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Improved estimates and a limit case for the electrostatic Klein–Gordon–Maxwell system

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We study the class of nonlinear Klein–Gordon–Maxwell systems describing a standing wave (charged matter field) in equilibrium with a purely electrostatic field. We improve some previous existence results in the case of an homogeneous nonlinearity. Moreover, we deal with a limit case, namely when the frequency of the standing wave is equal to the mass of the charged field; this case shows analogous features of the well-known 'zero-mass case' for scalar field equations.

1. Introduction

This paper is concerned with a class of Klein–Gordon–Maxwell systems written as follows:

$$\left.\begin{aligned} -\Delta u + [m^2 - (e\phi - \omega)^2]u - f'(u) = 0 \quad & \text{in } \mathbb{R}^3, \\ \Delta\phi = e(e\phi - \omega)u^2 \quad & \text{in } \mathbb{R}^3. \end{aligned}\right\} \tag{1.1}$$

This system was introduced in the pioneering work of Benci and Fortunato [5] in 2002. It represents a standing wave $\psi = u(x)e^{i\omega t}$ (charged matter field) in equilibrium with a purely electrostatic field $\boldsymbol{E} = -\nabla\phi(x)$. The constant $m \geqslant 0$ represents the mass of the charged field and e is the coupling constant introduced in the minimal coupling rule [20].

It is immediately seen that (1.1) deserves some interest as system if and only if $e \neq 0$ and $\omega \neq 0$; otherwise we get $\phi = 0$. Throughout the paper we are looking for *non-trivial solutions*, i.e. solutions such that $\phi \neq 0$.

Moreover, we point out that the sign of ω is not relevant for the existence of solutions. Indeed, if (u, ϕ) is a solution of (1.1) with a certain value of ω, then $(u, -\phi)$ is a solution corresponding to $-\omega$. So, without loss of generality, we shall

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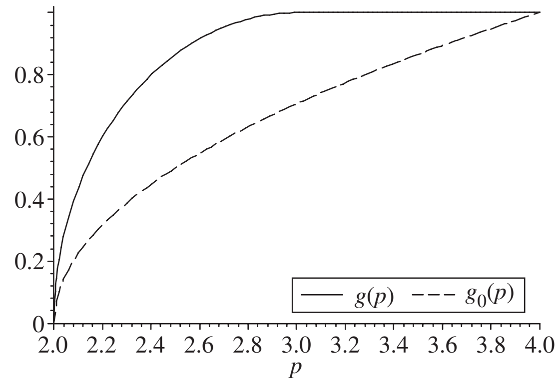

Figure 1. Comparison between $g(p)$ and $g_0(p)$.

assume $\omega > 0$. Analogously, the sign of e is not relevant, so we assume $e > 0$. Actually, the results we shall prove do not depend on the value of e.

Let us recall some previous results that led us to the present research. The first results are concerned with an homogeneous nonlinearity $f(t) = |t|^p/p$. Therefore, (1.1) becomes

$$\left.\begin{aligned} -\Delta u + [m^2 - (e\phi - \omega)^2]u - |u|^{p-2}u = 0 \quad & \text{in } \mathbb{R}^3, \\ \Delta\phi = e(e\phi - \omega)u^2 \quad & \text{in } \mathbb{R}^3. \end{aligned}\right\} \tag{1.2}$$

As stated above, the first result is due to Benci and Fortunato [5]. They showed the existence of infinitely many solutions whenever $p \in (4, 6)$ and $0 < \omega < m$.

In 2004, D'Aprile and Mugnai published two papers on this topic. In [16] they proved the existence of non-trivial solutions of (1.1) when $p \in (2, 4]$ and ω varies in a certain range depending on p:

$$0 < \omega < mg_0(p),$$

where

$$g_0(p) = \sqrt{\frac{p-2}{2}}.$$

Afterwards, in [17], the same authors showed that (1.1) has no non-trivial solutions if $p \geqslant 6$ and $\omega \in (0, m]$ (or $p \leqslant 2$).

Our first result gives a little improvement on problem (1.1) with $p \in (2, 4)$.

THEOREM 1.1. *Let $p \in (2, 4)$. Assume that $0 < \omega < mg(p)$, where*

$$g(p) = \begin{cases} \sqrt{(p-2)(4-p)} & \text{if } 2 < p < 3, \\ 1 & \text{if } 3 \leqslant p < 4. \end{cases}$$

Then (1.2) admits a non-trivial weak solution $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$.

It can immediately be seen that $g_0(p) < g(p)$, for any $p \in (2, 4)$; this is shown in Figure 1.

Under the above assumptions, the problem (1.2) is of a variational nature. Indeed, its weak solutions $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ can be characterized as critical points of the functional $\mathcal{S}\colon H^1(\mathbb{R}^3)\times\mathcal{D}^{1,2}(\mathbb{R}^3)\to\mathbb{R}$ defined as

$$\mathcal{S}(u,\phi)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2-|\nabla\phi|^2+[m_0^2-(\omega+e\phi)^2]u^2-\frac{1}{p}\int_{\mathbb{R}^3}|u|^p.$$

The first difficulty in dealing with the functional $\mathcal{S}$ is that it is strongly indefinite, namely it is unbounded both from below and from above on infinite-dimensional subspaces.

To avoid this indefiniteness, we will use a well-known reduction argument, stated in theorem 2.2. The finite energy solutions of (1.1) are pairs $(u,\phi_u)\in H^1(\mathbb{R}^3)\times\mathcal{D}^{1,2}(\mathbb{R}^3)$, where $\phi_u\in\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the unique solution of

$$\Delta\phi=e(e\phi-\omega)u^2\quad\text{in }\mathbb{R}^3 \tag{1.3}$$

(see lemma 2.1) and $u\in H^1(\mathbb{R}^3)$ is a critical point of

$$I(u)=\mathcal{S}(u,\phi_u).$$

The functional I no longer presents the strong indefiniteness. Under the assumptions of theorem 1.1, it will be studied by using an indirect method developed by Struwe [26] and Jeanjean [22].

In the second part of the paper we consider a more general nonlinearity $f(u)$.

Under usual assumptions, which describe behaviours analogous to $|t|^p$ (with $p\in(4,6)$), it is easy to obtain a generalization of the existence result [5] of Benci and Fortunato; we state this generalization in lemma 3.1. However, we point out that all the quoted results share the assumption $\omega<m$.

We are mainly interested in studying the limit case $\omega=m$, when (1.1) becomes

$$\left.\begin{aligned}-\Delta u+(2e\omega\phi-e^2\phi^2)u-f'(u)=0\quad&\text{in }\mathbb{R}^3,\\ \Delta\phi=e(e\phi-\omega)u^2\quad&\text{in }\mathbb{R}^3.\end{aligned}\right\} \tag{1.4}$$

If we assume, as above, $f''(0)=0$, we notice that the first equation in (1.4) has the form of a nonlinear Schrödinger equation with a potential vanishing at infinity. Indeed, if $\phi\in\mathcal{D}^{1,2}(\mathbb{R}^3)$ (and is radial), we have

$$\lim_{|x|\to\infty}(2e\omega\phi-e^2\phi^2)=0.$$

So we are in the so-called *zero-mass case* for nonlinear field equations (see, for example, [11] and [10]).

As in the cited papers, in order to get solutions, we need some stronger hypotheses on f, which force it to be inhomogeneous, with a supercritical growth near the origin and subcritical at infinity. More precisely, we assume that $f\colon\mathbb{R}\to\mathbb{R}$ satisfies the following assumptions:

(f1) $f\in C^1(\mathbb{R},\mathbb{R})$;

(f2) $\forall t\in\mathbb{R}\setminus\{0\}\colon\ \alpha f(t)\leqslant f'(t)t$;

(f3) $\forall t\in\mathbb{R}\colon\ f(t)\geqslant C_1\min(|t|^p,|t|^q)$;

(f4) $\forall t\in\mathbb{R}\colon\ |f'(t)|\leqslant C_2\min(|t|^{p-1},|t|^{q-1})$;

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with $4 < \alpha \leqslant p < 6 < q$ and C_1, C_2, positive constants. We shall prove the following result.

THEOREM 1.2. *Assume that f satisfies the above hypotheses. Then there exists a couple $(u_0, \phi_0) \in \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ that is a weak solution of (1.4).*

Under the assumptions of theorem 1.2, standard arguments (again lemma 3.1) yield the existence of $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ weak solutions of (1.1) in the case $\omega < m$. The limit case $\omega = m$ is trickier.

Even if the claim of theorem 1.2 is analogous to the cited existence results (e.g. theorem 1.1) and the meaning of 'weak solution' is the same, the approach in the proof is completely different. More precisely, in the zero-mass case, there exists no functional $\mathcal{S}$ defined on $\mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that its critical points are weak solutions of (1.4).

As above, we could consider a functional $\mathcal{S}\colon H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ whose critical points are finite energy weak solutions. For every $u \in H^1(\mathbb{R}^3)$ we can find $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$, the solution of (1.3). Then we could consider the reduced functional $I(u) = \mathcal{S}(u, \phi_u)$. The reduced functional I has the form Word added – OK?

$$I(u) = \mathcal{S}(u, \phi_u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + e\omega\phi_u u^2 - \int_{\mathbb{R}^3} f(u).$$

For such a functional the mountain-pass geometry in $H^1(\mathbb{R}^3)$ is not immediately available.

The solution $(u_0, \phi_0) \in \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ will be found as limit of solutions of approximating problems

$$\left.\begin{aligned} -\Delta u + (\varepsilon + 2e\omega\phi - e^2\phi^2)u - f'(u) = 0 \qquad & \text{in } \mathbb{R}^3, \\ \Delta\phi = e(e\phi - \omega)u^2 \quad & \text{in } \mathbb{R}^3. \end{aligned}\right\} \tag{1.5}$$

For every $\varepsilon > 0$, lemma 3.1 yields a solution $(u_\varepsilon, \phi_\varepsilon) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$. The stronger assumptions on f (subcritical at infinity, supercritical at zero) give rise to uniform estimate in $\mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ which allows one to pass to the limit as $\varepsilon \to 0$. A similar approach has been used in [4, 12].

Before giving the proof of theorems 1.1 and 1.2, let us recall some other results concerning (1.1). In [15] and [14] there are existence and non-existence results when f has a critical growth at infinity. The existence of a ground state for (1.2), under the existence assumptions of [16], is proved in [2]. Other recent papers (see, for example, [7, 25]) are concerned with the Klein–Gordon–Maxwell system with a completely different kind of nonlinearity, satisfying

$$\tfrac{1}{2}m^2t^2 - f(t) \geqslant 0.$$

The solutions in this case are called 'non-topological solitons'. In [7] the existence is proved of a non-trivial solution if the coupling constant e is sufficiently small. A perturbation theory, using the $e = 0$ as starting point, was developed in [23, 24]. There are also some results for the system (1.1) in a bounded spatial domain [13, 18, 19]. In this situation, existence and non-existence of non-trivial solutions depend on the boundary conditions, the boundary data, the kind of nonlinearity and the value of e. Lastly, we refer the reader to the review paper [21], which contains a large number of references on this topic, including existence results for other classes of Klein–Gordon–Maxwell systems, obtained with a more general *ansatz* (see, for example, [8, 9]).

In the following sections we shall prove theorems 1.1 and 1.2, respectively. The appendix contains the proof of a certain inequality, used in § 2, which involves only elementary calculus arguments.

2. Proof of theorem 1.1

We need the following.

LEMMA 2.1. *For any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ which satisfies*

$$\Delta\phi = e(e\phi - \omega)u^2 \quad \textit{in } \mathbb{R}^3.$$

Moreover, the map $\Phi\colon u \in H^1(\mathbb{R}^3) \mapsto \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ is continuously differentiable, and on the set $\{x \in \mathbb{R}^3 \mid u(x) \neq 0\}$

$$0 \leqslant \phi_u \leqslant \frac{\omega}{e}. \tag{2.1}$$

Proof. The proof can be found in [5, 17]. □

THEOREM 2.2. *The pair $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (1.2) if and only if u is a critical point of*

$$I(u) = \mathcal{S}(u, \phi_u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + (m^2 - \omega^2)u^2 + e\omega\phi_u u^2 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p,$$

and $\phi = \phi_u$.

For the sake of simplicity we set $\Omega = m^2 - \omega^2 > 0$.

With our assumptions, it is a hard task to find bounded Palais–Smale sequences of functional I. Therefore, we use an indirect method developed by Struwe [26] and Jeanjean [22]. We look for the critical points of the functional $I_\lambda \in C^1(H^1_r(\mathbb{R}^3), \mathbb{R})$

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \Omega u^2 + e\omega\phi_u u^2 - \frac{\lambda}{p}\int_{\mathbb{R}^3} |u|^p,$$

for λ close to 1, where

$$H^1_r(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) \mid u \text{ is radially symmetric}\}.$$

Set a positive number $\delta < 1$ (which we will estimate later), $J = [\delta, 1]$ and

$$\Gamma := \{\gamma \in C([0, 1], H^1_r(\mathbb{R}^3)) \mid \gamma(0) = 0,\ I_\lambda(\gamma(1)) < 0 \text{ for all } \lambda \in J\}.$$

Using a slightly modified version of [22, theorem 1.1], the following can be proved.

LEMMA 2.3. *If $\Gamma \neq \varnothing$ and, for every $\lambda \in J$,*

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > 0, \tag{2.2}$$

then for almost every $\lambda \in J$ there is a sequence $(v_n^\lambda)_n \subset H_r^1(\mathbb{R}^3)$ such that

(i) $(v_n^\lambda)_n$ *is bounded,*

(ii) $I_\lambda(v_n^\lambda) \to c_\lambda$,

(iii) $I'_\lambda(v_n^\lambda) \to 0$.

In order to apply theorem 2.3, we need only to verify that $\Gamma \neq \varnothing$ and (2.2) holds.

LEMMA 2.4. *For any $\lambda \in J$, we have that $\Gamma \neq \varnothing$.*

Proof. Let $u \in H_r^1(\mathbb{R}^3) \setminus \{0\}$ and let $\theta > 0$. Define $\gamma\colon [0,1] \to H_r^1(\mathbb{R}^3)$ such that $\gamma(t) = t\theta u$ for all $t \in [0,1]$. By (2.1), for any $\lambda \in J$, we have that

$$I_\lambda(\gamma(1)) = I_\lambda(\theta u) \leqslant \frac{\theta^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \Omega u^2 + \omega^2 u^2 - \delta\frac{\theta^p}{p}\int_{\mathbb{R}^3} |u|^p,$$

and then certainly $\gamma \in \Gamma$ for a suitable choice of θ. □

LEMMA 2.5. *For any $\lambda \in J$, we have that $c_\lambda > 0$.*

Proof. Observe that, for any $u \in H_r^1(\mathbb{R}^3)$ and $\lambda \in J$, by (2.1), we have

$$I_\lambda(u) \geqslant \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \Omega u^2 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p,$$

and then, by Sobolev embeddings, we conclude that there exists $\rho > 0$ such that, for any $\lambda \in J$ and $u \in H_r^1(\mathbb{R}^3)$ with $u \neq 0$ and $\|u\| \leqslant \rho$, we obtain $I_\lambda(u) > 0$. In particular, for any $\|u\| = \rho$, we have $I_\lambda(u) \geqslant \tilde{c} > 0$. Now fix $\lambda \in J$ and $\gamma \in \Gamma$. Since $\gamma(0) = 0 \neq \gamma(1)$ and $I_\lambda(\gamma(1)) \leqslant 0$, certainly $\|\gamma(1)\| > \rho$. By continuity, we deduce that there exists $t_\gamma \in (0,1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Therefore, for any $\lambda \in J$,

$$c_\lambda \geqslant \inf_{\gamma \in \Gamma} I_\lambda(\gamma(t_\gamma)) \geqslant \tilde{c} > 0.$$

□

Proof of theorem 1.1. Let $\lambda \in J$, for which there exists a bounded Palais–Smale sequence $(v_\lambda^n)_n$ in $H_r^1(\mathbb{R}^3)$ for functional I_λ at level c_λ, namely

$$I_\lambda(v_\lambda^n) \to c_\lambda;$$
$$I'_\lambda(v_\lambda^n) \to 0 \quad \text{in } (H_r^1(\mathbb{R}^3))'.$$

Up to a subsequence, we can suppose that there exists $v_\lambda \in H_r^1(\mathbb{R}^3)$ such that

$$v_\lambda^n \rightharpoonup v_\lambda \text{ weakly in } H_r^1(\mathbb{R}^3) \tag{2.3}$$

and

$$v_\lambda^n(x) \to v_\lambda(x) \text{ a.e. in } \mathbb{R}^N.$$

We make the following claims:

$$I'_\lambda(v_\lambda) = 0, \tag{2.4}$$
$$v_\lambda \neq 0,$$
$$I_\lambda(v_\lambda) \leqslant c_\lambda. \tag{2.5}$$

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Claim (2.4) follows immediately by [2, lemma 2.7].

Suppose by contradiction that $v_\lambda = 0$. Then, since $v_n^\lambda \to v_\lambda (\equiv 0)$ in $L^p(\mathbb{R}^3)$ and $I'_\lambda(v_n^\lambda)[v_n^\lambda] = o_n(1)\|v_n^\lambda\|$, we have

$$\int_{\mathbb{R}^3} |\nabla v_n^\lambda|^2 + \Omega(v_n^\lambda)^2 \leqslant \int_{\mathbb{R}^3} |\nabla v_n^\lambda|^2 + \Omega(v_n^\lambda)^2 + 2e\omega\phi_{v_n^\lambda}(v_n^\lambda)^2 - e^2\phi_{v_n^\lambda}^2(v_n^\lambda)^2$$
$$= \lambda \int_{\mathbb{R}^3} |v_n^\lambda|^p + o_n(1)\|v_n^\lambda\| = o_n(1).$$

Hence, $v_n^\lambda \to 0$ in $H^1(\mathbb{R}^3)$ and we get a contradiction with (2.2).

We pass to proving (2.5). Since $v_n^\lambda \to v_\lambda$ in $L^p(\mathbb{R}^3)$, by (2.3), by the weak lower semicontinuity of the $H^1(\mathbb{R}^3)$-norm and by Fatou's lemma, we get $I_\lambda(v_\lambda) \leqslant c_\lambda$.

Now we may consider a suitable $\lambda_n \nearrow 1$ such that, for any $n \geqslant 1$, there exists $v_n \in H_r^1(\mathbb{R}^3) \setminus \{0\}$ satisfying

$$I'_{\lambda_n}(v_n) = 0 \quad \text{in } (H_r^1(\mathbb{R}^3))', \tag{2.6}$$
$$I_{\lambda_n}(v_n) \leqslant c_{\lambda_n}. \tag{2.7}$$

We want to prove that such a sequence is bounded.

By [17], v_n satisfies the Pohozaev equality

$$\int_{\mathbb{R}^3} |\nabla v_n|^2 + 3\Omega v_n^2 + 5e\omega\phi_{v_n} v_n^2 - 2e^2\phi_{v_n}^2 v_n^2 - \frac{6\lambda_n}{p}\int_{\mathbb{R}^3} |v_n|^p = 0. \tag{2.8}$$

Therefore, by (2.6)–(2.8), the following system holds:

$$\int_{\mathbb{R}^3} \tfrac{1}{2}|\nabla v_n|^2 + \tfrac{1}{2}\Omega v_n^2 + \tfrac{1}{2}e\omega\phi_{v_n} v_n^2 - \frac{\lambda_n}{p}|v_n|^p \leqslant c_{\lambda_n}, \tag{2.9}$$
$$\int_{\mathbb{R}^3} |\nabla v_n|^2 + 3\Omega v_n^2 + 5e\omega\phi_{v_n} v_n^2 - 2e^2\phi_{v_n}^2 v_n^2 - \frac{6\lambda_n}{p}|v_n|^p = 0, \tag{2.10}$$
$$\int_{\mathbb{R}^3} |\nabla v_n|^2 + \Omega v_n^2 + 2e\omega\phi_{v_n} v_n^2 - e^2\phi_{v_n}^2 v_n^2 - \lambda_n|v_n|^p = 0. \tag{2.11}$$

Subtracting from (2.9) equation (2.10) multiplied by α and (2.11) multiplied by $(1-6\alpha)/p$, we get

$$\frac{p - 2\alpha p - 2 + 12\alpha}{2p}\int_{\mathbb{R}^3} |\nabla v_n|^2 + \int_{\mathbb{R}^3} [C_{p,\alpha}\Omega + B_{p,\alpha}e\omega\phi_{v_n} + A_{p,\alpha}e^2\phi_{v_n}^2]v_n^2 \leqslant c_{\lambda_n},$$

where

$$C_{p,\alpha} = \frac{(p-2)(1-6\alpha)}{2p},$$
$$B_{p,\alpha} = \frac{p - 10\alpha p - 4 + 24\alpha}{2p},$$
$$A_{p,\alpha} = \frac{1 + 2\alpha(p-3)}{p}.$$

It is easy to see that

$$\frac{p - 2\alpha p - 2 + 12\alpha}{2p} > 0,$$

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if and only if

$$\alpha > \frac{2-p}{2(6-p)}.$$

In the appendix (see lemma A.1) we will prove that there exists

$$\alpha \in \left(\frac{2-p}{2(6-p)}, \frac{1}{6}\right) : C_{p,\alpha}\Omega + B_{p,\alpha} e\omega\phi_{v_n} + A_{p,\alpha} e^2 \phi_{v_n}^2 \geqslant 0.$$

Then we can argue that

$$\|\nabla v_n\|_2 \leqslant C \quad \text{for all } n \geqslant 1. \tag{2.12}$$

Moreover, by (2.6), we have

$$\Omega \int_{\mathbb{R}^3} v_n^2 \leqslant \int_{\mathbb{R}^3} |\nabla v_n|^2 + \Omega v_n^2 + 2e\omega\phi_{v_n} v_n^2 - e^2\phi_{v_n}^2 v_n^2 = \lambda_n \int_{\mathbb{R}^3} |v_n|^p. \tag{2.13}$$

Since for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $t^p \leqslant C_\varepsilon t^6 + \varepsilon t^2$, for all $t \geqslant 0$, taking $\varepsilon = \Omega/2$, by (2.13) we get

$$\tfrac{1}{2}\Omega \int_{\mathbb{R}^3} v_n^2 \leqslant C_\varepsilon \int_{\mathbb{R}^3} v_n^6.$$

Therefore, by the Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and (2.12) we deduce that $(v_n)_n$ is bounded in $H^1(\mathbb{R}^3)$.

Up to a subsequence, there exists $v_0 \in H_r^1(\mathbb{R}^3)$ such that

$$v_n \rightharpoonup v_0 \text{ weakly in } H_r^1(\mathbb{R}^3).$$

By (2.6), we have that

$$I'(v_n) = (I_{\lambda_n})'(v_n) + (\lambda_n - 1)|v_n|^{p-2} v_n = (\lambda_n - 1)|v_n|^{p-2} v_n$$

so $(v_n)_n$ is a Palais–Smale sequence for the functional $I|_{H_r^1}$, since the sequence $(|v_n|^{p-2} v_n)_n$ is bounded in $(H_r^1(\mathbb{R}^3))'$.

By [2, lemma 2.7], we have that $I'(v_0) = 0$.

To conclude the proof, it remains to check that $v_0 \neq 0$.

By (2.6), we have

$$\int_{\mathbb{R}^3} |\nabla v_n|^2 + \Omega v_n^2 \leqslant \int_{\mathbb{R}^3} |\nabla v_n|^2 + \Omega v_n^2 + 2e\omega\phi_{v_n} v_n^2 - e^2\phi_{v_n}^2 v_n^2 \leqslant \int_{\mathbb{R}^3} |v_n|^p.$$

Then, there exists $C > 0$ such that $\|v_n\|_p \geqslant C$. Since $v_n \to v_0$ in $L^p(\mathbb{R}^3)$, the proof is concluded. □

3. Proof of theorem 1.2

The following lemma generalizes the existence result of [5].

Lemma 3.1. *Let f satisfy the following hypotheses:*

(f1) $f \in C^1(\mathbb{R}, \mathbb{R})$*;*

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(f2) $\exists \alpha > 4$ *such that* $\forall t \in \mathbb{R} \setminus \{0\}$: $\alpha f(t) \leqslant f'(t)t$;

(f5) $f'(t) = o(|t|)$ *as* $t \to 0$;

(f6) $\exists C_1, C_2 \geqslant 0$ *and* $p < 6$ *such that* $\forall t \in \mathbb{R}$: $|f'(t)| \leqslant C_1 + C_2|t|^{p-1}$.

Assume that $0 < \omega < m$. *Then (1.1) admits a non-trivial weak solution* $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$.

We simply give an outline of the proof.

- Using the same reduction argument (lemma 2.1 and theorem 2.2) applied to (1.1), it is immediately seen that $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (1.1) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + (m^2 - \omega^2)u^2 + e\omega\phi_u u^2 - \int_{\mathbb{R}^3} f(u),$$

  and $\phi = \phi_u$.

- The functional I satisfies the Palais–Smale condition in $H^1_r(\mathbb{R}^3)$.

- The functional I shows the mountain-pass geometry.

Remark 3.2. If f is odd, just like in [5], the $\mathbb{Z}_2$-mountain-pass theorem [1] yields infinitely many solutions.

Now we can prove theorem 1.2.

As stated in § 1, for every $\varepsilon > 0$, we consider the approximating problem (1.5). The above lemma gives the solution $(u_\varepsilon, \phi_\varepsilon) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$. More precisely these solution are found by means of the mountain-pass theorem and they are radially symmetric, in the sense that $u_\varepsilon \in H^1_r(\mathbb{R}^3)$ is a critical point of

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \varepsilon u^2 + e\omega\phi_u u^2 - \int_{\mathbb{R}^3} f(u),$$

at the level

$$c_\varepsilon = \inf_{g \in \Gamma_\varepsilon} \max_{\theta \in [0,1]} I_\varepsilon(g(\theta)),$$

where

$$\Gamma_\varepsilon = \{g \in C([0,1], H^1(\mathbb{R}^3)) \mid g(0) = 0,\ I_\varepsilon(g(1)) \leqslant 0,\ g(1) \neq 0\}.$$

Moreover, u_ε belongs to the Nehari manifold of I_ε:

$$\mathcal{N}_\varepsilon = \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} \;\middle|\; \int_{\mathbb{R}^3} |\nabla u|^2 + \varepsilon u^2 + 2e\omega\phi_u u^2 - e^2\phi_u^2 u^2 = \int_{\mathbb{R}^3} f'(u)u \right\}.$$

In the following, we will refer to those approximating solutions as *ε-solutions*.

Lemma 3.3. *There exists* $C > 0$ *such that* $c_\varepsilon < C$, *for any* $0 < \varepsilon \leqslant 1$.

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Proof. Fix $g \in \Gamma_1$ and let $0 < \varepsilon \leqslant 1$. Then, for a suitable θ_ε and since $g \in \Gamma_\varepsilon$, we have

$$c_\varepsilon \leqslant \max_{\theta \in [0,1]} I_\varepsilon(g(\theta)) = I_\varepsilon(g(\theta_\varepsilon)) \leqslant I_1(g(\theta_\varepsilon)) \leqslant \max_{\theta \in [0,1]} I_1(g(\theta)).$$

□

Lemma 3.4. *There exists $C > 0$ such that $\|u_\varepsilon\|_{\mathcal{D}^{1,2}} \geqslant C$, for any $\varepsilon > 0$. Moreover, for any $\varepsilon > 0$,*

$$\int_{\mathbb{R}^3} f'(u_\varepsilon)u_\varepsilon \geqslant C. \tag{3.1}$$

Proof. Since u_ε is solution of (1.5), using (2.1), we have

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 &\leqslant \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 + \varepsilon u_\varepsilon^2 + 2e\omega\phi_{u_\varepsilon} u_\varepsilon^2 - e^2\phi_{u_\varepsilon}^2 u_\varepsilon^2 \\ &= \int_{\mathbb{R}^3} f'(u_\varepsilon)u_\varepsilon \\ &\leqslant C\int_{\mathbb{R}^3} |u_\varepsilon|^6 \\ &\leqslant C\bigg(\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2\bigg)^3\end{aligned}$$

and so we conclude the proof. □

We need a uniform boundedness estimate on the family of the ε-solutions, letting ε go to zero.

We obtain the following result.

Lemma 3.5. *There exists a positive constant C which is a uniform upper bound for the family $(u_\varepsilon, \phi_{u_\varepsilon})_{\varepsilon>0}$ in the $\mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$-norm.*

Proof. We have

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 + \varepsilon u_\varepsilon^2 + e\omega\phi_{u_\varepsilon} u_\varepsilon^2 - \int_{\mathbb{R}^3} f(u_\varepsilon) = c_\varepsilon,$$

$$\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 + \varepsilon u_\varepsilon^2 + 2e\omega\phi_{u_\varepsilon} u_\varepsilon^2 - e^2\phi_{u_\varepsilon}^2 u_\varepsilon^2 - \int_{\mathbb{R}^3} f'(u_\varepsilon)u_\varepsilon = 0.$$

By lemma 3.3 and (f2) we deduce that

$$\left(\frac{\alpha}{2} - 1\right)\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 + \varepsilon u_\varepsilon^2 + \left(\frac{\alpha}{2} - 2\right)\int_{\mathbb{R}^3} e\omega\phi_{u_\varepsilon} u_\varepsilon^2 \leqslant C, \tag{3.2}$$

while, by $(1.5)_2$, we have

$$\int_{\mathbb{R}^3} |\nabla \phi_{u_\varepsilon}|^2 + e^2\phi_\varepsilon^2 u_\varepsilon^2 = \int_{\mathbb{R}^3} e\omega\phi_{u_\varepsilon} {u_\varepsilon}^2. \tag{3.3}$$

Combining (3.2) and (3.3), we infer that $(u_\varepsilon, \phi_{u_\varepsilon})_{\varepsilon>0}$ is bounded in the $\mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$-norm. □

Now we deduce that, for any $\varepsilon_n \to 0$, there exist a subsequence of $(u_{\varepsilon_n}, \phi_{u_{\varepsilon_n}})_n$ (which we relabel in the same way), and $(u_0, \phi_0) \in \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that

$$u_{\varepsilon_n} \rightharpoonup u_0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^3),$$
$$\phi_{u_{\varepsilon_n}} \rightharpoonup \phi_0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^3).$$

We want to show that (u_0, ϕ_0) is a weak non-trivial solution of (1.4). From now on, we use u_n and ϕ_n in place of u_{ε_n} and $\phi_{u_{\varepsilon_n}}$.

Proof of theorem 1.2. By [6, lemma 13] and [3, § 3], and by (3.1), we have that

$$\int_{\mathbb{R}^3} f'(u_0)u_0 = \lim_n \int_{\mathbb{R}^3} f'(u_n)u_n \geqslant C > 0,$$

and so $u_0 \neq 0$.

Let us show that (u_0, ϕ_0) is a weak solution of (1.4), namely

$$\int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla \psi + 2e\omega\phi_0 u_0 \psi - e^2\phi_0^2 u_0 \psi = \int_{\mathbb{R}^3} f'(u_0)\psi,$$
$$\int_{\mathbb{R}^3} \nabla \phi_0 \cdot \nabla \psi + e^2\phi_0 u_0^2 \psi = \int_{\mathbb{R}^3} e\omega u_0^2 \psi,$$

for any ψ test function.

Since, for any $n \geqslant 1$, (u_n, ϕ_n) is a solution of (1.5), we have

$$\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla \psi + \varepsilon_n u_n \psi + 2e\omega\phi_n u_n \psi - e^2\phi_n^2 u_n \psi = \int_{\mathbb{R}^3} f'(u_n)\psi,$$
$$\int_{\mathbb{R}^3} \nabla \phi_n \cdot \nabla \psi + e^2\phi_n u_n^2 \psi = \int_{\mathbb{R}^3} e\omega u_n^2 \psi.$$

Let us prove that

$$\int_{\mathbb{R}^3} \phi_n u_n \psi \to \int_{\mathbb{R}^3} \phi_0 u_0 \psi. \tag{3.4}$$

Indeed, defining $K = \text{supp}(\psi)$, we observe that

$$\begin{aligned}\left| \int_{\mathbb{R}^3} \phi_n u_n \psi - \phi_0 u_0 \psi \right| &\leqslant \int_{\mathbb{R}^3} |\phi_n u_n \psi - \phi_n u_0 \psi| + \int_{\mathbb{R}^3} |\phi_n u_0 \psi - \phi_0 u_0 \psi| \\ &\leqslant \int_{\mathbb{R}^3} |\phi_n||u_n - u_0||\psi| + \int_{\mathbb{R}^3} |\phi_n - \phi_0||u_0||\psi| \\ &\leqslant \left(\int_{\mathbb{R}^3} |\phi_n|^6\right)^{1/6} \left(\int_K |u_n - u_0|^{6/5}\right)^{5/6} \sup|\psi| \\ &\quad + \left(\int_K |\phi_n - \phi_0|^{6/5}\right)^{5/6} \left(\int_{\mathbb{R}^3} |u_0|^6\right)^{1/6} \sup|\psi|,\end{aligned}$$

and so we get (3.4), since $u_n \rightharpoonup u_0$ and $\phi_n \rightharpoonup \phi_0$ in $H^1(K)$.

Let us prove that

$$\int_{\mathbb{R}^3} \phi_n^2 u_n \psi \to \int_{\mathbb{R}^3} \phi_0^2 u_0 \psi. \tag{3.5}$$

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Indeed, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\phi_n^2u_n\psi-\phi_0^2u_0\psi\right| &\leqslant \int_{\mathbb{R}^3}\phi_n^2|u_n-u_0||\psi|+\int_{\mathbb{R}^3}|\phi_n^2-\phi_0^2||u_0||\psi| \\ &\leqslant \left(\int_{\mathbb{R}^3}|\phi_n|^6\right)^{1/6}\left(\int_K|u_n-u_0|^{3/2}\right)^{2/3}\sup|\psi| \\ &\quad+\left(\int_K|\phi_n^2-\phi_0^2|^{6/5}\right)^{5/6}\left(\int_{\mathbb{R}^3}|u_0|^6\right)^{1/6}\sup|\psi| \\ &= o_n(1).\end{aligned}$$

Therefore, by (3.4) and (3.5) and since ψ has compact support, we have

$$\begin{array}{ccccccccc}\underbrace{\int_{\mathbb{R}^3}\nabla u_n\cdot\nabla\psi} & + & \underbrace{\int_{\mathbb{R}^3}\varepsilon_n u_n\psi} & + & \underbrace{\int_{\mathbb{R}^3}2e\omega\phi_n u_n\psi} & - & \underbrace{\int_{\mathbb{R}^3}e^2\phi_n^2u_n\psi} & = & \underbrace{\int_{\mathbb{R}^3}f'(u_n)\psi}, \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \int_{\mathbb{R}^3}\nabla u_0\cdot\nabla\psi & + & 0 & + & \int_{\mathbb{R}^3}2e\omega\phi_0u_0\psi & - & \int_{\mathbb{R}^3}e^2\phi_0^2u_0\psi & = & \int_{\mathbb{R}^3}f'(u_0)\psi.\end{array}$$

Analogously, we have

$$\begin{array}{ccccc}\underbrace{\int_{\mathbb{R}^3}\nabla\phi_n\cdot\nabla\psi} & + & \underbrace{\int_{\mathbb{R}^3}e^2\phi_n u_n^2\psi} & = & \underbrace{\int_{\mathbb{R}^3}u_n^2\psi}, \\ \downarrow & & \downarrow & & \downarrow \\ \int_{\mathbb{R}^3}\nabla\phi_0\cdot\nabla\psi & + & \int_{\mathbb{R}^3}e^2\phi_0u_0^2\psi & = & \int_{\mathbb{R}^3}u_0^2\psi.\end{array}$$

In particular, by the latter identity, we infer that $\phi_0 \neq 0$ and this concludes the proof. □

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Appendix A.

LEMMA A.1. *Let $p \in (2,4)$ and $\omega \in (0, g(p)m)$. Then there exists*

$$\alpha \in I_p = \left(\frac{2-p}{2(6-p)},\frac{1}{6}\right) : A_{p,\alpha}e^2\phi_{v_n}^2 + B_{p,\alpha}e\omega\phi_{v_n} + C_{p,\alpha}\Omega \geqslant 0,$$

where

$$A_{p,\alpha} = \frac{1+2\alpha(p-3)}{p},$$
$$B_{p,\alpha} = \frac{p-10\alpha p-4+24\alpha}{2p},$$
$$C_{p,\alpha} = \frac{(p-2)(1-6\alpha)}{2p}.$$

Proof. Keeping in mind (2.1), we must show that

$$f(t) = A_{p,\alpha} t^2 + B_{p,\alpha}\omega t + C_{p,\alpha}\Omega \geqslant 0 \quad \text{for any } t \in [0,\omega]. \tag{A 1}$$

First, we notice that, for any $\alpha \in I_p$,

$$A_{p,\alpha} > 0, \qquad C_{p,\alpha} > 0.$$

Now we have to distinguish two cases: $p \in (3,4)$ and $p \in (2,3]$.

Case 1 ($p \in (3,4)$). If $\alpha = (4-p)/(24-10p) \in I_p$, we have $B_{p,\alpha} = 0$ and so we have proved (A 1).

Case 2 ($p \in (2,3]$). Since f reaches its minimum in $-B_{p,\alpha}\omega/2A_{p,\alpha}$ and it belongs to $[0,\omega]$, f is non-negative in $[0,\omega]$ if and only if

$$f\left(-\frac{B_{p,\alpha}\omega}{2A_{p,\alpha}}\right) \geqslant 0,$$

and, with straightforward calculations and using the fact that $A_{p,\alpha} + B_{p,\alpha} = C_{p,\alpha}$, this is equivalent to

$$\frac{m^2}{\omega^2} \geqslant \frac{(A_{p,\alpha} + C_{p,\alpha})^2}{4A_{p,\alpha}C_{p,\alpha}}. \tag{A 2}$$

We set

$$K_p(\alpha) := \frac{(A_{p,\alpha} + C_{p,\alpha})^2}{4A_{p,\alpha}C_{p,\alpha}} = \frac{p^2}{8(p-2)}\frac{(1-2\alpha)^2}{1-6\alpha}\frac{1}{1+2\alpha(p-3)}$$

and we shall prove that

$$\inf_{\alpha\in I_p} K_p(\alpha) = \frac{1}{(p-2)(4-p)}, \tag{A 3}$$

and thus we may conclude. Indeed, if $\omega \in (0, g(p)m)$, then, by (A 3),

$$\frac{m^2}{\omega^2} > \inf_{\alpha\in I_p} K_p(\alpha),$$

and so there exists $\alpha \in I_p$ such that

$$\frac{m^2}{\omega^2} \geqslant K_p(\alpha),$$

by which we deduce (A 2).

Let us now prove (A 3). Consider the case $p = 3$: in such a situation

$$K_3(\alpha) = \frac{9}{8}\frac{(1-2\alpha)^2}{1-6\alpha} \quad \text{and} \quad I_3 = (-\tfrac{1}{6}, \tfrac{1}{6}).$$

Since the function K_3 is increasing in I_3, we have

$$\inf_{\alpha\in I_3} K_3 = K_3(-\tfrac{1}{6}) = 1$$

and so we have proved (A 3).

Now, let us consider the case $p \in (2,3)$. We write

$$K_p(\alpha) = \frac{p^2}{8(p-2)} H_1(\alpha) H_2(\alpha),$$

where

$$H_1(\alpha) := \frac{(1-2\alpha)^2}{1-6\alpha}, \qquad H_2(\alpha) := \frac{1}{1+2\alpha(p-3)}.$$

Since, for $i = 1, 2$, H_i is a positive and increasing function in I_p, we have

$$\inf_{\alpha \in I_p} K_p = K_p\left(\frac{2-p}{2(6-p)}\right) = \frac{1}{(p-2)(4-p)},$$

and so we obtain (A 3). □

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