above, the fundamental scalars (i.e. the composite vortices) comprise the minimal charge anyons, here possessing electric charge $Q = 1/(Mk+2)$. This is the expected result for the minimal charge anyon in the general RR states.

4.4 Generalization to Non-Abelian $SU(N_f)$-Singlet States

Having derived a LG theory for the RR states, we will now demonstrate how our construction can be naturally extended to the generalized non-Abelian $SU(N_f)$-singlet states occuring at fillings

$$\nu = \frac{kN_f}{N_f + 1 + kMN_f}\,, \qquad k, N_f, M \in \mathbb{Z}\,. \tag{4.42}$$

These are clustered states in which k represents the number of local particles (fermions or bosons for odd and even M, respectively) in a cluster, M the number of attached Abelian fluxes, and N_f the number of internal degrees of freedom. Like the RR states, which correspond to $N_f = 1$, we will show that these states can also be obtained by pairing starting from a parent multi-layer Abelian LG theory. The particular Abelian states we will target are the N_f-component generalizations of the Halperin $(2, 2, 1)$ states. In parallel to Section 4.3, we will show that the LG theories of these Abelian states satisfy a new non-Abelian bosonization duality. This duality relates the Abelian LG theory of the generalized Halperin states to an $SU(N_f + 1)_1$ Chern-Simons-matter theory. That this is possible is perhaps not surprising given that the N_f-component $(2, 2, 1)$ state is known to have an edge theory which furnishes a representation of the $SU(N_f + 1)_1$ Kac-Moody algebra, as we shall review below [197, 201, 218, 219]. The generalized NASS states are then obtained by singlet pairing of the dual non-Abelian bosons.

4.4.1 Motivation: "Projecting Down" to the Generalized NASS States

Just as the RR states are naturally understood starting with the $\nu = 1/2$ Laughlin state by way of "projecting down," the generalized NASS states can be built up from N_f-component generalizations of the Halperin $(2, 2, 1)$ spin-singlet state [227]. These (bosonic) states are *Abelian* and correspond to $M = 0, k = 1$. These states are described by the wave functions

$$\Psi^{(221)}_{N_f}(\{z^\sigma_i\}) = \prod_{\sigma=1}^{N_f}\prod_{i<j}(z^\sigma_i - z^\sigma_j)^2 \prod_{\sigma<\sigma'}^{N_f}\prod_{i,j}(z^\sigma_i - z^{\sigma'}_j)^1 e^{-\frac{1}{4}\sum_{\sigma,i}|z^\sigma_i|^2}, \tag{4.43}$$

where $z^\sigma_i = x^\sigma_i + iy^\sigma_i$ denotes the complex coordinates of the i^{th} boson with component index σ. In direct analogy with the $\nu = k/2$ RR states, the generalized NASS wave functions for general k (but still $M = 0$) may be obtained by symmetrizing over a product of k copies of the N_f-component $(2, 2, 1)$ wave function [217],

$$\Psi_{k,N_f} = \mathcal{S}_k \left[\prod_{i=0}^{k-1} \Psi^{(221)}_{N_f}(z_{1+iN/k}, \ldots, z_{(i+1)N/k}) \right], \tag{4.44}$$

where the symmetrization operation $\mathcal{S}_k$ is morally the same as the one defined in Section 4.2.1. Again, the form of the wave function makes explicit the clustering of bosons characteristic of non-Abelian states. The wave functions for general M are obtained by multiplying Ψ_{k,N_f} by a $\nu = \frac{1}{M}$ Laughlin factor. Note that setting $N_f = 1$ recovers the RR wave functions (4.4).

The generalized NASS wave functions (4.44) should also be expresssible as correlators of the $SU(N_f+1)_k$ WZW CFT for $M = 0$ and of the $[U(1)]^{N_f} \times SU(N_f+1)/[U(1)]^{N_f}$ coset CFT for $M > 0$. Although this appears to have only been discussed explicitly for $N_f = 1, 2, 3$ [198, 212, 228], we will assume that this holds true for general N_f. We thus expect the corresponding bulk theories for the generalized NASS states to be $SU(N_f+1)_k$ Chern-Simons theories.

For the N_f-component Halperin states ($k = 1$), the presence of this "hidden" $SU(N_f+1)$ representation can be motivated as follows. These states are described by a $N_f \times N_f$ K-matrix and N_f-component charge vector q,

$$K = \begin{pmatrix} 2 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 2 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 2 & & & 1 \\ \vdots & \vdots & & \ddots & & \vdots \\ 1 & 1 & & & 2 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 2 \end{pmatrix}, \qquad q = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{4.45}$$

The form of the charge vector reflects the fact that the physical bosonic excitations of each species each carry the same EM charge, and it can read off that the Hall conductivity is $\sigma_{xy} = q^T K^{-1} q \frac{e^2}{h} = \frac{N_f}{N_f+1} \frac{e^2}{h}$. Under a particular change of basis $\tilde{K} = G^T K G$ and $\tilde{q} = Gq$, $G \in SL(N_f, \mathbb{Z})$, K can be shown to be related to the Cartan matrix of $SU(N_f+1)$ [201, 219],

$$G = \begin{pmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & \ddots & \ddots & & \\ & & & & 1 & -1 \\ & & & & & 1 \end{pmatrix} \Rightarrow \tilde{K} = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & & & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & & & 2 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix}, \qquad \tilde{q} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{4.46}$$

Using this fact, one can show that the edge theory defined by $\tilde{K}$ supports a $SU(N_f+1)_1$ Kac-Moody algebra (see e.g. Refs. [201, 229] for a derivation), and hence is equivalent to the $SU(N_f+1)$ WZW CFT. Consequently, the corresponding bulk theory of the N_f-component $(2,2,1)$ Halperin state is a $SU(N_f+1)_1$ Chern-Simons theory. This is the N_f-component generalization of the level-rank duality $U(1)_2 \leftrightarrow SU(2)_1$ described in Section 4.2.

This discussion indicates that we should expect the LG theories of the generalized NASS states can be obtained from pairing k copies of the N_f-component $(2,2,1)$ Halperin state. Because this state is level-rank dual to a $SU(N_f+1)_1$ theory, we might expect that there is a non-Abelian Chern-Simons-matter theory duality also taking this shape, from which we can build a LG theory of the non-Abelian states. We now show that this is indeed the case.

4.4.2 Non-Abelian Duals of N_f-Component Halperin $(2,2,1)$ States

The necessary non-Abelian duality can be constructed by starting with the Abelian LG theory for the N_f-component Halperin state, which we again call **Theory A**. This theory consists of N_f species of Wilson-Fisher bosons Φ_I, $I=1,\ldots,N_f$, each coupled to a $U(1)$ Chern-Simons gauge fields a_I,

$$\mathcal{L}_A = \sum_{I=1}^{N_f}\left(|D_{a_I}\Phi_I|^2 - |\Phi_I|^4\right) + \frac{1}{4\pi}\sum_{I,J=1}^{N_f} K_{IJ}a_I da_J + \frac{1}{2\pi}\sum_{I=1}^{N_f} q_I A da_I, \tag{4.47}$$

where K and q are given in Eq. (4.45). The N_f-component Halperin state corresponds to the phase in which all of the Φ_I fields – the Laughlin quasiparticles – are gapped. We emphasize that there is no continuous $SU(N_f)$ global symmetry rotating the Φ_I fields manifest in **Theory A**. Instead, there is only a discrete exchange symmetry of the Φ_I fields.

Following the reasoning laid out in Section 4.3.2, we now show that this theory is dual to one of N_f Wilson-Fisher bosons coupled to a *single* $SU(N_f+1)$ gauge field. Similar dualities have also been described in Ref. [230]. We start by applying the Abelian boson-fermion duality of Eq. (4.17) to each scalar Φ_I, treating the a_I's as background fields, to obtain the Dirac fermion **Theory C**,

$$\begin{aligned}\mathcal{L}_A \longleftrightarrow \mathcal{L}_C =& \sum_{I=1}^{N_f} i\bar{\psi}_I \not{D}_{b_I}\psi_I + \sum_{I=1}^{N_f}\frac{1}{4\pi}a_I da_I + \sum_{I=1}^{N_f}\sum_{J=I+1}^{N_f}\frac{1}{2\pi}a_I da_J + \sum_{I=1}^{N_f}\frac{1}{2\pi}A da_I \\ &+ \sum_{I=1}^{N_f}\left[-\frac{1}{2}\frac{1}{4\pi}b_I db_I + \frac{1}{2\pi}b_I da_I\right].\end{aligned} \tag{4.48}$$

As in the example discussed in Section 4.3.2, the a_I fields can be safely integrated out while respecting the

Dirac flux quantization condition. This is because all of the Chern-Simons terms have coefficient equal to unity. On integrating out one of the a_I fields, the remaining ones become Lagrange multipliers enforcing the constraints $b_I = b_1 \equiv b$. Integrating out the remaining a_I's, we find that **Theory C** can be rewritten as one of fermions coupled to a single dynamical gauge field,

$$\mathcal{L}_C = \sum_{I=1}^{N_f} i\bar{\psi}_I \not{D}_b \psi_I - \frac{N_f+2}{2}\frac{1}{4\pi}bdb - \frac{1}{2\pi}bdA - \frac{1}{4\pi}AdA. \tag{4.49}$$

In contrast to **Theory A**, **Theory C** has a manifest $SU(N_f)$ global flavor symmetry[3] since the fermions all couple in the same way to the gauge field b. This symmetry is thus an *emergent* symmetry from the point of view of **Theory A**.

We may now apply the non-Abelian duality (4.14) to **Theory C**, leading to a non-Abelian bosonic **Theory B**,

$$\mathcal{L}_B = \sum_{I=1}^{N_f} |D_{u-\frac{1}{N_f+1}A\mathbf{1}}\phi_I|^2 - |\phi|^4 + \frac{1}{4\pi}\mathrm{Tr}\left[udu - \frac{2i}{3}u^3\right] - \frac{1}{4\pi}\frac{N_f}{N_f+1}AdA\,. \tag{4.50}$$

where $-|\phi|^4$ denotes tuning to the Wilson-Fisher fixed point consistent with a global $SU(N_f)$ symmetry. We will again refer to the ϕ_I fields as the non-Abelian composite vortices. It will be convenient in the subsection below to re-express this theory as a $U(N_f+1)$ gauge theory with a constraint,

$$\mathcal{L}_B = \sum_{I=1}^{N_f} |D_u\phi_I|^2 - |\phi|^4 + \frac{1}{4\pi}\mathrm{Tr}\left[udu - \frac{2i}{3}u^3\right] + \frac{1}{2\pi}\alpha d(\mathrm{Tr}\,[u] - A) - \frac{1}{4\pi}AdA\,, \tag{4.51}$$

where we have introduced a $U(1)$ gauge field α. We have thus obtained a *new* triality,

$$\begin{gathered}\textbf{Theory A: } N_f \text{ scalars} + U(1)\ K\text{-matrix theory of Eq. (4.45)} \\ \updownarrow \\ \textbf{Theory C: } N_f \text{ fermions} + U(1)_{-\frac{N_f+2}{2}} \longleftrightarrow \textbf{Theory B: } N_f \text{ scalars} + SU(N_f+1)_1\,.\end{gathered} \tag{4.52}$$

This is the main result of this subsection. It is interesting that, for our particular choice of K-matrix in **Theory A**, we have obtained a non-Abelian dual theory in which the rank of the gauge group depends on the number of matter species and in which an emergent $SU(N_f)$ symmetry appears. Such trialities can be extended by applying the modular transformation $\mathcal{S}\mathcal{T}^{P-1}\mathcal{S}$ (flux attachment) to each side, transforming the K matrix of **Theory A** to that of the N_f-component $(P+1, P+1, P)$ Halperin states. The family

[3] See Ref. [223] for a more detailed discussion of global symmetries in non-Abelian dualities.

of Abelian composite fermion theories obtained by this transformation has been conjectured to describe plateau transitions in fractional Chern insulators [231].

Notice that Eq. (4.52) does not contain a non-Abelian fermionic theory analogous to **Theory D** in Eq. (4.24). That is not to say such a theory does not exist. As with the RR states, we leave to future work a full inquiry into how the NASS states, to be discussed in the next section, may arise in a fermionic picture.

4.4.3 Generating the Non-Abelian $SU(N_f)$-Singlet Sequence from Clustering

With the non-Abelian composite vortex description of the N_f-component $(2,2,1)$ states in hand, we can follow the pairing procedure of Section 4.3.3 to generate the generalized NASS sequence. Unlike in Section 4.3, in this section we will consider LG theories for general k, M, and N_f from the outset. Our **Theory A** will thus consist of k layers of LG theories of the N_f-flavor Halperin $(2,2,1)$ states,

$$\mathcal{L}_A = \sum_{I,n} \left(|D_{a_{I,n}} \Phi_{I,n}|^2 - |\Phi_{I,n}|^4 \right) + \frac{1}{4\pi} \sum_{I,J,n} K_{IJ} a_{I,n} da_{J,n} + \frac{1}{2\pi} \sum_{I,n} q_I A da_{I,n}, \tag{4.53}$$

where again the K-matrix and charge vector are given by Eq. (4.45), and $n = 1, \ldots, k$ denotes the layer index. Applying the duality (4.51) to each layer, this theory is dual to the non-Abelian **Theory B**,

$$\mathcal{L}_B = \sum_{I,n} |D_{u_n} \phi_{I,n}|^2 - \sum_n |\phi_n|^4 + \sum_{I,n} \mathcal{L}_{U(N_f+1)}[u_n] + \frac{1}{2\pi} \sum_{I,n} \alpha_n d(\mathrm{Tr}\,[u_n] - A) - \frac{k}{4\pi} AdA. \tag{4.54}$$

Here, lower case Latin letters denote a layer index, upper case Latin letters a flavor index. We have also defined, for compactness,

$$\mathcal{L}_{U(N)}[u] \equiv \frac{1}{4\pi} \mathrm{Tr} \left[udu - \frac{2i}{3} u^3 \right]. \tag{4.55}$$

We introduce M via flux attachment, or application of the modular transformation $\mathcal{S}\mathcal{T}^M\mathcal{S}$, as in Section 4.3.4. This yields a sequence of descendant theories labelled by k, M, and N_f,

$$\begin{aligned} \tilde{\mathcal{L}}_B = &\sum_{I,n} |D_{u_n} \phi_{I,n}|^2 - \sum_n |\phi_n|^4 + \sum_n \mathcal{L}_{U(N_f+1)}[u_n] + \frac{1}{2\pi} \sum_n \alpha_n d(\mathrm{Tr}\,[u_n] - a) \\ &- \frac{k}{4\pi} ada + \frac{1}{2\pi} adb + \frac{M}{4\pi} bdb + \frac{1}{2\pi} bdA. \end{aligned} \tag{4.56}$$

We are now in a position to consider singlet pairing between the different layers. One can also consider quartetting the composite vortices, but this only leads to additional Abelian sectors, as in the RR case.

Singlet pairing between the fundamental scalars is again mediated via a dynamical scalar field, $\Sigma_{m,n}(x) =$

$\Sigma^\dagger_{n,m}(x)$, transforming in the bifundamental representation of the $SU(N_f+1)$ factor on layer m and on layer n, i.e. $\Sigma_{m,n} \mapsto U_m \Sigma_{m,n} U_n^\dagger$, where $U_n, U_m \in SU(N_f+1)$. Note that the $U(1)$ gauge transformations cancel out, as the α_n fields force all the $U(1)$ gauge fields $\text{Tr}[u_n]$ to be equal. If we require that $\Sigma_{m,n}$ be a flavor singlet, its coupling to the non-Abelian composite vortices is therefore

$$\mathcal{L}_{\text{singlet pair}} = -\sum_{m,n,I} \phi^\dagger_{I,m}\, \Sigma_{m,n}\, \phi_{I,n}\,. \tag{4.57}$$

As before, the off-diagonal terms induce inter-layer pairing, while the diagonal terms can be used to ensure that $\langle \phi_{I,n} \rangle = 0$. Thus, we obtain a non-Abelian state when $\Sigma_{m,n}$ condenses in such a way that it enforces the constraint $u_n \equiv u'$ for all n. Putting these pieces together, we find that the paired phase is governed by the TQFT

$$\mathcal{L}_{\text{eff}} = k\, \mathcal{L}_{U(N_f+1)}[u'] - \frac{k}{4\pi}\text{Tr}[u']d\text{Tr}[u'] + \frac{1}{2\pi}\text{Tr}[u']db + \frac{M}{4\pi}bdb + \frac{1}{2\pi}bdA \tag{4.58}$$

Integrating out the fluctuating gauge fields indeed yields the correct Hall response,

$$\sigma_{xy} = \frac{kN_f}{N_f + 1 + kMN_f}\frac{e^2}{h}, \tag{4.59}$$

which is the expected result for the generalized NASS states.

As in our LG theories of the RR states, the fundamental scalars $\phi_{I,n}$ correspond to the minimal charge anyons. Indeed, one can check from the equations of motion that the fundamental scalar fields each carry charge $Q = \frac{1}{N_f+1+MkN_f}$, which reduces to the expected result for the minimal charge anyons of the RR and non-Abelian spin singlet states for $N_f = 1$ and $N_f = 2$, respectively. Additionally, in the paired phase, the condensation of the bilinears $\phi^\dagger_{I,m}\phi_{I,n} + H.c.$ (no sum on I) ensures that all the $\phi_{I,n}$, for fixed I, are indistinguishable, removing the redundancy of the layer degree of freedom. In particular, because we took the pairing interaction to be diagonal in the flavor indices, there is no mixing between flavors on different layers. Hence the fundamental scalar excitations should still transform into each other under the diagonal $SU(N_f)$ subgroup of the original $SU(N_f) \times \cdots \times SU(N_f)$ global symmetry. Consequently, our theory reproduces the desired anyon spectrum, and we conclude that we have obtained a LG theory for the generalized NASS states.

4.5 Discussion

Using non-Abelian boson-fermion dualities, we have presented a physical pairing mechanism by which the non-Abelian Read-Rezayi states and their generalizations, the non-Abelian $SU(N_f)$-singlet states, may be obtained by "projecting down" from parent Abelian states. These dualities relate the usual Abelian LG theories of the parent state to theories of non-Abelian "composite vortices," which pair to form the non-Abelian FQH state. While this pairing amounts to condensing local operators in the non-Abelian theory, this is not the case in the original Abelian LG theory of Laughlin quasiparticles, in which the composite vortices are monopoles. In the process of developing these theories, we have described a new triality (4.52) which parallels a level-rank duality apparent from CFT/ideal wave function considerations and which has the interesting property that it involves a non-Abelian gauge theory with rank depending on the number of matter species. We believe that this approach for obtaining physically motivated bulk descriptions of non-trivial gapped phases represents a promising direction for future applications of duality to condensed matter physics which has thus far been under-explored.

Our construction contrasts with earlier bulk descriptions of non-Abelian FQH states in important ways. The use of non-Abelian boson-fermion dualities, which relate parent quantum critical points, or Landau-Ginzburg effective field theories, provides a clear mapping to theories of non-Abelian "composite vortex" variables which are manifestly gauge invariant, unlike in earlier approaches that invoked level-rank duality deep in the topological phase [196, 197]. Additionally, we showed that these earlier approaches in fact lead to a superfluous Abelian sector on top of the desired non-Abelian topological order. The use of non-Abelian dualities also avoids the issues inherent to parton constructions [186–188], which provide a perhaps larger class of fractionalized descriptions but rely on the assumption that the fractionalized particles are not confined. This is in spite of the fact that they are generally charged under non-Abelian gauge fields without Chern-Simons terms and, as such, are known to be confining in 2+1 dimensions. Consequently, it is likely that many partonic descriptions are on unstable dynamical footing.

We anticipate that many more exotic FQH and otherwise topologically ordered states can be targeted with our approach. Again, we can draw inspiration from edge CFT and ideal wave function approaches. For instance, the spin-charge separated spin-singlet states of Ref. [232] can both be related to a parent bilayer Abelian state and be obtained from conformal blocks of an $SO(5)$ WZW theory. There exist, in fact, Chern-Simons-matter dualities involving precisely $SO(N)$ (and many other) gauge groups [221, 233], which suggests that it may be possible to formulate non-Abelian Landau-Ginzburg theories of these states. It is perhaps also possible to apply our approach to generating bulk parent descriptions of the orbifold FQH states [194], which can involve an interesting interplay of usual gauge symmetries with gauged higher-form symmetries [234, 235].

In this Chapter, we have focused on understanding non-Abelian states via pairing of non-Abelian bosonic matter. However, as described in Section 4.3, a non-Abelian composite fermion description is available for the $\nu = \frac{1}{2}$ Laughlin states. In the parent Abelian phase, these fermions feel a magnetic field and fill an integer number of Landau levels. Pairing across layers of these integer quantum Hall states appears to lead in fact to $SU(2)_{-k}$ theories, which may be connected to the particle-hole conjugates of the RR states (note the sign of k). One may also consider starting not from multiple layers of FQH phases but instead of the (fermionic) compressible states at filling $\nu = 1/2n$, for which Dirac fermion theories have been proposed [81, 88]. It is possible that applying non-Abelian dualities to these theories may provide an avenue for developing exotic non-Abelian *excitonic* phases. We plan to provide a general discussion of composite fermion approaches to generating non-Abelian states in future work.

We lastly comment on the possible connection of the theories presented here to numerical studies of transitions between Abelian and non-Abelian states in bilayers [202–205, 207, 208]. To the extent that these transitions are continuous, it is an exciting possibility that they are in the universality class of the quantum critical theories presented here. However, since these theories are very strongly coupled, the only analytic techniques against which this can be checked are large-N approaches, which may describe a wholly different fixed point. Perhaps eventually the conformal bootstrap will be able to shed light on this issue.

Chapter 5

Disorder, Strong Correlations, and Duality: The Dirty Boson Problem and Beyond

This Chapter is reproduced with the permission of the coauthors from Hart Goldman, Alex Thomson, Laimei Nie, and Zhen Bi, arXiv preprint 1909.09167 (2019), publication in progress in Phys. Rev. B. This paper is also cited as Ref. [90] in the References section of the thesis.

5.1 Introduction

Many of the most challenging questions in condensed matter physics involve an interplay of quenched disorder and strong interactions in two spatial dimensions at zero temperature. A prominent example is the problem of understanding the nature of the field-tuned superconductor to insulator transition in thin films. This transition not only appears to have the same critical exponents as the famously superuniversal quantum Hall plateau transitions [43, 44, 46, 47, 49–51], but also broadens into a finite metallic region in cleaner samples [52–56]. Crucially, the universal data of this quantum phase transition has failed to appear in any theoretical construction involving disorder or interactions exclusively, indicating that both must play important roles.

In spite of decades of effort, few organizing principles have been developed for understanding quantum critical systems with interactions and disorder, and analytically tractable models have proven rare. This problem is particularly acute in bosonic systems undergoing superconductor-insulator or superfluid-insulator transitions. While examples of quantum critical points and phases have been constructed in fermionic systems using perturbative and non-perturbative techniques [85, 175, 176, 236], few analogous examples exist for bosonic systems. At zero temperature, the only known examples of disordered-interacting fixed points of bosons in $2d$ arise in the context of the superfluid-insulator transition of bosons with random mass disorder and ϕ^4 interactions. These fixed points are obtained using a double-ϵ expansion about the free (Gaussian) fixed point in four spatial dimensions, perturbed with classical (finite temperature) disorder. This peculiar expansion, taken very far from the physical, quantum disordered situation of 2+1 spacetime dimensions, was introduced by Dorogovtsev [237] and by Boyanovsky and Cardy [238], who found a stable

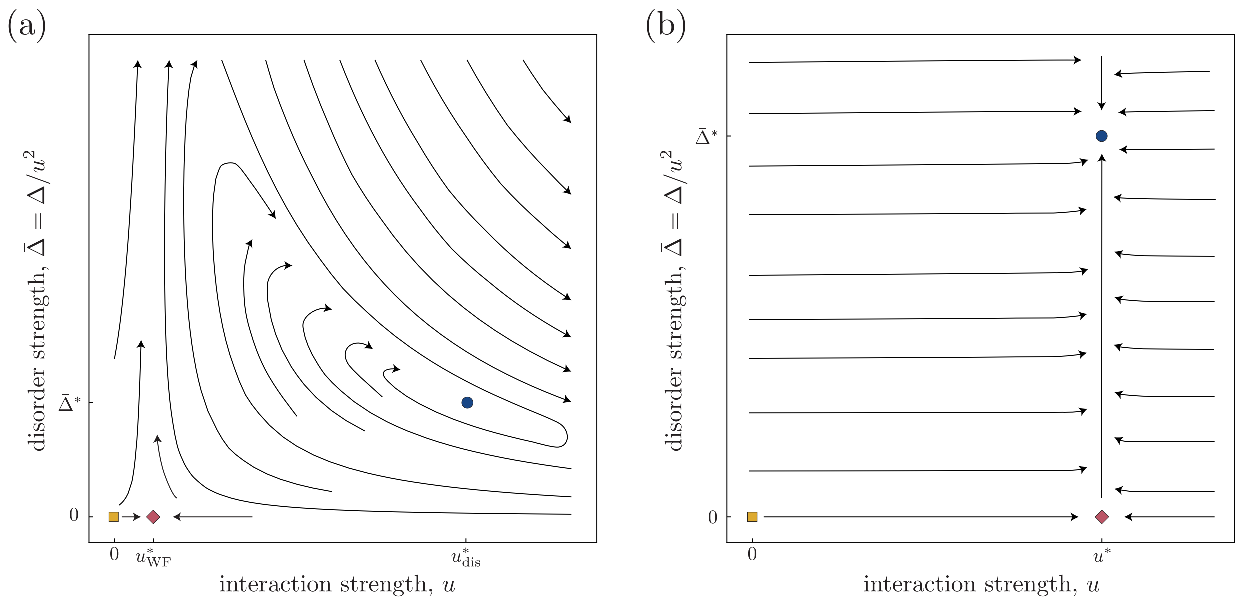

Figure 5.1: RG flow diagrams of the Gaussian fixed point (yellow square) as a function of the interaction coupling constant u and running disorder strength $\bar{\Delta} = \Delta/u^2$. The clean Wilson-Fisher fixed point is denoted by a red diamond, while the dirty fixed point is shown with a blue circle. (a) Spiralling RG flows are obtained in the double-ϵ expansion for small numbers N of bosons. (b) In the large-N limit, we show that the Wilson-Fisher fixed point flows directly to a dirty, interacting quantum critical point.

fixed point characterized by finite disorder and interactions (see also Ref. [239]). However, the character of this fixed point is very strange and is not obviously of direct physical significance: the renormalization group (RG) flows in its vicinity are spirals. As Fig. 5.1(a) demonstrates, it therefore takes a long time to approach this fixed point, and the critical regime may in fact be physically inaccessible. Fixed points with similar RG flows have been obtained in systems of bosons with $z = 2$ [240] as well as in holographic constructions [241, 242].

The view we take in this Chapter is that the unusual character of the double-ϵ expansion fixed point may be understood as an artifact of perturbing the free, classical fixed point. Near such a fixed point, disorder can prematurely take control of the physics, obscuring the true fate of the strongly interacting, disordered theory. Indeed, the technical reason[1] for the appearance of spiralling flows is that at the free, classical fixed point, the ϕ^4 operator and the operator associated with the quenched disorder (in the replica formalism) have the same scaling dimension. As a result, these operators can immediately mix along the RG flow in such a way that their scaling dimensions enter the complex plane, leading to the spirals in Fig. 5.1(a). In contrast, the appearance of complex scaling dimensions is not expected to occur near the Wilson-Fisher fixed point, where these operators do not have the same scaling dimensions. A hint that this is the case comes

[1] See Refs. 243 and 244 for a more detailed discussion.

	z	$[\phi]$	$[\phi^2]$
disordered WF	$1 + \frac{16}{3\pi^2 N}$	$\frac{1}{2} + \frac{2}{3\pi^2 N}$	$2 + \frac{16}{3\pi^2 N}$
clean WF	1	$\frac{1}{2} + \frac{2}{3\pi^2 N}$	$2 - \frac{16}{3\pi^2 N}$

Table 5.1: Scaling dimensions at the dirty, interacting QCP obtained in the large-N expansion, compared with the results the clean Wilson-Fisher fixed point at large-N. Here ϕ denotes the boson field, and ϕ^2 denotes the mass operator. The correlation length exponent ν is obtained through $\nu^{-1} = 2 + z - [\phi^2]$.

from studying the double-ϵ expansion RG equations in the limit of a large number N of boson species. In this limit, the scaling dimensions of these operators near the Wilson-Fisher fixed point are far from degenerate, and there is a crossover into a regime in which complex scaling dimensions no longer occur.

In this Chapter, we demonstrate that the strongly coupled Wilson-Fisher fixed point gives way to a quantum critical point (QCP) distinguished by both finite disorder and interactions using a large-N expansion. Instead of simultaneously perturbing the free, classical fixed point with both disorder and interactions, as in the double-ϵ expansion, we introduce weak disorder directly at the quantum[2], interacting Wilson-Fisher fixed point. While this fixed point saturates the Harris criterion in the $N \to \infty$ limit (disorder is marginal), we find that it is destabilized at $\mathcal{O}(1/N)$, resulting in flows of the type shown in Fig. 5.1(b). This fixed point is characterized by a correlation length exponent ν and a dynamical scaling exponent z given at $\mathcal{O}(1/N)$ by

$$\nu = 1\,, \qquad z = 1 + \frac{16}{3\pi^2 N}. \tag{5.1}$$

Extrapolation to $N = 1$ therefore yields a value $z \approx 1.5$ for the O(2) model. The associated operator scaling dimensions are presented alongside the critical exponents of the clean fixed point in Table 5.1.

The values these exponents take have several noteworthy implications. The correlation length exponent ν at the disordered fixed point is the same as at the clean Wilson-Fisher fixed point in the large-N limit. This absence of $1/N$ corrections may be interpreted as a physical consequence of the balancing that occurs between disorder and interaction effects. On the other hand, the fact that $1 < z < 2$ signals that the fixed point is neither clean nor conventionally diffusive ($z = 2$), a feature common amongst the dirty-interacting quantum critical states obtained in the literature thus far. A similar physical story occurs in the earlier studies of disorder in QED_3 [175, 176].

The QCP we obtain may be relevant to superfluid-insulator transitions in ^{4}He absorbed in porous Vycor [245–247], Josephson junction arrays [248, 249], doped quantum magnets [250–252], and cold atomic systems

[2]By quantum, we mean that that the theory has a time direction that is invisible to the quenched disorder, which only has correlations in space. This is the case for any problem of quenched disorder at zero temperature.

[253–255]. Superfluid-insulator transitions with similar exponents have also been observed numerically [256–262]. Indeed, the values we obtain at $\mathcal{O}(1/N)$ for ν, z, and the correlation function exponent $\eta \approx -0.47$ are strikingly close to those obtained in the most recent Monte Carlo study of the dirty O(2) model [262]. Moreover, the more germane RG flows we obtain are consistent with the numerical observation of a direct transition with universal features, while the spiralling flows of the double-ϵ expansion would have predicted the presence of oscillatory, non-universal behavior out to large system sizes. This achievement is all the more surprising given that it comes from extrapolating the small parameter $1/N$ to 1, a move which always carries a risk of being problematic. We note that in these numerical approaches the insulating phase is either a "Mott glass," which is incompressible [263, 264], or a "Bose glass," which has finite compressibility. While it is generally believed that the superfluid state always gives way to a glassy insulator in $2d$ [265, 266], assessing whether this is the case in the theory examined here requires the inclusion of non-perturbative effects, which are beyond the scope of our discussion here.

Similar large-N approaches to the study of quenched disorder at the Wilson-Fisher fixed point have been applied in the past by Kim and Wen [267] and by Hastings [268]. In the latter case, $1/N$ corrections were not considered, while in the former runaway flows were obtained. We believe that these runaway flows are the result of a redundant summation of diagrams.

We proceed as follows. In Sec. 5.2, we present a stability criterion for theories of interacting bosons to quenched disorder. We next perform the large-N analysis and describe the nature of the QCP we obtain in Sec. 5.3. We follow in Sec. 5.4 with a discussion of the effects of scale and vector potential disorder. In Sec. 5.5, the implications our result has for two dual descriptions of the single species ($N = 1$) theory, following earlier work coauthored by one of us [85]. We conclude in Sec. 5.6.

5.2 Stability Criterion for Free and Interacting Bosons

We begin this section by describing the criteria for the stability of theories of relativistic scalar bosons to quenched disorder at zero temperature, often referred to as quantum disorder. After presenting our conventions and the global symmetries, we derive a criterion for the free, Gaussian fixed point. We then generalize this criterion to the strongly interacting, Wilson-Fisher fixed point, where anomalous scaling dimensions appear. These stability criteria are quantum bosonic versions of the celebrated Harris criterion [269] and its generalization by Chayes *et al.* [270].

5.2.1 Degrees of Freedom and Global Symmetries

We consider one of the simplest families of quantum field theories: those describing massless, complex scalar fields transforming in the fundamental representation of U(N). Writing the bosonic degrees of freedom as N-component complex vectors $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_N)$, this global symmetry acts as $\boldsymbol{\phi} \to U\boldsymbol{\phi}$, $U \in \mathrm{U}(N)$. Throughout this Chapter, we restrict our attention to disorder and interactions that respect the full U(N) symmetry.

For the majority of this Chapter, we also impose two additional discrete, anti-unitary symmetries: time reversal, **T**, and particle-hole symmetry, **PH**. They act on the fields as

$$\mathbf{T} : \boldsymbol{\phi} \mapsto \boldsymbol{\phi} \,, \tag{5.2}$$

$$\mathbf{PH} : \boldsymbol{\phi} \mapsto \boldsymbol{\phi}^\dagger \,, \tag{5.3}$$

and both map $i \mapsto -i$. We eventually consider types of disorder that break these within each realization while preserving them on average in Sec. 5.4.

When the above global symmetries are imposed, the theory of $\boldsymbol{\phi}$ fields is also invariant under the larger symmetry group, O($2N$). Its action is obtained by defining $2N$ real fields, φ_I, from the complex fields: $\phi_I = \varphi_{2I-1} + i\varphi_{2I}$. The theory we discuss below is found to be invariant under the action of $\boldsymbol{\varphi} \to O\boldsymbol{\varphi}$ where $O \in \mathrm{O}(2N)$ and $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_{2N})$. Actually, the orthogonal global symmetry need not only arise as an enhanced symmetry, but can exist as a true global symmetry even away from the critical point. For such cases, there is no reason to the restrict the number of flavors to be even. Hence, while we primarily discuss the complex fields $\boldsymbol{\phi}$, we allow N to take half-integer values.

5.2.2 Free Bosons with Disorder

We begin with a free, or Gaussian, theory of N complex bosons,

$$\mathcal{L}_0[\boldsymbol{\phi}] = |\partial\boldsymbol{\phi}|^2 \,, \tag{5.4}$$

in $d+1$ spacetime dimensions. Throughout this Chapter, 'd' exclusively denotes the spatial dimension. Dimensional analysis sets the scaling dimension of $\boldsymbol{\phi}$ to $[\boldsymbol{\phi}] = (d-1)/2$, and the scaling of all operators in the free theory follows directly from this relation. The stability of the Gaussian theory is determined by assessing the relevance of all operators respecting the global symmetries described above. The most relevant such perturbation is the mass term, $r\,|\boldsymbol{\phi}|^2$, since $[r] = 2$ for all dimensions, and the requirement that the theory be massless is therefore predicated on the fine-tuning of r to zero. The next-most relevant, symmetry-preserving operator is the interaction term $u\,|\boldsymbol{\phi}|^4 = u\left(|\boldsymbol{\phi}|^2\right)^2$. Because $[|\boldsymbol{\phi}|^4] = 2(d-1)$, we have $[u] = 3-d$, implying that the Gaussian theory becomes unstable to this interaction when $d < 3$. In the next section, we discuss the effect of adding this term.

Disorder is introduced by perturbing $\mathcal{L}_0$ with an operator whose coefficient is a spatially varying, static field with values drawn from a probability distribution. Similar to the clean case, the most relevant, symmetry-preserving perturbation couples to the mass operator $|\boldsymbol{\phi}|^2$:

$$\mathcal{L}_0[\boldsymbol{\phi}, R] = |\partial\boldsymbol{\phi}|^2 + R(\mathbf{x})|\boldsymbol{\phi}|^2, \tag{5.5}$$

where bold face denotes purely spatial coordinates. We define $R(\mathbf{x})$ to have moments,

$$\overline{R(\mathbf{x})R(\mathbf{0})} \sim \frac{\Delta}{|\mathbf{x}|^{\chi}}\,, \qquad \overline{R(\mathbf{x})} = 0\,. \tag{5.6}$$

where $\chi \to d$ corresponds to Gaussian white noise[3]. As it couples to $|\boldsymbol{\phi}|^2$, the dimension of $R(\mathbf{x})$ is 2, just like the constant mass coefficient, r. From Eq. (5.6), it follows that the engineering dimension of the disorder strength Δ at the Gaussian fixed point is

$$[\Delta] = 4 - \chi\,. \tag{5.7}$$

For Gaussian white noise disorder, $\chi \to d$, implying that the theory is stable to random mass disorder provided that

$$d > 4\,, \tag{5.8}$$

which is the Harris criterion for free (relativistic) scalar fields.

Comparing against our brief analysis of the clean theory, we observe that mass disorder is marginal when $d = 4$, whereas the $|\boldsymbol{\phi}|^4$ interaction term is marginal when $d = 3$. This mismatch between the marginal dimensions associated with the disorder and interactions has been one of the major sources of difficulty in studying the dirty boson problem in two dimensions.

We note that while the disorder perturbation $R(\mathbf{x})\,|\boldsymbol{\phi}|^2$ and interaction term $|\boldsymbol{\phi}|^4$ were chosen as the most relevant operators preserving the U(N), **T**, and **PH** symmetries, they are also invariant under the

[3] More precisely, one writes the disorder correlations as a Riesz potential,

$$\overline{R(\mathbf{x})R(\mathbf{0})} = \frac{\Gamma\left(\frac{\chi}{2}\right)}{2^{d-\chi}\pi^{d/2}\Gamma\left(\frac{d-\chi}{2}\right)}\frac{\Delta}{|\mathbf{x}|^{\chi}}\,.$$

It is this function that reproduces Gaussian white noise (delta function) correlations in the limit $\chi \to d$. In this Chapter, we will generally suppress the additional gamma functions, as these do not impact scaling.

O($2N$) symmetry discussed in the previous section. When the discrete symmetries, **T** and **PH**, are no longer imposed, additional O($2N$)-breaking perturbations are allowed. We leave this discussion to Sec. 5.4.

5.2.3 Wilson-Fisher Bosons with Disorder

When $d < 3$, the Gaussian fixed point is unstable to *both* disorder and $|\boldsymbol{\phi}|^4$ interactions. In the clean limit, this leads to the famous Wilson-Fisher fixed point,

$$\mathcal{L}[\boldsymbol{\phi}] = |\partial\boldsymbol{\phi}|^2 + r_c\,|\boldsymbol{\phi}|^2 + \frac{u}{2N}\,|\boldsymbol{\phi}|^4\,. \tag{5.9}$$

Here $u = \Lambda^{3-d}\,\bar{u}$, $\bar{u} \sim \mathcal{O}(1)$, where Λ is a UV cutoff scale. The mass r_c tunes the theory to criticality. Its exact value is not physically meaningful, and we set it to zero throughout this Chapter. At the Wilson-Fisher fixed point, the dimension of $|\boldsymbol{\phi}|^2$ differs from its engineering dimension (*i.e.* scaling dimension in the free theory) by an anomalous dimension $\eta_{|\phi|^2}$,

$$\left\langle|\boldsymbol{\phi}(x)|^2|\boldsymbol{\phi}(0)|^2\right\rangle \sim \frac{1}{|x|^{2(d-1+\eta_{|\phi|^2})}}. \tag{5.10}$$

That is, the scaling dimension of $|\boldsymbol{\phi}|^2$ is $[|\boldsymbol{\phi}|^2] = d - 1 + \eta_{|\phi|^2}$. Importantly, the anomalous dimension $\eta_{|\phi|^2}$ is a function of the number of fields (and hence the symmetry of the theory).

We now perturb this fixed point with disorder,

$$\mathcal{L}[\boldsymbol{\phi}, R] = |\partial\boldsymbol{\phi}|^2 + R(\mathbf{x})|\boldsymbol{\phi}|^2 + \frac{u}{2N}|\boldsymbol{\phi}|^4, \tag{5.11}$$

where $R(\mathbf{x})$ continues to be defined as in Eq. (5.6). The dimension of R is related to the scaling dimension of $|\boldsymbol{\phi}|^2$ as follows,

$$\left[|\boldsymbol{\phi}|^2\right] = d - 1 + \eta_{|\phi|^2} = d + 1 - [R]\,. \tag{5.12}$$

With Eq. (5.6), we can now read off the scaling dimension of the disorder strength:

$$[\Delta] = 2[R] - \chi = 4 - 2\eta_{|\phi|^2} - \chi\,. \tag{5.13}$$

We conclude that the Wilson-Fisher fixed point is stable to Gaussian white noise disorder ($\chi \to d$) if

$$d > 4 - 2\eta_{|\phi|^2}\,. \tag{5.14}$$

5.2.4 Large-N Wilson-Fisher in $(2+1)d$

We now adapt this discussion to the particular case of a theory of $N \to \infty$ species of complex bosons in $d = 2$ spatial dimensions. In this limit, the stability criterion derived above becomes,

$$\eta_{|\phi|^2} - 1 > 0. \tag{5.15}$$

For a single species of complex boson, it is known from the conformal bootstrap that $\eta_{|\phi|^2} \sim 0.5$ in $2d$ [271], implying that disorder is a relevant perturbation when $N = 1$. Conversely, in the limit $N \to \infty$, with u held fixed, it turns out that $\eta_{|\phi|^2} \to 1$, as we will review in the next section. As a result, Gaussian white noise disorder ($\chi = 2$) is *marginal* at the Wilson-Fisher fixed point in the large-N limit! The interacting dirty boson problem can therefore be studied by first flowing to the $N \to \infty$ Wilson-Fisher fixed point and subsequently performing a perturbative RG calculation, with $1/N$ corrections entering as marginal perturbations of the $N \to \infty$ fixed point. This will be the goal of the next section.

Interpolating between the $N = 1$ limit, where $\eta_{|\phi|^2} \sim 0.5$, and the $N \to \infty$ limit, where $\eta_{|\phi|^2} \to 1$, we expect $1/N$ corrections to $[|\boldsymbol{\phi}|^2]$ to be *negative*, indicating that the Wilson-Fisher fixed point is ultimately unstable to disorder for finite N. Nevertheless, disorder generates additional corrections to scaling dimensions as well. Provided these quantum corrections to $[|\boldsymbol{\phi}|^2]$ are *positive*, they may be able to balance the corrections from interactions, thus resulting in a perturbatively accessible, disordered quantum critical point. In contrast, if the quantum corrections due to disorder are also negative, no such fixed point can exist, and all perturbations result in a flow to strong disorder. Serendipitously, we find that it is the former scenario which is played out.

5.3 The O($2N$) Model with a Random Mass

This section presents the primary technical content of the Chapter. We begin by describing the disorder-averaged theory and its replicated analogue. Next, the number of bosons N is taken to infinity, leaving us with a theory in which disorder is exactly marginal. We subsequently derive the β function for the running disorder strength at $\mathcal{O}(1/N)$ and demonstrate the existence of the fixed point and RG flow shown in Fig. 5.1(b). The section concludes with a comparison of the fixed point obtained here with the results from the double-ϵ expansion.

5.3.1 Disorder Averaging and the Replica Trick

We now describe how to systematically study the dirty Lagrangian in Eq. (5.11) in the $N \to \infty$ limit. While the addition of the quenched degree of freedom $R(\mathbf{x})$ strongly breaks translation invariance, seemingly rendering the theory intractable, we are interested in the disorder-averaged correlation functions, for which translation symmetry remains. Hence, all quantities of interest in the disordered theory may be calculated from the disorder-averaged free energy:

$$\bar{F} = -\overline{\log Z[R]} = -\int \mathcal{D}R\,\mathcal{P}[R]\,\log Z[R]\,, \tag{5.16}$$

where $\mathcal{P}[R]$ is the probability distribution that gives rise to the moments in Eq. (5.6). Specifying to Gaussian white noise disorder, the appropriate probability functional is

$$\mathcal{P}[R] = \frac{1}{\mathcal{N}}\,\exp\left(-\int d^2\mathbf{x}\,\frac{1}{2\Delta}R^2(\mathbf{x})\right), \tag{5.17}$$

where $\mathcal{N}$ is a normalization constant.

While directly disorder averaging the logarithm is prohibitively difficult, the problem can be made tractable by utilizing the so-called replica trick, in which one applies the identity,

$$\log Z = \lim_{n_r \to 0} \frac{Z^{n_r} - 1}{n_r}. \tag{5.18}$$

Upon inserting this expression into the definition of $\bar{F}$, we obtain

$$\begin{aligned} \bar{F} &= -\lim_{n_r \to 0} \frac{1}{n_r} \int \mathcal{D}R\,\mathcal{P}[R] \prod_{n=1}^{n_r} \int \mathcal{D}\boldsymbol{\phi}_n\, e^{-S[\boldsymbol{\phi}_n, R]} \\ &= -\lim_{n_r \to 0} \int \mathcal{D}R\,\mathcal{D}\boldsymbol{\phi}_n\, e^{-S_r[\boldsymbol{\phi}_n, R]}\,, \end{aligned} \tag{5.19}$$

where $S = \int d^2\mathbf{x}\,d\tau\,\mathcal{L}[\boldsymbol{\phi}_n, R]$ and n_r "replicas," $\boldsymbol{\phi}_n$, $n = 1, \ldots, n_r$, have been introduced. We remind the reader that each replica is associated with N physical species of bosons. The full replicated action for Gaussian white noise disorder is

$$\begin{aligned} S_r = &\int d^2\mathbf{x}\,d\tau \sum_{n=1}^{n_r} \left[|\partial \boldsymbol{\phi}_n(\mathbf{x},\tau)|^2 + \frac{R(\mathbf{x})}{\sqrt{N}} |\boldsymbol{\phi}_n(\mathbf{x},\tau)|^2 + \frac{u}{2N}\,|\boldsymbol{\phi}_n(\mathbf{x},\tau)|^4 \right] \\ &+ \int d^2\mathbf{x}\,\frac{1}{2\Delta} R^2(\mathbf{x}). \end{aligned} \tag{5.20}$$

Here, R has been rescaled by $\sqrt{N}$, equivalent to rescaling Δ by $1/N$. In summary, the replica trick has produced an action amenable to the standard tools of perturbative field theory through the addition of n_r replica fields, with the caveat that we must eventually take the limit $n_r \to 0$. It remains an open problem to determine the general conditions under which this limit exists.

5.3.2 The Large-N Limit

Fixing the value of Δ and u, we are now able to take the large-N limit. It is convenient to introduce a Hubbard-Stratonovich field $i\tilde{\sigma}$ (the reason for the tilde will become apparent shortly) to mediate the scalar self-interaction:

$$S_r = \int d^2\mathbf{x}\, d\tau \sum_n \left[|\partial \boldsymbol{\phi}_n|^2 + \frac{1}{\sqrt{N}} \Big(i\tilde{\sigma}_n + R(\mathbf{x}) \Big) |\boldsymbol{\phi}_n|^2 + \frac{1}{2u} \tilde{\sigma}_n^2 \right] + \int d^2\mathbf{x}\, \frac{1}{2\Delta} R^2(\mathbf{x})\,. \tag{5.21}$$

The equations of motion for $i\tilde{\sigma}$ directly relate it to the mass operator

$$i\tilde{\sigma}_n = \frac{u}{\sqrt{N}}\, |\boldsymbol{\phi}_n|^2\,, \tag{5.22}$$

and it follows that correlation functions containing $i\tilde{\sigma}$ will reproduce correlation functions containing $|\boldsymbol{\phi}|^2$ up to an contact term. Next, we shift $i\tilde{\sigma}_n \to i\sigma_n = i\tilde{\sigma}_n + R$ so that the coupling between R and the $\boldsymbol{\phi}$ fields is replaced with a coupling between R and σ,

$$S_r = \int d^2\mathbf{x}\, d\tau \sum_n \left[|\partial \boldsymbol{\phi}_n|^2 + \frac{i}{\sqrt{N}}\, \sigma_n |\boldsymbol{\phi}_n|^2 + \frac{i}{u}\, R(\mathbf{x})\, \sigma_n + \frac{1}{2u} \sigma_n^2 \right] + \int d^2\mathbf{x} \frac{1}{2\Delta} R^2(\mathbf{x}). \tag{5.23}$$

Here, an extra term quadratic in R is not included because it is proportional to the number of replicas and therefore vanishes in the replica limit. Finally, integrating out the quenched degree of freedom $R(\mathbf{x})$ yields

$$\begin{aligned} S_r = &\int d^2\mathbf{x}\, d\tau \sum_n \left[|\partial \boldsymbol{\phi}_n|^2 + \frac{i}{\sqrt{N}}\, \sigma_n |\boldsymbol{\phi}_n|^2 + \frac{1}{2u} \sigma_n^2 \right] \\ &+ \int d^2\mathbf{x}\, d\tau d\tau' \sum_{n,m} \frac{\Delta}{2u^2}\, \sigma_n(\mathbf{x},\tau) \sigma_m(\mathbf{x},\tau')\,. \end{aligned} \tag{5.24}$$

Equipped with this Lagrangian, we are now prepared to take the large-N limit following the standard procedure. For a more detailed review, see Refs. 272 and 273.

We begin by noting that the action S_r is quadratic with the exception of the $\sigma|\boldsymbol{\phi}|^2$ interaction. While σ couples more and more weakly to the $\boldsymbol{\phi}$'s as N approaches infinity, it also couples to increasingly many such fields. The result of these opposing effects can be understood in the language of Feynman diagrams.

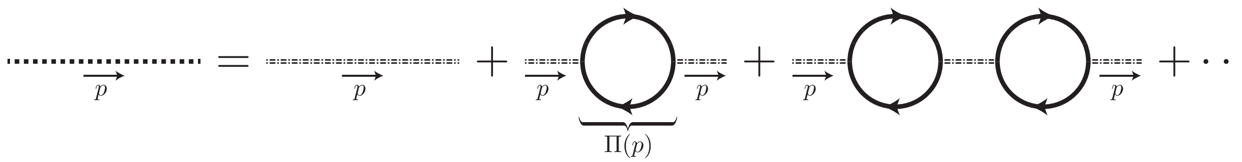

Figure 5.2: In the $N \to \infty$ limit, the propagator of σ may be represented as a geometric series of polarization bubbles $\Pi(p)$. The dash-dotted lines on the right-hand side represent the 'bare' σ propagator $\sim 1/u$, whereas the solid lines represent the ϕ propagator. The dotted line corresponds to the large-N σ Green's function.

In particular, the one-loop contribution to the σ propagator is the polarization bubble shown in Fig. 5.2. Because the internal boson lines must be summed over all N fields while each vertex contributes a factor of $1/\sqrt{N}$, this diagram is $\mathcal{O}(1)$. It evaluates to

$$\Pi(p) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2(p-k)^2} = \frac{1}{8|p|}. \tag{5.25}$$

Of course, if a diagram containing a single bubble is $\mathcal{O}(1)$, a diagram containing an arbitrary number of bubbles is also $\mathcal{O}(1)$, and so it should be include as well. The sum over bubble diagrams forms the geometric series shown in Fig. 5.2, which may be familiar to readers trained in the random phase approximation. The large-N σ propagator is therefore

$$G^\sigma(p) = \frac{u}{1 + u\,\Pi(p)} \to 8|p| \qquad \text{for } p \ll u\,, \tag{5.26}$$

where we have taken $u \sim \Lambda$ as our UV cutoff.

The physical meaning of these bubble diagrams can be understood by considering the real space representation of G^σ, which has been "screened" to be

$$G^\sigma(x) = \langle \sigma(x)\sigma(0) \rangle \sim \frac{1}{|x|^4}. \tag{5.27}$$

The large-N σ propagator makes it clear that $[\sigma] = 2$ when $N \to \infty$, implying that σ has acquired an anomalous dimension $\eta_\sigma = \eta_{|\phi|^2} = 1$, as claimed in the previous section.

Having accounted for the effect of bubble diagrams, the interaction between $\boldsymbol{\phi}$ and σ may be safely discarded in the limit $N \to \infty$. It is possible to access $1/N$ corrections by reintroducing the coupling between $\boldsymbol{\phi}$ and σ and using the screened σ propagator in Eq. (5.26) on the condition that bubble diagrams are not redundantly included in any subsequent calculation. Keeping this is mind, we obtain the effective action

$$
\begin{aligned}
S_{\text{eff}} &= S_\phi + S_{\sigma\phi} + S_{\text{dis}} && (5.28)\\
S_\phi &= \sum_n \int d^2\mathbf{x}\, d\tau\, |\partial \boldsymbol{\phi}_n|^2 && (5.29)\\
S_{\sigma\phi} &= \sum_n \int d^2\mathbf{x}\, d\tau \left[\frac{i}{\sqrt{N}}\,\sigma_n |\boldsymbol{\phi}_n|^2 + \frac{1}{16}\,\sigma_n(-\partial^2)^{-1/2}\sigma_n\right] && (5.30)\\
S_{\text{dis}} &= \sum_{n,m} \int d^2\mathbf{x}\, d\tau d\tau'\, \frac{\bar{\Delta}}{2}\,\sigma_n(\mathbf{x},\tau)\,\sigma_m(\mathbf{x},\tau')\,, && (5.31)
\end{aligned}
$$

where we have defined the dimensionless disorder strength $\bar{\Delta} \equiv \Delta/u^2$.

We are interested in the effect nonzero $\bar{\Delta}$ has on this theory, which we emphasize is now a *marginal* perturbation at tree level. Indeed, the disorder-mediated potential between two $\boldsymbol{\phi}$ fields has been screened to be

$$
V(\mathbf{x}-\mathbf{y}) \sim \frac{\bar{\Delta}}{|\mathbf{x}-\mathbf{y}|^4}. \tag{5.32}
$$

5.3.3 $1/N$ Corrections: Introducing Disorder at the Interacting Fixed Point

Philosophy and Scaling Conventions

We include the effects of disorder and interactions at $\mathcal{O}(1/N)$ via a Wilsonian momentum shell RG procedure. To begin, we present our tree-level scaling conventions. The action in Eq. (5.28), including the disorder, is scale invariant under

$$
\mathbf{x} \mapsto e^{\delta\ell}\mathbf{x}, \qquad \tau \mapsto e^{z\delta\ell}\tau, \qquad \boldsymbol{\phi} \mapsto e^{-\delta\ell/2}\boldsymbol{\phi}, \qquad \sigma \mapsto e^{-2\delta\ell}\sigma\,. \tag{5.33}
$$

Lorentz invariance dictates that space and time scale in the same way at the clean Wilson-Fisher fixed point; hence, $z=1$. The scaling prescriptions for $\boldsymbol{\phi}$ and σ are in agreement with our earlier statement that $[\boldsymbol{\phi}] = 1/2$ and $[\sigma] = 2$ in the $N \to \infty$ limit of the Wilson-Fisher fixed point. At $\mathcal{O}(1/N)$, these relations must be updated to account for anomalous dimensions generated by disorder and interactions, which we denote η_ϕ and η_σ for the $\boldsymbol{\phi}$ and σ fields, respectively. Similarly, because disorder breaks Lorentz invariance, the dynamical exponent is corrected to a value $z > 1$. We systematically compute these corrections to scaling by integrating out modes in a momentum shell $(1-\delta\ell)\Lambda < |\mathbf{p}| < \Lambda$, where $\Lambda \sim u$ is a hard cutoff. Note that because of the large-N limit, we may take $\bar{\Delta} \sim \mathcal{O}(1)$, as our perturbation theory continues to be controlled in powers of $1/N$.

Before presenting the details of our calculation, we remark on some idiosyncrasies of the theory (5.28) that ultimately serve to simplify our analysis. We first comment on the clean limit, $\bar{\Delta} = 0$. Quantum corrections are typically organized into self energy corrections and vertex corrections, which modify the scaling of the fields and affect the running of the interactions. In the theory (5.28), we would therefore expect $\sigma|\boldsymbol{\phi}|^2$ to enter in the Lagrangian alongside a running coupling constant. However, because σ was defined through a Hubbard-Stratonovich transformation, it is not independent from $|\boldsymbol{\phi}|^2$, as indicated by the operator identity of Eq. (5.22). It follows that the $\sigma|\boldsymbol{\phi}|^2$ vertex remains exactly marginal under the RG, making the renormalization of this vertex sufficient to determine η_σ, the anomalous dimension of σ. This observation is advantageous because the corrections to $\sigma|\boldsymbol{\phi}|^2$ all occur at one loop, whereas a direct calculation of the σ self energy involves the computation of two loop diagrams.

The introduction of disorder results in both a running disorder strength $\bar{\Delta}$ and the aforementioned dynamical scaling exponent z. It turns out that these are the only additional objects to be renormalized in our problem at $\mathcal{O}(1/N)$. Further, we find that the running of both may be obtained solely through the $\boldsymbol{\phi}$ self energy and the $\sigma|\boldsymbol{\phi}|^2$ vertex correction, similar to the clean case discussed above. The key consequence of this assertion is that under the modified scaling relations $\tau \mapsto e^{z\delta\ell}\tau, \mathbf{x} \mapsto e^{\delta\ell}\mathbf{x}$, and $\sigma \mapsto e^{-(2+\eta_\sigma)\delta\ell}\sigma$,

$$\beta_{\bar{\Delta}} = -\frac{\delta\bar{\Delta}}{\delta\ell} = 2\big(1 - z + \eta_\sigma\big)\bar{\Delta}\,. \tag{5.34}$$

The remainder of the section is dedicated to the calculation of z and η_σ.

We emphasize that this simplification is not a generic feature of the problem. It is possible for logarithmically divergent diagrams to generate operators containing[4] $\sum_n \int d\tau\, \sigma_n(\mathbf{x},\tau)$ independently from $\sigma_n(\mathbf{x},\tau)$. Such mixing would invalidate Eq. (5.34), as well as contribute to a running velocity for σ. For this reason, the σ self energy must also be computed. These considerations are reflected by the modification of Eq. (5.22) in the presence of disorder, which now involves this new, linearly independent operator,

$$i\sigma_n = \frac{u}{\sqrt{N}}|\boldsymbol{\phi}_n|^2 - iu\,\bar{\Delta}\sum_m \int d\tau\, \sigma_m(\mathbf{x},\tau)\,. \tag{5.35}$$

We evaluate the σ self energy in Appendix C.1 using a dimensional regularization scheme, a more natural method for higher loop calculations. This calculation confirms that no such diagrams occur at $\mathcal{O}(1/N)$, although they may appear at higher orders.

[4]For a more general discussion of this point, see Refs. 243 and 244.

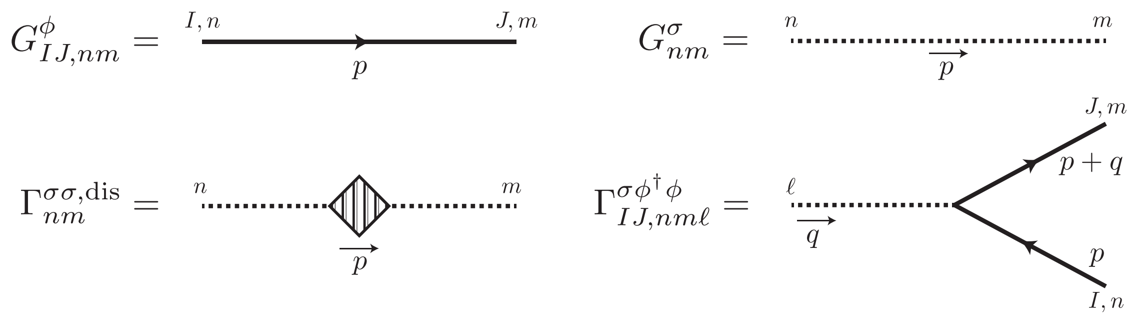

Figure 5.3: Feynman rules for the theory (5.28). Here, $p = (\mathbf{p}, \omega)$, $q = (\mathbf{q}, \nu)$, where $\mathbf{p}, \mathbf{q}$ are spatial momenta and ω, ν are frequencies.

Feynman Rules

The Feynman rules for the theory in Eq. (5.28) are shown in Fig. 5.3, where

$$G^{\phi}_{IJ,nm}(p) = \frac{1}{p^2}\,\delta_{IJ}\,\delta_{mn}\,, \tag{5.36}$$

$$G^{\sigma}_{nm}(p) = 8|p|\,\delta_{mn}\,, \tag{5.37}$$

$$\Gamma^{\sigma\phi^\dagger\phi}_{IJ,nm\ell} = -\frac{i}{\sqrt{N}}\,\delta_{IJ}\,\delta_{mn}\delta_{n\ell}\,, \tag{5.38}$$

$$\Gamma^{\sigma\sigma,\text{dis}}_{nm} = -2\pi\bar{\Delta}\,\delta(\omega)\,. \tag{5.39}$$

Here, we have suppressed the momenta-conserving delta functions and use $I, J = 1, \ldots, N$ to denote flavor indices. Below, we suppress the U(N) and replica indices in the three-point vertex functions: $\Gamma^{\sigma\phi^\dagger\phi}_{IJ,nm\ell} = \Gamma^{\sigma\phi^\dagger\phi}$. We also emphasize that the quenched disorder is capable of transferring momentum, but not frequency, as indicated with the frequency of δ-function.

We remark that disorder is being treated as a two-point vertex even though it appears as a quadratic field term in the action. While such terms are typically incorporated directly into the propagator, in our problem σ lines with multiple disorder insertions necessarily vanish in the replica limit, leaving only the contribution from the two-point vertex. We underscore that this is a non-perturbative statement, as $\bar{\Delta} \sim \mathcal{O}(1)$.

Momentum Shell RG

We first focus on the ϕ self energy, as shown in Fig. 5.4. After the momentum shell integration, we obtain

$$\Sigma(\mathbf{p},\omega) = \Sigma_{\text{int}}(\mathbf{p},\omega) + \Sigma_{\text{dis}}(\mathbf{p},\omega), \tag{5.40}$$

$$\Sigma_{\text{int}}(\mathbf{p},\omega) = -\frac{8}{N}\int_{(1-\delta\ell)\Lambda}^{\Lambda}\frac{d^2\mathbf{k}}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{|k-p|}{k^2} = -\frac{4}{3\pi^2 N}p^2\delta\ell, \tag{5.41}$$

$$\Sigma_{\text{dis}}(\mathbf{p},\omega) = \frac{64\bar{\Delta}}{N}\int_{(1-\delta\ell)\Lambda}^{\Lambda}\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{(\mathbf{k}-\mathbf{p})^2}{\omega^2+|\mathbf{k}|^2} = \frac{32\bar{\Delta}}{\pi N}(-\omega^2+|\mathbf{p}|^2)\delta\ell. \tag{5.42}$$

These correct the kinetic term of S_ϕ, Eq. (5.29); the mass renormalization has been suppressed. To maintain the scale invariance of the action, we correct the tree level scaling in Eq. (5.33) as follows,

$$\mathbf{x} \mapsto e^{\delta\ell}\mathbf{x}, \qquad \tau \mapsto e^{z\delta\ell}\tau, \qquad \boldsymbol{\phi} \mapsto e^{-\delta\ell/2}Z_\phi^{-1/2}\boldsymbol{\phi} = e^{-(1/2+\eta_\phi)\delta\ell}\boldsymbol{\phi}, \tag{5.43}$$

where η_ϕ and z are chosen to cancel the self energy corrections of Eqs. (5.41) and (5.42) respectively,

$$\eta_\phi = \frac{1}{2}\frac{\delta}{\delta\ell}\log Z_\phi = \frac{2}{3\pi^2 N}\,, \qquad z = 1 + \frac{32\bar{\Delta}}{\pi N}\,. \tag{5.44}$$

Here, $\eta_\phi > 0$ is the usual anomalous dimension of $\boldsymbol{\phi}$ arising from its interaction with σ at the clean Wilson-Fisher fixed point [274]. The deviation of the dynamical exponent z from unity signals the breaking of Lorentz invariance by quenched disorder. In Appendix C.2, we check our result for z against a general expression derived in Refs. 243 and 244 for dirty fixed points accessible through conformal perturbation theory. The agreement between this result and the value of z shown above serves as confirmation of our diagrammatic calculation.

We now study the remaining one-loop diagrams, which correct the vertex $\Gamma_{\sigma\phi^\dagger\phi}(\omega = 0, |\mathbf{p}| = 0)$. As shown on the second line of Fig. 5.4, there are contributions from both interactions and disorder,

$$\delta\Gamma_{\sigma\phi^\dagger\phi} = \delta\Gamma_{\text{int}} + \delta\Gamma_{\text{dis}}, \tag{5.45}$$

$$\delta\Gamma_{\text{int}} = \frac{i}{\sqrt{N}}\frac{8}{N}\int_{(1-\delta\ell)\Lambda}^{\Lambda}\frac{d^2\mathbf{k}}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{|k|}{k^4} = \frac{i}{\sqrt{N}}\frac{4}{\pi^2 N}\delta\ell\,, \tag{5.46}$$

$$\delta\Gamma_{\text{dis}} = -\frac{i}{\sqrt{N}}\frac{64\bar{\Delta}}{N}\int_{(1-\delta\ell)\Lambda}^{\Lambda}\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{|\mathbf{k}|^2}{|\mathbf{k}|^4} = -\frac{i}{\sqrt{N}}\frac{32\bar{\Delta}}{\pi N}\delta\ell\,. \tag{5.47}$$

Additional $\mathcal{O}(1/N)$ vertex diagrams do exist, but are not logarithmically divergent, as verified in Appendix C.1. The corrections obtained above must be added to the action S_r. Imposing scale invariance

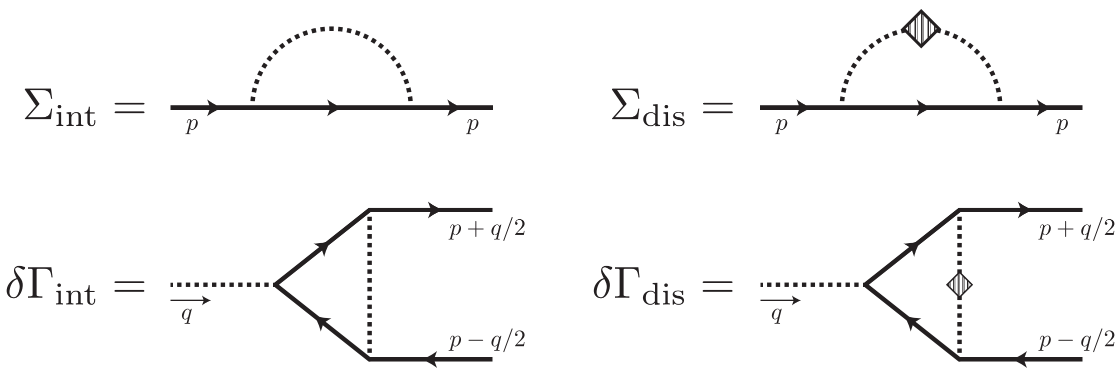

Figure 5.4: Quantum corrections at $\mathcal{O}(1/N)$. (Top) ϕ self-energy corrections, Σ_{int} (left) and Σ_{dis} (right). (Bottom) Logarithmically divergent vertex corrections, $\delta\Gamma_{\text{int}}$ (left) and $\delta\Gamma_{\text{dis}}$ (right). The full set of $\mathcal{O}(1/N)$ diagrams are shown in Fig. C.2 in Appendix C.1.

and the marginality of the $\sigma|\boldsymbol{\phi}|^2$ vertex requires updating Eq. (5.33) once more to include the anomalous dimension η_σ:

$$\sigma \mapsto e^{-2\delta\ell} Z_\sigma^{-1/2} \sigma = e^{-(2+\eta_\sigma)\delta\ell} \sigma \,. \tag{5.48}$$

Together with the results for z and η_ϕ in Eq. (5.44) , we find

$$\eta_\sigma = \frac{1}{2}\frac{\delta \log Z_\sigma}{\delta\ell} = z - 1 - 2\eta_\phi + \frac{32\bar{\Delta}}{\pi N} - \frac{4}{\pi^2 N} = \frac{64\bar{\Delta}}{\pi N} - \frac{16}{3\pi^2 N} \cdot \tag{5.49}$$

We verify that the second term is the known value of the $\mathcal{O}(1/N)$ anomalous dimension of σ at the clean Wilson-Fisher fixed point [273].

A Dirty Quantum Critical Point

In light of the comments in Sec. 5.3.3, the information obtained in the previous section allows us to calculate the running of $\bar{\Delta}$ directly from Eq. (5.31), which yields

$$\beta_{\bar{\Delta}} = -\frac{\delta\bar{\Delta}}{\delta\ell} = 2(1 - z + \eta_\sigma)\bar{\Delta} = \left(\frac{64}{\pi}\bar{\Delta} - \frac{32}{3\pi^2}\right)\frac{\bar{\Delta}}{N} \cdot \tag{5.50}$$

The flows exhibited by this β function are shown in Fig. 5.1(b). In particular, a fixed point with both finite disorder and interactions occurs at

$$\bar{\Delta}_* = \frac{1}{6\pi} \cdot \tag{5.51}$$

This fixed point constitutes a disordered, interacting quantum critical point! It is *attractive* (IR stable) in $\bar{\Delta}$ and u, but is unstable to perturbations in the mass of the boson, $\delta r\, |\boldsymbol{\phi}|^2$, which are allowed by symmetry[5]. For $\delta r < 0$, the theory flows to a phase in which the global O($2N$) symmetry is spontaneously broken, and the ground state hosts Goldstone bosons. On the other hand, for $\delta r > 0$, the theory flows to an insulating phase.

The QCP we have obtained is characterized by universal dynamical and correlation length exponents,

$$\nu = 1\,, \qquad z = 1 + \frac{16}{3\pi^2 N}\,, \tag{5.52}$$

where the correlation length exponent ν is defined via

$$\xi \sim |\delta r|^{-\nu}. \tag{5.53}$$

From dimensional analysis, this implies

$$\nu^{-1} = z + d - [|\boldsymbol{\phi}|^2] = z - \eta_\sigma = 1 - \frac{1}{2}\,\beta_{\bar{\Delta}}\,. \tag{5.54}$$

As we have demonstrated at $\mathcal{O}(1/N)$ [see Appendix C.1], so long as no additional anomalous dimensions are associated with disorder, the $\beta_{\bar{\Delta}}$ is given by Eq. (5.34), implying that the fixed point condition is identical to the statement $\nu = 1$, *i.e.* ν receives no quantum corrections. We can view this as the physical manifestation of the counterbalancing between disorder and interactions at the QCP. On the other hand, having $1 < z < 2$ is a reflection of the fact that this is a disordered quantum critical point – Lorentz invariance is broken, and the compressibility $\kappa \sim |\delta r|^{\nu(d-z)}$ vanishes as $\delta r \to 0$ at the transition.

Specifying to $N = 1$, the symmetry-broken state is a superfluid. The gapped, symmetry-preserving phase may be the "Mott glass" phase [263, 264], which is an insulating, glassy state with *vanishing* compressibility. This is in contrast to the perhaps more famous Bose glass phase, which includes disorder that does not respect particle-hole (**PH**) symmetry and has finite compresibility. We comment further on this case in the next subsection, although we emphasize that the glassy nature (or lack thereof) of the disordered state accessible through the dirty QCP derived here cannot be confirmed using our perturbative approach. Extrapolation of Eq. (5.52) to $N = 1$ yields

$$\nu = 1\,, \qquad z \approx 1.5\,. \tag{5.55}$$

[5] We define a quantum critical point as being a fixed point of a RG flow that can be perturbed by relevant operators without explicitly breaking a symmetry. This is in contrast to a quantum critical *phase*, for which any relevant perturbation breaks a symmetry.

Remarkably, these results are both consistent with recent numerical studies of the dirty superfluid-Mott glass transition [256, 262]. To our knowledge, the quantum critical point we describe here is the only analytic result to unambiguously achieve this agreement. It is therefore a tantalizing possibility that the fixed point we obtain is in the same universality class as this transition. However, as with any large-N expansion, we caution the reader that the extrapolation of the small parameter $1/N$ to unity may be problematic, as we have not proven that the $1/N$ expansion converges quickly enough for this to be truly reliable. We comment further on this and related issues in the subsection below.

5.3.4 Comparison with the Double-ϵ Expansion

It is important to understand the relationship the dirty QCP examined here has with those obtained in earlier approaches to the dirty boson problem. As mentioned in the Introduction, theories of bosons with self-interactions and random mass disorder have been considered before using an expansion in the number of spatial dimensions, $\epsilon = 4 - d$, and the number of time dimensions, $\epsilon_\tau = d_\tau$ [237–239]. This expansion involves perturbing the Gaussian fixed point in $d = 4$ dimensions with classical ($d_\tau = 0$) disorder, a situation far-removed from the physically relevant case of $d = 2$, $d_\tau = 1$. While this approach also yields a fixed point with finite disorder and interaction strengths, it exhibits some potentially pathological irregularities.

As Fig. 5.1(a) demonstrates, upon extrapolating back to $d = 2$, $d_\tau = 1$, the RG flows in the critical point's vicinity are spirals for the case of a single species of complex bosons ($N = 1$). In contrast, the results obtained in this Chapter through a large-N expansion show no indication of spiralling flows. This is not necessarily incompatible with the double-ϵ expansion since more germane, direct flows similar to Fig. 5.1(b) do appear when $N > N_c = 11 + 6\sqrt{3} \approx 21.4$. Therefore, while we must remain open to the possibility that spiralling flows may appear at a higher order in $1/N$, we argue here that they are instead an artifact of the double-ϵ expansion, implying that our results may be more physically relevant even for relatively small values of N.

We first note that the peculiar flows that appear in the double-ϵ theory follow from the appearance of complex anomalous dimensions, a signature of non-unitarity [243, 244]: unlike a unitary theory, the operator dimensions of a disorder-averaged theory are not constrained to the real line[6]. Nevertheless, in a perturbative expansion about a unitary theory, operators can only acquire complex scaling dimensions in conjugate pairs, implying that the (real) scaling dimensions of these operators became identical at some point along the RG flow. Since the ϕ^4 operator and the operator associated with the quenched disorder have the same scaling dimension at the free, classical fixed point in ($d = 4$, $d_\tau = 0$) being expanded about in

[6] For example, replica field theories have central charges which vanish in the replica limit, breaking unitarity, despite the fact that each disorder realization is itself a unitary quantum field theory.

the double-ϵ formalism, they can *immediately* mix in such a way that their anomalous dimensions enter the complex plane when disorder is added. Conversely, at the large-N fixed point, the scaling of $|\boldsymbol{\phi}|^2$ and thus the disorder operator is non-perturbatively altered, as indicated by a correlation length exponent $\nu = 1$ — a substantial modification from its free value, $\nu = 1/2$. Our expansion accordingly returns no indication of spiralling flows.

The absence of complex scaling dimensions in our theory may be interpreted as the result of balancing between interactions and disorder at the Wilson-Fisher fixed point. From this perspective, the ubiquity of strong interactions at the Wilson-Fisher fixed point should always deter (though not completely preclude) the formation of complex scaling dimensions. Indeed, the critical exponent ν differs significantly from its free value even for $N = 1$ where $\nu \approx 0.67$ [271]. It is therefore plausible that the propensity for spiralling flows displayed in the double-ϵ formalism is an unphysical consequence of starting from a degenerate point and that the value of N_c obtained by expanding in ϵ and ϵ_τ is greatly exaggerated compared to the true critical number of species for the onset of spiralling flows.

The failure of the ϵ expansion to capture the small-N behavior in such situations is not unprecedented. The Abelian Higgs model, a theory of complex scalar fields coupled to a fluctuating gauge field, appears to lack a (real) fixed point for $N \leq 182$ in $D = 4 - \epsilon$ spacetime dimensions [67]. However, lattice duality with the $3d$ XY model [64, 275, 276], for which the critical theory is the Wilson-Fisher fixed point discussed here, and numerical results [277, 278] place that critical number at values as small as one. As in the dirty boson problem, this phenomenon can be traced to the presence of two operators having the same scaling dimension.

We caution that while the agreement of our results with numerics is indeed remarkable, the arguments outlined by no means constitute a proof that the large-N expansion offers any advantage over the double-ϵ treatment or even that it is physically relevant. For $N = 1$, both methods are predicated on the disconcerting assignment of a small expansion parameter to an $\mathcal{O}(1)$ value, and both are therefore fundamentally suspect in this regime. We acknowledge that the absence of spiralling flows and complex dimensions in our study may simply follow from the fact we are perturbing about the regime where the flows from the Wilson-Fisher fixed point are regular. Nevertheless, even were this the case, our treatment and the fixed point should remain valid at least for sufficiently large-N.

5.4 Scalar and Vector Potential Disorder

We have so far focused exclusively on theories that preserve a global U(N), time-reversal (**T**), and particle-hole (**PH**) symmetry for each realization of disorder, and we have shown that this is equivalent to imposing a global O($2N$) symmetry. In this section, we relax this constraint by only imposing the discrete **T** and **PH** symmetries on average, allowing for additional disorder perturbations. Such perturbations can be chosen to preserve the U(N) symmetry for each disorder realization, but not the O($2N$) symmetry.

The symmetries **PH** and **T** are broken respectively by random scalar and vector potentials, which we denote $\mathcal{V}(\mathbf{x})$ and $\mathcal{A}_i(\mathbf{x})$,

$$\mathcal{L}_{J\text{-dis}} = \mathcal{V}(\mathbf{x})\, J_\tau(\mathbf{x},\tau) + \sum_{i=x,y} \mathcal{A}_i(\mathbf{x})\, J_i(\mathbf{x},\tau) \tag{5.56}$$

where

$$J_\tau = \boldsymbol{\phi}^\dagger \partial_\tau \boldsymbol{\phi} - \partial_\tau \boldsymbol{\phi}^\dagger\, \boldsymbol{\phi}, \qquad\qquad J_i = i\left(\boldsymbol{\phi}^\dagger \partial_i \boldsymbol{\phi} - \partial_i \boldsymbol{\phi}^\dagger\, \boldsymbol{\phi}\right). \tag{5.57}$$

Here, the scalar potential disorder may be interpreted as a random chemical potential that breaks **PH**, while vector potential disorder can be associated with a random magnetic flux that breaks **T** and parity ($\mathscr{P}$). The current J_μ is the global current corresponding to the electromagnetic charge, a U(1) subgroup of the global U(N) symmetry. While it may also be interesting to study disorder that couples to non-Abelian U(N) currents, such disorder breaks the U(N) symmetry within each realization, so we do not consider it.

As for the random mass disorder discussed in the previous section, we assume that scalar and vector potential disorder is drawn from a Gaussian white noise distribution with zero mean,

$$\overline{\mathcal{V}(\mathbf{x})\mathcal{V}(\mathbf{x}')} = \Delta_{\mathcal{V}}\, \delta(\mathbf{x}-\mathbf{x}')\,, \qquad \overline{\mathcal{A}_i(\mathbf{x})\mathcal{A}_j(\mathbf{x}')} = \Delta_{\mathcal{A}}\, \delta_{ij}\, \delta(\mathbf{x}-\mathbf{x}')\,, \qquad \overline{\mathcal{V}(\mathbf{x})} = \overline{\mathcal{A}_i(\mathbf{x})} = 0. \tag{5.58}$$

The case of general disorder correlations can also be studied, although we limit ourselves to the Gaussian white noise case for clarity.

Because $\mathcal{V}$ and $\mathcal{A}_{x,y}$ respectively couple to the temporal and spatial components of a conserved (Abelian) global current, their scaling dimensions satisfy certain non-perturbative constraints, and we use these to derive stability criteria that hold even away from a critical point. While $[J_\tau] = [J_i] = 2$ for relativistic ($z=1$) theories in 2+1 dimensions, these relations are modified in the absence of Lorentz symmetry. To see how, we recall that the currents' dimensions are fixed by their conservation,

$$\partial_\mu J^\mu = 0\,, \tag{5.59}$$

which implies a conserved, *dimensionless* charge

$$Q = \int d^2\mathbf{x}\, J^\tau\,. \tag{5.60}$$

More precisely, in the quantum theory, current conservation is the statement that correlation functions of J_μ satisfy Ward identities that embody the condition (5.59). The requirement that Q in Eq. (5.60) be dimensionless returns

$$[J_\tau] = 2. \tag{5.61}$$

while the continuity equation, Eq. (5.59), indicates that $\partial_\tau J^\tau$ and $\partial_i J^i$ must have the same scaling dimension, which gives

$$[J_i] = 1 + z\,. \tag{5.62}$$

Armed with the knowledge that any disorder leads to a deviation of z above unity, we use these relations to deduce the running of $\Delta_{\mathcal{V}}$ and $\Delta_{\mathcal{A}}$, both near the clean Wilson-Fisher fixed point and the dirty quantum critical point obtained in the previous subsection.

We first consider the case of vector potential disorder in the absence of scalar potential disorder. From Eq. (5.62), dimensional analysis indicates that $[\mathcal{A}] = 1$, which should be familiar as the usual scaling dimension of a vector potential. We conclude from Eq. (5.58) that $[\Delta_{\mathcal{A}}] = 0$ to *all* orders. Phrased in terms of β-functions, this reads simply as

$$\beta_{\Delta_{\mathcal{A}}} = 0\,. \tag{5.63}$$

In other words, the random vector potential is *exactly* marginal, both at the clean Wilson-Fisher fixed point and at our dirty quantum critical point. No matter how the dynamical exponent z is renormalized, $\Delta_{\mathcal{A}}$ will not run, resulting in a fixed line parameterized by z.

We now turn to the random scalar potential, following the same logic as we did for vector potential disorder. Using the fact that Eq. (5.61) implies $[\mathcal{V}] = z$, together with Eq. (5.58), we find $[\Delta_{\mathcal{V}}] = 2z - 2$, which is equivalent to

$$\beta_{\Delta_{\mathcal{V}}} = -(2z-2)\Delta_{\mathcal{V}}\,. \tag{5.64}$$

Hence, $\Delta_{\mathcal{V}}$ is *relevant* for any $z > 1$: both the clean Wilson-Fisher fixed point and our dirty quantum critical point are unstable to $\Delta_{\mathcal{V}}$, regardless of the strength of the mass or vector potential disorder.

Although the theory flows to strong disorder, and its ultimate fate cannot be understood perturbatively, one can speculate that the theory flows to a glassy state. Since **PH** is broken in each realization, this may be the Bose glass, which has finite compressibility despite being an insulator [265, 266]. Indeed, the exponents we obtain in Eq. (5.55) are fairly close to those obtained for the disorder-tuned transition between a superfluid and Bose glass if **PH** is only imposed on average [257–259, 261]. In particular, $\nu = 1$ is always seen, although there appears to be some disagreement in z[7]. This indicates that the quantum critical point obtained in the previous subsection may at least be in a similar universality class to these transitions.

The conclusions of this section hold in general for quenched disorder that couples to conserved Abelian global currents. The exact marginality of the random vector potential and the relevance of the random scalar potential for $z > 1$ are already well-known in the context of dirty non-interacting Dirac fermion systems [280–282]. They were also understood in the strongly interacting context of QED_3; there, the global U(1) current is actually a monopole current, $j^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda/2\pi$, where a is the fluctuating gauge field, and so random density and random flux exchange roles [85, 175, 176]. Note that if we had introduced disorder in the non-Abelian $\mathrm{U}(N)$ currents, this would have broken the $\mathrm{U}(N)$ symmetry explicitly in each realization, invalidating the non-perturbative conclusions of this section.

5.5 Boson-Fermion Duality and the $N = 1$ Theory

The proposal of a web of dualities connecting a menagerie of quantum critical points and phases in 2+1 spacetime dimensions [79, 80] has resulted in progress on several condensed matter problems [13, 81–89]. These dualities are non-perturbative tools that enable one to determine the low-energy behavior of a strongly-coupled quantum field theory by instead considering the physics of a dual theory that may be more tractable. In this section, we continue the results of Sections 5.3 and 5.4 to the case of $N = 1$ and explore their implications for the duals of this theory, following the philosophy of Ref. 85. In particular, we focus on the particular case of boson-fermion duality [79, 80, 104], in which the dual theory consists of Dirac fermions coupled to an emergent Chern-Simons gauge field. In Appendix C.3, we also consider the case of boson-vortex duality [64, 275, 276], in which the dual theory, known as the Abelian Higgs model, consists of bosonic vortices coupled to a fluctuating emergent gauge field. In both cases, an immediate consequence of the duality is that, in the presence of a random mass, the dual theory flows to a dirty, interacting QCP with

[7] For many years, it was expected that the superfluid-Bose glass transition in d spatial dimensions should have $z = d$ because both phases have finite compressibility, which scales in temperature like $\kappa = \partial n/\partial\mu \sim T^{(d-z)/z}$ [265, 266]. However, this expectation relies on the assumption that the measured compressibility is determined by the singular part of the free energy, which is not the case here when **PH** symmetry is broken [279].

the same exponents as those obtained in Section 5.3,

$$\nu = 1\,, \qquad z = 1 + \frac{16}{3\pi^2} \approx 1.5\,. \tag{5.65}$$

We emphasize, however, that this result relies on the extrapolation of N to unity, which may not be valid.

Although many of the results presented in this section are based on conjecture, they nevertheless represent progress in our understanding of dirty Chern-Simons-Dirac fermion theories. While disorder has been studied in such theories in the limit of a large number of Dirac fermion species [283, 284], such expansions suppress the role of the Chern-Simons term to sub-leading orders in $1/N$. The resulting analysis may therefore miss some of the important global effects of a $\mathcal{O}(1)$ Chern-Simons term. Using duality with the Wilson-Fisher theory circumvents the difficulties of developing a perturbative approach that treats both the disorder and the Chern-Simons gauge field equitably. However, we note that recent progress in studying the large-N Chern-Simons-Dirac problem with disorder [285] has yielded results for critical exponents which are impressively close to those we predict using duality.

We organize this section as follows. We begin with a brief review of the boson-fermion duality. We next apply the results of Section 5.3 for Wilson-Fisher bosons with random mass disorder to the Dirac fermion theory. Finally, we use the non-perturbative results of Section 5.4 to comment on the fate of the Dirac theory in the presence of random scalar and vector potentials.

5.5.1 Review of the Duality

We consider the boson-fermion duality [79, 80, 104] that relates the Wilson-Fisher theory of the boson ϕ to a theory of a Dirac fermion, ψ, coupled[8] to a fluctuating U(1) Chern-Simons gauge field, b_μ,

$$\mathcal{L}_\phi = |D_A\phi|^2 - |\phi|^4 \longleftrightarrow \mathcal{L}_\psi = i\bar{\psi}\not{D}_b\psi + \frac{1}{8\pi}bdb - \frac{1}{4g^2}f_{\mu\nu}f^{\mu\nu} + \frac{1}{2\pi}bdA + \frac{1}{4\pi}AdA\,, \tag{5.66}$$

The expressions D_B, $A\,dB$, $f_{\mu\nu}$, and $\not{D}$ are shorthand for $\partial - iB$, $\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu B_\lambda$, and $\partial_\mu b_\nu - \partial_\nu b_\mu$, and $D_\mu\gamma^\mu$, respectively. The double arrow, '$\longleftrightarrow$,' denotes duality. Since the duality holds only at energy scales much smaller than g^2, we omit the Maxwell term, $-\frac{1}{4g^2}f_{\mu\nu}f^{\mu\nu}$, below. For convenience, throughout this section we work with theories in Minkowski spacetime, which are related to the theories considered in earlier sections through a Wick rotation. Note that while **T** and **PH** are manifestly global symmetries of the bosonic theory, $\mathcal{L}_\phi$, they are not immediately apparent in the Dirac fermion theory, $\mathcal{L}_\psi$. Instead, they are to be viewed

[8] Note that we approximate the Atiyah-Patodi-Singer η-invariant by a level-1/2 Chern-Simons term and include it in the Lagrangian.

as emergent IR symmetries of the fermionic theory. Indeed, under this duality, the **T** symmetry actually manifests as fermion-vortex self-duality [79].

Varying both sides of Eq. (5.66) with respect to A, we see that charge in the bosonic theory maps to flux in the fermionic theory,

$$J^\mu_\phi = i(\phi^\dagger \partial^\mu \phi - \partial^\mu \phi^\dagger \phi) \longleftrightarrow \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu \left(b_\lambda + A_\lambda\right) , \tag{5.67}$$

where we have introduced the subscript on J^μ_ϕ for clarity. The physical interpretation of this relation is informed by the flux attachment implemented by the Chern-Simons gauge field. In the fermion theory, charge and flux are slaved to one another through the Chern-Simons gauge field, as are current and electric field. Indeed, differentiating the fermion Lagrangrian $\mathcal{L}_\psi$ with respect to b_μ one finds the mean field equations

$$\langle\bar{\psi}\gamma^\mu\psi\rangle + \frac{1}{2}\frac{1}{2\pi}\langle\varepsilon^{\mu\nu\lambda}\partial_\nu b_\lambda\rangle = -\frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda\,, \tag{5.68}$$

where brackets are used to emphasize that the right-hand side is not an operator, but a c-number. By defining the emergent and background electromagnetic fields $b_* = \varepsilon^{ij}\partial_i b_j, e_i = f_{it}(b), B = \varepsilon^{ij}\partial_i A_j, E_i = \partial_i A_t - \partial_t A_i$, and the Dirac fermion density and current, $\rho_\psi = J^t_\psi = \psi^\dagger\psi, J^i_\psi = \bar{\psi}\gamma^i\psi$, we re-express this relation as

$$\langle\rho_\psi\rangle + \frac{1}{2}\frac{1}{2\pi}\langle b_*\rangle = -\frac{1}{2\pi}\,B\,, \tag{5.69}$$

$$\langle J^i_\psi\rangle + \frac{1}{2}\frac{1}{2\pi}\varepsilon^{ij}\langle e_j\rangle = -\frac{1}{2\pi}\varepsilon^{ij}E_j\,. \tag{5.70}$$

The first equation relates the Dirac fermion charge density, ρ_ψ, to the sum of the emergent and background magnetic fields, while the second relates the Dirac fermion current to the sum of the emergent and background electric fields. In contrast, in a typical electromagnetic theory, vector potentials are associated with currents and scalar potentials are associated with charge.

It is helpful to determine the relationship between the conductivities of the bosons and fermions, defined via $\langle J^i_\phi\rangle = \sigma^\phi_{ij}E_j$ and $\langle J^i_\psi\rangle = \sigma^\psi_{ij}\langle e^j\rangle$. Combining these definitions with Eqs. (5.67) and (5.70), we obtain

$$\sigma^\psi = -\frac{1}{2}\frac{1}{2\pi}\,\varepsilon - \frac{1}{(2\pi)^2}\,\varepsilon\left(\sigma^\phi - \frac{1}{2\pi}\varepsilon\right)^{-1}\varepsilon\,, \tag{5.71}$$

where tensor indices have been suppressed to reduce clutter. Assuming rotational invariance and expanding in components, this relation becomes

$$\sigma^\psi_{xx} = \frac{1}{(2\pi)^2}\frac{\sigma^\phi_{xx}}{(\sigma^\phi_{xx})^2 + (\sigma^\phi_{xy} - 1/2\pi)^2}\,, \quad \sigma^\psi_{xy} = -\frac{1}{2}\frac{1}{2\pi} + \frac{1}{(2\pi)^2}\frac{1/2\pi - \sigma^\phi_{xy}}{(\sigma^\phi_{xx})^2 + (\sigma^\phi_{xy} - 1/2\pi)^2}. \tag{5.72}$$

Since we consider the bosonic theory in the absence of background magnetic fields, we take $\sigma_{xy}^{\phi} = 0$ below.

In terms of the Dirac fermion variables, the superfluid-insulator transition of the bosonic theory is experienced as a quantum Hall plateau transition tuned by the mass term, $-M\bar{\psi}\psi$. Integrating out the fermions yields a parity anomaly term for the emergent gauge field, $\mathrm{sgn}(M)\frac{1}{8\pi}b\,db$. For $M > 0$, the anomaly adds to the Chern-Simons term already in the Lagrangian, which gives the gauge field a so-called 'topological mass.' By integrating out the gauge field, we see that this state is a trivial, gapped insulator. To verify that the bosonic dual is also a trivial insulator, we set $\sigma_{xx}^{\phi} = \sigma_{xy}^{\phi} = 0$ in Eq. (5.72), which implies the expected response $\sigma_{xx}^{\psi} = 0$, $\sigma_{xy}^{\psi} = {}^{+1}\!/_{2\cdot 2\pi}$. On the other hand, for $M < 0$, the Chern-Simons terms cancel. The resulting Lagrangian consists of a gapless gauge field b, which Higgses the background fields A through the BF term, suggesting that this side of the transition corresponds to the superfluid phase, with b acting as the dual to the Goldstone mode of the bosonic theory. The insertion of the expected bosonic response, $\sigma_{xx}^{\phi} \to \infty$, $\sigma_{xy}^{\phi} = 0$, into Eq. (5.72) accordingly yields $\sigma_{xx}^{\psi} = 0$, $\sigma_{xy}^{\psi} = {}^{-1}\!/_{2\cdot 2\pi}$. We therefore conclude that, as in boson-vortex duality, the mass operators of the two theories are dual to one another,

$$|\phi|^2 \longleftrightarrow \bar{\psi}\psi\,. \tag{5.73}$$

This operator duality is highly non-trivial: it implies that $\bar{\psi}\psi$ has the same dimension as $|\phi|^2$ at the Wilson-Fisher fixed point, $[|\phi|^2] \approx 1.5$, meaning that interactions with the Chern-Simons gauge field lead to a *negative* anomalous dimension at the clean fixed point, $\eta_{\bar{\psi}\psi} \sim -0.5$.

5.5.2 Random Mass

Having reviewed the boson-fermion duality in the clean case, we now consider the effects of quenched disorder (again with Gaussian white noise correlations) in the Dirac fermion theory in the absence of the background field A. We mention that, since the boson-fermion duality is valid only in the IR, we require the disorder to be sufficiently long-wavelength that it may be considered a perturbation of the IR fixed point.

We first study the effect of a random mass. From Eq. (5.73), we again find that mass disorder maps to mass disorder

$$R(\mathbf{x})\,|\phi|^2(\mathbf{x},t) \longleftrightarrow R(\mathbf{x})\,\bar{\psi}\psi(\mathbf{x},t)\,. \tag{5.74}$$

As described in Section 5.3, a random mass causes the bosonic theory in the large-N limit to flow to a disordered, interacting QCP. Provided this remains true for $N = 1$, duality implies that the Dirac fermion theory also flows to such a QCP and that at this fixed point, the Dirac fermion mass operator has scaling dimension,

$$[\bar{\psi}\psi] = [|\phi|^2] = 2 + \frac{3}{16\pi^2}. \tag{5.75}$$

Moreover, the identification of the QCPs across the duality also implies that the correlation length and dynamical scaling exponents of the Dirac theory, respectively denoted ν_ψ and z_ψ, are identical to those obtained in Section 5.3,

$$\nu_\psi = \nu = 1\,, \qquad z_\psi = z = 1 + \frac{16}{3\pi^2} \approx 1.5\,. \tag{5.76}$$

The problem of mass disorder in Chern-Simons-Dirac fermion theories was recently revisited in a large-N expansion by Lee and Mulligan [284], who reproduced a fixed point of this type and found results for ν and z fairly close to those featured here when N is set to 1.

Since the QCP studied here is characterized by a universal DC conductivity, it would be very interesting to determine the DC transport properties of the Dirac fermions by applying the transport dictionary, Eq. (5.72), utilizing the DC response of the Wilson-Fisher bosons with a random mass. However, we leave this calculation, which is possible both using a large-N approach and numerical techniques, for future work.

5.5.3 Random Scalar and Vector Potentials

We now introduce random scalar and vector potentials, as in Eq. (5.57). We emphasize that the conclusions of this section are non-perturbative, and so are valid for $N = 1$. They are also consistent with the results of Ye [283] when Coulomb interactions are turned off. From the current mapping, Eq. (5.67), we first see that a random chemical potential in the bosonic theory maps to a randomly sourced flux in the Dirac fermion theory,

$$\mathcal{V}(\mathbf{x})\, J_0(\mathbf{x}, t) \longleftrightarrow \frac{1}{2\pi}\mathcal{V}(\mathbf{x})\, \varepsilon^{ij}\partial_i b_j(\mathbf{x}, t)\,. \tag{5.77}$$

Importantly, the flux attachment constraint, Eq. (5.69) implies that randomly sourcing the emergent magnetic field is equivalent to randomly sourcing the Dirac fermion density since the two operators are identical in the absence of an external magnetic field, $B = 0$. In other words, this disorder should be simultaneously understood as a random current and a random chemical potential (electric field), as can be seen from Eq. (5.70) by noting that a random scalar potential corresponds to $E_j = \partial_j \mathcal{V}/2\pi$.

From Section 5.4, we recognize that a random scalar potential is relevant, and we expect its addition to push the bosonic theory towards an insulating and possibly glassy phase. If this is true, then the DC response of the bosons is $\sigma^\phi_{xx} = \sigma^\phi_{xy} = 0$. The dual fermions therefore exhibit the same Hall effect as in the clean insulating state, $\sigma^\psi_{xx} = 0, \sigma^\psi_{xy} = +\frac{1}{2}\frac{1}{2\pi}$. It would be interesting to improve our understanding of this state in future work.

We conclude this section by considering a random vector potential,

$$\mathcal{A}^i(\mathbf{x})\, J_i(\mathbf{x},t) \longleftrightarrow \frac{1}{2\pi}\,\mathcal{B}(\mathbf{x})\, a_t(\mathbf{x},t) \tag{5.78}$$

where $\mathcal{B} = \varepsilon^{ij}\partial_i\mathcal{A}_j$. From Eq. (5.69), the random field $\mathcal{B}(\mathbf{x})$ should be interpreted both as a random density and a random random vector potential (magnetic field). As we observed in Section 5.4, this kind of perturbation is exactly marginal in the bosonic theory, and so the same should hold in the fermionic dual.

5.6 Discussion

In this Chapter, we have revisited the problem of quenched disorder at the quantum superfluid-insulator transition by directly introducing disorder at the strongly coupled Wilson-Fisher fixed point of the $\mathrm{O}(2N)$ model in $2+1$ spacetime dimensions. Using a controlled large-N expansion, we showed that, in the presence of a quenched random mass, the Wilson-Fisher fixed point flows directly to a QCP characterized by finite disorder and interaction strengths. When N is extrapolated to unity, the critical exponents for this transition are strikingly close to recent numerical results for the superfluid-Mott glass transition, although we again emphasize that this extrapolation may not be innocuous. As far as we are aware, ours is the first construction to achieve this, suggesting that the QCP we obtain may be in the same universality class as the superfluid-Mott glass transition. This is in contrast to earlier approaches using the double-ϵ expansions about the non-interacting fixed point, which returns spiralling RG flows that are not of obvious physical significance. Indeed, the relative simplicity of our result is a testament to the important roles played by both strong interactions and disorder in $2d$ quantum critical systems.

In addition, we presented non-perturbative results for the stability of this QCP to random scalar and vector potentials. While a random vector potential is exactly marginal, a random scalar potential is relevant, leading to what is likely a kind of compressible, glassy state referred to as a Bose glass. Understanding the nature of this glassy state and its relationship to the phenomenology of the Bose glass is an interesting direction for future exploration, although it requires accounting for non-perturbative, rare region effects. The theories considered in this Chapter may provide interesting platforms for the study of such non-perturbative effects when both disorder and interactions are present.

By setting N to unity and applying our results to dual theories of a Dirac fermion coupled to a fluctuating Chern-Simons gauge field, as well as the Abelian Higgs model (in Appendix C.3), we were able to make conjectures regarding the behavior of these theories to quenched disorder. Our conclusions constitute significant progress in the study of both of these historically difficult problems. The results of these approaches can then be compared to our conjecture from duality.

In addition to the critical exponents computed here, the QCP we discuss possesses universal DC and optical conductivities. Examining the universal transport properties of this theory via analytic of numerical techniques is important for understanding randomness at the Wilson-Fisher fixed point, as well as its duals. Such information may shed light on universal features of both superconductor-insulator transitions (the Abelian Higgs model) and plateau transitions (the Chern-Simons-Dirac theory).

Appendix A

Supplement to Chapter 2

A.1 Derivation of Self-Duality

A.1.1 Euclidean Lattice Model

In this appendix, we derive the self-duality of the Euclidean lattice model of Ref. [12],

$$Z = \sum_{\{J_\mu\}} \delta(\Delta_\mu J^\mu) e^{-S}\,, \tag{A.1}$$

where

$$\begin{aligned} S =& \frac{1}{2}\sum_{r,r'} J^\mu(r) G_{\mu\nu}(r-r') J^\nu(r') + \frac{i}{2}\sum_{r,R} J^\mu(r) K_{\mu\nu}(r,R) J^\nu(R) \\ &+ i\sum_{r,r'} e(r-r') J^\mu(r) A_\mu(r') + \sum_{R,R'} h(R-R')\epsilon^{\mu\nu\rho} J_\mu(R)\Delta_\nu A_\rho(R') \\ &+ \frac{1}{2}\sum_{r,r'} A_\mu(r)\Pi^{\mu\nu}(r,r') A_\nu(r')\,. \end{aligned} \tag{A.2}$$

Our conventions and notation are described in Section 2.2. We first consider the case in which the background fields $A_\mu = 0$. Following Ref. [65], we invoke the Poisson summation formula to make the formerly integer-valued J_μ real-valued

$$Z = \sum_{\{m_\mu\}} \int \mathcal{D}J\, \delta(\Delta_\mu J_\mu) e^{-S[J] + 2\pi i \sum_r m_\mu(r) J^\mu(r)}\,, \tag{A.3}$$

where m_μ is a new integer-valued variable. We can impose the delta function-imposed Gauss' Law $\Delta_\mu J^\mu = 0$ by rewriting J^μ as the curl of an emergent gauge field a_μ

$$J^\mu(r) = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\Delta_\nu a_\lambda(r)\,. \tag{A.4}$$

Plugging this into the action, we obtain

$$S = \frac{1}{2(2\pi)^2}\sum_{r,r'} a^\mu(r)G_{\mu\nu}(r-r')a^\nu(r') + \frac{i}{2(2\pi)^2}\sum_{r,R} a^\mu(r)K_{\mu\nu}(r-R)a^\nu(R) + i\sum_R a_\mu(R)\epsilon^{\mu\nu\lambda}\Delta_\nu m_\lambda(R)\,. \quad \text{(A.5)}$$

We can now change to vortex loop variables,

$$\tilde{J}^\mu = \epsilon^{\mu\nu\lambda}\Delta_\nu m_\lambda\,, \tag{A.6}$$

which satisfy their own Gauss' law $\Delta_\mu \tilde{J}^\mu = 0$. The partition function in these variables is thus

$$Z = \sum_{\{\tilde{J}_\mu\}} \int \mathcal{D}a\, \delta(\Delta_\mu \tilde{J}^\mu) e^{-S[\tilde{J},a]}\,. \tag{A.7}$$

We now proceed to integrate out a. In the long distance limit, we can use Eqs. (2.4)-(2.5) to write the propagator for a_μ as

$$\mathcal{G}^{\mu\nu}(p) = (2\pi)^2\left[\frac{g^2}{g^4+4\theta^2}\frac{1}{|p|}(\delta^{\mu\nu} - p^\mu p^\nu/p^2) + \frac{2\theta}{g^4+4\theta^2}\epsilon^{\mu\nu\lambda}\frac{p_\lambda}{p^2}\right], \tag{A.8}$$

where we have added a gauge fixing term $\frac{1}{2\xi}(\Delta_\mu a^\mu)^2$ in the limit $\xi \to 0$ (Landau gauge). Integrating out a gives the dual loop model action

$$S_D[\tilde{J}\,] = \frac{1}{2}\sum_p \tilde{J}^\mu(-p)\mathcal{G}_{\mu\nu}(p)\tilde{J}^\nu(p)\,. \tag{A.9}$$

In position space, this has the same form as Eq. (A.2), but with

$$g^2 \mapsto g_D^2 = \frac{g^2}{g^4/(2\pi)^2 + \theta^2/\pi^2}\,,\ \theta \mapsto \theta_D = -\frac{\theta}{g^4/(2\pi)^2+\theta^2/\pi^2}\,. \tag{A.10}$$

In other words, duality maps $\tau = \frac{\theta}{\pi} + i\frac{g^2}{2\pi}$ as a modular $\mathcal{S}$ transformation

$$\tau \mapsto -\frac{1}{\tau}\,. \tag{A.11}$$

Now consider the case with background fields turned on, $A_\mu \neq 0$. Then a_μ couples to

$$i\tilde{J}_\mu + i\frac{e}{2\pi}\,\epsilon_{\mu\nu\lambda}\Delta^\nu A^\lambda + \frac{h}{2\pi}(\Delta^2\delta_{\mu\nu} - \Delta_\mu\Delta_\nu)A^\nu\,. \tag{A.12}$$

When a_μ is integrated out, this leads to couplings between $\tilde{J}_\mu$ and A_μ which in momentum space take the form

$$\begin{aligned} & i\tilde{J}_\rho \mathcal{G}^{\rho\mu}\left(-\frac{e}{2\pi}\,\epsilon_{\mu\nu\lambda}p^\nu A^\lambda + \frac{h}{2\pi}(p^2\delta_{\mu\nu} - p_\mu p_\nu)A^\nu\right) \\ = \; & \frac{i}{2\pi}\left[\theta_D e + |p| g_D^2 h\right]\tilde{J}_\rho A^\rho + \frac{i}{2\pi}\left[-\frac{1}{|p|}g_D^2 e + \theta_D h\right]\epsilon^{\mu\nu\rho}\tilde{J}_\mu p_\nu A_\rho\,. \end{aligned} \tag{A.13}$$

So under duality the charges map as

$$e \mapsto \frac{1}{2\pi}\left[\theta_D e + |p| g_D^2 h\right],\; h \mapsto \frac{1}{2\pi}\left[-\frac{1}{|p|}g_D^2 e + \theta_D h\right], \tag{A.14}$$

meaning that, under particle-vortex duality, electric and magnetic charges are mapped to dyons!

Similarly, the background polarization tensor is shifted under duality. If we define it in momentum space as

$$\Pi_{\mu\nu} = \Pi_{\text{even}}(p)(\delta_{\mu\nu} - p_\mu p_\nu/p^2) + \Pi_{\text{odd}}(p)\epsilon_{\mu\nu\lambda}\frac{p^\lambda}{|p|}\,, \tag{A.15}$$

then

$$\Pi_{\text{even}} \;\mapsto\; \Pi_{\text{even}} + |p|(e^2 - h^2p^2)\frac{g_D^2}{(2\pi)^2} - 2p^2 he\frac{\theta_D}{(2\pi)^2}\,, \tag{A.16}$$

$$\Pi_{\text{odd}} \;\mapsto\; \Pi_{\text{odd}} - |p|(e^2 - h^2p^2)\frac{\theta_D}{(2\pi)^2} - 2p^2 eh\frac{g^2}{(2\pi)^2}\,. \tag{A.17}$$

A.1.2 Conformal Field Theory

The above lattice derivation carries over to conformal field theories of the form (now working in Minkowski space)

$$\begin{aligned} \mathcal{L} \;=\; & |D_a\phi|^2 - |\phi|^4 - \frac{1}{4g_D^2}f_{\mu\nu}\frac{i}{\sqrt{\partial^2}}f^{\mu\nu} + \frac{1}{4\theta_D}ada \\ & + e(x)J_\mu A^\mu + h(x)JdA + A_\mu(x)\Pi^{\mu\nu}(x,x')A_\nu(x')\,, \end{aligned} \tag{A.18}$$

where ϕ is a complex scalar field at its Wilson-Fisher fixed point (thus the notation $-|\phi|^4$), $D_a = \partial_\mu - ia_\mu$, $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, A_μ is a background $U(1)$ gauge field, and J_μ is the global $U(1)$ current $J_\mu = i(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*)$. We also use the notation $AdB = \epsilon^{\mu\nu\rho}A_\mu\partial_\nu B_\rho$. For a discussion of the self-duality of this theory with Dirac fermion matter, see Ref. [92].

The coupling constants g_D^2 and θ_D are indeed the corresponding quantities in the loop model description (A.10). If we deform this model into its symmetric phase (the trivial insulator) with the operator $-m^2|\phi|^2$, we can construct a loop model with Lagrangian

$$J_\mu a^\mu - \frac{1}{4g_D^2} f_{\mu\nu} \frac{i}{\sqrt{\partial^2}} f^{\mu\nu} + \frac{1}{4\theta_D} ada \,, \tag{A.19}$$

where we have set $A_\mu = 0$ for clarity. Integrating out a_μ results in a loop model characterized by a modular parameter $\tau = -\left(\frac{\theta_D}{\pi} + i\frac{g_D^2}{2\pi}\right)^{-1} = \frac{\theta}{\pi} + i\frac{g^2}{2\pi}$.

We can derive the self-duality of the theory (A.18) by assuming the bosonic particle-vortex duality relating a Wilson-Fisher fixed point to the critical point of the Abelian Higgs model [65, 66],

$$|D_A\phi| - |\phi|^2 \longleftrightarrow |D_a\tilde{\phi}|^2 - |\tilde{\phi}|^4 + \frac{1}{2\pi} adA \,, \tag{A.20}$$

and applying it to the Lagrangian, Eq. (A.18). Again turning off background fields for clarity, we obtain

$$\mathcal{L} \longleftrightarrow |D_b\tilde{\phi}|^2 - |\tilde{\phi}|^4 + \frac{1}{2\pi} adb - \frac{1}{4g_D^2} f_{\mu\nu} \frac{i}{\sqrt{\partial^2}} f^{\mu\nu} + \frac{1}{4\theta_D} ada \,. \tag{A.21}$$

Integrating out a in the dual theory gives an action

$$\tilde{\mathcal{L}} = |D_b\tilde{\phi}|^2 - |\tilde{\phi}|^4 - \frac{1}{4g^2} f'_{\mu\nu} \frac{i}{\sqrt{\partial^2}} f'^{\mu\nu} + \frac{1}{4\theta} bdb \,, \tag{A.22}$$

where $f'_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$. g^2, θ are related to g_D^2, θ_D by the modular $\mathcal{S}$ transformation (A.10). The analogous transformation laws for the source terms Eq. (A.14) and Eqs. (A.16)-(A.17) can be obtained in the same way.

A.2 Derivation of Polyakov's Duality

In this appendix, we review Polyakov's argument for the duality (2.43), which relates a theory of *non-intersecting* bosonic loops (i.e. bosonic loops with strong short-ranged repulsion) to a theory of a single free species of Dirac fermions [104]. Some of the finer points of this argument were ironed out in Refs. [141, 142, 146–148], to which we point the reader interested in a more detailed analysis. For a review of how to construct world line partition functions from gapped quantum field theories, see Refs. [63, 139, 140, 142].

We start from the bosonic loop model on the right hand side of Eq. (2.43). The amplitude for a path of length L with tangent vector $\hat{e}(s)$ between two points x and x' is

$$G(x - x') = \int_0^\infty dL \int \mathcal{D}\hat{e}\, \delta\left(1 - |\hat{e}|^2\right) \delta\left(x' - x - \int_0^L ds\, \hat{e}(s)\right) e^{-|m|L \pm i\pi\mathcal{W}[\hat{e}]} \,, \tag{A.23}$$

where $\mathcal{W}[\hat{e}]$ is the Berry phase term in the twist, Eq. (2.42),

$$\mathcal{W}[\hat{e}] = \frac{1}{2\pi} \int_0^L ds \int_0^1 du \, \hat{e} \cdot (\partial_s \hat{e} \times \partial_u \hat{e}) \, . \tag{A.24}$$

The momentum space representation is obtained via Fourier transform

$$G(p) = \int_0^\infty dL \int \mathcal{D}\hat{e} \, \delta\left(1 - |\hat{e}|^2\right) e^{-|m|L \pm i\pi \mathcal{W}[\hat{e}]} e^{ip^\mu \int_0^L ds \, \hat{e}_\mu(s)} \, . \tag{A.25}$$

This is none other than the coherent state path integral for a spin-1/2 particle in a magnetic field $b^\mu = \pm 2p^\mu$. The equation of motion for $\hat{e}$ is

$$\partial_s \hat{e}_\mu = \pm 2\epsilon_{\mu\nu\lambda} \hat{e}^\nu p^\lambda = i[H, \hat{e}_\mu] \, , \tag{A.26}$$

where H is the Hamiltonian operator. This implies that, since the Hamiltonian for the spin is $H = -\vec{b} \cdot \vec{S} = \mp p^\mu \hat{e}_\mu$, upon quantization $\hat{e}$ should satisfy commutation relations

$$[\hat{e}_\mu, \hat{e}_\nu] = 2i\epsilon_{\mu\nu\rho} \hat{e}^\rho \, , \tag{A.27}$$

meaning that we can perform the path integral over $\hat{e}$ by identifying it with the Pauli matrices $\hat{e}_\mu \to \sigma_\mu$. There is one slight problem with the above discussion, however: in reality, the magnetic field is not $2p^\mu$ but $2p^\mu/3$, meaning that this identification involves the appearance of an extra constant factor which can be eliminated by rescaling $m \to M$ [148]. Performing the integral over L finally gives the propagator of a free Dirac fermion of mass M

$$G(p) \propto \frac{1}{ip^\mu \sigma_\mu - M} \, , \tag{A.28}$$

where we have written $\pm|M| \equiv M$. One thus expects the partition function (2.43) to reproduce the correlation functions of a free Dirac fermion of mass M at sufficiently long distances.

Note that the theory we have discussed here is not coupled to gauge fields. As is well known, when coupled to a gauge field, theories of a single Dirac fermion exhibit the parity anomaly. See Section 2.4.1 for a discussion of how the parity anomaly appears in the context of this duality.

Appendix B

Supplement to Chapter 4

B.1 Chern-Simons conventions

In this appendix, we lay out our conventions for non-Abelian Chern-Simons gauge theories. We define $U(N)$ gauge fields $a_\mu = a_\mu^b t^b$, where t^b are the (Hermitian) generators of the Lie algebra of $U(N)$, which satisfy $[t^a, t^b] = if^{abc}t^c$, where f^{abc} are the structure constants of $U(N)$. The generators are normalized so that $\mathrm{Tr}[t^b t^c] = \frac{1}{2}\delta^{bc}$. The trace of a is a $U(1)$ gauge field, which we require to satisfy the Dirac quantization condition,

$$\int_{S^2} \frac{d\,\mathrm{Tr}[a]}{2\pi} \in \mathbb{Z}\,. \tag{B.1}$$

In general, the Chern-Simons levels for the $SU(N)$ and $U(1)$ components of a can be different. We therefore adopt the standard notation [78],

$$U(N)_{k,k'} = \frac{SU(N)_k \times U(1)_{Nk'}}{\mathbb{Z}_N}\,. \tag{B.2}$$

By taking the quotient with $\mathbb{Z}_N$, we are restricting the difference of the $SU(N)$ and $U(1)$ levels to be an integer multiple of N,

$$k' = k + nN\,, n \in \mathbb{Z}\,. \tag{B.3}$$

This enables us to glue the $U(1)$ and $SU(N)$ gauge fields together to form a gauge invariant theory of a single $U(N)$ gauge field $a = a_{SU(N)} + \tilde{a}\,\mathbf{1}$, with $\mathrm{Tr}[a] = N\tilde{a}$ having quantized fluxes as in Eq. (B.1). The Lagrangian for the $U(N)_{k,k'}$ theory can be written as

$$\mathcal{L}_{U(N)_{k,k'}} = \frac{k}{4\pi}\,\mathrm{Tr}\left[a_{SU(N)}da_{SU(N)} - \frac{2i}{3}a^3_{SU(N)}\right] + \frac{Nk'}{4\pi}\tilde{a}d\tilde{a}\,. \tag{B.4}$$

For the case $k = k'$, we simply refer to the theory as $U(N)_k$.

Throughout this paper, we implicitly regulate non-Abelian (Abelian) gauge theories using Yang-Mills (Maxwell) terms, as opposed to dimensional regularization [69, 286]. In Yang-Mills regularization, there is a one-loop exact shift of the $SU(N)$ level, $k \to k + \mathrm{sgn}(k)N$, that does not appear in dimensional regularization. Consequently, to describe the same theory in dimensional regularization, one must start with a $SU(N)$ level $k_{\mathrm{DR}} = k + \mathrm{sgn}(k)N$. The dualities discussed in this paper, e.g. Eqs.(4.13)-(4.15), therefore would take a somewhat different form in dimensional regularization.

Appendix C

Supplement to Chapter 5

C.1 RG Calculation using Dimensional Regularization

C.1.1 Renormalization

Dimensional regularization is a more natural scheme when considering higher loop diagrams, as needed to calculate the σ self energy at $\mathcal{O}(1/N)$. Our method is as follows. The action given in Eq. (5.28) is the bare action. For convenience, we reproduce it here:

$$S_r^B = \sum_n \int d^d\mathbf{x}\, d\tau_B \Bigg[\boldsymbol{\phi}^\dagger_{B,n}\left(-\frac{\partial^2}{\partial\tau_B^2} - \frac{\partial^2}{\partial\mathbf{x}^2}\right)\boldsymbol{\phi}_{B,n} + \frac{1}{2\cdot 8}\,\sigma_{B,n}\left(-\frac{\partial^2}{\partial\tau_B^2} - \frac{\partial^2}{\partial\mathbf{x}^2}\right)^{-1/2}\sigma_{B,n} \\ + \frac{i}{\sqrt{N}}\sigma_{B,n}\left|\boldsymbol{\phi}_{B,n}\right|^2\Bigg] + \frac{\bar{\Delta}_B}{2}\int d^d\mathbf{x}\sum_n\int d\tau_B\,\sigma_{B,n}(\mathbf{x},\tau_B)\sum_m\int d\tau_B'\,\sigma_{B,m}(\mathbf{x},\tau_B). \tag{C.1}$$

Notably, we have added a subscript or superscript 'B' to the fields, coupling constants, and time coordinate to highlight that these are the bare objects and thus not physical. The spatial dimension is $d = 2-\epsilon$. The Feynman rules are the same as those shown in Fig. 5.3 and given in Eq. (5.36) save that these objects should now include a 'B' subscript (or superscript).

The physical object is the generating functional Γ, and the theory is renormalized by ensuring its finiteness at each order in $1/N$. To guarantee that the time direction is being renormalized correctly, it is useful to rederive the relation between the bare and renormalized vertex functions explicitly. In doing so, we can suppress both replica and U(N) vector indices since we assume that neither symmetry is broken. The generating functional is a function of the bare field configuration $\bar{\boldsymbol{\phi}}_B$ and $\bar{\sigma}_B$:

$$\Gamma[\bar{\boldsymbol{\phi}}_B, \bar{\sigma}_B] = \sum_{\mathcal{N},\mathcal{M}=0}^{\infty}\frac{1}{\mathcal{N}!\mathcal{M}!}\int\prod_{i=1}^{\mathcal{N}+\mathcal{M}}\left(d^d\mathbf{x}_i\, d\tau_i^B\right)\Gamma_B^{(\mathcal{N},\mathcal{M})}\left(\{\mathbf{x}_i,\tau_i^B\}\right) \\ \times\prod_{j=1}^{\mathcal{N}}\bar{\boldsymbol{\phi}}_B\left(\mathbf{x}_j,\tau_j^B\right)\cdot\prod_{k=\mathcal{N}+1}^{\mathcal{N}+\mathcal{M}}\bar{\sigma}_B\left(\mathbf{x}_k,\tau_k^B\right), \tag{C.2}$$

where $\Gamma^{(0,0)} = 0$ and the Δ_B dependence is left implicit. To make contact with the notation of the main text, we note that the vertices $\Gamma_{\sigma\phi^\dagger\phi} = \Gamma^{(1,2)}$, $\Gamma^{(2,0)} = -(G^\phi)^{-1}$ and $\Gamma^{(0,2)} = -(G^\sigma)^{-1}$.

As emphasized, the vertex functions Γ_B are not finite in the limit that the UV cutoff $\Lambda \to \infty$. We define the renormalized fields and time as

$$\boldsymbol{\phi}(\mathbf{x},\tau) = Z_\phi^{1/2}\boldsymbol{\phi}_B\left(\mathbf{x},\tau^B\right), \qquad \sigma(\mathbf{x},\tau) = Z_\sigma^{1/2}\sigma_B\left(\mathbf{x},\tau^B\right), \qquad \tau^B = Z_\tau\tau, \qquad \bar{\Delta}_B = Z_{\bar{\Delta}}\bar{\Delta}. \tag{C.3}$$

The renormalization constants can be written as $Z_x = 1 + \delta_x$, $x = \phi, \sigma, \tau, \Delta$, where δ_x is $\mathcal{O}(1/N)$, allowing for a perturbative treatment. Inserting the renormalized fields into the functional returns

$$\begin{aligned}\Gamma[\bar{\boldsymbol{\phi}}_B, \bar{\sigma}_B] = \sum_{\mathcal{N},\mathcal{M}=0}^{\infty} \frac{1}{\mathcal{N}!\mathcal{M}!} \int \prod_{\ell=1}^{\mathcal{N}+\mathcal{M}} \left(d^d\mathbf{x}_i\, d\tau_i\right) Z_\phi^{\mathcal{N}/2} Z_\sigma^{\mathcal{M}/2} Z_\tau^{\mathcal{N}+\mathcal{M}} \Gamma_B^{(\mathcal{N},\mathcal{M})}\left(\{\mathbf{x}_\ell, \tau_\ell^B\}\right) \\ \times \prod_{n=1}^{\mathcal{N}} \bar{\phi}\left(\mathbf{x}_n, \tau_n\right) \cdot \prod_{m=\mathcal{N}+1}^{\mathcal{N}+\mathcal{M}} \bar{\sigma}\left(\mathbf{x}_m, \tau_m\right).\end{aligned} \tag{C.4}$$

The renormalized vertex functions are obtained by differentiating Γ with respect to $\bar{\phi}$ and $\bar{\sigma}$. It follows that

$$\Gamma_R^{(\mathcal{N},\mathcal{M})}\left[\{\mathbf{x}_i, \tau_i\}\right] = Z_\phi^{\mathcal{N}/2} Z_\sigma^{\mathcal{M}/2} Z_\tau^{\mathcal{N}+\mathcal{M}} \Gamma_B^{(\mathcal{N},\mathcal{M})}\left[\left\{\mathbf{x}_i, \tau_i^B\right\}\right]. \tag{C.5}$$

Finally, since perturbation theory is more efficiently done in momentum space, we Fourier transform to obtain

$$\begin{aligned}&(2\pi)^{d+1}\delta^d\left(\textstyle\sum_i \mathbf{p}_i\right)\delta\left(\textstyle\sum_i p_{0,i}\right)\Gamma_R^{(\mathcal{N},\mathcal{M})}\left[\{\mathbf{p}_i, p_{0,i}\}\right] \\ &\qquad = (2\pi)^{d+1}\delta^d\left(\textstyle\sum_\ell p_\ell\right)\delta\left(\textstyle\sum_\ell p_{0,\ell}^B\right) Z_\phi^{\mathcal{N}/2} Z_\sigma^{\mathcal{M}/2} \Gamma_B^{(\mathcal{N},\mathcal{M})}\left[\left\{\mathbf{p}_\ell, p_{0,\ell}^B\right\}\right] \\ &\qquad = (2\pi)^{d+1}\delta^d\left(\textstyle\sum_\ell p_\ell\right)\delta\left(\textstyle\sum_\ell p_{0,\ell}\right) Z_\phi^{\mathcal{N}/2} Z_\sigma^{\mathcal{M}/2} Z_\tau \Gamma_B^{(\mathcal{N},\mathcal{M})}\left[\left\{\mathbf{p}_\ell, p_{0,\ell}^B\right\}\right],\end{aligned} \tag{C.6}$$

where in the second line we used $p_{0,B} = p_0/Z_\tau$. Cancelling the δ-functions, we are left with

$$\Gamma_R^{(\mathcal{N},\mathcal{M})}\left[\{\mathbf{p}_\ell, p_{0,\ell}\}\right] = Z_\phi^{\mathcal{N}/2} Z_\sigma^{\mathcal{M}/2} Z_\tau \Gamma_B^{(\mathcal{N},\mathcal{M})}\left[\left\{\mathbf{p}_\ell, p_{0,\ell}^B\right\}\right]. \tag{C.7}$$

The renormalization constants are determined by first calculating the bare vertex functions and cancelling all divergences in Γ_B with the counterterms Z_x. Since we use a dimensional regularization scheme ($D = 3 - \epsilon$), this is done by defining the Z's such that all $1/\epsilon$ poles cancel. (We express these $1/\epsilon$ poles in terms of the cutoff Λ and renormalization scale μ in Appendix C.1.3.)

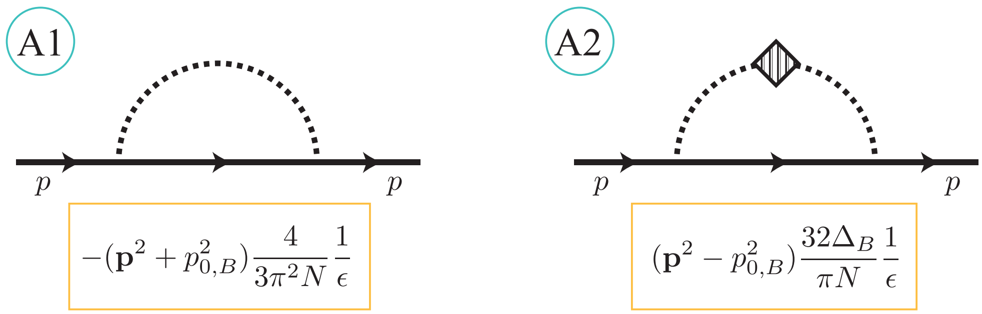

Figure C.1: Divergences corresponding to the ϕ self energy. In the notation of this appendeix, they contribute to $\Gamma_B^{(2,0)}$. The time-component of momenta, p, q is considered to be bare, $q = (\mathbf{q}, q_{0,B})$, etc.

We emphasize that the bare vertex functions must be computed entirely using the bare propagators and vertex functions, as well as time (frequency). If this is not done, there is a risk of double counting some of the divergences, as we believe was done in Ref. 267.

At $\mathcal{O}(1/N)$, only three vertex functions, $\Gamma_B^{(2,0)}$, $\Gamma_B^{(2,1)}$, and $\Gamma_B^{(0,2)}$, need be considered. We compute these below. In what follows, all non-log-divergent contributions (*e.g.* all divergences that do not contribute a $1/\epsilon$ pole) are ignored.

C.1.2 Diagrams

$\Gamma_R^{(2,0)}$: $\phi\phi$ self-energy

The log-divergent contributions to the ϕ propagator are shown in Fig. C.1. Summing them, we find

$$\Sigma_\phi = \text{A1} + \text{A2} + \text{finite} = -\boldsymbol{p}^2\left(\frac{4}{3\pi^2 N}\frac{1}{\epsilon} - \frac{32\bar{\Delta}_B}{\pi N}\frac{1}{\epsilon}\right) - p_{0,B}^2\left(\frac{4}{3\pi^2 N}\frac{1}{\epsilon} + \frac{32\bar{\Delta}_B}{\pi N}\frac{1}{\epsilon}\right), \tag{C.8}$$

and then using $\Gamma_B^{(2,0)} = \mathbf{p}^2 + p_{0,B}^2 - \Sigma_{\phi,B}$, gives

$$\begin{aligned}\Gamma_R^{(2,0)} &= Z_\phi Z_\tau\left[\mathbf{p}^2\left(1 + \frac{4}{3\pi^2 N}\frac{1}{\epsilon} - \frac{32\bar{\Delta}}{\pi N}\frac{1}{\epsilon}\right) + p_{0,B}^2\left(1 + \frac{4}{3\pi^2 N}\frac{1}{\epsilon} + \frac{32\bar{\Delta}}{\pi N}\frac{1}{\epsilon}\right)\right] \\ &= \mathbf{p}^2\left(1 + \delta_\phi + \delta_\tau + \frac{4}{3\pi^2 N}\frac{1}{\epsilon} - \frac{32\bar{\Delta}}{\pi N}\frac{1}{\epsilon}\right) + p_0^2\left(1 + \delta_\phi - \delta_\tau + \frac{4}{3\pi^2 N}\frac{1}{\epsilon} + \frac{32\bar{\Delta}}{\pi N}\frac{1}{\epsilon}\right).\end{aligned} \tag{C.9}$$

From this we conclude

$$\delta_\phi = -\frac{4}{3\pi^2 N}\frac{1}{\epsilon}, \qquad \delta_\tau = \frac{32\bar{\Delta}}{\pi N}\frac{1}{\epsilon}. \tag{C.10}$$

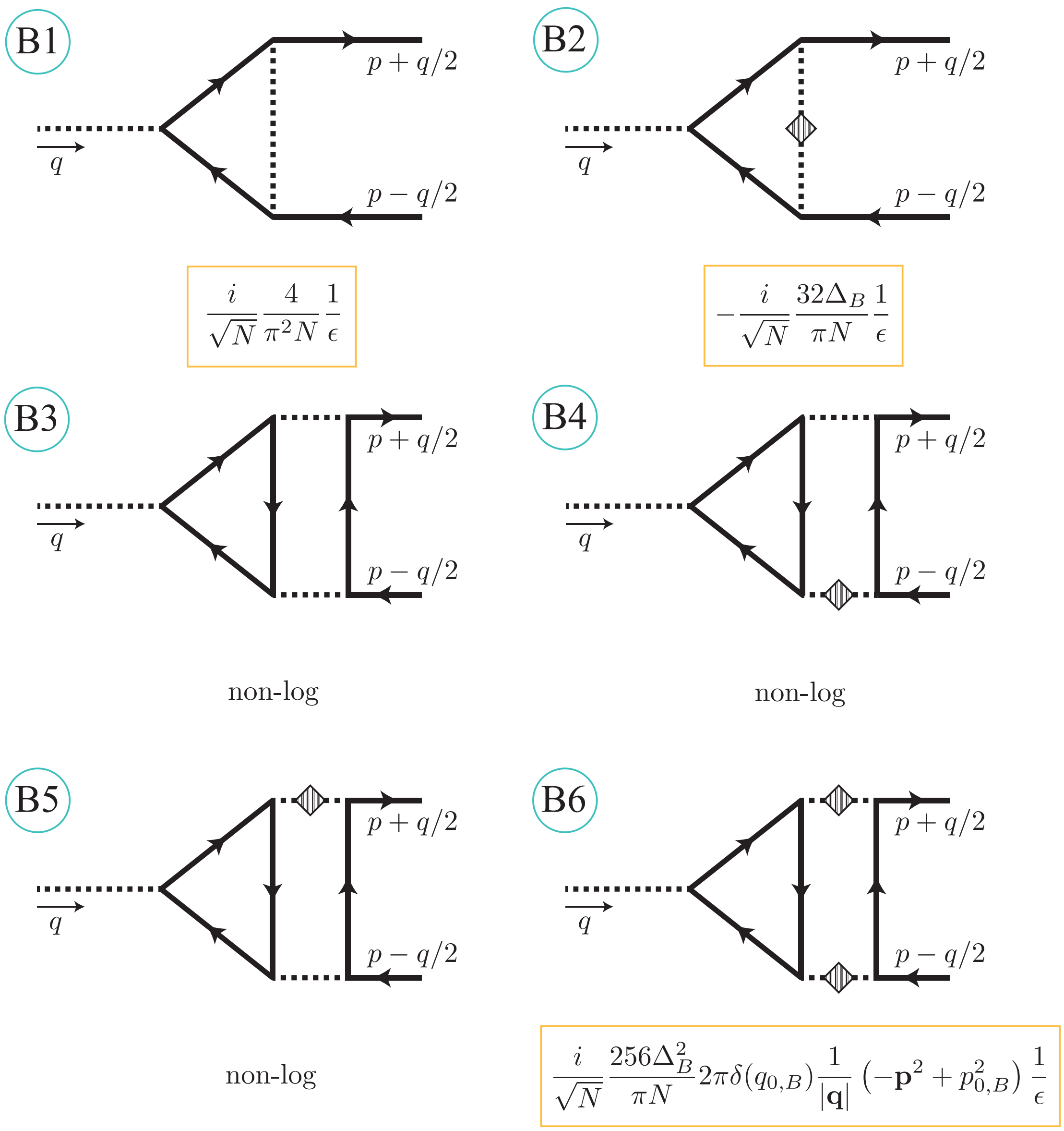

Figure C.2: All three-point diagrams correcting the $\sigma\,|\phi|^2$ vertex, $\Gamma_B^{(2,1)}$ at $\mathcal{O}(1/N)$. Diagrams B3-B6 possess partners where the ϕ fields traverse the loop in the converse direction. The time-component of momenta, p, q is considered to be bare, $q = (\mathbf{q}, q_{0,B})$, etc.

$\Gamma_R^{(2,1)}$: 3-point vertex

We summarize the divergent contributions to the 3-point vertex in Fig. C.2. We note that the diagram B6 indicates that $\overline{\langle\sigma\,|\,\boldsymbol{\phi}\,|^{\,2}\rangle}$ mixes with

$$\overline{\left\langle \int_{\tau} (-\boldsymbol{\nabla}^2)^{-1/2}\,\sigma \int_{\tau'} \left[\left(\boldsymbol{\phi}^{\dagger}\boldsymbol{\nabla}^2\boldsymbol{\phi} - \boldsymbol{\nabla}\boldsymbol{\phi}^{\dagger}\cdot\boldsymbol{\nabla}\boldsymbol{\phi}\right) - \left(\boldsymbol{\phi}^{\dagger}\partial_0\boldsymbol{\phi} - \partial_0\boldsymbol{\phi}^{\dagger}\partial_0\boldsymbol{\phi}\right)\right]\right\rangle}, \tag{C.11}$$

This is a consequence of the fact that the disordered theory is nonrenormalizable. For the purpose of determining the renormalization constant Z_σ, it is not necessary to consider this mixing.

Ignoring these terms, we find that the bare 3-point is

$$\Gamma_B^{(2,1)} \sim -\frac{i}{\sqrt{N}} + \mathrm{B1} + \mathrm{B2} = -\frac{i}{\sqrt{N}}\left(1 - \frac{4}{\pi^2 N}\frac{1}{\epsilon} + \frac{32\bar{\Delta}_B}{\pi N}\frac{1}{\epsilon}\right), \tag{C.12}$$

implying that the renormalized vertex function is

$$\begin{aligned}
\Gamma_R^{(2,1)} &= Z_\sigma^{1/2} Z_\phi Z_\tau \Gamma_B^{(2,1)} \\
&\sim -\frac{i}{\sqrt{N}}\left(1 + \frac{1}{2}\delta_\sigma + \delta_\phi + \delta_\tau - \frac{4}{\pi^2 N}\frac{1}{\epsilon} + \frac{32\bar{\Delta}}{\pi N}\frac{1}{\epsilon}\right) \\
&= -\frac{i}{\sqrt{N}}\left(1 + \frac{1}{2}\delta_\sigma - \frac{4}{3\pi^2 N}\frac{1}{\epsilon} + \frac{32\bar{\Delta}}{\pi N}\frac{1}{\epsilon} - \frac{4}{\pi^2 N}\frac{1}{\epsilon} + \frac{32\bar{\Delta}}{\pi N}\frac{1}{\epsilon}\right),
\end{aligned} \tag{C.13}$$

where the results of Eq. (C.10) have been inserted in the third line. Enforcing the finiteness of $\Gamma_R^{(2,1)}$ requires

$$\delta_\sigma = \left(\frac{32}{3\pi^2 N} - \frac{128\bar{\Delta}}{\pi N}\right)\frac{1}{\epsilon}. \tag{C.14}$$

$\Gamma_R^{(0,2)}$: σ self energy

In order to determine whether $\int d\tau\, \sigma(\mathbf{x},\tau)$ is renormalized differently than $\sigma(\mathbf{x},\tau)$, we directly calculate the σ self energy. We remark that the renormalization scheme [Eq. (C.3)] cannot account for these types of divergences – new counterterms would be required. Our ability to renormalize $\Gamma_R^{(0,2)}$ with the current set of counterterms is proof that our scheme is sufficient at $\mathcal{O}(1/N)$. It also serves as verification of our results for δ_σ and δ_τ above.

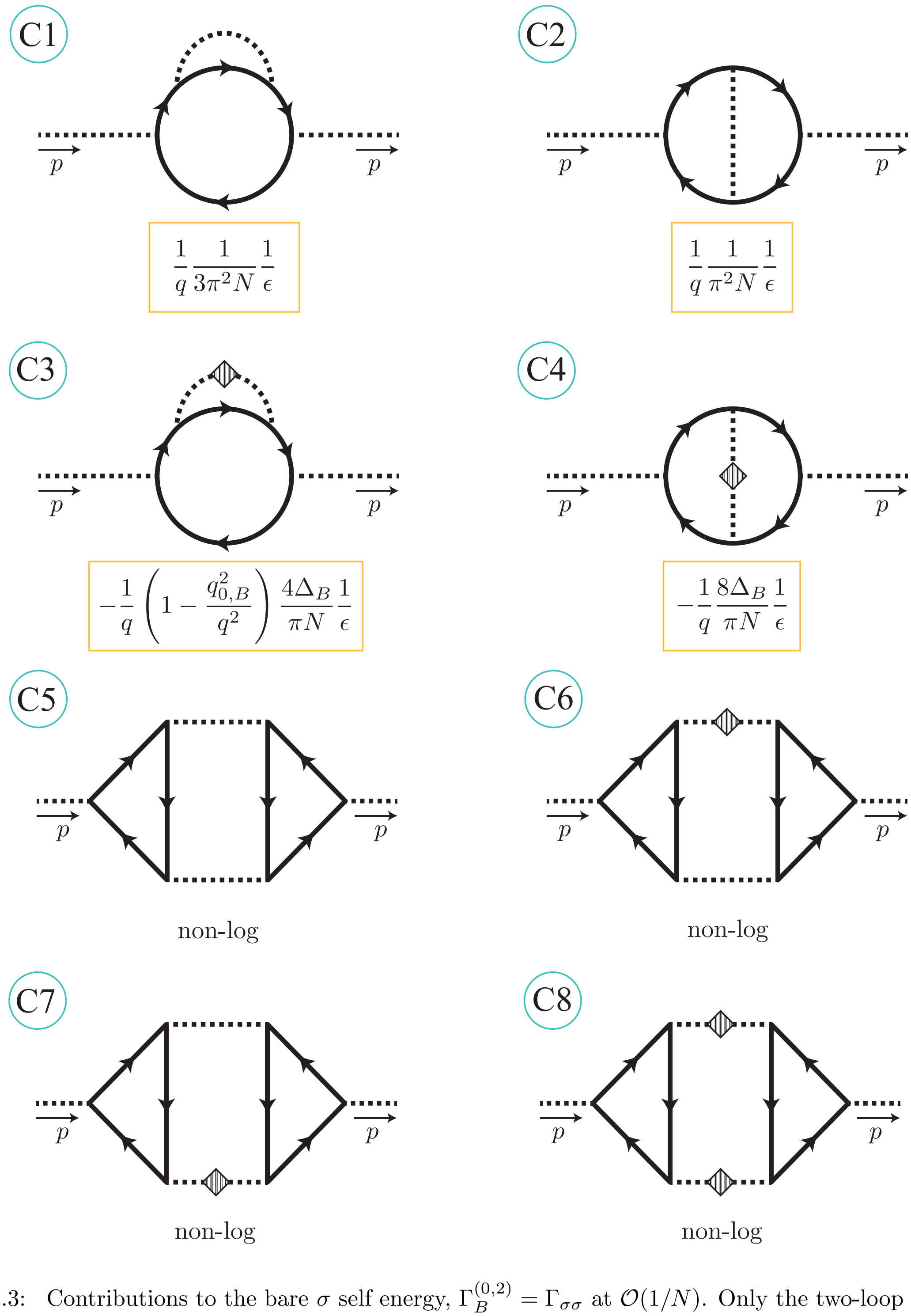

Figure C.3: Contributions to the bare σ self energy, $\Gamma_B^{(0,2)} = \Gamma_{\sigma\sigma}$ at $\mathcal{O}(1/N)$. Only the two-loop diagrams, C1-C4, are found to contain log divergences. We note that each of these diagrams has a partner where the internal G^ϕ lines in the converse direction. These diagrams have been included in the divergent terms shown beneath each diagram. The time-component of the momenta, p, q is considered to be bare, $q = (\mathbf{q}, q_{0,B})$, etc.

The log-divergent contributions are shown in Fig. C.3. Adding them, we find

$$\begin{aligned}\Sigma_{\sigma,B}[\mathbf{q}, q_{0,B}] &= \mathrm{C1} + \mathrm{C2} + \mathrm{C3} + \mathrm{C4} \\ &= \frac{1}{\sqrt{\mathbf{q}^2 + q_{0,B}^2}} \left(\frac{1}{\pi^2 N} + \frac{1}{3\pi^2 N} - \frac{8\bar{\Delta}_B}{\pi N} - \frac{4\bar{\Delta}_B}{\pi N} \right) \frac{1}{\epsilon} + \frac{q_{0,B}^2}{\left(\mathbf{q}^2 + q_{0,B}^2\right)^{3/2}} \frac{4\bar{\Delta}_B}{\pi N} \frac{1}{\epsilon}.\end{aligned} \tag{C.15}$$

The bare 2-point σ vertex is therefore

$$\Gamma_B^{(0,2)}[\mathbf{q}, q_{0,B}] = \frac{1}{8\sqrt{\mathbf{q}^2 + q_{0,B}^2}} \left(1 - \frac{32}{3\pi^2 N} \frac{1}{\epsilon} + \frac{96\bar{\Delta}_B}{\pi N} \frac{1}{\epsilon} - \frac{q_{0,B}^2}{\mathbf{q}^2 + q_{0,B}^2} \frac{32\bar{\Delta}_B}{\pi N} \frac{1}{\epsilon} \right) + 2\pi\delta(q_{0,B})\bar{\Delta}_B. \tag{C.16}$$

To renormalize, we write

$$\begin{aligned}\Gamma_R^{(0,2)}[\mathbf{q}, q_0] &= Z_\sigma Z_\tau \Gamma_B^{(0,2)}[\mathbf{q}, q_{0,B}] \\ &= \frac{1}{8\sqrt{\mathbf{q}^2 + q_0^2}} \left(1 + \delta_\sigma + \delta_\tau - \frac{32}{3\pi^2 N} \frac{1}{\epsilon} + \frac{96\bar{\Delta}}{\pi N} \frac{1}{\epsilon} + \frac{q_0^2}{\mathbf{q}^2 + q_0^2} \left[\delta_\tau - \frac{32\bar{\Delta}}{\pi N} \frac{1}{\epsilon} \right] \right) \\ &\quad + 2\pi\delta(q_0)\bar{\Delta}(1 + \delta_\sigma + 2\delta_\tau + \delta_{\bar{\Delta}}).\end{aligned} \tag{C.17}$$

Ensuring finiteness returns

$$\delta_\sigma = \left(\frac{32}{3\pi^2 N} - \frac{128\bar{\Delta}}{\pi N} \right) \frac{1}{\epsilon}, \qquad \delta_\tau = \frac{32\bar{\Delta}}{\pi N} \frac{1}{\epsilon}, \qquad \delta_{\bar{\Delta}} = \left(-\frac{32}{3\pi^2 N} + \frac{64\bar{\Delta}}{\pi N} \right) \frac{1}{\epsilon}. \tag{C.18}$$

Our results for δ_σ and δ_τ are notably in agreement with what we obtained from the ϕ self-energy and the three-point vertex in Eqs. (C.10) and (C.14).

In Ref. 267, the renormalized ϕ propagator was instead used to compute the diagram C3. As a result, its divergence cancels out and does not appear in Eq. (C.17).

C.1.3 Scaling functions

Summarizing our results from Eqs. (C.10), (C.14), and (C.18), we have

$$\begin{aligned}\delta_\phi &= -\frac{4}{3\pi^2 N} \left[\frac{1}{\epsilon} + \log\left(\frac{\Lambda}{\mu}\right) \right], & \delta_\tau &= \frac{32\bar{\Delta}}{\pi N} \left[\frac{1}{\epsilon} + \log\left(\frac{\Lambda}{\mu}\right) \right], \\ \delta_\sigma &= \left(\frac{32}{3\pi^2 N} - \frac{128\bar{\Delta}}{\pi N} \right) \left[\frac{1}{\epsilon} + \log\left(\frac{\Lambda}{\mu}\right) \right], & \delta_{\bar{\Delta}} &= \left(-\frac{32}{3\pi^2 N} + \frac{64\bar{\Delta}}{\pi N} \right) \left[\frac{1}{\epsilon} + \log\left(\frac{\Lambda}{\mu}\right) \right].\end{aligned} \tag{C.19}$$

Here, we have taken ${}^1\!/\!_\epsilon \to {}^1\!/\!_\epsilon + \log\left({}^\Lambda\!/\!_\mu\right)$ through the following reasoning. In the Feynamn diagrams calculated, factor of $1/\epsilon$ is always accompanied by $-\log p$. Since p is dimensionful, the logarithm should actually be a fraction of p to some other scale. The only other scale in the theory is the UV cutoff Λ, and it follows that these diagrams should be interpreted as $\# \left[{}^1\!/\!{}_\epsilon + \log\left({}^\Lambda\!/\!{}_p\right)\right]$ where '#' represents the coefficients we just calculated. Hence, in order to ensure that the renormalized diagram is finite as $\Lambda \to \infty$, the ${}^1\!/\!{}_\epsilon$ of the counterterm should be accompanied by $\log\left({}^\Lambda\!/\!{}_\mu\right)$, where μ is the renormalization scale: $\delta_j = -\# \left[{}^1\!/\!{}_\epsilon + \log\left({}^\Lambda\!/\!{}_\mu\right)\right]$.

With these counterterms, we can now calculate the primary quantities of interest: the dynamical critical exponent z, the anomalous dimension for ϕ, the anomalous dimension for σ, and the β-function for the disorder strength Δ.

Dynamical critical exponent z

The dynamical critical exponent is defined through

$$\mu \frac{d}{d\mu} \tau = z\tau \tag{C.20}$$

The bare time, conversely, scales as

$$\mu \frac{d}{d\mu} \tau_B = \tau_B. \tag{C.21}$$

Inserting $\tau_B = Z_\tau \tau$, we find

$$z = 1 - \mu \frac{d}{d\mu} Z_\tau = 1 + \frac{32\bar{\Delta}}{\pi N}. \tag{C.22}$$

Anomalous dimensions of ϕ and σ

We define the anomalous dimension of an operator $\mathcal{O}$ as $\eta_{\mathcal{O}}$ such that $[\mathcal{O}] = [\mathcal{O}]_0 + \eta_{\mathcal{O}}$, where $[\mathcal{O}]_0$ is the engineering dimension of $\mathcal{O}$. It follows from the definitions of Eq. (C.3) that the anomalous dimension of ϕ is

$$\eta_\phi = \frac{1}{2} \mu \frac{d}{d\mu} \log Z_\phi = \frac{2}{3\pi^2 N}, \tag{C.23}$$

and of σ is

$$\eta_\sigma = \frac{1}{2} \mu \frac{d}{d\mu} \log Z_\sigma = -\frac{16}{3\pi^2 N} + \frac{64\bar{\Delta}}{\pi N}. \tag{C.24}$$

Recall that the operator identity of Eq. (5.22) implies $\eta_\sigma = \eta_{|\phi|^2}$.

β function of $\bar{\Delta}$

Finally, $\beta_{\bar{\Delta}}$ is defined through the requirement that the bare coupling constant be invariant under RG:

$$\mu \frac{d}{d\mu} \Delta_B = 0. \tag{C.25}$$

From this we find

$$\beta_{\bar{\Delta}} = \mu \frac{d}{d\mu} \Delta = -\bar{\Delta} \frac{d \log Z_{\bar{\Delta}}}{d \log \mu} = -\frac{32\bar{\Delta}}{3\pi^2 N} + \frac{64\bar{\Delta}^2}{\pi N}. \tag{C.26}$$

We note that here we are using the high energy convention, so that $\beta_{\bar{\Delta}} < 0$ implies a flow to strong coupling.

C.2 Check of dynamical critical exponent

The authors of Ref. 243 derive a formula for the leading order correction to the dynamical critical exponent z of a a generic theory with (quantum) disorder of strength Δ coupling to an operator $\mathcal{O}$. In Eq. (4.36) of their paper, they state

$$z - 1 = \frac{\Delta}{2} \frac{c_{\mathcal{O}\mathcal{O}}}{c_T} \frac{D(D+1)}{D-1} \frac{\Gamma(D/2)}{2\pi^{D/2}} \overset{D \to 3}{=} \frac{\Delta}{2} \frac{c_{\mathcal{O}\mathcal{O}}}{c_T} \frac{3}{2\pi}. \tag{C.27}$$

Here, $c_{\mathcal{O}\mathcal{O}}$ is the coefficient of the two-point $\mathcal{O}$ correlator and c_T is the central charge (the coefficient of the two-point correlator of the stress energy tensor). We show that this is consistent with our results.

From Eq. (5.23), we see that the disorder couples to $i\sigma(\mathbf{x}, \tau)/u$, and it follows that for us $c_{\mathcal{O}\mathcal{O}} = -c_{\sigma\sigma}/u^2$. This coefficient is determined by the real space σ Green's function:

$$G_\sigma(r) = \int \frac{d^D p}{(2\pi)^D} e^{ip \cdot r}\, 8\, |p| = -\frac{8}{\pi^2} \frac{1}{r^4}, \tag{C.28}$$

implying that

$$-\frac{c_{\sigma\sigma}}{u^2} = \frac{1}{u^2} \frac{8}{\pi^2}. \tag{C.29}$$

The leading contribution to the central charge of the O($2N$) Wilson-Fisher fixed point corresponds simply to the central charge of $2N$ real, free bosons, which is given by [287, 288]

$$c_T \cong 2N \left(\frac{1}{2\pi^{D/2}/\Gamma(D/2)}\right)^2 \frac{D}{D-1} \overset{D\to 3}{=} \frac{3N}{16\pi^2}, \tag{C.30}$$

where $D = d + 1$ is the total number of spacetime dimensions. Putting this together, we find

$$z - 1 = \frac{1}{2}\frac{\Delta}{u^2}\frac{8/\pi^2}{3N/16\pi^2}\frac{3}{2\pi} = \frac{32\bar{\Delta}}{\pi N}, \tag{C.31}$$

in perfect agreement with Eq. (5.44) (as well as Eq. (C.22) in Appendix C.1).

C.3 Boson-Vortex Duality

C.3.1 Review of the Duality

The first duality we consider [64, 275, 276] relates a single complex scalar field, ϕ (we drop the boldface since $N = 1$), at its Wilson-Fisher fixed point to the Abelian Higgs model, a theory of complex bosonic vortices, $\tilde{\phi}$, also at their Wilson-Fisher fixed point. These vortices additionally interact through a logarithmic potential mediated by an emergent U(1) gauge field, a_μ,

$$\mathcal{L}_\phi = |D_A\phi|^2 - |\phi|^4 \longleftrightarrow \mathcal{L}_{\tilde{\phi}} = |D_a\tilde{\phi}|^2 - |\tilde{\phi}|^4 + \frac{1}{2\pi}Ada - \frac{1}{4g^2}f_{\mu\nu}f^{\mu\nu}, \tag{C.32}$$

where A_μ is a background gauge field. Here the interaction terms $-|\phi|^4$, $-|\tilde{\phi}|^4$ imply that the theories are tuned to the Wilson-Fished fixed point. As in the case of the boson-fermion duality, we only consider physics at energy scales much smaller than g^2, allowing us to omit the Maxwell term, $-\frac{1}{4g^2}f_{\mu\nu}f^{\mu\nu}$. We again work in Minkowski spacetime.

By differentiating each theory in Eq. (C.32) with respect to A_μ, one sees that this duality relates charge in the Wilson-Fisher theory to flux in the Abelian Higgs model,

$$J^\mu = i(\phi^\dagger\partial^\mu\phi - \partial^\mu\,\phi^\dagger\phi) \longleftrightarrow j^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda\,, \tag{C.33}$$

By considering the equations of motion for a_μ in the Abelian Higgs model, it follows that the converse is also true,

$$\frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda = \langle\tilde{J}^\mu\rangle = \langle i(\tilde{\phi}^\dagger\partial^\mu\tilde{\phi} - \partial^\mu\,\tilde{\phi}^\dagger\tilde{\phi})\rangle\,. \tag{C.34}$$

In terms of global symmetries, the mapping of charge to flux across the duality implies an exchange of **T** and

PH symmetries (here defined with appropriate transformation laws for the gauge fields). Since current and voltage exchange roles across the duality, the conductivity of the particles ϕ corresponds to the resistivity of the vortices $\tilde{\phi}$ and vice versa

$$\sigma^{\phi}_{ij} = \frac{1}{(2\pi)^2}\varepsilon^{ik}\varepsilon^{jl}\rho^{\tilde{\phi}}_{kl}\,, \tag{C.35}$$

where we write conductivity (resistivity) in units of $e^2/\hbar$ ($\hbar/e^2$). This dictionary is obtained using the charge-flux relations, Eqs. (C.33)-(C.34), and the definition of the conductivities $\langle J_i\rangle = \sigma^{\phi}_{ij}E^j$, $\langle\tilde{J}_i\rangle = \sigma^{\tilde{\phi}}_{ij}\langle e^j\rangle$, where $E_i = \partial_i A_t - \partial_t A_i$ and $e(a) = f_{it}(a)$ are the electric fields associated with A and a respectively, and $\rho = \sigma^{-1}$.

The duality, Eq. (C.32), can be verified by considering the phase diagrams of each of the dual theories. As discussed earlier, the Wilson-Fisher theory is tuned through the addition of a mass, $\delta r\,|\phi|^2$. For $\delta r > 0$, ϕ is gapped, and the ground state is insulating, while for $\delta r < 0$, ϕ condenses, and the ground state hosts a Goldstone mode. On the other hand, when a mass term $-\delta\tilde{r}|\tilde{\phi}|^2$ with $\delta\tilde{r} > 0$ is added to the dual theory, $\mathcal{L}_{\tilde{\phi}}$, $\tilde{\phi}$ is gapped out, but the ground state contains a gapless gauge field. This is the superfluid phase seen in in the Wilson-Fisher theory: the gauge field is the dual of the Goldstone mode. Similarly, for $\delta\tilde{r} < 0$, $\tilde{\phi}$ condenses and the gauge field is Higgsed, forming a superconductor. The conductivity dictionary of Eq. (C.35) indicates that a superconductor of vortices ($\rho^{\tilde{\phi}} = 0$) is an insulator of ϕ particles, making it the dual of the insulating phase of ϕ's. This mapping of the phase diagrams suggests that the mass operators in the two theories are dual to one another up to a sign,

$$|\phi|^2 \longleftrightarrow -|\tilde{\phi}|^2\,. \tag{C.36}$$

In summary, when the charge in one theory is gapped, the vortices of the dual theory condense, and vice versa.

C.3.2 Random Mass

We now use the results of Sections 5.3 and 5.4 and the operator dictionaries, Eqs. (C.33) and (C.36), to determine the effects of disorder on the Abelian Higgs model (setting the background field, A_μ, to zero). We begin by considering the effect of a random mass with Gaussian white noise correlations, as discussed in Section 5.3. From Eq. (C.36), we see that a random mass at the $N = 1$ Wilson-Fisher fixed point is dual to a random mass in the Abelian Higgs model,

$$R(\mathbf{x})|\phi|^2(\mathbf{x},\tau) \longleftrightarrow -R(\mathbf{x})|\tilde{\phi}|^2(\mathbf{x},\tau)\,. \tag{C.37}$$

Since R is a random variable which can take positive and negative values, the change in sign is immaterial. In the large-N limit, we observed that the Wilson-Fisher fixed point gives way to a QCP with finite disorder and interaction strengths. Under the assumption that this story continues to hold down to $N = 1$, the Abelian Higgs model with a random mass must also flow to such a QCP. Moreover, since the mass operators in the two theories are dual to one another, they have the same scaling dimension at the fixed point,

$$[|\tilde{\phi}|^2] = [|\phi|^2] = 2 + \frac{3}{16\pi^2}\,. \tag{C.38}$$

As in the boson-fermion duality, the dynamical scaling exponent, $\tilde{z}$, and correlation length exponent, $\tilde{\nu}$, remain unchanged across the duality,

$$\tilde{\nu} = \nu = 1\,, \qquad \tilde{z} = z = 1 + \frac{16}{3\pi^2} \approx 1.5\,. \tag{C.39}$$

It should be possible to compute these exponents in a large-N expansion of the Abelian Higgs model as well, and it would be interesting to compare the two results. However, we caution that for $N > 1$ the theories are no longer dual, and one limit may be more similar to the $N = 1$ behavior than the other. It may also be possible to obtain exponents numerically for the dirty Abelian Higgs model with $N = 1$.

Should the Abelian Higgs model with a random mass flow to such a QCP, this QCP will be characterized by a universal conductivity, which would be related to the universal conductivity of the fixed point we developed in Section 5.3 via Eq. (C.35). We leave the calculation of the DC response of the Wilson-Fisher bosons with a random mass, both using a large-N approach and numerical techniques, for future work.

C.3.3 Random Scalar and Vector Potentials

We now consider the effects of perturbing by random scalar and vector potentials, as in Eq. (5.57). The conclusion reached in that section only necessitated the preservation of a U(1) symmetry so our results remain valid even if the continuation to $N = 1$ is invalid. By the mapping of charge to flux in Eq. (C.33), the vortices $\tilde{\phi}$ experience a random scalar potential as a randomly sourced flux of a_i,

$$\mathcal{V}(\mathbf{x})\, J_0(\mathbf{x}, t) \longleftrightarrow \frac{1}{2\pi}\mathcal{V}(\mathbf{x})\, \varepsilon^{ij}\partial_i a_j(\mathbf{x}, t)\,. \tag{C.40}$$

Integrating by parts, we see that the disorder takes the form of a random current, $\mathcal{J}_i(\mathbf{x}) = \partial_i\mathcal{V}/2\pi$. As demonstrated in Section 5.4, the $\mathcal{V}(\mathbf{x})$ disorder is always relevant since it involves the temporal component of a conserved current, the flux $j^t = \varepsilon^{ij}\partial_i a_j/2\pi$. The ultimate fate of the Abelian Higgs theory is inaccessible through the perturbative RG approach employed throughout this paper. Nevertheless, since we expect the ϕ bosons form a (perhaps glassy) insulating state in the presence of a random scalar potential, the conductivty dictionary in Eq. (C.35) indicates that the $\tilde{\phi}$ vortices have DC resistivity $\rho_{xx}(T/\omega \to 0) \to 0$. The vortices therefore appear to form a superconducting state. It would be interesting to better characterize this state in future work, using the conductivity dictionary and making suitable assumptions regarding fate of the Wilson-Fisher theory with a random scalar potential.

In keeping with the exchange of flux and charge, a random vector potential in the Wilson-Fisher theory maps to a random magnetic field $\mathcal{B}(\mathbf{x}) = \varepsilon^{ij}\partial_i\mathcal{A}_j(\mathbf{x})$, which manifests as a random charge density in the Abelian Higgs model,

$$\mathcal{A}^i(\mathbf{x})\, J_i(\mathbf{x},t) \longleftrightarrow \frac{1}{2\pi}\mathcal{B}(\mathbf{x})\, a_t(\mathbf{x},t)\,. \tag{C.41}$$

As discussed in Section 5.4, this type of disorder is exactly marginal, leading to a line of fixed points parameterized by the dynamical exponent z, which depends on the disorder variance $\Delta_{\mathcal{A}}$.

References

[1] H. P. Wei, D. C. Tsui, M. A. Paalanen, and A. M. M. Pruisken, "Experiments on delocalization and university in the integral quantum hall effect," Phys. Rev. Lett. **61**, 1294–1296 (1988).

[2] L. Engel, H. Wei, D. Tsui, and M. Shayegan, "Critical exponent in the fractional quantum hall effect," Surface Science **229**, 13 – 15 (1990).

[3] S. Koch, R. J. Haug, K. v. Klitzing, and K. Ploog, "Size-dependent analysis of the metal-insulator transition in the integral quantum hall effect," Phys. Rev. Lett. **67**, 883–886 (1991).

[4] L. W. Engel, D. Shahar, i. m. c. Kurdak, and D. C. Tsui, "Microwave frequency dependence of integer quantum hall effect: Evidence for finite-frequency scaling," Phys. Rev. Lett. **71**, 2638–2641 (1993).

[5] H. P. Wei, L. W. Engel, and D. C. Tsui, "Current scaling in the integer quantum hall effect," Phys. Rev. B **50**, 14609–14612 (1994).

[6] L. W. Wong, H. W. Jiang, N. Trivedi, and E. Palm, "Disorder-tuned transition between a quantum hall liquid and hall insulator," Phys. Rev. B **51**, 18033–18036 (1995).

[7] W. Li, G. A. Csáthy, D. C. Tsui, L. N. Pfeiffer, and K. W. West, "Scaling and universality of integer quantum hall plateau-to-plateau transitions," Phys. Rev. Lett. **94**, 206807 (2005).

[8] W. Li, C. L. Vicente, J. S. Xia, W. Pan, D. C. Tsui, L. N. Pfeiffer, and K. W. West, "Scaling in plateau-to-plateau transition: A direct connection of quantum hall systems with the anderson localization model," Phys. Rev. Lett. **102**, 216801 (2009).

[9] W. Li, J. S. Xia, C. Vicente, N. S. Sullivan, W. Pan, D. C. Tsui, L. N. Pfeiffer, and K. W. West, "Crossover from the nonuniversal scaling regime to the universal scaling regime in quantum hall plateau transitions," Phys. Rev. B **81**, 033305 (2010).

[10] S. Kivelson, D.-H. Lee, and S.-C. Zhang, "Global phase diagram in the quantum hall effect," Phys. Rev. B **46**, 2223–2238 (1992).

[11] E. Shimshoni, S. L. Sondhi, and D. Shahar, "Duality near quantum Hall transitions," Phys. Rev. B **55**, 13730–13738 (1997).

[12] E. Fradkin and S. Kivelson, "Modular invariance, self-duality and the phase transition between quantum Hall plateaus," Nucl. Phys. B **474**, 543–574 (1996).

[13] A. Hui, E.-A. Kim, and M. Mulligan, "Non-abelian bosonization and modular transformation approach to superuniversality," Phys. Rev. B **99**, 125135 (2019).

[14] J. T. Chalker and P. D. Coddington, "Percolation, quantum tunnelling and the integer hall effect," Journal of Physics C: Solid State Physics **21**, 2665–2679 (1988).

[15] B. Huckestein, "Scaling theory of the integer quantum hall effect," Rev. Mod. Phys. **67**, 357–396 (1995).

[16] K. Slevin and T. Ohtsuki, "Critical exponent for the quantum hall transition," Phys. Rev. B **80**, 041304 (2009).

[17] M. P. A. Fisher, G. Grinstein, and S. M. Girvin, "Presence of quantum diffusion in two dimensions: Universal resistance at the superconductor-insulator transition," Phys. Rev. Lett. **64**, 587–590 (1990).

[18] D. Shahar, D. C. Tsui, M. Shayegan, R. N. Bhatt, and J. E. Cunningham, "Universal conductivity at the quantum hall liquid to insulator transition," Phys. Rev. Lett. **74**, 4511–4514 (1995).

[19] D. Shahar, D. C. Tsui, M. Shayegan, E. Shimshoni, and S. L. Sondhi, "Evidence for charge-flux duality near the quantum hall liquid-to-insulator transition," Science **274**, 589–592 (1996).

[20] E. Shimshoni, S. L. Sondhi, and D. Shahar, "Duality near quantum hall transitions," Phys. Rev. B **55**, 13730–13738 (1997).

[21] B. I. Halperin, P. A. Lee, and N. Read, "Theory of the half-filled Landau level," Phys. Rev. B **47**, 7312 (1993).

[22] P. Kumar, S. Raghu, and M. Mulligan, "Composite fermion Hall conductivity and the half-filled Landau level," ArXiv e-prints (2018), arXiv:1803.07767 [cond-mat.str-el] .

[23] P. Kumar, M. Mulligan, and S. Raghu, "Why is the HLR theory particle-hole symmetric?" (2018), arXiv:1805.06462 [cond-mat.str-el] .

[24] P. Kumar, M. Mulligan, and S. Raghu, "Emergent reflection symmetry from nonrelativistic composite fermions," Phys. Rev. B **99**, 205151 (2019).

[25] P. Kumar, Y. B. Kim, and S. Raghu, "Self-duality of the integer quantum hall to insulator transition: Composite fermion description," Phys. Rev. B **100**, 235124 (2019).

[26] Y. B. Kim, A. Furusaki, X.-G. Wen, and P. A. Lee, "Gauge-invariant response functions of fermions coupled to a gauge field," Phys. Rev. B **50**, 17917–17932 (1994).

[27] S. A. Kivelson, D.-H. Lee, Y. Krotov, and J. Gan, "Composite-fermion Hall conductance at $\nu = \frac{1}{2}$," Phys. Rev. B **55**, 15552–15561 (1997).

[28] H. W. Jiang, H. L. Stormer, D. C. Isui, L. N. Pfeiffer, and K. W. West, "Transport anomalies in the lowest landau level of two-dimensional electrons at half-filling," Phys. Rev. B **40**, 12013–12016 (1989).

[29] W. Kang, H. L. Stormer, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, "How real are composite fermions?" Phys. Rev. Lett. **71**, 3850–3853 (1993).

[30] J. H. Smet, K. von Klitzing, D. Weiss, and W. Wegscheider, "dc transport of composite fermions in weak periodic potentials," Phys. Rev. Lett. **80**, 4538–4541 (1998).

[31] J. H. Smet, S. Jobst, K. von Klitzing, D. Weiss, W. Wegscheider, and V. Umansky, "Commensurate composite fermions in weak periodic electrostatic potentials: Direct evidence of a periodic effective magnetic field," Phys. Rev. Lett. **83**, 2620–2623 (1999).

[32] R. L. Willett, K. W. West, and L. N. Pfeiffer, "Geometric resonance of composite fermion cyclotron orbits with a fictitious magnetic field modulation," Phys. Rev. Lett. **83**, 2624–2627 (1999).

[33] D. Kamburov, M. Shayegan, L. N. Pfeiffer, K. W. West, and K. W. Baldwin, "Commensurability oscillations of hole-flux composite fermions," Phys. Rev. Lett. **109**, 236401 (2012).

[34] D. Kamburov, Y. Liu, M. A. Mueed, M. Shayegan, L. N. Pfeiffer, K. W. West, and K. W. Baldwin, "What determines the fermi wave vector of composite fermions?" Phys. Rev. Lett. **113**, 196801 (2014).

[35] H. Deng, Y. Liu, I. Jo, L. N. Pfeiffer, K. W. West, K. W. Baldwin, and M. Shayegan, "Commensurability oscillations of composite fermions induced by the periodic potential of a wigner crystal," Phys. Rev. Lett. **117**, 096601 (2016).

[36] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, "Scaling theory of localization: Absence of quantum diffusion in two dimensions," Phys. Rev. Lett. **42**, 673–676 (1979).

[37] S.-S. Lee, "Low-energy effective theory of fermi surface coupled with u(1) gauge field in $2+1$ dimensions," Phys. Rev. B **80**, 165102 (2009).

[38] C. Nayak and F. Wilczek, "Renormalization group approach to low temperature properties of a non-Fermi liquid metal," Nuclear Physics B **430**, 534 – 562 (1994).

[39] D. F. Mross, J. McGreevy, H. Liu, and T. Senthil, "Controlled expansion for certain non-fermi-liquid metals," Phys. Rev. B **82**, 045121 (2010).

[40] D. Dalidovich and S.-S. Lee, "Perturbative non-Fermi liquids from dimensional regularization," Phys. Rev. **B88**, 245106 (2013), arXiv:1307.3170 [cond-mat.str-el] .

[41] A. Schlief, P. Lunts, and S.-S. Lee, "Exact critical exponents for the antiferromagnetic quantum critical metal in two dimensions," Phys. Rev. X **7**, 021010 (2017).

[42] J. A. Damia, S. Kachru, S. Raghu, and G. Torroba, "Two-dimensional non-fermi-liquid metals: A solvable large-n limit," Phys. Rev. Lett. **123**, 096402 (2019).

[43] D. B. Haviland, Y. Liu, and A. M. Goldman, "Onset of superconductivity in the two-dimensional limit," Phys. Rev. Lett. **62**, 2180–2183 (1989).

[44] H. M. Jaeger, D. B. Haviland, B. G. Orr, and A. M. Goldman, "Onset of superconductivity in ultrathin granular metal films," Phys. Rev. B **40**, 182–196 (1989).

[45] A. F. Hebard and M. A. Paalanen, "Magnetic-field-tuned superconductor-insulator transition in two-dimensional films," Phys. Rev. Lett. **65**, 927–930 (1990).

[46] M. A. Paalanen, A. F. Hebard, and R. R. Ruel, "Low-temperature insulating phases of uniformly disordered two-dimensional superconductors," Phys. Rev. Lett. **69**, 1604–1607 (1992).

[47] M. Steiner and A. Kapitulnik, "Superconductivity in the insulating phase above the field-tuned superconductor–insulator transition in disordered indium oxide films," Physica C: Superconductivity **422**, 16–26 (2005).

[48] M. A. Steiner, N. P. Breznay, and A. Kapitulnik, "Approach to a superconductor-to-bose-insulator transition in disordered films," Phys. Rev. B **77**, 212501 (2008).

[49] A. M. Goldman, "Superconductor-insulator transitions," International Journal of Modern Physics B **24**, 4081–4101 (2010).

[50] N. P. Breznay, M. A. Steiner, S. A. Kivelson, and A. Kapitulnik, "Self-duality and a hall-insulator phase near the superconductor-to-insulator transition in indium-oxide films," Proceedings of the National Academy of Sciences **113**, 280–285 (2016), https://www.pnas.org/content/113/2/280.full.pdf .

[51] N. P. Breznay and A. Kapitulnik, "Particle-hole symmetry reveals failed superconductivity in the metallic phase of two-dimensional superconducting films," Science Advances **3**, e1700612 (2017).

[52] A. Yazdani and A. Kapitulnik, "Superconducting-insulating transition in two-dimensional a-moge thin films," Phys. Rev. Lett. **74**, 3037–3040 (1995).

[53] N. Mason and A. Kapitulnik, "Dissipation effects on the superconductor-insulator transition in 2d superconductors," Phys. Rev. Lett. **82**, 5341–5344 (1999).

[54] N. Mason and A. Kapitulnik, "True superconductivity in a two-dimensional superconducting-insulating system," Phys. Rev. B **64**, 060504 (2001).

[55] Y. Wang, I. Tamir, D. Shahar, and N. P. Armitage, "A Bose metal has no cyclotron resonance," ArXiv e-prints (2017), arXiv:1708.01908 [cond-mat.supr-con] .

[56] A. Kapitulnik, S. A. Kivelson, and B. Spivak, "Colloquium: Anomalous metals: Failed superconductors," Rev. Mod. Phys. **91**, 011002 (2019).

[57] P. A. M. Dirac, "Quantised singularities in the electromagnetic field,," Proc. Roy. Soc. Lond. **A133**, 60–72 (1931).

[58] T. T. Wu and C. N. Yang, "Dirac Monopole Without Strings: Monopole Harmonics," Nucl. Phys. **B107**, 365 (1976).

[59] H. A. Kramers and G. H. Wannier, "Statistics of the two-dimensional ferromagnet. part i," Phys. Rev. **60**, 252–262 (1941).

[60] L. P. Kadanoff and H. Ceva, "Determination of an operator algebra for the two-dimensional ising model," Phys. Rev. B **3**, 3918–3939 (1971).

[61] F. J. Wegner, "Duality in generalized ising models and phase transitions without local order parameters," Journal of Mathematical Physics **12**, 2259–2272 (1971), https://doi.org/10.1063/1.1665530 .

[62] K. G. Wilson, "Confinement of quarks," Phys. Rev. D **10**, 2445–2459 (1974).

[63] E. Fradkin, *Field Theories of Condensed Matter Systems, Second Edition* (Cambridge University Press, Cambridge, U.K., 2013).

[64] P. R. Thomas and M. Stone, "Nature of the phase transition in a non-linear O(2) $_3$ model," Nuclear Physics B **144**, 513–524 (1978).

[65] M. E. Peskin, "Mandelstam-'t Hooft duality in abelian lattice models," Annals of Physics **113**, 122–152 (1978).

[66] C. Dasgupta and B. I. Halperin, "Phase Transition in a Lattice Model of Superconductivity," Phys. Rev. Lett. **47**, 1556–1560 (1981).

[67] B. I. Halperin, T. C. Lubensky, and S.-k. Ma, "First-order phase transitions in superconductors and smectic-a liquid crystals," Phys. Rev. Lett. **32**, 292–295 (1974).

[68] F. Wilczek, "Magnetic Flux, Angular Momentum, and Statistics," Phys. Rev. Lett. **48**, 1144 (1982).

[69] E. Witten, "Quantum field theory and the Jones polynomial," Commun. Math. Phys. **121**, 351–399 (1989).

[70] J. K. Jain, "Composite-fermion approach for the fractional quantum Hall effect," Phys. Rev. Lett. **63**, 199 (1989).

[71] A. López and E. Fradkin, "Fractional quantum Hall effect and Chern-Simons gauge theories," Phys. Rev. B **44**, 5246 (1991).

[72] B. I. Halperin, P. A. Lee, and N. Read, "Theory of the half-filled Landau level," Phys. Rev. B **47**, 7312 (1993).

[73] S. C. Zhang, T. H. Hansson, and S. Kivelson, "Effective-Field-Theory Model for the Fractional Quantum Hall Effect," Phys. Rev. Lett. **62**, 82 (1989).

[74] X.-G. Wen, "Topological Orders and Edge Excitations in FQH States," Adv. Phys. **44**, 405 (1995).

[75] E. H. Fradkin, C. Nayak, A. Tsvelik, and F. Wilczek, "A Chern-Simons effective field theory for the Pfaffian quantum Hall state," Nucl. Phys. **B516**, 704–718 (1998), arXiv:cond-mat/9711087 [cond-mat] .

[76] S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, S. R. Wadia, and X. Yin, "Chern-Simons theory with vector fermion matter," Eur. Phys. J. C **72**, 1–65 (2012).

[77] O. Aharony, G. Gur-Ari, and R. Yacoby, "d = 3 bosonic vector models coupled to Chern-Simons gauge theories," J. High Energy Phys. **03**, 37 (2012).

[78] O. Aharony, Z. Komargodski, and S. Yankielowicz, "Disorder in large-n theories," Journal of High Energy Physics **2016**, 13 (2016).

[79] N. Seiberg, T. Senthil, C. Wang, and E. Witten, "A duality web in 2 + 1 dimensions and condensed matter physics," Annals of Physics **374**, 395–433 (2016), arXiv:1606.01989 [hep-th] .

[80] A. Karch and D. Tong, "Particle-Vortex Duality from 3D Bosonization," Phys. Rev. X **6**, 031043 (2016).

[81] D. T. Son, "Is the composite fermion a dirac particle?" Phys. Rev. X **5**, 031027 (2015).

[82] C. Wang and T. Senthil, "Dual Dirac liquid on the surface of the electron topological insulator," Phys. Rev. X **5**, 041031 (2015).

[83] M. A. Metlitski and A. Vishwanath, "Particle-vortex duality of two-dimensional Dirac fermion from electric-magnetic duality of three-dimensional topological insulators," Phys. Rev. B **93**, 245151 (2016).

[84] D. Radicevic, D. Tong, and C. Turner, "Non-Abelian 3d Bosonization and Quantum Hall States," JHEP **12**, 067 (2016), arXiv:1608.04732 [hep-th] .

[85] H. Goldman, M. Mulligan, S. Raghu, G. Torroba, and M. Zimet, "Two-dimensional conductors with interactions and disorder from particle-vortex duality," Phys. Rev. B **96**, 245140 (2017).

[86] C. Wang, A. Nahum, M. A. Metlitski, C. Xu, and T. Senthil, "Deconfined quantum critical points: Symmetries and dualities," Phys. Rev. X **7**, 031051 (2017).

[87] A. Thomson and S. Sachdev, "Fermionic spinon theory of square lattice spin liquids near the néel state," Phys. Rev. X **8**, 011012 (2018).

[88] H. Goldman and E. Fradkin, "Dirac Composite Fermions and Emergent Reflection Symmetry about Even Denominator Filling Fractions," Phys. Rev. **B98**, 165137 (2018), arXiv:1808.09314 [cond-mat.str-el] .

[89] H. Goldman, R. Sohal, and E. Fradkin, "Landau-ginzburg theories of non-abelian quantum hall states from non-abelian bosonization," Phys. Rev. B **100**, 115111 (2019).

[90] H. Goldman, A. Thomson, L. Nie, and Z. Bi, "Collusion of Interactions and Disorder at the Superfluid-Insulator Transition: A Dirty 2d Quantum Critical Point," (2019), arXiv:1909.09167 [cond-mat.str-el] .

[91] D. F. Mross, J. Alicea, and O. I. Motrunich, "Explicit Derivation of Duality between a Free Dirac Cone and Quantum Electrodynamics in (2+1) Dimensions," Phys. Rev. Lett. **117**, 016802 (2016).

[92] D. F. Mross, J. Alicea, and O. I. Motrunich, "Symmetry and duality in bosonization of two-dimensional Dirac fermions," Phys. Rev. X **7**, 041016 (2017).

[93] J.-Y. Chen, J. H. Son, C. Wang, and S. Raghu, "Exact Boson-Fermion Duality on a 3D Euclidean Lattice," Phys. Rev. Lett. **120**, 016602 (2018).

[94] J. H. Son, J.-Y. Chen, and S. Raghu, "Duality Web on a 3D Euclidean Lattice and Manifestation of Hidden Symmetries," JHEP **06**, 038 (2019), arXiv:1811.11367 [hep-th] .

[95] S. Kachru, M. Mulligan, G. Torroba, and H. Wang, "Mirror symmetry and the half-filled Landau level," Phys. Rev. B **92**, 235105 (2015).

[96] S. Kachru, M. Mulligan, G. Torroba, and H. Wang, "Bosonization and Mirror Symmetry," Phys. Rev. **D94**, 085009 (2016).

[97] S. Kachru, M. Mulligan, G. Torroba, and H. Wang, "Nonsupersymmetric dualities from mirror symmetry," Phys. Rev. Lett. **118**, 011602 (2017).

[98] P.-S. Hsin and N. Seiberg, "Level/rank Duality and Chern-Simons-Matter Theories," JHEP **09**, 095 (2016), arXiv:1607.07457 [hep-th] .

[99] H. Goldman and E. Fradkin, "Loop Models, Modular Invariance, and Three Dimensional Bosonization," Phys. Rev. B **97**, 195112 (2018), arXiv:1801.04936 [cond-mat.str-el] .

[100] X. G. Wen and Q. Niu, "Ground-state degeneracy of the fractional quantum Hall states in the presence of a random potential and on high-genus Riemann surfaces," Phys. Rev. B **41**, 9377 (1990).

[101] X. G. Wen, "Topological Orders in Rigid States," Int. J. Mod. Phys. B **4**, 239 (1990).

[102] X. G. Wen, "Topological Orders and Edge Excitations in Fractional Quantum Hall States," Adv. Phys. **44**, 405 (1995).

[103] G. Moore and N. Read, "Non-Abelions in the Fractional Quantum Hall Effect," Nucl. Phys. B **360**, 362 (1991).

[104] A. M. Polyakov, "Fermi-Bose Transmutations Induced By Gauge Fields," Mod. Phys. Lett. A **3**, 325–328 (1988).

[105] C. P. Burgess and F. Quevedo, "Bosonization as duality," Nucl. Phys. B **421**, 373 (1993).

[106] E. Fradkin and F. A. Schaposnik, "The fermion-boson mapping in three-dimensional quantum field theory," Physics Letters B **338**, 253 – 258 (1994).

[107] J. C. Le Guillou, E. Moreno, F. A. Schaposnik, and C. Núñez, "On Three Dimensional Bosonization," Physics Letters B **409**, 257 (1997).

[108] K. Symanzik, "Euclidean Quantum Field Theory. I. Equations for a Scalar Model," Journal of Mathematical Physics **7**, 510–525 (1966).

[109] K. Symanzik, *Euclidean Quantum Field Theory* (Academic Press, New York, London, 1969) Chapter in *Local Quantum Field Theory*, R. Jost ed.

[110] J. Fröhlich, "On the triviality of $\lambda\phi_d^4$ theories and the approach of the critical point in $d > 4$ dimensions," Nuclear Physics B **200**, 281 (1982).

[111] C. A. Lütken and G. G. Ross, "Duality in the Quantum Hall System," Phys. Rev. B **45**, 11837 (1992).

[112] C. A. Lütken and G. G. Ross, "Delocalization, duality, and scaling in the quantum Hall system," Phys. Rev. B **48**, 2500 (1993).

[113] C. P. Burgess and B. P. Dolan, "Particle-vortex duality and the modular group: Applications to the quantum Hall effect and other two-dimensional systems," Phys. Rev. B **63**, 155309 (2001).

[114] A. Hui, E.-A. Kim, and M. Mulligan, "Superuniversality and non-abelian bosonization in $2 + 1$ dimensions," (2017), arXiv:1712.04942 [cond-mat.str-el] .

[115] G. 't Hooft, "Topology of the gauge conditions and new confinement phases in non-abelian gauge theories," Nucl. Phys. B **190**, 455 (1981).

[116] J. L. Cardy and E. Rabinovici, "Phase structure of Zp models in the presence of a θ parameter," Nuclear Physics B **205**, 1 – 16 (1982).

[117] J. L. Cardy, "Duality and the θ parameter in Abelian lattice models," Nuclear Physics B **205**, 17 – 26 (1982).

[118] S. D. Geraedts and O. I. Motrunich, "Phases and phase transitions in a $U(1) \times U(1)$ system with $\theta = 2\pi/3$ mutual statistics," Phys. Rev. B **86**, 045106 (2012).

[119] S. D. Geraedts and O. I. Motrunich, "Monte Carlo study of a $U(1) \times U(1)$ loop model with modular invariance," Phys. Rev. B **86**, 245121 (2012).

[120] S. D. Geraedts and O. I. Motrunich, "Exact realization of integer and fractional quantum Hall phases in $U(1) \times U(1)$ models in (2+1)d," Annals of Physics **334**, 288 – 315 (2013).

[121] E. Witten, "SL(2,Z) Action On Three-Dimensional Conformal Field Theories With Abelian Symmetry," (2003), published in *From Fields to Strings* vol. 2, p. 1173, M. Shifman *et al*, eds., World Scientific, 2005, arXiv:0307041 .

[122] R. G. Leigh and A. C. Petkou, "SL(2,Z) action on three-dimensional CFTs and holography," J. High Energy Phys. **12**, 020 (2003).

[123] J. K. Jain, S. A. Kivelson, and N. Trivedi, "Scaling theory of the fractional quantum Hall effect," Phys. Rev. Lett. **64**, 1297–1300 (1990).

[124] E. Fradkin, C. Nayak, A. Tsvelik, and F. Wilczek, "A Chern-Simons effective field theory for the Pfaffian quantum Hall state," Nucl. Phys. B **516**, 704 (1998).

[125] K. Damle and S. Sachdev, "Nonzero-temperature transport near quantum critical points," Phys. Rev. B **56**, 8714–8733 (1997).

[126] W.-H. Hsiao and D. T. Son, "Duality and Universal Transport in a Mixed-Dimension Electrodynamics," Phys. Rev. B **96**, 075127 (2017).

[127] J. Maciejko, X.-L. Qi, A. Karch, and S.-C. Zhang, "Fractional Topological Insulators in Three Dimensions," Phys. Rev. Lett. **105**, 246809 (2010).

[128] J. Maciejko, X.-L. Qi, A. Karch, and S.-C. Zhang, "Models of three-dimensional fractional topological insulators," Phys. Rev. B **86**, 235128 (2012).

[129] J. Maciejko and G. A. Fiete, "Fractionalized topological insulators," Nat Phys **11**, 385–388 (2015).

[130] B. Swingle, M. Barkeshli, J. McGreevy, and T. Senthil, "Correlated topological insulators and the fractional magnetoelectric effect," Phys. Rev. B **83**, 195139 (2011).

[131] A.-M. Chan, T. L. Hughes, S. Ryu, and E. Fradkin, "Effective field theories for topological insulators by functional bosonization," Phys. Rev. B **87**, 085132 (2013).

[132] P. Ye and X.-G. Wen, "Constructing symmetric topological phases of bosons in three dimensions via fermionic projective construction and dyon condensation," Phys. Rev. B **89**, 045127 (2014).

[133] P. Ye, T. L. Hughes, J. Maciejko, and E. Fradkin, "Composite particle theory of three-dimensional gapped fermionic phases: Fractional topological insulators and charge-loop excitation symmetry," Phys. Rev. B **94**, 115104 (2016).

[134] G. Y. Cho, J. C. Y. Teo, and E. Fradkin, "Symmetric-gapped surface states of fractional topological insulators," Phys. Rev. B **96**, 161109 (2017).

[135] P. Ye, M. Cheng, and E. Fradkin, "Fractional S-duality, classification of fractional topological insulators, and surface topological order," Phys. Rev. B **96**, 085125 (2017).

[136] S. D. Geraedts and O. I. Motrunich, "Line of continuous phase transitions in a three-dimensional U(1) loop model with $1/r^2$ current-current interactions," Phys. Rev. B **85**, 144303 (2012).

[137] C. P. Herzog, P. Kovtun, S. Sachdev, and D. T. Son, "Quantum critical transport, duality, and m theory," Phys. Rev. D **75**, 085020 (2007).

[138] C. Closset, T. T. Dumitrescu, G. Festuccia, Z. Komargodski, and N. Seiberg, "Comments on Chern-Simons contact terms in three dimensions," J. High Energy Phys. **9**, 91 (2012).

[139] A. M. Polyakov, *Gauge Fields and Strings*, Contemporary Concepts in Physics, Vol. 3 (Harwood Academic Publishers, Chur, Switzerland, 1987).

[140] A. M. Polyakov, "Two-dimensional quantum gravity: Superconductivity at high T_c," in *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena Les Houches, France, June 28-August 5, 1988* (1988) pp. 305–368.

[141] J. Grundberg, T. Hansson, and A. Karlhede, "On Polyakov's spin factors," Nuclear Physics B **347**, 420 – 440 (1990).

[142] N. Shaji, R. Shankar, and M. Sivakumar, "On Bose-Fermi Equivalence in a U(1) Gauge Theory with Chern-Simons Action," Mod. Phys. Lett. A **5**, 593–603 (1990).

[143] D. Bar-Natan and E. Witten, "Perturbative expansion of Chern-Simons theory with noncompact gauge group," Comm. Math. Phys. **141**, 423–440 (1991).

[144] J. H. White, "Self-Linking and the Gauss Integral in Higher Dimensions," American Journal of Mathematics **91**, 693–728 (1969).

[145] Y. Ferreirós and E. Fradkin, "Boson-Fermion Duality in Three Dimenions," (2018), in preparation.

[146] A. Y. Alekseev and S. Shatashvili, "Propagator for the Relativistic Spinning Particle via Functional Integral Over Trajectories," Modern Physics Letters A **03**, 1551–1559 (1988).

[147] J. Grundberg, T. Hansson, A. Karlhede, and U. Lindström, "Spin, statistics and linked loops," Physics Letters B **218**, 321 – 325 (1989).

[148] P. Orland, "Bosonic Path Integrals for Four-dimensional Dirac Particles," Int. J. Mod. Phys. **A4**, 3615 (1989).

[149] T. Hansson, A. Karlhede, and M. Roček, "On Wilson loops in Abelian Chern-Simons theories," Physics Letters B **225**, 92 – 94 (1989).

[150] R. E. Gamboa Saraví, G. L. Rossini, and F. A. Schaposnik, "The ζ function answer to parity violation in three-dimensional gauge theories," Int. J. Mod. Phys. A **11**, 2643 (1996).

[151] C. Wang, N. R. Cooper, B. I. Halperin, and A. Stern, "Particle-hole symmetry in the fermion-chern-simons and dirac descriptions of a half-filled landau level," Phys. Rev. X **7**, 031029 (2017).

[152] N. Read, "Theory of the half-filled landau level," Semiconductor Science and Technology **9**, 1859 (1994).

[153] S. M. Girvin, "Particle-hole symmetry in the anomalous quantum hall effect," Phys. Rev. B **29**, 6012–6014 (1984).

[154] L. W. Wong, H. W. Jiang, and W. J. Schaff, "Universality and phase diagram around half-filled landau levels," Phys. Rev. B **54**, R17323–R17326 (1996).

[155] A. Hui, E.-A. Kim, and M. Mulligan, "Superuniversality and non-abelian bosonization in 2 + 1 dimensions," (2017), arXiv:1712.04942 [cond-mat.str-el] .

[156] M. Mulligan, S. Raghu, and M. P. A. Fisher, "Emergent particle-hole symmetry in the half-filled landau level," Phys. Rev. B **94**, 075101 (2016).

[157] S. D. Geraedts, M. P. Zaletel, R. S. K. Mong, M. A. Metlitski, A. Vishwanath, and O. I. Motrunich, "The half-filled Landau level: The case for Dirac composite fermions," Science **352**, 197–201 (2016).

[158] S. D. Geraedts, J. Wang, E. Rezayi, and F. D. M. Haldane, "Berry phase and model wavefunction in the half-filled Landau Level," ArXiv e-prints (2017), arXiv:1711.07864 [cond-mat.str-el] .

[159] Z. Wang and S. Chakravarty, "Pairing of particle-hole symmetric composite fermions in half-filled landau level," Phys. Rev. B **94**, 165138 (2016).

[160] Y. You, "Partially filled landau level at even denominators: A vortex metal with a berry phase," Phys. Rev. B **97**, 165115 (2018).

[161] J. Wang, "A Dirac Fermion Hierarchy of Composite Fermi Liquids," (2018), arXiv:1808.07529 [cond-mat.mes-hall] .

[162] F. D. M. Haldane, "Berry curvature on the fermi surface: Anomalous hall effect as a topological fermi-liquid property," Phys. Rev. Lett. **93**, 206602 (2004).

[163] A. Pruisken, "On localization in the theory of the quantized hall effect: A two-dimensional realization of the θ-vacuum," Nuclear Physics B **235**, 277 – 298 (1984).

[164] A. M. M. Pruisken, "Dilute instanton gas as the precursor to the integral quantum hall effect," Phys. Rev. B **32**, 2636–2639 (1985).

[165] X. G. Wen and A. Zee, "Shift and spin vector: New topological quantum numbers for the hall fluids," Phys. Rev. Lett. **69**, 953–956 (1992).

[166] N. Read and E. H. Rezayi, "Hall viscosity, orbital spin, and geometry: Paired superfluids and quantum hall systems," Phys. Rev. B **84**, 085316 (2011).

[167] C. Hoyos and D. T. Son, "Hall Viscosity and Electromagnetic Response," Phys. Rev. Lett. **108**, 066805 (2012), arXiv:1109.2651 [cond-mat.mes-hall] .

[168] B. Bradlyn, M. Goldstein, and N. Read, "Kubo formulas for viscosity: Hall viscosity, ward identities, and the relation with conductivity," Phys. Rev. B **86**, 245309 (2012).

[169] G. Y. Cho, Y. You, and E. Fradkin, "Geometry of fractional quantum hall fluids," Phys. Rev. B **90**, 115139 (2014).

[170] A. Gromov, G. Y. Cho, Y. You, A. G. Abanov, and E. Fradkin, "Framing anomaly in the effective theory of the fractional quantum hall effect," Phys. Rev. Lett. **114**, 016805 (2015).

[171] M. Levin and D. T. Son, "Particle-hole symmetry and electromagnetic response of a half-filled landau level," Phys. Rev. B **95**, 125120 (2017).

[172] Y. You, G. Y. Cho, and E. Fradkin, "Nematic quantum phase transition of composite fermi liquids in half-filled landau levels and their geometric response," Phys. Rev. B **93**, 205401 (2016).

[173] D. X. Nguyen, A. Gromov, and D. T. Son, "Fractional quantum hall systems near nematicity: Bimetric theory, composite fermions, and dirac brackets," Phys. Rev. B **97**, 195103 (2018).

[174] L. V. Delacrétaz and A. Gromov, "Transport signatures of the hall viscosity," Phys. Rev. Lett. **119**, 226602 (2017).

[175] P. Goswami, H. Goldman, and S. Raghu, "Metallic phases from disordered (2+1)-dimensional quantum electrodynamics," Phys. Rev. B **95**, 235145 (2017), arXiv:1701.07828 [cond-mat.str-el] .

[176] A. Thomson and S. Sachdev, "Quantum electrodynamics in 2+1 dimensions with quenched disorder: Quantum critical states with interactions and disorder," Phys. Rev. B **95**, 235146 (2017).

[177] N. Read and D. Green, "Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum hall effect," Phys. Rev. B **61**, 10267–10297 (2000).

[178] M. Banerjee, M. Heiblum, V. Umansky, D. E. Feldman, Y. Oreg, and A. Stern, "Observation of half-integer thermal Hall conductance," Nature **559**, 205–210 (2018), arXiv:1710.00492 [cond-mat.mes-hall] .

[179] M. Storni, R. H. Morf, and S. Das Sarma, "Fractional quantum hall state at $\nu = \frac{5}{2}$ and the moore-read pfaffian," Phys. Rev. Lett. **104**, 076803 (2010).

[180] M. P. Zaletel, R. S. K. Mong, F. Pollmann, and E. H. Rezayi, "Infinite density matrix renormalization group for multicomponent quantum hall systems," Phys. Rev. B **91**, 045115 (2015).

[181] E. H. Rezayi, "Landau level mixing and the ground state of the $\nu = 5/2$ quantum hall effect," Phys. Rev. Lett. **119**, 026801 (2017).

[182] R. V. Mishmash, D. F. Mross, J. Alicea, and O. I. Motrunich, "Numerical exploration of trial wave functions for the particle-hole-symmetric pfaffian," Phys. Rev. B **98**, 081107(R) (2018).

[183] G. Moore and N. Read, "Nonabelions in the fractional quantum hall effect," Nuclear Physics B **360**, 362 – 396 (1991).

[184] N. Read and E. Rezayi, "Beyond paired quantum hall states: Parafermions and incompressible states in the first excited landau level," Phys. Rev. B **59**, 8084–8092 (1999).

[185] E. Ardonne, R. Kedem, and M. Stone, "Filling the Bose sea: symmetric quantum Hall edge states and affine characters," Journal of Physics A: Mathematical and General **38**, 617–636 (2004).

[186] X. G. Wen, "Non-abelian statistics in the fractional quantum hall states," Phys. Rev. Lett. **66**, 802–805 (1991).

[187] X.-G. Wen, "Projective construction of non-abelian quantum hall liquids," Phys. Rev. B **60**, 8827–8838 (1999).

[188] M. Barkeshli and X.-G. Wen, "Effective field theory and projective construction for Z_k parafermion fractional quantum hall states," Phys. Rev. B **81**, 155302 (2010).

[189] X.-G. Wen, "Continuous topological phase transitions between clean quantum hall states," Phys. Rev. Lett. **84**, 3950–3953 (2000).

[190] D. C. Cabra, E. Fradkin, G. L. Rossini, and F. A. Schaposnik, "Non-Abelian fractional quantum Hall states and chiral coset conformal field theories," International Journal of Modern Physics A **15**, 4857–4870 (2000).

[191] D. Cabra, A. Lopez, and G. Rossini, "Transition from abelian to non-abelian fqhe states," The European Physical Journal B **19**, 21–24 (2001).

[192] E. Rezayi, X.-G. Wen, and N. Read, "Condensation of fractional excitons, non-Abelian states in double-layer quantum Hall systems and Z_4 parafermions," arXiv e-prints , arXiv:1007.2022 (2010), arXiv:1007.2022 [cond-mat.mes-hall] .

[193] M. Barkeshli and X.-G. Wen, "Anyon condensation and continuous topological phase transitions in non-abelian fractional quantum hall states," Phys. Rev. Lett. **105**, 216804 (2010).

[194] M. Barkeshli and X.-G. Wen, "Bilayer quantum hall phase transitions and the orbifold non-abelian fractional quantum hall states," Phys. Rev. B **84**, 115121 (2011).

[195] A. Vaezi and M. Barkeshli, "Fibonacci anyons from abelian bilayer quantum hall states," Phys. Rev. Lett. **113**, 236804 (2014).

[196] E. H. Fradkin, C. Nayak, and K. Schoutens, "Landau-Ginzburg theories for nonAbelian quantum Hall states," Nucl. Phys. **B546**, 711–730 (1999), arXiv:cond-mat/9811005 [cond-mat] .

[197] E. Ardonne and K. Schoutens, "New class of non-abelian spin-singlet quantum hall states," Phys. Rev. Lett. **82**, 5096–5099 (1999).

[198] A. Cappelli, L. S. Georgiev, and I. T. Todorov, "Parafermion hall states from coset projections of abelian conformal theories," Nuclear Physics B **599**, 499 – 530 (2001).

[199] E. Ardonne, P. Bouwknegt, and P. Dawson, "K-matrices for 2d conformal field theories," Nuclear Physics B **660**, 473 – 531 (2003).

[200] J. C. Y. Teo and C. L. Kane, "From luttinger liquid to non-abelian quantum hall states," Phys. Rev. B **89**, 085101 (2014).

[201] Y. Fuji and P. Lecheminant, "Non-abelian $su(n-1)$-singlet fractional quantum hall states from coupled wires," Phys. Rev. B **95**, 125130 (2017).

[202] Z. Papić, M. O. Goerbig, N. Regnault, and M. V. Milovanović, "Tunneling-driven breakdown of the 331 state and the emergent pfaffian and composite fermi liquid phases," Phys. Rev. B **82**, 075302 (2010).

[203] M. R. Peterson, Y.-L. Wu, M. Cheng, M. Barkeshli, Z. Wang, and S. Das Sarma, "Abelian and non-abelian states in $\nu = 2/3$ bilayer fractional quantum hall systems," Phys. Rev. B **92**, 035103 (2015).

[204] S. Geraedts, M. P. Zaletel, Z. Papić, and R. S. K. Mong, "Competing abelian and non-abelian topological orders in $\nu = 1/3 + 1/3$ quantum hall bilayers," Phys. Rev. B **91**, 205139 (2015).

[205] Z. Liu, A. Vaezi, K. Lee, and E.-A. Kim, "Non-abelian phases in two-component $\nu = 2/3$ fractional quantum hall states: Emergence of fibonacci anyons," Phys. Rev. B **92**, 081102(R) (2015).

[206] W. Zhu, S. S. Gong, D. N. Sheng, and L. Sheng, "Possible non-abelian moore-read state in double-layer bosonic fractional quantum hall system," Phys. Rev. B **91**, 245126 (2015).

[207] W. Zhu, Z. Liu, F. D. M. Haldane, and D. N. Sheng, "Fractional quantum hall bilayers at half filling: Tunneling-driven non-abelian phase," Phys. Rev. B **94**, 245147 (2016).

[208] V. Crépel, B. Estienne, and N. Regnault, "Variational Ansatz for an Abelian to non-Abelian Topological Phase Transition in $\nu = 1/2 + 1/2$ Bilayers," (2019), arXiv:1904.01589 .

[209] S. G. Naculich and H. J. Schnitzer, "Duality between $SU(N)_k$ and $SU(k)_N$ WZW models," Nuclear Physics B **347**, 687 – 742 (1990).

[210] S. Naculich, H. Riggs, and H. Schnitzer, "Group-level duality in WZW models and Chern-Simons theory," Physics Letters B **246**, 417 – 422 (1990).

[211] M. Camperi, F. Levstein, and G. Zemba, "The large N limit of Chern-Simons gauge theory," Physics Letters B **247**, 549 – 554 (1990).

[212] J. W. Reijnders, F. J. M. van Lankvelt, K. Schoutens, and N. Read, "Quantum hall states and boson triplet condensate for rotating spin-1 bosons," Phys. Rev. Lett. **89**, 120401 (2002).

[213] E. Ardonne, N. Read, E. Rezayi, and K. Schoutens, "Non-abelian spin-singlet quantum hall states: wave functions and quasihole state counting," Nuclear Physics B **607**, 549 – 576 (2001).

[214] D. Gepner, "New conformal field theories associated with lie algebras and their partition functions," Nuclear Physics B **290**, 10 – 24 (1987).

[215] E. Fradkin and L. P. Kadanoff, "Disorder variables and para-fermions in two-dimensional statistical mechanics," Nuclear Physics B **170**, 1 (1980).

[216] J. W. Reijnders, F. J. M. van Lankvelt, K. Schoutens, and N. Read, "Rotating spin-1 bosons in the lowest landau level," Phys. Rev. A **69**, 023612 (2004).

[217] A. Sterdyniak, C. Repellin, B. A. Bernevig, and N. Regnault, "Series of abelian and non-abelian states in $c > 1$ fractional chern insulators," Phys. Rev. B **87**, 205137 (2013).

[218] N. Read, "Excitation structure of the hierarchy scheme in the fractional quantum hall effect," Phys. Rev. Lett. **65**, 1502–1505 (1990).

[219] J. Fröhlich and A. Zee, "Large scale physics of the quantum Hall fluid," Nuclear Physics B **364**, 517 (1991).

[220] R. Jackiw and P. Rossi, "Zero modes of the vortex-fermion system," Nuclear Physics B **190**, 681 – 691 (1981).

[221] O. Aharony, F. Benini, P.-S. Hsin, and N. Seiberg, "Chern-Simons-matter dualities with SO and USp gauge groups," JHEP **02**, 072 (2017), arXiv:1611.07874 .

[222] M. Barkeshli and J. McGreevy, "Continuous transition between fractional quantum hall and superfluid states," Phys. Rev. B **89**, 235116 (2014).

[223] F. Benini, P.-S. Hsin, and N. Seiberg, "Comments on global symmetries, anomalies, and duality in (2 + 1)d," JHEP **04**, 135 (2017), arXiv:1702.07035 [cond-mat.str-el] .

[224] C. Cordova, P.-S. Hsin, and N. Seiberg, "Global Symmetries, Counterterms, and Duality in Chern-Simons Matter Theories with Orthogonal Gauge Groups," SciPost Phys. **4**, 021 (2018), arXiv:1711.10008 [hep-th] .

[225] S. Chatterjee, H. Guo, S. Sachdev, R. Samajdar, M. S. Scheurer, N. Seiberg, and C. Xu, "Field-induced transition to semion topological order from the square-lattice Néel state," (2019), arXiv:1903.01992 [cond-mat.str-el] .

[226] N. Seiberg and E. Witten, "Gapped boundary phases of topological insulators via weak coupling," Progress of Theoretical and Experimental Physics **2016**, 12C101 (2016).

[227] B. I. Halperin, "Theory of the Quantized Hall Conductance," Helv. Phys. Acta **56**, 75 (1983).

[228] K. Schoutens, E. Ardonne, and F. J. M. van Lankvelt, "Paired and clustered quantum hall states," in *Statistical Field Theories*, edited by A. Cappelli and G. Mussardo (Springer Netherlands, Dordrecht, 2002) pp. 305–316.

[229] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer-Verlag, Berlin, 1997).

[230] A. Karch, B. Robinson, and D. Tong, "More Abelian Dualities in 2+1 Dimensions," JHEP **01**, 017 (2017), arXiv:1609.04012 [hep-th] .

[231] J. Y. Lee, C. Wang, M. P. Zaletel, A. Vishwanath, and Y.-C. He, "Emergent multi-flavor qed$_3$ at the plateau transition between fractional chern insulators: Applications to graphene heterostructures," Phys. Rev. X **8**, 031015 (2018).

[232] E. Ardonne, F. J. M. van Lankvelt, A. W. W. Ludwig, and K. Schoutens, "Separation of spin and charge in paired spin-singlet quantum hall states," Phys. Rev. B **65**, 041305(R) (2002).

[233] C. Cordova, P.-S. Hsin, and K. Ohmori, "Exceptional Chern-Simons-Matter Dualities," arXiv e-prints , arXiv:1812.11705 (2018), arXiv:1812.11705 [hep-th] .

[234] A. Kapustin and N. Seiberg, "Coupling a QFT to a TQFT and duality," J. High Energy Phys. **2014**, 1 (2014).

[235] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, "Generalized global symmetries," Journal of High Energy Physics **2015**, 172 (2015).

[236] H. Yerzhakov and J. Maciejko, "Disordered fermionic quantum critical points," Phys. Rev. B **98**, 195142 (2018).

[237] S. Dorogovtsev, "Critical exponents of magnets with lengthy defects," Physics Letters A **76**, 169 – 170 (1980).

[238] D. Boyanovsky and J. L. Cardy, "Critical behavior of m-component magnets with correlated impurities," Phys. Rev. B **26**, 154–170 (1982).

[239] I. D. Lawrie and V. V. Prudnikov, "Static and dynamic properties of systems with extended defects: two-loop approximation," Journal of Physics C: Solid State Physics **17**, 1655–1668 (1984).

[240] T. R. Kirkpatrick and D. Belitz, "Long-range order versus random-singlet phases in quantum antiferromagnetic systems with quenched disorder," Phys. Rev. Lett. **76**, 2571–2574 (1996).

[241] S. A. Hartnoll and J. E. Santos, "Disordered horizons: Holography of randomly disordered fixed points," Phys. Rev. Lett. **112**, 231601 (2014).

[242] S. A. Hartnoll, D. M. Ramirez, and J. E. Santos, "Thermal conductivity at a disordered quantum critical point," Journal of High Energy Physics **2016**, 22 (2016).

[243] O. Aharony and V. Narovlansky, "Renormalization group flow in field theories with quenched disorder," Phys. Rev. D **98**, 045012 (2018).

[244] V. Narovlansky and O. Aharony, "Renormalization group in field theories with quantum quenched disorder," Phys. Rev. Lett. **121**, 071601 (2018).

[245] B. C. Crooker, B. Hebral, E. N. Smith, Y. Takano, and J. D. Reppy, "Superfluidity in a dilute bose gas," Phys. Rev. Lett. **51**, 666–669 (1983).

[246] J. Reppy, "4he as a dilute bose gas," Physica B+C **126**, 335 – 341 (1984).

[247] M. H. W. Chan, K. I. Blum, S. Q. Murphy, G. K. S. Wong, and J. D. Reppy, "Disorder and the superfluid transition in liquid ^{4}He," Phys. Rev. Lett. **61**, 1950–1953 (1988).

[248] H. S. J. van der Zant, F. C. Fritschy, W. J. Elion, L. J. Geerligs, and J. E. Mooij, "Field-induced superconductor-to-insulator transitions in josephson-junction arrays," Phys. Rev. Lett. **69**, 2971–2974 (1992).

[249] H. S. J. van der Zant, W. J. Elion, L. J. Geerligs, and J. E. Mooij, "Quantum phase transitions in two dimensions: Experiments in josephson-junction arrays," Phys. Rev. B **54**, 10081–10093 (1996).

[250] A. Oosawa and H. Tanaka, "Random bond effect in the quantum spin system $(\mathrm{tl}_{1-x}\mathrm{k}_x)\mathrm{cucl}_3$," Phys. Rev. B **65**, 184437 (2002).

[251] T. Hong, A. Zheludev, H. Manaka, and L.-P. Regnault, "Evidence of a magnetic bose glass in $(\mathrm{CH}_3)_2\mathrm{chnh}_3\mathrm{Cu}(\mathrm{Cl}_{0.95}\mathrm{Br}_{0.05})_3$ from neutron diffraction," Phys. Rev. B **81**, 060410(R) (2010).

[252] R. Yu, L. Yin, N. S. Sullivan, J. Xia, C. Huan, A. Paduan-Filho, N. F. Oliveira, S. Haas, A. Steppke, C. F. Miclea, F. Weickert, R. Movshovich, E. Mun, B. L. Scott, V. Zapf, and T. Roscilde, "Bose glass and mott glass of quasiparticles in a doped quantum magnet," Nature **489**, 379–384 (2012).

[253] M. White, M. Pasienski, D. McKay, S. Q. Zhou, D. Ceperley, and B. DeMarco, "Strongly interacting bosons in a disordered optical lattice," Phys. Rev. Lett. **102**, 055301 (2009).

[254] S. Krinner, D. Stadler, J. Meineke, J.-P. Brantut, and T. Esslinger, "Superfluidity with disorder in a thin film of quantum gas," Phys. Rev. Lett. **110**, 100601 (2013).

[255] C. D'Errico, E. Lucioni, L. Tanzi, L. Gori, G. Roux, I. P. McCulloch, T. Giamarchi, M. Inguscio, and G. Modugno, "Observation of a disordered bosonic insulator from weak to strong interactions," Phys. Rev. Lett. **113**, 095301 (2014).

[256] N. Prokof'ev and B. Svistunov, "Superfluid-insulator transition in commensurate disordered bosonic systems: Large-scale worm algorithm simulations," Phys. Rev. Lett. **92**, 015703 (2004).

[257] A. Priyadarshee, S. Chandrasekharan, J.-W. Lee, and H. U. Baranger, "Quantum phase transitions of hard-core bosons in background potentials," Phys. Rev. Lett. **97**, 115703 (2006).

[258] L. Pollet, N. V. Prokof'ev, B. V. Svistunov, and M. Troyer, "Absence of a direct superfluid to mott insulator transition in disordered bose systems," Phys. Rev. Lett. **103**, 140402 (2009).

[259] H. Meier and M. Wallin, "Quantum critical dynamics simulation of dirty boson systems," Phys. Rev. Lett. **108**, 055701 (2012).

[260] S. Iyer, D. Pekker, and G. Refael, "Mott glass to superfluid transition for random bosons in two dimensions," Phys. Rev. B **85**, 094202 (2012).

[261] R. Ng and E. S. Sørensen, "Quantum critical scaling of dirty bosons in two dimensions," Phys. Rev. Lett. **114**, 255701 (2015).

[262] T. Vojta, J. Crewse, M. Puschmann, D. Arovas, and Y. Kiselev, "Quantum critical behavior of the superfluid-mott glass transition," Phys. Rev. B **94**, 134501 (2016).

[263] T. Giamarchi, P. Le Doussal, and E. Orignac, "Competition of random and periodic potentials in interacting fermionic systems and classical equivalents: The mott glass," Phys. Rev. B **64**, 245119 (2001).

[264] P. B. Weichman and R. Mukhopadhyay, "Particle-hole symmetry and the dirty boson problem," Phys. Rev. B **77**, 214516 (2008).

[265] D. S. Fisher and M. P. A. Fisher, "Onset of superfluidity in random media," Phys. Rev. Lett. **61**, 1847–1850 (1988).

[266] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, "Boson localization and the superfluid-insulator transition," Phys. Rev. B **40**, 546–570 (1989).

[267] Y. B. Kim and X.-G. Wen, "Large-n renormalization-group study of the commensurate dirty-boson problem," Phys. Rev. B **49**, 4043–4052 (1994).

[268] M. B. Hastings, "Renormalization group for the large-n strongly commensurate dirty boson model," Phys. Rev. B **60**, 9755–9762 (1999).

[269] A. B. Harris, "Effect of random defects on the critical behaviour of ising models," Journal of Physics C: Solid State Physics **7**, 1671–1692 (1974).

[270] J. T. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, "Finite-size scaling and correlation lengths for disordered systems," Phys. Rev. Lett. **57**, 2999–3002 (1986).

[271] F. Kos, D. Poland, and D. Simmons-Duffin, "Bootstrapping the $o(n)$ vector models," Journal of High Energy Physics **2014**, 91 (2014).

[272] A. M. Polyakov, "Gauge Fields and Strings," Contemp. Concepts Phys. **3**, 1–301 (1987).

[273] M. Moshe and J. Zinn-Justin, "Quantum field theory in the large N limit: A Review," Phys. Rept. **385**, 69–228 (2003), arXiv:hep-th/0306133 [hep-th] .

[274] J. Zinn-Justin, "Quantum field theory and critical phenomena," Int. Ser. Monogr. Phys. **113**, 1–1054 (2002).

[275] M. E. Peskin, "Mandelstam 't Hooft Duality in Abelian Lattice Models," Annals Phys. **113**, 122 (1978).

[276] C. Dasgupta and B. I. Halperin, "Phase transition in a lattice model of superconductivity," Phys. Rev. Lett. **47**, 1556–1560 (1981).

[277] A. K. Nguyen and A. Sudbo, "Topological phase fluctuations, amplitude fluctuations, and criticality in extreme type-ii superconductors," Phys. Rev. B **60**, 15307–15331 (1999).

[278] K. Kajantie, M. Laine, T. Neuhaus, A. Rajantie, and K. Rummukainen, "Duality and scaling in 3-dimensional scalar electrodynamics," Nuclear Physics B **699**, 632 – 656 (2004).

[279] P. B. Weichman and R. Mukhopadhyay, "Critical dynamics of the dirty boson problem: Revisiting the equality $z = d$," Phys. Rev. Lett. **98**, 245701 (2007).

[280] E. Fradkin, "Critical behavior of disordered degenerate semiconductors. I. Models, symmetries, and formalism," Phys. Rev. B **33**, 3257–3262 (1986).

[281] E. Fradkin, "Critical behavior of disordered degenerate semiconductors. II. Spectrum and transport properties in mean-field theory," Phys. Rev. B **33**, 3263–3268 (1986).

[282] A. W. W. Ludwig, M. P. A. Fisher, R. Shankar, and G. Grinstein, "Integer quantum hall transition: An alternative approach and exact results," Phys. Rev. B **50**, 7526–7552 (1994).

[283] J. Ye, "Effects of weak disorders on quantum hall critical points," Phys. Rev. B **60**, 8290–8303 (1999).

[284] C.-J. Lee and M. Mulligan, "Scaling and Diffusion of Dirac Composite Fermions," (2019), arXiv:1912.12303 [cond-mat.str-el] .

[285] P. A. Lee and T. V. Ramakrishnan, "Disordered electronic systems," Rev. Mod. Phys. **57**, 287–337 (1985).

[286] W. Chen, G. W. Semenoff, and Y.-S. Wu, "Two loop analysis of nonAbelian Chern-Simons theory," Phys. Rev. **D46**, 5521–5539 (1992), arXiv:hep-th/9209005 [hep-th] .

[287] A. C. Petkou, "C_T and C_J up to next-to-leading order in {1}/{N} in the conformally invariant O(N) vector model for $2 < d < 4$," Physics Letters B **359**, 101–107 (1995), arXiv:hep-th/9506116 [hep-th] .

[288] A. Petkou, "Conserved Currents, Consistency Relations, and Operator Product Expansions in the Conformally Invariant O(N) Vector Model," Annals of Physics **249**, 180–221 (1996), arXiv:hep-th/9410093 [hep-th] .