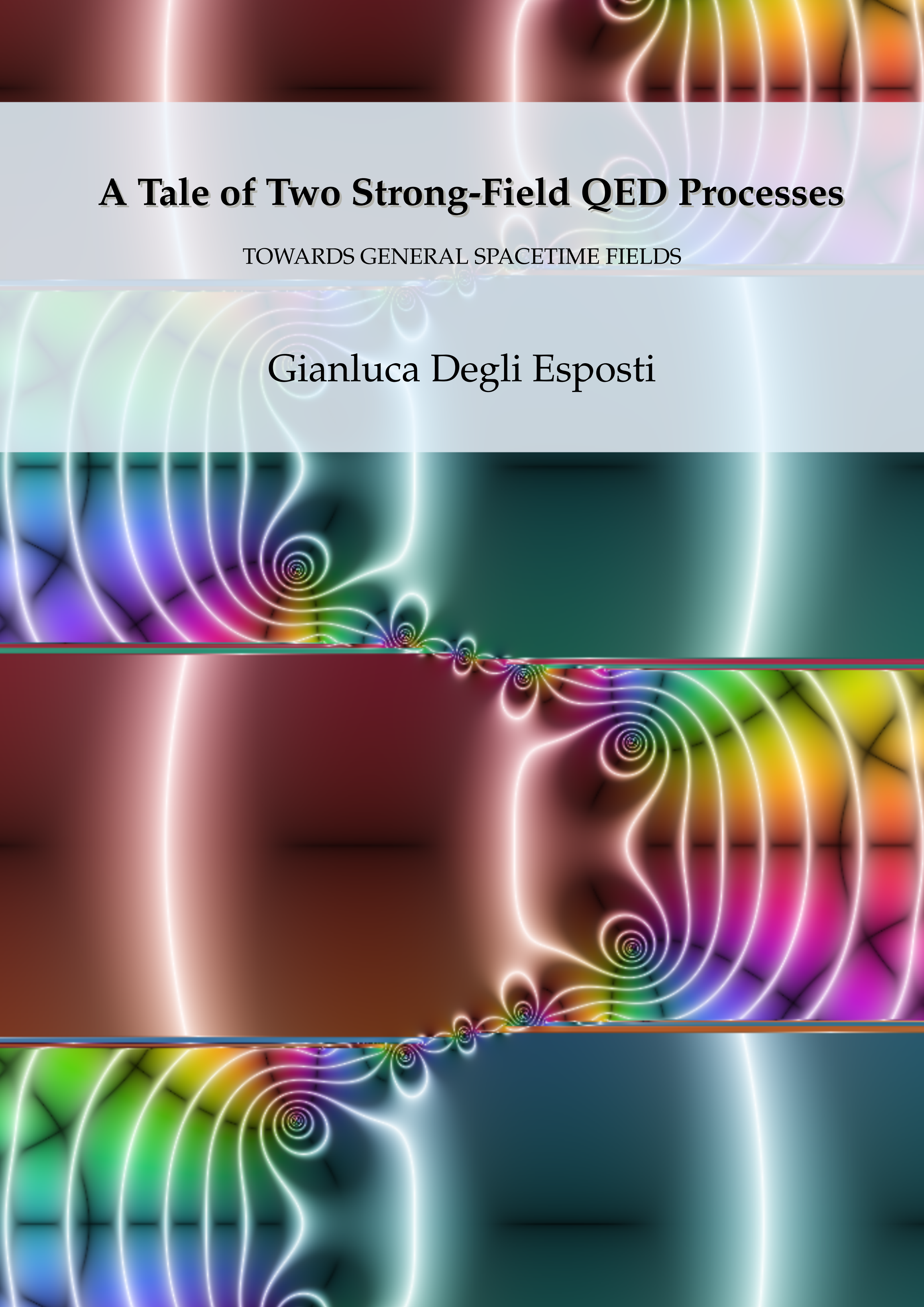

A Tale of Two Strong-Field QED Processes

TOWARDS GENERAL SPACETIME FIELDS

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TECHNISCHE UNIVERSITÄT DRESDEN

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presented by

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Preface

One of the most striking predictions of Quantum Electrodynamics (QED) is the production of matter from light. In the present thesis the focus is on two processes: Schwinger and Breit-Wheeler pair production. Schwinger pair production occurs when the vacuum, under the influence of an extremely strong electric field, generates particle-antiparticle pairs. Heuristically, it can be viewed as due to the field pulling apart the virtual electron-positron pairs and, if strong enough, turning them into real particles. With the addition of high-energy photons to catalyze the creation, the process is known as Breit-Wheeler pair production. Due to its exponential scaling $\mathbb{P} \sim \exp\{-E_S/E\}$ with critical field strength $E_S \sim 10^{18}$ V/m much higher than those available at current strong laser facilities such as the European X-Ray Free-Electron Laser Facility (XFEL) or the Extreme Light Infrastructure (ELI), Schwinger pair production remains unobserved to this day. However, it has been possible to observe Breit-Wheeler pair production at the E-144 SLAC experiment, where the photons were produced from the scattering of a 46.6 GeV electron beam with a strong laser field. There are several theoretical methods to tackle these problems, all of them with their upsides and downsides, but one barrier common to most of them is the difficulty one faces when considering fields with both space and time structure. This work is an attempt to break down that barrier by providing quantitative results for the spectra of particles produced by spacetime fields. While the focus is mostly on $2D$ fields, an instructive example of a more realistic $4D$ field with optimal spacetime focusing is also shown. Since future experiments will reasonably involve highly focused pulses in both space and time, it is useful to have a theoretical toolkit that allows us to access the spectra of spacetime fields.

List of publications

This thesis is based on the following papers:

- G. Degli Esposti and G. Torgrimsson
  Momentum spectrum of nonlinear Breit-Wheeler pair production in space-time fields
  Phys. Rev. D **110**, no.7, 076017 (2024)

  Contributions: Sections on the analytic structure and patching of the instantons on the complex plane, prefactor, phase transition, and critical behavior with input on many aspects from GT, numerical calculations

- G. Degli Esposti and G. Torgrimsson
  Momentum spectrum of Schwinger pair production in four-dimensional e-dipole fields
  Phys. Rev. D **109**, no.1, 016013 (2024)

  Contributions: Sections on the analytic structure of the instantons on the complex plane, calculations of prefactor and widths in parallel with GT, calculation of LCF exponent, numerical calculations

- G. Degli Esposti and G. Torgrimsson
  Worldline instantons for the momentum spectrum of Schwinger pair production in space-time dependent fields
  Phys. Rev. D **107**, no.5, 056019 (2023)

  Contributions: Analytic calculations of exponent/prefactor performed in parallel with GT, development of the numerical instanton-finding method together with GT, numerical calculations

- G. Degli Esposti and G. Torgrimsson
  Worldline instantons for nonlinear Breit-Wheeler pair production and Compton scattering
  Phys. Rev. D **105**, no.9, 096036 (2022)
  Contributions: Calculations for plane waves and constant fields

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I am thankful to Christian Schneider for providing his Mathematica code for discrete worldline instantons. Being able to compare the results of the present thesis for the spectrum against the integrated probability from the effective action was crucial to back up the validity of many results of the present work.

Vorrei ringraziare i miei genitori per il loro sostegno e avermi dato la possibilità di arrivare a intraprendere questo percorso. Senza il vostro aiuto, nulla di ciò sarebbe stato possibile.

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Por último em ordem, ainda que não em importância, vem o agradecimento do coração. Eu queria agradecer à Danielle Maass, a rosa do meu coração, sem a qual esta tese teria sido concluída vários meses antes, por mudar minha vida aqui em Dresden e por acreditar em mim mais do que eu mesmo. *Tecum, quos dederint annos mihi fila sororum, vivere contingat*.

Contents

Chapter 1

Introduction

We give a general introduction to the field of Strong-Field QED and list some known results about plane waves that have been known for a long time. Using the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula with open worldlines explained in the present work, we rederive them to test the validity of this method. Then, we discuss briefly the two processes studied throughout the rest of this work, i.e. Schwinger and Breit-Wheeler pair production. In the end we give a short outline of the rest of the thesis.

1.1 Strong-Field QED

Without a doubt, Quantum Electrodynamics (QED), i.e. the quantum theory of matter and light, is one of the most successful physical theories due its many accurate predictions including the well-known agreement of the anomalous magnetic moment of the electron up to many digits of precision [1, 2]. This particular high-precision agreement, arising from a perturbative calculation at several loop orders, could in principle allow for the discovery of unknown highly suppressed physics from small discrepancies between the calculated and the experimental values. However, while in the weak-field and high-energy regime perturbation theory is a powerful tool, it is not the best option if we consider the interaction of matter or hard photons with strong electromagnetic fields given by a huge number of coherent low-energy photons. The prototypical example of this is the interaction of matter with lasers. The standard method to get around this is splitting the electromagnetic field into a classical background field given by the coherent sum of (typically optical) photons, and a quantized radiation field describing the interaction with individual incoherent photons (typically of much higher energy than the optical laser, such as X-rays or higher energy). This is known as the Furry picture [3], and the study of the interaction of matter or high-energy photons with strong background fields is then known as Strong-Field QED. In the past decades, partly because the technological advances will allow the experimental test of the predictions of Strong-Field QED, there has been a great effort on the theoretical side [4, 5, 6, 7, 8].

We give now a very brief introduction to the main ideas necessary for this thesis. Throughout the present work, unless stated otherwise, we work in natural units $c = \hbar = 1$ and absorb the electric charge into the background field $qA \to A$. In Quantum Field Theory language, the splitting mentioned above implies in practice that we have a Lagrangian (before gauge fixing)

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - A\!\!\!/ - m)\Psi - \frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} - q\mathcal{A}_\mu\bar{\Psi}\gamma^\mu\Psi \tag{1.1}$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$ with the quantized photon field $\mathcal{A}_\mu$, and A_μ the classical background. Since the field is very strong and we want to treat it exactly, only the last term is an interaction term to be treated perturbatively in powers of α. This is the so-called Furry picture [9].

Given a field $A_\mu(x)$, one can show that any field invariant can be expressed in terms of

$$\mathscr{F} = \frac{1}{4} F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(E^2 - B^2) \qquad \mathscr{G} = \frac{1}{4} \star F_{\mu\nu}F^{\mu\nu} = -\mathbf{E}\cdot\mathbf{B} \tag{1.2}$$

where $\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ is the Hodge dual of F and ϵ is the Levi-Civita symbol. A well-known class of fields used in practical calculations due to their simplicity are plane waves, namely fields that depend only on $\phi = kx$ where k is a null vector $k^2 = 0$. For these fields

$$\mathscr{F} = \mathscr{G} = 0 \ . \tag{1.3}$$

Plane waves are widely used because the solutions to the Dirac equation and the propagator have a simple form and the effective field felt by a high energy particle is typically a plane wave. Letting

$$\begin{aligned} S_p(x) &= -px - \frac{1}{2pk}\int_0^\phi d\varphi\,\big(2pA(\varphi) - A^2(\varphi)\big) \\ K_p(\phi) &= 1 + \frac{\not{k}\not{A}(\phi)}{2pk} \end{aligned} \tag{1.4}$$

and $u_{p,s}$ be the free Dirac spinor with momentum p, one can see that

$$\Psi_{p,s}(x) = K_p(\phi)e^{iS_p(x)}\,u_{p,s} \tag{1.5}$$

is an exact solution to

$$(i\not{\partial} - \not{A} - m)\Psi_{p,s}(x) = 0 \ . \tag{1.6}$$

These functions are known as Volkov states [10]. To obtain the corresponding positron solution we let $p \to -p$ and $u_p \to v_p$. Although one could in principle find other solutions to the Dirac equation, the Volkov states represent a basis, i.e. they are orthogonal according to a suitable scalar product and they give a spectral resolution of the identity [11].

Furthermore, one can see that

$$S(y,x) = \int \frac{d^4p}{(2\pi)^4}\,K_p(\phi_y)\,\frac{\not{p}+m}{p^2-m^2+i\varepsilon}\,\bar{K}_p(\phi_x)\,e^{-ip\cdot(y-x)-i\int_{kx}^{ky}V_p} \tag{1.7}$$

is the exact (Volkov) propagator for a generic plane wave background [12, 13] with

$$V_p = \frac{2pA - A^2}{2pk} \ . \tag{1.8}$$

One thing to point out is that the expression above looks like a Fourier transform but it is not. In fact, the actual Fourier transform of the Volkov propagator has a much more complicated pole structure [14].

Although in this work we focus on spin-$\frac{1}{2}$ particles because we are interested in processes involving electrons and positrons, one can derive similar results also for scalar particles. In particular the Volkov states are obtained by simply dropping anything involving spin

$$\Phi_p(x) = e^{iS_p(x)} \tag{1.9}$$

and similarly for the propagator

$$G(y,x) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot(y-x) - i\int_{kx}^{ky} V_p} . \tag{1.10}$$

Armed with these exact solutions, one can derive the Feynman rules (in position space) for Strong-Field QED in a plane wave background by simply replacing free propagators with S (or G) and free external states with Volkov states. However, there is no Schwinger pair production for fields with vanishing field invariants such as plane waves, so the Volkov states/propagator cannot be used for the Schwinger effect.

1.2 Schwinger and Breit-Wheeler pair production

It has been well-known for a long time that the presence of a background field changes the structure of QED in drastic ways. One of the most fascinating and at the same time elusive phenomena is Schwinger pair production. It has been speculated since about a century ago [15, 16, 17] that, given a sufficiently strong electromagnetic field, the vacuum becomes unstable and pairs of charged particles are created. Due to the extremely large field intensity required, of the order[1] of $E_S \sim 10^{18}\,\mathrm{V/m}$ or $I_S \sim 10^{29}\,\mathrm{W/cm^2}$, it has yet not been possible to verify the existence of such process experimentally despite the significant effort in the development of strong lasers. To this day, this is perhaps the most remarkable yet untested prediction of QED. The process is challenging both experimentally (for said reasons) and theoretically due to its nonperturbative nature. For a constant field, the rate of pair production per unit volume scales as[2]

$$\mathbb{P} \sim \exp\left\{-\frac{\pi}{E}\right\} \tag{1.11}$$

so it is highly suppressed for E much smaller than the Schwinger limit $E_S = 1$. It is also interesting to point out that this is the same exponential scaling of e.g. atomic ionization in an electric field [20]

$$\mathbb{P}_{\text{ion}} \sim \exp\left\{-\frac{4}{3}\frac{\sqrt{2}\,E_b^{3/2}}{E}\right\} \tag{1.12}$$

where E_b is the binding energy, which also suggests that Schwinger pair production is a *tunneling* process. In chapter 3, we will corroborate this interpretation using an instanton picture. For comparison, if we consider for example the binding energy of hydrogen [21], we get a critical field of the order of $10^9\,\mathrm{V/m}$, which is many orders of magnitude smaller than the Schwinger limit.

[1] The Schwinger effect is an electric phenomenon, although it has been speculated the existence of an analogous effect where pairs of magnetic monopoles are created in a strong magnetic field [18, 19].

[2] We have also set the electron mass to one, $m = 1$, so that the critical Schwinger field is $E_S = 1$.

There are various ways to generalize the calculation of of (1.11) for more general fields. One can solve numerically the Dirac equation in a background field [22, 23], but this is very complicated for realistic spacetime fields. Other numerical methods are the Wigner formalism [24, 25, 26] and the quantum Vlaslov equation [27, 28, 29, 30]. Alternatively, there are also analytical semiclassical techniques such as Wentzel-Kramers-Brillouin (WKB) [31, 32, 33, 34, 35, 36, 37] and closed instantons for the effective action [38, 39, 40, 41, 42, 43, 44]. For slowly varying fields, one can also use the locally constant field method [40, 45] to obtain information about the total probability. There are upsides and downsides to each of these. The WKB approximation is simple and very powerful for one-dimensional fields, but it does not generalize in an obvious way to higher dimensional inhomogeneities (which is the scope of the present work), although recently some progress has been made [46, 47]. The effective action method works well analytically for one-dimensional fields but also numerically for higher dimensional fields, as demonstrated in [41, 44]. However, the output is the total integrated probability and nothing is known about the spectrum of particles produced (although for time-dependent fields there is a trick that allows one to find the spectrum as well [48]). The Wigner formalism does give information about the spectrum, but it is numerically challenging to work with spacetime fields (especially more than $2D$). Spacetime fields have been considered with such quantum kinetic approaches in [26, 49, 50, 51]. The approach we present here, which uses open worldline instantons at the amplitude level, allows us to work with spacetime fields and produces information about the spectrum at generic electron/positron momenta independently of each other, while the Wigner formalism considers only one momentum variable. The only assumption we make everywhere is that the field strength must be much smaller than the critical Schwinger limit $E \ll 1$ so that we can work in the semiclassical/saddle-point approximation. This regime will be interesting experimentally in the future because the first tests of the Schwinger effect will reasonably be at field strengths smaller than the Schwinger limit. Open worldlines have also been considered in [52, 53].

Being able to have quantitative results for spacetime fields is also useful because in order to obtain the highest possible field strength at the peak it is essential to have a highly focused beam in both space and time [54, 55], so the constant field or time-dependent approximations would not be the best options if $\omega/E \sim \mathcal{O}(1)$ where ω is the laser frequency. It is also necessary to tell apart the signal coming from Schwinger pair production from any other background noise inevitably present in the data. To do this, it is very useful to predict what the spectrum of emitted particles is. While for fields with only one peak the spectrum is always Gaussian in the $E \ll 1$ regime, a realistic pulse will have a nontrivial subcycle structure giving rise to a rich spectrum made out of several peaks and valleys from the interference of the multiple peaks in the pulse [37, 48, 56, 57, 58]. The method presented here also allows us to consider fields with multiple peaks. In principle, one could also tweak the field parameters in order to produce a desired spectrum [59].

If one works with the effective action $\Gamma[A]$ defined from the vacuum persistence amplitude in the presence of a field A

$$e^{i\Gamma[A]} := {}_{\text{out}}\langle 0|0\rangle_{\text{in}} \tag{1.13}$$

one can obtain the total pair production probability from the complement of the vacuum to vacuum probability[3]

$$\mathbb{P}_{\text{pair}} \simeq 1 - |{}_{\text{out}}\langle 0|0\rangle_{\text{in}}|^2 \simeq 2\text{Im}\,\Gamma[A] \tag{1.14}$$

thus any information about the spectrum and spin is lost. If instead we obtain the amplitude from the amputation of the propagator using the LSZ reduction formula [60], we can access the spectrum and the spin as well. However, the usual LSZ is not suitable for this purpose because we would have to work off the mass shell and take the on-shell limit at some point during the calculation, which is not necessarily easy. An alternative way is to use the LSZ as in [52] in which an asymptotic time limit is present instead of a time integral and one can work on the mass shell from the very beginning. The amplitude for the production of an electron with momentum p and a positron with momentum p' is given by

$$M = {}_{\text{out}}\langle p, p'|0\rangle_{\text{in}} = \lim_{t_\pm \to \infty} \int d^3x_+\, d^3x_-\, e^{ipx_+ + ip'x_-}\, \bar{u}_s(p)\, \gamma^0\, S(x_+, x_-)\, \gamma^0\, v_{s'}(p') \tag{1.15}$$

where $S(x_+, x_-)$ is the Dirac propagator in a background field A and $\bar{u}$ and v are the free Dirac spinors. The spectrum is then obtained by the modulus squared of the amplitude. For fields with several peaks, the amplitude will be the sum of multiple terms, which can give interference.

We use the worldline representation of the propagator S in terms of a path integral over trajectories [61, 62, 63]

$$S(x_+, x_-) = (i\not{\partial}_{x_+} - \not{A}(x_+) + 1)\frac{1}{2}\int_0^\infty dT \int_{x_-}^{x_+} \mathcal{D}q\, e^{-i\int_0^1 d\tau\, \frac{\dot{q}^2}{2T} + \frac{T}{2} + \dot{q}\cdot A}\, \mathcal{P}\, e^{-\frac{iT}{4}\int \sigma^{\mu\nu} F_{\mu\nu}}\,. \tag{1.16}$$

where T can be interpreted as the total proper time, $\mathcal{P}$ is the proper-time ordering operator, and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. For electromagnetic fields that depend on space and/or time, some of the integrals in (1.15) and (1.16) cannot be done analytically, so we must use an approximation method. In the $E \ll 1$ regime, we can use the saddle-point method. The saddle-point equation for the path integral in (1.16) is the Lorentz force equation

$$\ddot{q}^\mu = T F^{\mu\nu} \dot{q}_\nu \tag{1.17}$$

and the solutions are known as the *instantons*. A similar representation in terms of a single particle path integral is used in [38, 39, 40, 41] for the effective action as well, but in that case the path integral is over trajectories with periodic boundary conditions $q(0) = q(1)$, so the corresponding instantons are closed as well, while in the case considered here they are open $q(0) \neq q(1)$.

[3] We neglect the highly suppressed production of multiple pairs.

From the saddle-point equations with respect to the variables T and $\mathbf{x}_\pm$ we get respectively[4]

$$\dot{q}^2 = T^2 \qquad \dot{q}_i(1) = Tp_i \qquad \dot{q}_i(0) = -Tp_i' \;, \tag{1.18}$$

where the first one is an on-shell condition and the other two fix the initial/final derivatives of the instanton in terms of the asymptotic momenta. Since in the asymptotic time limit $t_\pm \to \infty$ the saddle point value of the total proper time T_s also goes to infinity, and τ represents the normalized proper time, it is convenient to change the variable to $u = T_s(\tau - 1/2)$ so that (1.17) and (1.18) do not contain divergent variables. The boundary conditions can make it challenging to find the instantons numerically because we have to use the shooting method [41] varying some initial condition (at $u = 0$ for convenience) until (1.18) are satisfied.

Considering the single pulse case for simplicity, at the end of the calculation we find [64, 65] that the spectrum can be written as

$$\mathbb{P}(p, p') = E^a F(p, p')\, e^{-\frac{1}{E}\mathcal{A}(p,p')} \tag{1.19}$$

where $\mathcal{A}(p, p')$ is an integral involving the instantons

$$\mathcal{A}(p, p') = 2\mathrm{Im} \int du\, q^\mu \partial_\mu A_\nu \,\frac{dq^\nu}{du} \;, \tag{1.20}$$

$F(p, p')$ is a numerical prefactor that can be calculated solving some second order linear ODE describing fluctuations of the Lorentz force equation (1.17) around the instanton, and a is some real number. We have extracted the dependence on E to make the point more clear. While this method in principle allows us to find the spectrum at a generic momentum by finding the corresponding instanton that satisfies (1.18), there is a simpler way. Since the integral $\mathcal{A}(p, p')$ will have some maximum (p_s, p_s') and does not depend on the field strength E, for $E \ll 1$ the spectrum is approximately a simple Gaussian centered at (p_s, p_s') with a narrower peak the smaller E is. This can be seen by Taylor expanding $\mathcal{A}$ around (p_s, p_s'). Since the peak is very narrow, we can approximate the prefactor $F(p, p')$ by considering its value at the maximum $F(p_s, p_s')$.

The particular form of the spectrum will depend on the field considered. For time-dependent fields, for example, the spatial momentum is conserved, so the spectrum will be proportional to a delta function $\delta(\mathbf{p} + \mathbf{p}')$. If, on the other hand, we consider a field of the form $E_3(t, z)$ and all other components equal to zero, the spatial dependence breaks momentum conservation along z so there are more degrees of freedom. For such fields, it is convenient to change the variables to

$$P = \frac{1}{2}(p_3' - p_3) \qquad \Delta = p_3 + p_3' \tag{1.21}$$

and denote by $p_\perp$, $p_\perp'$ the momentum values transverse to the polarization of the field x and y. Since the field only depends on z, the transverse momentum is still conserved $p_\perp + p_\perp' = 0$ and the maximum is at $p_\perp = p_\perp' = 0$. For fields symmetric with respect to t and z the spectrum is symmetric with

[4] $\mathbf{x}_\pm$ denoting the space components of the initial/final points $x_\pm$ of the propagator (1.16).

respect to $\Delta \to -\Delta$, so the saddle point is $\Delta_s = 0$. Denoting $\mathscr{P}$ the nontrivial saddle point of P we have

$$\mathbb{P}(p,p') \sim \exp\left\{-\frac{\mathcal{A}}{E} - \frac{p_\perp^2}{E\,d_\perp^2} - \frac{(P-\mathscr{P})^2}{E\,d_P^2} - \frac{\Delta^2}{E\,d_\Delta^2}\right\} \,, \tag{1.22}$$

where $\mathcal{A} = \mathcal{A}(p_s, p'_s)$. The advantage is that the three widths $d_\perp$, d_P, and d_Δ in the spectrum can be calculated numerically in a simple way by solving some ODE involving the instantons at the maximum (p_s, p'_s), therefore, instead of calculating many instantons and integrating them for each value of the momenta, one has to do this only once. Furthermore, the instantons at the maximum are numerically easier to find due to symmetry.

To lower the threshold for Schwinger pair production, one possibility is using colliding laser pulses [66]. If we consider an e-dipole field [67, 68], a $4D$ field with optimal focusing, the spectrum has more degrees of freedom because no momentum component is conserved. Defining variables as before

$$P_i = \frac{1}{2}(p'_i - p_i) \qquad \Delta_i = p_i + p'_i \tag{1.23}$$

we have, due to symmetry, $P_{\perp s} = \Delta_{\perp s} = \Delta_{zs} = 0$, so the only nontrivial momentum maximum is $\mathscr{P} = P_{zs}$. The spectrum now has the form

$$\mathbb{P}(p,p') \sim \exp\left\{-\frac{\mathcal{A}}{E} - \frac{\Delta_\perp^2}{E\,d_{\Delta,\perp}^2} - \frac{\Delta_z^2}{E\,d_{\Delta,z}^2} - \frac{P_\perp^2}{E\,d_{P,\perp}^2} - \frac{(P_z-\mathscr{P})^2}{E\,d_{P,z}^2}\right\} \,. \tag{1.24}$$

The punchline is that we can find the spectrum in the semiclassical approximation using the following simple recipe:

1. find the instanton $q(u)$ at the momentum maximum
2. perform an integral involving $q(u)$ to find $\mathcal{A}$
3. solve some linear ODEs to compute the widths

We point out that, from a numerical point of view, this method is quite efficient and typically stable. The point where we need to be the most careful is step 1 because, since we use the shooting method, choosing a clever initial guess is essential, and even more so for very fast pulses with imaginary poles [69], where the instantons have tricky branch cuts that require some care. Step 3 does not require any shooting since the ODEs in question have simple initial conditions at $u = 0$. Once we have all the widths, we find immediately the integrated probability and we can compare with the effective action method.

The method outlined here is a considerable simplification compared to finding the spectrum on some grid points around the maximum. If we have a more realistic spacetime field with several peaks this method is even more powerful because the spectrum is no longer a simple Gaussian, so to resolve the multiple peaks properly we would need a great number of sample points, which is numerically very intensive. For such fields the expansion above is slightly more general, but the idea is the same–we find the instantons and then use them to calculate a small number of numerical parameters that give the full spectrum in the saddle-point approximation.

These instantons also have a physical interpretation. The saddle-point method is a semiclassical approximation in which we only consider contributions from the solution to the classical equation of motion (1.17) plus Gaussian fluctuations around it. However, since we are dealing with a classically forbidden process, namely particles tunneling from the vacuum, the instantons are complex functions. Allowing also the proper time to be complex, i.e. choosing a complex contour $u(r)$ in (1.17), we find a particular contour which splits the instantons into a "creation" region and an "acceleration" region. In the creation region the spatial components are real but time is imaginary so the particles are tunneling, while in the acceleration region all components are real and the solution describes the propagation of real particles accelerated by the field. Once we allow proper time to be complex, we can interpret the instantons as analytic functions defined on some region of the complex u-plane. A way to visualize them is to use domain coloring on the complex u-plane and add contour lines for some desired quantity (e.g. modulus or real/imaginary parts) as in Fig. 1.1. As we can see, analytical continuation of the instantons on the complex plane encounters branch points, so in the plot we show only one Riemann sheet. We show that such branch points are due to the singularities of the electromagnetic field $F^{\mu\nu}$. In Fig. 1.1 we can also see the physical contour mentioned above represented by the green line.

The other process we consider is nonlinear Breit-Wheeler pair production, $\gamma \to e^- e^+$, i.e. we add a high-energy photon to catalyze the process. The energy provided by the hard photon significantly enhances the probability [70] and the process remains nonperturbative if we are away from the regime $a_0 = E/\omega \ll 1$ where ω is the laser frequency. Similarly to the Schwinger effect, we can write the spectrum in terms of some widths and saddle points. In addition, we study the effect of a Gaussian wave packet for the photon momentum k on the spectrum and the total integrated probability.

We consider a $2D$ electric field $E_3(t, z)$ and a photon wave packet

$$f(k) \sim \exp\left\{ -\sum_{j=1}^{3} \frac{(k_j - l_j)^2}{2\lambda_j^2} \right\} , \tag{1.25}$$

centered around $l_1 > 0$, $l_2 = l_3 = 0$ with very narrow widths $\lambda_1, \lambda_2 \ll 1$ for the x and y momenta but a generic λ_3 for the z momentum. We find that there is a phase transition at some finite critical size of the wave packet $\lambda = \lambda_c$ where the spectrum splits from one to two peaks as shown in Fig. 1.2 and the distance Δ_s of the two peaks behaves like an order parameter for a continuous phase transition.

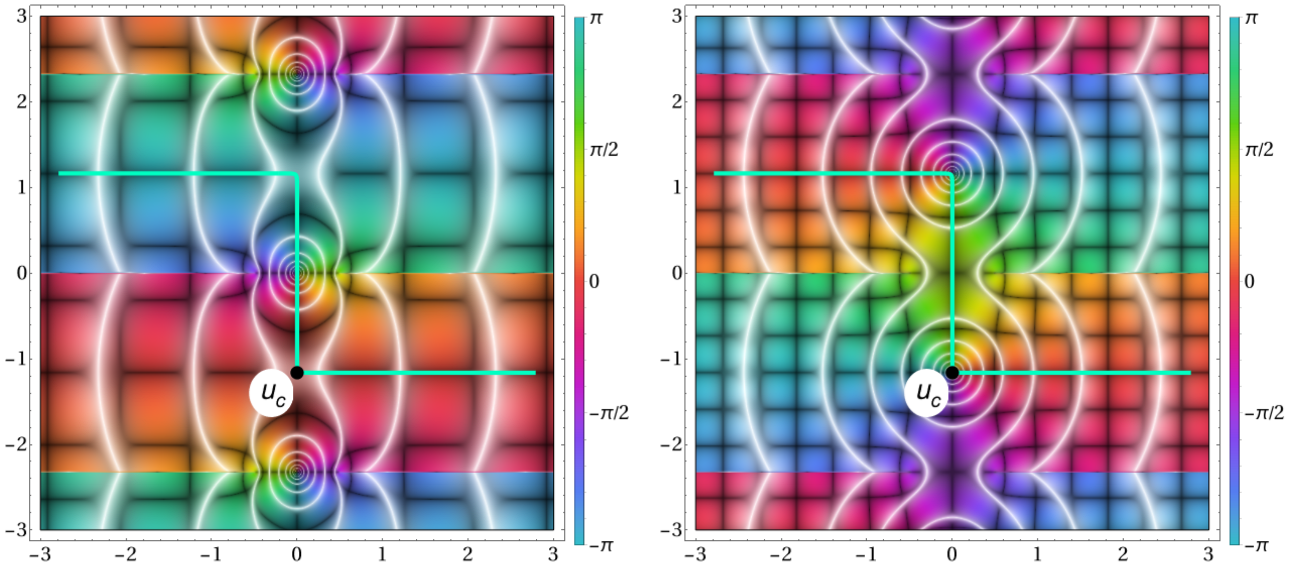


Figure 1.1: Component $z(u)$ (left) and $t(u)$ (right) of the instanton for Schwinger pair production in a double Sauter pulse $E_3(t,z) = E\,\mathrm{sech}(\omega t)^2\,\mathrm{sech}(\kappa z)^2$. The parameters are given by $\gamma_\omega = \omega/E = 1$ and $\gamma_k = \kappa/E = 1$. The green line is the physical contour that produces the split into formation (vertical) and acceleration (horizontal) regions. The color represents the phase according to the legend on the right. In particular, red indicates a region with a large positive real part and light blue a region with large negative real part. The white lines represent contour lines of $|q|$ and the black ones contour lines of Re(q) and Im(q). u_c indicates the zero of $t(u)$, which is also a stationary point for the spatial component $z'(u_c) = 0$.

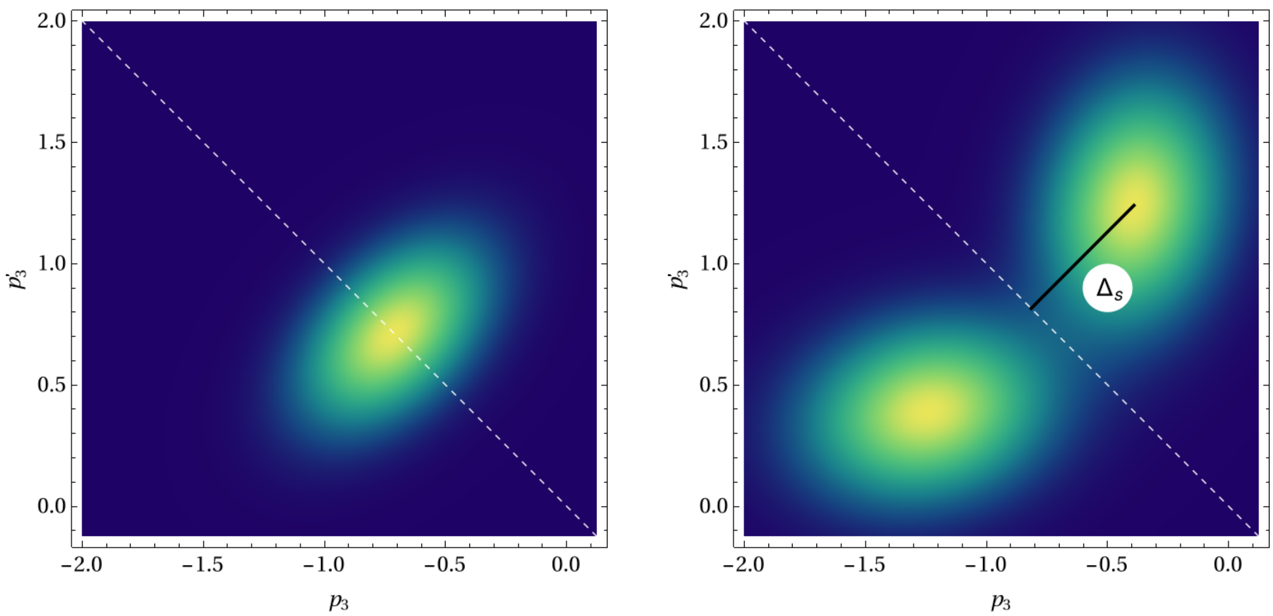


Figure 1.2: Longitudinal spectrum before (left) and after (right) the phase transition at $p_\perp = p_{\perp s}$ with parameters $E = 1/20$ and $\Omega = \gamma_t = \gamma_z = 1$. The plot on the left is at $\lambda = 2\sqrt{E}$ and the one on the right at $\lambda = 3.2\sqrt{E}$. The critical point is $\lambda_c \simeq 2.23\sqrt{E}$. The dashed line represents the set of points with $\Delta = 0$.

Each peak of the spectrum has the form

$$\mathbb{P}(p,p') \sim \exp\left\{-\frac{\mathcal{A}}{E} - \frac{p_2^2}{E\,d_2^2} - \frac{1}{E}(\Pi - \Pi_s)\cdot \mathbf{d}^{-2}\cdot(\Pi - \Pi_s)\right\} \tag{1.26}$$

where $\Pi = (p_1, \Delta, P)$ is a collective variable for all particle momenta. Compared to the previous case, the spectrum has a more complicated structure encoded in the matrix $\mathbf{d}^{-2}$, but the general idea is the same. One does not have to find the spectrum on a grid of momenta, but only the widths.

In this case, the Lorentz force equation receives an extra term $k^\mu \delta(u)$ from the photon

$$\frac{d^2 q^\mu}{du^2} = F^{\mu\nu} \frac{dq_\nu}{du} + k^\mu \delta(u) \tag{1.27}$$

which begs the question about the meaning of $\delta(u)$ when we choose a complex contour. The interpretation is in fact quite simple. The delta function splits the instanton into two: at $u > 0$ (before contour deformation) we denote the instanton by $q_{(+)}(u)$ and at $u < 0$ by $q_{(-)}(u)$ with a discontinuity in the derivative at $u = 0$ due to the $k^\mu \delta(u)$ term

$$\frac{dq^\mu_{(+)}}{du}(0) - \frac{dq^\mu_{(-)}}{du}(0) = k^\mu \ . \tag{1.28}$$

We can view $u = 0$ as the value of proper time where the photon absorption happens and $q_{(+)}(u)$ and $q_{(-)}(u)$ as describing the electron/positron respectively. Both functions can be defined over the whole complex plane (up to branch cuts), so if we want a single-valued instanton we can glue them together defining

$$q(u) := \begin{cases} q_{(-)}(u) & \text{Im}(u) > 0 \\ q_{(+)}(u) & \text{Im}(u) < 0 \end{cases} \tag{1.29}$$

obtaining Fig. 1.3. In this case we choose the contour at an angle because the zeros $u_c^\pm$ of $t(u)$ are not along the imaginary axis.

1.3 Outline of the thesis

We begin with a brief introduction to the worldline formalism in chapter 2, used throughout the following chapters, providing a representation of the spinor propagator in terms of a particle path integral rather than an integral over fields. In chapter 3 we calculate the Schwinger pair production spectrum for spacetime fields, first for $2D$ and then $4D$ fields. We compute probabilities and the spectrum widths at various field parameters. We also explain the method for finding the instantons and study their analytic properties as functions of a complex variable. We also show how the interpretation of the pair production as a tunneling phenomenon arises naturally from the instantons. Finally, for dipole fields we derive an analytic slow pulse approximation beyond the locally constant field method. In chapter 4 we study the Breit-Wheeler spectrum for $2D$ fields and the properties of the phase transition of the spectrum. We find that the spectrum becomes infinitely spread out in one direction as we reach the critical point, indicating that the semiclassical approximation breaks down.

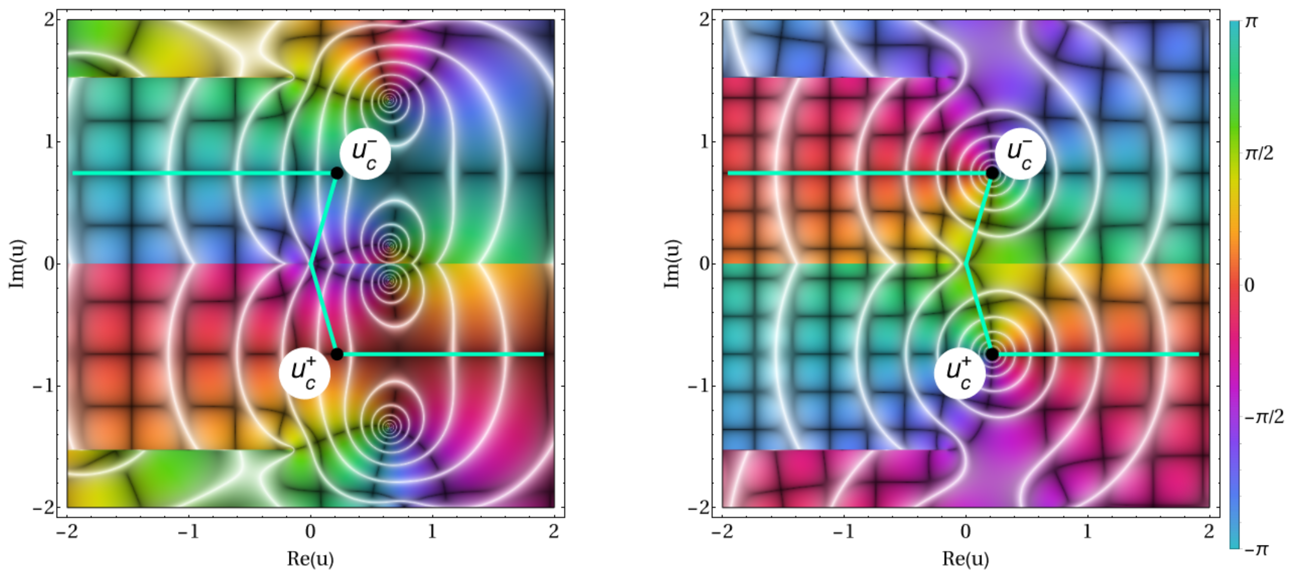


Figure 1.3: Components $z(u)$ (left) and $t(u)$ (right) of the instanton for Breit-Wheeler pair production defined with (1.29) in a double Sauter pulse $E_3(t,z) = E\,\mathrm{sech}(\omega t)^2\,\mathrm{sech}(\kappa z)^2$. The parameters are given by $\gamma_\omega = \omega/E = 1$ and $\gamma_k = \kappa/E = 1$ for the field, and $\{k_1, k_2, k_3\} = \{1, 0, 1.15\}$, $\lambda = 3\sqrt{E}$ for the incoming photon wave packet. The green line is the physical contour that produces the split into formation and acceleration regions.

We look also at the spacetime trajectory of the pair after particle creation. In appendix A we explain briefly the saddle-point ideas and calculate the worldline Green's function for a time-dependent field. In Appendix B we provide some basic notions of complex analysis and explain how we use them to study the rich structure of the worldline instantons. In Appendices C and D we use the aforementioned representation of the propagator in the LSZ reduction formula to derive the exact solutions of the Dirac equation and the propagator in a plane wave and in a constant electric field. We show that, in a constant field, one has to choose the asymptotic states carefully to obtain the correct result, because the field extends infinitely and the particles are not free asymptotically. We also derive the amplitude for Schwinger pair production (for the constant field) and Breit-Wheeler (for both the plane wave and the constant field).

Chapter 2

Worldline formalism

IN THIS CHAPTER WE derive some basic results of the worldline formalism. The usual way to make quantum mechanics a relativistic theory is to treat space and time on the same footing as labels on quantum fields constructing a quantum field theory. In this framework, we either deal with operators or represent propagators as path integrals over field configurations. However, in the worldline formalism approach, more similar to string theory, we represent propagators and amplitudes in terms of single particle path integrals [61, 62, 63] as in nonrelativistic quantum mechanics. The standard material is reviewed in [71, 72]. In this thesis we do not examine the foundations of the worldline formalism; we simply use worldline representations of the propagators in the LSZ reduction formula to compute scattering amplitudes. While this is not the simplest method to work with for plane waves, constant fields or one-dimensional fields, for which either exact or WKB solutions are known, the main advantage is that it allows us to go beyond and study spacetime fields quite effectively.

2.1 Scalar particle

Assuming that the Hamiltonian for a relativistic particle is the well-known energy

$$H = \sqrt{\mathbf{p}^2 + m^2}\ , \tag{2.1}$$

performing a Legendre transform we find

$$\mathcal{L} = \mathbf{p}\cdot\dot{\mathbf{x}} - H = -m\sqrt{1-\dot{\mathbf{x}}^2} \tag{2.2}$$

hence the action that describes a massive relativistic scalar particle is the integral of the line element $ds = \sqrt{dt^2 - d\mathbf{x}^2}$

$$S = m\int_{t_0}^{t_1} dt\,\sqrt{1-\dot{\mathbf{x}}^2}\ . \tag{2.3}$$

However, this is not nice because it is not manifestly Lorentz invariant. We can perform a change of variable to τ with $t \to t(\tau)$

$$S = m\int_0^1 d\tau\,\sqrt{\dot{x}^2} \tag{2.4}$$

where now $\dot{x}(\tau)^2 = \dot{t}(\tau)^2 - \dot{\mathbf{x}}(\tau)^2$. Although the notation τ resembles proper time, we can perform a change of variable $\tau \to \tau' = F(\tau)$ where $F(\tau)$ is any diffeomorphism from $[0,1]$ to $[0,1]$. This reparametrization invariance is a gauge symmetry because it reflects a redundancy in our description.

While (2.4) is manifestly Lorentz invariant, using it to perform a path integral is quite a challenge since it is not quadratic. The trick to make it quadratic is straightforward: we add an extra degree of freedom that is not dynamical (the action does not depend on its derivative) such that, if we solve the Euler-Lagrange equations for it, we obtain (2.4). Let us call this degree of freedom $e(\tau)$. We must reobtain (2.4) when we solve

$$\frac{\partial \mathcal{L}}{\partial e} = 0 \tag{2.5}$$

and $\mathcal{L}$ shall be quadratic. We can easily see that

$$\mathcal{L} = \frac{\dot{x}^2}{2e} + \frac{m^2 e}{2} \tag{2.6}$$

does the trick, since $e = \sqrt{\dot{x}^2}/m$. The field $e(\tau)$ is called an einbein. Interestingly, (2.6) also describes a particle propagating in 0+1 dimensional spacetime with external gravity $g_{00}(\tau) = e(\tau)$. The corresponding Hamiltonian is obtained immediately with a Legendre transform

$$H = \frac{e}{2}\left(p^2 - m^2\right) . \tag{2.7}$$

We can now compute a path integral with (2.6). Analogously to QED, integrating over all configurations of $x(\tau)$ and $e(\tau)$ would produce an infinite result due to gauge invariance. One can show [72] that the correct gauge fixed path integral is (denoting the integration variable path as $q(\tau)$ and the initial/final points x and y)

$$G(y,x) := \frac{1}{2}\int_0^\infty dT \int_x^y \mathcal{D}q\, e^{-i\int d\tau\, \frac{\dot{q}^2}{2T} + \frac{m^2}{2}T} . \tag{2.8}$$

To check that this gives the Klein-Gordon propagator, the easiest way is in momentum space

$$\tilde{G}(p,p') = \int d^4x d^4y\, e^{ipx+ip'y} G(y,x) . \tag{2.9}$$

The path integral can be computed by changing the variable to $q(\tau) \to x + (y-x)\tau + q(\tau)$ so that $q(0) = q(1) = 0$ using the normalization

$$\int \mathcal{D}q\, e^{-\frac{i}{2T}\int d\tau\, \dot{q}^2} = \frac{1}{(2\pi T)^2} . \tag{2.10}$$

The last integral over T is defined in terms of the analytic continuation of

$$\int_0^\infty dT\, e^{iTz} = \frac{i}{z} \tag{2.11}$$

from Re$(z) > 0$ to a purely imaginary exponent Re$(z) = 0$. This is equivalent to integrating along a slightly tilted contour close to the real axis and letting $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0^+} \int_0^{(1+i\varepsilon)\infty} dT\, e^{iT} = i . \tag{2.12}$$

Up to a phase we obtain the Klein-Gordon propagator

$$\tilde{G}(p,p') = \frac{1}{p^2 - m^2}\, \delta(p+p') . \tag{2.13}$$

We can now couple the scalar particle to an external electromagnetic field A by minimal coupling[1] $p \to p - A$ in the phase space action

$$H = \frac{e}{2}\big((p-A)^2 - m^2\big) \tag{2.14}$$

from which we Legendre transform it back to obtain

$$\mathcal{L} = \frac{\dot{x}^2}{2e} + \dot{x}^\mu A_\mu + \frac{m^2 e}{2}\ . \tag{2.15}$$

This gives us the representation for the exact dressed propagator

$$G(y,x) = \int_0^\infty dT\, e^{-i\frac{m^2}{2}T} \int_x^y \mathcal{D}q\, e^{-i\int d\tau\, \frac{\dot{q}^2}{2T} + A\cdot\dot{q}} \tag{2.16}$$

which will keep us company in the next chapters. Unlike the Quantum Field Theory representation, here we have a path integral over real trajectories, which are functions of only one parameter τ.

We can derive this representation by brute force as well. By definition

$$G(y,x) = -\langle y|\frac{1}{-\partial^2 - m^2}|x\rangle = \frac{i}{2}\int_0^\infty dT\, \langle y|e^{-\frac{iT}{2}(\partial^2+m^2)}|x\rangle \tag{2.17}$$

so if we compare this expression with the propagator[2] (with $m = 1$)

$$\langle \mathbf{y}, t|e^{-it\frac{-\nabla^2}{2}}|\mathbf{x}, 0\rangle = \int_{\mathbf{x}}^{\mathbf{y}} \mathcal{D}\mathbf{q}\, e^{-i\int_0^1 -\frac{\dot{\mathbf{q}}^2}{2t}} \tag{2.18}$$

we simply replace $t \to T$ and

$$-\nabla^2 \to \partial_0^2 - \nabla^2 = \partial^2 \qquad\qquad -\dot{\mathbf{q}}^2 \to (\dot{q}^0)^2 - \dot{\mathbf{q}}^2 = \dot{q}^2 \tag{2.19}$$

to obtain (up to a phase)

$$G(y,x) = \frac{1}{2}\int_0^\infty dT \int_x^y \mathcal{D}q\, e^{-i\int d\tau\, \frac{\dot{q}^2}{2T} + \frac{m^2}{2}T} \tag{2.20}$$

which is precisely (2.8).

There is a nice way to incorporate photons into this treatment. Usually, in Quantum Field Theory, we add photons in the LSZ reduction formula as new external states and amputate to derive the Feynman rules. In the Worldline Formalism, they appear as plane waves in the *classical* external field $A(x)$ [71, 73, 74], as we are going to show.

[1] The electric charge is absorbed into the definition of the field $qA \to A$.

[2] $\mathbf{q}$ denoting only the space components since in nonrelativistic quantum mechanics one uses a path integral over the spatial coordinates only.

We begin with the LSZ formula for scalar QED with one outgoing photon[3] with momentum l and polarization ε attached to a scalar line, i.e. with one incoming and one outgoing scalar particle with momenta respectively p' and p. In the LSZ we have an amputation for each external particle, i.e.

$$\langle p; l, \varepsilon | p' \rangle = \int d^4x\, d^4y\, e^{ipy - ip'x}\, (\partial_x^2 + m^2)(\partial_y^2 + m^2) \int d^4z\, e^{ilz}\, \varepsilon_\mu \partial_z^2 \langle \Omega | \mathbb{T} \phi^\dagger(x) \phi(y) A^\mu(z) | \Omega \rangle . \tag{2.21}$$

Using the quantum field theory representation

$$\langle \Omega | \mathbb{T} \phi^\dagger(x) \phi(y) A^\mu(z) | \Omega \rangle = \int \mathcal{D}\phi \mathcal{D}\phi^\dagger \mathcal{D}A\, e^{i \int d^4w\, \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{M}}} \phi^\dagger(x) \phi(y) A^\mu(z) \tag{2.22}$$

where $\mathcal{L}_0 = \partial_\mu \phi^\dagger\, \partial^\mu \phi - m^2 \phi^\dagger \phi$, $\mathcal{L}_{\text{int}} = i A^\mu (\partial_\mu \phi^\dagger\, \phi - \phi^\dagger \partial_\mu \phi) + A^2 \phi^\dagger \phi$, and $\mathcal{L}_{\text{M}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, we get scattering at tree level by expanding the interacting Lagrangian at first order

$$e^{i \int d^4w\, \mathcal{L}_{\text{int}}} \sim 1 + i \int d^4w\, A^\nu(w) (\partial_\nu \phi^\dagger\, \phi - \phi^\dagger \partial_\nu \phi) \, . \tag{2.23}$$

The zeroth order is of course zero, and at first order we contract the electromagnetic fields $A^\mu(z)$ and $A^\nu(w)$ obtaining

$$\begin{aligned} &\int d^4z\, e^{ilz}\, \varepsilon_\mu \partial_z^2 \langle \Omega | \mathbb{T} \phi^\dagger(x) \phi(y) A^\mu(z) | \Omega \rangle = \\ &\qquad \int d^4z d^4w\, e^{ilz}\, \varepsilon_\mu\, \delta^{\mu\nu} \partial_z^2 \Delta(w - z) \langle 0 | \mathbb{T} \phi^\dagger(x) \phi(y) (\partial_\nu \phi(w)^\dagger\, \phi(w) - \phi^\dagger(w) \partial_\nu \phi(w)) | 0 \rangle \end{aligned} \tag{2.24}$$

where

$$\langle 0 | \mathbb{T}\, F(\phi, \phi^\dagger)\, | 0 \rangle = \int \mathcal{D}\phi \mathcal{D}\phi^\dagger\, e^{i \int d^4w\, \mathcal{L}_0}\, F(\phi, \phi^\dagger) \tag{2.25}$$

and

$$\delta^{\mu\nu} \Delta(z - w) = \int \mathcal{D}A\, A^\mu(z) A^\nu(w)\, e^{i \int \mathcal{L}_M} \tag{2.26}$$

is the photon propagator in real coordinates. Note that difference between (2.22) and (2.25) is that $|\Omega\rangle$ is the vacuum of the interacting theory, while $|0\rangle$ is the vacuum of the free scalar theory. This is a great simplification because expectations values of the form $\langle 0 | \phi \ldots \phi^\dagger \ldots | 0 \rangle$ are combinations of the free scalar propagator that follow from Wick's theorem.

Using $\partial_z^2 \Delta(w - z) = \delta(w - z)$ (2.24) becomes

$$\begin{aligned} i \int d^4z\, e^{ilz}\, \varepsilon^\mu\, \langle 0 | \mathbb{T} \phi^\dagger(x) \phi(y) \big(\partial_\mu \phi^\dagger(z)\, \phi(z) - \phi^\dagger(z) \partial_\mu \phi(z) \big) | 0 \rangle = \\ \int d^4z\, e^{ilz}\, \varepsilon^\mu\, \frac{\delta}{\delta A^\mu(z)}\, \langle 0 | \mathbb{T} \phi^\dagger(x) \phi(y) | 0 \rangle_{A=0} \end{aligned} \tag{2.27}$$

where

$$\frac{\delta}{\delta A^\mu(z)} \langle 0 | \mathbb{T} \phi^\dagger(x) \phi(y) | 0 \rangle_{A=0} \tag{2.28}$$

[3] We work in Feynman gauge for simplicity.

is the functional derivative of the scalar propagator dressed with a *classical* field, subsequently evaluated at $A = 0$. The functional derivative pulls down a vertex.

At this point we can use the worldline representation of the dressed scalar propagator

$$\langle 0|\mathbb{T}\phi^\dagger(x)\phi(y)|0\rangle_A = \frac{1}{2}\int_0^\infty dT \int_x^y \mathcal{D}q\, e^{-i\int \frac{\dot{q}^2}{2T}+\frac{m^2 T}{2}+A\cdot\dot{q}} \tag{2.29}$$

so clearly

$$\begin{aligned}\int d^4z\, e^{ilz}\, \varepsilon^\mu\, &\frac{\delta}{\delta A^\mu(z)} \langle 0|\mathbb{T}\phi^\dagger(x)\phi(y)|0\rangle_{A=0} \\ &= \frac{-i}{2}\int_0^\infty dT \int_x^y \mathcal{D}q \int_0^1 d\tau_1\, e^{il\cdot q(\tau_1)} \varepsilon\cdot\dot{q}(\tau_1) e^{-i\int \frac{\dot{q}^2}{2T}+\frac{m^2T}{2}} \\ &= \frac{1}{2}\int_0^\infty dT \int_x^y \mathcal{D}q\, e^{-i\int \frac{\dot{q}^2}{2T}+\frac{m^2T}{2}+A(q)\cdot\dot{q}}\Big|_{\text{lin}}\end{aligned} \tag{2.30}$$

where

$$A_\mu(q) = \varepsilon_\mu\, e^{ilq} \tag{2.31}$$

and "lin" means that we take the linear part of the exponent in (2.30) when Taylor expanded in ε, i.e.

$$e^{-i\int A(q)\cdot\dot{q}}\Big|_{\text{lin}} = -i\int_0^1 d\tau_1\, e^{il\cdot q(\tau_1)}\varepsilon\cdot\dot{q}(\tau_1)\ . \tag{2.32}$$

Putting everything together, Eq. (2.21) reduces to

$$\langle p; l, \varepsilon|p'\rangle = \frac{1}{2}\int d^4x\, d^4y\, e^{ipy-ip'x}\,(\partial_x^2+m^2)(\partial_y^2+m^2)\int_0^\infty dT\int_x^y \mathcal{D}q\, e^{-i\int \frac{\dot{q}^2}{2T}+\frac{m^2T}{2}+A(q)\cdot\dot{q}}\Big|_{\text{lin}}\ . \tag{2.33}$$

To summarize, going from (2.21) to (2.33) we have dropped the photon amputation at the price of adding a plane wave background field $A_\mu(q) = \varepsilon_\mu\, e^{ilq}$ to the scalar propagator. This is a convenient trick to add a photon because we now have only one scalar propagator (hence one path integral) and no z-integral as in (2.21). This method generalizes for an arbitrary number of photons, and if we want to have a background field on top of that, we perform the usual modifications to the scalar lines of the LSZ and simply evaluate at $A = A_{\text{ext}}$ at the end. This allows us to deal with Breit-Wheeler with the worldline formalism in a simpler way. A general formula for the N-photon scattering amplitude is called a *master formula*, but we shall not pursue this here [73, 74, 75, 76, 77, 78, 79].

2.2 Spinor particle

We have a number of different representations for the spinor propagator. The most straightforward representation can be quickly derived analogously to the previous section

$$
\begin{aligned}
S(y,x) = & \Big\langle y \Big| \frac{1}{m - i\hat{\not{D}}} \Big| x \Big\rangle = \Big\langle y \Big| \frac{m + i\hat{\not{D}}}{m^2 + \hat{\not{D}}^2} \Big| x \Big\rangle \\
& = (i\not{D}_y + m) \Big\langle y \Big| \frac{1}{\hat{D}^2 + m^2 + \frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}} \Big| x \Big\rangle \\
& = (i\not{D}_y + m) K_A(y,x)
\end{aligned}
\tag{2.34}
$$

where D is the covariant derivative $D = \partial + iA$ (the electric charge being absorbed into the field) and the heat kernel K_A is the scalar propagator with an extra matrix valued potential, for which we already know a worldline path integral representation. The matrix valued potential however forces us to add a path ordering at the exponential, analogously to the Dyson series

$$
\Big\langle y \Big| \frac{1}{\hat{D}^2 + m^2 + \frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}} \Big| x \Big\rangle = \frac{1}{2} \int_0^\infty dT \int_x^y \mathcal{D}q \, e^{-i\int_0^1 d\tau \, \frac{\dot{q}^2}{2T} + \frac{m^2 T}{2} + A\cdot\dot{q}} \, \mathcal{P} \, e^{-\frac{iT}{4}\int \sigma^{\mu\nu} F_{\mu\nu}} . \tag{2.35}
$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and we used $i\gamma^\mu\gamma^\nu F_{\mu\nu} = \sigma^{\mu\nu} F_{\mu\nu}$ thanks to the antisymmetry of F.

The path-ordered exponent is a spin factor. In principle, we could rewrite it in terms of a path integral over Grassmann variables as in [71], but for us it will be more convenient to keep it as it is.

2.3 Effective action

We take now a quick detour into the world of closed loops. The effective action method with closed worldline instantons has proved itself to be an effective tool for perturbative loop calculations in QFT [71] and for nonperturbative Strong-Field QED calculations with plane waves [80, 81, 82], constant fields [70, 83, 84, 85, 86, 87], one-dimensional fields [39, 40], and more general configurations [41, 43, 44, 88]. The vacuum to vacuum amplitude in quantum field theory for a scalar particle in the presence of an external electromagnetic field is given by

$$
e^{i\Gamma[A]} := \langle 0|0\rangle_A = \int \mathcal{D}\phi\mathcal{D}\phi^* \, e^{iS[\phi,\phi^*;A]} \tag{2.36}
$$

with

$$
S[\phi,\phi^*;A] = \int d^4x \, |D\phi|^2 - m^2|\phi|^2 \tag{2.37}
$$

where $D = \partial + iA$. This quantity is very useful for the computation of pair production. Since probabilities must add up to one, we have the approximate relation

$$
P_{\text{pair}} \simeq 1 - |\langle 0|0\rangle_A|^2 = 1 - |e^{i\Gamma[A]}|^2 \simeq 2\text{Im}\,\Gamma[A] \tag{2.38}
$$

thus from the imaginary part of the effective action we immediately obtain the total integrated probability. This method, however, does not allow one to obtain the spectrum of emitted pairs, since P_{pair} is the overall probability that a pair is created, i.e. the integrated spectrum. In the next chapters we will show how the spectrum can be obtained using open worldlines.

Using a well known property of Gaussian integral we simply find

$$e^{i\Gamma[A]} \sim \det \frac{1}{-D^2 + m^2} \tag{2.39}$$

and, with the normalization $e^{i\Gamma[0]} = 1$ due to the fact that the vacuum does not produce any pairs, (2.39) becomes an equality. The effective action is readily found

$$\Gamma[A] = i \ln\det(-D^2 + m^2) = i\,\mathrm{Tr}\ln(-D^2 + m^2) \tag{2.40}$$

and, since the trace is independent of the basis, we can choose a set of position eigenstates

$$\mathrm{Tr}\,O = \int d^4x\,\langle x|O|x\rangle \tag{2.41}$$

and use

$$\ln a = -\int_0^\infty dT\,\frac{e^{-\frac{ia}{2}T} - e^{-\frac{i}{2}T}}{T} \tag{2.42}$$

dropping the second term, we obtain

$$\Gamma[A] = -i\int_0^\infty \frac{dT}{T}\int d^4x\,\langle x|e^{-\frac{iT}{2}(-D^2+m^2)}|x\rangle\ . \tag{2.43}$$

The transition amplitude between two position eigenstates can be expressed in term of a particle path integral [61], giving a final expression

$$\Gamma[A] = -i\int_0^\infty \frac{dT}{T}\oint_{PBC}\mathcal{D}q\,e^{-i\int_0^1 d\tau\,\frac{\dot{q}^2}{2T}+A\cdot\dot{q}} \tag{2.44}$$

with

$$\oint_{PBC}\mathcal{D}q = \int d^4x\int_{x(0)=x}^{x(1)=x}\mathcal{D}q\ . \tag{2.45}$$

The spinor QED analog is simply multiplied by a spin factor

$$\Gamma_{\text{spin}}[A] = -\frac{i}{2}\int_0^\infty \frac{dT}{T}\oint_{PBC}\mathcal{D}q\,e^{iS[q;A]}\,\mathrm{Tr}\,\mathcal{P}\,e^{-\frac{iT}{4}\int\sigma^{\mu\nu}F_{\mu\nu}}\ . \tag{2.46}$$

This spin factor (now including a trace) can be expressed in terms of a fermionic path integral

$$\int_{\psi(0)=-\psi(1)}\mathcal{D}\psi\,e^{-\frac{1}{2}\int_0^1 d\tau\,\psi_\mu\dot{\psi}^\mu+T\psi^\mu F_{\mu\nu}\psi^\nu} \tag{2.47}$$

where ψ is a Grassmann, fermionic variable that makes the action supersymmetric [71].

Chapter 3

Schwinger pair production

WE PRESENT A WORLDLINE APPROACH with open instantons to compute the spectrum of pairs produced in spacetime fields. While this method can be used for simpler fields such as plane waves, constant fields, and 1D fields, its true potential is allowing us to make the leap to a general spacetime field. The calculation is performed using the LSZ reduction formula with the worldline propagator and the saddle-point method to approximate the nontrivial integrals. We show how to obtain simple expressions for the spectrum widths and use them to compare with the effective action method for a few example fields such as a simple product of Sauter pulses and the e-dipole pulse. The instantons play a major role in finding the spectrum, and developing a systematic way to find them is crucial. With a particular choice of parametrization for proper time, we find a natural interpretation of the instantons as describing tunneling in the central region and free particles asymptotically. The material in this chapter is taken from [64, 65].

3.1 Pair production amplitude in a 2D field

3.1.1 Definitions

We begin by writing the amplitude with the LSZ reduction formula

$$M = \lim_{t_\pm\to\infty} \int d^3x_+\, d^3x_-\, e^{ipx_+ + ip'x_-}\, \bar{u}_s(p)\, \gamma^0\, S(x_+, x_-)\, \gamma^0\, v_{s'}(p') \tag{3.1}$$

where p and p' are respectively the electron/positron momenta, $\bar{u}$ and v the free Dirac spinors in the basis

$$\begin{aligned} u_s(p) &= \frac{1}{\sqrt{2p_0(p_0+p_3)}}(\not{p}+1)R_s \\ v_s(p) &= \frac{1}{\sqrt{2p_0(p_0-p_3)}}(-\not{p}+1)R_s \end{aligned} \tag{3.2}$$

with R_s chosen such that $\gamma^0\gamma^3 R_s = R_s$ ($s = 1, 2$), S the Dirac propagator in a background field A_μ in the worldline representation (2.34)

$$S(x_+, x_-) = (i\not{D}_{x_+} + 1)\frac{1}{2}\int_0^\infty dT \int_{x_-}^{x_+} \mathcal{D}q\, e^{-i\int_0^1 d\tau\, \frac{\dot{q}^2}{2T}+\frac{T}{2}+\dot{q}\cdot A}\, \mathcal{P}\, e^{-\frac{iT}{4}\int \sigma^{\mu\nu}F_{\mu\nu}} \tag{3.3}$$

where D is the covariant derivative $D = \partial + iA$, $\mathcal{P}$ is the proper-time ordering symbol, $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and we have set $m = 1$ so that all energy scales are relative to the electron mass. In this section we consider an electric field $E_3(t, z)$ polarized along the z direction with gauge potential $A_3(t, z)$. For numerical computations, we specialize to a product of Sauter pulses

$$A_3(t, z) = \frac{E}{\omega}\tanh(\omega t)\,\text{sech}^2(\kappa z)\ . \tag{3.4}$$

The asymptotic states are free because, although in this gauge A_3 is nontrivial when $t \to \infty$, the instantons will satisfy $|z| \to \infty$ as well. We may also choose a gauge where the potential manifestly vanishes when $t \to \infty$ such as $A_0(t, z) = -\frac{E}{\kappa}\tanh(\kappa z)\,\text{sech}^2(\omega t)$. Since the electromagnetic potential

$A(q)$ is a nonlinear function of the path integral variable $q(\tau)$, the integrand is no longer Gaussian so some of the integrals cannot be performed exactly. Rescaling $q \to q/E$, $x_\pm \to x_\pm/E$, and $T \to T/E$ we see that the scalar part of the exponent is proportional to $1/E$

$$\frac{1}{E}\left(ipx_+ + ip'x_- - i\int_0^1 d\tau\,\frac{\dot{q}^2}{2T} + \frac{T}{2} + \dot{q}\cdot A(q/E)\right) \tag{3.5}$$

so for field strengths much smaller than the Schwinger limit $E \ll 1$, which is what we consider in this work, we use the saddle-point method [39] to compute the integrals. Since the spin factor in (3.1) is of order $\mathcal{O}(E^0)$, for the saddle point equations we only consider the scalar terms (3.5).

If we have a field that depends on some characteristic frequencies $\{\omega_1, \ldots, \omega_n\}$, before computing any integrals we already see that the terms inside the parenthesis in (3.5) only depend on the field E and the ω_i through the Keldysh parameters γ_i

$$\gamma_i := \frac{\omega_i}{E} \tag{3.6}$$

so in what follows we always regard the γ_i as independent of E. In the saddle-point approximation we consider the leading order contribution in E, so the spectrum and the integrated probability will have a scaling of the form[1]

$$\mathbb{P}(p,p') = E^a\,\mathcal{S}(\gamma_i)\,e^{-\frac{\mathcal{A}(\gamma_i)}{E}-\frac{1}{E}(\mathbf{\Pi}-\mathbf{\Pi}_s)\cdot\mathbf{d}^{-2}(\gamma_i)\cdot(\mathbf{\Pi}-\mathbf{\Pi}_s)} \qquad \mathbb{P} = E^b\,\mathcal{F}(\gamma_i)\,e^{-\frac{\mathcal{A}(\gamma_i)}{E}} \tag{3.7}$$

where a, b are some constants and $\mathcal{A}, \mathcal{S}, \mathcal{F}$, and $\mathbf{d}^{-2}$ are functions of the field parameters γ_i. Since the dependence on E is trivial (3.7), in the present work we almost always focus on how the maximum $\mathcal{A}$, prefactor, widths, etc. change with the Keldysh parameters γ_i without the overall factors of E for convenience. In some cases, for example if we want to plot the spectrum for fields with several peaks, we resurrect the factor of E and fix it to some numerical value. For the 2D fields we consider in this section we define

$$\gamma_\omega = \frac{\omega}{E} \qquad \gamma_k = \frac{\kappa}{E}\;. \tag{3.8}$$

3.1.2 Exponent

To compute the exponent, we make no assumptions on the field other than the fact that it vanishes asymptotically. The saddle point equations for the path integral are the Lorentz force equations

$$\ddot{q}^\mu = TF^{\mu\nu}\dot{q}_\nu\;. \tag{3.9}$$

Multiplying (3.9) by $\dot{q}_\mu$ we also see that $\dot{q}^2$ is independent of τ. For the ordinary variables T and $\mathbf{x}_\pm$ respectively

$$T^2 = q^2 \qquad \dot{q}_i(1) = Tp_i \qquad \dot{q}_i(0) = -Tp'_i\;. \tag{3.10}$$

[1] Extracting all factors of E for the sake of clarity.

The first is an on-shell condition, the other two fix initial/final derivatives of the instanton in terms of the momenta. They can be shown by considering a variation δq which is only nonzero at the boundary. We can now evaluate the exponent using these instanton equations. We integrate $\int \dot{q}^2$ by parts and use

$$\ddot{q}_\mu = TF_{\mu\nu}\dot{q}^\nu = T\partial_\mu A_\nu \dot{q}^\nu - T\dot{A}_\mu \tag{3.11}$$

from which

$$-\frac{iT}{2} - i\int_0^1 \frac{\dot{q}^2}{2T} + A\,\dot{q} = -i\left[\frac{\dot{q}}{T}\right]\cdot q\Big|_0^1 + i\int_0^1 q^\mu\,\partial_\mu A_\nu \dot{q}^\nu\ . \tag{3.12}$$

Using (3.10), these boundary terms cancel against the contributions from the asymptotic states $ipx_+ + ip'x_-$ in (3.1). Changing the variable to $u = T_s(\tau - \sigma)$, where T_s is the saddle-point value of the T-integral and σ is chosen so that at $u \sim 0$ the worldline passes through the field, the final exponent becomes

$$\boxed{\psi = i\int_{\mathcal{C}} du\, q^\mu\,\partial_\mu A_\nu \frac{dq^\nu}{du}} \tag{3.13}$$

where the contour extends from $-\infty$ to $+\infty$ because T_s, i.e. the total proper time, goes to infinity in the asymptotic limit as we show later.

The difference with the effective action [39] and open instanton results [89] in time-dependent fields is that the final expression (3.13) for the exponent explicitly contains the instantons. This will motivate us to study their properties in detail and to develop a method to find them efficiently. For now it is enough to point out that Cauchy's theorem tells us that (3.13) is manifestly independent of the parametrization as long as $\mathcal{C}$ goes from $-\infty$ to $+\infty$ and we do not cross any poles or branch cuts of the integrand. If we start with a real parametrization, it is not only convenient but sometimes necessary to change it in order to find "good" instantons at all. Apart from the computational point of view, some parametrizations allow for a clearer picture of what the instantons represent. Note that the instanton appears to be traveling backward in time in the region $(-\infty, 0)$, i.e. where it describes a positron, which agrees with the interpretation of positrons [90, 91].

As a final remark, one might be tempted to interpret the result above as a line integral of a differential form. If such form is closed, one might hope that it is also exact and use Stokes' theorem. Defining $f_\mu = q^\rho\partial_\rho A_\mu$ the integrand becomes $f_\mu dq^\mu$, which indeed has the structure of a 1-form. But such form has no chance of being exact, since its differential

$$\partial_\mu f_\nu - \partial_\nu f_\mu = F_{\mu\nu} + q^\rho\partial_\rho F_{\mu\nu} \tag{3.14}$$

is gauge invariant, and it is zero when $F_{\mu\nu} = 0$. Assuming that our field is gauge equivalent to zero, $A_\mu = \partial_\mu\alpha$, we immediately see that the integrand is exact

$$f_\mu = q^\rho\partial_\rho\partial_\mu\alpha = \partial_\mu(q^\rho\partial_\rho\alpha - \alpha) \tag{3.15}$$

therefore defining $\omega = q^\rho \partial_\rho \alpha - \alpha$ we find

$$\int_{\mathcal{C}} dq^\mu \, \partial_\mu \omega = \omega(\infty) - \omega(-\infty) \tag{3.16}$$

which gives a pure phase.

3.1.3 Path integral

As to the prefactor, one needs to be careful about the order in which one computes the integrals when using the saddle-point method. Given an integral with respect to some variables $x_1, \dots x_n$, if we compute e.g. x_1 first then the exponent is evaluated at the saddle point for x_1 as a function of all the other variables $x_{1s}(x_2, \dots x_n)$, hence when taking derivatives of the exponent with respect to $x_2, \dots x_n$ to compute the other integrals the dependence in $x_{1s}(x_2, \dots x_n)$ must be taken into account.

We start from the path integral with the Gelfand-Yaglom method [40]. Expanding the exponent of (3.1) around the instantons $q \to q + \delta q$, where q are now the instantons, we find

$$\exp\left\{ -\frac{i}{2T} \int_0^1 \begin{pmatrix} \delta t & \delta z \end{pmatrix} \Lambda \begin{pmatrix} \delta t \\ \delta z \end{pmatrix} \right\} , \tag{3.17}$$

with

$$\Lambda = T \begin{pmatrix} -\frac{1}{T}\partial_\tau^2 + A_{tt}\dot{z} & A_{tz}\dot{z} + A_t \partial_\tau \\ A_{tz}\dot{z} - \partial_\tau A_t & \frac{1}{T}\partial_\tau^2 - A_{tz}\dot{t} \end{pmatrix} , \tag{3.18}$$

where $A_{tz} = \partial_t \partial_z A$ and so on. The path integral over transverse coordinates gives the free normalization (2.10)

$$\int \mathcal{D}q_\perp e^{i\int \frac{q_\perp^2}{2T}} = \frac{1}{2\pi T} \tag{3.19}$$

while for the t, z components we have (3.1)

$$\int \mathcal{D}\delta q_t \mathcal{D}\delta q_z \, e^{-\frac{i}{2T}\delta q \cdot \Lambda \cdot \delta q} = \frac{1}{2\pi T} \frac{1}{\sqrt{\det \Lambda}} \tag{3.20}$$

with $\det \Lambda$ given by

$$\det \Lambda = \det \begin{pmatrix} \phi_1^{(1)} & \phi_1^{(2)} \\ \phi_2^{(1)} & \phi_2^{(2)} \end{pmatrix} (\tau = 1) \tag{3.21}$$

where the $\phi^{(i)}$ are solutions to

$$\Lambda \phi^{(i)} = 0 \qquad \phi_j^{(i)}(0) = 0 \qquad \dot{\phi}_j^{(i)}(0) = \delta_j^i \, . \tag{3.22}$$

Since we do not need the result for general $t_\pm$ but only in the asymptotic limit, there are considerable simplifications. We expect to be able to write $\det \Lambda$ as

$$\det \Lambda = (\text{divergent terms})(\text{finite numerical terms}) \tag{3.23}$$

where the divergent terms contain some powers of $t_\pm$ and perhaps T. In other words, we want to analytically extract the divergent contributions in the asymptotic limit $t_\pm \to \infty$ and obtain a finite numerical contribution. We expect this to be possible because the field is only nonzero in a bounded region around $u = 0$. We briefly outline the idea but more details can be found in [64].

We begin by changing variable to $u = T(\tau - \sigma)$ and letting $u_0 = -T\sigma$, $u_1 = T(1-\sigma)$, i.e. the initial and final point of the worldline. In the asymptotic limit we thus have $u_0 \to -\infty$ and $u_1 \to +\infty$. We also define $(\tilde{u}_0, \tilde{u}_1)$ to be the region where the field can be considered nonzero, which is finite as $t_\pm \to \infty$. Since the field is zero in the region $(u_0, \tilde{u}_0)$ and we have initial conditions (3.22) the solutions $\phi^{(i)}$ are straight lines until $\tilde{u}_0$

$$\phi^{(1)}(\tilde{u}_0) \simeq \begin{pmatrix} (\tilde{u}_0 - u_0)/T \\ 0 \end{pmatrix} \qquad \frac{d\phi^{(1)}}{du}(\tilde{u}_0) \simeq \begin{pmatrix} 1/T \\ 0 \end{pmatrix} \tag{3.24}$$

and similarly for $\phi^{(2)}$. Then, using

$$\tilde{u}_0 - u_0 = \int_{t_0}^{\tilde{t}_0} \frac{dt}{t'} = \frac{t_0}{p_0'} + \mathcal{O}(1) \tag{3.25}$$

we can rewrite is as

$$\phi^{(1)}(\tilde{u}_0) \simeq \begin{pmatrix} t_0/(Tp_0') \\ 0 \end{pmatrix} + \mathcal{O}(1/T) \; . \tag{3.26}$$

The subtle point, now, is where we are allowed to take the $T \to \infty$ limit, since one might be tempted to just set $d\phi/du$ to zero. However, doing so, we miss a necessary contribution. We can thus split ϕ into

$$\phi^{(j)} = \frac{t_0}{Tp_0'}\phi_d^{(j)} + \frac{1}{T}\phi_n^{(j)} \tag{3.27}$$

with

$$\begin{aligned} \phi_d^{(1)}(\tilde{u}_0) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad & \frac{d\phi_d^{(1)}}{du}(\tilde{u}_0) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \phi_n^{(1)}(\tilde{u}_0) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad & \frac{d\phi_n^{(1)}}{du}(\tilde{u}_0) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; . \end{aligned} \tag{3.28}$$

Plugging the expression for ϕ in terms of ϕ_d and ϕ_n into

$$\det \Lambda = (\phi_1^{(1)}\phi_2^{(2)} - \phi_2^{(1)}\phi_1^{(2)})|_{\tau=1} \tag{3.29}$$

we get contributions of the form $\phi_d\phi_d$, $\phi_d\phi_n$, and $\phi_n\phi_n$. One can show that the last type of terms can be neglected, while the first two combine to give

$$\det \Lambda = \frac{t_+ t_-}{Tp_0' p_0} h(\tilde{u}_1) \tag{3.30}$$

where $h(u) = \eta'(u)$ with $\eta(u)$ a solution to

$$\eta'' = (E^2 + \nabla E \cdot \{z', t'\})\eta \tag{3.31}$$

with initial conditions

$$\eta(\tilde{u}_0) = 1 \qquad \eta'(\tilde{u}_0) = 0 \; . \tag{3.32}$$

The solution $\eta(u)$ is therefore the only contribution we have to find numerically, and it is finite in the asymptotic limit.

3.1.4 Ordinary integrals

After computing the path integral, the exponent is evaluated at the instantons $q(u)$ for generic T and initial/final points $z_\pm$, $t_\pm$. This makes the calculation of the integral with respect to T and $z_\pm$ nontrivial because we do not know the analytical dependence of the instantons with respect to those variables, and for the saddle point method we need to take derivatives of the exponent. Nevertheless, since we only need to compute them in the asymptotic limit $t_\pm \to +\infty$, there are considerable simplifications as for the path integral. We consider a contour $u(r)$ with initial and final points u_0 and u_1, which we will eventually send to infinity. The transverse components are trivial. Shifting $q^\perp(\tau) \to x^\perp_- + (x^\perp_+ - x^\perp_-)\tau + q^\perp(\tau)$ so that the path integral has boundary conditions $q^\perp(0) = q^\perp(1) = 0$ and defining

$$\varphi^\perp = \frac{1}{2}(x^\perp_+ + x^\perp_-) \qquad \theta^\perp = x^\perp_+ - x^\perp_- \tag{3.33}$$

we simply get an overall momentum conserving delta function at the prefactor

$$\int \mathcal{D}q^\perp \int d^\perp x_+ \, d^\perp x_- \to (2\pi)^2 \, \delta_\perp(p + p') \; . \tag{3.34}$$

Note that we only obtain a momentum conserving delta function along the transverse components because the z dependence of the field breaks translation invariance over z.

Let us consider the remaining nontrivial integrals over $z_- = z(u_0)$, $z_+ = z(u_1)$, and T. As we show in Appendix A, for the first derivative we only have to consider the *explicit* dependence of the exponent on the variables since the implicit dependence due to the instantons drops out, hence

$$\frac{\partial \psi}{\partial z_-} = i[p'_3 - z'(u_0)] \qquad \frac{\partial \psi}{\partial z_+} = i[p_3 + z'(u_1)] \qquad \frac{\partial \psi}{\partial T} = \frac{i}{2}(a^2 - m_\perp^2) \tag{3.35}$$

where $a^2 = t'^2 - z'^2$. Note that the reason why these integrals are hard is that the three quantities $z'(u_0)$, $z'(u_1)$ and a^2 are functions of the integration variables $\mathbf{X} := (z_-, z_+, T)$ that we do not know. This also implies that we do not have explicit expressions of the saddle points. Fortunately, the asymptotic limit saves the day thanks to the following trick

$$\begin{aligned} z_- - \tilde{z}_- &= \int_{\tilde{z}_-}^{z_-} dz = \int_{\tilde{t}_-}^{t_-} dt\, \frac{z'}{t'} = -\frac{t_-\, z'(u_0)}{\sqrt{a^2 + z'(u_0)^2}} + \mathcal{O}(1) \\ z_+ - \tilde{z}_+ &= \int_{\tilde{z}_+}^{z_+} dz = \int_{\tilde{t}_+}^{t_+} dt\, \frac{z'}{t'} = \frac{t_+\, z'(u_1)}{\sqrt{a^2 + z'(u_1)^2}} + \mathcal{O}(1) \\ T &= \int_{t_-}^{t_+} \frac{dt}{t'} = \frac{t_-}{\sqrt{a^2 + z'(u_0)^2}} + \frac{t_+}{\sqrt{a^2 + z'(u_1)^2}} + \mathcal{O}(1) \end{aligned} \tag{3.36}$$

obtained exploiting the fact that the instantons are nontrivial only in a bounded interval $(\tilde{u}_0, \tilde{u}_1)$ and straight lines elsewhere. These equations can be inverted to find the saddle points themselves

$$\begin{gathered} z'(u_0) = -\frac{z_-}{T}\left(1 + \frac{\sqrt{t_+^2 - z_+^2}}{\sqrt{t_-^2 - z_-^2}}\right) \qquad z'(u_1) = \frac{z_+}{T}\left(1 + \frac{\sqrt{t_-^2 - z_-^2}}{\sqrt{t_+^2 - z_+^2}}\right) \\ a^2 = \frac{1}{T^2}\left(\sqrt{t_-^2 - z_-^2} + \sqrt{t_+^2 - z_+^2}\right)^2 \end{gathered} \tag{3.37}$$

and, evaluating (3.36) at $a^2 = m_\perp^2$ etc, we find

$$z_{-s} = -\frac{p_3' t_-}{\sqrt{m_\perp^2 + p_3'^2}} \qquad z_{+s} = -\frac{p_3 t_+}{\sqrt{m_\perp^2 + p_3^2}} \qquad T_s = \frac{t_-}{\sqrt{m_\perp^2 + p_3'^2}} + \frac{t_+}{\sqrt{m_\perp^2 + p_3^2}} \; . \tag{3.38}$$

We can now compute all the second derivatives, but their expressions look complicated on their own. If we group them into a 3x3 Hessian matrix

$$\int dz_-\, dz_+\, dT \to \int d^3\mathbf{X}\, e^{-\mathbf{X}\cdot\mathbf{H}\cdot\mathbf{X}} = \sqrt{\frac{\pi^3}{\det \mathbf{H}}} \tag{3.39}$$

we find a nice and simple determinant

$$\det \mathbf{H} = \frac{p_0^3 p_0'^3}{8 m_\perp^2 t_+ t_- T} \; . \tag{3.40}$$

We note that in the end we can take the limit analytically as all factors of t_-, t_+ and T cancel

$$\frac{1}{2\pi T} \frac{1}{\sqrt{\det \Lambda}} \sqrt{\frac{\pi^3}{\det \mathbf{H}}} = \sqrt{\frac{2\pi m_\perp^2}{p_0 p_0' h(\tilde{u}_1)}} \tag{3.41}$$

where we have used (3.30).

3.1.5 Spin

The spin part is also very simple. With

$$\frac{d}{du} \ln(z' + t') = E \tag{3.42}$$

we find

$$-\frac{iT}{4}\sigma^{\mu\nu} \int_0^1 d\tau\, F_{\mu\nu} = \frac{1}{2}\gamma^0\gamma^3 \ln\rho \; , \tag{3.43}$$

where

$$\rho = -\frac{p_0 - p_3}{p'_0 - p'_3} \ . \tag{3.44}$$

Finally, integrating by parts the spatial derivatives and acting on the exponent with the remaining time derivative we find

$$(i\not{D}_{x_+} + 1) \to \not{p} + 1 \tag{3.45}$$

therefore

$$\text{Spin}_{ss'} = \frac{1}{2}\bar{R}_s(\not{p}+1)\gamma^0(\not{p}+1)\frac{1}{2\sqrt{\rho}}\left(\rho[1+\gamma^0\gamma^3] + 1 - \gamma^0\gamma^3\right)\gamma^0(-\not{p}'+1)R_{s'} = \delta_{ss'}\sqrt{p_0\, p'_0} \ . \tag{3.46}$$

In this case the spin dependence is trivial, but for more general fields this is not the case. Once we take the modulus squared and sum over spins we find

$$\sum_{ss'} |\text{Spin}_{ss'}|^2 = 2p_0 p'_0 \ . \tag{3.47}$$

3.2 Instantons

3.2.1 Pursuing instantons

We have seen that, in order to find the exponent and the prefactor, the key point is finding the instantons. For the 2D electric fields we are considering, the Lorentz force equation (3.9) reduces to

$$t''(u) = E_3(t,z)z'(u) \qquad z''(u) = E_3(t,z)t'(u) \ . \tag{3.48}$$

From the previous section we saw that we find the spectrum at momenta (p_3, p'_3) by solving the Lorentz force equation (3.48) with nonlocal boundary conditions on the derivatives

$$\begin{aligned} z'(u_0) &= p'_3 \qquad & z'(u_1) &= -p_3 \\ t'(u_0) &= -p'_0 \qquad & t'(u_1) &= p_0 \ . \end{aligned} \tag{3.49}$$

However, numerically we need initial conditions from which we evolve the solutions in discrete steps using (3.48), so we use the so-called *shooting method* [41]. Guessing some initial conditions $q(0)$ and $q'(0)$ at $u = 0$, we "shoot" the solution by evolving it with the Lorentz force equation numerically until the instantons are outside the field and the derivatives become constant. Looking at the resulting values of the asymptotic derivatives, we adjust the initial conditions in a suitable way to obtain momenta closer to the desired values, proceeding until a certain level of numerical precision is obtained. In principle this method allows us to find the spectrum on a grid of momentum points, but it is computationally intensive.

Since we work in the saddle-point approximation, the peaks are very narrow so we can Taylor expand the exponent up to second order around the maximum and find the widths $\mathbf{d}^{-2}$ directly. The spectrum then has the form (3.7). But in order to do so, we need to know the maximum. Again, we could in principle start at some initial guess $\mathbf{\Pi}_{\text{guess}}$, find the instantons, integrate them to find the exponent (3.13) Re $\psi(\mathbf{\Pi}_{\text{guess}})$, and iterate until the maximum is found. This is also very computationally intensive because at every step we need to use the shooting method and compute an integral.

Luckily, we can use asymptotic conditions that do not involve the momenta at all and find the instantons at the maximum *without knowing where the maximum is*. First of all, we note that for symmetric fields the maximum is at equal momentum values $-p_3 = p_3' = P$. This is a significant simplification because from (3.49) and the symmetry of $E_3(t,z)$ the instanton has odd z-component and even t-component, thus two conditions follow immediately $z(0) = t'(0) = 0$. Furthermore, from $q'^2 = 1$ we find $z'(0) = i$, reducing the number of free parameters to one complex value $\tilde{t} = t(0)$, i.e. the turning point. In the next section we show that at the maximum the instantons satisfy

$$\frac{\partial}{\partial P}\text{Re}\,\psi = \text{Im}\left[\frac{z'(u_1)}{t'(u_1)}t(u_1) - z(u_1)\right] = 0 \tag{3.50}$$

which we use an asymptotic condition in place of (3.49). We still need to use the shooting method, but only once. The asymptotic momentum P at the maximum then follows as a *consequence* when we evaluate the instantons simply from $P = z'(u_1)$. Since the turning point $t(0)$ is a complex parameter, (3.50) is actually not enough to determine the instantons uniquely. There are in fact many solutions that satisfy (3.50) but have complex momentum, so we get rid of them by adding Im $z'(u_1) = 0$. To summarize, we have

Initial conditions	Constraints
$\begin{cases} t(0) = \tilde{t} \\ t'(0) = 0 \\ z(0) = 0 \\ z'(0) = i \end{cases}$	$\begin{cases} \text{Im}\left[\frac{z'(u_1)}{t'(u_1)}t(u_1) - z(u_1)\right] = 0 \\ \text{Im}\, z'(u_1) = 0 \end{cases}$

For symmetric fields such as a product of Sauter pulses $E_3(t,z) = \text{sech}^2(\omega t)\,\text{sech}^2(\kappa z)$, the turning point is in addition always imaginary for all values of ω and κ, therefore Im $[z'(u_1)] = 0$ is actually enough to find the instantons. However, one has to be careful because there are spurious solutions which the algorithm might find. For better stability, it is convenient to start at $\gamma_k = 0$ where we have an exact expression of the turning point

$$\tilde{t} = \frac{i}{\gamma_\omega}\arctan(\gamma_\omega) \tag{3.51}$$

and perform a numerical continuation [44], i.e. increase γ_k in small steps using the known turning point at a given γ_k as an initial guess at $\gamma_k + \Delta\gamma_k$. This allows us to go up to large values of γ_k.

So far we have not discussed the choice of contour. For practical calculations it is often convenient to use a tilted straight line

$$u(r) = e^{-i\theta}\, r \;\rightarrow\; f(r) = \frac{du}{dr} = e^{-i\theta} \tag{3.52}$$

with $\theta > 0$ and sufficiently small. The reason why we tilt it clockwise will be explained in the next section. The drawback is that the asymptotic velocities as functions of r are not real due to the phase. To fix this, another possibility is to consider an affine combination of a tilted parametrization when $r \sim 0$ and let $f \to 1$ at large values of r. We can construct it by letting $\psi(r)$ be a normalized bump function such as

$$\psi(r) = \frac{1}{2\tanh\left[\frac{L}{W}\right]} \left(\tanh\left[\frac{r+L}{W}\right] + \tanh\left[\frac{-r+L}{W}\right] \right) \tag{3.53}$$

shown in Fig. 3.1 with adjustable parameters L and W, then define an affine combination of the trivial einbein $f = 1$ and $f = e^{-i\theta}$

$$f(r) = 1 \cdot \big(1 - \psi(r)\big) + e^{-i\theta}\, \psi(r) \tag{3.54}$$

so that $f(r \sim 0) \sim e^{-i\theta}$ and $f \to 1$ after some value in both directions. If we compare the instantons obtained using (3.54) with the ones obtained using the tilted parametrization, we see that they are equal for small r and separate when ψ goes to zero as in Fig. 3.1.

Although with this parametrization it is guaranteed that the imaginary parts do not grow linearly with r, one might wonder if we can find a set of parameters such that one or both components are also purely real. Assuming that either z or t becomes real, (3.50) implies that the other component is real as well. This is very promising. Since $\tilde{t}$ is purely imaginary, first of all we set $\theta = \frac{\pi}{2}$ so that $t(r)$ travels along the imaginary axis towards the origin and fix W equal to some small value. At this point we only need to fine-tune the size of the bump L so that it makes a turn right at the origin and becomes real. Doing so we obtain the instantons in Fig. 3.2.

We can interpret the region $r \sim 0$ as the *formation region*, i.e. where the particles are not real yet, and the asymptotic parts as the regions after the particles have been created but are still inside the field and keep being accelerated for a little longer. Note that for a time-dependent Sauter pulse the turning point $\tilde{t}(\gamma_\omega)$ in (3.51) gets closer to the origin as γ_ω grows, which means that the formation region shrinks and the probability is enhanced [39].

While one could in principle try different contours until the one that gives the instantons in Fig. 3.2 is found, we can directly use the shooting method with L as a free parameter as well. The conditions that work best numerically depend on the field and the parameters, but in general for symmetric fields we use the two real parameters L (real) and $\tilde{t}$ (imaginary) and conditions

$$\text{Im}\, t(r_a) = \text{Im}\, z(r_a) = 0 \tag{3.55}$$

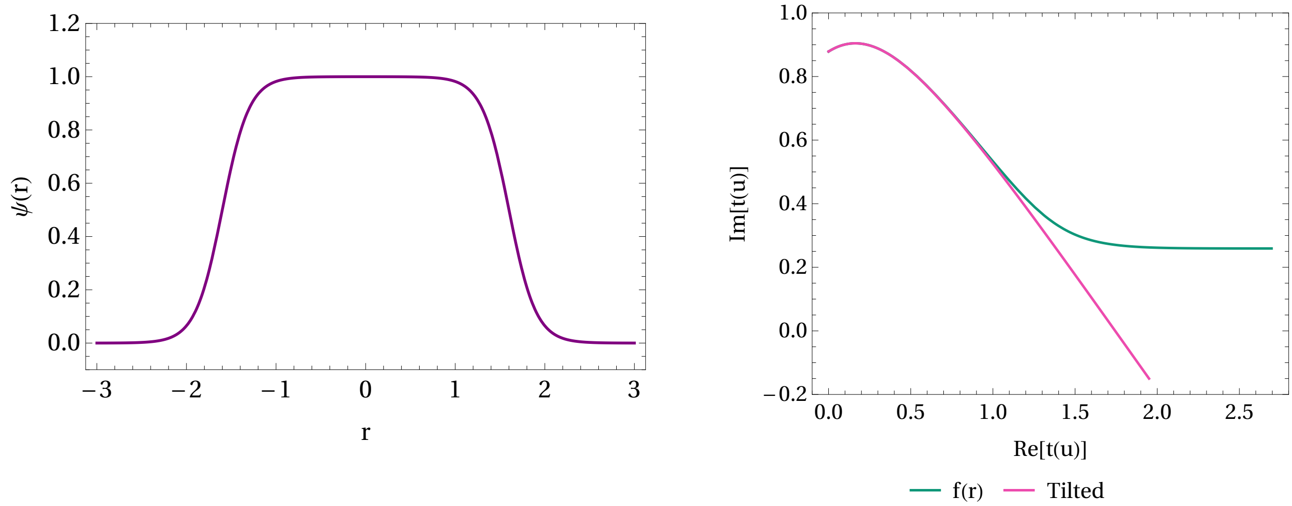


Figure 3.1: Bump function with parameters $L = 1.6$ and $W = 0.3$ (left) and comparison of the t-component obtained using the tilted and the affine contour (right).

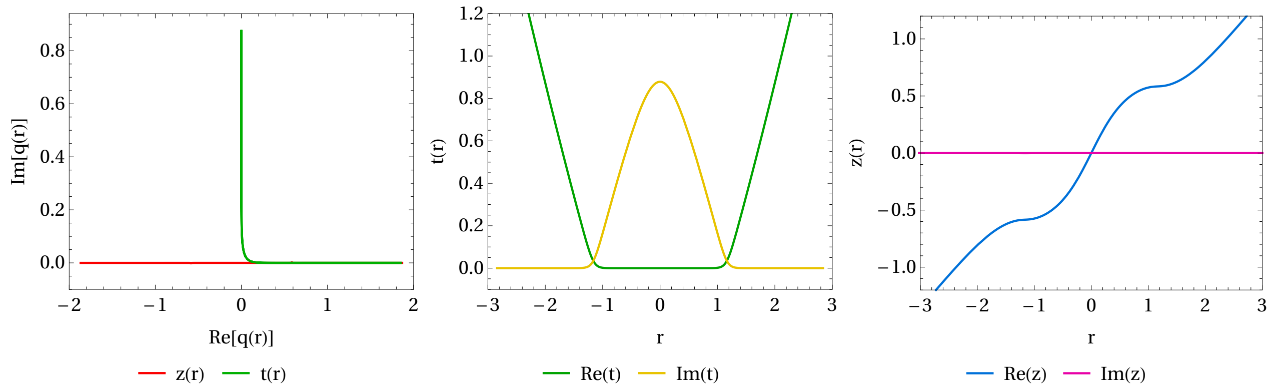


Figure 3.2: Complex instantons with fine-tuned parameters $L = 1.1636$ and $W = 0.1$ (left), real and imaginary parts of t (center) and z (right). The field parameters are $\gamma_k = \gamma_\omega = 1$.

or

$$\operatorname{Im} t(r_a) = \operatorname{Im} t(r_b) = 0 \tag{3.56}$$

where $r_a, r_b = \mathcal{O}(1)$ just need to be large enough to be in the acceleration region, i.e. where the instantons are both real.

3.2.2 Instantons on the complex plane

Besides looking at individual contours, it would be nice to get a global picture of the instantons viewed as complex variable solutions $q(u)$ to the Lorentz force equations with fixed initial conditions. An interesting feature of complex differential equations is that their solutions can have *branch points*. Since there are in general an infinite number of contours that go from the origin to a given complex value u, $q(u)$ is not necessarily single-valued, even if the function is analytic. For this to happen, letting $\mathcal{C}_1$ and $\mathcal{C}_2$ be two simple paths from the origin to u, the area enclosed by $\mathcal{C}_1 \cup \mathcal{C}_2^-$ must contain a branch point. It is tempting to say that they are caused by singularities of the field, but we might wonder what happens when the field is an entire function such as the Gaussian field $E(t) \sim e^{-t^2}$. In this case the singularity arises due to the fact that along a real trajectory $u(r) = r$ the component $t(r)$ is imaginary and keeps growing faster and faster, but the exponential function has an essential singularity at infinity. Due to Liouville there are no analytic functions on the Riemann sphere apart from constant functions, so we either have poles or essential singularities at infinity (the only entire functions that have poles at infinity are polynomials, but a realistic field is bounded along the real axis). Before discussing branch points, we notice that for a field with singularities like a Sauter pulse or a Lorentz pulse

$$E(t) \sim \frac{1}{1+t^2} \tag{3.57}$$

the instantons cannot be entire functions. This is because, due to Picard's little theorem, their image would be $\mathbb{C}$ without at most a single point, therefore they would cross a pole of the field. At that point however they cannot be analytic, so we have a contradiction.

We now show explicitly that the poles of the field are the source of the branch points, and for simplicity, let us consider a time-dependent field. Let $E(t)$ be be a field with a pole of order β at t_p and expand the instantons around a point u_B with an ansatz

$$E(t) \sim \frac{R}{(t-t_p)^\beta}, \qquad t(u) \sim t_p + c_t(u-u_B)^\alpha \tag{3.58}$$

and similarly for z. Plugging this into the Lorentz force equation we see that $\alpha = 1/\beta$, therefore for a field like a Sauter pulse with a double pole the branch point is like a square root $t(u) \sim t_p + c_t\sqrt{u-u_B}$, so the instantons remain finite. This method does not give the correct result for a field with a simple pole like a Lorentz pulse, indicating that near the branch point the instanton is not approximated by $(u-u_B)^\alpha$ for any fractional power α. This is related to the fact that $A(t)$ itself has a branch point of log-type when $A'(t) = E(t)$ has a simple pole. On the other hand, one also sees that for the Gaussian pulse we have $t(u) \sim \sqrt{\ln(u-u_B)}$, so the instantons are divergent at branch points. Due do Liouville's theorem, we always have singularities except for constant fields. Indeed the constant field instantons are trivially entire functions.

We can also obtain an explicit expression for the branch points. Let us use the initial condition $t(0) = \tilde{t}$, starting with a Gaussian field. Letting $u(r)$ be a contour such that $t(r) = \tilde{t} + ir$, we have

$$\frac{dt}{du} = \sqrt{1+\left(\frac{dz}{du}\right)^2} \;\rightarrow\; u'(r) = \frac{i}{\sqrt{1+A(\tilde{t}+ir)^2}} \tag{3.59}$$

where we have used $t'(r) = i$ and $z'(u) = A(\tilde{t}+ir)$. Integrating we obtain

$$u_B = i\int_0^\infty \frac{dr}{\sqrt{1+A(\tilde{t}+ir)^2}}\,. \tag{3.60}$$

Since this integral is finite, $t(u)$ reaches infinity along the imaginary axis for a finite imaginary value u_B. Similarly, if the field has a pole t_p, we simply integrate from 0 to the value r such that $t = t_p$, i.e.

$$u_B = i\int_0^{\frac{t_p-\tilde{t}}{i}} \frac{dr}{\sqrt{1+A(\tilde{t}+ir)^2}} \ . \tag{3.61}$$

Note that these equations are valid not only when $\tilde{t}$ is a turning point: if we consider e.g. particles initially at rest $z'(0) = z(0) = t(0) = 0$, $t'(0) = 1$, we have the same integrals but without $\tilde{t}$. Evaluating them explicitly we see that in this case the branch points are not on the real axis. This last observation is nontrivial; from the initial conditions $z'(0) = i$ and $t(0) = \tilde{t} \in i\mathbb{R}$, if we consider a contour along the real axis, the instantons grow in absolute value but remain imaginary because $E(t)$ is always real when t is imaginary, therefore it seems obvious that at some point $t(u)$ either hits the pole or diverges. But the initial conditions above are all real, so it is not so obvious.

To get an intuitive picture of the analytic structure of the instantons, such as the position of the branch points, we want to plot them in the complex plane. To this end, we must make a choice regarding the plotting method. Since we want to see the regions where the instantons are real or imaginary and the cuts on the complex plane, we use domain coloring, i.e. we color the complex u-plane according to the phase of $q(u)$ (either t or z). In addition, we use contour lines to keep track of $|q(u)|$ and $\mathrm{Re}\, q(u)$, $\mathrm{Im}\, q(u)$. For fields where the instantons are known exactly [39], this allows us to obtain plots right away. However, for spacetime fields we do not have an analytic expression of the instantons, so we must solve the Lorentz force equation over a dense enough set of contours. Before doing so, we use the shooting method so that $\tilde{t} = t(0)$ is known. Then, let us say that we want to find the value of $q(\bar{u})$ for some $\bar{u} \neq 0$. Writing $\bar{u} = iR + r$ with $r, R \in \mathbb{R}$, we first solve numerically from $u = 0$ to $u = iR$ along the imaginary axis and then from $u = iR$ to $u = iR + r$ parallel to the real axis. Since we want to find $q(u)$ for many values of u, we need to solve along the imaginary axis only once so that $q(iR)$ is known for all R in some domain and then solve parallel to the real axis using $q(iR)$ (and its derivatives) as the initial conditions. The result is that we find $q(u)$ on a grid of points with an implicit contour choice for every u. This choice of contours is arbitrary but it is motivated by the periodic structure of the branch points.

We see the instantons obtained with this method in Fig. 3.4 for a spacetime Sauter pulse

$$E_3(t,z)/E = \mathrm{sech}(\omega t)^2\,\mathrm{sech}(\kappa z)^2 \ , \tag{3.62}$$

which is field with imaginary poles, and a Gaussian pulse

$$E_3(t,z)/E = e^{-(\omega t)^2-(\kappa z)^2} \tag{3.63}$$

in Fig. 3.5, which is an entire function. We clearly see in both cases branch points along the real axis and periodicity along the imaginary axis. If we solve only along the imaginary axis we obtain the

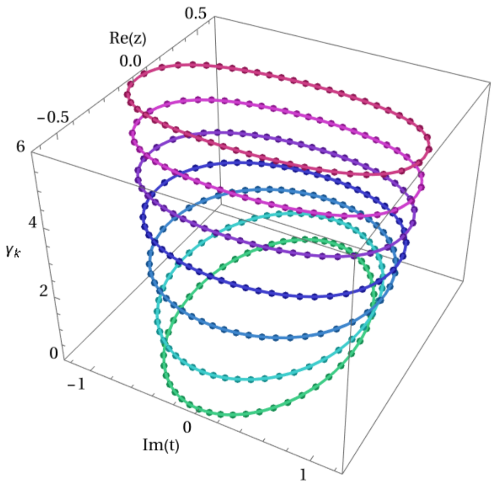


Figure 3.3: Comparison between the instantons at different γ_k for a double Sauter pulse (3.62) obtained by solving along the imaginary axis using the turning point $\tilde{t}$ found from the shooting method with (3.50) (continuous lines) and the discrete instantons from [44]. We also see that increasing γ_k tends to expand the instanton along the t-axis.

same closed instantons used in the effective action method [39]. The comparison in Fig. 3.3 with the discrete instantons from [44] indeed shows exact agreement. If we use the tilted contour $u = e^{i\theta}r$ to find the turning point, from the plots 3.4 it is now clear that we have to use $\theta < 0$ because rays with $\theta > 0$ (and small) go towards the region with the wrong asymptotic conditions $t_\pm \to -\infty$ (light blue region of $t(u)$, on the right of Fig. 3.4). We want contours that travel in the $t_\pm \to +\infty$ branches, i.e. the red regions in the plot of $t(u)$.

Besides the slightly different behavior near the branch point u_B, the three plots look remarkably similar. One might thus be tempted to think that for every pulse shape with a single peak the instanton will look like Fig. 3.4. However, this is not the case. The reason why the instantons look similar is that the shapes (3.62) and (3.63) are similar not only along the real axis but also along the imaginary one apart from (3.62) having a singularity at finite t and (3.63) at infinity. To see this, let us consider a field that has a completely different behavior along the imaginary axis such as the double supergaussian

$$E_3(t,z)/E = e^{-(\omega t)^4-(\kappa z)^4} \,. \tag{3.64}$$

As we can see Fig. 3.6, the instanton now looks quite different. Note that in this case using a tilted contour is not possible because the good asymptotic regions (red in the plot of t) are very thin, so a contour of the form (3.54) is necessary.

Notably, the Sauter and Gaussian pulses grow very fast along the imaginary t-axis so there is a branch point at real u, while the supergaussian is bounded for $t = i\tau$

$$E_3(t,z) \sim e^{-(\omega\tau)^4} \tag{3.65}$$

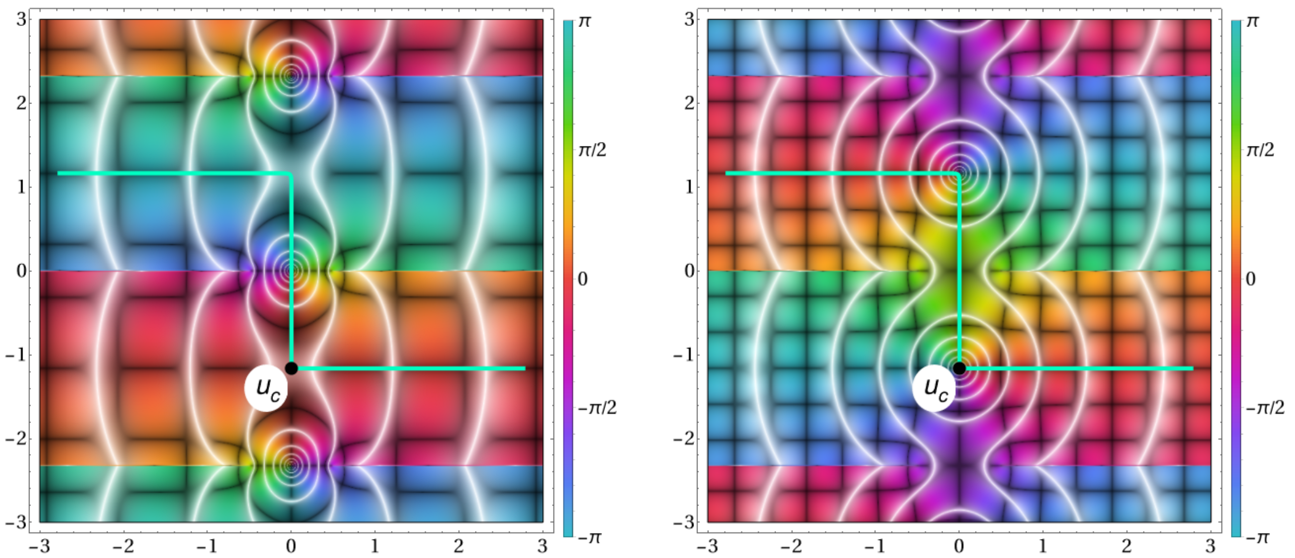


Figure 3.4: Space (left) and time (right) component of the instanton with a double Sauter pulse (3.62) $\gamma_\omega = \gamma_k = 1$. The green line is the physical contour that produces the split into formation (vertical) and acceleration (horizontal) regions. u_c and $-u_c$ represent electron and positron creation as they are at boundary between creation/acceleration regions.

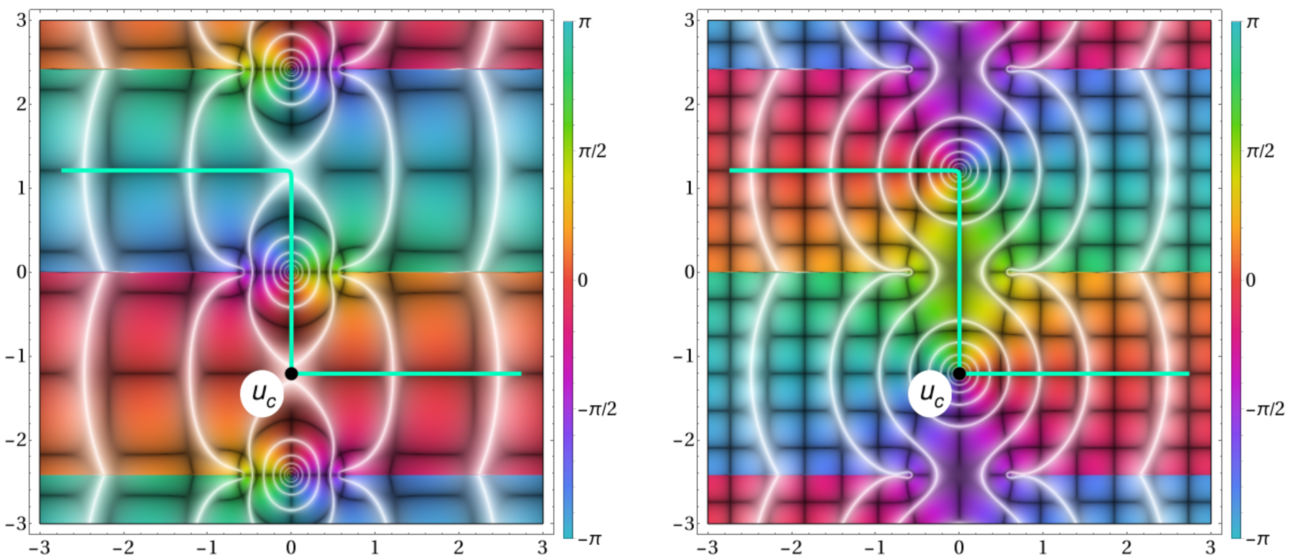


Figure 3.5: Space (left) and time (right) component of the instanton with a double Gaussian pulse (3.63) $\gamma_\omega = \gamma_k = 1$.

and goes to zero very fast when $\tau \gg 1$ so the instanton is perfectly regular for $u \in \mathbb{R}$.

3.3 Spectrum and integrated probability

To summarize, we have shown that in order to find the spectrum for a particular momentum, we only need to find the respective instanton numerically and solve an ordinary differential equation (3.31)

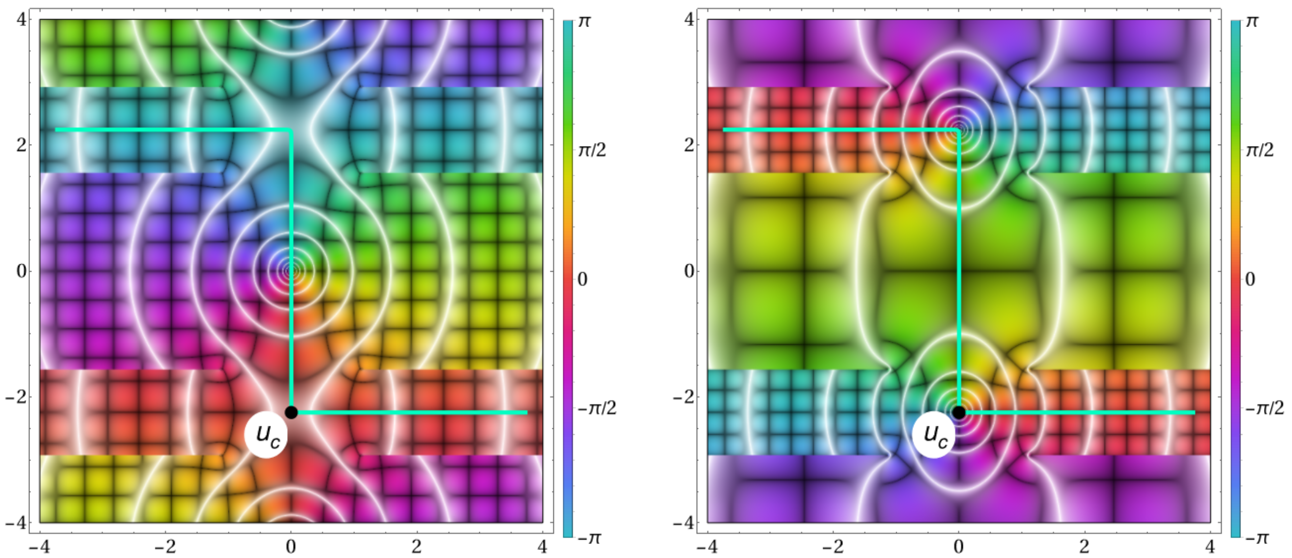


Figure 3.6: Space (left) and time (right) component of the instanton with a double supergaussian (3.64) with $\gamma_\omega = 0.8, \gamma_k = 0.32$.

with initial conditions (3.32) to find $h(\tilde{u}_1)$. Since the exponent has the form

$$\sim e^{\frac{1}{E}(\dots)} \tag{3.66}$$

we can integrate the spectrum using the saddle-point method to find the total probability. This allows us to check with the discrete instanton method from [44]. We will see that the momentum widths can be computed using *only the instantons on the maximum*. This is very good news because, due to symmetry, it is in general much easier to find instantons on the maximum of the spectrum. Furthermore, we also gain information about the small γ_k behavior by taking the time-dependent limit.

3.3.1 Momentum widths

We want to calculate

$$\mathbb{P} = \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\Big|\dots\Big|^2 = V_\perp \int \frac{dp_\perp}{(2\pi)^2}\frac{dp'}{2\pi}\frac{dp}{2\pi}\frac{4\pi m_\perp^2}{p_0\,p_0'\,|h(\tilde{u}_1)|}e^{-\frac{1}{E}\mathcal{A}} \tag{3.67}$$

where the perpendicular volume factor comes from the delta function and

$$\mathcal{A} = 2\,\mathrm{Im}\int_{\mathcal{C}} du\, q^\mu\,\partial_\mu A_\nu \frac{dq^\nu}{du}\ . \tag{3.68}$$

Note that in order to have a nontrivial spectrum the integral above must have an imaginary component, which is the case if and only if the instantons themselves do.

We begin with the simple transverse width. It is much simpler to consider $\mathcal{A}$ before the simplifications. With

$$\mathcal{A} = 2\,\mathrm{Im}\left(px_+ + p'x_- - \frac{T}{2} - \int_0^1 d\tau\,\frac{\dot{q}^2}{2T} + A\cdot\dot{q}\right) \tag{3.69}$$

we only need to consider the *explicit* dependence on the momentum because all the rest will cancel (see Appendix A). Furthermore, the maximum is $p_\perp = 0$, so we have a transverse spectrum of the form

$$\mathbb{P} \sim e^{-\frac{p_\perp^2}{d_\perp^2}} . \tag{3.70}$$

Taking the derivative of the exponent with respect to $p_\perp$ (i.e. either p_1 or p_2) we get

$$\frac{\partial \mathcal{A}}{\partial p_\perp} = 2 \lim_{-u_0, u_1 \to \infty} \text{Im} \left(\frac{t(u_1)}{p_0} p_\perp + \frac{t(u_0)}{p_0'} p_\perp + q^\perp(u_1) - q^\perp(u_0) \right) \tag{3.71}$$

but the transverse component of the instanton is trivially $q^\perp(u) = p^\perp u = -p_\perp u$, therefore taking the derivative again and evaluating at $p_\perp = 0$

$$d_\perp^{-2} = \frac{1}{2} \frac{\partial^2 \mathcal{A}}{\partial p_\perp^2} = \lim_{-u_0, u_1 \to \infty} \text{Im} \left(\frac{t(u_0)}{p_0'} + \frac{t(u_1)}{p_0} - u_1 + u_0 \right) . \tag{3.72}$$

Such equation is valid for every contour, but we typically choose a symmetric contour and have symmetric momenta[2] $p_0 = p_0'$. For a nontrivial contour such as (3.54), $u_1 - u_0$ can be found using

$$u_1 - u_0 = \int_{r_0}^{r_1} dr\, f(r) \tag{3.73}$$

where the parameters r_i satisfy $u(r_i) = u_i$.

As to the longitudinal part, the symmetry under exchange of $-\mathbf{p}$ and $\mathbf{p}'$ suggests to define some new variables

$$p_3 = -P + \frac{\Delta}{2} \qquad p_3' = P + \frac{\Delta}{2} . \tag{3.74}$$

Note that the symmetry implies that there are no mixed terms $P\Delta$, therefore we have two independent widths. We did not lose any degrees of freedom because in the Hessian matrix for (p_3, p_3') we have two independent parameters (it is symmetric and the diagonal coefficients are equal).

We begin with the Δ width; since the instantons are functions of the momenta, we define

$$\delta q_\Delta(u) := \left. \frac{\partial q(u)}{\partial \Delta} \right|_{\Delta=0} \tag{3.75}$$

with boundary conditions obtained by taking the derivatives of both sides of

$$z'(u_1) = P - \frac{\Delta}{2} \qquad z'(u_0) = P + \frac{\Delta}{2} \tag{3.76}$$

[2] Note that the widths are evaluated at the momentum saddle point because we expand $f(x) \sim f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$ and the width corresponds to $f''(x_0)$.

and similarly for $t'(u_{0,1})$, obtaining

$$\begin{aligned} \delta t'_\Delta(u_0) &= \delta t'_\Delta(u_1) = -\frac{P}{2p_0} \\ \delta z'_\Delta(u_0) &= -\delta z'_\Delta(u_1) = \frac{1}{2} \end{aligned} \tag{3.77}$$

i.e. δt_Δ is odd and δz_Δ even, satisfying

$$\begin{aligned} \delta t''_\Delta &= E_z \delta z_\Delta\, z' + E_t \delta t_\Delta\, z' + E \delta z'_\Delta \\ \delta z''_\Delta &= E_z \delta z_\Delta\, t' + E_t \delta t_\Delta\, t' + E \delta t'_\Delta\ . \end{aligned} \tag{3.78}$$

Taking two derivatives with respect two Δ we find

$$d_\Delta^{-2} := \frac{1}{2}\frac{\partial^2 \mathcal{A}}{\partial \Delta^2} = \text{Im}\left[\frac{t}{2p_0^3} - \frac{P}{p_0}\delta t_\Delta + \delta z_\Delta\right]_{u\to\infty} \tag{3.79}$$

or, defining $\eta_a := 2p_0(z'\delta t_\Delta - t'\delta z_\Delta)$

$$d_\Delta^{-2} = \frac{1}{2p_0^2}\text{Im}\left[\frac{t}{p_0} - \eta_a\right]_{u\to\infty} \tag{3.80}$$

such that $\eta'_a(u_0) = \eta'_a(u_1) = 1$. Note that η_a is an antisymmetric solution which satisfies

$$\eta''_a = \left(E^2 + E_z\, t' + E_t\, z'\right)\eta_a \tag{3.81}$$

which is a *homogeneous* equation, hence given some solution $\eta_{a,n}$ with $\eta_{a,n}(0) = 0$ and $\eta'_{a,n}(0) = 1$, the asymptotic derivative will be some value $\eta'_{a,n}(u_1)$ in general different from one, but if we define

$$\eta_a(u) := \frac{\eta_{a,n}(u)}{\eta'_{a,n}(u_1)} \tag{3.82}$$

then it is clearly a solution and it has by construction the desired asymptotic condition. The punchline is that we can find the width numerically with simple initial conditions, i.e. *without the shooting method*.

Now we consider an independent variation of P. From the first derivative

$$\frac{\partial \mathcal{A}}{\partial P} = \text{Im}\left[\frac{P}{p_0}t(u_1) - z(u_1)\right] \tag{3.83}$$

we can now calculate the width defining as before

$$\delta q_P(u) := \frac{\partial q(u)}{\partial P}\Big|_{P=\mathscr{P}} \tag{3.84}$$

with boundary conditions

$$\begin{aligned} -\delta t'_P(u_0) &= \delta t'_P(u_1) = \frac{1}{2} \\ \delta z'_P(u_0) &= \delta z'_P(u_1) = \frac{1}{2} \end{aligned} \tag{3.85}$$

combined into $\eta_s := p_0(t'\delta_P z - z'\delta_P t)$ with $-\eta_s'(u_0) = \eta_s'(u_1) = 1$ to give

$$d_P^{-2} = \frac{2}{p_0^2}\text{Im}\left[\frac{t}{p_0} - \eta_s\right]_{u\to\infty} . \tag{3.86}$$

Of course as before we can use initial conditions $\eta_{s,n}(0) = 1$, $\eta_{s,n}'(0) = 0$ and divide by the asymptotic derivative at the end.

3.3.2 Integrated probability and comparisons

It is now finally time to check these results with the closed worldline/effective action method. The exponent can be compared directly, whereas we cannot check the individual widths as the effective action produces only the total integrated probability. We can thus check the spectrum by comparing its integral (3.67)

$$\begin{aligned} \mathbb{P} &= V_\perp \int dp_\perp\, dP\, d\Delta\, \mathbb{P}(p_\perp, P, \Delta) \\ \mathbb{P}(p_\perp, P, \Delta) &= \frac{1}{4\pi^3|h(\tilde{u}_1)|p_0 p_0'} \exp\left\{-\mathcal{A} - \frac{p_\perp^2}{d_\perp^2} - \frac{(P-\mathscr{P})^2}{d_P^2} - \frac{\Delta^2}{d_\Delta^2}\right\} \end{aligned} \tag{3.87}$$

which gives

$$\mathbb{P} = V_\perp \frac{1}{4\pi|h(\tilde{u}_1)|p_0 p_0'} \frac{e^{-\mathcal{A}}}{d_\perp^{-2}\sqrt{d_P^{-2} d_\Delta^{-2}}} . \tag{3.88}$$

As mentioned above, we always start at $\gamma_k = 0$ and increase in small steps $\Delta\gamma_k$ to obtain better convergence of the shooting method. The comparisons of the exponent and the prefactor of (3.88) are shown in Fig. 3.7, the widths in Fig 3.8.

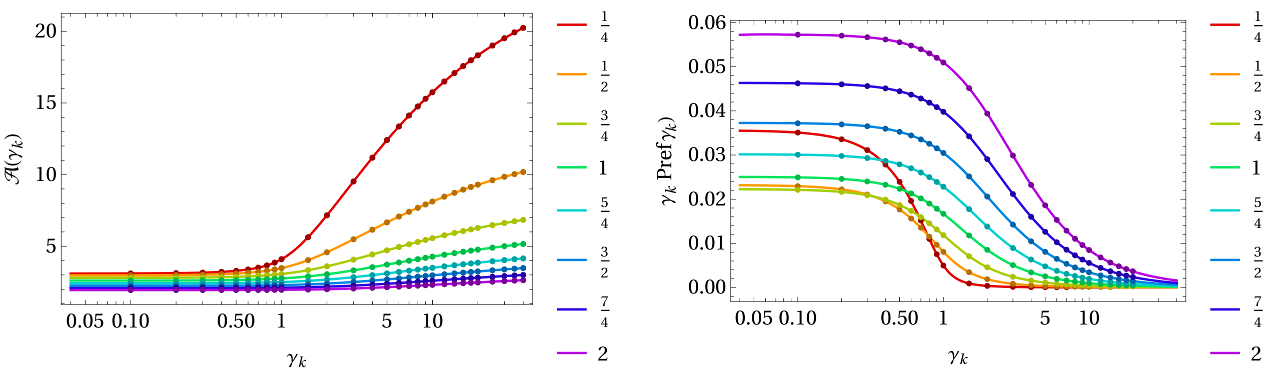


Figure 3.7: Comparison of the exponent and the prefactor (without the overall factors of E). The latter has been multiplied by γ_k to obtain a finite expression at $\gamma_k \to 0$. The different colors of the curves represent different values of γ_ω from $\frac{1}{4}$ to 2 in equal steps of $\Delta\gamma_\omega = \frac{1}{4}$. The solid lines are obtained using the methods explain in this chapter, the darker dots from the effective action. We see perfect agreement between the two for a wide range of parameters.

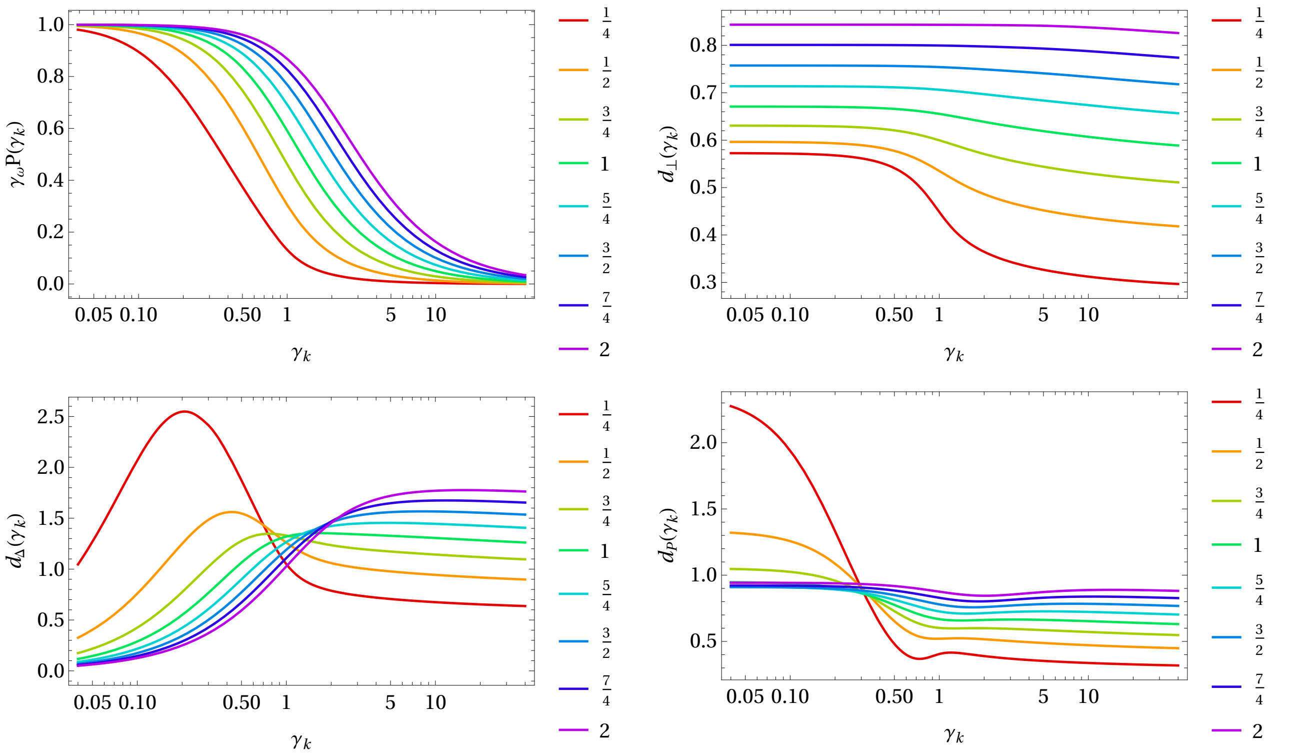


Figure 3.8: Saddle point value of the transverse momentum and all three widths (without the overall factors of E). We see that as γ_k increases the maximum of the longitudinal momentum decreases; this matches with our intuition that, as γ_k increases, the physical size of the field becomes smaller, so the pairs are accelerated less.

In general we observe, as expected, that the probability is enhanced when γ_ω increases (faster pulse) but suppressed when γ_k increases (smaller field). This is consistent with the fact that time inhomogeneities increase the probability, spatial ones decrease it [39].

3.3.3 Time-dependent limit

We look at the time-dependent limits. Letting $\gamma_k \to 0$ we expect first of all that the exponent reduces to [32, 34]

$$\mathcal{A} \to \frac{1}{E}\frac{2\pi}{1+\sqrt{1+\gamma_\omega^2}} \tag{3.89}$$

and indeed, if we divide the left side curves in 3.7 by such value they all converge to 1. This also happens for the two widths

$$d_\perp^{-2} \to \frac{1}{E}\frac{\pi}{\sqrt{1+\gamma_\omega^2}}\,, \qquad d_P^{-2} \to \frac{1}{E}\frac{\pi\gamma_\omega^2}{(1+\gamma_\omega)^{3/2}}\,. \tag{3.90}$$

For the last width d_Δ, however, we cannot check its time-dependent limit analytically because $d_\Delta \sim \gamma_k$ (see [64]), and as $\gamma_k \to 0$ it produces the momentum-conserving delta function. Since also $h \sim \gamma_k^2$, we can define $d_\Delta = \hat{d}_\Delta \gamma_k$ and $h = \hat{h}\gamma_k^2$ so that the Δ part of the spectrum (3.87) gives

$$\mathbb{P}(\Delta) \sim \frac{1}{2\pi\hat{h}\gamma_k^2} e^{-\frac{\Delta^2}{\hat{d}_\Delta^2 \gamma_k^2}} \to \frac{\hat{d}_\Delta}{2\sqrt{\pi}\hat{h}\gamma_k}\delta(\Delta) \tag{3.91}$$

therefore the overall prefactor goes like $1/\gamma_k$ at small γ_k. This gives the regularized volume factor V_3.

We also see that in the time dependent case (3.13) reduces to the result obtained with the saddle-point approximation of the Riccati equation [37]. In fact, let us consider a field $A_3(t)$ and all other components equal to zero. We assume for simplicity $A_3(t)$ odd. We define u_1 to be a point in the contour with $\mathrm{Re}\, u_1 \gg 1$ and $t(u_1) = t_+$. In the time-dependent case the instanton satisfy [39] (when $\mathrm{Re}\, u_1 > 0$)

$$t'(u) = \pi_0(\mathbf{p}) := \sqrt{m_\perp^2 + (p_3 - A(t))^2} \qquad t(0) = \tilde{t} \tag{3.92}$$

where $\tilde{t}$ is the turning point, so using the Lorentz force equation and integrating by parts we see that

$$\begin{aligned} \psi = 2i \int_0^{u_1} du\, t(u) t''(u) &= \big[t(u)t'(u)\big]_0^{u_1} - 2i \int_0^{u_1} du\, t'(u)^2 \\ &= 2i \int^{\tilde{t}} dt\, \pi_0(\mathbf{p}) \qquad (+\text{phase}) \end{aligned} \tag{3.93}$$

where the lower integration boundary is an arbitrary real number which just affects the phase.

3.4 Other pulse shapes and dynamical assistance

3.4.1 Comparison with other pulses

We look at the shape dependence of the exponent $\mathcal{A}$, the prefactor, and the widths. We define

$$F_S(x) = \mathrm{sech}(x)^2 \qquad F_G(x) = e^{-x^2} \qquad F_L(x) = \frac{1}{1+x^2} \tag{3.94}$$

and we refer to them respectively as the Sauter, Gaussian, and Lorentz pulse. They all have a maximum at $x = 0$ and are normalized so that

$$F(0) = 1 \qquad F''(0) = -2\ . \tag{3.95}$$

For example, in the previous sections we considered a field $E_3(t,z)/E = F_S(\omega t)\, F_S(\kappa z)$. If we change F_S for either F_G or F_L, the normalized action and prefactor change as in Fig. 3.9 while the widths as in Fig. 3.10. We see that, while the time-dependent limit of the prefactor is the same for all pulses, the d_Δ width for the Lorentz pulse is very different from the others for $\gamma_k \to 0$. This is because the Sauter and Gaussian pulses decrease exponentially for $t, z \gg 1$ while the Lorentz pulse only like a power, so besides the $1/\gamma_k$ factor there is an additional $\log(\gamma_k)$. Indeed, the scaling $d_\Delta \sim \gamma_k$ holds for fields that decrease faster than

$$E(t) \sim \frac{1}{t^2}\,. \tag{3.96}$$

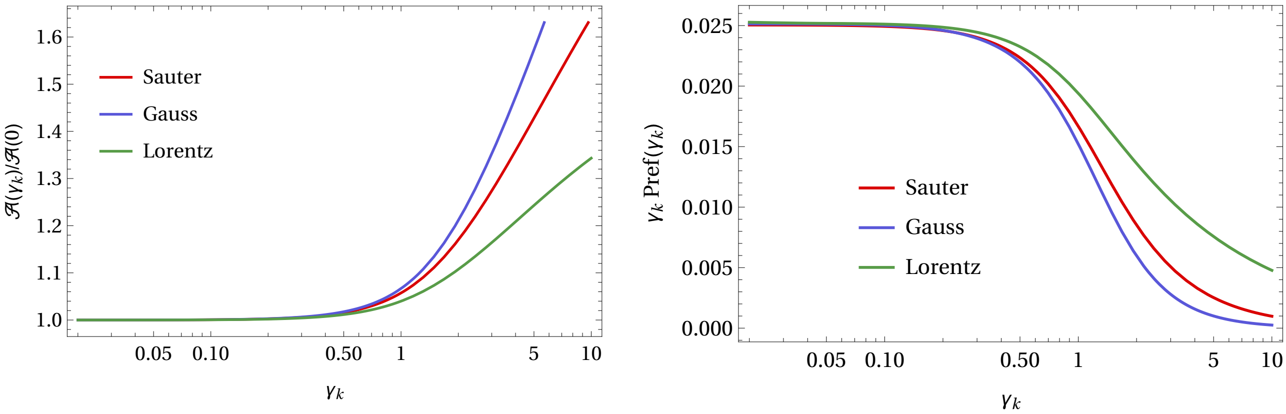


Figure 3.9: Normalized exponents and prefactors (without the overall factor of E) for the pulses $E_3(t,z)/E = F(\gamma_\omega t)F(\gamma_k z)$ with $F(x)$ given by (3.94) at $\gamma_\omega = 1$ and different values of γ_k.

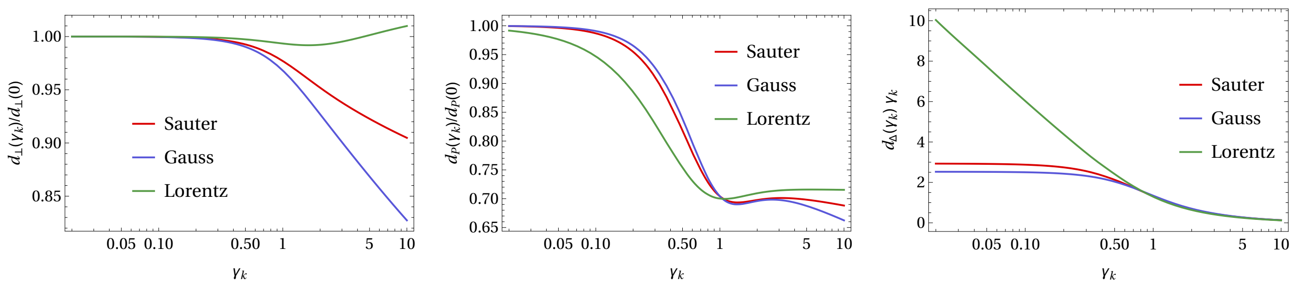


Figure 3.10: Normalized widths (d_Δ without the factor of E) for the pulses $E_3(t,z)/E = F(\omega t)F(\kappa z)$ with $F(x)$ given by (3.94) at $\gamma_\omega = 1$ and different values of γ_k.

3.4.2 Asymmetric field

All the pulses considered so far are even with respect to t and z independently, i.e.

$$E_3(t,z) = E_3(-t,z) = E_3(t,-z) \tag{3.97}$$

which implies that the Lorentz force equation (3.48) is consistent with solutions that are t-even and z-odd. This is still true if we relax this condition and consider fields that are only symmetric with respect to z, namely $E_3(t,z) = E_3(t,-z)$. However, as soon as we break this symmetry the instantons are no longer symmetric. First of all, since $z(u)$ is no longer odd, $\Delta_s \neq 0$ and we can have mixed terms in the spectrum $\sim P\Delta$. Furthermore, since the δq defined in (3.75), (3.84) are not symmetric as well, the expressions for the widths are more complicated. The exact form is discussed in [64] and here we just focus on the spectrum. Letting $\Pi = \{\Delta, P\}$ and

$$d^{-2}_{\alpha\beta} := \frac{1}{2}\frac{\partial^2 \mathcal{A}}{\partial\Pi_\alpha \partial\Pi_\beta} = \begin{pmatrix} d^{-2}_{\Delta\Delta} & d^{-2}_{\Delta P} \\ d^{-2}_{\Delta P} & d^{-2}_{PP} \end{pmatrix} \tag{3.98}$$

we have a spectrum

$$\exp\left\{-\mathcal{A} - \frac{p_\perp^2}{d_\perp^2} - (\mathbf{\Pi} - \mathbf{\Pi}_s)\cdot \mathbf{d}^{-2}\cdot(\mathbf{\Pi} - \mathbf{\Pi}_s)\right\} \tag{3.99}$$

and the prefactor (3.88) is slightly changed because the matrix of coefficients (3.98) is not diagonal, thus

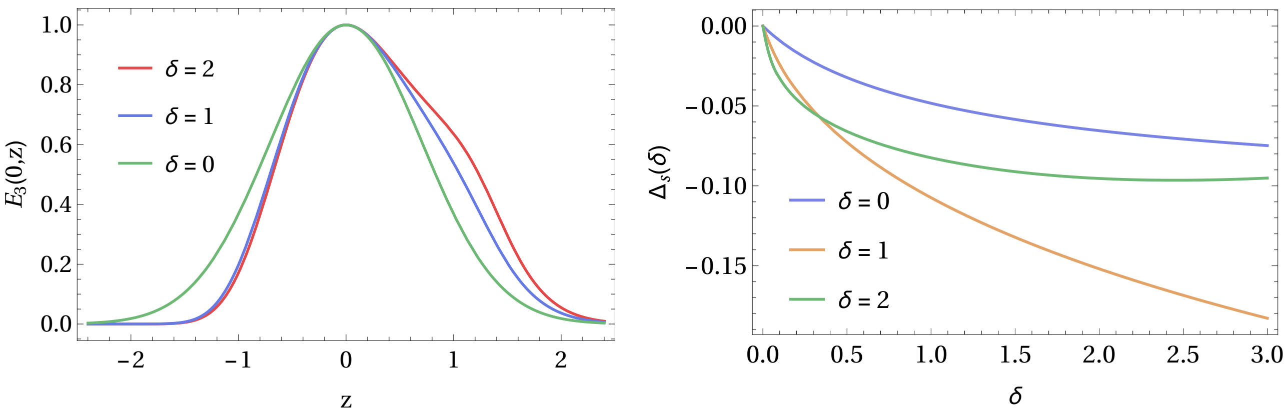


Figure 3.11: (Left) field (3.100) at $t = 0$ for some values of δ. (Right) saddle point value Δ_s as δ is increased at $\gamma_\omega = 1$ and some values of γ_k.

For the numerical calculations we want some $F(x;\delta)$ with a parameter δ such that $F(x;0)$ is even and $F(0;\delta) = 1$ for all δ. We consider for example

$$E_3(t,z)/E = e^{-(\omega t)^2}\frac{(1+\delta)e^{-(\kappa z)^2}}{1+\delta\, e^{-(\kappa z)^3}} \tag{3.100}$$

shown in Fig. 3.11 for some values of δ. As $\delta \to 0$, everything reduces to the previous cases, and in particular $\Delta_s(\delta = 0) = 0$. For increasing δ, i.e. as field becomes more skewed, $\Delta_s(\delta)$ increases as shown in Fig. 3.11. Since the field is skewed toward $z > 0$, more pairs are created in the $z > 0$ region than $z < 0$, so positrons are accelerated more than electrons. Thus, the sign of Δ_s turns out to be negative. In Figs. 3.12 and 3.13 we also see how the exponent, prefactor, and widths change as δ is increased. For this field and the parameters considered

$$d^{-2}_{\Delta P} << d^{-2}_{\Delta\Delta}, d^{-2}_{PP} \tag{3.101}$$

so the matrix is approximately diagonal and we simply rename $d^{-2}_{\Delta\Delta} \to d^{-2}_{\Delta}$, $d^{-2}_{PP} \to d^{-2}_{P}$.

$$\mathbb{P} = V_\perp \frac{1}{4\pi|h(\tilde{u}_1)|p_0 p_0'}\frac{e^{-\mathcal{A}}}{d_\perp^{-2}\sqrt{\det \mathbf{d}^{-2}}}\,. \tag{3.102}$$

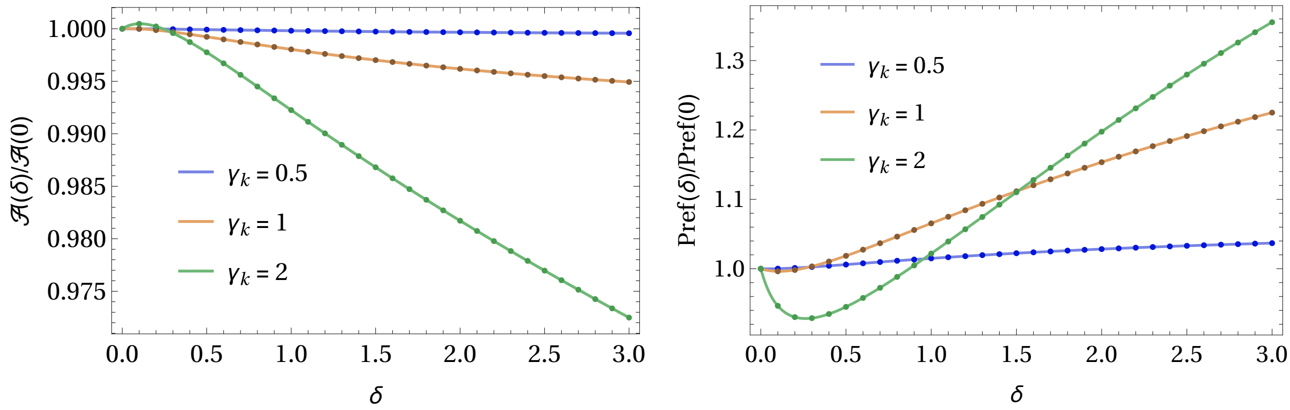


Figure 3.12: Exponent and prefactor (without the overall factors of E) as δ is increased at $\gamma_\omega = 1$ and some values of γ_k.

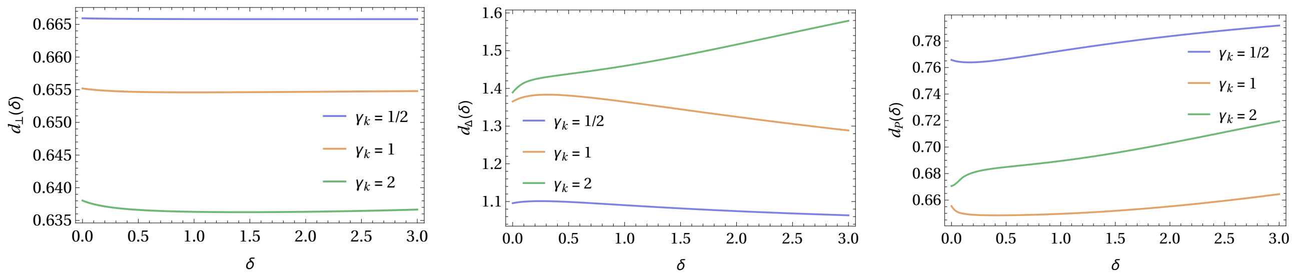


Figure 3.13: Widths as δ is increased at $\gamma_\omega = 1$ and some values of γ_k (without the overall factors of E).

3.4.3 Dynamical assistance

The Schwinger pair production probability can be enhanced by superimposing a strong, slowly varying field and a weak, faster pulse, as shown in [92]. With the methods outlined here, we consider a spacetime version of dynamical assistance with a field

$$E_3(t,z)/E = \operatorname{sech}(\Omega t)^2 + \varepsilon \operatorname{sech}(\omega t)^2 \operatorname{sech}(\kappa z)^2 \tag{3.103}$$

with $\varepsilon, E \ll 1$, $\Omega \ll \omega \ll 1$ and $\kappa \ll 1$. Defining

$$\gamma_\Omega = \frac{\Omega}{E} \tag{3.104}$$

we fix $\varepsilon = \gamma_\Omega = 1/10$ for the numerical calculations. As demonstrated in [92], for fields with poles, dynamical assistance can be thought of as due to the additional imaginary pole of the fast field

$$t_p = \frac{i\pi}{2\omega} \tag{3.105}$$

acting like a barrier and squeezing the time component of the instanton. On the other hand, as noted in [39], adding a space dependence to the field tends to stretch the instantons along the time component. In this case, it is stretched toward the pole (3.105) as we see in Fig. 3.14. This suggests that the threshold should be sharper for larger γ_k and fixed ε, and this indeed is the case as we see in Fig. 3.14. Furthermore, from Fig. 3.15 we see that the widths $d_\perp$ and d_P are not very sensitive to changes in γ_k, while d_Δ changes significantly as γ_k is increased.

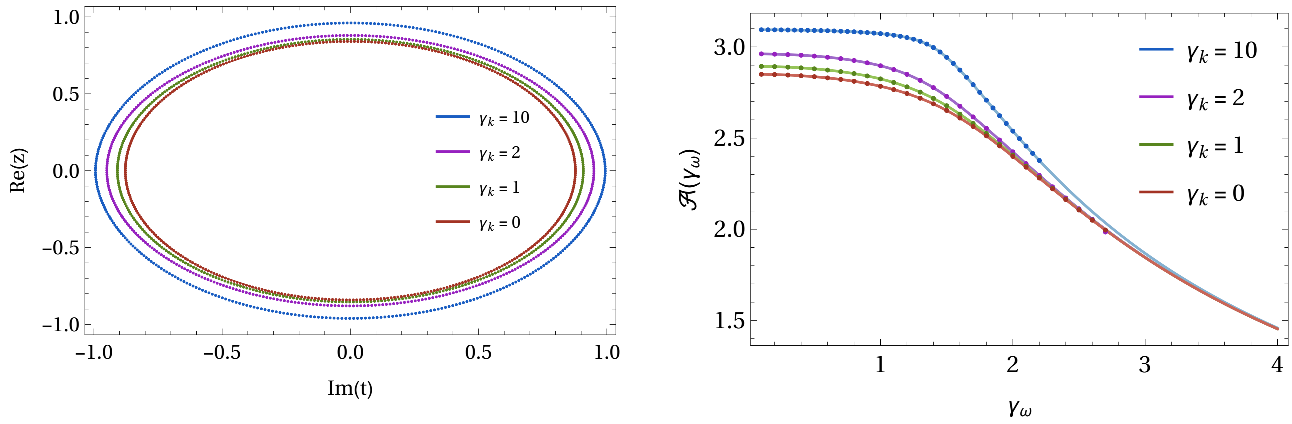


Figure 3.14: Closed instantons (left) and exponent (right) as a function of γ_ω and a few different values of γ_k. In the exponent plot, the continuous lines represent the result with the methods outlined here and the darker dots the result obtained from the effective action with discrete instantons, showing perfect agreement.

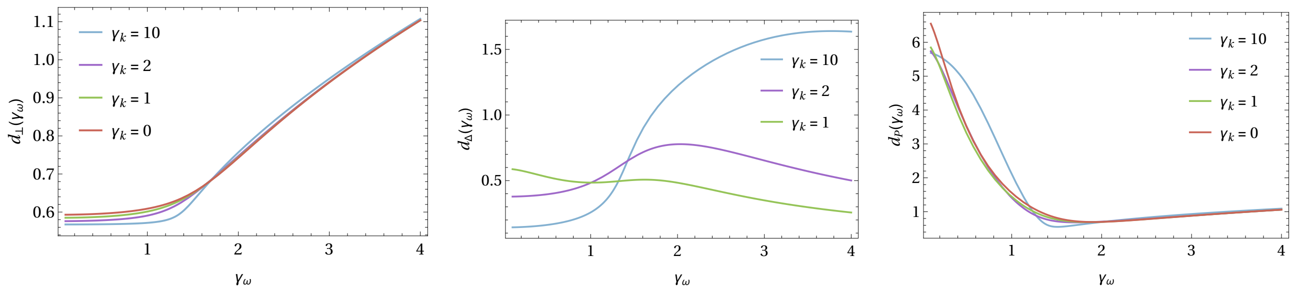


Figure 3.15: Widths as functions of γ_ω for a few fixed values of γ_k (without the overall factors of E).

3.5 Spectrum for 4D e-dipole fields

3.5.1 Introduction

While it seems challenging to extend the WKB method even beyond 1D fields, we have seen how the worldline formalism provides a powerful alternative way to access not only the integrated probability but also the spectrum of Schwinger pairs produced in spacetime fields. However, the 2D fields considered so far are not exact solutions of Maxwell's equations without a current[3]. In this section we turn to a realistic, exact, 4D class of solutions of Maxwell's equations in a vacuum, called e-dipole fields [67, 68], which are also optimally focused in the sense that they maximize the peak electric field for a given total energy. This class of fields is characterized by a single parameter γ, apart from the maximum field strength E. For slowly varying pulses $\gamma \ll 1$ we can use the locally constant field approximation as in [68]

$$\mathbb{P}_{LCF} = 2 \int d^4x \, \frac{\mathcal{E}^2(x)}{(2\pi)^3} \, e^{-\frac{\pi}{\mathcal{E}(x)}} \tag{3.106}$$

[3] Although this may not be such a tremendous problem [69].

where $\mathcal{E}^2 = -F_{\mu\nu}F^{\mu\nu} = \mathbf{E}^2 - \mathbf{B}^2$ (for these fields $\mathbf{E}\cdot\mathbf{B} = 0$). When $E \ll 1$ we can perform the integrals with the saddle-point method and obtain the leading order exponent and prefactor, but for $\gamma \sim \mathcal{O}(1)$ we cannot use this method. Here we show that one can obtain analytic corrections to the LCF approximation in powers of γ to the exponents and for the prefactor using open worldline instantons. We will see that just from the first couple of terms we obtain a good approximation even for values of the parameter that are not small.

An e-dipole field is defined as follows. Let g be an arbitrary function, $r = \sqrt{x^2+y^2+z^2}$, and

$$\mathbf{Z} = \mathbf{e}_z \, \frac{3E}{4r}[g(t+r) - g(t-r)] \; . \tag{3.107}$$

One can show that the electromagnetic field given by

$$\mathbf{E} = -\nabla\times\nabla\times\mathbf{Z} \, , \qquad \mathbf{B} = -\nabla\times\partial_t\mathbf{Z} \tag{3.108}$$

is an exact solutions to Maxwell's equations. As pointed out in [67], this field has the property of having finite energy and a finite limit when $r\to 0$ given by

$$\mathbf{E} = E\,\mathbf{e}_z\, g'''(t) \, , \qquad \mathbf{B} = 0 \; . \tag{3.109}$$

In practice, this means that we choose $g(t)$ such that $g'''(t)$ has a particular shape, such as Gaussian, Lorentzian, Sauter, or even more "exotic" examples such as a supergaussian field $g'''(t) = e^{-(\omega t)^4}$. For the moment, we only assume that $g(t)$ has a single real maximum.

The calculation is more similar to the one in the previous section than one might initially imagine. In this case, however, since the field is 4D, we cannot perform any of the integrals exactly, so we use the saddle-point method for all of them. In the previous section we have derived the saddle-point equation and exponent for a general spacetime field, obtaining in particular the Lorentz force equation from the path integral. At first sight, it might seem that finding the instantons becomes significantly more challenging in this case. However, we have already seen that we only need the instantons at the saddle point of the momentum even for the widths, and for this class of fields we have $p_\perp = p'_\perp = 0$ at the saddle point due to symmetry, which means we also have trivial transverse components of the instanton $q^\perp = 0$. Not only are we left with just two nontrivial components, but we also have on this subspace $\mathbf{B} = 0$, $E_1 = E_2 = 0$, which means that at least the method for finding the instantons fully reduces to the 2D case.

This is enough to obtain the exponent, but before moving on to the prefactor, we make an important remark on contours. In the previous section we found it more convenient to use a tilted contour $u(r) = e^{-i\theta}r$ but also mentioned that there exists always a contour such that t is purely imaginary in a small interval and real everywhere else. We called these two regions respectively the *formation* region and the *acceleration* region.

For dipoles, the field becomes wide when $\gamma \ll 1$, and so does the region we have to consider in order to see convergence to the asymptotics. Solving the Lorentz force equation over a large domain involves many time steps. However, using the physical contour (3.56) is much more convenient because the search is restricted to the formation region, which is bounded by $\tilde{t}(\gamma = 0) = i$ and $L = \pi/2$ for a constant field. We can see the real and imaginary parts of the instantons for a small and large value of γ in Fig. 3.16.

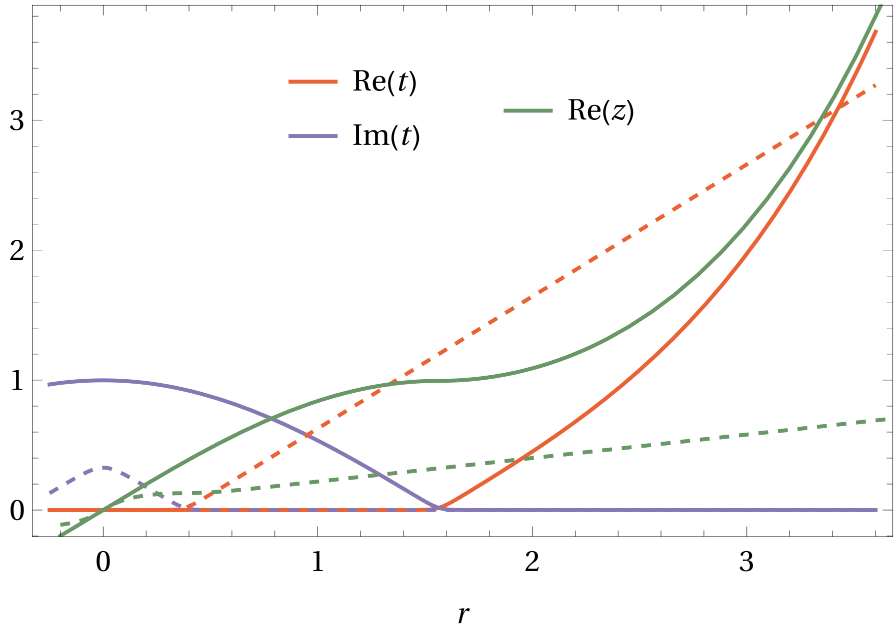


Figure 3.16: Real and imaginary parts of t and real part of z for $\gamma = 1/10$ (solid line) and $\gamma = 5$ (dashed). For small γ the formation region is larger, approximately $r_c \simeq \pi/2 \simeq 1.57$, while for lager γ it becomes tiny. Furthermore, at γ small the instantons also converge at large r.

This splitting is not just helpful for finding the instantons; it also allows us show that, while the individual widths depend on the acceleration region, the total integrated probability depends only on the formation region and not on what happens when the particles are accelerated. Similar contours have been used in other contexts as well, including particles experiencing a time-dependent mass [93], nonrelativistic tunneling [94, 95], and saddle points of fields [96, 97, 98, 99].

Before moving on to the calculation, let us look closer at some properties of e-dipole fields. Letting $g_1(t)$ and $g_2(t)$ be analytic functions, the field obtained by the sum of the two is simply the sum of the fields obtained by the two functions alone, therefore we can view $g(t) \to (\mathbf{E}, \mathbf{B})$ as a linear operator. Furthermore, a second order polynomial at^2+bt+c lies in the kernel of this operator, therefore we can add or subtract any such function without changing the fields, which in practice means that we can freely choose $g(0) = g'(0) = g''(0) = 0$. We further restrict to symmetric fields $g'''(t) = g'''(-t)$.

In terms of the function g, the electric field on the relevant subspace $x = y = 0$ simplifies to

$$E_3(t, z) = \frac{3E}{2z^2}\left[g(t-z) - g(t+z) + z\left(g'(t-z) + g'(t+z)\right)\right] \tag{3.110}$$

and taking $z \to 0$ one immediately sees $E(t, z = 0) = Eg'''(t)$. We can see in Fig. 3.17 what $E_3(t, z)$ looks like as a function of z and different times t. We remark in particular that for $|z| \gg 1$ the field goes to zero like $1/|z|$. From Fig. 3.17 it seems like there are multiple stationary points and thus multiple instantons, but the additional stationary points with respect to z are at $t \neq 0$ and thus satisfy $\partial_z E_3 = 0$ but $\partial_t E_3 \neq 0$. The only true stationary point is at $t = z = 0$ if $g'''(t)$ has a single maximum.

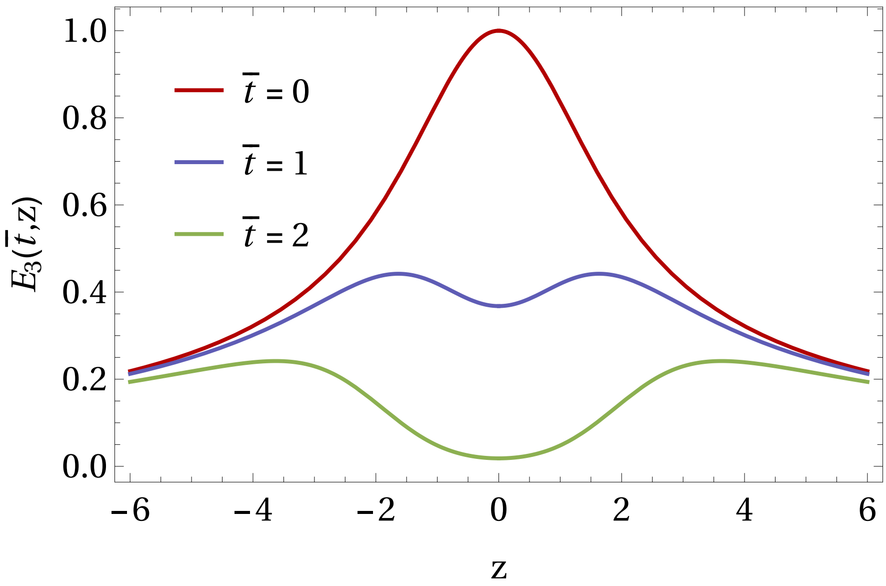


Figure 3.17: $E_3(t, z)/E$ as a function of z and fixed times with $g(t)$ chosen such that $g'''(t) = e^{-(\gamma t)^2}$ at $\gamma = 1$.

Finally, we note that the fields (3.108) are defined in terms of $\mathbf{Z}$ which is however not a gauge potential, so we need to construct some potential A_μ to calculate the exponent. Since $\mathbf{B} = -\nabla \times \partial_t \mathbf{Z}$ and the space components satisfy $\mathbf{B} = \nabla \times \mathbf{A}$, it is natural to identify $-\partial_t \mathbf{Z} = \mathbf{A}$ (where $\{0, 0, 1\} \cdot \mathbf{A} = -A_3$). The relative A_0, if we define $\mathbf{Z} = Z\mathbf{e}_z$, is simply $\partial_z Z$, from which

$$A_\mu = \{\partial_z Z, 0, 0, \partial_t Z\} \,. \tag{3.111}$$

The aforementioned gauge is advantageous with respect to the coordinate gauge [44, 100] in that it avoids integrals that are not necessarily simple. Furthermore, the two components $A_\perp$ are automatically zero.

3.5.2 Complex instantons

As in the previous section, once we have used the shooting method to compute the turning point $\tilde{t} = t(0)$, we can use the initial conditions to solve along a set of contours to visualize the instantons on the complex plane. Although dipoles have a different structure (see Fig. 3.17) compared to the 2D fields in the previous section, the instantons in the complex plane are very similar as we can see by comparing with Figs. 3.18 and 3.19.

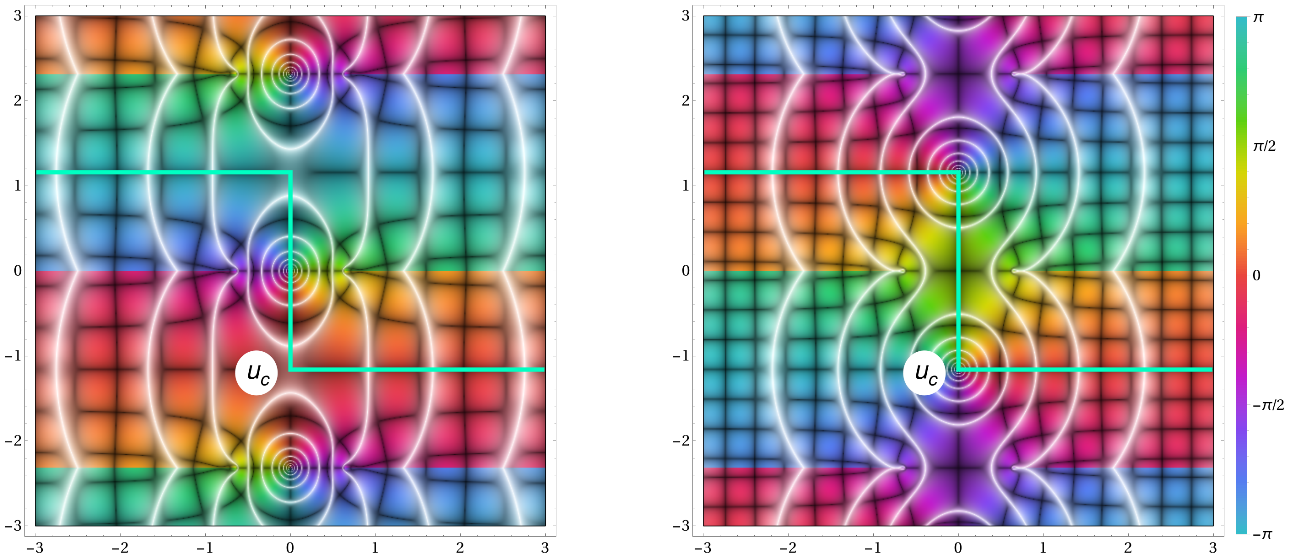


Figure 3.18: Space z (left) and time t (right) components of the instantons for the Gaussian dipole $g'''(t) = e^{-(\omega t)^2}$ with $\gamma = 1$.

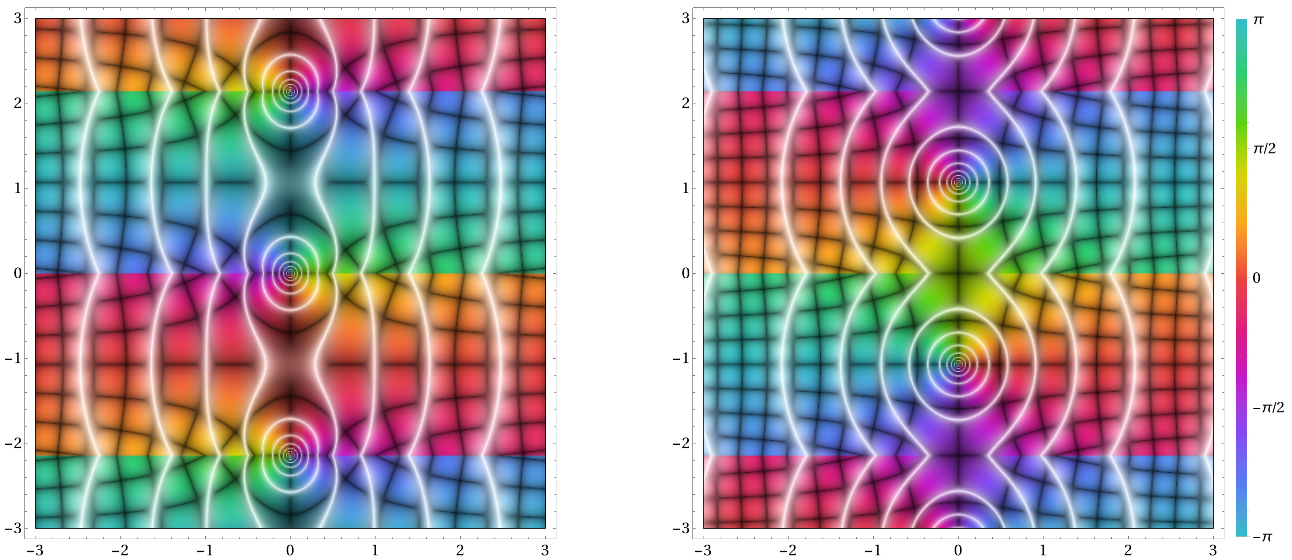


Figure 3.19: Space z (left) and time t (right) components of the instantons for the Lorentzian dipole $g'''(t) = (1 + (\omega t)^2)^{-1}$ with $\gamma = 1$.

3.5.3 Amplitude

Since the result for the exponent of the previous section

$$\psi = i \int du\, q^\mu \partial_\mu A_\nu \frac{dq^\nu}{du} \tag{3.112}$$

was already valid for any field that is asymptotically zero, we just need to deal with the prefactor, starting from the path integral. Expanding the exponent up to quadratic order in $\delta q = q - q_{\text{inst}}$ we get

$$-\frac{i}{2T}\int du\,\delta q\,\Lambda\,\delta q \tag{3.113}$$

where

$$\Lambda_{\mu\nu} = T^2(-\eta_{\mu\nu}\partial_u^2 + F_{\mu\nu}\partial_u + q'^{\rho}\partial_\nu F_{\mu\rho})\ . \tag{3.114}$$

However, the calculation simplifies greatly because the field is to be evaluated at the instanton, which has zero transverse components. However, while the only nonzero component of the field is E_3, the derivatives are to be computed first and evaluated at $x = y = 0$ only after, thus terms such as $\partial_x E_x$ are not necessarily zero even though $E_x = 0$. At the end one finds the following structure

$$\Lambda = \begin{pmatrix} \Lambda_{2D} & 0 & 0 \\ 0 & \Lambda_\perp & 0 \\ 0 & 0 & \Lambda_\perp \end{pmatrix} \tag{3.115}$$

where Λ_{2D} is the submatrix relative to the (t,z) components identical to (3.18) and

$$\Lambda_\perp = T^2(\partial_u^2 - t'\partial_x E_x + z'\partial_x B_y)\ . \tag{3.116}$$

Because the matrix is block diagonal, the determinant is the product of the determinants of the blocks

$$\det\Lambda = \det\Lambda_{2D}\,(\det\Lambda_\perp)^2 \tag{3.117}$$

with Λ_{2D} as in (3.18) and $\det\Lambda_\perp$ given by

$$\det\Lambda_\perp = \phi(u_1)\ , \tag{3.118}$$

with [101] ϕ a solution of

$$\Lambda_\perp\phi = 0 \qquad \phi(u_0) = 0 \qquad \phi'(u_0) = \frac{1}{T}\ . \tag{3.119}$$

As in the previous chapter, we would like to extract the dependence on the variables $t_\pm$ and T that go to infinity.

Let $(\tilde{u}_0, \tilde{u}_1)$ be some bounded interval where the field is nonzero. Since from u_0 to $\tilde{u}_0$ the solution is a straight line, we get

$$\phi(\tilde{u}_0) \simeq \frac{\tilde{u}_0 - u_0}{T} \simeq \frac{t_-}{Tp_0'} \tag{3.120}$$

so defining $\phi = \bar{\phi}\,t_-/Tp_0'$ we have new simple initial conditions for $\bar{\phi}$

$$\bar{\phi}(\tilde{u}_0) = 1 \qquad \bar{\phi}'(\tilde{u}_0) = 0\ , \tag{3.121}$$

where the second conditions follows from $T \to \infty$. The determinant follows evaluating $\phi(u_1)$ and separating the other asymptotic region where $\bar{\phi}$ is a straight line

$$\det \Lambda_\perp = \phi(u_1) \simeq \bar{\phi}'(\tilde{u}_1)(u_1 - \tilde{u}_1)\frac{t_-}{Tp'_0} \simeq \bar{\phi}'(\tilde{u}_1)\frac{t_+ t_-}{Tp_0 p'_0} \; . \tag{3.122}$$

Thus, the overall determinant is given by

$$\det \Lambda = \left(\frac{t_- t_+}{Tp'_0 p_0}\right)^3 \eta'(\tilde{u}_1) \left(\bar{\phi}'(\tilde{u}_1)\right)^2 \; . \tag{3.123}$$

Concerning the ordinary integral, we use a similar procedure as in the $2D$ case. Writing the exponent in its original form, we take first order derivatives with respect to $\mathbf{x}_\pm$ and T

$$\frac{\partial \psi}{\partial x^j_+} = i[p_j + q'^j(u_1)] \qquad \frac{\partial \psi}{\partial x^j_-} = i[p'_j - q'^j(u_0)] \qquad \frac{\partial \psi}{\partial T} = i[a^2 - 1] \tag{3.124}$$

where $q'^2 = a^2$. Eqs. (3.36) and (3.37) hold also in the $4D$ case for every spatial variable and with the replacement $t^2_\pm - z^2_\pm \to x^2_\pm = t^2_\pm - \mathbf{x}^2_\pm$. Using

$$q^{j\prime}(u_0) = -\frac{x^j_-}{T}\left(1 + \frac{\sqrt{x^2_+}}{\sqrt{x^2_-}}\right) \qquad q^{j\prime}(u_1) = \frac{x^j_+}{T}\left(1 + \frac{\sqrt{x^2_-}}{\sqrt{x^2_+}}\right) \qquad a^2 = \frac{\sqrt{x^2_-} + \sqrt{x^2_+}}{T} \; , \tag{3.125}$$

we can calculate the second derivatives of the exponent and evaluate them at the saddle points

$$x^j_{-s} = -\frac{p'_j}{p'_0} t_- \qquad x^j_{+s} = -\frac{p_j}{p_0} t_+ \qquad T_s = \frac{t_+}{p_0} + \frac{t_-}{p'_0} \; . \tag{3.126}$$

Defining a collective coordinate $\mathbf{X} := (T, \mathbf{x}_+, x_-)$ and $\delta\mathbf{X} = \mathbf{X} - \mathbf{X}_s$ we expand the exponent up to second order around $\mathbf{X}_s$ and obtain

$$\int d^7\mathbf{X}\, e^{-\delta\mathbf{X}\cdot\mathbf{H}\cdot\delta\mathbf{X}} = \sqrt{\frac{\pi^7}{\det \mathbf{H}}} \; . \tag{3.127}$$

The Hessian matrix itself

$$H_{ij} = -\frac{1}{2}\frac{d^2\psi}{\partial X_i \partial X_j} \tag{3.128}$$

does not have a nice form, but its determinant in the asymptotic $t_\pm \to \infty$ limit is

$$\det \mathbf{H} = \frac{p_0^5 p_0'^5}{2^7 t_-^3 t_+^3 T} \; . \tag{3.129}$$

Finally, regarding spin, since we evaluate the spin factor at the instantons, i.e. in particular on the subspace where $x = y = 0$, from $\mathbf{B} = 0$ and $E_x = E_y = 0$ we immediately see that the structure is the same as in Eq. (3.46), i.e.

$$\text{Spin}_{ss'} = \frac{1}{2}\bar{R}_s(\dots)R_{s'} = \delta_{ss'}\sqrt{p_0\, p'_0} \; . \tag{3.130}$$

Putting together all the contributions we find a spectrum

$$\mathbb{P} = 2\int \frac{d^3p\, d^3p'}{(2\pi)^3} \frac{1}{|h(\tilde{u}_1)|\, |\phi'(\tilde{u}_1)|^2\, p_0 p_0'}\, e^{-\mathcal{A}} , \qquad \mathcal{A} = 2\text{Im}\int du\, q^\mu \partial_\mu A_\nu \frac{dq^\nu}{du} \tag{3.131}$$

where we remark that the only numerical contributions to the prefactor are due to the functions $h(u)$ and $\bar{\phi}(u)$, which are solutions to (3.31) and (3.119), thus require no shooting.

3.5.4 Fantastic widths and how to find them

In the same spirit as in the last chapter, we now compute the widths in terms of the variables $p_j = -P_j + \frac{\Delta_j}{2}$ and $p'_j = P_j + \frac{\Delta_j}{2}$. Expanding around the saddle point $\Delta_j = 0$ we get

$$\mathcal{A}(\Delta) \simeq \mathcal{A}(0) + \Delta_i\, \Delta_j\, \mathcal{A}^{\Delta}_{ij} \tag{3.132}$$

where $\mathcal{A}^{\Delta}_{ij}$ is obtained as follows. Since the first order derivatives of the exponent are simply

$$\begin{aligned} \frac{\partial \mathcal{A}}{\partial p_j} &= 2\text{Im}\left(q^j(u_1) + \frac{p_j}{p_0} t(u_1)\right) \\ \frac{\partial \mathcal{A}}{\partial p'_j} &= 2\text{Im}\left(q^j(u_0) + \frac{p'_j}{p'_0} t(u_0)\right) . \end{aligned} \tag{3.133}$$

we get, using the chain rule,

$$\frac{\partial \mathcal{A}}{\partial \Delta_j} = \text{Im}\left(q^j(u_1) + \frac{p_j}{p_0} t(u_1)\right) + \text{Im}\left(q^j(u_0) + \frac{p'_j}{p'_0} t(u_0)\right) . \tag{3.134}$$

Now we simply need to perform a variation of the instantons (no sum over the index j)

$$q^\mu \to q^\mu + \Delta_j\, \delta q^\mu_{(j)} + \mathcal{O}(\Delta^2) , \tag{3.135}$$

with first order variations satisfying

$$\frac{d^2}{du^2}\delta q^\mu_{(j)} = F^{\mu\nu} \frac{d}{du}\delta q_{(j),\nu} + \partial_\rho F^{\mu\nu}\, q'_\nu\, \delta q^\rho_{(j)} \tag{3.136}$$

with boundary conditions

$$\begin{aligned} -\delta q'^i_{(j)}(u_0) = \delta q'^i_{(j)}(u_1) &= -\frac{\delta^i_j}{2} \\ \delta t'_{(j)}(u_0) = \delta t'_{(j)}(u_1) &= -\frac{P_j}{2p_0} \end{aligned} \tag{3.137}$$

to obtain

$$\mathcal{A}^{\Delta}_{ij} := \frac{1}{2}\frac{\partial^2 \mathcal{A}}{\partial \Delta_i\, \partial \Delta_j} = \text{Im}\left[\delta q^i_{(j)}(u_1) - \delta t_{(j)}(u_1)\frac{P_i}{p_0} + \frac{t(u_1)}{2p_0}\left(\delta^i_j - \frac{P_i P_j}{p_0^2}\right)\right] . \tag{3.138}$$

Once we have computed the integrals over Δ_j, we can take the derivative with respect to P_j to find the maximum value

$$\frac{\partial \mathcal{A}}{\partial P_i} = 4\,\text{Im}\left[\frac{P_i}{p_0} t(u_1) - q^i(u_1)\right] \stackrel{!}{=} 0 \tag{3.139}$$

analogous to the 2D case; we can now perform a variation

$$q^\mu \to q^\mu + \Delta_j\, \delta q^\mu_{(j)} \tag{3.140}$$

and find

$$\mathcal{A}^P_{ij} := \frac{1}{2}\frac{\partial^2 \mathcal{A}}{\partial P_i\, \partial P_j} = 2\,\mathrm{Im}\left[-\delta q^i_{(j)}(u_1) + \delta t_{(j)}\frac{P_i}{p_0} + \frac{t(u_1)}{p_0}\left(\delta^i_j - \frac{P_i P_j}{p_0^2}\right)\right] \tag{3.141}$$

with boundary conditions

$$\begin{aligned} \delta q'^i_{(j)}(u_0) &= \delta q'^i_{(j)}(u_1) = \delta^i_j \\ -\delta t'_{(j)}(u_0) &= \delta t'_{(j)}(u_1) = \frac{P_j}{p_0}\ . \end{aligned} \tag{3.142}$$

Focusing now on dipole fields, we expect most coefficients of the widths to vanish. Indeed, since the only nonzero derivatives are[4]

$$\begin{aligned} \partial_x E_x &= \partial_y E_y = -\frac{1}{2}\partial_z E_z \\ \partial_x B_y &= -\partial_y B_x = \frac{1}{2}\partial_t E_z \end{aligned} \tag{3.143}$$

and since the variations satisfy the boundary conditions above, all mixed terms cancel $A_{12} = \mathcal{A}_{13} = \mathcal{A}_{23} = 0$ and the perpendicular diagonal terms are equal $\mathcal{A}_{11} = \mathcal{A}_{22}$ (equalities holding for both Δ and P terms).

In summary, we have only four independent nontrivial widths, namely $\mathcal{A}^\Delta_{11}$ and $\mathcal{A}^\Delta_{33}$ for the Δ components and A^P_{11} and $\mathcal{A}^P_{33}$ for P. The terms $\mathcal{A}^\Delta_{33}$ and $\mathcal{A}^P_{33}$ can be treated identically to the 2D case, defining new η functions to reduce the degrees of freedom. As to the perpendicular ones we can avoid using the shooting method as well as follows. Noting that $P_\perp = 0$, the only function we have to evaluate to find the widths is $\delta x_{(1)}$ with certain boundary conditions that make it symmetric for Δ and antisymmetric for P. Since the equation it satisfies is homogeneous

$$\delta x''_{(1)} = (t'\partial_x E_x - z'\partial_x B_y)\delta x_{(1)} = -\frac{1}{2}\nabla E \cdot \{z', t'\}\delta x_{(1)} \tag{3.144}$$

we can express a general solution as

$$\delta x(u) = c_a \delta x_a(u) + c_s \delta x_s(u)\ , \tag{3.145}$$

where the subscripts a, s denote respectively antisymmetric/symmetric solutions with *initial* conditions

$$\delta x_a(0) = 0 \qquad \delta x'_a(0) = 1 \qquad \delta x_s(0) = 1 \qquad \delta x'_s(0) = 0\ . \tag{3.146}$$

For $\mathcal{A}^\Delta_{11}$, the solution will be given by

$$\delta x_{(1)}(u) = -\frac{1}{2}\frac{\delta x_s(u)}{\delta x'_s(\infty)} \tag{3.147}$$

[4] Equalities between them following from Maxwell's equation plus symmetry.

because it satisfies the differential equation and it has by construction the appropriate boundary conditions. Similarly, for $\mathcal{A}^P$ we have

$$\delta x_{(1)}(u) = \frac{\delta x_a(u)}{\delta x'_a(\infty)} \ . \tag{3.148}$$

To summarize, the four widths are given by

$$\begin{aligned} d_{\Delta,z}^{-2} &= \frac{1}{2p_0^2}\text{Im}\left(\frac{t}{p_0} - \frac{\eta_a}{\eta'_a}\right) \qquad & d_{P,z}^{-2} &= \frac{2}{p_0^2}\text{Im}\left(\frac{t}{p_0} - \frac{\eta_s}{\eta'_s}\right) \\ d_{\Delta,\perp}^{-2} &= \frac{1}{2}\text{Im}\left(\frac{t}{p_0} - \frac{\delta x_s}{\delta x'_s}\right) \qquad & d_{P,\perp}^{-2} &= 2\text{Im}\left(\frac{t}{p_0} - \frac{\delta x_a}{\delta x'_a}\right) \end{aligned} \tag{3.149}$$

with all functions evaluated at $u_1 \to \infty$.

While the expressions above are valid for any contour, i.e. with u_1 going to infinity along any path as long as we are on the appropriate branch, it is convenient to choose the contour with $\text{Im}t = 0$ for $r > r_c$. While it does not look like a big improvement to just drop the first addendum in all four widths, we show later that this contour simplifies greatly the calculations of the $\gamma \ll 1$ limits of the widths and the total probability. In particular, we expect the total probability to depend only on the *formation* region and independent of what happens in the asymptotic regions, i.e. when the particles have already been created and are simply accelerated. On the other hand, the momentum widths should be influenced by the *acceleration* region as well, therefore there must be some cancellations between the various contributions which are not obvious at the moment.

Writing the real/imaginary parts explicitly such as $\eta_s = \eta_{sr} + i\eta_{si}$ we get

$$\begin{aligned} d_{\Delta,z}^{-2} &= \frac{1}{2p_0^2}\text{Im}\left(-\frac{\eta_a}{\eta'_a}\right) = \frac{W(\eta_{ar},\eta_{ai})}{2p_0^2|\eta'_a|^2} \qquad & d_{P,z}^{-2} &= \frac{2}{p_0^2}\text{Im}\left(-\frac{\eta_s}{\eta'_s}\right) = 2\frac{W(\eta_{sr},\eta_{si})}{p_0^2|\eta'_s|^2} \\ d_{\Delta,\perp}^{-2} &= \frac{1}{2}\text{Im}\left(-\frac{\delta x_s}{\delta x'_s}\right) = \frac{W(\delta x_{sr},\delta x_{si})}{2|\delta x'_s|^2} \qquad & d_{P,\perp}^{-2} &= 2\text{Im}\left(-\frac{\delta x_a}{\delta x'_a}\right) = 2\frac{W(\delta x_{ar},\delta x_{ai})}{|\delta x'_a|^2} \ , \end{aligned} \tag{3.150}$$

with $W(f,g) = fg' - f'g$ the Wronskian of the functions f and g.

Furthermore, since $h(u_1) = \eta'(u_1)$, we can express η as a linear combination of η_s and η_a and find

$$|h| = 2\left|\frac{\eta_s}{\eta'_s} - \frac{\eta_a}{\eta'_a}\right|^{-1} \tag{3.151}$$

but since the Wronskian is constant $(\eta_s\eta'_a - \eta_a\eta'_s) = 1$ we have

$$|h(u_1)| = 2|\eta'_s\eta'_a| \ . \tag{3.152}$$

Similarly, writing $\bar{\phi} = c_a\delta x_a + c_s\delta x_s$ we find

$$|\bar{\phi}'(u_1)| = 2|\delta x'_s\delta x'_a| \ . \tag{3.153}$$

The reason why it is useful to express them in this form is not merely to make the result computationally more economic. This expressions (3.150) are effectively split into a *local* contribution, given by the Wronskians, and a *nonlocal* one, given by the denominators. The Wronskian is "local" because at $r > r_c$ the real/imaginary parts are separately solutions so the Wronskian becomes constant. This in turn implies that we can evaluate it at $r \gtrsim r_c$, i.e. when the particles are created, instead of when the field goes to zero (which is large when $\gamma \ll 1$).

3.5.5 Spectrum and integrated probability

In terms of the widths calculated above the spectrum has the form

$$\boxed{\mathbb{P}(p,p') = \frac{2e^{-\mathcal{A}}}{(2\pi)^3|h\bar{\phi}'^2|p_0^2}\exp\left\{-\frac{\Delta_\perp^2}{d_{\Delta,\perp}^2}-\frac{\Delta_z^2}{d_{\Delta,z}^2}-\frac{P_\perp^2}{d_{P,\perp}^2}-\frac{(P_z-\mathscr{P})^2}{d_{P,z}^2}\right\}} \tag{3.154}$$

where $\mathcal{A}$ is the exponent at the saddle point and $\mathscr{P}$ is the saddle point of the longitudinal momentum P_z. The total probability

$$\mathbb{P} = \int d^3p\, d^3p'\; \mathbb{P}(p,p') \tag{3.155}$$

is hence given by

$$\boxed{\mathbb{P} = \frac{[W(\eta_{ar},\eta_{ai})W(\eta_{sr},\eta_{si})]^{-1/2}e^{-\mathcal{A}}}{32W(\delta x_{ar},\delta x_{ai})W(\delta x_{sr},\delta x_{si})}}\,. \tag{3.156}$$

The reason why (3.156) is significant is that all the nonlocal contributions cancel and we are left with the local ones. As we mentioned, the widths themselves are affected by what happens in the whole region where the field is nonzero, whereas the total probability only depends on the formation region and not on what happens after $r > r_c$, where the particles are real and are just being accelerated. The exponent and prefactor are shown in Fig. 3.20.

3.6 Locally constant field approximation and beyond

In this section we show how to compute the limit $\gamma \to 0$ of the exponent and the prefactor of the total probability and obtain analytic corrections to the locally constant field approximation in powers of γ for both. In the LCFA, the probability is given by

$$\mathbb{P}_{LCF} = 2\int d^4x\, \frac{\mathcal{E}^2(x)}{(2\pi)^3}\, e^{-\frac{\pi}{\mathcal{E}(x)}} \tag{3.157}$$

and one can show that, for any e-dipole field with a single peak, the integral above gives, in the $E \ll 1$ limit,

$$\frac{5\sqrt{5}}{(2\pi)^3\gamma^4}e^{-\frac{\pi}{E}} \tag{3.158}$$

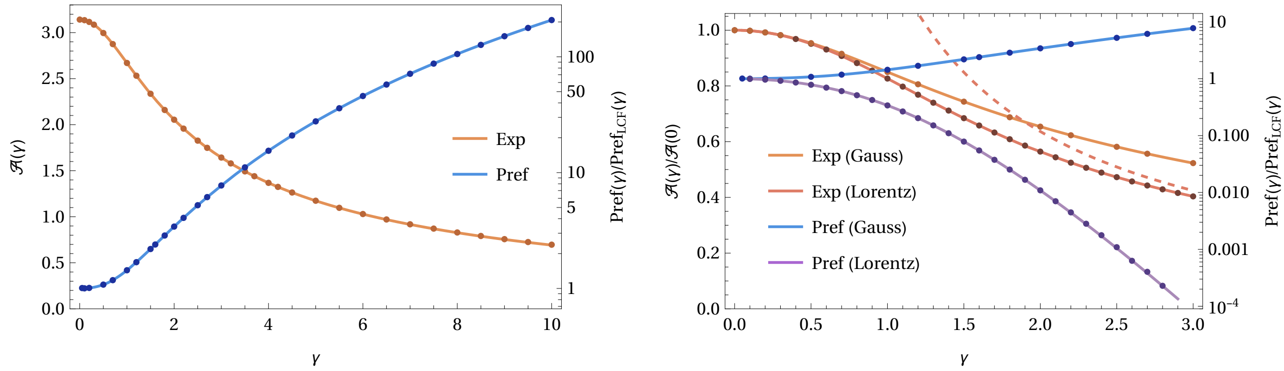


Figure 3.20: Exponent and prefactor obtained using the method outlined in this section and comparison with the effective action method. On the left we see both for a Gaussian pulse, on the right the comparison with a Lorentz pulse. The prefactor is normalized using the locally constant field result (3.158). The solid lines in the plot on the left are obtained with open instantons and the LSZ formula, while the darker dots with discrete closed instantons and the effective action. The dashed line represents the perturbative result $\gamma \gg 1$ in (3.179).

where the exponent is simply the constant field result. However, using this method, one cannot go beyond the leading order result and it does not give information about the spectrum. We now show how to obtain higher order corrections in γ using open worldline instantons. First of all, we note that in the constant field limit the field becomes larger so the pair is accelerated more the smaller γ becomes. In particular, letting

$$g(u) = G(\omega u)/\omega^3 \tag{3.159}$$

the leading order approximation to the asymptotic momenta is given by [65]

$$\mathscr{P} \simeq p_0 = t'(\infty) \simeq \frac{3G''(\infty)}{2\gamma} , \tag{3.160}$$

so for the pulses we consider

$$\mathscr{P}_{\text{Gauss}} \simeq \frac{3\sqrt{\pi}}{4\gamma} , \qquad \mathscr{P}_{\text{Lorentz}} \simeq \frac{3\pi}{4\gamma} . \tag{3.161}$$

The normalized longitudinal momentum at various γ is shown in Fig. 3.21.

To calculate the small γ expansion of the exponent and the prefactor we express the instantons as power series

$$t(u) = \sum_{n=0}^{\infty} t_n(u)\gamma^{2n} \qquad z(u) = \sum_{n=0}^{\infty} z_n(u)\gamma^{2n} \tag{3.162}$$

with $t_0(u) = i\cosh(u)$ and $z_0(u) = i\sinh(u)$ the constant field instantons. However, we must be careful because the $\gamma \to 0$ limit is not *uniform*, i.e. when u is larger the convergence is slower (thus the approximation is worse). This might be troublesome because, unlike the time-dependent limit

$\gamma \to \gamma_k$ in previous sections, the domain wherein the instantons are nontrivial grows when γ becomes smaller. Furthermore, while the derivatives of the instantons are bounded for all values of γ, this is clearly not true for the constant field instantons (and for the higher order corrections as well). At any rate, we can dismiss this subtlety by assuming that we work in a bounded domain, which is good if we focus on the exponent and on the total probability. Since for a constant field $\gamma = 0$ the formation region is $u_c = -\frac{i\pi}{2}$, it is bounded for any γ.

We can find the next-to-leading correction to the instantons using the initial conditions $z_1(0) = z_1'(0) = t_1'(0) = 0$, and $t_1(0)$ a constant to be found. Let $u_c(\gamma)$ be the point such that $t(u) = 0$ for a given γ; we can expand it

$$u_c \simeq -\frac{i\pi}{2} + \delta u\, \gamma^2 \tag{3.163}$$

and determine the two constants δu and $t_1(0)$ by demanding $t(u_c) = z'(u_c) = 0$

$$t_1(0) = -\frac{i}{5} \qquad \delta u = \frac{i\pi}{5} \tag{3.164}$$

from which

$$\begin{aligned} t_1(u) &= \frac{i}{20}[8u\sinh(u) - 5\cosh(u) + \cosh(3u)] \\ z_1(u) &= \frac{i}{20}[8u\cosh(u) - 11\sinh(u) + \sinh(3u)] \,. \end{aligned} \tag{3.165}$$

To obtain the next-to-leading order (NLO) term in the exponent

$$\mathcal{A}(\gamma) \simeq \mathcal{A}(0) + \frac{1}{2}\mathcal{A}''(0)\gamma^2 \tag{3.166}$$

we begin by writing the exponent as

$$\mathcal{A}(\gamma) = 2\,\mathrm{Im}\left[px_+ + p'x_- - \frac{T}{2} - \int_0^1 d\tau\, A\dot{q}\right] \,. \tag{3.167}$$

(3.167) is evaluated at all the saddle points as functions of γ, but taking the derivative we only have to consider the *explicit* dependence on γ in the field A (cf. Appendix A). Furthermore, since there is no $\mathcal{O}(\gamma)$ term due to symmetry we have

$$\frac{1}{2}\mathcal{A}''(0) = -\mathrm{Im}\lim_{\gamma\to 0}\frac{1}{\gamma}\int du\, \frac{dA^\mu}{d\gamma}q'_\mu \tag{3.168}$$

where the instanton q_μ is the constant field one. For all dipole fields, one finds (without the overall factor of $1/E$)

$$\frac{1}{2}\mathcal{A}''(0) = -\frac{\pi}{5} \,. \tag{3.169}$$

We note that, to obtain beyond the next-to-leading order order term we only use the local behavior of the field $A(q)$ near $q = 0$ and the constant field instantons, so the correction is the same for all dipole fields. To go beyond NLO and obtain the next-to-next-to-leading order (NNLO), we also need (3.165). One can generalize the previous argument and find (without the overall factor of $1/E$)

$$\mathcal{A} \simeq \pi \left(1 - \frac{\gamma^2}{5} - [G^{(7)}(0) - 28]\frac{\gamma^4}{280}\right) \tag{3.170}$$

which shows that the NNLO correction depends on the shape of the field through $G^{(7)}(0)$.

Similarly, expanding the η's and δx's one can show

$$\text{Pref} \simeq \frac{5\sqrt{5}}{(2\pi)^3\gamma^4}\left[1 + \frac{4557 - 224\pi^2 - 162G^{(7)}(0)}{1680}\gamma^2\right] . \tag{3.171}$$

From (3.171) we see that, unlike the exponent, the prefactor becomes larger or smaller depending on the field shape. For example, for a Gaussian pulse we have

$$\frac{\text{Pref}}{\text{Pref}_{LO}} \simeq 1 + 0.24\gamma^2 \tag{3.172}$$

while for a Lorentzian pulse

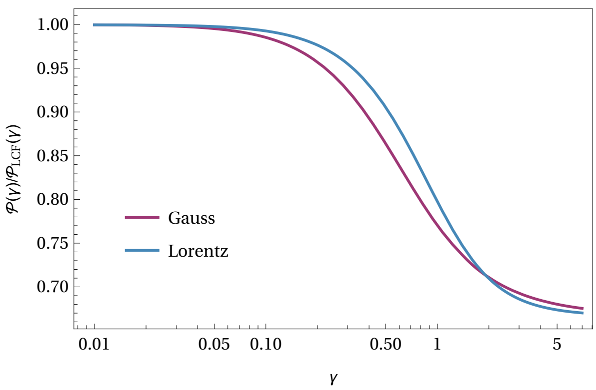
$$\frac{\text{Pref}}{\text{Pref}_{LO}} \simeq 1 - 0.92\gamma^2 \tag{3.173}$$

as we can see in Fig. 3.22.

We stress that the next-to-leading corrections cannot be computed using the locally constant field approximation, which only gives the trivial constant field result for the exponent and the $\sim 1/\gamma^4$ term at the prefactor. A comparison with the various orders of approximation of the exponent can be found in Fig. 3.21 by plotting the relative error for the generic quantity Q

$$\Delta Q := \left|\frac{Q_{\text{exact}}}{Q_{\text{approx}}} - 1\right| \tag{3.174}$$

for a Gaussian as well as for a Lorentzian pulse.



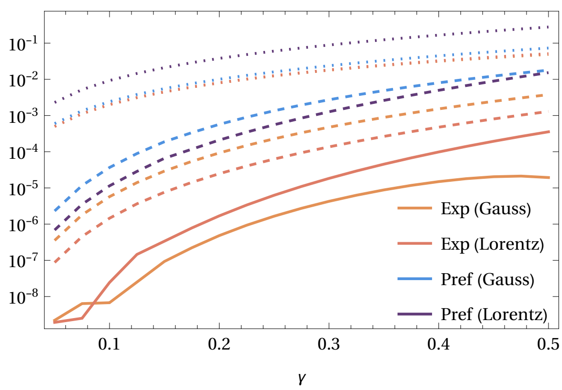


Figure 3.21: Normalized longitudinal momentum for Gaussian and Lorentzian pulse (left) and relative error at various order of approximation. The dotted line are LO, dashed NLO, solid NNLO.

The computation of the nonlocal contributions to the widths is much more complicated and can be found in [65]. As to the longitudinal widths, one can show that

$$\eta_a'(\infty) \simeq -\frac{i\gamma}{2\phi'(\infty)} \qquad \eta_s'(\infty) \simeq -\frac{3i\gamma}{2\phi'(\infty)} \tag{3.175}$$

from which these simple results follow

$$p_0^2|\eta_a'(\infty)|^2 \simeq \frac{1}{4} \qquad p_0^2|\eta_s'(\infty)|^2 \simeq \frac{9}{4} \qquad p_0^2|h| \simeq \frac{3}{2} \tag{3.176}$$

hence the widths become (without the factor of E)

$$d_{P,z} \simeq \frac{3}{2\pi\gamma} \qquad d_{\Delta,z} \simeq \frac{\sqrt{5}}{\pi\gamma} \tag{3.177}$$

which, remarkably, is the same for all e-dipole fields. For the transverse widths we find

$$d_{P,\perp} \simeq \left|c_1 \ln\left[\frac{1}{\gamma}\right] + c_2\right| \qquad d_{\Delta,\perp} \simeq \frac{c_3}{\gamma} \tag{3.178}$$

where the constants c_i are to be determined numerically as explained in [65]. Using these results we normalize the widths obtained numerically in Figs. 3.20 and 3.22. The opposite limit, $\gamma \gg 1$, can be found in [65]. For a Lorentzian pulse we find

$$\mathcal{A} \simeq \frac{4}{\gamma} \qquad d_P \simeq \sqrt{\gamma/2} \qquad d_\Delta \simeq \sqrt{2\gamma}\,. \tag{3.179}$$

Notably, the transverse and longitudinal widths converge as we see in Fig. 3.22.

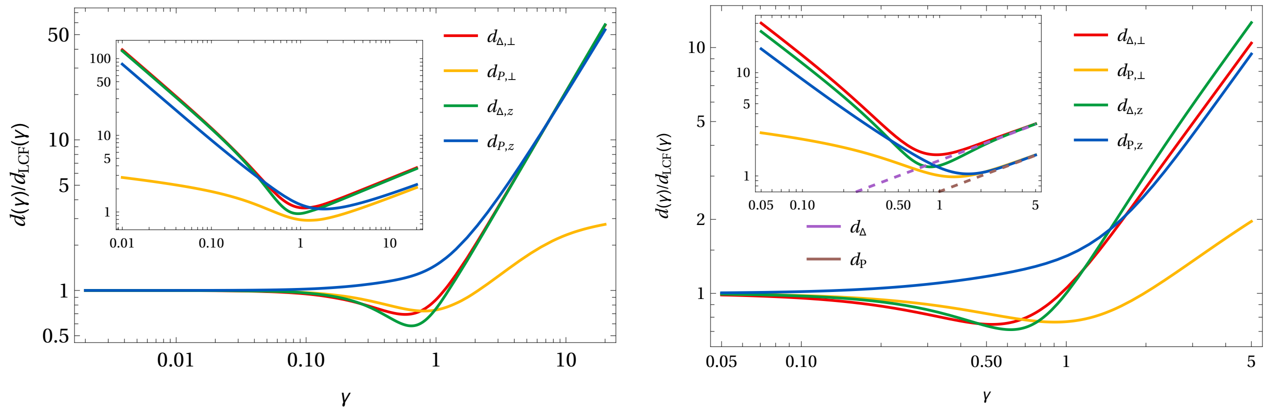


Figure 3.22: Normalized and unnormalized widths for a Gaussian (left) and Lorentzian pulse (right). The dashed lines on the right are the perturbative limits (3.179) at $\gamma \gg 1$.

Chapter 4

Breit-Wheeler catastrophes

After the study of Schwinger pair production for $2D$ and $4D$ fields in the previous chapter, we now include an incoming photon and apply the instanton techniques to Breit-Wheeler pair production. In the previous chapter we used a semiclassical approximation to obtain the pair production spectrum from the worldline instantons. In this chapter, using the same techniques, we show that there is a phase transition in the momentum spectrum as the incoming photon wave packet becomes more spread out in momentum space. From a mathematical perspective, this is analogous to Landau's theory of continuous phase transitions. The material in this chapter is taken from [102].

4.1 Amplitude

We consider the LSZ formula from the previous chapter (3.1) with the replacement

$$A_\mu \to A_\mu + \varepsilon_\mu \, e^{-ikq} \tag{4.1}$$

in the Dirac propagator (3.3) to include the photon in the incoming state. We define $\Gamma(x_+, x_-; \varepsilon, k)$ as the spinor propagator after applying (4.1) and taking the linear part in ε and

$$M(k) = \lim_{t_\pm \to \infty} \int d^3x_+ \, d^3x_- \, e^{ipx_+ + ip'x_-} \bar{u}_s(p) \, \gamma^0 \, \Gamma(x_+, x_-; \varepsilon, k) \, \gamma^0 \, v_{s'}(p') \tag{4.2}$$

the amplitude at photon momentum k. We also call σ the proper-time point in τ when the photon is absorbed. Including the effects of a Gaussian wave packet $f(k)$ the amplitude is given by

$$M = \int \frac{d^3k}{(2\pi)^3 2k_0} f(k) M(k) \tag{4.3}$$

with

$$f(k) = \rho(k) \exp\left\{ \sum_{j=1}^{3} -\frac{(k_j - l_j)^2}{2\lambda_j^2} + i b^j k_j \right\} \tag{4.4}$$

normalized according to

$$\int \frac{d^3k}{(2\pi)^3 2k_0} |f(k)|^2 = 1 \, . \tag{4.5}$$

We consider a wave packet centered around $l_3 = 0$ and with no impact parameter $b^j = 0$. Furthermore, without loss of generality we set $l_2 = 0$ and define $\Omega = l_1$.

We consider a $2D$ field of the form $E_3(t, z)$ as in the previous chapter. For the numerical calculations and the results shown in the plots we use a double Sauter pulse

$$A_3(t, z) = \frac{E}{\omega} \tanh(\omega t) \, \mathrm{sech}(\kappa z)^2 \tag{4.6}$$

with

$$\gamma_t = \frac{\omega}{E} \qquad \gamma_z = \frac{\kappa}{E} \, . \tag{4.7}$$

As in the previous chapter, defining $\varphi^\perp = x_+^\perp - x_-^\perp$ and shifting $q^\perp(\tau) \to q^\perp(\tau) + \varphi^\perp$ the $\varphi^\perp$ integral gives a delta function

$$\int d^2\varphi\, e^{i(p+p'-k)_\perp \varphi^\perp} = (2\pi)^2 \delta_\perp(p + p' - k) \tag{4.8}$$

which we use to perform the $k_\perp$ integrals in (4.3) so that $k_\perp = p_\perp + p'_\perp$. Assuming λ_1, λ_2 to be very small, we can perform the $p'_\perp$ integrals as

$$\int \frac{d^2p'_\perp}{(2\pi)^2} \exp\left\{-\sum_{j=1}^{2} \frac{(p_\perp + p'_\perp - l_\perp)^2}{\lambda_j^2}\right\} F(p'_\perp) \simeq \frac{\lambda_1\lambda_2}{4\pi} F(l_\perp - p_\perp)\ . \tag{4.9}$$

For time-dependent fields, if we assume the wave packet to be very narrow in the z direction as well, we can completely neglect wave packet effects. However, the 2D fields we consider have a finite size along z, so if $\lambda := \lambda_3$ is small as well then the wave packet is highly spread out in space and the overlapping with the field is small.

After calculating all the transverse integrals we arrive at our starting point for the calculation

$$\mathbb{P} = \frac{e^2}{4\pi^{3/2}\lambda} \int \frac{d^2p_\perp}{(2\pi)^2} \frac{dp_3\, dp'_3}{(2\pi)^2} \left| \int \frac{dk_3}{\sqrt{k_0}} \exp\left[-\frac{k_3^2}{2\lambda^2}\right] \mathbb{M} \right|^2 \tag{4.10}$$

with $\mathbb{M}$ defined as the amplitude stripped off of the perpendicular delta function (4.8).

4.1.1 Exponent

The calculation of the wave packet independent exponent is nearly the same as in the previous chapter. From

$$\psi = ipx_+ + ip'x_- - \frac{iT}{2} - i\int \frac{\dot{q}^2}{2T} + A\dot{q} + \delta(\tau - \sigma)\, kq \tag{4.11}$$

we see that saddle point equations for T and $\mathbf{x}_\pm$ are the same as for Schwinger pair production (3.10), and from the additional $d\sigma$ integral we have

$$\dot{q}(\sigma) \cdot k = 0\ . \tag{4.12}$$

However, the Lorentz force equation have an additional term due to (4.1)

$$\ddot{q}^\mu = TF^{\mu\nu}\dot{q}_\nu + Tk^\mu\delta(\tau - \sigma) \tag{4.13}$$

which amounts to a jump in the derivative $\dot{q}(\sigma^+) - \dot{q}(\sigma^-) = Tk$. Using the saddle-point equations (3.10), (4.12), (4.13) and integrating by parts one sees

$$-\frac{iT}{2} - i\int_0^1 d\tau\, \frac{\dot{q}^2}{2T} + A\dot{q} = -ipx_+ - ip'x_- + i\int_0^1 d\tau\, q^\mu \partial_\mu A_\nu \dot{q}^\nu - \delta(\tau - \sigma)\, kq \tag{4.14}$$

from which almost everything cancels and we are left with

$$\psi = i \int du\, q^\mu \partial_\mu A_\nu \frac{dq^\nu}{du} \tag{4.15}$$

where $u = T_s(\tau - \sigma)$ extends from $-\infty$ to ∞ since $T_s \to \infty$. A plot of the exponent at the probability level

$$\mathcal{A} = -2\text{Re}\,\psi \tag{4.16}$$

is shown in Fig. 4.1.

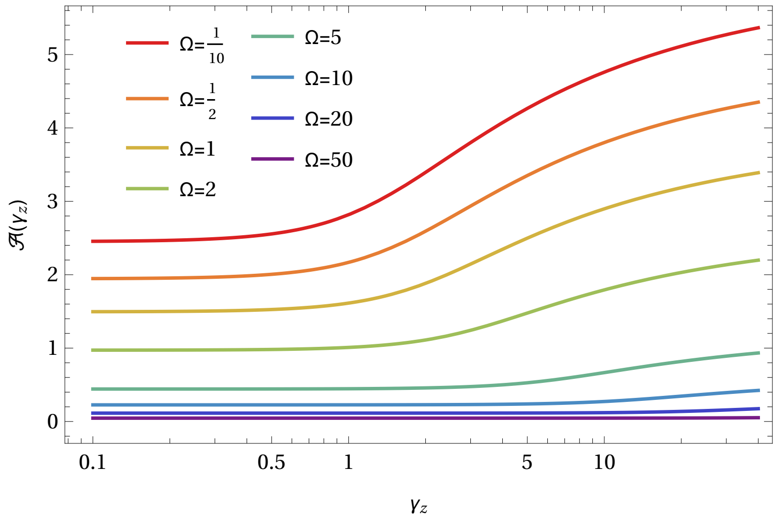


Figure 4.1: Exponent (without the overall factor of 1/E) as a function of the size of the field γ_z for different values of the photon energy Ω and $\gamma_t = 1$ in the subcritical regime

Unlike in Schwinger pair production, where the only non-analyticity is given by the branch points, the Breit-Wheeler instantons have now a kink at the origin $u = 0$ due to the delta function in the Lorentz force equation (4.13). If we perform a change of contour and make u complex, it might not seem obvious what the meaning of $\delta(u)$ should be. As we discuss in more detail in the next section, this simply splits each component of the instanton into two parts.

4.1.2 Scalar prefactor

The ordinary and path integrals are very similar to the previous chapter, and in some sense even simpler. Starting with the path integral, we use the Gelfand-Yaglom method as in the previous chapter to find the determinant of the operator Λ defined as in (3.18) with the only difference that the instantons are not smooth at one point.

Since the field is zero in the asymptotic regions $(u_0, \tilde{u}_0)$ and $(\tilde{u}_1, u_1)$ with some finite $\tilde{u}_0, \tilde{u}_1$, we can write

$$\phi_j^{(i)}(\tilde{u}_0) \simeq \frac{1}{T}(\tilde{u}_0 - u_0)\delta_j^i \tag{4.17}$$

and

$$\begin{aligned} \tilde{u}_0 - u_0 &\simeq \frac{t_-}{p'_0} \\ u_1 - \tilde{u}_1 &\simeq \frac{t_+}{p_0} \ . \end{aligned} \tag{4.18}$$

Then, rescaling

$$\bar{\phi} := \phi \frac{Tp'_0}{t_-} \tag{4.19}$$

we have new initial conditions at $\tilde{u}_0$ independent of t_- and T

$$\bar{\phi}_j^{(i)}(\tilde{u}_0) = \delta_j^i \qquad \frac{d\bar{\phi}_j^{(i)}}{du}(\tilde{u}_0) = 0 \ . \tag{4.20}$$

Since the $\bar{\phi}$ solutions are linear in the asymptotic region $(\tilde{u}_1, u_1)$ as well, the determinant scales quadratically with u

$$D(u) := \left(\bar{\phi}_1^{(1)}\,\bar{\phi}_2^{(2)} - \bar{\phi}_2^{(1)}\,\bar{\phi}_1^{(2)}\right)(u) \simeq \frac{1}{2}D''(\tilde{u}_1)(u-\tilde{u}_1)^2 \tag{4.21}$$

where

$$\frac{1}{2}D''(u) = \left(\bar{\phi}_1'^{(1)}\,\bar{\phi}_2'^{(2)} - \bar{\phi}_2'^{(1)}\,\bar{\phi}_1'^{(2)}\right)(u) \ . \tag{4.22}$$

Using the relations above, the determinant follows immediately

$$\begin{aligned} \det\Lambda &= \left(\frac{t_-}{Tp'_0}\right)^2 \frac{1}{2}D''(\tilde{u}_1)(u_1-\tilde{u}_1)^2 \\ &= \left(\frac{t_+t_-}{Tp_0p'_0}\right)^2 \frac{1}{2}D''(\tilde{u}_1) \ . \end{aligned} \tag{4.23}$$

As to the ordinary integrals, the difference with the Schwinger $2D$ case is that we have the extra σ integral and $t'^2 - z'^2$ is equal to two different constants before vs after absorption

$$\begin{aligned} \tau < \sigma \qquad & t'^2 - z'^2 = a_-^2 \\ \tau > \sigma \qquad & t'^2 - z'^2 = a_+^2 \ . \end{aligned} \tag{4.24}$$

Starting with the transverse integrals we get

$$\begin{aligned} \int d^2\varphi_\perp\,(\dots) &= (2\pi)^2\delta_\perp(p+p'-k) \\ \int d^2\theta_\perp\,\mathcal{D}q_\perp(\dots) &= e^{-\frac{iT}{2}[\sigma p_+'^2 + (1-\sigma)p_\perp^2]} \ . \end{aligned} \tag{4.25}$$

The remaining nontrivial integrals over $\mathbf{X} = (T, \sigma, z_\pm)$ are treated as in the Schwinger $2D$ case where we extracted the formally divergent contributions analytically. Since we have performed the path integral, the exponent is evaluated at the instantons as functions of $\mathbf{X}$, which we can only find numerically. However, as shown in Appendix A, for the first derivative the additional terms cancel and we get

$$\begin{aligned} \frac{\partial\psi}{\partial z_-} &= i[p'_3 - z'(u_0)] \qquad \frac{\partial\psi}{\partial z_+} = i[p_3 + z'(u_1)] \qquad \frac{\partial\psi}{\partial\sigma} = -ik\dot{q}(\sigma) \\ \frac{\partial\psi}{\partial T} &= \frac{i}{2}\left\{\sigma a_-^2 + (1-\sigma)a_+^2 - [\sigma p_+'^2 + (1-\sigma)p_\perp^2]\right\} \ . \end{aligned} \tag{4.26}$$

Due to the kink of $\dot{q}$ at $\tau = \sigma$, our first task is to make sense of the derivative with respect to σ. Using

$$k\dot{q}(\sigma^+) - k\dot{q}(\sigma^-) = Tk^2 = 0 \tag{4.27}$$

we find a well-defined expression

$$k\dot{q}(\sigma) = \frac{T}{2}[q'^2(0^+) - q'^2(0^-)] = \frac{T}{2}(a_+^2 - a_-^2 - p_\perp^2 + p_\perp'^2)\ . \tag{4.28}$$

To compute the second derivatives we use

$$\begin{aligned} T\sigma = T\int_0^\sigma d\tau &= \int_{t_-}^{\tilde{t}} \frac{dt}{t'} \simeq \frac{t_-}{\sqrt{a_-^2 + z'(u_0)^2}} \\ T(1-\sigma) = T\int_\sigma^1 d\tau &= \int_{\tilde{t}}^{t_+} \frac{dt}{t'} \simeq \frac{t_+}{\sqrt{a_+^2 + z'(u_1)^2}} \\ z_- = \tilde{z} &+ \int_{\tilde{t}}^{t_-} dt\frac{z'}{t'} \simeq -\frac{z'(u_0)t_-}{\sqrt{a_-^2 + z'(u_0)^2}} \\ z_+ = \tilde{z} &+ \int_{\tilde{t}}^{t_+} dt\frac{z'}{t'} \simeq \frac{z'(u_1)t_+}{\sqrt{a_+^2 + z'(u_1)^2}}\ , \end{aligned} \tag{4.29}$$

to rewrite

$$\begin{aligned} z'(u_0) &= -\frac{z_-}{T\sigma} \qquad & z'(u_1) &= \frac{z_+}{T(1-\sigma)} \\ a_-^2 &= \frac{t_-^2 - z_-^2}{T^2\sigma^2} \qquad & a_+^2 &= \frac{t_+^2 - z_+^2}{T^2(1-\sigma)^2}\ . \end{aligned} \tag{4.30}$$

and to find the saddle-point values of the integration variables $\mathbf{X}$

$$\begin{aligned} z_{-s} &= -\frac{p_3' t_-}{p_0'} \qquad & z_{+s} &= -\frac{p_3 t_+}{p_0} \\ T_s &= \frac{t_-}{p_0'} + \frac{t_+}{p_0} \qquad & \sigma_s &= \frac{t_-}{p_0'}\left(\frac{t_-}{p_0'} + \frac{t_+}{p_0}\right)^{-1}\ . \end{aligned} \tag{4.31}$$

Letting $\delta\mathbf{X} = \mathbf{X} - \mathbf{X}_s$ and expanding the exponent up to second order we find

$$\int d^4\mathbf{X}\, e^{-\delta\mathbf{X}\cdot\mathbf{H}\cdot\delta\mathbf{X}} = \frac{\pi^2}{\sqrt{\det\mathbf{H}}}\ , \tag{4.32}$$

where $\mathbf{H}_{ij} = -(1/2)\partial^2\psi/(\partial X_i \partial X_j)$ is the Hessian matrix with determinant

$$\det\mathbf{H} = \left(\frac{p_0^2 p_0'^2 T}{4t_+t_-}\right)^2\ . \tag{4.33}$$

We note that (4.33) has the exact opposite asymptotic contributions with respect to (4.23), so the factors of $t_\pm$ will cancel.

4.1.3 Spin

As in the previous chapter, there is no on-shell contribution from the shift $A \to A + \varepsilon^{-ikq}$ in the $\not{A}$ term at the prefactor of Γ in (4.2). The two contributions, T_1 and T_2, are respectively from ε in the $A\dot{q}$ and $\sigma^{\mu\nu}F_{\mu\nu}$ terms. Defining

$$\varepsilon_\mu^{(\|)} = \frac{1}{k_0}\,(0, -k_3, 0, k_1)\ , \qquad \varepsilon_\mu^{(\perp)} = (0, 0, 1, 0) \tag{4.34}$$

we express a general polarization state as

$$\varepsilon_\mu = \cos\left(\frac{\rho}{2}\right)\varepsilon_\mu^{(\|)} + \sin\left(\frac{\rho}{2}\right)e^{i\lambda}\varepsilon_\mu^{(\perp)}\ , \tag{4.35}$$

where ρ and λ are two constants. The probability can be expressed in terms of the Stokes vector

$$\mathbf{N} = \{1, \cos(\lambda)\sin(\rho), \sin(\lambda)\sin(\rho), \cos(\rho)\}\ . \tag{4.36}$$

In the perpendicular case we have $\varepsilon^{(\perp)}q'(0) = 0$ and hence no contribution from T_1, whereas in the parallel case we find

$$-i\varepsilon^{(\|)}q'(0) = 1\ . \tag{4.37}$$

For T_1 we have

$$\begin{aligned}&\mathcal{P}\exp\left\{-i\frac{T}{4}\int_0^1 d\tau\,\sigma^{\mu\nu}F_{\mu\nu}\right\}\\ &= \exp\left(\frac{1}{2}\gamma^0\gamma^3\int_{-\infty}^{\infty} du\,E\right)\ .\end{aligned} \tag{4.38}$$

From the Lorentz force equation we find

$$\frac{d}{du}\ln[\pm t'(u) + z'(u)] = \pm E\ , \tag{4.39}$$

from which it follows that

$$\int_{-\infty}^{0} du\,E = -\ln\left(\frac{-t'(0^-) + z'(0^-)}{p_0' + p_3'}\right) \tag{4.40}$$

and

$$\int_{0}^{\infty} du\,E = -\ln\left(\frac{t'(0^+) + z'(0^+)}{p_0 - p_3}\right)\ . \tag{4.41}$$

The values of the derivatives at zero are now functions of k_3 which we show later to be

$$-t'(0^-) + z'(0^-) = (k_0 + k_3)\left(\frac{1}{2} + \frac{i}{k_1}\right) \tag{4.42}$$

and

$$t'(0^+) + z'(0^+) = (k_0 - k_3)\left(\frac{1}{2} + \frac{i}{k_1}\right)\ . \tag{4.43}$$

T_2 is proportional to

$$\exp\left(\frac{1}{2}\gamma^0\gamma^3\int_0^\infty du\, E\right)\frac{\not{k}\not{\epsilon}}{2}\exp\left(\frac{1}{2}\gamma^0\gamma^3\int_{-\infty}^0 du\, E\right) \ . \tag{4.44}$$

Combining T_1 and T_2 and using $\exp(\gamma^0\gamma^3 x) = \cosh(x) + \sinh(x)\gamma^0\gamma^3$ we define

$$\frac{p_0 p_0'}{m_\perp^2} S := \left|\bar{u}\gamma^0(\not{p}+1)[\varepsilon q'(0)(4.38) + (4.44)]\gamma^0 v\right|^2 \ . \tag{4.45}$$

Although the rest of the calculation is linear algebra and can be done without choosing a representation of the Dirac matrices, it is convenient to use the spin basis in Eq. (8) in [89] for u and v. We find

$$\frac{1}{4}\sum_{\text{spins}} S = \mathbf{N}^\gamma \cdot \mathbf{m} \ , \tag{4.46}$$

where $\mathbf{N}^\gamma$ is the Stokes vector for the photon (4.36) and

$$\mathbf{m} = \{1 + 3p_1^2, 0, 0, 1 - p_1^2\} \ , \tag{4.47}$$

which is the same as in Eq. (154) in [89] or Eq. (44) in [103].

4.2 Kinky instantons

In Sec. 4.1.1 we mentioned that contour changes are perfectly fine despite the presence of the delta function in the Lorentz force equation and postponed the discussion, so now we justify this statement. From previous studies [104, 105] with simpler fields, it has been shown that the maximum probability is obtained when the photon momentum is shared equally between the electron and the positron, $\mathbf{p} = \mathbf{p}' = \mathbf{k}/2$. However, since we consider a spacetime dependent field which depends on t and z, we do not expect the z components of the asymptotic momenta to satisfy this property because the particles are accelerated along different trajectories thus feel a different field. Indeed, in the previous chapter we showed that already for Schwinger pair production (which amounts to setting $k = 0$ in the Lorentz force equation) this is not true for longitudinal momenta, so we expect only the transverse component of the photon to be shared equally, $p_\perp = p'_\perp = k_\perp/2$. In the next section, we show that this assumption is correct so we look at the properties of the instantons setting $(p_1, p_2) = (k_1/2, 0)$. From the discontinuity in the derivative

$$\begin{aligned} t'(0^+) - t'(0^-) &= k_0 \\ z'(0^+) - z'(0^-) &= -k_3 \end{aligned} \tag{4.48}$$

and the on-shell conditions $q'(0^\pm)^2 = 1$ we find

$$
\begin{aligned}
z'(0^+) &= i\frac{k_0}{k_1} - \frac{k_3}{2} \\
z'(0^-) &= i\frac{k_0}{k_1} + \frac{k_3}{2}
\end{aligned}
\tag{4.49}
$$

and

$$
\begin{aligned}
t'(0^+) &= \frac{k_0}{2} - i\frac{k_3}{k_1} \\
t'(0^-) &= -\frac{k_0}{2} - i\frac{k_3}{k_1} \; .
\end{aligned}
\tag{4.50}
$$

Since the initial velocities are already determined, the free parameters we have to vary to find the instantons at some generic asymptotic momenta are only $t(0)$ and $z(0)$. Unlike Schwinger pair production, the instanton components are now not even/odd, so $z(0)$ is a free parameter.

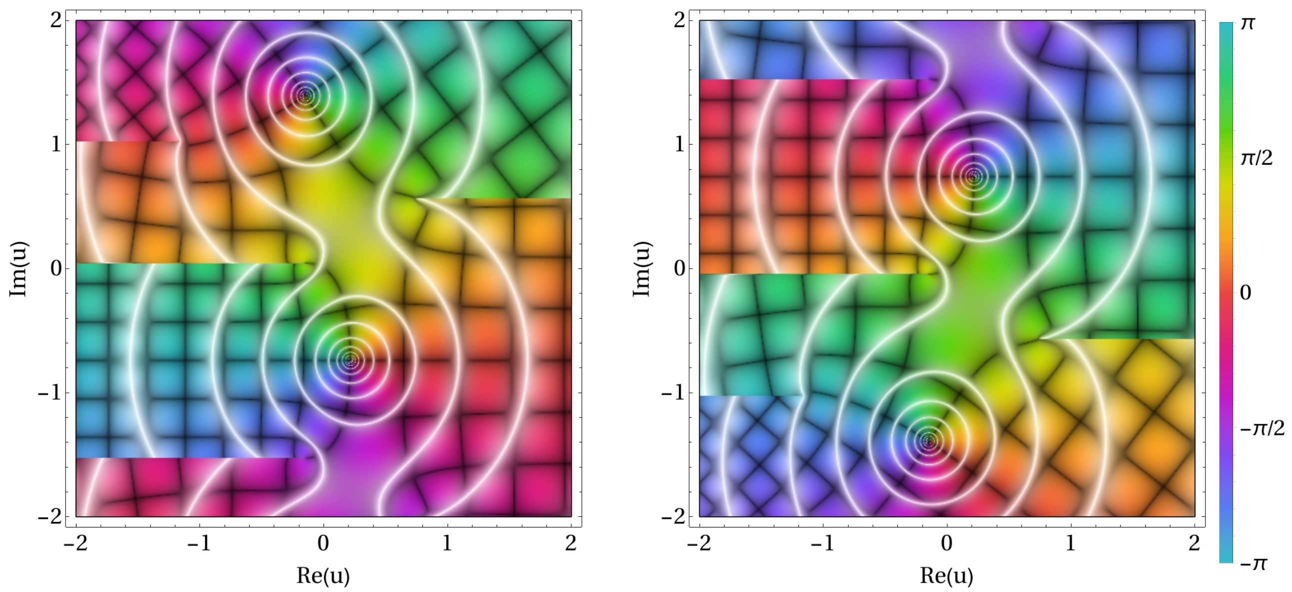


Figure 4.2: Time components of $q_{(+)}$ and $q_{(-)}$ with parameters $k_1 = \Omega = \gamma_t = \gamma_z = 1$, $k_3 \simeq 1.15$, and $\lambda = 3\sqrt{E}$ (i.e. in the supercritical regime, since $\lambda_c \simeq 2.23\sqrt{E}$. Gluing these two functions we obtain Fig. 4.3.

To perform contour deformations, we note that the delta function in the Lorentz force equation simply tells us that there is a discontinuity in the derivative according to (4.48), therefore we can regard the instantons before vs after absorption as distinct functions, $q_{(+)}$ and $q_{(-)}$, given by the solutions to the Lorentz force equation with initial conditions

$$
\begin{aligned}
t'_{(+)}(0) &= \frac{k_0}{2} - i\frac{k_3}{k_1} \\
z'_{(+)}(0) &= i\frac{k_0}{k_1} - \frac{k_3}{2}
\end{aligned}
\tag{4.51}
$$

and

$$
\begin{aligned}
t'_{(-)}(0) &= -\frac{k_0}{2} - i\frac{k_3}{k_1} \\
z'_{(-)}(0) &= i\frac{k_0}{k_1} + \frac{k_3}{2}
\end{aligned}
\tag{4.52}
$$

defined over the whole complex plane. Their plots are shown in Fig. 4.2. However, since we want the asymptotic region $u \to \infty$ to represent the solution with $q_{(+)}$ and vice-versa, and we want a single valued function $q(u)$, we define

$$q(u) = \begin{cases} q_{(+)}(u) & \text{Im}(u) < 0 \\ q_{(-)}(u) & \text{Im}(u) > 0 \end{cases} \tag{4.53}$$

obtaining Fig. 4.3.

When $p_1 = k_1/2$, the initial conditions satisfy $q'_{(+)}(0) = -q'_{(-)}(0)^*$. Furthermore, when the impact parameter is zero, $b = 0$, the initial conditions are imaginary $q(0) = -q(0)^*$ as well. From the uniqueness theorem of ODEs

$$q_{(+)}(u^*) = -q_{(-)}(u)^* \tag{4.54}$$

which, in terms of the instanton defined by (4.53), implies

$$q(u^*) = -q(u)^* \tag{4.55}$$

as shown graphically in Fig. 4.3. For Breit-Wheeler, the physical contour (green line) that produces the formation/acceleration region split is different from the Schwinger pair production case. In particular, the formation region is not along the imaginary axis because the zeros $t(u_c^\pm) = 0$ are not imaginary. Of course the contour choice is arbitrary, but we want the acceleration regions to start at $u_c^\pm$ because $t = 0$ represent the moments when the particles are actually created and time becomes real hence physical rather than imaginary during tunneling. In the formation region shown in Fig. 4.3 time is actually not imaginary, but since $t(0) \in i\mathbb{R}$ we could in principle achieve this by deforming the contour. However, there are no practical advantages and it is very difficult because the contour is some nontrivial curve on the complex plane.

4.3 Spectrum and phase transition

4.3.1 Saddle points

At the amplitude level, since the derivatives of the saddle points do not contribute, we find

$$\frac{\partial \psi}{\partial p_\perp} = ip_\perp \left(\frac{t(u_1)}{p_0} - u_1 \right) - ip'_\perp \left(\frac{t(u_0)}{p'_0} + u_0 \right) \tag{4.56}$$

for the transverse momenta, and

$$\begin{aligned} \frac{\partial \psi}{\partial p_3} &= i \left[z(u_1) + \frac{p_3}{p_0} t(u_1) \right] \\ \frac{\partial \psi}{\partial p'_3} &= i \left[z(u_0) + \frac{p'_3}{p'_0} t(u_0) \right] \end{aligned} \tag{4.57}$$

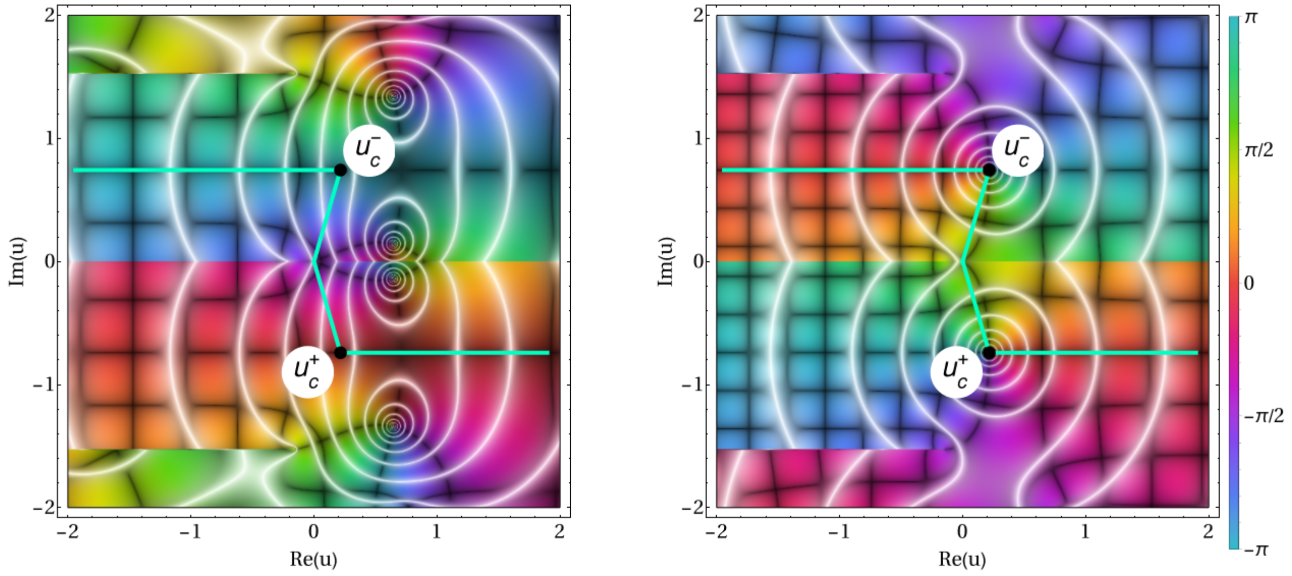


Figure 4.3: Space (left) and right (right) components of the instanton defined in (4.53). The time component is obtained gluing the two functions shown in Fig. 4.2. We denote by $u_c^{\pm}$ the zeros of $t(u)$.

for the longitudinal ones. Writing the exponent at the probability level as

$$\mathbb{P} \sim e^{-\mathcal{A}} \tag{4.58}$$

we then have the saddle point equations

$$\begin{aligned} \frac{\partial \mathcal{A}}{\partial p_3} &= \text{Im}\left[z(u_1) + \frac{p_3}{p_0} t(u_1)\right] = 0 \\ \frac{\partial \mathcal{A}}{\partial p_3'} &= \text{Im}\left[z(u_0) + \frac{p_3'}{p_0'} t(u_0)\right] = 0 \end{aligned} \tag{4.59}$$

$$\frac{\partial \mathcal{A}}{\partial p_\perp} = p_\perp \text{Im}\left(\frac{t(u_1)}{p_0} - u_1\right) - p_\perp' \text{Im}\left(\frac{t(u_0)}{p_0'} + u_0\right) = 0 \,. \tag{4.60}$$

In the previous section we have considered $p_1 = k_1/2$ and $p_2 = 0$, but since the instantons are not symmetric, it was not so obvious that $p_1 = k_1/2$ is also the momentum saddle point. Setting this momentum value and considering the contour shown in Fig. 4.3 we have $t(u_0), t(u_1) \in \mathbb{R}$, thus

$$\frac{\partial \mathcal{A}}{\partial p_\perp} = -\frac{k_1}{2} \text{Im}(u_1 + u_0) \,. \tag{4.61}$$

From the symmetry $q(u^*) = -q(u)^*$ it follows that the contour has the form

$$\begin{aligned} u_1 &= u_c + r \\ u_0 &= u_c^* - r \end{aligned} \tag{4.62}$$

with $r \in \mathbb{R}$, therefore

$$\frac{\partial \mathcal{A}}{\partial p_\perp} = -\frac{k_1}{2} \text{Im}(u_c + u_c^*) = 0 \tag{4.63}$$

hence $p_1 = k_1/2$ is a saddle point.

Finally, from

$$\frac{\partial\psi}{\partial k_3} = -i\left[z(0) + \frac{k_3}{k_0}t(0)\right] \tag{4.64}$$

the saddle point equation for the k_3 integral is given by

$$-\frac{k_s}{\lambda^2} - i\left[z(0) + \frac{k_s}{k_0}t(0)\right] = 0 \tag{4.65}$$

which is of course a function of λ. On the momentum saddle point, the initial conditions $q(0)$ are both imaginary, therefore the solution k_s is real.

4.3.2 Instanton variations

Since the spectrum at some momentum value is obtained after integrating over dk_3, the computation is trickier than in the previous cases due to the wave packet integral. We begin by defining the partial derivatives of the instantons at the saddle points as

$$\delta q_k = \frac{\partial q}{\partial k_3} \qquad \delta q_\alpha = \frac{\partial q}{\partial \Pi_\alpha} \tag{4.66}$$

where $\alpha \in \{p_1, \Delta, P\}$, which amounts to considering variations $q \to q + \delta k_3\, \delta q_k$ or $q \to q + \delta\Pi_\alpha\, \delta q_\alpha$. All the instanton variations satisfy

$$\begin{aligned} \delta t'' &= E\delta z' + \nabla E \cdot \{\delta t, \delta z\} z' \\ \delta z'' &= E\delta t' + \nabla E \cdot \{\delta t, \delta z\} t' \end{aligned} \tag{4.67}$$

but with different initial conditions. Since the equations are homogeneous, any solution can be written as a superposition

$$\delta q(u) = \sum_{j=1}^{4} a_j\, \delta q_{[j]}(u) \tag{4.68}$$

of some basis solutions $\delta q_{[j]}(u)$ with initial conditions

$$\delta t_{[1]}(u) = \delta z_{[2]}(u) = \delta t'_{[3]}(u) = \delta z'_{[4]}(u) = 1 \tag{4.69}$$

and all others zero. With some algebra, one can find the coefficients for all the solutions appearing in the widths.

We can write the coefficients

$$X_{kk} = \frac{\partial^2\psi}{\partial k_3^2} \qquad X_{k\alpha} = \frac{\partial^2\psi}{\partial k_3 \partial \Pi_\alpha} \qquad X_{\alpha\beta} = \frac{\partial^2\psi}{\partial \Pi_\alpha \partial \Pi_\beta} \tag{4.70}$$

in terms of the δq. We refer to [102] for their explicit expressions.

4.3.3 Intermezzo: time-dependent field with large wave packets

Let us consider a time-dependent field. We focus only on the Δ integral, i.e. the one that receives a contribution from the wave packet. The others components are not directly affected. After computing the integrals at the amplitude level except k_3 we are left with a delta function $\delta(p_3+p_3'-k_3) = \delta(\Delta-k_3)$ from the spatial integral $\varphi^3 = \frac{1}{2}(z_+ + z_-)$ and an exponent

$$\psi_a(k_3, \Delta) := -\frac{k_3^2}{2\lambda^2} + \psi(k_3, \Delta) \tag{4.71}$$

where ψ is everything apart from the wave packet contribution evaluated at all the saddle points of the ordinary and path integrals, which are functions of k_3 and Δ. Due to the delta function, the k_3 integral is trivial and it amounts to the replacement $k_3 = \Delta$ at the exponent. Unlike Schwinger pair production, we now have a nonzero d_Δ width even for a time-dependent field unless λ is so small that we can approximate the exponential with a delta function

$$e^{-\frac{\Delta^2}{\lambda^2}} \sim \delta(\Delta) \ . \tag{4.72}$$

This is because a finite width λ for the photon momentum k_3 allows the absorption of photons with $k_3 \neq 0$, thus the creation of pairs with different asymptotic momenta

$$p_3 + p_3' = \Delta = k_3 \neq 0 \ . \tag{4.73}$$

Let us now look at the width

$$d_\Delta^{-2} = -\text{Re}\, \frac{d^2\psi_a}{d\Delta^2} \tag{4.74}$$

where d denotes total derivative with respect to Δ. Defining partial derivatives as before

$$X_{\Delta\Delta} = \frac{\partial^2\psi_a}{\partial\Delta^2} \,, \qquad X_{k\Delta} = \frac{\partial^2\psi_a}{\partial k_3\partial\Delta} \,, \qquad X_{kk} = \frac{\partial^2\psi_a}{\partial k_3^2} \tag{4.75}$$

we find, applying the chain rule trivially to $\psi_a(k_3 = \Delta, \Delta)$ with $k_3'(\Delta) = 1$,

$$X_0 := \frac{d^2\psi}{d\Delta^2} = X_{\Delta\Delta} + 2X_{k\Delta} + X_{kk} \tag{4.76}$$

thus, defining $X_{0r} := \text{Re}\, X_0$,

$$d_\Delta^{-2} = \frac{1}{\lambda^2} - X_{0r} \ . \tag{4.77}$$

If $X_{0r} < 0$ then the expression above is always positive so there is no problem, but since $X_{0r} > 0$ there exists a finite critical value of the wave packet size λ_c such that

$$d_\Delta^{-2} = \frac{1}{\lambda_c^2} - X_{0r} = 0 \tag{4.78}$$

given by

$$\lambda_c = \frac{1}{\sqrt{X_{0r}}} \ . \tag{4.79}$$

Furthermore, it seems like the width should actually be negative $d_\Delta^{-2} < 0$ at $\lambda > \lambda_c$, which would give a negative determinant of the Hessian matrix of the momentum integrals and thus an imaginary prefactor. However, the width becoming negative simply means that the value $k_3 = \Delta = 0$ is no longer a maximum when $\lambda > \lambda_c$ and that the saddle point $\Delta = 0$ is degenerate at $\lambda = \lambda_c$. To see this in a simpler example, consider the integral

$$I(\lambda) := \int dx\, e^{-f(x)} \tag{4.80}$$

with

$$f(x) = \epsilon x^4 + (\lambda_c - \lambda)x^2 \,. \tag{4.81}$$

When $\epsilon \ll 1$ we can use the saddle point method and neglect the fourth order coefficient, obtaining a width

$$d^{-2} := \frac{1}{2}\frac{d^2 f(x)}{dx^2} \tag{4.82}$$

evaluated at $x = 0$, which is always a saddle point of $f(x)$, given by

$$d^{-2} = \lambda_c - \lambda \,, \tag{4.83}$$

so that

$$I(\lambda) = \sqrt{\frac{\pi}{d^{-2}}} = \sqrt{\frac{\pi}{\lambda_c - \lambda}} \,. \tag{4.84}$$

Here we have the analogous situation, namely a well defined result when $\lambda < \lambda_c$, a divergence when $\lambda = \lambda_c$, and then an imaginary integral when $\lambda > \lambda_c$. What has happened? Looking back at the original function $f(x)$ we see where the problem was: at $\lambda < \lambda_c$ the saddle point $x = 0$ is a global maximum of the exponent, therefore it gives the largest contribution to the integral, but then when $\lambda > \lambda_c$ it becomes a local minimum instead, and there are two distinct maxima given by

$$x = \pm\sqrt{\frac{\lambda - \lambda_c}{2\epsilon}} \,. \tag{4.85}$$

therefore the result above is only valid when $\lambda < \lambda_c$ simply because we have chosen the wrong saddle point. When $\lambda > \lambda_c$ we evaluate the width at the two maxima above and sum the results obtaining

$$I(\lambda) = \sqrt{\frac{\pi}{d^{-2}}} = \sqrt{\frac{\pi}{\lambda - \lambda_c}} \,. \tag{4.86}$$

The neighborhood of $\lambda = \lambda_c$ is the trickiest regime: here the quadratic term vanishes or it is comparable in size to the quartic one, depending on the relative sizes of ϵ and $\lambda - \lambda_c$, so the saddle point approximation breaks down and we have to consider the quartic term. Performing the integral exactly we obtain Bessel functions, which in the limit $(\lambda - \lambda_c)^2 \gg \epsilon$ reduce to either of the two results above, depending on the sign of $\lambda - \lambda_c$. Going back to our original problem, the change in the sign of d_Δ^{-2} tells us that we are simply evaluating the width at the wrong saddle point.

4.3.4 Phase transition

Letting

$$\psi_a = \psi - \frac{k_3^2}{2\lambda^2} \qquad \psi_r = 2\mathrm{Re}\,\psi_a \tag{4.87}$$

be the full exponent at the amplitude/probability level, the (in principle several) saddle points (k_s, Π_s) are the solutions to the set of equations

$$\begin{aligned} &\frac{\partial\psi}{\partial k_3}(k_s, \Pi_s) - \frac{k_s}{\lambda^2} = 0 \\ &\frac{\partial\psi_r}{\partial\Pi_\alpha}(k_s, \Pi_s) = 0 \; . \end{aligned} \tag{4.88}$$

Using the ideas of Appendix A, from the implicit function theorem we find a local solution to the second equation $\Pi(k_3)$ in a neighborhood of k_s with $\Pi(k_s) = \Pi_s$. Plugging it into the equation and taking the total derivative with respect to k_3 we get

$$R_{\alpha\beta}\frac{d\Pi_\beta}{dk_3} + R_{\alpha k} = 0 \;\rightarrow\; \frac{d\Pi_\alpha}{dk_3} = R^{-1}_{\alpha\beta}R_{\beta k} \tag{4.89}$$

where

$$R_{\alpha\beta} = 2\mathrm{Re}\,X_{\alpha\beta} \qquad R_{\alpha k} = 2\mathrm{Re}\,X_{\alpha k} \; . \tag{4.90}$$

Plugging now $\Pi(k_3)$ into the first equation and defining

$$\chi(k_3) := \frac{\partial\psi}{\partial k_3}\big(k_3, \Pi(k_3)\big) \tag{4.91}$$

we get the single variable equation

$$\chi(k_3) = \frac{k_3}{\lambda^2} \tag{4.92}$$

with both sides real. When λ is sufficiently small, the left hand side grows slower than k_3/λ^2, therefore $k_3 = 0$ is the only intersection point because

$$|\chi(k_3)| < \left|\frac{k_3}{\lambda^2}\right| \tag{4.93}$$

$\forall k_3 \neq 0$, but above some critical size $\lambda > \lambda_c$ the function $\chi(k_3)$ grows faster near $k_3 \sim 0$ as we can see in Fig. 4.4 so there are two new nonzero saddle points. In the language of catastrophe theory, this is known as a cusp catastrophe or pitchfork bifurcation [106]. The critical size λ_c is thus the value where the two functions have the same slope at the origin, or

$$\frac{1}{\lambda_c^2} = \chi'(0) \tag{4.94}$$

from which we get

$$\lambda_c = \frac{1}{\sqrt{X_{kk} - X_{\alpha k}R^{-1}_{\alpha\beta}R_{\beta k}}} \tag{4.95}$$

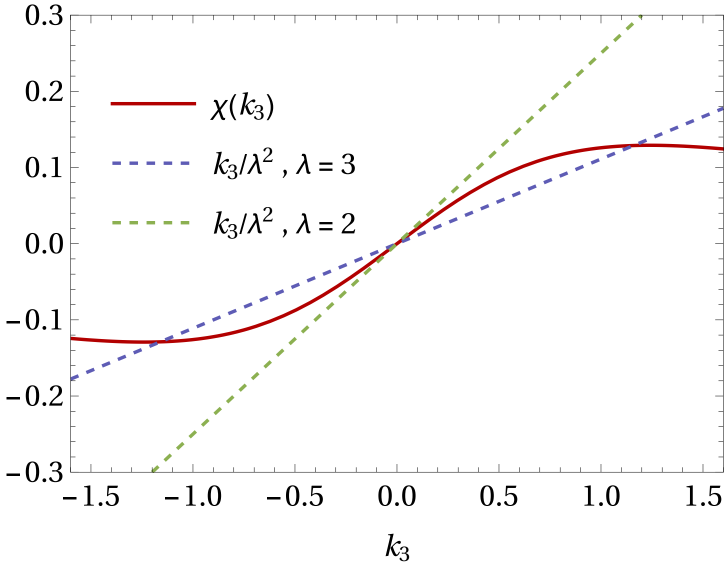


Figure 4.4: $\chi(k_3)$ (solid red line) and $k_3/2\lambda^2$ for $\lambda = 2$ and $\lambda = 3$ in units of $\sqrt{E}$ (blue and green line). We see that at $\lambda = 2$ only $k_3 = 0$ is a solution to (4.92), while at $\lambda = 3$ there are two additional intersection points. With these parameters, $\Omega = \gamma_t = \gamma_z = 1$, the critical wave packet size is $\lambda_c \approx 2.23\sqrt{E}$.

where all coefficients are evaluated at $k_3 = 0$. As we have mentioned in the previous section, the critical point λ_c is finite even for a purely time-dependent field, which means that the existence of the phase transition is not related to the fact that we have a spacetime-dependent field.

Since χ is odd (Fig. 4.4), we have two nonzero saddle points $k_3 = \pm k_s \neq 0$ in the supercritical regime. We stress that from the argument above we can only infer the existence of the nonzero saddle points, but we know nothing about their nature, i.e. whether a particular k_3 corresponds to a maximum of the spectrum or not. To achieve this, we have to look at the eigenvalues of the Hessian matrix of the spectrum, since some saddle point Π_s is a maximum if and only if all eigenvalues are positive. For this we first need to find the instantons at the saddle points k_s.

While in principle one could calculate the instantons at several k_3 values until a solution to the saddle-point equation is found, a better strategy is to use the saddle point equation for k_3 (4.92) to find

$$z(0) = k_3 \left(\frac{i}{\lambda^2} - \frac{t(0)}{k_0} \right) \tag{4.96}$$

and then let $t(0)$ and k_3 be free parameters to vary using the shooting method until the particle momenta saddle point equations (4.59) are satisfied as well. As we can see in Fig. 4.5, the saddle point splits from zero to $\pm k_s$ at $\lambda = \lambda_c$. The instantons at $\pm k_s$ are related by

$$\{z(u), t(u)\} \longleftrightarrow \{-z(-u), t(-u)\} \tag{4.97}$$

so that, when $k_s = 0$, the instanton components are respectively odd/even so the transformation (4.97) is a symmetry. (4.97) implies that the momenta of the two saddle points $\pm k_s$ are related by

$$\{p_{3s}, p'_{3s}\} \longleftrightarrow \{-p'_{3s}, -p_{3s}\} \tag{4.98}$$

or equivalently $P_s \longleftrightarrow P_s$ and $\Delta_s \longleftrightarrow -\Delta_s$, so the spectrum is still symmetric with respect to Δ. Since the different k_3 saddle points correspond to different momenta saddle points, there is no interference term.

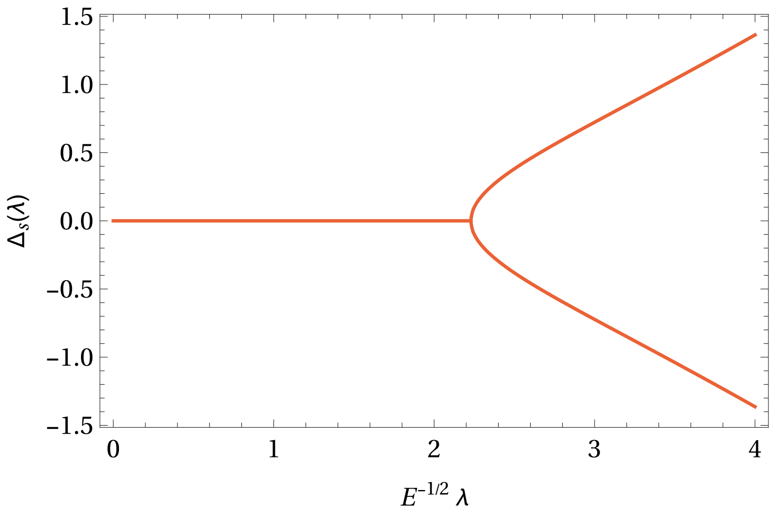


Figure 4.5: Pitchfork bifurcation of the saddle-point value $\Delta_s(\lambda)$ with parameters $\Omega = \gamma_t = \gamma_z = 1$. The critical point is $\lambda_c \simeq 2.23\sqrt{E}$. A similar behavior can be seen in $k_s(\lambda)$.

To compute the widths, i.e. the spectrum, we begin with the k_3 integral. For a given value of the momenta Π, we find in general a complex saddle point $k_s(\Pi)$ satisfying $k_s(\Pi_s) = k_s$, which means that

$$\frac{\partial \psi_a}{\partial k_3}\big(k_s(\Pi), \Pi\big) \stackrel{!}{=} 0 \tag{4.99}$$

for every Π, thus taking the partial derivative with respect to some component Π_α and rearranging we get

$$\frac{\partial k_s}{\partial \Pi_\alpha} = -\frac{\partial^2 \psi_a}{\partial k_3 \partial \Pi_\alpha}\left(\frac{\partial^2 \psi_a}{\partial k_3^2}\right)^{-1} . \tag{4.100}$$

To find the widths, we first take the derivative of the exponent at the probability level

$$\frac{d\psi_r}{d\Pi_\alpha} = \frac{\partial \psi_r}{\partial \Pi_\alpha} + 2\mathrm{Re}\,\frac{\partial k_s}{\partial \Pi_\alpha}\frac{\partial \psi_a}{\partial k_3} = \frac{\partial \psi_r}{\partial \Pi_\alpha} \tag{4.101}$$

then with the handy relation above we take another derivative and find a matrix of coefficients

$$\begin{aligned} 2d_{\alpha\beta}^{-2} &:= -\frac{d^2\psi_r}{d\Pi_\alpha d\Pi_\beta} \\ &= -\frac{\partial^2 \psi_r}{\partial \Pi_\alpha \partial \Pi_\beta} - 2\mathrm{Re}\frac{\partial k_s}{\partial \Pi_\alpha} X_{\beta k} \\ &= -\frac{\partial^2 \psi_r}{\partial \Pi_\alpha \partial \Pi_\beta} - 2\lambda^2\mathrm{Re}\frac{X_{\alpha k}X_{\beta k}}{1 - X_{kk}\lambda^2} \end{aligned} \tag{4.102}$$

thus we get the usual widths without the wave packet contribution, $\partial_\alpha \partial_\beta \psi_r$, and an extra term that vanishes as $\lambda \to 0$. In the subcritical regime ($\lambda < \lambda_c$ and $k_3 = 0$) there is no mixing with the P component[1] due to $X_{kP} = X_{\Delta P} = X_{\Delta p_1} = 0$, so the spectrum has the form

$$\mathbf{d}^{-2} := \begin{pmatrix} d^{-2}_{p_1 p_1} & d^{-2}_{p_1 \Delta} & 0 \\ d^{-2}_{p_1 \Delta} & d^{-2}_{\Delta\Delta} & 0 \\ 0 & 0 & d^{-2}_{PP} \end{pmatrix} . \tag{4.103}$$

4.3.5 Critical behavior

Besides the splitting of the spectrum as shown in Fig. 4.6, the behavior near the critical point shows another interesting feature. Let us recalculate the prefactor at $k_3 = 0$ in a different way rewriting the integral inside the modulus squared as

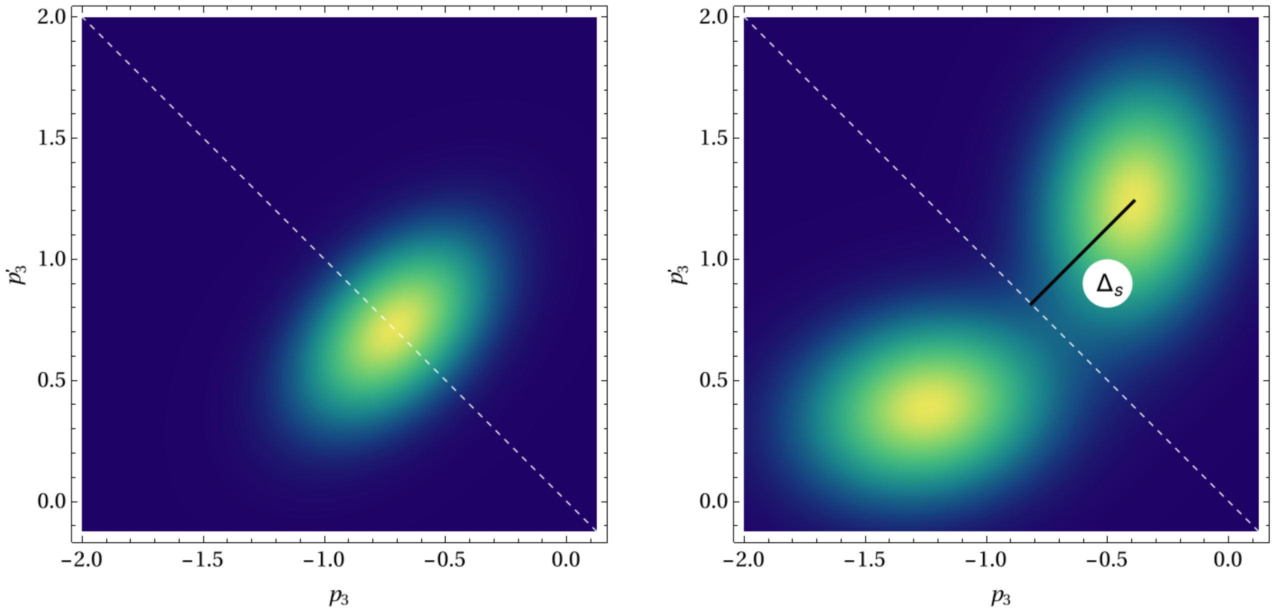


Figure 4.6: Longitudinal spectrum before (left) and after (right) the phase transition at $p_\perp = p_{\perp s}$ with parameters $E = 1/20$ and $\Omega = \gamma_t = \gamma_z = 1$. The plot on the left is at $\lambda = 2\sqrt{E}$ and the one on the right at $\lambda = 3.2\sqrt{E}$. The critical point is $\lambda_c \simeq 2.23\sqrt{E}$. The dashed line represents the set of points with $\Delta = 0$.

$$\left| \int dk_3 \, e^{\psi_a(k_3, \Pi)} \right|^2 = \int dk_3 \, d\tilde{k}_3 \, e^{\psi_a(k_3,\Pi) + \psi_a(\tilde{k}_3,\Pi)^*} \tag{4.104}$$

and then calculating all the integrals, namely $(k_3, \tilde{k}_3, \Pi)$, together. The saddle points are the same as before, with $\tilde{k}_s = k_s$. The total prefactor is then proportional to the determinant of the Hessian matrix

$$H_{ij} = -\frac{\partial^2}{\partial \xi_i \partial \xi_j} \left[\psi_a(k_3, \Pi) + \psi_a(\tilde{k}_3, \Pi)^* \right] \tag{4.105}$$

[1] This can be seen from the explicit expressions of the X in [102].

where $\xi_i, \xi_j \in \{k_3, \tilde{k}_3, p_1, \Delta, P\}$.

The Hessian (4.105) has a natural block matrix structure consisting of the 2x2 $(k_3, \tilde{k}_3)$ block, the 3x3 momentum block, and the mixed blocks. From $\tilde{k}_s = k_s$ we find

$$\begin{aligned} H_{kk} &= \frac{1}{\lambda^2} - X_{kk} \\ H_{\tilde{k}\tilde{k}} &= \frac{1}{\lambda^2} - X^*_{kk} \\ H_{k\tilde{k}} &= H_{\tilde{k}k} = 0 \end{aligned} \tag{4.106}$$

for the 2x2 block, then

$$H_{\alpha\beta} = -(X_{\alpha\beta} + X^*_{\alpha\beta}) = -R_{\alpha\beta} \tag{4.107}$$

with R given by (4.90) and $\alpha, \beta \in \{p_1, \Delta, P\}$, and also

$$H_{\alpha k} = -X_{\alpha k} \qquad H_{\alpha\tilde{k}} = -X^*_{\alpha k}\ . \tag{4.108}$$

To visualize the next step better, we express $H_{\alpha k}$ as a column vector $\mathbf{v}$ and denote $\mathbf{R}$ the matrix of coefficients $R_{\alpha\beta}$, leading to

$$\mathbf{H} = \begin{pmatrix} \frac{1}{\lambda^2} - X_{kk} & 0 & \mathbf{v}^{\mathrm{T}} \\ 0 & \frac{1}{\lambda^2} - X^*_{kk} & \left(\mathbf{v}^{\mathrm{T}}\right)^* \\ \mathbf{v} & \mathbf{v}* & -\mathbf{R} \end{pmatrix} \tag{4.109}$$

which shows the block structure

$$\mathbf{H} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\ . \tag{4.110}$$

A corresponds to the $(k_3, \tilde{k}_3)$ block (4.106), $D = -\mathbf{R}$ the momentum block (4.107), and $B^{\mathrm{T}} = C$ the mixed blocks (4.108). From the property of determinants of block matrices, we have

$$\begin{aligned} \det(4.110) &= \det(A)\ \det(D - CA^{-1}B) \\ &= \det(D)\ \det(A - BD^{-1}C) \end{aligned} \tag{4.111}$$

corresponding to, respectively,

$$\left|\frac{1}{\lambda^2} - X_{kk}\right| \det \mathbf{d}^{-2} \tag{4.112}$$

and (with some algebra)

$$\det(-\mathbf{R}) \left(\left| \frac{1}{\lambda^2} - X_{kk} + X_{\alpha k} R^{-1}_{\alpha\beta} X_{\beta k} \right|^2 - \left| X_{\alpha k} R^{-1}_{\alpha\beta} X^*_{\beta k} \right|^2 \right)\ . \tag{4.113}$$

The two results (4.112) and (4.113) are, respectively, what we would obtain if we integrated with respect to k_3 or particle momenta Π first and they are equal

$$\left|\frac{1}{\lambda^2} - X_{kk}\right| \det \mathbf{d}^{-2} = \det(-\mathbf{R}) \left(\left| \frac{1}{\lambda^2} - X_{kk} + X_{\alpha k} R^{-1}_{\alpha\beta} X_{\beta k} \right|^2 - \left| X_{\alpha k} R^{-1}_{\alpha\beta} X^*_{\beta k} \right|^2 \right)\ . \tag{4.114}$$

Noting that $-\mathbf{R}$ is the spectrum at $\lambda = 0$, i.e. when the wave packet is a delta function around $l_3 = 0$, we conclude that $\det(-\mathbf{R}) > 0$ because only $k_3 = 0$ can contribute, so it is a maximum. Furthermore, since the instantons only depend on λ through k_3 and $k_3 = 0$ in the subcritical regime $\lambda < \lambda_c$, $\det(-\mathbf{R})$ is constant as we increase λ all the way until the critical point $\lambda = \lambda_c$. Thus, since the two equations above must be equal, the determinant of the spectrum $\det \mathbf{d}^{-2}$ vanishes if and only if

$$\left|\frac{1}{\lambda^2} - X_{kk} + X_{\alpha k} R^{-1}_{\alpha\beta} X_{\beta k}\right|^2 = \left|X_{\alpha k} R^{-1}_{\alpha\beta} X^*_{\beta k}\right|^2 . \tag{4.115}$$

From the definition of λ_c it follows immediately that the equality above is satisfied at $\lambda = \lambda_c$, therefore

$$\boxed{\det \mathbf{d}^{-2}(\lambda_c) = 0} \tag{4.116}$$

so there exists at least one vanishing eigenvalue of $\mathbf{d}^{-2}$ corresponding to the momentum direction where the spectrum spreads out. The eigenvalues as functions of λ are shown in Fig. 4.7 for some set of parameter values. Furthermore, $\det \mathbf{d}^{-2} < 0$ at $\lambda \gtrsim \lambda_c$, i.e. $k_3 = 0$ no longer corresponds to maximum of the spectrum. Furthermore, from numerical considerations using a product of Sauter pulse we also see that $d^{-2}_{p_1\Delta}, d_{\Delta\Delta} \to 0$ as $\lambda \to \lambda_c$ from below, which means that the spectrum spreads out in the Δ direction.

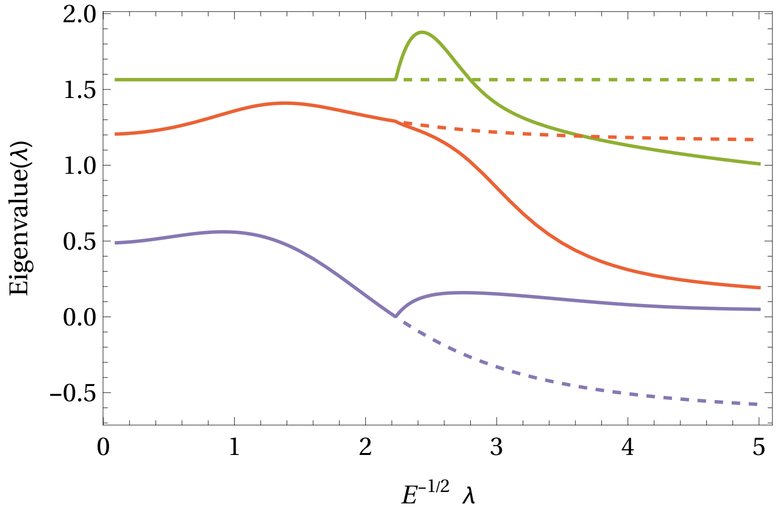


Figure 4.7: Eigenvalues of $d^{-2}_{\alpha\beta}$ without the overall factor of $1/E$ as functions of λ at parameters $\Omega = \gamma_t = \gamma_z = 1$. The dashed lines represent the eigenvalues at $k_3 = 0$ after the phase transition. The lilac line shows that one eigenvalue becomes negative at the critical point $\lambda_c \simeq 2.23\sqrt{E}$.

Nevertheless, this only gives us local information about the spectrum close to the critical point, but there is no reason why $k_3 = 0$ should not become a maximum again. In fact, for some values of the parameters, there is a second solution to (4.115)

$$\widetilde{\lambda}_c = \frac{1}{\sqrt{X_{kk} - X_{\alpha k} R^{-1}_{\alpha\beta} I_{\beta k}}} \tag{4.117}$$

where $I_{\beta k} = 2i\text{Im}\, X_{\beta k}$, such that at $\lambda > \widetilde{\lambda}_c$ all eigenvalues are positive and thus $k_3 = 0$ corresponds to a maximum again.

In conclusion of this section we point out an analogy between the calculation considered here and Landau's theory of continuous phase transitions [107]. If we look at the behavior of Δ_s in Fig. 4.5 as a function of λ we might be reminded of the Ising model of ferromagnets, where the system undergoes a continuous phase transition into one of two magnetization states as we lower the temperature. To make this more precise, let us look at the exponent at $\lambda \simeq \lambda_c$ for a generic Δ but on the saddle points of P and k_3. We have an expansion at $b, \Delta \ll 1$

$$\psi_r(\Delta) \sim a_1\Delta + a_2(\lambda)\Delta^2 + a_4\Delta^4 \tag{4.118}$$

with $a_1 \sim b$ and $a_2(\lambda) \sim \lambda - \lambda_c$. A plot of $\psi_r(\Delta)$ is shown in Fig. 4.8. The order parameter Δ_s is given by

$$\left.\frac{\partial\psi_r}{\partial\Delta}\right|_{\Delta=\Delta_s} = 0\,. \tag{4.119}$$

Since $\psi_r(\Delta)$ has the form of the free energy in Landau theory of continuous phase transitions, we

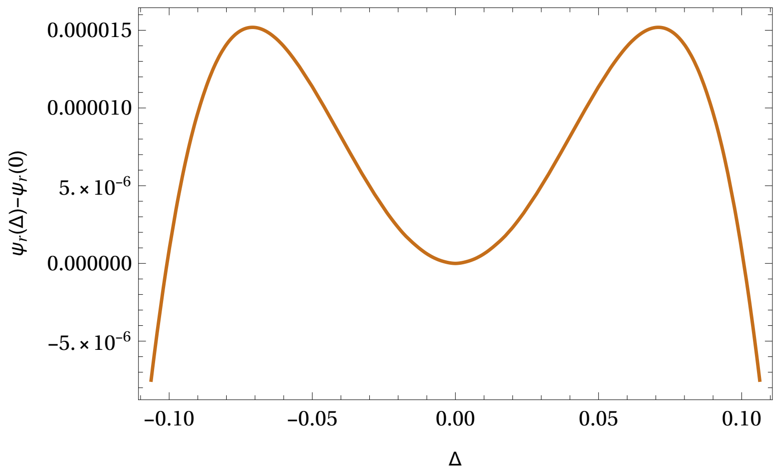


Figure 4.8: $\psi_r(\Delta)$ defined in (4.118) for parameters $\lambda = 2.24\sqrt{E}$, $\Omega = \gamma_t = \gamma_z = 1$. The value at the minimum is $\psi_r(0) \simeq 1.535$.

immediately conclude that the critical exponents defined as

$$\begin{aligned}
&\Delta_s(b=0) \sim \bar{\lambda}^{\beta} \qquad \Delta_s(\lambda=\lambda_c) \sim b^{\frac{1}{\delta}} \\
&\left.\frac{\partial\Delta_s}{\partial b}\right|_{b=0} \sim \begin{cases} \bar{\lambda}^{-\gamma} & \lambda < \lambda_c \\ \bar{\lambda}^{-\gamma'} & \lambda > \lambda_c \end{cases} \\
&C = -\lambda\frac{d^2g}{d\lambda^2} \sim \begin{cases} (-\bar{\lambda})^{-\alpha} & \lambda < \lambda_c \\ (-\bar{\lambda})^{-\alpha'} & \lambda > \lambda_c \end{cases}
\end{aligned} \tag{4.120}$$

where $g(\lambda) = \psi_r\big(\Delta_s(\lambda)\big)$ and $\bar{\lambda} = (\lambda - \lambda_c)/\lambda_c$, are given by

$$\alpha = \alpha' = 0 \qquad \beta = \frac{1}{2} \qquad \gamma = \gamma' = 1 \qquad \delta = 3 \tag{4.121}$$

thus this phase transition falls in the universality class of mean field theory.

4.4 Integrated probability

To obtain the integrated probability we evaluate the exponent at the saddle points and collect all the contributions to the prefactor. We write

$$\mathbb{P} = \int d^2p_\perp dp_3 dp_3' \, \mathbb{P}(p_\perp, p_3, p_3') \tag{4.122}$$

for the integrated probability and the spectrum. Writing the spectrum as $\mathbb{P}(p, p') = \mathbf{N} \cdot M(p, p')$ with $\mathbf{N}$ given by (4.36), we have

$$\mathbf{M}(p, p') = \frac{\alpha}{\sqrt{\pi}\lambda(2\pi)^4} \frac{p_0 p_0' \mathbf{m}}{m_\perp^2} \left| \int \frac{dk_3}{\sqrt{k_0}} \frac{e^{-\frac{k_3^2}{2\lambda^2}+\psi}}{2\pi T} \frac{T}{\sqrt{\det \Lambda}} \frac{\pi^2}{\sqrt{\det \mathbf{H}}} \right|^2 . \tag{4.123}$$

where $\det \mathbf{H}$ is the determinant of the Hessian matrix of the $\{T, \sigma, z_\pm\}$ integrals (4.33). Using the results from the previous sections we obtain

$$\mathbf{M}(p, p') = \frac{\alpha\{1 + 3p_1^2, 0, 0, 1 - p_1^2\}}{2\pi^{3/2} k_0 p_0 p_0' m_\perp^2 \lambda |D''/2|} \frac{e^{-\mathcal{A}(p,p')}}{\left|\frac{1}{\lambda^2} - X_{kk}\right|} . \tag{4.124}$$

Since for $\lambda > \lambda_c$ the spectrum has more than one peak, the probability is the sum of the contributions of the individual peaks

$$\mathbb{P} = \sum_n \mathbb{P}_n . \tag{4.125}$$

Integrating over momenta we obtain

$$\mathbf{M} = \frac{\alpha\{1 + 3p_1^2, 0, 0, 1 - p_1^2\}\sqrt{\pi}}{2\lambda k_0 m_\perp^2 p_0 p_0' |D''/2|} \frac{1}{\left|X_{kk} - \frac{1}{\lambda^2}\right|} \frac{e^{-\mathcal{A}}}{\sqrt{d_2^{-2} \det \mathbf{d}^{-2}}} , \tag{4.126}$$

where d_2^{-2} comes from the integration over p_2 and $\det \mathbf{d}^{-2}$ from (p_1, Δ, P).

We show the total probability as a function of λ and all other parameters fixed (photon energy, field strength, and γ's) in Fig. 4.9. At the phase transition, we clearly see a peak in the probability.

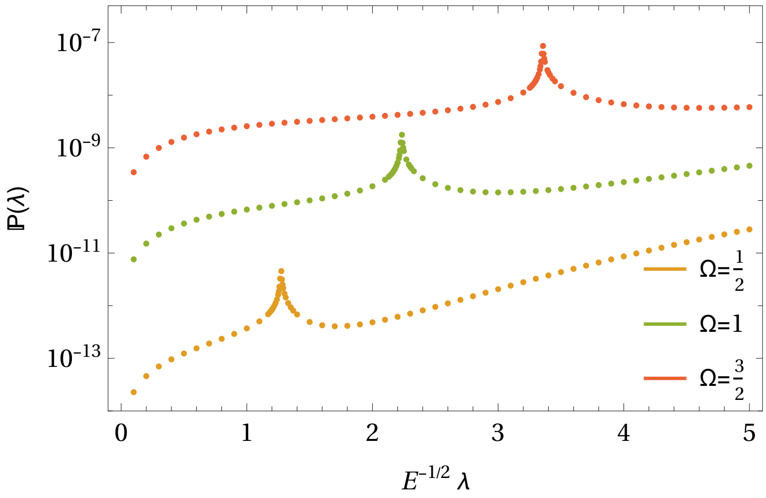


Figure 4.9: Integrated probability $\mathbb{P} = \mathbf{M} \cdot \mathbf{N}$ with $\mathbf{M}$ given by (4.126) as a function of λ at $E = 1/10$, $\Omega = \gamma_t = \gamma_z = 1$ and for perpendicular polarization $\mathbf{N} = \{1, 0, 0, -1\}$.

4.5 Momentum at particle creation

For Schwinger pair production we see from Fig. 3.4 that along the imaginary axis there are zeros $\pm u_c$ of $t(u)$ corresponding to the point where the particles become real. From the plot of z we also see that they are stationary points of $z(u)$, namely $z'(\pm u_c) = 0$, which suggests that the particles are created at rest. If we now look at the Breit-Wheeler complex plots 4.3, we might expect that the absorption of a photon with nonzero longitudinal component $k_3 \neq 0$ gives an initial momentum to the pair. Let us define u_c^+ the zero of $t(u)$ in the lower half plane (corresponding to the creation of the electron) and u_c^- the zero on the upper half plane (corresponding to the creation of the positron) as in Fig. 4.3. Let us denote by

$$p_e := -z'(u_c^+) \qquad p_p := z'(u_c^-) \tag{4.127}$$

the electron/positron momenta at creation. In Sec. 4.2 we showed that the instantons satisfy $q(u^*) = -q(u)^*$ for a generic k_3 (not necessarily on the saddle point), therefore from $z(u^*) = -z(u)^*$ and $u_c^+ = (u_c^-)^*$ and from the definition (4.127) we see that

$$p_e = p_p \tag{4.128}$$

i.e. the longitudinal momentum is shared equally between the electron and the positron so that the particles initially travel along the same direction. Note however that they are not created in the same position because $z(u_c^+) \neq z(u_c^-)$. After creation, since the field accelerates one of them and decelerates the other, the asymptotic momenta become different $p_3 \neq p_3'$ when they are outside the field, and their differences are given by the integral of $E(t, z)$ along the respective worldline

$$\begin{aligned} p_3 - p_e &= \int_{u_c^+}^{\infty} du\, t'(u) E(t, z) \\ p_3' - p_p &= -\int_{-\infty}^{u_c^-} du\, t'(u) E(t, z) \,. \end{aligned} \tag{4.129}$$

For time-dependent fields we have

$$p_e = \frac{k_3}{2} \tag{4.130}$$

or, defining the fraction of momentum given to each particle $\bar{p}$

$$\bar{p} := \frac{p_e}{k_3} \tag{4.131}$$

we have $\bar{p} = 1/2$. As γ_k is increased, thus as the field becomes smaller, $\bar{p}$ becomes smaller as well as we see in Fig. 4.10.

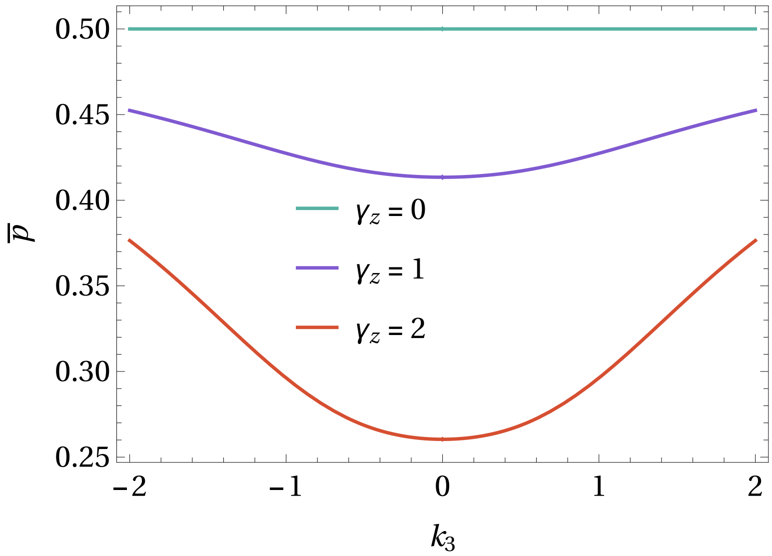


Figure 4.10: Particle momenta (4.131) normalized by k_3 at creation with $\gamma_t = 1$ and $\Omega = 1$ as a function of the longitudinal momentum at different fixed values of γ_z.

We can also visualize the worldline trajectory of the particles after creation as follows. Let us define the electron worldline as

$$\{t_e(r), z_e(r)\} = \{t(u_c^+ + r), z(u_c^+ + r)\} \tag{4.132}$$

and similarly for the positron

$$\{t_p(r), z_p(r)\} = \{t(u_c^- - r), z(u_c^- - r)\} \,. \tag{4.133}$$

For $r > 0$, the functions defined above are always real because they are simply the instantons evaluated in the two acceleration regions. We can see them in Fig. 4.11. At $k_3 = 0$ (dashed lines), the two particles are created with no longitudinal momentum[2] then they accelerate for a while because of the field until the velocities become constant when they exit the field. At $k_3 = 2$ (solid lines), the longitudinal component of the absorbed photon gives an initial and equal momentum to the two particles. The positron however (blue line) is accelerated by the field while the electron (red line) is decelerated until it changes direction completely. If we make k_3 larger, the field does not manage to change the direction of the electron and so both particles travel toward $z < 0$.

[2] But with equal transverse momenta $k_1/2$.

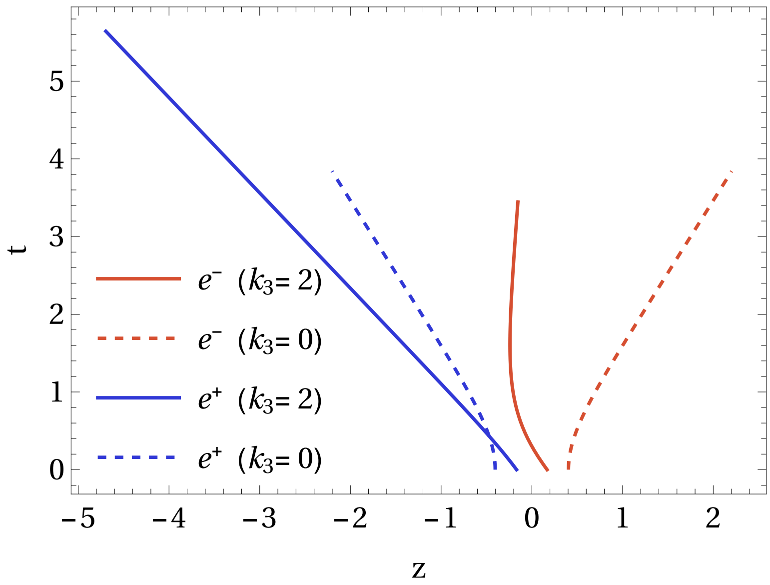


Figure 4.11: Worldlines defined by Eq. (4.132) and (4.133) for $k_3 = 0$ (dashed) and $k_3 = 2$ (solid). The red line represents the electron and the blue one the positron. The parameters of the field are $\gamma_t = \gamma_z = 1$ and the incoming photon has transverse momentum $\Omega = 1$.

Conclusions

IN THIS THESIS, the focus has been on Schwinger and Breit-Wheeler pair production making use of an open worldline instanton formalism. While simpler field configurations such as constant fields, plane waves, and 1D fields have been considered for a long time, methods that work for multidimensional fields are harder to come by. A similar formalism to the one considered here, based on closed worldline instantons [41, 44], allows one to find the total pair production probability for fields with both a spatial and temporal structure, but it does not give information about the momentum spectrum or spin of the produced particles, which was the scope of the present work. The first part was dedicated to Schwinger pair production in 2D electric fields and 4D e-dipole fields [67, 68], and the second one to Breit-Wheeler in 2D electric fields. Emphasis has been placed on worldline instantons, as they play a key role in the calculation of the momentum spectrum. In previous works, the instantons were considered along the imaginary axis [39, 40] or along along a zig-zag contour [48]. In this thesis, the instantons are considered along arbitrary contours and they are plotted on the complex proper-time plane. This allows us to visualize the instantons and get an intuition of their analytic properties such as the existence of branch points. Furthermore, it shows that there exists a special contour consistent with the interpretation of particles tunneling from the vacuum at the beginning and then accelerating due to the background field. Besides the physical interpretation, this contour is useful in practice when the instantons are found with the shooting method [41].

A natural extension would be to study nonlinear Compton scattering in spacetime fields with open worldline instantons, generalizing the results we obtained for time-dependent fields in [89]. As shown in [102] for Breit-Wheeler, the incoming wave packet plays a significant role, so it would be interesting to study the implications of a nontrivial electron wave packet for Compton scattering. The worldline formalism has also been shown to be able to deal with multiloop amplitudes quite efficiently [108, 109, 110], thus one could consider loop corrections to the probability. One could also consider subleading contributions in the semiclassical approximation

$$\mathbb{P} = e^{-\frac{a}{E}} \sum_{n=0}^{\infty} c_n \, E^n$$

using Green's function techniques analogous to Appendix A. In this work we considered for the most part purely electric field where the spin contribution is trivial. It would be interesting to add a magnetic component and study the particle spins. Dipole fields, although having a magnetic component as well, are very special in that at leading order there is a trivial spin structure. Another possible path would be to consider a nonabelian background field such as a color field in SU(3) [111] or even a gravitational field to investigate Hawking radiation [112, 113, 114, 115, 116]. Finally, since instantons are objects that arise naturally when one studies tunneling phenomena, one could use techniques analogous to the ones showcased here in the context of dynamically assisted nuclear fusion [117, 118].

Hic sunt Dracones

Appendices

Appendix A

Saddle points and Green's functions

A.1 Saddle-point method

IN THE CALCULATIONS INVOLVING spacetime fields, many integrals were approximated using the so-called saddle-point or Laplace or steepest descent method [119]. In essence, what we have done is Taylor expanding the exponent part of the integrand up to quadratic order, obtaining thus a Gaussian integral. In this appendix we provide some motivation for why this can be done.

Let us consider an integral of the form

$$I(\epsilon) := \int_{\mathbb{R}} dx\, e^{-\frac{1}{\epsilon}f(x)} \tag{A.1}$$

where ϵ is small and $f(x)$ has a unique maximum $x = 0$ which is also nondegenerate $f''(x) \neq 0$. If we Taylor expand and rescale $x \to \sqrt{\epsilon}x$ we get an exponent

$$\frac{f(0)}{\epsilon} + \frac{f''(0)}{2}x^2 + \frac{f'''(0)\sqrt{\epsilon}}{6}x^3 + \mathcal{O}(\epsilon) \tag{A.2}$$

so we see that all the terms higher than quadratic are multiplied by positive powers of ϵ, and are therefore suppressed. Neglecting them we obtain at leading order a simple Gaussian integral

$$I(\epsilon) = \sqrt{\frac{2\pi\epsilon}{f''(0)}}e^{-\frac{1}{\epsilon}f(0)}\ . \tag{A.3}$$

In principle one could also expand the higher order terms at the exponent, obtaining at the prefactor a power series in ϵ.

We can generalize this method to integrals of the form

$$\int_{\mathbb{R}} dx\, h(x)\, e^{-\frac{1}{\epsilon}f(x)} = h(0)\sqrt{\frac{2\pi\epsilon}{f''(0)}}e^{-\frac{1}{\epsilon}f(0)}\ , \tag{A.4}$$

where $h(x)$ is a slowly varying function, simply evaluating $h(x)$ at the maximum $x = 0$. This is because the domain where the exponential term of (A.4) is nonzero gets smaller and smaller as $\epsilon \to 0^+$, thus $h(x)$ is roughly constant in such interval. The generalization in D dimensions is simple: if $\nabla f(0) = 0$, we get

$$\int_{\mathbb{R}^D} d^D x\, h(x)\, e^{-\frac{1}{\epsilon} f(x)} = h(0) \sqrt{\frac{(2\pi\epsilon)^D}{\det \mathbf{H}_f(0)}} e^{-\frac{1}{\epsilon} f(0)} \tag{A.5}$$

where $\mathbf{H}_f(0)$ is the Hessian matrix of f evaluated at zero.

A further, less obvious, generalization is for complex-valued functions

$$\int_{\mathbb{R}} dx\, e^{\frac{i}{\epsilon} f(x)} \tag{A.6}$$

with again $\epsilon \to 0$ and $f'(0) = 0$. Now the integrand does not go to zero as $|x| \to \infty$, but the oscillations become faster and faster and cancel each other out in a sort of destructive interference. The only region where this does not happen is near the stationary point $x = 0$, therefore

$$\int_{\mathbb{R}} dx\, e^{\frac{i}{\epsilon} f(x)} = \sqrt{\frac{2\pi i\epsilon}{f''(0)}} e^{\frac{i}{\epsilon} f(0)} \ . \tag{A.7}$$

In quantum mechanics, where we identify the small constant ϵ with Planck's constant $\epsilon = \hbar$, transition amplitudes are given by path integrals of the form

$$\mathcal{M} = \int \mathcal{D}q\, e^{\frac{i}{\hbar} S[q]} \tag{A.8}$$

and using the saddle-point method we get

$$\mathcal{M} \sim e^{\frac{i}{\hbar} S[q_{\text{cl}}]} \tag{A.9}$$

where q_{cl} satisfies the classical equations of motion

$$\left.\frac{\delta S}{\delta q}\right|_{q=q_{\text{cl}}} = 0 \tag{A.10}$$

so this is a semiclassical approximation because it takes into account the classical limit $\hbar \to 0$ plus quantum fluctuations up to quadratic order.

If the saddle point is complex, the situation is trickier. One can show that the integral can be split into a sum of steepest descent contours (i.e. with constant imaginary part) known as Lefschetz thimbles [120].

Addendum: perturbative corrections

The result above can be seen as the first order in an expansion of the form

$$\sqrt{\frac{2\pi\epsilon}{f''(0)}} e^{-\frac{1}{\epsilon} f(0)} \sum_{n=0}^{\infty} c_n \epsilon^n \tag{A.11}$$

where the exponential term is the leading order *nonperturbative* result, and the power series in ϵ are *perturbative* corrections to the leading order exponential. The zeroth term is obviously $c_0 = 1$, and the first term can be obtained expanding the cubic and quartic powers

$$e^{\frac{f^{(3)}(0)\sqrt{\epsilon}}{3!}x^3+\frac{f^{(4)}(0)\epsilon}{4!}x^4} \to \frac{f^{(3)}(0)^2\epsilon}{2(3!)^2}x^6 + \frac{f^{(4)}(0)\epsilon}{4!}x^4 \tag{A.12}$$

where we have neglected the cubic term proportional to $\sqrt{\epsilon}$ because its integral is zero. Performing the Gaussian integrals we get

$$\sqrt{\frac{2\pi\epsilon}{f''(0)}}e^{-\frac{1}{\epsilon}f(0)}\left[1+\epsilon\left(\frac{5f^{(3)}(0)^2}{24f''(0)}-\frac{f^{(4)}(0)}{8f''(0)^2}\right)+\mathcal{O}(\epsilon^2)\right] . \tag{A.13}$$

A.2 Generalized saddle-point method

Now we construct a general way to compute an arbitrary number of path integrals and ordinary integrals in the saddle-point approximation.

Let $f(x,y)$ be a function with a unique minimum (x_0, y_0). The saddle point method tells us

$$\int dxdy\, e^{-f(x,y)} \simeq \frac{2\pi}{\sqrt{H_{xy}(x_0,y_0)}}\, e^{-f(x_0,y_0)} \tag{A.14}$$

with

$$\begin{aligned} f_x(x_0,y_0) &= 0 \\ f_y(x_0,y_0) &= 0\,. \end{aligned} \tag{A.15}$$

However, suppose we want to compute the integrals one at a time. If we perform the x integral first, we find a saddle point given by a function $\tilde{x}(y)$

$$f_x(\tilde{x}(y),y) = 0 \tag{A.16}$$

where the equality holds for every y. In particular, we can take the derivative of (A.16) with respect to y and find

$$\tilde{x}' = -\frac{f_{xy}}{f_{xx}} . \tag{A.17}$$

The saddle-point equation for y is the same because the extra term $\tilde{x}'(y)\, f_x(\tilde{x}(y),y)$ due to the implicit differentiation drops using (A.16)

$$d_y f(\tilde{x}(y),y) = \tilde{x}'(y)\, f_x(\tilde{x}(y),y) + f_y(\tilde{x}(y),y) = f_y(\tilde{x}(y),y) = 0 \tag{A.18}$$

therefore the saddle point is still (x_0, y_0). The second derivative gives

$$d_y^2 f(\tilde{x}(y),y) = f_{yy} + \tilde{x}'\, f_{xy} = f_{yy} - \frac{f_{xy}^2}{f_{xx}} \tag{A.19}$$

therefore the prefactor is the expression above times f_{xx} from the initial x integral

$$f_{xx}\left[f_{yy}-\frac{f_{xy}^2}{f_{xx}}\right]=f_{xx}f_{yy}-f_{xy}^2 \tag{A.20}$$

but this is precisely the determinant of the hessian matrix H_{xy}, therefore the result is independent of the integration order.

We can easily generalize this result to an n-dimensional x integral. Let us define

$$\begin{aligned} f_{ij} &= \partial_{x_i x_j} f \\ f_{iy} &= \partial_{x_i y} f \\ f_{yy} &= \partial_{yy} f \end{aligned} \tag{A.21}$$

and $\tilde{x}_i(y)$ to be the n saddle points as functions of y, which are solutions to $f_j(\tilde{x},y)=0$ and satisfy (taking the y derivative)

$$\tilde{x}_i' = -f_{ij}^{-1} f_{jy}\ . \tag{A.22}$$

The second derivative with respect to y then gives

$$d_y^2 f(\tilde{x}(y),y) = f_{yy} + \tilde{x}_i' f_{iy} = f_{yy} - f_{iy} f_{ij}^{-1} f_{jy}\ . \tag{A.23}$$

If we first integrate over $d^n x$ and obtain an n-dimensional Hessian $H_x = \det(f_{ij})$, the global prefactor is $\det(f_{ij})$ times the contribution from the y integral (A.23), therefore

$$\det(H_{xy}) = \det(f_{ij})\ \left[f_{yy} - f_{iy} f_{ij}^{-1} f_{jy}\right]. \tag{A.24}$$

This is not surprising – in fact this is a well-known property of determinants in linear algebra. Suppose we want to compute the determinant of the $n+1$-dimensional hessian matrix

$$\begin{bmatrix} f_{11} & f_{12} & \dots & f_{1n} & f_{1y} \\ \vdots & \vdots & & \vdots & \vdots \\ f_{n1} & f_{n2} & \dots & f_{nn} & f_{ny} \\ f_{y1} & f_{y2} & \dots & f_{yn} & f_{yy} \end{bmatrix} \tag{A.25}$$

then we have a block matrix of the form

$$M = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \tag{A.26}$$

where $A = f_{ij}$ is an $n\times n$ matrix, $B = f_{iy}$ is an n-dimensional vector, and $D = f_{yy}$ is a scalar. From linear algebra we know

$$\det(M) = \det(A)\ (D - BA^{-1}B) \tag{A.27}$$

which is precisely (A.24).

Suppose that we would rather compute the single integral over y first. The first contribution to the prefactor is

$$\int d^n x\, dy\, e^{-f(x,y)} = \int d^n x \sqrt{\frac{2\pi}{f_{yy}}} e^{-f(x,\tilde{y}(x))} \tag{A.28}$$

where $\tilde{y}(x)$ is the y saddle point as a function of the n dimensional variable x (which in the continuous limit becomes a functional $\tilde{y}[x(\tau)]$). This function is in general very complicated, but as before we do not need to find it explicitly, we simply compute the n dimensional integral with the saddle-point method. We must distinguish total derivatives with respect to x_i and partial derivatives, since at the prefactor we have total derivatives (using partial derivatives would amount to having a block diagonal hessian matrix (A.25)). Nonetheless, having computed the integral over y first, we have

$$\partial_y f(x, \tilde{y}(x)) = 0 \tag{A.29}$$

identically for every x, therefore

$$\begin{aligned} f_i(x,\tilde{y}(x)) &= \frac{\partial f}{\partial x_i}(x,\tilde{y}(x)) \\ d_i f(x,\tilde{y}(x)) &= \frac{df}{dx_i}(x,\tilde{y}(x)) = f_i(x,\tilde{y}(x)) + \partial_y f(x,\tilde{y}(x))\, \tilde{y}_i = f_i(x,\tilde{y}(x)) \end{aligned} \tag{A.30}$$

so

$$d_{ij} f = f_{ij} + \tilde{y}_j\, f_{iy} = f_{ij} - \frac{f_{iy} f_{jy}}{f_{yy}}. \tag{A.31}$$

We can now compute the determinant of (A.31) using the matrix determinant lemma, which tells us that

$$\det(A + v^T u) = \det(A)(1 + v^T A^{-1} u) \tag{A.32}$$

where $A = f_{ij}$

$$\det\left(f_{ij} - \frac{f_{iy} f_{jy}}{f_{yy}}\right) = \det(f_{ij}) \left(1 - \frac{f_{iy} f_{ij}^{-1} f_{jy}}{f_{yy}}\right) \tag{A.33}$$

but if we include the contribution from the y integral, i.e. f_{yy}, we have

$$f_{yy} \det\left(f_{ij} - \frac{f_{iy} f_{jy}}{f_{yy}}\right) = \det(f_{ij}) \left(f_{yy} - f_{iy} f_{ij}^{-1} f_{jy}\right) \tag{A.34}$$

but notice that this is exactly (A.24). The generalization to several y-variables is straightforward: the scalar terms of (A.24) become matrices, and the term $f_{yy} - \ldots$ becomes the determinant of such matrix.

In summary, we have shown in three different ways that the prefactor is (A.24): integrating over $d^n x$ first (which represents a discretized path integral), integrating over dy first then using the matrix determinant lemma, and computing them together and using the property (A.27) of determinants of block matrices.

This is useful if we have a path integral plus ordinary integrals. The determinant of the path integral part A can be found using the Gelfand-Yaglom method, but the biggest problem is the computation of the inverse $G = A^{-1}$, which is difficult in general. If we have an explicit expression then

$$f_{iy} f_{ij}^{-1} f_{jy} \Longrightarrow \int_{I\times I} d\tau d\tau' \, \partial_y \Big(\frac{\delta S}{\delta q(\tau)} \Big) \, G(\tau, \tau') \, \partial_y \Big(\frac{\delta S}{\delta q(\tau')} \Big) \, . \tag{A.35}$$

Summary

We can summarize this result in the following proposition.

Given a path integral over the variable $q(\tau)$ and and ordinary integral over y, the saddle point approximation of the combined integral is

$$\boxed{\int \mathcal{D}q \int dy \, e^{-f[q(\tau),y]} = \sqrt{\frac{1}{2\pi T}} \sqrt{\frac{\det(\Lambda_{\text{free}})}{\det(\Lambda)}} \sqrt{\frac{2\pi}{f_{yy} - f_{y\tau} \cdot G_{\tau\tau'} \cdot f_{y\tau'}}}} \tag{A.36}$$

where Λ is the quadratic operator, G is its green's function, and $\Lambda_{\text{free}} = \frac{1}{T}\partial_\tau^2$.

A.3 Time-dependent Green's function

In this section we use the method outlined above to compute the fluctuation prefactor in a time-dependent field. Let $A(t)$ be an odd function such that $A(\infty) = A_\infty$. Expanding the exponent up to quadratic order around the instanton $q \to q + \delta q$

$$\int \frac{\dot{q}^2}{2T} + A(t)\dot{z} \to \int \frac{\dot{q}^2}{2T} + A(t)\dot{z} + \frac{1}{2}\delta q \cdot \Lambda \cdot \delta q \tag{A.37}$$

one finds a kinetic operator [89]

$$\Lambda = \begin{pmatrix} -\frac{1}{T}\partial_\tau^2 + A''\dot{z} & +A'\partial_\tau \\ -\partial_\tau (A' \,) & \frac{1}{T}\partial_\tau^2 \end{pmatrix} \, . \tag{A.38}$$

We want to find the inverse the operator Λ, i.e. a matrix of Green's function that satisfies

$$\begin{pmatrix} -\frac{1}{T}\partial_\tau^2 + A''\dot{z} & +A'\partial_\tau \\ -\partial_\tau (A' \,) & \frac{1}{T}\partial_\tau^2 \end{pmatrix} \begin{pmatrix} a(\tau,\tau') & b(\tau,\tau') \\ c(\tau,\tau') & d(\tau,\tau') \end{pmatrix} = \begin{pmatrix} \delta_{\tau\tau'} & 0 \\ 0 & \delta_{\tau\tau'} \end{pmatrix} \tag{A.39}$$

with boundary conditions $G_i = 0$ for all four components at the boundary of the square $[0,1] \times [0,1]$

$$G_i(\tau, 0) = G_i(\tau, 1) = G_i(0, \tau') = G_i(1, \tau') = 0 \, . \tag{A.40}$$

A.3.1 Computation of a and c

We want to solve

$$-\frac{1}{T}\partial^2 a + A''\dot{z}a + A'\partial c = \delta_{\tau\tau'}$$
$$\frac{1}{T}\partial^2 c - \partial(A'a) = 0. \tag{A.41}$$

The delta function in the first equation tells us that either a or c have a discontinuity in the derivative of some order. From the second equation $\partial_\tau(A'a) \sim \partial_\tau^2 c$ we note that c is more regular than a, thus a must be continuous with a derivative jump and c continuously differentiable. We begin by integrating the second equation

$$\partial c = TA'a + k_1 \tag{A.42}$$

and plugging it into the first one

$$-\frac{1}{T}\partial^2 a + (A''\dot{z} + TA'^2)a + k_1 A' = \delta_{\tau\tau'}. \tag{A.43}$$

We now multiply both sides by $\dot{t}(\tau)$ and use

$$-\frac{\dot{t}}{T}\partial^2 a = -\frac{1}{T}\partial\big(\dot{t}\,\partial a - a\ddot{t}\big) - \dot{t}(A''\dot{z} + TA'^2)a \tag{A.44}$$

to rewrite it as

$$-\frac{1}{T}\partial\big(\dot{t}\,\partial a - a\ddot{t} - k_1 TA\big) = \dot{t}\delta_{\tau\tau'}. \tag{A.45}$$

We integrate it

$$\dot{t}\,\partial a - a\ddot{t} = -T\theta_{\tau\tau'}\dot{t}(\tau') + TAk_1 + k_2 \tag{A.46}$$

and use $\dot{t}^2\partial\left(a/\dot{t}\right) = \dot{t}\,\partial a - a\ddot{t}$. The solution is finally

$$a(\tau,\tau') = \dot{t}(\tau)\int_0^\tau d\tilde{\tau}\,\frac{TAk_1 + k_2 - T\theta_{\tilde{\tau}\tau'}\dot{t}(\tau')}{\dot{t}^2}. \tag{A.47}$$

where k_1, k_2 are actually functions of τ' which we fix by imposing $a(1,\tau') = 0$ and $c(1,\tau') = 0$. We define for convenience

$$I_n := \int_0^1 \frac{(TA)^n}{\dot{t}^2}\,. \tag{A.48}$$

From $a(1,\tau') = 0$ we find

$$k_1 I_1 + k_2 I_0 = T\dot{t}(\tau')\int_{\tau'}^1 \frac{1}{\dot{t}^2}. \tag{A.49}$$

We can now integrate (A.42)

$$c(\tau,\tau') = k_1\tau + \int_0^\tau d\tilde{\tau}\,T(A_\tau - A)\frac{TAk_1 + k_2 - T\theta_{\tilde{\tau}\tau'}\dot{t}(\tau')}{\dot{t}^2} \tag{A.50}$$

and from $c(1,\tau')=0$ we find

$$k_1 + TA_\infty\Big(k_1 I_1 + k_2 I_0 - T\dot{t}(\tau')\int_{\tau'}^1 \frac{1}{\dot{t}^2}\Big) - k_1 I_2 - k_1 I_1 + T\dot{t}(\tau')\int_{\tau'}^1 \frac{TA}{\dot{t}^2} = 0. \tag{A.51}$$

Putting them together with $\lambda := I_0(1-I_2)+I_1^2$

$$\begin{aligned} k_1(\tau') &= \frac{T\dot{t}(\tau')}{\lambda}\left(I_1\int_{\tau'}^1 \frac{1}{\dot{t}^2} - I_0\int_{\tau'}^1 \frac{TA}{\dot{t}^2}\right) = \frac{T\dot{t}(\tau')}{\lambda}\left[I_0 I_1(\tau') - I_1 I_0(\tau')\right] \\ k_2(\tau') &= \frac{T\dot{t}(\tau')}{\lambda}\left(I_1\int_{\tau'}^1 \frac{TA}{\dot{t}^2} + (1-I_2)\int_{\tau'}^1 \frac{1}{\dot{t}^2}\right) = \frac{T\dot{t}(\tau')}{\lambda}\left[\lambda - (1-I_2)I_0(\tau') - I_1 I_1(\tau')\right]. \end{aligned} \tag{A.52}$$

It is easy to see that $k_1(0)=0$, $k_2(0)=T\dot{t}(0)$ and $k_1(1)=0$, $k_2(1)=0$, which imply $a(\tau,0)=a(\tau,1)=0$ as well, thus our Green's functions vanish over the boundary of the square $[0,1]\times[0,1]$ as they should. $a(\tau,\tau')$ is also symmetric, but it is not obvious in this form. A nicer way to write our Green's functions using (A.48) is

$$\begin{aligned} a(\tau,\tau') = T\dot{t}(\tau)\dot{t}(\tau')\Big[&\frac{1}{\lambda}\left(I_0 I_1(\tau)I_1(\tau') + (I_2-1)I_0(\tau)I_0(\tau') - I_1 I_0(\tau)I_1(\tau') - I_1 I_1(\tau)I_0(\tau')\right) \\ &+ \theta_{\tau\tau'}\, I_0(\tau') + \theta_{\tau'\tau}\, I_0(\tau)\Big] \end{aligned} \tag{A.53}$$

$$\begin{aligned} c(\tau,\tau') = &\frac{T}{\lambda}\dot{t}(\tau')TA(\tau)\Big[I_1(\tau)\left(I_0 I_1(\tau') - I_1 I_0(\tau')\right) + I_0(\tau)\left((I_2-1)I_0(\tau') - I_1 I_1(\tau')\right)\Big] \\ &- \frac{T}{\lambda}\dot{t}(\tau')\Big[I_2(\tau)\left(I_0 I_1(\tau') - I_1 I_0(\tau')\right) + I_1(\tau)\left((I_2-1)I_0(\tau') - I_1 I_1(\tau')\right)\Big] \\ &+ T\dot{t}(\tau')\Big[\theta_{\tau\tau'}\left(TA(\tau)I_0(\tau') - I_1(\tau')\right) + \theta_{\tau'\tau}\left(TA(\tau)I_0(\tau) - I_1(\tau)\right)\Big] \\ &+ \tau\frac{T}{\lambda}\dot{t}(\tau')\left[I_0 I_1(\tau') - I_1 I_0(\tau')\right] \end{aligned} \tag{A.54}$$

Notice that $a(\tau,\tau')$ is proportional to $\dot{t}(\tau)\dot{t}(\tau')$. At the end, we want to integrate by parts functions like

$$\int d\tau d\tau'\, A'(\tau)\, A'(\tau')\, a(\tau,\tau') \tag{A.55}$$

therefore it is convenient to define *reduced* green's functions such as

$$\tilde{a}(\tau,\tau') := \frac{a(\tau,\tau')}{\dot{t}(\tau)\dot{t}(\tau')} \tag{A.56}$$

so that

$$\int d\tau d\tau'\, A'(\tau)\, A'(\tau')\, a(\tau,\tau') = \int d\tau d\tau'\, \dot{A}(\tau)\, \dot{A}(\tau')\, \tilde{a}(\tau,\tau') = \int d\tau d\tau'\, A(\tau)\, A(\tau')\, {}^\bullet\tilde{a}^\bullet(\tau,\tau') \tag{A.57}$$

where the dots denote derivatives with respect to τ and τ'. It is easy to see that

$${}^\bullet\tilde{a}^\bullet(\tau,\tau') = \frac{T}{\dot{t}^2(\tau)}\delta_{\tau\tau'} + \frac{T}{\dot{t}^2(\tau)\dot{t}^2(\tau')\lambda}\left(I_0\, T^2 A(\tau)A(\tau') - TI_1\left(A(\tau)+A(\tau')\right) - 1 + I_2\right). \tag{A.58}$$

A.3.2 Computation of b and d

We want to solve

$$\begin{aligned}
&-\frac{1}{T}\partial^2 b + A''\dot{z}b + A'\partial d = 0\\
&\frac{1}{T}\partial^2 d - \partial(A'b) = \delta_{\tau\tau'}.
\end{aligned} \tag{A.59}$$

We integrate the second equation

$$\partial d = \alpha_1 + T\theta_{\tau\tau'} + TA'b \tag{A.60}$$

and plug it in the first, finding

$$-\frac{1}{T}\partial\left(\dot{t}\partial b - \ddot{t}b\right) + \alpha_1\partial A = -T\partial A\theta_{\tau\tau'}. \tag{A.61}$$

Using

$$\int f'(x)\theta(x-x_0)\,dx = \left[f(x) - f(x_0)\right]\theta(x-x_0) + c \tag{A.62}$$

the expression for b follows

$$b(\tau,\tau') = \dot{t}(\tau)\int_0^\tau d\tilde{\tau}\left(TA\alpha_1 + \alpha_2 + T^2(A - A_{\tau'})\theta_{\tilde{\tau}\tau'}\right)\frac{1}{\dot{t}^2} \tag{A.63}$$

from which we also find d

$$d(\tau,\tau') = T(\tau-\tau')\theta_{\tau\tau'} + \alpha_1\tau + \int_0^\tau T(A_\tau - A)\frac{1}{\dot{t}^2}\left(TA\alpha_1 + \alpha_2 + T^2(A - A_{\tau'})\theta_{\tilde{\tau}\tau'}\right) \tag{A.64}$$

where α_1, α_2 are functions of τ'. Setting $b(1,\tau') = d(1,\tau') = 0$ we find

$$\begin{aligned}
\alpha_1(\tau') &= \frac{T}{\lambda}\left[I_1\int_{\tau'}^1 \frac{T(A_{\tau'} - A)}{\dot{t}^2} + I_0\left(\tau' - 1 - \int_{\tau'}^1 TA\frac{T(A_{\tau'} - A)}{\dot{t}^2}\right)\right]\\
&= \frac{T}{\lambda}\left[TA_{\tau'}\left(I_0 I_1(\tau') - I_0(\tau')I_1\right) - \lambda + I_0\,\tau' + I_1 I_1(\tau') - I_0 I_2(\tau')\right]\\
\\
\alpha_2(\tau') &= \frac{T}{\lambda}\left[(1 - I_2)\int_{\tau'}^1 \frac{T(A_{\tau'} - A)}{\dot{t}^2} - I_1\left(\tau' - 1 - \int_{\tau'}^1 TA\frac{T(A_{\tau'} - A)}{\dot{t}^2}\right)\right]\\
&= \frac{T}{\lambda}\left[TA_{\tau'}\left(\lambda - I_0(\tau')[1 - I_2] - I_1(\tau')I_1\right) - I_1\,\tau' + (1 - I_2)I_1(\tau') + I_1 I_2(\tau')\right]
\end{aligned} \tag{A.65}$$

which satisfy $\alpha_1(0) = -T$, $\alpha_2(0) = T^2 A_\infty$ and $\alpha_1(1) = 0$, $\alpha_2(1) = 0$, from which $b(\tau,0) = b(\tau,1) = d(\tau,0) = d(\tau,1) = 0$.

One can show that $c(\tau,\tau') = b(\tau',\tau)$ and $d(\tau,\tau')$ is symmetric with an expression

$$\begin{aligned}
&d(\tau,\tau') = \qquad\qquad (A.66)\\
&= \frac{T^3 A_\tau A_{\tau'}}{\lambda}\left[I_0 I_1(\tau) I_1(\tau') - I_0(\tau) I_0(\tau')(1-I_2) - I_1 I_0(\tau) I_1(\tau') - I_1 I_1(\tau) I_0(\tau')\right]\\
&+ \frac{T^2 A_\tau}{\lambda}\Big[I_1(\tau) I_1(\tau') I_1 - I_1(\tau) I_2(\tau') I_0 + I_0(\tau) I_2(\tau')(1-I_0) + I_1(\tau) I_0(\tau') I_1 + I_1(\tau) I_0 \tau' - I_0(\tau) I_1 \tau'\Big]\\
&+ \frac{T^2 A_{\tau'}}{\lambda}\Big[I_1(\tau) I_1(\tau') I_1 - I_2(\tau) I_1(\tau') I_0 + I_2(\tau) I_0(\tau')(1-I_0) + I_0(\tau) I_1(\tau') I_1 + I_1(\tau') I_0 \tau - I_0(\tau') I_1 \tau\Big]\\
&+ \frac{T}{\lambda}\Big[I_0 I_2(\tau) I_2(\tau') - I_1 I_2(\tau) I_1(\tau') - I_1 I_1(\tau) I_2(\tau') - I_1(\tau) I_1(\tau')(1-I_2) + I_1(\tau) I_1 \tau' + I_1(\tau') I_1 \tau\\
&- I_2(\tau) I_0 \tau' - I_2(\tau') I_0 \tau + I_0\, \tau\, \tau'\Big] + T[(\tau-\tau')\theta_{\tau\tau'} - \tau]\\
&+ T\theta_{\tau\tau'}\left[TA_\tau\, TA_{\tau'} I_0(\tau') + I_2(\tau') - (TA_\tau + TA_{\tau'}) I_1(\tau')\right]\\
&+ T\theta_{\tau'\tau}\left[TA_\tau\, TA_{\tau'} I_0(\tau) + I_2(\tau) - (TA_\tau + TA_{\tau'}) I_1(\tau)\right] \qquad (A.67)
\end{aligned}$$

A.3.3 Integrating

We use the result from the previous section (A.36) with the Green's function matrix to compute the prefactor of Schwinger pair production in time-dependent fields from [89], where we have a path integral over q_z and q_t and ordinary integrals over T and space variables $\theta^i = x^i_+ - x^i_-$. For simplicity we neglect the transverse integrals, which are trivial anyway. Letting $\theta = z_+ - z_-$, $\dot z(\tau) = \theta + q_z(\tau)$, we have an exponent

$$f = \frac{\theta^2}{2T} - \int d\tau\, \frac{\dot q_t^2}{2T} - \frac{\dot q_z^2}{2T} + A(q_t)(\theta + \dot q_z) \tag{A.68}$$

and mixed partial derivatives

$$f_{Tt} = \partial_T\Big(\frac{\delta f}{\delta q_t(\tau)}\Big) = -\frac{\ddot q_t}{T^2}\,, \qquad f_{Tz} = \partial_T\Big(\frac{\delta f}{\delta q_z(\tau)}\Big) = \frac{\ddot q_z}{T^2}. \tag{A.69}$$

Using $b(\tau,\tau') = c(\tau',\tau)$ we want to compute

$$\begin{aligned}
f_{Ti} f^{-1}_{ij} f_{Tj} &= \int d\tau d\tau' \begin{bmatrix} f_{Tt}(\tau) & f_{Tz}(\tau)\end{bmatrix} \begin{bmatrix} a(\tau,\tau') & b(\tau,\tau') \\ c(\tau,\tau') & d(\tau,\tau')\end{bmatrix} \begin{bmatrix} f_{Tt}(\tau') \\ f_{Tz}(\tau')\end{bmatrix} \qquad (A.70)\\
&= \int d\tau d\tau'\, f_{Tt}(\tau)\, a(\tau,\tau')\, f_{Tt}(\tau') + 2 f_{Tz}(\tau)\, c(\tau,\tau')\, f_{Tt}(\tau') + f_{Tz}(\tau)\, d(\tau,\tau')\, f_{Tz}(\tau').
\end{aligned}$$

Term with a. With $a(\tau,\tau') = \tilde a(\tau,\tau')\, \dot t(\tau)\, \dot t(\tau')$ we have

$$\begin{aligned}
f_{Tt}\cdot a \cdot f_{Tt} &= \frac{1}{T^4}\int \ddot t(\tau) \dot t(\tau)\, \tilde a(\tau,\tau')\, \ddot t(\tau') \dot t(\tau') = \frac{1}{4T^4}\int \dot t^2(\tau)\, {}^\bullet\tilde a^\bullet(\tau,\tau')\, \dot t^2(\tau')\\
&= \frac{1}{4T^3}\left[I_4 + \frac{1}{\lambda}(I_0 I_3^2 - 2 I_1 I_2 I_3 - I_2^2(1-I_2)\right] \qquad (A.71)
\end{aligned}$$

Term with c. Define $c(\tau,\tau') = \tilde{c}(\tau,\tau')\,\dot{t}(\tau')$, $k_1(\tau') = \tilde{k}_1(\tau')\,\dot{t}(\tau')$. The second term gives

$$\begin{aligned} 2f_{Tz}\cdot c\cdot f_{Tt} &= -\frac{2}{T^4}\int \ddot{z}(\tau)\,\tilde{c}(\tau,\tau')\dot{t}(\tau')\,\ddot{t}(\tau') = -\frac{1}{T^4}\int \dot{z}(\tau)\,{}^\bullet\tilde{c}^\bullet(\tau,\tau')\,\dot{t}^2(\tau') \\ &= -\frac{1}{T^3}\int A(\tau)\,\left[T\dot{A}(\tau)\,\tilde{a}^\bullet(\tau,\tau') + \tilde{k}_1^\bullet(\tau')\right]\,\dot{t}^2(\tau') \end{aligned} \tag{A.72}$$

which receives two contributions from

$$\begin{aligned} -\frac{1}{2T^3}\int T\partial_\tau\left(A(\tau)\right)\,\tilde{a}^\bullet(\tau,\tau')\,\dot{t}^2(\tau') &= \frac{1}{2T}\int TA^2(\tau)\,{}^\bullet\tilde{a}^\bullet(\tau,\tau')\,A^2(\tau') \\ &= \frac{1}{2T^3}\left[I_4 + \frac{1}{\lambda}(I_0 I_3^2 - 2I_1 I_2 I_3 - I_2^2(1-I_2)\right] \end{aligned} \tag{A.73}$$

and

$$-\frac{1}{T^3}\int A(\tau)\,\tilde{k}_1^\bullet(\tau')\,\dot{t}^2(\tau') = -\frac{1}{T^3\lambda}\left(I_3 + T^2 I_1\right)\left(I_0 I_3 - I_1 I_2\right)\ . \tag{A.74}$$

Term with d. We have

$$\begin{aligned} f_{Tz}\cdot d\cdot f_{Tz} &= \frac{1}{T^4}\int \ddot{z}(\tau)\,d(\tau,\tau')\,\ddot{z}(\tau') = \frac{1}{T^4}\int \dot{z}(\tau)\,{}^\bullet d^\bullet(\tau,\tau')\,\dot{z}(\tau') \\ &= \frac{1}{T^2}\int A(\tau)\,A(\tau')\left[TA'\,b^\bullet(\tau,\tau') + \dot{\alpha}_1(\tau') - T\delta_{\tau\tau'}\right] \\ &= \frac{1}{T^2}\int A(\tau)\,A(\tau')\left[T\dot{A}\,{}^\bullet\tilde{c}(\tau',\tau) + \dot{\alpha}_1(\tau') - T\delta_{\tau\tau'}\right] \\ &= \frac{1}{2T^3\lambda}\left(I_3 + T^2 I_1\right)\left(I_0 I_3 + 2I_0 I_1 T^2 + I_1 I_2\right) - \frac{1}{T^4}\left(I_4 + I_2 T^2\right) \\ &\quad + \frac{1}{4T^3}\left[I_4 + \frac{1}{\lambda}(I_0 I_3^2 - 2I_1 I_2 I_3 - I_2^2(1-I_2)\right] + \frac{1}{2T^3\lambda}\left(I_3 + T^2 I_1\right)\left(I_0 I_3 - I_1 I_2\right) \end{aligned} \tag{A.75}$$

where we have used the derivative of (A.60) with respect to τ' to simplify the first line.

Final result

The final result for the total derivative with respect to T is then

$$\boxed{d_T^2 f = \partial_T^2 f + f_{Ti} f_{ij}^{-1} f_{Tj} = -\frac{1}{T}\frac{I_0}{I_0^2 T^2 + I_1^2}}\,. \tag{A.76}$$

which matches with Eq.(84) in [89]

$$-\frac{1}{4}\left(G_{00} - G_{0j}G_{jk}^{-1}G_{0k}\right)^{-1} = -\frac{1}{T}\frac{I_0}{I_0^2 T^2 + I_1^2}\,. \tag{A.77}$$

Let us include the integral over θ as well (the other spatial integrals are trivial). First of all, for convenience we define a product $S_{uv} := f_{uv} + f_{ui} f_{ij}^{-1} f_{vj}$ so that the result for the T integral is simply S_{TT}. For the spatial contribution we must calculate

$$\det\begin{bmatrix} S_{\theta\theta} & S_{T\theta} \\ S_{T\theta} & S_{TT}\end{bmatrix} = S_{TT}\left[S_{\theta\theta} - \frac{S_{T\theta}^2}{S_{TT}}\right]. \tag{A.78}$$

One can show that

$$S_{\theta\theta} = f_{\theta\theta} + f_{\theta t} \cdot a \cdot f_{\theta t} = \frac{I_0}{T\lambda} \tag{A.79}$$

$$S_{T\theta} = f_{T\theta} + (f_{Tt} \cdot a + f_{Tz} \cdot c) \cdot f_{\theta t} = -\frac{I_1}{\lambda T^2} \tag{A.80}$$

therefore

$$S_{\theta\theta} - \frac{S_{T\theta}^2}{S_{TT}} = \frac{1}{T^3 I_0} = \frac{1}{G_{33}} \tag{A.81}$$

which matches with Eq. (91) in [89].

Appendix B

Complex Analysis

THROUGHOUT THE PRESENT THESIS, complex analysis plays a vital role. The central objects of study, used to obtain everything else, are complex solutions to ordinary differential equations called instantons. Viewing them as complex variable functions shows their analytical properties (branch points, zeros, regions where they are real/imaginary) and also allows us to see directly the physical contour leading to the formation/acceleration regions and the tunneling interpretation. To compute them in practice, we need to specify a particular contour and use the shooting method to find the appropriate initial conditions, and in some cases, visualizing them on the complex plane is helpful to come up with sensible asymptotic constraints. Many of the assumptions made in the following, such as the smoothness of contours, are too strong and could be relaxed, but for the purpose of this thesis this is not necessary.

B.1 A dive into the complex world

In this section we review some of the main definitions and theorems used in this thesis. Proofs and technical details are not addressed, but can be found in any complex analysis book such as [121].

Throughout the following, all functions of complex variable

$$f : D \longrightarrow \mathbb{C} \tag{B.1}$$

defined in some open subset $D \subseteq \mathbb{C}$ are continuous with respect to the complex norm. Furthermore, we assume them to be differentiable almost everywhere in the complex sense for all $a \in D$, i.e.

$$f'(a) := \lim_{z \to a} \frac{f(z) - f(a)}{z - a} \tag{B.2}$$

exists. Equivalently, $f(z)$ is said to be analytic or holomorphic.

Complex differentiable functions are very different from real ones. In the complex world, differentiability at a point is enough to imply the existence of a convergent power series in an open neighborhood of the point. Furthermore, analytic functions are also conformal mappings[1] (they preserve angles) in some $U \subseteq D$ if $f'(z) \neq 0 \; \forall z \in U$. What is most relevant for this thesis, however, are the properties of integrals over complex contours.

Given a smooth contour α parametrized by

$$u : \mathbb{R} \longrightarrow \mathbb{C} \tag{B.3}$$

the integral over α is defined as

$$\int_\alpha f(z)\, dz := \int_{\mathbb{R}} f\big(u(r)\big) u'(r)\, dr \; . \tag{B.4}$$

One can easily show that the integral above is independent of the parametrization $u(r)$, so it is well-defined.

The following theorem is the most important result of this section:

Theorem (Cauchy). *Let f be an analytic function on a simply connected open $D \subseteq \mathbb{C}$ and α a simple contour contained in D. The integral*

$$\int_\alpha f(z)\, dz \tag{B.5}$$

depends only on the initial and final points of the contour. Furthermore, integrals along closed loops always vanish.

The assumption of D being simply connected is a crucial one. To put it another way, given two contours α, β with the same initial and final point, we have

$$\int_\alpha f(z)\, dz = \int_\beta f(z)\, dz \tag{B.6}$$

if α can be continuously deformed into β without going outside of the domain D where f is analytic.

B.2 Singularities as obstructions

The takeaway message here is that the Cauchy theorem is a *topological* one. If we deform the domain D without altering its topology, i.e. without creating/destroying holes, two contours α and β continue to be homotopic. If, on the other hand, there is some singularity z_0 that prevent such deformation, the integral might in general be different. The singularities of $f(z)$ behave as topological

[1] This is also visible in the complex plots shown later.

obstructions in the deformation. The study of singularities of complex functions is a very fascinating one. Perhaps the most deep and surprising result is a theorem by Liouville:

Theorem (Liouville). *Let f be an analytic function over the whole complex plane $\mathbb{C}$. If f is bounded, i.e. there exists $M > 0$ such that $|f(z)| < M \; \forall z \in \mathbb{C}$, then f is constant.*

In other words, non-constant complex functions either have singularities or they diverge at infinity. If a singularity z_0 of f is a zero of $1/f$ and f is analytic in a neighborhood of z_0, it is called a pole. Intuitively, this means that near the singularity we have

$$f(z) \sim \frac{1}{(z - z_0)^n} \tag{B.7}$$

for some positive integer n. If a function has only poles, we call it *meromorphic*. This rules out for example compactly supported bump functions.

Functions of complex variables can have another, radically different, type of singular point. Consider the function $f(z) = z^{\frac{1}{2}}$ and a contour $z(r) = e^{2\pi i r}$. At the beginning of the contour, $r = 0$, we have $z = 1$ and $f = 1$. As we run along the contour, when $r = 1$, we are back at $z = 1$, but the function now gives $f = -1$. In this case, we say that $z = 0$ is a branch point because a contour that runs around zero leads to a different value of f. Of course this is due to the fact that $f(z)$ is the inverse of a noninjective function, namely z^2, therefore we can regard f as a multi-valued function. In order to get a single valued function, we can restrict to a single value, i.e. for this function we choose $\arg z \in (-\pi, \pi)$ and throw away the negative real axis. This is called a "branch cut", i.e. a restriction of the domain in order to avoid loops around the origin causing troubles. Letting

$$\mathcal{B} = \{z \in \mathbb{C} \mid \text{Im}(z) = 0, \, \text{Re}(z) < 0\} \tag{B.8}$$

be the branch cut, we have a well-defined, single-valued function

$$f : \mathbb{C} \backslash \mathcal{B} \longrightarrow \mathbb{C} \, . \tag{B.9}$$

While this is the approach taken in this thesis, a more sophisticated way of dealing with such situations is to extend rather than restrict the domain. Since we run into problems insisting that the domain should be the whole set of complex numbers, we can relax this condition and let

$$f : \Sigma \longrightarrow \mathbb{C} \tag{B.10}$$

for some surface Σ called a Riemann surface. The domain is now a complex manifold.

Depending on the approach one chooses, the two distinct values of $f(1) = \pm 1$ are said to lie on different branches or different Riemann sheets.

Singularities of complex functions are important for the study of instantons because, as we see in the main text, the singularities of the electromagnetic field have a profound effect on the instantons as they lead to branch points. If we consider for simplicity a time-dependent field $E(t)$, even if it is a well-behaved function when t is real, such as a Sauter pulse $E(t) = E\ \mathrm{sech}^2(\omega t)$, as a function of complex variable it can have poles. But even if it does not have poles, somewhere in the complex plane it must become very large due to Liouville's theorem. Since the instantons are necessarily complex, they can (and typically do) travel towards a pole or a region where the field grows very quickly.

B.3 Instantons as functions of complex variable

It is customary to regard the worldline instantons, namely solutions to[2]

$$\frac{d^2q^\mu}{du^2} = F^{\mu\nu}\frac{dq_\nu}{du}\ , \tag{B.11}$$

as complex valued functions of a real variable $q : \mathbb{R} \to \mathbb{C}$ by fixing a particular contour $u(r)$. In the context of the widely studied closed instantons, the contour is a straight line along imaginary axis $u = -ir$. Starting from the simplest case, namely a constant electric field, we would like to generalize this idea and treat the worldline instantons as functions of a complex variable.

For a constant field we get the simple set of equations

$$\begin{aligned} t''(u) &= z'(u) \\ z''(u) &= t'(u)\ . \end{aligned} \tag{B.12}$$

With initial conditions $t'(0) = z(0) = 0$, $z'(0) = i$, $t(0) = i$, one readily verifies that the solutions are given by

$$t(u) = i\cosh(u) \qquad z(u) = i\sinh(u)\ . \tag{B.13}$$

Along the imaginary axis, $u = -ir$, the instantons become the well-known parametrization of a circle in the $\big(\mathrm{Re}(z), \mathrm{Im}(t)\big)$ plane as shown in [39]

$$t(u = -ir) = i\cos(r) \qquad z(u = -ir) = \sin(r)\ . \tag{B.14}$$

One can use such closed instantons to calculate the Schwinger pair production probability using the effective action and then consider more general time-dependent [39] or spacetime-dependent fields [41, 44].

[2] We work with the rescaled variables $q \to q/E$ and $u \to u/E$ so there are no factors of E.

In this work, on the other hand, we deal with open instantons. Furthermore, since the exponent of final result at the amplitude level

$$i \int du\, q^{\mu} \partial_{\mu} A_{\nu} \frac{dq^{\nu}}{du} \tag{B.15}$$

is the integral of an almost everywhere analytic function, it is natural to consider contour deformations in u.

In addition, if we consider the instantons above as functions of a complex variable and interpret the contour as a particular choice of gauge, we obtain much more information. A useful way to visualize them is using domain coloring, i.e. coloring the complex u-plane according to the phase of $t(u)$ or $z(u)$ and adding contour lines of their moduli. For convenience, we can also add contour lines of the real/imaginary parts. Starting from some zero, which can be seen in Fig. B.1 as sets of white circles, they allow us to visualize the contours along which the instantons are purely real or purely imaginary.

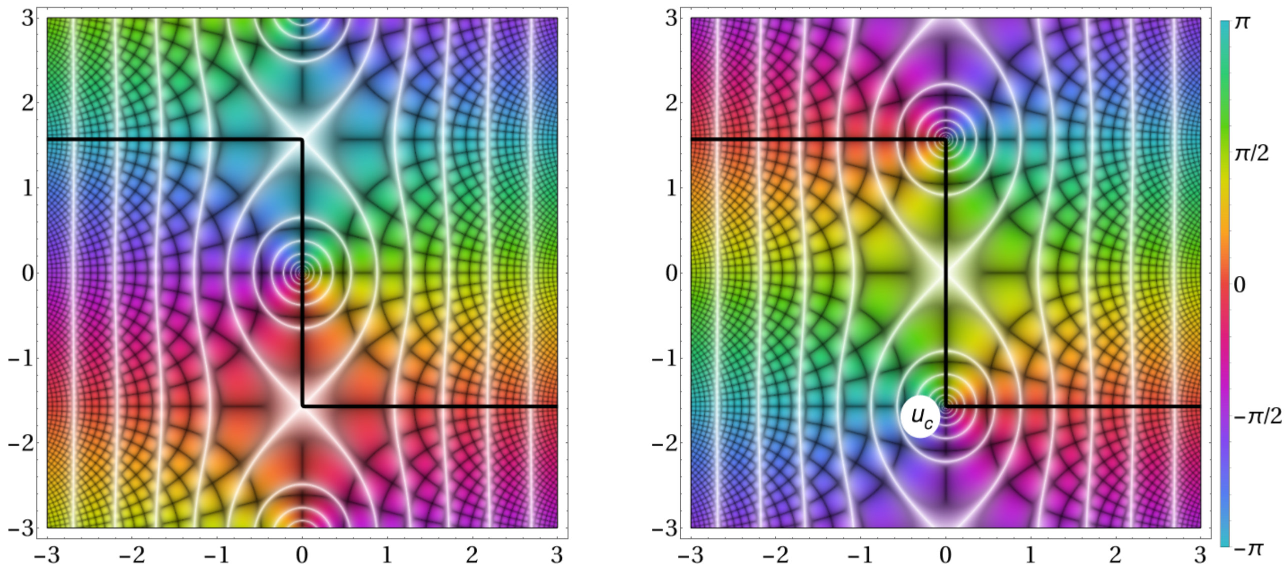


Figure B.1: Constant field instantons (B.13), $z(u)$ on the left and $t(u)$ on the right. The black contour, given by (B.16), makes z always real and t real asymptotically and imaginary in the formation region $-\frac{\pi}{2} < r < \frac{\pi}{2}$. The legend on the right tells us the phase at each complex value u.

Looking at the plots, one might notice a special choice of contour. Letting

$$u(r) = \begin{cases} \frac{i\pi}{2} + r + \frac{\pi}{2} & r < -\frac{\pi}{2} \\ -ir & -\frac{\pi}{2} < r < \frac{\pi}{2} \\ -\frac{i\pi}{2} + r - \frac{\pi}{2} & r > \frac{\pi}{2} \end{cases} \tag{B.16}$$

we obtain the black zigzag contour starting at $-\infty + i\pi/2$ running until $i\pi/2$ parallel to the real axis, then until $-i\pi/2$ along the imaginary axis, and finally to $+\infty - i\pi/2$ parallel to the real axis again. We call the vertical segment the *formation region* and horizontal segments the *acceleration regions*.

While $z(u)$ is always real, $t(u)$ is purely imaginary along the creation region and purely real along the acceleration regions. Along the creation region the particles are not real yet, then they are created at $u = \pm u_c$, and finally travel accelerated by the field forever (since we consider a constant field). The imaginary time in the creation process reinforces the interpretation of Schwinger pair production as a tunneling process. With this contour, the integral above (B.15) gives (restoring the factor of $1/E$)

$$\frac{i}{E}\int du\, z'(u)t(u) = -\frac{\pi}{2E} + \text{(imaginary terms)} \tag{B.17}$$

where the formally divergent imaginary contribution cancels at the probability level, leading to

$$\mathbb{P} \sim e^{-\frac{\pi}{E}} \ . \tag{B.18}$$

Note that, since at the end we take the modulus squared, we can restrict the integral to the formation region, as the instantons are purely real in the acceleration regions. This suggests that the total number of particles only depends on the small region where the particles are created and not on what happens when the particles are accelerated by the field.

The constant field is useful to understand the analytic properties of the instantons and the existence of the tunneling contour, and almost everything generalizes to a time-dependent and even spacetime-dependent field. We consider only fields with one stationary point for simplicity. For a time-dependent Sauter pulse we have for example [39]

$$\begin{aligned} t(u) &= \frac{i}{\gamma}\arcsin\left(\frac{\gamma\,\cosh\left(\sqrt{1+\gamma^2}u\right)}{\sqrt{1+\gamma^2}}\right) \\ z(u) &= i\frac{\arcsin\left(\gamma\sinh\left(\sqrt{1+\gamma^2}u\right)\right)}{\gamma\sqrt{1+\gamma^2}} \end{aligned} \tag{B.19}$$

with still z real and t imaginary along the imaginary axis, $t(u_c) = 0$ now at

$$u_c = -\frac{i\pi}{2\sqrt{1+\gamma^2}} \tag{B.20}$$

and both components real in the acceleration regions. However, the instantons are now no longer entire functions due to the branch points where the argument of $\arcsin$ is equal to plus or minus one. Such branch points are related to poles in the field, since we have

$$t(u_B) = \frac{i\pi}{2\gamma} \tag{B.21}$$

which is exactly the pole of $\mathrm{sech}(\gamma t)^2$. Such branch points can be seen explicitly in Fig. B.2. Note in particular that this implies that the instantons are multivalued (see also [122]).

Unfortunately, in general we cannot find an analytic expression for the instantons for spacetime fields, but from the saddle point equation for the longitudinal momentum

$$\mathrm{Im}\left[t(u_1) - z(u_1)\frac{P}{p_0}\right] = 0 \tag{B.22}$$

we know that, if we find a contour such that one of the two components of the instanton become real asymptotically, the other must also be real. In the main text we show that such contour is especially useful for dipole fields to find an approximation in the locally constant field limit.

We now show how to plot the instantons on the complex plane. Since we do not have an explicit expression for the instantons, we must solve the Lorentz force equation along a large set of contours. If $\mathcal{B}$ is the set of branch cuts, we regard the instantons as functions on the complex plane minus such set $q_{\mathbb{C}} : \mathbb{C}\backslash\mathcal{B} \to \mathbb{C}$. Hidden in the definition there is also a choice of contour from the origin to any $u \in \mathbb{C}\backslash\mathcal{B}$. From the exact Sauter pulse solution we find a periodic set of pairs of branch points, therefore one possible choice can be the following: we start with a single contour along the imaginary axis $u_i(r) = ir$, so that we obtain solutions $t_i(r) := t(ir)$, $z_i(r) := z(ir)$. Now we treat $t_i(r)$ and $z_i(r)$ as a set of initial conditions to solve parallel to the real axis along a contour $u_R(r) = iR + r$ with conditions

$$\begin{aligned} t_R(0) &= t_c(R) \qquad & t'_R(0) &= -i\, t'_c(R) \\ z_R(0) &= z_c(R) \qquad & z'_R(0) &= -i\, z'_c(R) \end{aligned} \tag{B.23}$$

so that effectively $q_R(r) = q(iR + r)$ because we first go from $u = 0$ to $u = iR$, then from $u = iR$ to $u = iR + r$.

With these contours we obtain the complex plot in Fig B.2. We can immediately see both the "tunneling contour" and a periodic one. The physical contour that gives the tunneling interpretation is the one traveling along the red region until the zero, then along the imaginary axis until the other zero, then to the right in the other red region. Note that with this coloring red means real and positive and yellow-green imaginary with positive imaginary part. If we instead look at what happens along the imaginary part we see an alternating set of turning points and zeros. This is in fact no coincidence: if we solve along a purely imaginary contour we obtain precisely the closed instantons used in the effective action method. Surprisingly, we see that adding the space dependence does not change the qualitative behavior of the field; it merely shifts the turning points and the branch points.

Finally, from these plots we can also interpret the numerical phenomenon discussed in [48]. Let us compare the results obtained with two very similar contours such as $u(r) = e^{i\theta}r$ for $|\theta| \ll 1$ either positive or negative. In both cases $t(u)$ goes very close to the pole of the Sauter pulse, but when $\theta > 0$ it then makes a turn to the left towards $-\infty$ (light blue region in the u-domain in B.2), whereas if $\theta < 0$ it turns to the right towards $+\infty$ (red region in the u-domain). We can now see that we need to consider a phase that tilts the contour clockwise because, if we were to do the opposite, we would obtain the wrong asymptotic behavior $t \to -\infty$.

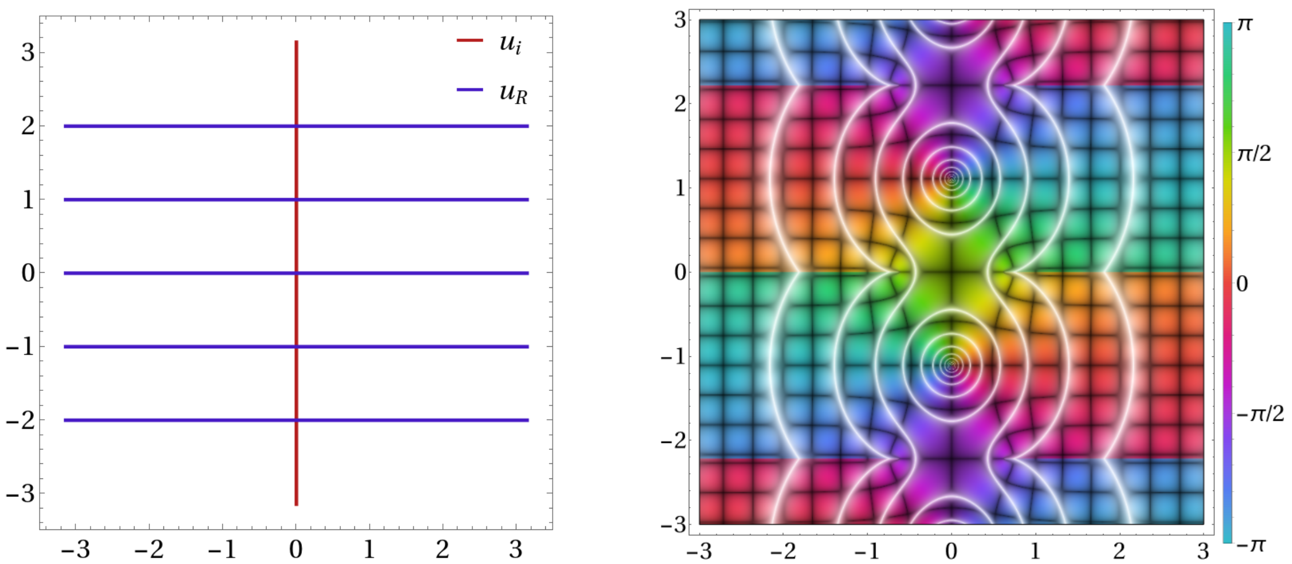


Figure B.2: Imaginary contour and a few contours parallel to the real axis (left). Complex plot of $t(u)$ for the Sauter pulse with $\gamma = 1$ (right). The color represents the phase, the white lines are contour lines of $|t(u)|$, the black ones constant real or imaginary part. The latter reveal a lot of information: starting from the zeros on the imaginary axis we see that the real part of $t(u)$ is constant and zero along the imaginary u axis, therefore $t(u)$ is purely imaginary. On the other hand, starting again from a zero but along a horizontal contour, $t(u)$ is purely real because they are contours of constant imaginary part. Lastly, the coloring tells us that the zeros are simple zeros, i.e. $t(u) \sim u - u_c$, and there are no poles.

Appendix C

Plane waves

WE PROVIDE A WORLDLINE derivation of basic Strong-Field QED results in a plane wave for both a scalar and a spinor particle. After a warm-up derivation of the Volkov propagators and the exact solutions of the Klein-Gordon and Dirac equations, we compute the amplitude for nonlinear Compton scattering. Finally, since the amplitude for vacuum pair production is zero for plane waves, we conclude with a derivation of Breit-Wheeler pair production. We will start every calculation off the mass shell and take the on-shell limit only towards the end. The calculations shown in this appendix were performed as a preliminary check to compute known quantities using the LSZ formula with the worldline representation of the scalar/spinor propagator. During the course of the present work, the authors of [82] used the same techniques to generalize the nonlinear Compton scattering result to a master formula with N external photons.

C.1 Asymptotic states and amputation

We would like to construct an LSZ representation for generic processes involving plane waves. In general, we want to study processes of the form

$$ {}_{\text{out}}\langle p, p'|0\rangle_{\text{in}} = {}_{\text{out}}\langle 0|a_{p'}(\infty)\, a_p(\infty)\, |0\rangle_{\text{in}} \tag{C.1} $$

for pair production, or

$$ {}_{\text{out}}\langle p'|p\rangle_{\text{in}} = {}_{\text{out}}\langle 0|\, a_{p'}(\infty)\, a_p^\dagger(-\infty)\, |0\rangle_{\text{in}} \tag{C.2} $$

for nonlinear Compton scattering. Photons will always be added at the end using the substitution

$$ A_\mu \to A_\mu + \varepsilon_\mu\, e^{ilq} \,. \tag{C.3} $$

As a warm-up, we provide a worldline derivation of the Volkov propagator. We define lightfront coordinates by

$$ x^\pm = x^0 \pm x^3 \qquad x^\perp = \{x^1, x^2\} \tag{C.4} $$

and consider a plane wave $A_\perp(x^+)$ with $A_0 = A_3 = 0$, $A(-\infty) = 0$ and $A(\infty) = A_\infty$. If the electromagnetic field is zero asymptotically, we can always shift the gauge such that this holds. We start from the worldline representation of the exact propagator (see (2.16) in the main text)

$$ G(y,x) = \frac{1}{2}\int_0^\infty dT\, e^{-\frac{iT}{2}m^2}\int_x^y \mathcal{D}q\, e^{-i\int_0^1 d\tau\, \frac{\dot q^2}{2T} + A\cdot\dot q} \,. \tag{C.5} $$

We rewrite a generic trajectory as $q(\tau) \to x + (y-x)\tau + q(\tau)$ such that $q(0) = q(1) = 0$. Since the perpendicular part of the path integral is Gaussian, it can be calculated by evaluating the exponent at its saddle point

$$ \ddot q^\perp(\tau) = T\dot A^\perp(\tau) \Longrightarrow \dot q^\perp(\tau) = TA^\perp(\tau) - \int_0^1 A^\perp \tag{C.6} $$

and using the normalization[1] up to a phase

$$\int \mathcal{D}q_\perp e^{i\int_0^1 d\tau\, \frac{\dot{q}_\perp^2}{2T}} = \frac{1}{2\pi T} \tag{C.7}$$

giving

$$\int \mathcal{D}q_\perp\, e^{-\int_0^1 d\tau\, \frac{\dot{q}^2}{2T} + A\cdot\dot{q}} = \frac{1}{2\pi T} e^{\frac{iT}{2}\left(\int A^2 - (\int A)^2\right)} \,. \tag{C.8}$$

The path integral over $q^0\, q^3$ looks very hard, but it is in fact simple. Since the field only depends on q^+, we can change the variables to q^+ and q^- and integrate over q^-, obtaining an infinite dimensional delta function $\delta(q^+)$ which sets $q^+(\tau) = 0$ [80, 123]. We have now

$$\frac{1}{2}\int_0^\infty dT\, \frac{1}{(2\pi T)^2}\, e^{-\frac{iT}{2}m^2 - i\frac{(y-x)^2}{2T} - i(y-x)\cdot\int_0^1 A + \frac{iT}{2}\left(\int A^2 - (\int A)^2\right)} \tag{C.9}$$

where the field evaluated at $A\big(x^+ + (y^+ - x^+)\tau\big)$. We can rewrite some parts of it as

$$\frac{1}{(2\pi T)^2}\, e^{-i\frac{(y-x)^2}{2T} - i(y-x)\cdot\int_0^1 A - \frac{iT}{2}(\int A)^2} = \int \frac{d^4p}{(2\pi)^4} e^{\frac{iT}{2}p^2 - ip\cdot(y-x) - i\frac{1}{p\cdot k}p\cdot\int_{kx}^{ky} A} \tag{C.10}$$

To see this, if we consider the $p^+\, p^-$ part of the exponent

$$\frac{i}{2}\Big(Tp^+p^- - (y^+ - x^+)p^- - (y^- - x^-)p^+\Big) \tag{C.11}$$

we integrate over p^- first, giving a delta function $\delta\Big(Tp^+ - (y^+ - x^+)\Big)$, so $p\cdot k = k\cdot(y-x)/T$, and the last integral becomes

$$-i\frac{1}{p\cdot k}\, p\cdot\int_{k\cdot x}^{k\cdot y} d\phi\, A = -iTp\cdot\int_0^1 d\tau\, A\big(x^+ + (y^+ - x^+)\tau\big) \tag{C.12}$$

changing the variable with $\phi = k_+\big(x^+ + (y^+ - x^+)\tau\big)$. Similarly

$$\frac{iT}{2}\int_0^1 d\tau\, A^2 = \frac{i}{2p\cdot k}\int_{k\cdot x}^{k\cdot y} d\phi\, A^2 \tag{C.13}$$

so we have now

$$\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\int_0^\infty dT\, e^{\frac{iT}{2}(p^2 - m^2) - ip\cdot(y-x) - i\int_{kx}^{ky} V_p} \,. \tag{C.14}$$

Finally, calculating the remaining trivial T integral and adding the $i\varepsilon$ prescription we find

$$G(y,x) = \int \frac{d^4p}{(2\pi)^4}\, \frac{e^{-ip\cdot(y-x) - i\int_{kx}^{ky} V_p}}{p^2 - m^2 + i\varepsilon} \,. \tag{C.15}$$

Adding spin is easy: looking at

$$S(y,x) = (i\not{D}_y + m)\frac{1}{2}\int_0^\infty dT \int_x^y \mathcal{D}q\, e^{-i\int_0^1 d\tau\, \frac{\dot{q}^2}{2T} + \frac{m^2T}{2} + \dot{q}\cdot A}\, \mathcal{P}\, e^{-\frac{iT}{4}\int \sigma^{\mu\nu}F_{\mu\nu}} \tag{C.16}$$

[1] We omit the boundaries of the path integral whenever they are zero.

we can see that the spin factor exponent becomes[2]

$$-\frac{iT}{4}\int_0^1 d\tau\,\sigma^{\mu\nu}F_{\mu\nu}\big(x+(y-x)\tau)\big) = \frac{T}{2}\int_0^1 d\tau \not{k}\not{A}'\big(x+(y-x)\tau)\big)$$
$$= \frac{\not{k}\not{A}(\phi_y)}{2p\cdot k} - \frac{\not{k}\not{A}(\phi_x)}{2p\cdot k} \tag{C.17}$$

where A' denotes derivative with respect to ϕ. Since only the first term of the expansion of the spin factor in (C.16) is nonzero the expansion is simply

$$\mathcal{P}\,e^{-\frac{iT}{4}\int\sigma^{\mu\nu}F_{\mu\nu}} = \Big(1+\frac{\not{k}\not{A}(\phi_y)}{2p\cdot k}\Big)\Big(1-\frac{\not{k}\not{A}(\phi_x)}{2p\cdot k}\Big)\,. \tag{C.18}$$

We perform the integrals as before and introduce an integration over d^4p

$$\int\frac{d^4p}{(2\pi)^4}\,(i\not{D}_y+m)\frac{e^{-ip\cdot(y-x)-i\int_{kx}^{ky}V_p}}{p^2-m^2+i\varepsilon}\,\Big(1+\frac{\not{k}\not{A}(\phi_y)}{2p\cdot k}\Big)\Big(1-\frac{\not{k}\not{A}(\phi_x)}{2p\cdot k}\Big)\,. \tag{C.19}$$

When we compute the covariant derivative, since $\not{\partial}_y\,\not{k}\not{A}(\phi_y)\sim k^2=0$ it only acts on the exponent and with some algebra with gamma matrices we obtain

$$S(y,x) = \int\frac{d^4p}{(2\pi)^4}\,K_{py}\,\frac{\not{p}+m}{p^2-m^2+i\varepsilon}\,\bar{K}_{px}\,e^{-ip\cdot(y-x)-i\int_{kx}^{ky}V_p} \tag{C.20}$$

where

$$K_{py} = 1+\frac{\not{k}\not{A}(\phi_y)}{2p\cdot k} \qquad\qquad \bar{K}_{px} = \gamma^0K_{px}^\dagger\gamma^0 = 1-\frac{\not{k}\not{A}(\phi_x)}{2p\cdot k} \tag{C.21}$$

which matches with [12, 13].

We want to show now how the propagator can be amputated to derive the Volkov solutions. We have the general form

$$\text{(exact)} = \int \text{(asymptotic) (amputation) (exact propagator)} \tag{C.22}$$

which we can see, in a sense, as extracting the exact state from the propagator by "projecting" it, i.e. taking the scalar product with its asymptotic version. Note that the propagator contains information about the exact states because it can be expanded precisely in terms of such solutions [124].

Since $A(-\infty)=0$ asymptotically, we have an amputation of the form

$$\varphi_p(y) = \int d^4x\,e^{-ipx}(\partial_x^2+m^2)G(y,x)\,. \tag{C.23}$$

Of course, one could in principle amputate (C.15) directly to obtain φ_p, but we want to show how this can be done with the scalar propagator in its worldline representation (C.5) without going through the usual form (C.15). In doing so, we are compelled to work off the mass shell $p\neq m^2$ and take the

[2] Recall that the path integral gives $q^+=0$.

on-shell limit at the end as in [82]. Integrating by parts and defining $\delta = p^2 - m^2$ gives us (up to a sign)

$$\frac{1}{2}\int d^4x\,\delta\,e^{-ipx}\int_0^\infty dT\,e^{-\frac{iT}{2}m^2}\int_x^y \mathcal{D}q\,e^{-i\int_0^1 d\tau\,\frac{\dot{q}^2}{2T}+A\cdot\dot{q}}\ . \tag{C.24}$$

As before, we shift the variable in the path integral $q(\tau)$ and calculate it identically

$$\frac{1}{2}\int_0^\infty dT\int d^4x\,\delta\,e^{-ipx-\frac{iT}{2}m^2-i\frac{(y-x)^2}{2T}-i(y-x)\cdot\int_0^1 A+\frac{iT}{2}\left(\int A^2-(\int A)^2\right)} \tag{C.25}$$

after which we change the variable to $x - y = \theta$ and perform the θ integrals. The perpendicular ones are trivial, and from the θ^- integral we get $\theta^+ = -Tp^+$. Since the field is now evaluated at $A\big(y^+ + Tp^+(1-\tau)\big)$, we change the variable to $\phi = k_+\big(y^+ + Tp^+(1-\tau)\big)$

$$e^{-ipy}\frac{1}{2}\int_0^\infty dT\,\delta\,e^{\frac{iT}{2}\delta-i\int_{\phi_y-Tp\cdot k}^{\phi_y}V_p} \tag{C.26}$$

which is now the right moment to take the on-shell limit. Given an integral of the form, we will show that

$$\lim_{\delta\to 0^+}\int_0^\infty dT\,\delta\,e^{iT\,\delta}f(T) = if(\infty) \tag{C.27}$$

in two ways. If we integrate by parts

$$\begin{aligned}\lim_{\delta\to 0^+}\int_0^\infty dT\,\delta\,e^{iT\,\delta}f(T) &= -i\lim_{\delta\to 0^+}e^{iT\,\delta}f(T)\Big|_0^\infty - i\lim_{\delta\to 0^+}\int_0^\infty dT\,e^{iT\,\delta}f'(T)\\ &= if(0) - i\lim_{\delta\to 0^+}\int_0^\infty dT\,e^{iT\,\delta}f'(T)\\ &= if(0) - i\int_0^\infty dT\,f'(T) = if(\infty)\end{aligned} \tag{C.28}$$

or we can simply change the variable to $T \to T/\delta$

$$\begin{aligned}\lim_{\delta\to 0^+}\int_0^\infty dT\,\delta\,e^{iT\,\delta}f(T) &= f(\infty)\int_0^\infty dT\,e^{iT}\\ &= if(\infty)\ .\end{aligned} \tag{C.29}$$

Note that the on-shell limit simplifies and makes the T integral possible: if we tried amputating the propagator off the mass shell, we would get stuck at this point.

Let us assume that the field is essentially zero for $\phi < \phi_1$, so that

$$-i\int_{\phi_y-Tp\cdot k}^{\phi_y}V_p \to -i\int_{\phi_1}^{\phi_y}V_p \tag{C.30}$$

giving us finally

$$e^{-ipy-i\int^{\phi_y}V_p} \tag{C.31}$$

where we have dropped the lower integration variable because changing it only changes the overall phase.

Adding spin is not too difficult at this point. The amputations are slightly different

$$\int d^4x\, e^{-ipx} S(y,x)(-i\overleftarrow{\not\partial}_x + m)u_s(p) \tag{C.32}$$

and the propagator becomes

$$S(y,x) = (i\not{D}_y + m)\frac{1}{2}\int_0^\infty dT \int_x^y \mathcal{D}x\, e^{-i\int_0^1 d\tau\, \frac{\dot{x}^2}{2T} + \frac{m^2 T}{2} + \dot{x}\cdot A}\, \mathcal{P}\, e^{-\frac{iT}{4}\int \sigma^{\mu\nu}F_{\mu\nu}} \ . \tag{C.33}$$

$$-\frac{iT}{4}\int \sigma^{\mu\nu}F_{\mu\nu} = \frac{1}{2p\cdot k}\int_{k\cdot y - Tp\cdot k}^{k\cdot y} \not{k}\not{A}' \to \frac{\not{k}\not{A}(\phi_y)}{2p\cdot k} \tag{C.34}$$

and since $k^2 = k\cdot A = 0$ the exponent is truncated at first order

$$\mathcal{P}e^{-\frac{iT}{4}\int \sigma^{\mu\nu}F_{\mu\nu}} = 1 + \frac{\not{k}\not{A}(\phi_y)}{2p\cdot k} \ . \tag{C.35}$$

The amputation becomes $-\not{p} + m$, which is zero on-shell because $\not{p}\, u_s(p) = m\, u_s(p)$. It is nonetheless convenient to do the following

$$(-\not{p} + m)u_s(p) = \frac{1}{2m}(-\not{p} + m)(\not{p} + m)u_s(p) = \frac{1}{2m}(-p^2 + m^2)u_s(p) \to -\frac{\delta}{2m}u_s(p) \tag{C.36}$$

so that we can perform the on-shell limit as before. Since $(i\not{D}_y + m)$ can be pulled out of the integral, we perform the same steps as before to obtain

$$(i\not{D}_y + m)\Big[e^{-ipy - i\int^{\phi_y} V_p}\Big(1 + \frac{\not{k}\not{A}(\phi_y)}{2p\cdot k}\Big)\Big] = e^{-ipy - i\int^{\phi_y} V_p}\Big(1 + \frac{\not{k}\not{A}(\phi_y)}{2p\cdot k}\Big)(\not{p} + m) \ . \tag{C.37}$$

The last term finally acts on the spinor

$$\frac{1}{2m}(\not{p} + m)u_s(p) = u_s(p) \tag{C.38}$$

and we obtain

$$e^{-ipy - i\int^{\phi_y} V_p}\Big(1 + \frac{\not{k}\not{A}(\phi_y)}{2p\cdot k}\Big)u_s(p) \ . \tag{C.39}$$

which agrees with the amputation performed in [14].

The other amputation is essentially identical, but here the outgoing particle classically has a Lorentz momentum

$$\pi' = p' - A_\infty + kV_{p'}^\infty \qquad\qquad V_{p'}^\infty = \frac{2p'\cdot A_\infty - A_\infty^2}{2p'\cdot k} \ . \tag{C.40}$$

To guess the asymptotic form, we just take the $x^+ \to \infty$ limit of the exact Volkov solution

$$\text{outgoing Volkov} \sim e^{i(\pi' + A_\infty)y} \tag{C.41}$$

and at the prefactor we cannot simply have $\partial_y^2 + m^2$, because we would not get zero on-shell. We can simply amend it by switching to the asymptotic covariant derivative $\mathcal{D}_\infty^2 + m^2$, i.e. we obtain the asymptotic Klein-Gordon operator. To summarize, we want to calculate

$$\int d^4y\, e^{i(\pi'+A_\infty)y}(\mathcal{D}_\infty^2 + m^2)G(y,x) \tag{C.42}$$

similarly to before, going off the mass shell first and taking the limit at the end. Since $\pi'^2 = p'^2$, we can simply define $\delta = p'^2 - m^2$. As before, we integrate over $\mathcal{D}q$ first

$$\frac{1}{2}\int_0^\infty dT \int d^4y\, \delta\, e^{i(\pi'+A_\infty)y - \frac{iT}{2}m^2 - i\frac{(y-x)^2}{2T} - i(y-x)\cdot\int_0^1 A + \frac{iT}{2}\left(\int A^2 - (\int A)^2\right)} \tag{C.43}$$

and afterward over the variable $y - x = \theta$

$$e^{i(\pi'+A_\infty)x}\frac{1}{2}\int_0^\infty dT\, \delta\, e^{\frac{iT}{2}\delta - \frac{iT}{2}\int_0^1 d\tau\, \delta V_{p'}} \tag{C.44}$$

with the field now evaluated at $A\big(x^+ + Tp'^+\tau\big)$, where

$$\delta V_{p'} = V_{p'} - V_{p'}^\infty\ . \tag{C.45}$$

We change the variable in the integral at the exponent to $\phi = k_+\big(x^+ + Tp'^+\tau\big)$ and use (C.27) to conclude

$$e^{i(\pi'+A_\infty)x - i\int_{\phi_x}\delta V_{p'}}\ . \tag{C.46}$$

In the spinor case we obtain

$$\int d^4y\, e^{i(\pi'+A_\infty)y}\bar{u}_{s'}(\pi')(-i\not{D}_{y,\infty} + m)S(y,x) \tag{C.47}$$

so, proceeding as before, we get a spin factor (truncated at first order)

$$\mathcal{P}e^{-\frac{iT}{4}\int\sigma^{\mu\nu}F_{\mu\nu}} = 1 + \frac{\not{k}\not{A}_\infty - \not{k}\not{A}(\phi_x)}{2p'\cdot k}\ . \tag{C.48}$$

We also integrate by parts to obtain

$$(-i\not{D}_{y,\infty} + m)(i\not{D}_y + m) \to (-\not{\pi}' + m)(\not{\pi}' + \not{A}_\infty - \not{A}(y) + m) \tag{C.49}$$

but $A(y) = A(\theta + x)$ and $\theta^+ = Tp'^+ \to \infty$, so effectively $\not{A}_\infty - \not{A}(y) \to 0$. At the end we have

$$\bar{u}_{s'}(\pi')\delta\bar{K}_{p'x}\, e^{i(\pi'+A_\infty)x - i\int_{\phi_x}\delta V_{p'}} \tag{C.50}$$

where

$$\delta\bar{K}_{p'x} = 1 + \frac{\not{k}\not{A}_\infty - \not{k}\not{A}(\phi_x)}{2p'\cdot k}\ . \tag{C.51}$$

C.2 Scalar nonlinear Compton scattering

As before, one could in principle calculate the amplitude for Compton scattering using the Volkov states derived above, and the result would follow very quickly. However, our goal is to show that this can be done without using the Volkov states at all because when we consider more complicated fields we do not have access to such solutions. What we want to do instead is computing the amplitude using the inclusion of the photon with the replacement

$$A_\mu \to A_\mu + \varepsilon_\mu \, e^{ilq} \tag{C.52}$$

in the worldline propagator, as mentioned at the beginning of the chapter.

The method outline here is equivalent but slightly different to that used in [82] in that the authors use a representation of the spin factor in terms of a Grassmann path integral (see previous chapter). We want to calculate

$$M = \int d^4x d^4y \, e^{-ipx+i(\pi'+A_\infty)y} (\partial_x^2 + m^2)(\mathcal{D}_y^2(\infty) + m^2) \int_0^\infty \frac{dT}{2} \int_0^1 d\sigma \int_x^y \mathcal{D}x \, e^{-i\int_0^1 d\tau \, \frac{\dot{q}^2}{2T} + A\cdot\dot{q} + J\cdot q} \Big|_{\text{lin}} \tag{C.53}$$

where

$$J(\tau) = -l\,\delta(\tau-\sigma) - \varepsilon\,\dot{\delta}(\tau-\sigma) \tag{C.54}$$

such that $\varepsilon\cdot l = l^2 = \varepsilon\cdot k = 0$. The last condition is satisfied simply by choosing the so-called lightfront gauge $\varepsilon^+ = 0$. The first step is to integrate by parts with both momenta off-shell, so that

$$(\partial_x^2 + m^2)(\mathcal{D}_y^2(\infty) + m^2) \to (p^2 - m^2)(p'^2 - m^2) \tag{C.55}$$

and perform the path integral. We write as usual $q(\tau) \to x + (y-x)\tau + q(\tau)$ and get rid of the linear terms in q by shifting $q \to q + q_{cl}$ where q_{cl} satisfies $q_{cl}^- = 0$ and

$$\ddot{q}_{cl}^{\perp,+} = T\tilde{J}^{\perp,+} \tag{C.56}$$

where $\tilde{J}^{\perp,+} = J^{\perp,+} - \dot{A}$. Since $\tilde{J}^+ = J^+ = -l^+\delta(\tau-\sigma)$, this part is actually trivial. In general we can use the standard method of Green's functions to solve (C.56): given a differential equation of the form

$$L_y\, u(y) = f(y) \tag{C.57}$$

if we find a function $G(y,x)$ such that $L_y\, G(y,x) = \delta(y-x)$, then the solution is given by

$$u(y) = \int dy\, G(y,x) f(x) \; . \tag{C.58}$$

In our case, the Green's function of the operator ∂_τ^2 is well-known [74, 71]

$$G(\tau,\tau') = \tau\tau' + \frac{|\tau-\tau'|}{2} - \frac{\tau+\tau'}{2} \tag{C.59}$$

so we have

$$q_{cl}^{\perp,+}(\tau) = T\int_0^1 d\tau'\, G(\tau,\tau')\,\tilde{J}^{\perp,+}(\tau')\;. \tag{C.60}$$

With this substitution

$$-i\int_0^1 d\tau\,\frac{\dot{q}^2}{2T} + \tilde{J}\cdot q \to -\frac{iT}{2}\tilde{J}\cdot G\cdot\tilde{J} - i\int_0^1 d\tau\,\frac{\dot{q}^2}{2T} \tag{C.61}$$

where the dot stands for τ integral and spacetime scalar product at once. We can now perform the path integral and expand the first term. Since G has a trivial diagonal structure proportional to $\eta^{\mu\nu}$, we use $J^2 = 0$ and integrate by parts some terms to find

$$\begin{aligned}&-\frac{iT}{2}\tilde{J}\cdot G\cdot\tilde{J} = \\ &\quad\frac{iT}{2}\Big[\int A^2 - \Big(\int A\Big)^2 + 2\int_0^1 d\tau\,A\cdot l\,\big(\sigma+\theta(\tau-\sigma)-1\big) - 2\int_0^1 d\tau\,A\cdot\varepsilon\,\big(1-\delta(\tau-\sigma)\big)\Big]\end{aligned} \tag{C.62}$$

and recognize the first two terms, already present in the amputations. Note that the field is evaluated at $x^+ + (y^+ - x^+)\tau + q_{cl}^+(\tau)$ where

$$q_{cl}^+(\tau) = -Tl^+G(\tau,\sigma)\;. \tag{C.63}$$

The ordinary integrals are very simple, we simply shift the y variable $y = x + \theta$ and perform everything except x^+. We have[3]

$$\begin{aligned}\int dx^\perp dx^-\,e^{i(\pi'+A_\infty+l-p)x} &= (2\pi)^3\delta_{\perp,-}(p'+l-p)e^{i(\pi'+A_\infty+l-p)_+x^+}\\ \int d^4\theta\,e^{(\dots)} &= (2\pi T)^2 e^{\frac{iT}{2}(\int\pi'+A_\infty-\varepsilon+\sigma l-A)^2}\end{aligned} \tag{C.64}$$

Adding together the path integral and θ contributions we find

$$\begin{aligned}&\frac{iT}{2}\Big[\int_0^1 d\tau\,\Big(A^2 + 2(\theta(\tau-\sigma)-1)A\cdot l - 2A\cdot p'\Big)\\ &\quad - 2p'\cdot\varepsilon + 2\sigma\big(p'+kV_{p'}(\infty)\big)\cdot l + 2A(\sigma)\cdot\varepsilon + p'^2 + 2A_\infty\cdot p' - A_\infty^2\Big]\end{aligned} \tag{C.65}$$

with the field now evaluated at $\theta^+ = T(p'^+ + \sigma l^+)$. Shifting $x^+ \to x^+ - Tp^+\sigma$ the argument becomes $x^+ + T(\tau-\sigma)[p^+ - \theta(\tau-\sigma)l^+]$, which looks like the worldline with momentum that jumps from p^+ to $p^+ - l^+$ (hence emits a photon) at $\tau = \sigma$. We can now also take the linear part in ε

$$-iT\varepsilon\cdot(p' - 2A(\sigma)) =: -iT\pi(\sigma)\cdot\varepsilon\;. \tag{C.66}$$

Collecting everything else together and performing some algebraic manipulations, we are faced with the following expression (up to the delta function)

$$\begin{aligned}&(p^2-m^2)(p'^2-m^2)\int dx^+\int_0^\infty dT\int_0^1 d\sigma\,\frac{T}{4}\varepsilon\cdot\pi(\sigma)e^{i(\pi'+A_\infty-p+l)_+x^+}\\ &e^{\frac{iT}{2}\left[(1-\sigma)(2A_\infty\cdot p'-A_\infty^2)+\sigma(p^2-m^2)+(1-\sigma)(p'^2-m^2)-\int_0^\sigma d\tau\,\left(2A\cdot p-A^2\right)-\int_\sigma^1 d\tau\,\left(2A\cdot p'-A^2\right)\right]}\;.\end{aligned} \tag{C.67}$$

[3] There is an overall factor of two from the Jacobian when we go to lightfront coordinates.

Before taking the on-shell limit, it is convenient to change the variable in the last integrals similarly to the previous section $\varphi = k_+[x^+ + T(\tau-\sigma)(p^+ - \theta(\tau-\sigma)l^+]$ so that the integrals become

$$-\frac{iT}{2}\int_0^\sigma d\tau\left(2A\cdot p - A^2\right) = -i\int_{kx-T\sigma kp}^{kx} V_p$$
$$-\frac{iT}{2}\int_\sigma^1 d\tau\left(2A\cdot p' - A^2\right) = -i\int_{kx}^{kx+T(1-\sigma)kp'} V_{p'}\,. \tag{C.68}$$

We are now ready to go on the mass shell. The expression looks like we should change the variables to σT and $T(1-\sigma)$ instead, so we do that but also include a factor of two

$$v = \frac{T\sigma}{2} \qquad w = \frac{T(1-\sigma)}{2}\,. \tag{C.69}$$

We have the following structure

$$(p^2-m^2)(p'^2-m^2)\int_0^\infty dT\int_0^1 d\sigma\,\frac{T}{4}(\dots) = \int_0^\infty dv\int_0^\infty dw\,\delta\,\delta' e^{i\delta v + i\delta' w} f(v,w) \tag{C.70}$$

that we immediately recognize to be very similar to what we had before, but with more variables. We shift $v\to v/\delta$ and $w\to w/\delta'$ and perform the on shell limit

$$\lim_{\delta,\delta'\to 0^+}\int_0^\infty dv\int_0^\infty dw\,\delta\,\delta' e^{i\delta v+i\delta' w} f(v,w) = -f(\infty,\infty) \tag{C.71}$$

with the usual analytic continuation.

As before we define ϕ_1 and ϕ_2 such that A can be neglected for $\phi<\phi_1$ and $A(\phi)=A_\infty$ when $\phi>\phi_2$. Such values exist because we assume $A(\phi\to-\infty)=0$ and $A(\phi\to+\infty)=A_\infty$. Then

$$-i\int_{kx-T\sigma kp}^{kx} V_p \to -i\int_{\phi_1}^{kx} V_p$$
$$iV_{p'}(\infty)(kx+T(1-\sigma)p'\cdot k) - i\int_{kx}^{kx+T(1-\sigma)kx} V_{p'} \to iV_{p'}(\infty)\phi_2 - i\int_{kx}^{\phi_2} V_{p'} \tag{C.72}$$

where the $iV_{p'}(\infty)kx$ term is taken from $ix^+(\pi'+A_\infty)_+$. Performing some manipulations one can show

$$V_{p'} = -\frac{p\cdot l}{p'\cdot k} + V_p + \frac{\pi\cdot l}{p'\cdot k} \qquad x^+(p'_+ + l_+ - p_+) = \phi\frac{p\cdot l}{p'\cdot k} \tag{C.73}$$

which gives the final result (where $\phi = k\cdot x$)

$$\boxed{M = (2\pi)^3\delta_{\perp,-}(p'+l-p)\frac{1}{k_+}\int d\phi\, e^{i\int^\phi \frac{\pi\cdot l}{p'\cdot k}}\,\varepsilon\cdot\pi(\phi)}\,. \tag{C.74}$$

It is quite clear that this approach, at least for plane waves, is significantly more challenging, or at least more involved, than the approach with Volkov states. This is because the Volkov states are effectively already amputated, whereas here we perform the amputation and go on-shell during the calculation. But for spacetime fields where no exact states is known, this route is not available.

C.3 Spinor nonlinear Compton scattering

Most of the calculation in the previous section is identical for a spinor particle; in particular, the exponent is exactly identical. The part which requires the most care is the evaluation of the spin factor. To begin with, the inclusion of a photon via the replacement

$$A \to A + \varepsilon\, e^{ilx} \tag{C.75}$$

does not affect the prefactor since the extra term we would obtain from

$$i\not{\partial}_y - \not{A}(y) + m \to i\not{\partial}_y - \not{A}(y) - \not{\varepsilon} e^{ily} + m \tag{C.76}$$

is zero on-shell because it corresponds to a photon being absorbed outside the field.

Regarding the spin factor, we must now be more careful because the path ordering cannot be simply dropped because we have two contributions $F^A_{\mu\nu} + F^\gamma_{\mu\nu}$ with

$$\begin{aligned} &\frac{1}{4}[\gamma^\mu, \gamma^\nu] F^A_{\mu\nu} = \not{k}\not{A}'(\tau) \\ &\frac{1}{4}[\gamma^\mu, \gamma^\nu] F^\gamma_{\mu\nu} = i\not{l}\not{\varepsilon}\, e^{il\cdot x} \end{aligned} \tag{C.77}$$

although fortunately most terms cancel because $k^2 = 0$ and we must take the linear part in ε

$$\begin{aligned} \mathcal{P}e^{\frac{T}{2}\int_0^1 d\tau\, \not{k}\not{A}' + i\not{l}\not{\varepsilon}\, e^{ilx}} = 1 &+ \frac{T}{2}\int_0^1 d\tau\, (\not{k}\not{A}' + i\not{l}\not{\varepsilon}\, e^{ilx}) + \frac{iT^2}{4}\int_0^1 d\tau d\sigma\, \big(\not{k}\not{A}'_\tau \not{l}\not{\varepsilon}\theta_{\tau\sigma} + \not{l}\not{\varepsilon}\not{k}\not{A}'_\tau\theta_{\sigma\tau}\big)e^{ilx(\sigma)} \\ &+ \frac{iT^3}{8}\int_0^1 d\tau d\sigma d\tau'\, \theta_{\tau\sigma}\theta_{\sigma\tau'}\, \not{k}\not{A}_\tau \not{l}\not{\varepsilon}\not{k}\not{A}_{\tau'} e^{ilx(\sigma)}\,. \end{aligned} \tag{C.78}$$

This is not the end of the story, however, since we have the scalar contribution with $\varepsilon\cdot\dot{x}$ at the exponent, but this just means that at the end of the calculation we multiply the terms in the spin factor without $\not{\varepsilon}$ by $-iT\varepsilon\cdot\pi(\tau=\sigma)$

$$\begin{aligned} &-iT\Big(1 + \frac{T}{2}\int_0^1 d\tau'\, \not{k}\not{A}'_{\tau'}\Big)\pi(\phi)\cdot\varepsilon + \frac{iT}{2}\not{l}\not{\varepsilon} + \frac{iT^2}{4}\int_0^1 d\tau'\, \big(\not{k}\not{A}'_{\tau'}\not{l}\not{\varepsilon}\,\theta_{\tau'\sigma} + \not{l}\not{\varepsilon}\not{k}\not{A}'_{\tau'}\,\theta_{\sigma\tau'}\big) \\ &+ \frac{iT^3}{8}\int_0^1 d\tau' d\tau\, \big(\not{k}\not{A}'_{\tau'}\, \not{l}\not{\varepsilon}\, \not{k}\not{A}'_\tau\, \theta_{\tau'\sigma}\theta_{\sigma\tau}\big)\,. \end{aligned} \tag{C.79}$$

Expanding everything carefully, one sees that we always have

$$\text{lin}_\varepsilon \to \int_0^1 d\sigma\, (\dots)e^{il\cdot x(\sigma)}\,. \tag{C.80}$$

The only thing we need to deal with is now the prefactor in the limit $v, w \to \infty$ (with variables defined in the previous section); recall that after computing the path integral and ordinary integrals the argument of the field is $\varphi = k_+[x^+ + T(\tau-\sigma)(p^+ - \theta(\tau-\sigma)l^+]$ therefore we use this as the new variable at the prefactor as well, and in the on-shell limit we find

$$-2\Big(1 + \frac{\not{k}\not{A}}{2p\cdot k} + \frac{\not{k}\not{A}_\infty - \not{k}\not{A}}{2p'\cdot k}\Big)\pi(\phi)\cdot\varepsilon + \not{l}\not{\varepsilon} + \Big(\frac{\not{k}\not{A}_\infty - \not{k}\not{A}}{2p'\cdot k}\not{l}\not{\varepsilon} + \not{l}\not{\varepsilon}\frac{\not{k}\not{A}}{2p\cdot k}\Big) + \frac{\not{k}\not{A}_\infty - \not{k}\not{A}}{2p'\cdot k}\not{l}\not{\varepsilon}\frac{\not{k}\not{A}}{2p\cdot k}\,. \tag{C.81}$$

Regarding amputations, since $A(y^+) = A(\theta^+ + x^+) \to A_\infty$, the term from the propagator becomes

$$i\partial\!\!\!/_y - A\!\!\!/(y) + m \to \pi\!\!\!/' + m \tag{C.82}$$

so we combine it with $-\pi\!\!\!/' + m$ to get $-\delta'$, whereas for the other we simplify it as follows

$$(-p\!\!\!/ + m)u_s(p) = \frac{1}{2m}(-p\!\!\!/ + m)(p\!\!\!/ + m)u_s(p) \to -\frac{\delta}{2m}u_s(p)\ . \tag{C.83}$$

We can rewrite the prefactor (C.81) as

$$\delta \bar{K}_{p'} \left(2\pi(\phi)\cdot\varepsilon - l\!\!\!/\varepsilon\!\!\!/\right) K_p \tag{C.84}$$

and using the following identities, together with $a \cdot b = a\!\!\!/ b\!\!\!/ + b\!\!\!/ a\!\!\!/$

$$\pi\!\!\!/(\phi)K_p = K_p p\!\!\!/ \qquad \delta\bar{K}_{p'}\pi\!\!\!/'(\phi) = \pi\!\!\!/'_\infty \delta\bar{K}_{p'}$$

$$(p' + l - p)_+ = k_+ \frac{p\cdot l}{p'\cdot k} \qquad V_p = V_{p'} + \frac{p\cdot l}{p'\cdot k} - \frac{\pi\cdot l}{p'\cdot k}$$

we find

$$\delta\bar{K}\varepsilon\!\!\!/\pi\!\!\!/(\phi)K + \delta\bar{K}(\pi\!\!\!/(\phi) - l\!\!\!/)\varepsilon\!\!\!/ K = 2m\,\delta\bar{K}\varepsilon\!\!\!/ K + k\!\!\!/\varepsilon\!\!\!/\frac{\pi\cdot l}{p'\cdot k} \tag{C.85}$$

where we let $p\!\!\!/$ and $\pi\!\!\!/'_\infty$ act on the spinors to simplify the expression. The factor $2m$ cancels, and the last term is a total derivative and we can get rid of it. Finally, using

$$\bar{u}_{s'}(\pi'_\infty)\delta\bar{K}_{p'} = \bar{u}_{s'}(p')\bar{K}_{p'}. \tag{C.86}$$

we obtain the final amplitude

$$\boxed{M = (2\pi)^3\delta_{\perp,-}(p' + l - p)\frac{1}{2k_+}\int d\phi\ e^{i\int^\phi \frac{\pi\cdot l}{p'\cdot k}}\ u_{s'}(p')\bar{K}_{p'}\varepsilon\!\!\!/ K_p\, u_s(p)} \tag{C.87}$$

which agrees with [12].

C.4 Breit-Wheeler pair production

A minor variation of the nonlinear Compton calculation gives us the amplitude for Breit-Wheeler pair production $\gamma \to e^+e^-$. Intuitively, we are taking the Feynman diagram for Compton scattering and changing the orientation of two arrows. The photon becomes incoming, so we simply change its momentum $l \to -l$; we do this at the end of the calculation. The incoming fermion line, on the other hand, now becomes outgoing, so we change the spinor from u_s to v_s and take some care with its momentum. The outgoing positron, just like the electron, will receive some modifications due to A_∞.

We use the LSZ

$$\int d^4xd^4y\, e^{i(\pi'_\infty+A_\infty)y-i(\pi_\infty+A_\infty)x}\bar{u}_{s'}(\pi'_\infty)[-i\slashed{D}^\infty_y+m]S(y,x)[i\overleftarrow{\slashed{D}}^\infty_x+m]v_s(\pi_\infty) \tag{C.88}$$

where

$$\begin{aligned}\pi'_\infty =&p'-A_\infty+k\frac{2p'\cdot A_\infty-A_\infty^2}{2p'\cdot k}=p'-A_\infty+kV_{p'},\\ \pi_\infty =&-p-A_\infty-k\frac{-2p\cdot A_\infty-A_\infty^2}{2p\cdot k}=-p-A_\infty-kV_{-p}\end{aligned} \tag{C.89}$$

where the opposite sign for the positron follows from the fact that we have absorbed the charge of the electron in A, and $\pi_\infty=\pi(-p,\infty)$. Substituting $A\to A+\varepsilon e^{ilx}$ in the worldline representation of $S(y,x)$, shifting $y=\theta+x$ in the path integral, and performing a few more steps gives

$$\begin{aligned}&\frac{1}{4}\bar{\delta}(p'+p+l)\bar{u}_{s'}(\pi'_\infty)[-\slashed{\pi}'_\infty+m]\int_0^\infty dT\int_0^1 d\sigma\int dx^+d^4\theta\,\mathcal{D}q\,(\slashed{\pi}'_\infty+\slashed{A}_\infty-\slashed{A}(\theta+x)+m)\\ &e^{i(\pi'_\infty+\pi_\infty+l)_+x^+-i\frac{\theta^2}{2T}-i\frac{m^2T}{2}+i(\pi'_\infty-\pi_\infty+l\sigma-\varepsilon-\int A)\theta-i\int\frac{\dot{q}^2}{2T}+\tilde{J}\cdot q}\,\mathrm{Spin}[A]\,[-\slashed{\pi}_\infty+m]v_s(\pi_\infty)\end{aligned} \tag{C.90}$$

where the spin prefactor Spin[A] is given by

$$\begin{aligned}\mathrm{Spin}[A]:=&\Big(1+\frac{T}{2}\int_0^1 d\tau'\,\slashed{k}\slashed{A}'_{\tau'}\Big)\mathrm{lin}_\varepsilon+\frac{iT}{2}\slashed{l}\slashed{\epsilon}+\frac{iT^2}{4}\int_0^1\big(\slashed{k}\slashed{A}'_{\tau'}\slashed{l}\slashed{\epsilon}\,\theta_{\tau'\sigma}+\slashed{l}\slashed{\epsilon}\slashed{k}\slashed{A}'_{\tau'}\,\theta_{\sigma\tau'}\big)\\ &+\frac{iT^3}{8}\int_0^1 d\tau d\sigma d\tau'\,\theta_{\tau\sigma}\theta_{\sigma\tau'}\,\slashed{k}\slashed{A}_\tau\slashed{l}\slashed{\epsilon}\slashed{k}\slashed{A}_{\tau'}\end{aligned} \tag{C.91}$$

and

$$\bar{\delta}(p'+p+l)=(2\pi)^3\,\delta_{\perp,-}(p'+p+l) \tag{C.92}$$

Where there is no lin_ε we simply ignore the ε contribution that will come from the exponent. The integrations over $d\theta$ and $\mathcal{D}q$ are no different from Compton scattering. We have delta functions which set

$$\begin{aligned}y^+&\to T(l^+\sigma+p'^+)\\ q^+(\tau)&\to -Tl^+\Big[\tau\sigma+\frac{|\tau-\sigma|}{2}-\frac{\tau+\sigma}{2}\Big].\end{aligned} \tag{C.93}$$

We also shift $x^+\to x^++Tp^+\sigma$ (opposite sign with respect to Compton). By going on shell we have $T\sigma$, $T(1-\sigma)\to\infty$, so the exponential becomes

$$i(p'+p+l)_+x^+-i\int_\phi^{\phi_2}-V_{-p}-i\int_\phi^{\phi_2}V_p \tag{C.94}$$

so using

$$\begin{aligned}x^+(p'+l+p)_+=&-\phi\frac{p\cdot l}{p'\cdot k},\\ V_{p'}=&\phi\frac{p\cdot l}{p'\cdot k}+V_{-p}-\frac{\pi_{-p}\cdot l}{p'\cdot k}\end{aligned} \tag{C.95}$$

where $-\pi = p + A_\infty - kV_{-p}$, we get an exponential

$$-i\int^{\phi}\frac{\pi_{-p}\cdot l}{p'\cdot k}. \tag{C.96}$$

The spin prefactor on the other hand

$$2\pi_{-p}\cdot\varepsilon\,\delta\bar{K}_{p'}\,\delta K_{-p} + \delta\bar{K}_{p'}\,\not{\varepsilon}\not{l}\,\delta K_{-p} \tag{C.97}$$

can be manipulated using the following properties

$$\not{\pi}_{-p}(\phi)\,\delta K_{-p} = \delta K_{-p} \qquad \delta\bar{K}_{p'}\,\not{\pi}'(\phi) = \not{\pi}'_\infty\,\delta\bar{K}_{p'} \tag{C.98}$$

to give [4]

$$2\pi_{-p}\cdot\varepsilon\,\delta\bar{K}_{p'}\,\delta K_{-p} + \delta\bar{K}_{p'}\,\not{\varepsilon}\not{l}\,\delta K_{-p} = \delta\bar{K}_{p'}\not{\varepsilon}\delta K_{-p}(\not{\pi}_\infty + m). \tag{C.99}$$

Also using

$$\bar{u}_{s'}(\pi'_\infty)\,\delta\bar{K}_{p'}\bar{u}_{s'}(p')\,\bar{K}_{p'} \qquad \delta K_{-p}\,v_s(\pi_\infty) = K_{-p}\,v_s(p) \tag{C.100}$$

we get

$$\frac{1}{2k_+}\bar{\delta}(p'+l+p)\int d\phi\; e^{-i\int^{\phi}\frac{\pi_{-p}\cdot l}{p'\cdot k}}\;\bar{u}_{s'}(p')\,\bar{K}_{p'}\not{\varepsilon}K_{-p}\,v_s(p)\,. \tag{C.101}$$

We should also substitute $l \to -l$ such that the photon is incoming. Here $-\pi_{-p} = p + A - kV_{-p}$ because it is for the positron.

[4] The rightmost factor $(\not{\pi}_\infty + m)$ combines with the spinor truncation.

Appendix D

Constant field

ANOTHER SIMPLE EXAMPLE where we can perform the calculations analytically is a constant field. We focus for simplicity on a constant electric field polarized in the z direction, but in principle it is possible to generalize to a general constant electromagnetic field $F_{\mu\nu}$ with all non-zero electric and magnetic components. The biggest complication is that the asymptotic states are no longer free, so the usual LSZ reduction formula must be modified in a non-trivial way. We start with a derivation of the exact propagator, then move to the amputations and obtain the exact solutions to the Dirac equation in a constant electric field, which are well-known, in terms of parabolic cylinder functions. After these preliminary calculations, we compute the amplitudes for Schwinger pair production and Breit-Wheeler pair production. We consider only the spinor case, as it is the most relevant, since we are typically interested in studying electrons in an electromagnetic field.

D.1 Propagator

As in the previous appendix we start with a worldline derivation of the exact propagator. We consider a constant electric field $F_{03} = -F_{30} = E$ with a gauge $A_3(t) = Et$. From here on, we label x_- the initial point and x_+ the final point. The worldline representation is still the same

$$S(x_+, x_-) = (i\partial\!\!\!/_{x_+} - \gamma_3\, Et_+ + m)K_E(x_+, x_-) \tag{D.1}$$

and for the most part we focus on the heat kernel $K_E(x_+, x_-)$

$$K_E(x_+, x_-) = \int \frac{dT}{2} e^{-\frac{im^2T}{2}} \int_{x_-}^{x_+} \mathcal{D}q\, e^{-i\int \frac{\dot{q}^2}{2T} + A\cdot\dot{q}}\, \mathcal{P}\, e^{-\frac{iT}{4}\int \sigma^{\mu\nu}F_{\mu\nu}} \,. \tag{D.2}$$

Since $F_{\mu\nu} =$ const, the spin factor is easily expanded

$$e^{\frac{ET}{2}\gamma^0\gamma^3} = \cosh\Big(\frac{ET}{2}\Big)\Big(1 + \gamma^0\gamma^3 \tanh\Big(\frac{ET}{2}\Big)\Big) \,. \tag{D.3}$$

As to the path integral, we compute the exponent and the prefactor separately. Since it is Gaussian, we compute the exponent by evaluating it with the solutions to the equations of motion. The perpendicular components are trivial $\ddot{q}^\perp(\tau) = 0$, while two nontrivial components satisfy

$$\begin{aligned} \ddot{t}(\tau) &= TE\,\dot{z}(\tau) \\ \ddot{z}(\tau) &= TE\,\dot{t}(\tau) \end{aligned} \tag{D.4}$$

with boundary conditions $t(0) = t_-$, $t(1) = t_+$, $z(0) = z_-$, $z(1) = z_+$. Evaluating at the instantons we get

$$\begin{aligned} &\int_0^1 d\tau\, \frac{\dot{t}^2(\tau)}{2T} - \frac{\dot{z}^2(\tau)}{2T} + E\,t(\tau)\dot{z}(\tau) - \frac{(\dot{q}^\perp)^2}{2T} = \\ &\qquad \frac{E}{2}(t_+ + t_-)(z_+ - z_-) + \frac{E}{4}\coth\left(\frac{ET}{2}\right)\Big((t_+ - t_-)^2 - (z_+ - z_-)^2\Big) - \frac{(x_+^\perp - x_-^\perp)^2}{2T} \,. \end{aligned} \tag{D.5}$$

As to the prefactor, we can compute it in two ways. We can simply use the Gelfand-Yaglom method [40]: collecting the symmetrized second order terms at the exponent we can write it as

$$-\frac{i}{2T}\int d\tau\, q\cdot\Lambda\cdot q \tag{D.6}$$

where $q=\{t,z\}$

$$\Lambda=\begin{pmatrix}-\partial_\tau^2 & ET\partial_\tau\\ -ET\partial_\tau & \partial_\tau^2\end{pmatrix}\ . \tag{D.7}$$

We must find two solutions that satisfy

$$\Lambda\phi^{(i)}=0 \qquad \phi_j^{(i)}(0)=0 \qquad \dot{\phi}_j^{(i)}(0)=\delta_j^i \tag{D.8}$$

to obtain the prefactor from

$$\frac{1}{(2\pi T)^2}\sqrt{\frac{1}{\det\Lambda}} \tag{D.9}$$

with

$$\det\Lambda=\left(\phi_1^{(1)}\phi_2^{(2)}-\phi_2^{(1)}\phi_1^{(2)}\right)\Big|_{\tau=1}\ . \tag{D.10}$$

It is easy to verify that the solutions are given by

$$\phi^{(1)}(\tau)=\begin{pmatrix}\frac{\sinh(ET\tau)}{ET}\\ \frac{\cosh(ET\tau)-1}{ET}\end{pmatrix} \qquad \phi^{(2)}(\tau)=\begin{pmatrix}\frac{\cosh(ET\tau)-1}{ET}\\ \frac{\sinh(ET\tau)}{ET}\end{pmatrix} \tag{D.11}$$

from which we immediately see

$$\det\Lambda=\frac{\sinh^2\left(\frac{ET}{2}\right)}{\left(\frac{ET}{2}\right)^2} \tag{D.12}$$

so the prefactor is

$$\frac{1}{(2\pi T)^2}\frac{\frac{ET}{2}}{\sinh\left(\frac{ET}{2}\right)}\ . \tag{D.13}$$

Alternatively, we can find the eigenvalues of

$$\Lambda\,\varphi_n(\tau)=\lambda_n\,\varphi_n(\tau) \tag{D.14}$$

which we do by solving the equation above with $\varphi_n(0)=0$ and finding the values λ_n for which $\varphi_n(1)=0$. One can show that the eigenvalues are given by

$$\lambda_n=n^2\pi^2+\left(\frac{ET}{2}\right)^2\ . \tag{D.15}$$

The prefactor is therefore given by[1]

$$\frac{1}{(2\pi T)^2}\prod_n \frac{\lambda_n^0}{\lambda_n} = \frac{1}{(2\pi T)^2}\frac{1}{\prod_n\left(1+\frac{E^2T^2}{4n^2\pi^2}\right)} = \frac{1}{(2\pi T)^2}\frac{\frac{ET}{2}}{\sinh\left(\frac{ET}{2}\right)} \ . \tag{D.16}$$

The heat kernel K_E is finally

$$\begin{aligned} K_E(x_+,x_-) = &\int_0^\infty dT\, e^{-i\frac{m^2T}{2}-i\frac{E}{2}(z_+-z_-)(t_++t_-)+\frac{i(x_+^\perp-x_-^\perp)^2}{2T}-i\frac{E}{4}\coth\left(\frac{ET}{2}\right)\left((t_+-t_-)^2-(z_+-z_-)^2\right)} \\ &\times \frac{\frac{ET}{2}\coth\left(\frac{ET}{2}\right)}{2(2\pi T)^2}\left(1+\gamma^0\gamma^3\,\tanh\left(\frac{ET}{2}\right)\right) \ . \end{aligned} \tag{D.17}$$

In order to write it in terms of a Fourier transform, we use

$$\begin{aligned} &\frac{\frac{ET}{2}\coth\left(\frac{ET}{2}\right)}{(2\pi T)^2} e^{\frac{i(x_+^\perp-x_-^\perp)^2}{2T}-i\frac{E}{4}\coth\left(\frac{ET}{2}\right)\left((t_+-t_-)^2-(z_+-z_-)^2\right)} = \\ &\qquad \int\frac{d^4q}{(2\pi)^4}e^{-iq(x_+-x_-)-i\frac{q_\perp^2T}{2}+i(q_0^2-q_3^2)\frac{\tanh\left(\frac{ET}{2}\right)}{E}} \end{aligned} \tag{D.18}$$

and let the derivatives act

$$i\partial\!\!\!/_+ - \gamma_3\, Et_+ + m \to q\!\!\!/ + \frac{E}{2}\left((z_+-z_-)\gamma^0-(t_+-t_-)\gamma^3\right)+m \ . \tag{D.19}$$

Finally, since the extra E terms are precisely the saddle point values of the q^0 and q^3 integrals, we can substitute

$$\frac{E}{2}(t_+-t_-)\to q_0\,\tanh\left(\frac{ET}{2}\right) \qquad \frac{E}{2}(z_+-z_-)\to -q_3\,\tanh\left(\frac{ET}{2}\right) \tag{D.20}$$

so finally

$$\begin{aligned} S(x_+,x_-) =&\frac{1}{2}\, e^{-i\frac{E}{2}(z_+-z_-)(t_++t_-)}\int\frac{d^4q}{(2\pi)^4}\int_0^\infty dT\, e^{-iq(x_+-x_-)-i\frac{m^2T}{2}-i\frac{q_\perp^2T}{2}+i(q_0^2-q_3^2)\frac{\tanh\left(\frac{ET}{2}\right)}{E}} \\ &\left(q\!\!\!/ + m-(q_0\gamma^3+q_3\gamma^0)\tanh\left[\frac{ET}{2}\right]\right)\left(1+\gamma^0\gamma^3\,\tanh\left[\frac{ET}{2}\right]\right) \end{aligned} \tag{D.21}$$

which is the well-known result [17].

D.2 From asymptotic states to pair production

Analogously to the plane wave case, we derive the exact solutions from the asymptotic ones by amputating the propagator. From now on, we set $m = 1$ so that all energy scales are relative to the electron mass. For constant fields, the particles are never free asymptotically, so we cannot use

[1] There is no square root because the eigenvalues are doubly degenerate.

plane waves as outgoing states in the LSZ as we do for all the other cases. We use instead the WKB approximations [49, 125, 126]

$$\begin{aligned} U_s(t,\mathbf{p}) &= (\gamma^0\pi_0 + \gamma^i\pi_i + 1)G^+(\mathbf{p},t)R_s \\ V_s(t,-\mathbf{p}) &= (-\gamma^0\pi_0 + \gamma^i\pi_i + 1)G^-(t,\mathbf{p})R_s \end{aligned} \tag{D.22}$$

where $\pi_\perp = p_\perp$, $m_\perp^2 = 1 + \pi_\perp^2$, $\pi_3 = p_3 - A_3(t)$, $\pi_0 = \sqrt{m_\perp^2 + \pi_3^2}$, and

$$\begin{aligned} R_1 = (0, \cos\varphi, \sin\varphi, 0)\,, \qquad R_2 = (0, -\sin\varphi, \cos\varphi, 0) \\ G^\pm(t,\mathbf{q}) = \frac{1}{\sqrt{2\pi_0(\pi_0 \pm \pi_3)}}\, e^{\mp i\int^t \pi_0} \end{aligned} \tag{D.23}$$

with R_s chosen to satisfy $\gamma^0\gamma^3 R_s = R_s$. The LSZ for Schwinger pair production in a constant electric field will then look like

$$\lim_{t_\pm\to\infty} \int d^3x_+\, d^3x_-\, e^{i\mathbf{p}\cdot\mathbf{x}_+ + i\mathbf{p}'\cdot\mathbf{x}_-}\bar{U}_s(\mathbf{p},t_+)\,\gamma^0\,\langle 0|\mathbb{T}\Psi(x_+)\Psi(x_-)|0\rangle\,\gamma^0\, V_{s'}(\mathbf{p}',t_-)\;. \tag{D.24}$$

We consider one amputation first. We want to calculate

$$\lim_{t_+\to\infty} \int d^3x_+\, e^{i\mathbf{p}\cdot\mathbf{x}_+}\bar{u}_{s'}^{\infty}(\mathbf{p},t_+)\,\gamma^0\, G(x_+,x_-) \tag{D.25}$$

where we write $G(x_+,x_-)$ as

$$G(x_+,x_-) = K_E(x_+,x_-)(-i\overleftarrow{\not{\partial}}_{x_-} - \not{A}(x_-) + 1) \tag{D.26}$$

and

$$\begin{aligned} K_E(x_+,x_-) = \int_0^\infty dT\, \frac{1}{2}\frac{\frac{ET}{2}}{\sinh\!\left(\frac{ET}{2}\right) i(2\pi T)^2}\Big(\cosh\Big[\frac{ET}{2}\Big] + \gamma^0\gamma^3\sinh\Big[\frac{ET}{2}\Big]\Big) \\ \times\, e^{-i\frac{T}{2} - i\frac{E}{2}(z_+-z_-)(t_++t_-) + \frac{i(x_+^\perp - x_-^\perp)^2}{2T} - i\frac{E}{4}\coth\left(\frac{ET}{2}\right)\left((t_+-t_-)^2-(z_+-z_-)^2\right)}. \end{aligned} \tag{D.27}$$

For convenience, we can omit the operator $(-i\overleftarrow{\not{\partial}}_{x_-} - \not{A}(x_-) + 1)$, since it can simply act at the end

$$(-i\overleftarrow{\not{\partial}}_{x_-} - \not{A}(x_-) + 1) \Longrightarrow (-i\overleftarrow{\partial}_{t_-} + \pi_i\gamma^i + 1)\;. \tag{D.28}$$

We derive the asymptotic expansion of the WKB state

$$\bar{u}_s^{\infty}(\mathbf{p},t_+) = \lim_{t_+\to\infty} \bar{U}_s(\mathbf{p},t_+) \tag{D.29}$$

from the exponent

$$\int^{t_+} d\bar{t}\, \sqrt{m_\perp^2 + (E\bar{t} - p_3)} \tag{D.30}$$

defining a new variable $\bar{v} := (E\bar{t} - p_3)/m_\perp$ (and $v = (Et_+ - p_3)/m_\perp$) so that

$$\sqrt{1+\bar{v}^2} \sim \bar{v} + \frac{1}{2v} \tag{D.31}$$

gives us, defining $\eta = m_\perp^2/E$,

$$\eta \int^v d\bar{v}\, \sqrt{1+\bar{v}^2} \sim \frac{\eta}{2}\left(v^2 + \ln(v)\right) \ . \tag{D.32}$$

We also simplify the prefactor and find

$$\bar{u}_s^\infty(\mathbf{p}, t_+)\gamma^0 = \frac{1}{m_\perp} e^{\frac{i\eta}{2}\left(v^2+\ln(v)\right)}\ \bar{R}_s\,(\gamma^\perp \pi_\perp + 1)\gamma^0 \ . \tag{D.33}$$

We can immediately perform the d^3x_+ integrals as well and redefine $T \to T/E$, obtaining

$$\begin{aligned} &e^{i\mathbf{p}\cdot\mathbf{x}_-} \lim_{v\to\infty} \frac{1}{m_\perp} \bar{R}_s(\gamma^\perp\pi_\perp + 1)\gamma^0 \sqrt{\frac{1}{4\pi E}}\Big(1 + \gamma^0\gamma^3 \tanh\Big[\frac{T}{2}\Big]\Big) \\ &\qquad \times \int_0^\infty \frac{dT}{2\sqrt{\coth(T/2)}}\, e^{\frac{i\eta}{2}\left(v^2 + \log(v) - T - \frac{1}{2}\coth(T/2)(v'-v)^2 - \frac{1}{2}\tanh(T/2)(v'+v)^2\right)} \ . \end{aligned} \tag{D.34}$$

For the T integral, we have to figure out a way to take the limit explicitly, but once we find it, the T integral actually becomes doable. As it is now, it is very hard.

Notice that when $T \to \infty$ the v^2 terms simply cancel, and the infinite terms at the exponent are only $\ln(v) - T$. If we perform the substitution $T \to T + \ln(v)$ and take the $v \to \infty$ limit, we cancel the logarithmic divergence *and* obtain the desired $T \to \infty$ limit. We can expand the hyperbolic functions at the exponent as follows

$$\begin{aligned} \coth\left(\frac{x}{2}\right) &\simeq (1+e^{-x})(1+e^{-x}+e^{-2x}) \simeq 1 + 2e^{-x} + 2e^{-2x} \\ \tanh\left(\frac{x}{2}\right) &\simeq (1-e^{-x})(1-e^{-x}+e^{-2x}) \simeq 1 - 2e^{-x} + 2e^{-2x} \end{aligned} \tag{D.35}$$

and we are almost ready to take the limit. It is convenient to actually change the variable as $T = \ln(v/\tau)$ so that

$$\int_0^\infty dT = \int_0^v \frac{d\tau}{\tau} \to \int_0^\infty \frac{d\tau}{\tau} \tag{D.36}$$

obtaining a finite exponent when $v \to \infty$

$$\frac{i\eta}{2}\left(-v'^2 - 2\tau^2 + 4\tau v'\right) + \left(\frac{i\eta}{2} - 1\right)\ln(\tau) \ . \tag{D.37}$$

We already know that the result will contain parabolic cylinder functions, so we just have to find the integral representation that is most convenient. Changing the variables to

$$r := \sqrt{2\eta}e^{i\frac{\pi}{4}}\,\tau \qquad z := \sqrt{2\eta}e^{i\frac{\pi}{4}}\,v' \tag{D.38}$$

our exponent looks like (removing the phases)

$$-\frac{r^2}{2} + zr - \frac{z^2}{4} + \Big(\frac{i\eta}{2} - 1\Big)\ln(r) + \frac{\pi\eta}{8} + \frac{1}{2}\ln(2\eta) \tag{D.39}$$

and all together

$$e^{i\mathbf{p}\cdot\mathbf{x}_-}\,\bar{R}_s(\gamma^\perp\pi_\perp+1)\gamma^0\sqrt{\frac{1}{4\pi E}}\,e^{\frac{\pi\eta}{8}}\int_0^\infty dr\,e^{-\frac{r^2}{2}+zr-\frac{z^2}{4}}\;r^{\frac{i\eta}{2}-1}\;. \tag{D.40}$$

This is precisely the integral form [127]

$$D_\nu(z)=\frac{e^{-\frac{z^2}{4}}}{\Gamma(-\nu)}\int_0^\infty dx\,e^{-zx-\frac{x^2}{2}}x^{-\nu-1} \tag{D.41}$$

so we are almost done. Acting with

$$e^{i\mathbf{p}\cdot\mathbf{x}_-}\Big(-i\overleftarrow{\not\partial}_{x_-}-\not A(x_-)+1\Big)=e^{i\mathbf{p}\cdot\mathbf{x}_-}\Big(-i\overleftarrow{\partial}_0\gamma^0+\gamma^i\pi'_i+1\Big) \tag{D.42}$$

we have

$$e^{i\mathbf{p}\cdot\mathbf{x}_-}\;\frac{\Gamma\big(\frac{i\eta}{2}\big)}{2m_\perp\sqrt{\pi E}}\bar{R}_s(\gamma^\perp p_\perp+1)\gamma^0\,e^{\frac{\pi\eta}{8}}D_{-\frac{i\eta}{2}}\big(-\sqrt{2\eta}\,v'\,e^{i\frac{\pi}{4}}\big)\big(-i\overleftarrow{\partial}_0\gamma^0+\pi_i\gamma^i+1\big)\;. \tag{D.43}$$

and using

$$D'_\nu(z)=-\frac{1}{2}zD_\nu(z)+\nu D_{\nu-1}(z) \tag{D.44}$$

we obtain

$$\begin{aligned}&D_{-\frac{i\eta}{2}}\big(-\sqrt{2\eta}\,v'\,e^{i\frac{\pi}{4}}\big)\big(-i\overleftarrow{\partial}_0\gamma^0+\pi_i\gamma^i+1\big)=\\&\qquad D_{-\frac{i\eta}{2}}\big(-\sqrt{2\eta}\,v'\,e^{i\frac{\pi}{4}}\big)(p_\perp\gamma^\perp+1)+\sqrt{2E}e^{i\frac{\pi}{4}}\frac{\eta}{2}D_{-\frac{i\eta}{2}-1}\big(-\sqrt{2\eta}\,v'\,e^{i\frac{\pi}{4}}\big)\gamma^0\end{aligned} \tag{D.45}$$

therefore, up to a phase, the full result is

$$\frac{e^{i\mathbf{p}\cdot\mathbf{x}_-}}{{}_{\text{out}}\langle 0|0\rangle_{\text{in}}}\frac{1}{\sqrt{\eta E}}e^{-\frac{\pi\eta}{8}}\Big(m_\perp\bar{B}_0\,D_{-\frac{i\eta}{2}}\big(-\sqrt{2\eta}\,v'\,e^{i\frac{\pi}{4}}\big)+\sqrt{2E}\,e^{i\frac{\pi}{4}}\,\bar{B}_1\,\frac{\eta}{2}\,D_{-\frac{i\eta}{2}-1}\big(-\sqrt{2\eta}\,v'\,e^{i\frac{\pi}{4}}\big)\Big) \tag{D.46}$$

where

$$\begin{aligned}&B_0:=\gamma^0R_s\qquad\qquad B_1:=\frac{1}{m_\perp}(1+\gamma^\perp p_\perp)\,R_s\\&{}_{\text{out}}\langle 0|0\rangle_{\text{in}}=-2i\sqrt{\frac{\pi}{\eta}}\,\frac{e^{-\frac{\pi\eta}{4}}}{\Gamma\big(\frac{i\eta}{2}\big)}e^{-\frac{i\eta}{4}\big(\ln(2/\eta)+1\big)+i\frac{\pi}{4}}\end{aligned} \tag{D.47}$$

in agreement with [124].

At this point, most of the work to compute the the pair production amplitude is done. By definition we have

$$\lim_{t_\pm\to\infty}\int d^3x_+\,d^3x_-\,e^{i\mathbf{p}\cdot\mathbf{x}_++i\mathbf{p}'\cdot\mathbf{x}_-}\bar{U}_s(t_+,\mathbf{p})\,\gamma^0\,G(x_+,x_-)\,\gamma^0\,V_{s'}(t_-,\mathbf{p}') \tag{D.48}$$

but we have already carefully shown that

$$\lim_{t_+\to\infty}\int d^3x_+\,e^{i\mathbf{p}\cdot\mathbf{x}_+}\bar{U}_s(\mathbf{p},t_+)\,\gamma^0\,G(x_+,x_-)=\text{(D.46)} \tag{D.49}$$

so the integral over d^3x_- gives us a delta function

$$\int d^3x_- \to (2\pi)^3\,\delta(\mathbf{p}+\mathbf{p}') \tag{D.50}$$

and the asymptotic v^∞ state can be expanded as follows letting $v' = m_\perp(Et_+ + p_3') = m_\perp(Et_- - p_3)$

$$\frac{1}{\sqrt{2\pi_0'(\pi_0'-\pi_3')}} \to \frac{1}{2m_\perp v'} \qquad \gamma^0(-\gamma^0\pi_0+\gamma^3\pi_3)R_{s'} \to -2m_\perp v' R_{s'} \qquad e^{i\int^{t_-}\pi_0'} \to e^{\frac{i\eta}{2}(v'^2+\ln(v'))}\,. \tag{D.51}$$

Finally, the parabolic cylinder functions have an asymptotic expansion

$$D_\nu(z) \sim e^{-\frac{z^2}{4}}\,z^\nu \tag{D.52}$$

where $-\frac{5\pi}{4} \leq \arg(z) \leq \frac{3\pi}{4}$. In our case we have a phase $-e^{\frac{i\pi}{4}} = e^{-i\frac{3\pi}{4}}$, which matters because it gives a real contribution at the exponent

$$\left(e^{-i\frac{3\pi}{4}}\right)^{-\frac{i\eta}{2}} = e^{-\frac{3\pi\eta}{8}}\,. \tag{D.53}$$

The leading order $v' \to \infty$ of (D.46) is the first term, so up to a phase we have

$$e^{-\frac{\pi\eta}{8}}D_{-\frac{i\eta}{2}}\big(-\sqrt{2\eta}\,v'\,e^{i\frac{\pi}{4}}\big) \sim e^{-\frac{\pi\eta}{2}}e^{-\frac{i\eta}{2}(v'^2+\ln(v'))} \tag{D.54}$$

which precisely cancels the infinite part from v^∞; this gives us an amplitude

$$\boxed{\frac{1}{{}_{\text{out}}\langle 0|0\rangle_{\text{in}}}\delta_{ss'}\,(2\pi)^3\delta(\mathbf{p}'+\mathbf{p})\,e^{-\frac{\pi\eta}{2}}}\,. \tag{D.55}$$

The presence of the normalization factor ${}_{\text{out}}\langle 0|0\rangle_{\text{in}}$ tells us that the worldline propagator $S(x_+,x_-)$ is not the same as ${}_{\text{out}}\langle 0|\mathbb{T}\Psi(x_+)\bar{\Psi}(x_-)|0\rangle_{\text{in}}$ but rather [52, 124]

$$S(x_+,x_-) = \frac{{}_{\text{out}}\langle 0|\mathbb{T}\Psi(x_+)\bar{\Psi}(x_-)|0\rangle_{\text{in}}}{{}_{\text{out}}\langle 0|0\rangle_{\text{in}}}\,. \tag{D.56}$$

Since we want to use ${}_{\text{out}}\langle 0|\mathbb{T}\Psi(x_+)\bar{\Psi}(x_-)|0\rangle_{\text{in}}$ rather than S itself in the LSZ, we get rid of the factor ${}_{\text{out}}\langle 0|0\rangle_{\text{in}}$ and integrate over the momenta

$$\sum_{ss'}\int\frac{d^3p\,d^3p'}{(2\pi)^6}\,|{}_{\text{out}}\langle 0|0\rangle_{\text{in}}|^2\,|(\text{D.55})|^2 = EV_4\sum_{ss'}\int\frac{dp_1dp_2}{(2\pi)^3}\,e^{-\pi\eta} = V_4\frac{E^2}{4\pi^3}e^{-\frac{\pi}{E}} \tag{D.57}$$

where we used $\int dp_3 = EV_0$. Note that this gives the total probability to produce just one pair, so it is different from the vacuum decay rate, which includes the probability to produce any number of pairs [17].

Of course, one can show that amputating the propagator with the positron asymptotic state gives the same result. The exact solution produced in that case is given by

$$\frac{e^{i\mathbf{p}'\cdot\mathbf{x}_+}}{{}_{\text{out}}\langle 0|0\rangle_{\text{in}}}\frac{1}{\sqrt{2E}}e^{-\frac{\pi\eta}{8}+i\frac{\pi}{4}}\left(m_\perp B_0\,D_{-\frac{i\eta}{2}-1}\big(-\sqrt{2\eta}\,v\,e^{i\frac{\pi}{4}}\big)-\sqrt{2E}\,e^{-i\frac{\pi}{4}}\,B_1\,D_{-\frac{i\eta}{2}}\big(-\sqrt{2\eta}\,v\,e^{i\frac{\pi}{4}}\big)\right) \tag{D.58}$$

where

$$B_0 = \gamma^0 R_{s'} \qquad\qquad B_1 = \frac{1}{m_\perp}(m - \gamma^\perp p_\perp)\, R_{s'}. \tag{D.59}$$

D.3 Breit-Wheeler pair production

We consider a photon with momentum l_μ which decays into an electron with momentum p and a positron with momentum p' in a constant electric field E along the z axis. The gauge is always $A_3(t) = Et$. Explicitly, we take the photon momentum to be $l_\mu = \Omega(1, \sin\theta, 0, \cos\theta)$ and consider two polarization vectors $\varepsilon^{(\|)} = (0, -\sin\theta, 0, \cos\theta)$ and $\varepsilon^{(\perp)} = (0, 0, 1, 0)$. Since the maximum probability is when the photon is perpendicular to the electric field, we consider the case $\theta = \frac{\pi}{2}$ or $l_\mu = \Omega(1, 1, 0, 0)$. From the amplitude, we obtain the total probability by integrating the modulus squared over outgoing momenta and summed over spins. One momentum integral is solved by the delta function, and we can solve the other with the saddle point method. One can show [104, 105] that the biggest contribution is when $\mathbf{p}' = \mathbf{p} = \mathbf{l}/2$. In order to make things more precise, especially with the expectation of generalizing to spacetime fields, we consider a photon wave packet in the incoming state

$$|l, \varepsilon\rangle_{\text{in}} \;\to\; \int \frac{d^3k}{(2\pi)^3 2k_0} f(k)\epsilon(k) a^\dagger(k)|0\rangle_{\text{in}} \tag{D.60}$$

where $f(k)$ is a probability distribution peaked around $\mathbf{k} = \mathbf{l}$ and normalized as

$$\int \frac{d^3k}{(2\pi)^3 2k_0} |f(k)|^2 = 1\ . \tag{D.61}$$

Choosing a Gaussian wave packet

$$f(k) \sim \exp\left\{ \sum_{j=1}^{3} -\frac{(k_j - l_j)^2}{2\lambda_j^2} + i b^j k_j \right\} \tag{D.62}$$

we get a normalized distribution

$$f(k) = \sqrt{\frac{(2\pi)^3 2k_0}{\pi^{3/2}\lambda_1\lambda_2\lambda_3}} \exp\left\{ \sum_{j=1}^{3} -\frac{(k_j - l_j)^2}{2\lambda_j^2} + i b^j k_j \right\}\ . \tag{D.63}$$

In this chapter we choose the wave packet to be so sharply peaked that we can effectively neglect all wave packet effects. However, as we show in chapter 4, a large wave packet leads to interesting consequences.

Since the field is space-independent, from the spatial integral over $\varphi^i = \frac{1}{2}(x_+^i + x_-^i)$ we obtain a delta function

$$\int d^3\varphi\, e^{i(p_j + p'_j - k_j)\varphi^j} = (2\pi)^3\, \delta(\mathbf{p} + \mathbf{p}' - \mathbf{k}) \tag{D.64}$$

which we can use to perform the k integrals. Then, we assume that the λ_j are so small that

$$\int \frac{d^3p'}{(2\pi)^3} \exp\left\{\sum_{j=1}^{3} -\frac{(k_j - l_j)^2}{\lambda_j^2}\right\} F(\mathbf{p}') \simeq \frac{\lambda_1\lambda_2\lambda_3}{8\pi^{3/2}} F(\mathbf{l} - \mathbf{p}) \tag{D.65}$$

so that at the prefactor (D.65) cancels with the factor of $(2\pi)^3$ and the normalization of $f(k)$ to give in the end

$$P = \frac{e^2}{2\Omega} \sum_{\text{spin}} \int \frac{d^3p}{(2\pi)^3} |M|^2 \tag{D.66}$$

where $\Omega := l_0 = k_0$ and M is the amplitude stripped off of the delta function Defining the amplitude M as

$$(2\pi)^3 \delta(\mathbf{p}' + \mathbf{p} - \mathbf{l}) M := {}_{\text{out}}\langle p, s; p', s' | l, \varepsilon \rangle_{\text{in}} \ . \tag{D.67}$$

Eq. (D.66) is our starting point.

In principle we could find the stimulated pair production amplitude using the LSZ and the exact states

$$\langle 0|0\rangle^2 \lim_{t_\pm \to \infty} \int d^3x_+ d^3x_- d^4x \, e^{ip_i x_+^i + ip'_i x_-^i - ilx} \, \bar{U}_s(t_+, \mathbf{p}) \, G(x_+, x) \, \not{\varepsilon} \, G(x, x_-) \, V_{s'}(t_-, \mathbf{p}') \tag{D.68}$$

simply amputating both propagators to obtain

$$\int d^4x \, e^{-ilx} \, {}_+\bar{\psi}_p(x) \, \not{\varepsilon} \, {}_-\psi'_p(x) \tag{D.69}$$

where ${}_+\psi_{p'}(x)$ and ${}_-\psi_p(x)$ are the exact solutions which represent a single electron and a single positron at ∞, shown in the previous section. The resulting spatial integrals give a delta function, and the time integral can be computed using the saddle point method when $E \ll 1$. Since we use the saddle point method, it actually makes no difference if we use the WKB approximations instead of the exact solutions with parabolic cylinder functions.

Our goal, however, is to use the worldline representation with the photon introduced as a plane wave $A \to A + \varepsilon \, e^{-ilx}$

$$\Gamma(x_+, x_-; l, \varepsilon) = (i\not{D}_{x_+} + 1) \frac{1}{2} \int_0^\infty dT \int_{x_-}^{x_+} \mathcal{D}q \, e^{-i \int_0^1 d\tau \, \frac{\dot{q}^2}{2T} + \frac{T}{2} + A\cdot\dot{q} + \varepsilon\cdot\dot{q}\, e^{-ilq}} \, \mathcal{P} e^{\frac{E}{2} T\gamma^0\gamma^3 - iT\not{\varepsilon}\not{l} \int e^{-ilq}} \Big|_{\text{lin}_\varepsilon} . \tag{D.70}$$

where the ε term from $\not{D}_{x_+}$ gives zero as for plane waves.

D.3.1 Exponent

We want to calculate

$$\lim_{t_\pm\to\infty}\int d^3x_+\, d^3x_-\, e^{ip_ix_+ + ip'_ix_-}\bar{U}_s(t_+,\mathbf{p})\,\gamma^0\,\Gamma(x_+,x_-;l,\varepsilon)\,\gamma^0\,V_{s'}(t_-,\mathbf{p}')\ . \tag{D.71}$$

We begin with the path integral: the instantons now satisfy $\ddot{q}^\mu = TF^{\mu\nu}\dot{q}_\nu + TJ^\mu$ with $J^\mu = l^\mu\,\delta(\tau-\sigma)$, or in components

$$\begin{aligned}\ddot{q}^0 &= ET\dot{q}^3 + Tl^0\,\delta(\tau-\sigma)\\ \ddot{q}^3 &= ET\dot{q}^0 + Tl^3\,\delta(\tau-\sigma)\\ \ddot{q}^\perp &= Tl^\perp\,\delta(\tau-\sigma)\ .\end{aligned} \tag{D.72}$$

The path integral is Gaussian, so evaluating it at the instantons is exact.

We turn to the spatial integrals: changing the variables to $\varphi^i = \frac{1}{2}(x^i_+ + x^i_-)$ and $\theta^i = (x^i_+ - x^i_-)$ we immediately integrate over $d^3\varphi$ to obtain a delta function $(2\pi)^3\delta^3(p+p'-l)$. Since the $d^3\theta$ integral is Gaussian, we can solve it exactly by evaluating the exponent at the saddle point. We redefine the limit $t\to t-p_3/E$ so that p_3 drops from the expression.

As to the last two T and σ, we have to use the saddle point method. As we show in Appendix A, we can find the saddle points for T and σ from the initial exponent (i.e. not evaluated on the other saddle points)

$$\begin{aligned}T^2 &= \dot{q}^2\\ l\cdot\dot{q}(\sigma) &= 0\end{aligned} \tag{D.73}$$

but they are implicit equations; we can obtain a more useful asymptotic expression as follows: using $T^2=\dot{q}^2$, and since t goes to infinity, we must have

$$\dot{t}(\tau) = T\theta_{\tau\sigma}\sqrt{m_\perp^2 + (p_3 - Et(\tau))^2} - T\theta_{\sigma\tau}\sqrt{{m'_\perp}^2 + (p_3 - Et(\tau))^2}\ . \tag{D.74}$$

When $\tau=\sigma$ there is a turning point: comparing the equation above with $\dot{t}(\sigma+\varepsilon^+) - \dot{t}(\sigma-\varepsilon^+) = T\Omega$ we see that

$$\Omega = \sqrt{m_\perp^2 + (p_3 - E\tilde{t})^2} + \sqrt{{m'_\perp}^2 + (p_3 - E\tilde{t})^2} \tag{D.75}$$

where we have defined $\tilde{t} = t(\sigma)$. At the momentum saddle point and shifting t so that p_3 disappears we find a much simpler condition

$$\frac{\Omega}{2} = \sqrt{1 + \frac{\Omega^2}{4} + (E\tilde{t})^2} \to \tilde{t} = \frac{i}{E}\ . \tag{D.76}$$

From (D.74) we find (changing the variable to get rid of p_3)

$$\begin{aligned}T = T\int_0^\sigma + T\int_\sigma^1 &= \int_{\tilde{t}}^t d\bar{t}\,\frac{1}{\sqrt{m_\perp^2 + E^2\bar{t}^2}} + \frac{1}{\sqrt{{m'_\perp}^2 + E^2\bar{t}^2}}\\ &= \frac{\operatorname{arcsinh}\left(\frac{Et}{m_\perp}\right) - i\arcsin\left(\frac{1}{m_\perp}\right)}{E} + \frac{\operatorname{arcsinh}\left(\frac{Et}{m'_\perp}\right) - i\arcsin\left(\frac{1}{m'_\perp}\right)}{E}\ .\end{aligned} \tag{D.77}$$

In particular this tells us that T goes to infinity parallel to the real axis but slightly shifted. This imaginary contribution to T is important because at the end we take the modulus squared so the real phases get canceled.

Notice that, for generic momenta, even if we take the limit along $t = t_+ = t_-$, we do not have $\sigma = \frac{1}{2}$, since from

$$\begin{aligned} T\sigma &= \frac{\operatorname{arcsinh}\left(\frac{Et}{m_\perp}\right) - i \arcsin\left(\frac{m}{m_\perp}\right)}{E} \\ T(1-\sigma) &= \frac{\operatorname{arcsinh}\left(\frac{Et}{m'_\perp}\right) - i \arcsin\left(\frac{m}{m'_\perp}\right)}{E} \end{aligned} \tag{D.78}$$

we have $T\sigma = T(1-\sigma)$, hence $\sigma = \frac{1}{2}$, if and only if $m_\perp = m'_\perp$. This implies that, when evaluating the exponent at the saddle points for T and σ but before the momentum integrals, we cannot simply set $\sigma = \frac{1}{2}$.

Writing the exponent explicitly as a function of T and σ gives a complicated expression, but we only need to consider the asymptotic limit $T \to \infty$. Furthermore, when we integrate over perpendicular momenta, we find a saddle point at $p_\perp = p'_\perp = \frac{l_\perp}{2}$; at the probability level we have the exponent

$$\boxed{\frac{2}{E}\left(\frac{\Omega}{2} - \left(1 + \frac{\Omega^2}{4}\right) \arctan\left(\frac{2}{\Omega}\right)\right)} \tag{D.79}$$

which is the result obtained in [70].

D.3.2 Prefactor

As for the path integral, since the current is zero almost everywhere[2], we solve the Lorentz force equations in the regions $[0, \sigma)$ and $(\sigma, 1]$ separately and match the solutions at $\tau = \sigma$; we assume for simplicity $l_3 = 0$. Since $\ddot{q}^\mu$ contains a delta function, $\dot{q}^\mu$ makes a jump $\dot{q}^\mu(\tau + \varepsilon^+) - \dot{q}^\mu(\tau - \varepsilon^+) = T\, l^\mu$ and q^μ is continuous. The path integral normalization is unchanged with respect to the vacuum case since only the instantons change but the kinetic operator does not depend on them

$$\int_{x_-}^{x_+} \mathcal{D}q \to \frac{1}{(2\pi T)^2} \frac{ET/2}{\sinh\left(ET/2\right)} \, . \tag{D.80}$$

The spatial variables give

$$\int d^3\phi \, d^3\theta \longrightarrow (2\pi)^3 \delta(\mathbf{p}' + \mathbf{p} - \mathbf{l}) \; 2\pi T \sqrt{\frac{4\pi \tanh\left(\frac{ET}{2}\right)}{E}} \tag{D.81}$$

whereas the Hessian matrix for T and σ gives a very complicated expression for general time and momenta, but in the asymptotic limit and on the momentum saddle point

$$\int dT d\sigma \;\longrightarrow\; \frac{2\sqrt{2}\,\pi}{ET\left(i + \omega/2\right)\sqrt{\omega}} \, . \tag{D.82}$$

[2] In a measure-theoretic sense, the single point $\tau = \sigma$ has measure zero in the open interval $[0, 1]$.

At the prefactor the term $(i\not{D}_+ + 1)$ can be simplified by integrating the spatial derivatives by parts and letting ∂_{t_+} act on the exponent, giving us

$$i\partial_{t_+} e^{-i\int(\dots)} = E \int \dot{z}(\tau)\,\tau = \frac{\dot{t}(1)}{T} = p_0' \tag{D.83}$$

where after taking the derivative we set $t = t_+$ and integrate by parts the τ integral.

As to the spin factor, we have a scalar addendum

$$-i\varepsilon \cdot \dot{q}\; e^{\frac{ET}{2}\gamma^0\gamma^3} \tag{D.84}$$

and a spinor one

$$\mathcal{P}e^{\frac{ET}{2}\gamma^0\gamma^3 - \frac{iT}{2}\not{l}\not{\varepsilon}} = -\frac{iT}{2}\; e^{\frac{ET}{2}\gamma^0\gamma^3(1-\sigma)}\; \not{l}\not{\varepsilon}\; e^{\frac{ET}{2}\gamma^0\gamma^3\sigma}\;. \tag{D.85}$$

For the scalar one we simply have $-i\varepsilon \cdot \dot{q}(\sigma) = T\varepsilon_3$ and

$$\frac{1}{m_\perp}\bar{R}_i(p_\perp\gamma^\perp + 1)\gamma^0(i\not{D}_y + 1)(\dots)e^{\frac{ET}{2}\gamma^0\gamma^3} R_j \;\rightarrow\; \frac{1}{m_\perp} R_i^\dagger R_j\, m_\perp^2\, e^{\frac{ET}{2}} \tag{D.86}$$

whereas for the spinor part we have

$$\begin{aligned} -\frac{iT}{2}\frac{1}{m_\perp}\bar{R}_i(p_\perp\gamma^\perp + 1)\gamma^0(i\not{D}_y + 1)(\dots)e^{\frac{ET}{2}\gamma^0\gamma^3(1-\sigma)}\; \not{l}\not{\varepsilon}\; e^{\frac{ET}{2}\gamma^0\gamma^3\sigma} R_j \;\rightarrow \\ -\frac{iT\, e^{\frac{ET}{2}}}{2}\frac{1}{m_\perp} R_i^\dagger\Big(\gamma^0(p_\perp\gamma^\perp + 1)\Big(i + \frac{\omega}{2}\Big) + m_\perp^2\Big)\; \not{l}\not{\varepsilon}\; R_j \end{aligned} \tag{D.87}$$

where in the last line we have inverted the instanton relation (D.77)

$$2Et = \Big(i + \frac{\omega}{2}\Big)e^{\frac{ET}{2}}\,. \tag{D.88}$$

At this point one can see that all functions of T cancel in prefactor, so that the limit is finite.

Defining

$$M_{ij} := \frac{1}{m_\perp} R_i^\dagger\Big[-\frac{i}{2}\Big(\gamma^0(p_\perp\gamma^\perp + 1)\Big(i + \frac{\omega}{2}\Big) + m_\perp^2\Big)\; \not{l}\not{\varepsilon} + m_\perp^2\Big] R_j \tag{D.89}$$

we have the following sum over spins

$$\sum_{i,j} |M_{ij}|^2 = 2m_\perp^2\,. \tag{D.90}$$

Finally, integrating over momenta yields a contribution

$$\int dp_\perp \;\rightarrow\; \frac{\pi E}{2\sqrt{\arctan\left[\frac{1}{p}\right]\left(\arctan\left[\frac{1}{p}\right] - \frac{p}{1+p^2}\right)}} \tag{D.91}$$

for the transverse momenta, while since the integrand does not depend on p_3 we get a volume factor

$$\int dp_3 \;\rightarrow\; EV_0\,. \tag{D.92}$$

Putting everything together

$$
\begin{aligned}
P^{(\|)} &= \frac{\alpha E V_0}{4p\omega} \frac{e^{\frac{2m^2}{E}\left(p-(1+p^2)\arctan\left[\frac{1}{p}\right]\right)}}{\sqrt{\arctan\left[\frac{1}{p}\right]\left(\arctan\left[\frac{1}{p}\right]-\frac{p}{1+p^2}\right)}} \\
P^{(\perp)} &= \frac{2p^2}{1+p^2} P^{(\|)}.
\end{aligned}
\tag{D.93}
$$

where $p = \frac{\Omega}{2}$. The results agree with [70].

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe. Die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Diese Dissertation wurde am Helmholtz-Zentrum Dresden-Rossendorf unter der wissenschaftlichen Betreuung von Herrn Prof. Dr. Ralf Schützhold angefertigt. Ich habe bisher an keiner Institution, weder im Inland noch im Ausland, einen Antrag auf Eröffnung eines Promotionsverfahrens gestellt. Ferner erkläre ich, dass ich die Promotionsordnung der Fakultät Mathematik und Naturwissenschaften der Technischen Universität Dresden vom 23.02.2011 anerkenne.

Dresden, 19.07.2024 Gianluca Degli Esposti