INTEGRABILITY, CHAOS AND NONLINEAR SUPERPOSITION FORMULAS FOR DIFFERENTIAL MATRIX RICCATI EQUATIONS

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Abstract. We point out that nonlinear superposition principles can be used to identify integrable systems of nonlinear ordinary differential equations among families of nonintegrable ones. A superposition formula is then obtained for a class of integrable equations, namely the matrix Riccati equations.

1. INTEGRABLE AND NONINTEGRABLE SYSTEMS RICCATI EQUATIONS.

We shall call a system of first order nonlinear ordinary differential equations (ODE's) a system of Riccati equations if all the nonlinearities are quadratic

$$y'^{\mu} = a^{\mu} + b^{\mu}{}_{\alpha} y^{\alpha} + c^{\mu}{}_{\alpha\beta} y^{\alpha} y^{\beta}, \qquad \mu = 1,\dots,n. \tag{1}$$

The coefficients a^{μ}, $b^{\mu}{}_{\alpha}$, $c^{\mu}{}_{\alpha\beta}$ are, in general, arbitrary functions of t, the prime denotes differentiation with respect to time t and summation over repeated indices is to be understood.

Equations of the type (1) can serve as prototypes of systems demonstrating chaotic behavior. Examples of such systems with extremely sensitive dependence on the initial conditions are the Lorenz equations, various versions of the Volterra-Lotke equations, or the Henon-Heiles equations[1-3].

On the other hand, many particular cases of equations of type (1) exist which are integrable and hence have stable solutions with regular long term (global) behavior. Clearly it is of interest to identify such systems and to solve them. These systems can then serve as tools for studying "nearby" nonintegrable systems, obtained by perturbing the coefficients in the integrable systems. Several methods are commonly used for identifying integrable systems among families of nonintegrable ones. In addition to direct searches for first integrals, or for linearizing transformations, we mention the method of Painlevé analysis[4]. This is an investigation of the singularity structure of the solutions of the equations, the purpose of which is to find equations for which the solutions have no moving critical points.

In this contribution we present a different integrability test[5], based on the fact that certain systems of nonlinear ODE's admit **superposition formulas.** We use this term to indicate that the general solution of such a system of n ODE's can be expressed as a function of a finite number m of particular solutions and of n constants[6-10]. This approach makes use of the following theorem, due to S. Lie[11].

Theorem 1. The necessary and sufficient condition for a system of first order ODE's

$$y' = \eta\,(y, t) \tag{2}$$

to admit a superposition formula

$$y = F\,(y_1,\dots,y_m, c_1,\dots,c_n) \tag{3}$$

is that :

(i) The system (2) have the form

$$y' = \sum_{k=1}^{s} Z_k(t)\,\xi_k(y) \tag{4}$$

(ii) The vector fields

$$\hat{X}_k = \sum_{\mu=1}^{n} \xi_k^\mu(y)\frac{\partial}{\partial y^\mu}\,, \quad \xi_k = \left(\xi_k^1\,,\dots,\xi_k^n\right) \tag{5}$$

generate a finite dimensional Lie algebra.

All indecomposable systems of equations satisfying Lie's criteria have been recently classified, making use of the theory of transitive primitive Lie algebras[9]. All systems of n such equations with $n \leq 3$ have recently been integrated[5].

Equations satisfying Lie's criterion do not necessarily have polynomial nonlinearities; examples have been given of equations with rational and other nonlinearities[9,10]. On the other hand, families of Riccati type equations have been obtained that do admit superposition formulas and still have coefficients that are arbitrary functions of t. Among these we mention :

1. Projective Riccati equations[6,7]

$$y'^{\mu} = a^{\mu} + b^{\mu}{}_{\alpha} y^{\alpha} + y^{\mu} c_{\alpha} y^{\alpha} \qquad (6)$$

2. Conformal Riccati equations[7]

$$y'^{\mu} = a^{\mu} + \Lambda^{\mu}{}_{\nu} y^{\nu} + s\, y^{\mu} + y^{\mu} c_{\alpha} y^{\alpha} - c^{\mu} y_{\alpha} y^{\alpha}/2 \qquad (7)$$

$$\Lambda^{\mu}{}_{\nu} + \Lambda^{\nu}{}_{\mu} = 0$$

3. Matrix Riccati equations

$$W' = A + WB + CW + WDW \qquad (8)$$

$$W, A \in R^{n \times k},\ B \in R^{k \times k},\ C \in R^{n \times n},\ D \in R^{k \times n}$$

(equations (6) are a special case of the MRE (8) with $k = 1$).

In order to show the qualitative difference between Riccati equations with and without superposition formulas, consider a simple example[5]

$$x' = y + x(c_1 x + c_2 y) \qquad y' = -(2 + Q\cos 2t)y + y(c_1 x + c_3 y) \qquad (9)$$

where c_1, c_2, c_3 and Q are constants. For $c_3 = c_1$ (9) is a special case of projective Riccati equations (6), for $c_3 \neq c_1$ the conditions of Lie's theorem are not satisfied. For $c_3 = c_2$ we can integrate analytically, for $c_3 \neq c_2$ numerical integration is required. On Fig. 1 we show Poincaré surfaces of section for two different cases, namely $Q = 0.2$ and $c_3 = c_2$ and $Q = 0.2$, $c_3 = c_2 - 0.1 = 0.6$. In both cases the orbits are plotted for $t = k\pi$ (k = integer).

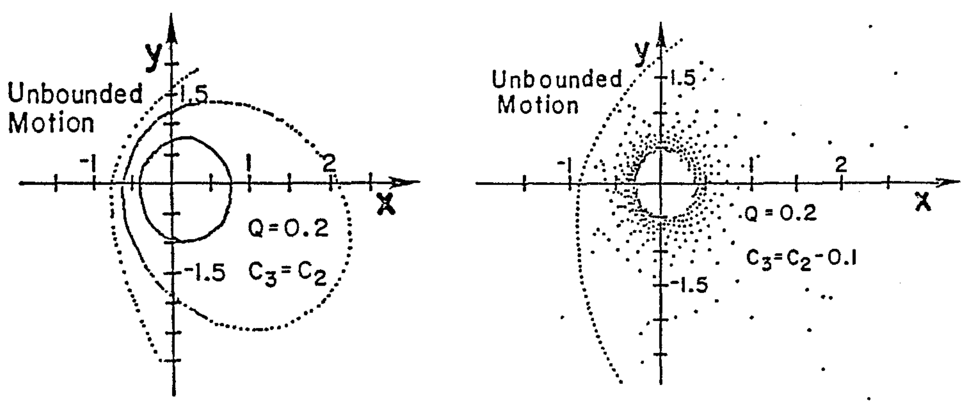

Fig. 1

In both cases we can have unbounded motion (for initial conditions to the left of the dotted line) or bounded motion (to the right of that boundary). In the case with a superposition formula we obtain regular closed orbits. For $c_3 \neq c_2$ we start the integration at a point ($x_0 \sim -1$, $y_0 = 0$, $t = 0$) close to the boundary. The trajectory starts out in a complicated manner that depends very sensitively on the exact value of x_0; it then settles into a spiralling motion towards the origin. We observe (for many different values of x_0, Q, and $c_3 - c_2 \neq 0$) a phenomenon of "transitory" chaos in a non Hamiltonian system.

In order to show how the nonlinear superposition formulas provide solutions, we consider the matrix Riccati equations (8).

2. SUPERPOSITION FORMULAS FOR THE MATRIX RICCATI EQUATION.

The case we present here corresponds to the MRE (8) based on the Lie group $SL(N, \mathbf{C})$ and a maximal parabolic subgroup $P(k)$, i.e. we consider $N = n+k$, and $n = rk$, where $n \geq k$ and $r \in \mathbf{Z}$. It has been shown that we only need 5 particular solutions to construct the general solution of $r=1$ (square MRE's[8]) and $n+2$ if $k=1$ (projective MRE's[6,7]).

Let $P(k)$ be a maximal subgroup of the Lie group $SL(n+k, \mathbf{C})$. We construct the homogeneous space $SL(n+k, \mathbf{C})/P(k) \simeq G_k(\mathbf{C}^{n+k})$, the Grassmannian of k-planes in $\mathbf{C}^{n+k}$. We introduce homogeneous coordinates on $G_k(\mathbf{C}^{n+k})$ as the components of a matrix ξ of rank k

$$\xi = \begin{pmatrix} X \\ Y \end{pmatrix} \sim \begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} XG \\ YG \end{pmatrix}, \quad X \in \mathbf{C}^{n\times k},\ Y \in \mathbf{C}^{k\times k},\ G \in GL(k,\mathbf{C}). \qquad (10)$$

The inherent redundancy in homogeneous coordinates is removed by introducing affine coordinates $W = XY^{-1}$ on $G_k(\mathbf{C}^{n+k})$ (for $\det Y \neq 0$).

The system of ODE's related to the action of $SL(n+k, \mathbf{C})$ on $G_k(\mathbf{C}^{n+k})$ is precisely the rectangular MRE (8) with $W, A \in \mathbf{C}^{n\times k}$, $B \in \mathbf{C}^{k\times k}$, $C \in \mathbf{C}^{n\times n}$, $D \in \mathbf{C}^{k\times n}$, where $A,\ldots,D$ are matrix functions of (time) t. The right hand side of (8) corresponds to a curve in the Lie algebra $sl(n+k, \mathbf{C})$. The general solution of this system is given by the action of $SL(n+k, \mathbf{C})$ on $G_k(\mathbf{C}^{n+k})$, i.e.

$$W(t) = (G_{11}(t)U + G_{12}(t))\,(G_{21}(t)U + G_{22}(t))^{-1}, \quad G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \in SL(n+k,\mathbf{C}). \quad (11)$$

Here $G(t)$ is a curve in $SL(n+k, \mathbf{C})$, and $U \in \mathbf{C}^{n\times k}$ is a constant matrix, specifying the initial conditions for $W(t)$. The problem is now to determine $G_{ik}(t)$ as functions of a (minimal) sufficient number of particular solutions $W_i(t)$ of the equation (8). These solutions are called a **fundamental set of solutions.**

In the following we will consider the case $n \geq k$ (the case $n < k$ is reduced to this one by transposition). The minimal number m of known particular solutions needed verifies $mnk \geq (n+k)^2 - 1$, where the right hand term is the number of independent matrix elements of $G(t)$. Then m particular solutions $W_1(t),\ldots,W_m(t)$ will determine $G(t)$, at least locally, if the stability group of m initial values $W_i(t_0)$ on the product of m copies of $G_k(\mathbf{C}^{n+k})$ is contained in the center of $SL(n+k, \mathbf{C})$. It is possible to choose a fundamental set of solutions genericly and to transform it into a "standard" particularly convenient set. These results can be expressed in homogeneous or in affine coordinates (see ref. 12 for all details and proofs of statements).

The points of $G_k(C^{n+k})$ are expressed as

$$\xi^T = (X_1{}^T,\dots,X_r{}^T, Y^T),\ W^T = (W_1{}^T,\dots,W_r{}^T),\ X_i,Y,W_i \in C^{k\times k},\ i = 1,\dots,r \qquad (12)$$

in homogeneous or affine coordinates, respectively (the superscript T denotes transposition). Correspondingly, we shall write the elements of $G \in SL(n+k, \mathbf{C})$ of (11) as

$$G_{11} = \begin{bmatrix} M_{11},\dots, M_{1r} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ M_{r1},\dots, M_{rr} \end{bmatrix},\ G_{12}^T = (N_1^T,\dots,N_2^T),\ G_{21} = (P_1,\dots, P_r),\ G_{22} = Q\,. \quad (13)$$

Theorem 2. The following "standard set" of r + 3 initial conditions of the MRE (8), given in homogeneous coordinates, has only the center of $SL(n+k, \mathbf{C})$ as its isotropy group :

$$\{\xi_1^S,\dots,\xi_{r+3}^S\} = \left\{ \begin{bmatrix} I_k \\ 0 \\ \vdots \\ 0 \\ - \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ I_k \\ \vdots \\ 0 \\ - \\ 0 \end{bmatrix},\dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_k \\ - \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ - \\ I_k \end{bmatrix} \begin{bmatrix} I_k \\ I_k \\ \vdots \\ I_k \\ - \\ I_k \end{bmatrix} \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_r \\ - \\ I_k \end{bmatrix} \right\}. \quad (14)$$

The blocks $\Lambda_i \in C^{k\times k}$ are such that one of them, say Λ_1 satisfies $\Lambda_1 = \mathrm{diag}(\lambda_1, \lambda_2,\dots, \lambda_k)$, with $\lambda_i \in \mathbf{C}$, $\lambda_i \neq \lambda_j$ for $i \neq j$ and another one, say Λ_2, has no irreducible invariant subspaces in common with Λ_1.

Theorem 3. Given a set of r + 3 initial conditions of the MRE (8) in affine coordinates $\{U_1,\dots,U_{r+3}\} \subset C^{rk\times k}$ satisfying the conditions

(i) $$\det \begin{bmatrix} U_1 & \dots & U_r & U_{r+1} \\ I_k & \dots & I_k & I_k \end{bmatrix} \neq 0,$$

(ii) $$\det \begin{bmatrix} U_1 & \dots & U_{j-1} & U_{j+1} & \dots & U_{r+1} & U_{r+2} \\ I_k & \dots & I_k & I_k & \dots & I_k & I_k \end{bmatrix} \neq 0,\ j = 2,\dots,r$$

(iii) $$\det \begin{bmatrix} U_1 & \dots & U_r & U_{r+3} \\ I_k & \dots & I_k & I_k \end{bmatrix} \neq 0.$$

(iv) The matrices

$T_i = S_i R_i (S_{r+1} R_{r+1})^{-1} \in C^{k\times k}, \quad i = 1,2$

have no common nontrivial irreducible eigenspaces and one of them, say T_1 has k distinct eigenvalues, where S_i and R_i are defined by

$$U = \begin{pmatrix} U_1 & \cdots & U_{r+1} \\ I_k & \cdots & I_k \end{pmatrix},$$

$$\begin{pmatrix} (S_1)^{-1} \\ \vdots \\ (S_{r+1})^{-1} \end{pmatrix} = U^{-1} \begin{pmatrix} U_{r+2} \\ \\ I_k \end{pmatrix}, \quad \begin{pmatrix} R_1 \\ \vdots \\ R_{r+1} \end{pmatrix} = U^{-1} \begin{pmatrix} U_{r+3} \\ \\ I_k \end{pmatrix}.$$

Then, there exists a transformation $G \in SL(n+k, C)$ transforming the set

$$\{\xi_i\} = \left\{ \begin{pmatrix} U_i \\ I_k \end{pmatrix} \right\}, \quad i = 1,\dots,r+3$$

into the standard set ξ^S_i of (14)

The superposition formula can be obtained by reconstructing the group element $G(t)$ in terms of $r+3$ particular solutions.

We parametrize the group element $G(t)$ as in (11) and (13). Writing (11) for the first $r+1$ "standard" solutions $W_i(t)$ we obtain

$M_{ij} = W_{ji} P_j, \; N_i = W_{r+1,i}\, Q, \; i, j = 1,\dots,r$

where we put

$$W_j(t) = \begin{pmatrix} W_{j1}(t) \\ \vdots \\ W_{jr}(t) \end{pmatrix}, \quad j = 1,\dots,r+3.$$

Using $W_{r+2}(t)$ we obtain a system of inhomogeneous linear equations for P_i in terms of the known solutions $W_j(t)$ $(j = 1,\dots,r+2)$ and the still unknown matrix $Q(t) \in C^{k\times k}$:

$$\tilde{W} \begin{pmatrix} P_1 \\ \vdots \\ P_r \end{pmatrix} = (W_{r+1} - W_{r+2})\, Q, \quad \tilde{W} = \begin{pmatrix} W_{r+2,1} - W_{11}, \dots, W_{r+2,1} - W_{r1} \\ \cdots\cdots\cdots\cdots \\ W_{r+2,r} - W_{1r}, \dots, W_{r+2,r} - W_{rr} \end{pmatrix}.$$

The solution exists and is unique as long as $\det \tilde{W} \neq 0$. Finally, to determine Q we use the remaining solution $W_{r+3}(t)$:

$$\tilde{\tilde{W}} \begin{pmatrix} F_1 Q \Lambda_1 \\ \vdots \\ F_r Q \Lambda_r \end{pmatrix} = (W_{r+1} - W_{r+3}) Q \tag{15}$$

where

$$\tilde{W} \begin{pmatrix} F_1 \\ \vdots \\ F_r \end{pmatrix} = (W_{r+1} - W_{r+2}) Q, \quad \tilde{\tilde{W}} = \begin{bmatrix} W_{r+3,1} - W_{11}, \ldots, W_{r+3,1} - W_{r1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ W_{r+3,r} - W_{1r}, \ldots, W_{r+3,r} - W_{rr} \end{bmatrix}.$$

and $\Lambda_1, \ldots, \Lambda_r$ are defined in Theorem 2.

Using (15) and

$$\begin{pmatrix} H_1 \\ \vdots \\ H_r \end{pmatrix} = \tilde{\tilde{W}}^{-1} (W_{r+1} - W_{r+3}),$$

we can write the following equations

$$Q\Lambda_i Q^{-1} = (F_i)^{-1} H_i, \quad i = 1, \ldots, r$$

which determine Q. Note that the matrices $F^{-1}{}_i H_i$ are conjugate to constant matrices. The existence of W^{-1} is assured by the conditions imposed in Theorem 3. For the same reasons $F^{-1}{}_i$ exists, $i = 1, \ldots, r$. (Theorem 3, (ii).)

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