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Padovan and Perrin hyperbolic spinors

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Abstract

In this study, we intend to bring together Padovan and Perrin number sequences, which are one of the most popular third-order recurrence sequences, and hyperbolic spinors, which are used in several disciplines from physics to mathematics, with the help of the split quaternions. This paper especially improves the relationship between hyperbolic spinors, both a physical and mathematical concept, and number theory. For this aim, we combine the hyperbolic spinors and Padovan and Perrin numbers concerning the split Padovan and Perrin quaternions, and we determine two new special recurrence sequences named Padovan and Perrin hyperbolic spinors. Then, we give Binet formulas, generating functions, exponential generating functions, Poisson generating functions, and summation formulas. Additionally, we present some matrix and determinant equations with respect to them. Besides, we construct some special equations that give relations between Padovan and Perrin hyperbolic spinors and Padovan and Perrin numbers. Further, we give a short introduction for (s,t)-Padovan and (s,t)-Perrin hyperbolic spinors in order to shed light on future studies.

Keywords Hyperbolic spinors · Padovan numbers · Perrin numbers · Split Padovan quaternions · Split Perrin quaternions

Mathematics Subject Classification 11B37 · 11K31 · 11R52 · 11Y55 · 15A66

1 Introduction

Among the innumerable concepts associated with mathematics and physics, spinors draw attention as a basic and extensively studied topic. One of the most important notions is spinors, which researchers study from mathematics to physics. Cartan introduced the notion of the

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spinor in 1913. In spite of the fact that the term "spinor" was coined by Ehrenfest in the 1920s, the intrinsic notion of a spinor is much older than that, as a spinor is a special linear structure that had been studied (and had been used in civil engineering in order to calculate statics) long before Ehrenfest used it in quantum theory Vaz and da Rocha (2016). Spinors are quite an important concept for quantum mechanics and, therefore, modern physics as a whole. Nearly at the starting point of quantum theory, in 1927, physicists Pauli and Dirac obtained spinors to express wave function, with the former for three-dimensional space and the latter for four-dimensional space-time. When studying the representation of groups in mathematics, Cartan found spinors and explained how spinors supply a linear depiction of a space's rotations in any dimension. Because of the fact that spinors are closely related to geometry, their presentation is commonly abstract and without any clear geometric interpretation Hladik (1999). The spinors in a geometrical sense are examined by Cartan Cartan (1966). Thanks to the study Cartan (1966), the set of isotropic vectors of the vector space $\mathbb{C}^3$ establishes a two-dimensional surface in the two-dimensional complex space $\mathbb{C}^2$. Conversely, these vectors in $\mathbb{C}^2$ represent the same isotropic vectors. Cartan expressed that these vectors are complex as two-dimensional in the space $\mathbb{C}^2$ Cartan (1966); Erişir (2021). In addition to these, the triads of unit vectors are orthogonal by twos and were expressed in terms of a single vector that has two complex components, which is called a spinor Cartan (1966); Torres del Castillo (2003); Torres del Castillo and Barrales (2004). To get more detailed information with respect to the spinors, we refer to the studies Vaz and da Rocha (2016); Cartan (1966); Torres del Castillo (2003); Torres del Castillo and Barrales (2004); Brauer and Weyl (1935); Lounesto (1986).

Torres del Castillo and Barrales gave the spinor representations of the Frenet-Serret frame in Torres del Castillo and Barrales (2004), which is a milestone for several researchers in order to construct the representations of moving frames associated with the spinors. Also, the spinor representations of involute-evolute curves Erişir and Kardağ (2019), Bertrand curves Erişir (2021), and successor curves Erişir and Öztaş (2022) were examined. Then, Doğan Yazıcı et al. Doğan Yazıcı et al. (2022) introduced the spinor representation of Mannheim framed curves, which can have singular points, and İşbilir et al. İşbilir et al. (2023a) determined the spinor representation of Bertrand framed curves. Then, İşbilir et al. İşbilir et al. (2023b) investigated the spinor representation of framed curves in 3-dimensional Lie groups.

Moreover, hyperbolic spinors, which are another type of spinor, were scrutinized and combined with the different and several frames. Balcı et al. determined the hyperbolic spinor representation of space-like curves associated with the Darboux frame in Minkowski space Balcı et al. (2015). Erişir et al. introduced the hyperbolic spinor equations of alternative frame Erişir et al. (2015). Also, Ketenci et al. investigated the spinor equations of curves in Minkowski space Ketenci et al. (2014), and gave the construction of hyperbolic spinors related to Frenet frame in Minkowski space Ketenci et al. (2015).

Number theory is a well-established and important topic for lots of disciplines, such as computer systems, engineering, architecture, and others. Several studies have been undertaken and are still in progress with respect to numbers and number systems. Quaternions are among the most important concepts in the number system. Quaternions were investigated by Hamilton in 1843 to extend the complex numbers, and the quaternion algebra is associative, non-commutative, and 4-dimensional Clifford algebra. The set of quaternions (real/Hamilton type) is denoted by $\mathbb{H}$ and defined as $\mathbb{H} = \{q \mid q = q_0 + q_1 i + q_2 j + q_3 k,\ q_0, q_1, q_2, q_3 \in \mathbb{R}\}$, where i, j, k are real quaternionic units that satisfy the multiplication rules Hamilton (1844, 1853, 1969):

$$\begin{aligned} i^2 = j^2 = k^2 = -1, \\ ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \end{aligned} \tag{1}$$

Table 1 Some values of Padovan and Perrin numbers

n	...	-5	-4	-3	-2	-1	0	1	2	3	4	5	...
P_n	...	1	0	0	1	0	1	1	1	2	2	3	...
R_n	...	4	-3	2	1	-1	3	0	2	3	2	5	...

The literature includes many works concerning quaternions, and many researchers have examined them and investigated other types of quaternions. Additionally, Cockle introduced the split quaternions Cockle (1849). The set of split quaternions is denoted by $\mathbb{H}_S$ and defined as $\mathbb{H}_S = \{q \mid q = q_0 + q_1 i + q_2 j + q_3 k,\ q_0, q_1, q_2, q_3 \in \mathbb{R}\}$, where i, j, k are split quaternionic units satisfy the following rules Cockle (1849); Dişkaya and Menken (2019b):

$$\begin{gathered} i^2 = -1, \quad j^2 = k^2 = 1, \\ ij = -ji = k, \quad jk = -kj = -i, \quad ki = -ik = j. \end{gathered} \tag{2}$$

On the other hand, special recurrence sequences are one of the most attractive concepts for researchers, and a great number of papers have been completed and ongoing with respect to them. If one examines the literature, it can be seen that there are lots of special sequences with different orders. In this study, we intend to examine the third-order recurrence sequences named Padovan and Perrin numbers, which are the special types of generalized Tribonacci numbers Cerda-Morales (2017a); Soykan (2020a). The Padovan sequence is a recurrence sequence of integers, and the n^{th} Padovan number is represented by P_n and satisfies the following recurrence relation:

$$P_{n+3} = P_{n+1} + P_n \quad \text{for all} \quad n \geq 0, \tag{3}$$

with the initial values $P_0 = P_1 = P_2 = 1$ Soykan (2023). Also, the Perrin sequence is a recurrence sequence of integers, and the n^{th} Perrin number is represented by R_n and satisfies the following recurrence relation:

$$R_{n+3} = R_{n+1} + R_n \quad \text{for all} \quad n \geq 0, \tag{4}$$

with the initial values $R_0 = 3, R_1 = 0, R_2 = 2$ Soykan (2023). In the existing literature, the recurrence relations of Padovan and Perrin numbers can also be written as $P_n = P_{n-2} + P_{n-3}$ and $R_n = R_{n-2} + R_{n-3}$ for all $n \geq 3$ Sokhuma (2013a, b), respectively. Special generalizations for Padovan and Perrin numbers are studied, named (s, t)-Padovan and (s, t)-Perrin numbers. For $s > 0$, $t \neq 0$ and $27t^2 - 4s^3 \neq 0$, (s, t)-Padovan and (s, t)-Perrin numbers satisfy the recurrence relations $\mathscr{P}_{n+3}(s, t) = s\mathscr{P}_{n+1}(s, t) + t\mathscr{P}_n(s, t)$, where $\mathscr{P}_0(s, t) = 0$, $\mathscr{P}_1(s, t) = 1$, $\mathscr{P}_2(s, t) = 0$ and $\mathscr{R}_{n+3}(s, t) = s\mathscr{R}_{n+1}(s, t) + t\mathscr{R}_n(s, t)$, where $\mathscr{R}_0(s, t) = 3$, $\mathscr{P}_1(s, t) = 0$, $\mathscr{P}_2(s, t) = 2s$ Cerda-Morales (2017b); Dişkaya and Menken (2019a, b). Moreover, Padovan and Perrin numbers can be extended to negative subscripts as follows Soykan (2023):

$$P_{-n} = P_{-(n-3)} - P_{-(n-1)} \quad \text{for all} \quad n > 0$$

and

$$R_{-n} = R_{-(n-3)} - R_{-(n-1)} \quad \text{for all} \quad n > 0,$$

respectively.

It should be noted that, throughout this study, we are interested in the nonnegative subscripted Padovan and Perrin numbers. Negative indices are only mentioned in some cases to make the indices easier to see and understand.

Fibonacci and Lucas numbers are one of quite popular second-order special number sequences, and one of the concepts that makes these sequences special is the golden ratio. Padovan and Perrin numbers are one of the most interesting third-order number sequences. The reason why these special sequences are so popular and interesting is due to their theoretical properties as well as their appearance in various application areas, especially in architecture and nature. Similar to the concept of the golden ratio in Fibonacci numbers, there is the concept of the plastic ratio in the Padovan and Perrin numbers. While the ratio of two successive Fibonacci numbers converges to the *golden ratio*[1] Dunlap (1997), and the ratio of two successive Padovan or Perrin numbers converges to the *plastic ratio*[2] Shannon et al. (2006a); Soykan (2023); Padovan (1994, 2002). These number sequences have been and are being studied with quaternions in many studies. In this study, we study Padovan and Perrin numbers together with hyperbolic spinors.

In addition to these, several researchers have brought together the special recurrence sequences and special type quaternions in the existing literature and in progress. Cerda-Morales determined the real quaternions with generalized Tribonacci numbers components in Cerda-Morales (2017a). Also, Taşcı defined the real quaternions with Padovan and Pell-Padovan numbers components in Taşcı (2018). Günay and Taşkara studied some properties of Padovan real quaternions in Günay and Taşkara (2019). Then, the generalized Padovan sequence, which is a subgroup of the generalized Tribonacci family, was taken, and real-type Padovan, Perrin, and Van der Laan quaternions were determined in Günay (2019). Also, Dişkaya and Menken defined the real quaternions with (s,t)-Padovan and (s,t)-Perrin numbers components Dişkaya and Menken (2019a), and split quaternions with (s,t)-Padovan and (s,t)-Perrin numbers components Dişkaya and Menken (2019b). The n^{th} split (s,t)-Padovan and split (s,t)-Perrin quaternion are denoted by $\breve{\mathscr{P}}_n$ and $\breve{\mathscr{R}}_n$ and also determined as follows:

$$\breve{\mathscr{P}}_n = \mathscr{P}_n + \mathscr{P}_{n+1}i + \mathscr{P}_{n+2}j + \mathscr{P}_{n+3}k$$

and

$$\breve{\mathscr{R}}_n = \mathscr{R}_n + \mathscr{R}_{n+1}i + \mathscr{R}_{n+2}j + \mathscr{R}_{n+3}k,$$

where $\mathscr{P}_n$ and $\mathscr{R}_n$ are the n^{th} (s,t)-Padovan and (s,t)-Perrin numbers and i, j, k are the split quaternionic units that satisfy the multiplication rules given in the Eq. (2). Then, the following recurrence relations are expressed for split (s,t)-Padovan and (s,t)-Perrin quaternions:

$$\breve{\mathscr{P}}_{n+3} = s\breve{\mathscr{P}}_{n+1} + t\breve{\mathscr{P}}_n \quad \text{for all} \quad n \geq 0$$

and

$$\breve{\mathscr{R}}_{n+3} = s\breve{\mathscr{R}}_{n+1} + t\breve{\mathscr{R}}_n \quad \text{for all} \quad n \geq 0.$$

The conjugate of the n^{th} (s,t)-Padovan and (s,t)-Perrin split quaternion is denoted by $\breve{\mathscr{P}}^*_n = \mathscr{P}_n - \mathscr{P}_{n+1}i - \mathscr{P}_{n+2}j - \mathscr{P}_{n+3}k$ and $\breve{\mathscr{R}}^*_n = \mathscr{R}_n - \mathscr{R}_{n+1}i - \mathscr{R}_{n+2}j - \mathscr{R}_{n+3}k$, respectively Dişkaya and Menken (2019b); Akyiğit et al. (2013). Also, Dişkaya and Menken Dişkaya and Menken (2019b) determined some properties, formulas, and equations with respect to (s,t)-Padovan and (s,t)-Perrin split quaternions. One can see that if $s = t = 1$,

[1] This ratio $((1+\sqrt{5})/2)$ plays a significant role in the growth of many biological systems, which can be detailed such as symmetry characters (from a flower, seed pattern on a sunflower to starfish and also capsomer of a virus exhibiting icosahedral symmetry), optimal spacing, Fibonacci growth spirals, architecture, and innumerable natural applications Dunlap (1997).

[2] This ratio (cf. α in Eq. (9)) was determined by Cordonnier in 1924 and has many uses and applications, from architecture to mathematics. This special ratio is also used in constructing chromatic scale temperament and music Dişkaya and Menken (2021); Marohnić and Strmečki (2012).

then split quaternions with Padovan and Perrin numbers components are obtained Dişkaya and Menken (2019b). Throughout this article, for the sake of clarity, we use these notations $\breve{P}_n$ and $\breve{R}_n$ for $s = t = 1$ with respect to the Padovan split quaternions and Perrin split quaternions, respectively. According to this, n^{th} split (s, t)-Padovan quaternion $\breve{P}_n$ and n^{th} split (s, t)-Perrin quaternion $\breve{R}_n$ are expressed as follows:

$$\breve{P}_n = P_n + P_{n+1}i + P_{n+2}j + P_{n+3}k$$

and

$$\breve{R}_n = R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k.$$

Additionally, generalized quaternions (which include real, split, semi, split semi, 1/4 quaternions) with Padovan and Perrin numbers components were given by İşbilir and Gürses in İşbilir and Gürses (2022). Also, (s, t)-Padovan and (s, t)-Perrin, and (s, t)-Lucas-Padovan (Ludovan) quaternions and matrix sequences Lee (2022); Vieira et al. (2020).

On the other hand, Vivarelli Vivarelli (1984) determined the relation between the spinors and quaternions, and with respect to the relation between quaternions and rotations in Euclidean 3-space, the spinor representations of these 3-dimensional rotations were given Erişir (2024). Tarakçıoğlu et al. investigated the relations between the hyperbolic spinors and split quaternions in Tarakçıoğlu et al. (2018). Recently, Erişir and Güngör introduced the Fibonacci and Lucas spinors in Erişir and Güngör (2020), and Erişir determined the Horadam spinors in Erişir (2024). Moreover, Kumari et al. defined the k-Fibonacci and k-Lucas spinors in Kumari et al. (2023), and Leonardo spinors in Kumari et al. (2026). Dişkaya and Menken introduced the Padovan and Perrin spinors Dişkaya (2024). Then, Cerda-Morales determined the generalized Tribonacci spinors Cerda-Morales (2024). Özçevik and Dertli determined the hyperbolic Jacobsthal spinor sequences in Özçevik and Dertli (2024).

In this paper, we strengthen the relationships of hyperbolic spinors and special recurrence sequences by using split quaternions. For this purpose, we determine the Padovan and Perrin hyperbolic spinors and examine some properties of them. Also, we present Binet formulas, generating functions, exponential generating functions, Poisson generating functions, and summation formulas. Moreover, we obtain some matrix and determinant equations concerning them. Then, we give a short introduction for (s, t)-Padovan hyperbolic spinors and (s, t)-Perrin hyperbolic spinors. These new sequences include the Padovan and Perrin hyperbolic spinors for the values of s and t. Consequently, we give conclusions and express our intention on how we can take this work to an even higher level in the future.

2 Basic concepts

In this section, we remind some required notions and notations with respect to the used concept throughout this study such as; hyperbolic spinors, split quaternions, and Padovan and Perrin numbers.

2.1 Hyperbolic spinors

Suppose that ε is an $n \times n$ matrix that is defined on the hyperbolic number system $\mathbb{H}$. $\varepsilon^{\dagger}$ defined as transposing and conjugating of ε, that is $\varepsilon^{\dagger} = \overline{\varepsilon^t}$, which is an $n \times n$ matrix. Provided that ε is a Hermitian matrix with respect to $\mathbb{H}$, then $\varepsilon^t = \varepsilon$. Also, if ε is an anti-Hermitian matrix with respect to $\mathbb{H}$, then $\varepsilon^t = -\varepsilon$. Let ε be a Hermitian matrix, the equation $UU^{\dagger} = U^{\dagger}U = 1$ is valid for $U = e^{j\varepsilon}$. The set of all $n \times n$ type matrices on $\mathbb{H}$ which satisfies the previous equation establishes a group called hyperbolic unitary group, and denoted by $U(n,\mathbb{H})$. If $\det U = 1$, then this type group is represented by $SU(n,\mathbb{H})$ Antonuccio (1998); Balcı et al. (2015); Erişir et al. (2015).

Additionally, Lorentz group is a group of all Lorentz transformations in the Minkowski space and it is a subgroup of the Poincaré group. Moreover, Poincaré group is determined as the group of all isometries in the Minkowski space. The term "orthochronous" is a Lorentz transformation that is kept in the direction of time. Then, the orthochronous Lorentz group is defined as that rigid transformation of Minkowski 3-space that kept both the direction of time and orientation. If they have the determinant $+1$, then this subgroup is represented as $SO(1,3)$ Carmeli (1977); Balcı et al. (2015); Erişir et al. (2015); Ketenci et al. (2015).

Further, there is a homomorphism between the group $SO(1,3)$, which is the group of the rotation along the origin, and $SU(2,\mathbb{H})$, which is the group of the unitary 2×2 type matrix. While the elements of the group $SU(2,\mathbb{H})$ present a fillip to the hyperbolic spinors, the elements of the group $SO(1,3)$ give a fillip to the vectors with three real components in Minkowski space Sattinger and Weaver (1986); Balcı et al. (2015); Erişir et al. (2015); Ketenci et al. (2015).

One can represent a hyperbolic spinor with two hyperbolic components as follows:

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

by using the vectors $a, b, c \in \mathbb{R}^3_1$ such that

$$\begin{aligned} a + jb &= \psi^t \sigma \psi, \\ c &= -\widehat{\psi}^t \sigma \psi, \end{aligned} \tag{5}$$

where "t" denotes the transposition, $\overline{\psi}$ is the conjugate of ψ, $\widehat{\psi}$ is the mate of ψ. Also, the followings can be expressed:

$$\widehat{\psi} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \overline{\psi} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \overline{\psi}_1 \\ \overline{\psi}_2 \end{bmatrix} = \begin{bmatrix} -\overline{\psi}_2 \\ \overline{\psi}_1 \end{bmatrix}.$$

Also, 2×2 hyperbolic symmetrimatrices,es which are cartesian components for the vector $\varsigma = (\varsigma_1, \varsigma_2, \varsigma_3)$

$$\varsigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \varsigma_2 = \begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}, \quad \varsigma_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{6}$$

are written Balcı et al. (2015); Erişir et al. (2015); Tarakçıoğlu et al. (2018); Ketenci et al. (2015, 2014). The ordered triads $\{a, b, c\}$, $\{b, c, a\}$, $\{c, a, b\}$ correspond to different hyperbolic spinors, and the hyperbolic spinors ψ and $-\psi$ correspond to the same ordered orthogonal basis. For the hyperbolic spinors ψ and ϕ, the following equations hold:

$$\begin{aligned} \psi^t \sigma \phi &= \phi^t \varsigma \psi, \\ \overline{\psi^t \sigma \phi} &= -\widehat{\psi}^t \varsigma \widehat{\phi}, \\ (\widehat{\nu_1 \psi + \nu_2 \phi}) &= \overline{\nu}_1 \widehat{\psi} + \overline{\nu}_2 \widehat{\phi}, \end{aligned}$$

where $\nu_1, \nu_2 \in \mathbb{H}$ Balcı et al. (2015); Ketenci et al. (2015); Erişir et al. (2015); Ketenci et al. (2014). Let $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{H}^3$ be an isotropic vector (namely, length of this vector is zero: $\langle \xi, \xi \rangle = 0, \xi \neq 0$) in $\mathbb{R}^3_1$. According to the above notions and notations, the following equations can be given:

$$\xi_1 = \eta_1^2 - \eta_2^2, \quad \xi_2 = j(\eta_1^2 + \eta_2^2), \quad \xi_3 = -2\eta_1\eta_2.$$

Also, the following equations

$$\psi_1 = \pm\sqrt{\frac{\xi_1 + j\xi_2}{2}} \quad \text{and} \quad \psi_2 = \pm\sqrt{\frac{-\xi_1 + j\xi_2}{2}}$$

can be given. In that case, $||a|| = ||b|| = ||c|| = \overline{\psi}^t \psi$. According to the (5) and (6), the followings

$$\xi_1 = \psi^t \sigma_1 \psi = \psi_1^2 - \psi_2^2, \quad \xi_2 = \psi^t \sigma_2 \psi = j(\eta_1^2 + \eta_2^2), \quad \xi_3 = \psi^t \sigma_3 \psi = -2\psi_1\psi_2$$

and

$$\begin{aligned} a + jb &= \left(\psi_1^2 - \psi_2^2, j(\psi_1^2 + \psi_2^2), -2\psi_1\psi_2\right), \\ c &= \left(\psi_1\overline{\psi}_2 + \overline{\psi}_1\psi_2, j(\psi_1\overline{\psi}_2 - \overline{\psi}_1\psi_2), |\psi_1|^2 - |\psi_2|^2\right), \end{aligned}$$

can be written Balcı et al. (2015); Ketenci et al. (2015); Erişir et al. (2015). For more detailed information with respect to the hyperbolic spinor (especially related to hyperbolic spinors and moving frames), we want to refer to the studies Balcı et al. (2015); Ketenci et al. (2015); Tarakçıoğlu et al. (2018); Erişir et al. (2015).

2.2 Split quaternions

The split quaternion $q \in \mathbb{H}_S$ can be written as $q = S_q + \overrightarrow{V}_q$, where $S_q = q_0$ is scalar part and $\overrightarrow{V}_q = q_1 i + q_2 j + q_3 k$ is vector part. For the split quaternions $q, p \in \mathbb{H}_S$ with respect to the Eq. (2), some algebraic properties can be expressed as follows Akyiğit et al. (2013); Dişkaya and Menken (2019b); Hacisalihoğlu (1983); Özdemir (2020):

- **Addition/Subtraction:**

$$q \pm p = q_0 \pm p_0 + (q_1 \pm p_1)\, i + (q_2 \pm p_2)\, j + (q_3 \pm p_3)\, k.$$

- **Multiplication by a scalar:**

$$\omega q = \omega q_0 + \omega q_1 i + \omega q_2 j + \omega q_3 k, \quad \omega \in \mathbb{R}.$$

- **Multiplication:**

$$\begin{aligned} qp &= (q_0 + q_1 i + q_2 j + q_3 k)\,(p_0 + p_1 i + p_2 j + p_3 k) \\ &= (q_0 p_0 - q_1 p_1 + q_2 p_2 + q_3 p_3) + (q_0 p_1 - q_1 p_0 - q_2 p_3 + q_3 p_2)\, i \\ &\quad + (q_0 p_2 - q_1 p_3 + q_2 p_0 + q_3 p_1)\, j + (q_0 p_3 - q_1 p_2 - q_2 p_1 + q_3 p_0)\, k \\ &= S_q S_p + g\left(\overrightarrow{V}_q, \overrightarrow{V}_p\right) + S_q \overrightarrow{V}_p + S_p \overrightarrow{V}_q + \overrightarrow{V}_q \wedge \overrightarrow{V}_p, \end{aligned}$$

  where

$$g\left(\overrightarrow{V}_q, \overrightarrow{V}_p\right) = -q_0 p_0 + q_1 p_1 + q_2 p_2 + q_3 p_3$$

  and

$$\overrightarrow{V}_q \wedge \overrightarrow{V}_p = \begin{vmatrix} -i & j & k \\ q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{vmatrix}.$$

- **Conjugate:** The conjugate of the split quaternion q is $q^* = q_0 - q_1 i - q_2 j - q_3 k$.
- **Norm:** The norm of q is: $N_q = qq^* = q_0^2 + q_1^2 - q_2^2 - q_3^2$.

2.3 Relations between the hyperbolic spinors and split quaternions

In Tarakçıoğlu et al. (2018); Tarakçıoğlu (2018), the relations between the hyperbolic spinors and split quaternions were examined. Let the split quaternion $q \in \mathbb{H}_S$ and the hyperbolic spinor ψ be given, then we have Tarakçıoğlu et al. (2018); Tarakçıoğlu (2018):

$$\begin{aligned} f &: \mathbb{H}_S \to \mathbb{S} \\ q &\to f(q_0 + q_1 i + q_2 j + q_3 k) = \begin{bmatrix} q_0 + q_3 j \\ -q_1 + q_2 j \end{bmatrix} \equiv \psi_n, \end{aligned} \tag{7}$$

where the function f is linear, one-to-one, and onto. Hence, $f(q+p) = f(q) + f(p)$ and $f(\omega q) = \omega f(q)$, where $\omega \in \mathbb{R}$ and $kerf = \{0\}$. According to the conjugation of the split quaternion q, the following is satisfied Tarakçıoğlu et al. (2018); Tarakçıoğlu (2018):

$$f(q^*) = f(q_0 - q_1 i - q_2 j - q_3 k) = \begin{bmatrix} q_0 - q_3 j \\ q_1 - q_2 j \end{bmatrix} \equiv \psi_n^*. \tag{8}$$

For more detailed information with respect to the relations and representations between the hyperbolic spinors and split quaternions, see Tarakçıoğlu et al. (2018); Tarakçıoğlu (2018).

2.4 Padovan and Perrin numbers

The characteristic equation of the Padovan and Perrin numbers is $x^3 - x - 1 = 0$ and the roots of it are as follows Soykan (2023):

$$\begin{cases} \alpha = \sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}} \approx 1.3247\ldots, \\ \beta = -\sqrt[3]{\frac{1}{16} + \frac{1}{48}\sqrt{\frac{23}{3}}} - \sqrt[3]{\frac{1}{16} - \frac{1}{48}\sqrt{\frac{23}{3}}} + \frac{i\sqrt{3}}{2}\left(\sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} - \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}}\right), \\ \gamma = -\sqrt[3]{\frac{1}{16} + \frac{1}{48}\sqrt{\frac{23}{3}}} - \sqrt[3]{\frac{1}{16} - \frac{1}{48}\sqrt{\frac{23}{3}}} - \frac{i\sqrt{3}}{2}\left(\sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} - \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}}\right), \end{cases} \tag{9}$$

where $\alpha + \beta + \gamma = 0$, $\alpha\beta + \alpha\gamma + \beta\gamma = -1$, and $\alpha\beta\gamma = 1$. In addition to these, for all $n \geq 0$, Binet formulas of Padovan and Perrin numbers are given as follows, respectively Cerda-Morales (2017b); Dişkaya and Menken (2019b); Yılmaz (2015):

$$P_n = \sigma_1 \alpha^n + \sigma_2 \beta^n + \sigma_3 \gamma^n \tag{10}$$

and

$$R_n = \alpha^n + \beta^n + \gamma^n$$

where

$$\sigma_1 = \frac{(\beta - 1)(\gamma - 1)}{(\alpha - \beta)(\alpha - \gamma)}, \quad \sigma_2 = \frac{(\alpha - 1)(\gamma - 1)}{(\beta - \alpha)(\beta - \gamma)}, \quad \sigma_3 = \frac{(\alpha - 1)(\beta - 1)}{(\gamma - \alpha)(\gamma - \beta)}.$$

Then, for all $n \geq 0$, the following relations between the Padovan and Perrin numbers hold Yılmaz (2015); Yılmaz and Taşkara (2013):

$$R_n = 3P_{n-5} + 2P_{n-4}, \tag{11}$$

$$P_{n-1} = \frac{1}{23}\left(R_{n-3} + 8R_{n-2} + 10R_{n-1}\right). \tag{12}$$

On the other hand, with the same initial conditions, the n^{th} power of 3×3 Padovan matrix Q was given as for all $n \in \mathbb{N}$ Yılmaz (2015); Yılmaz and Taşkara (2013):

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad Q^n = \begin{bmatrix} P_{n-5} & P_{n-3} & P_{n-4} \\ P_{n-4} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-1} & P_{n-2} \end{bmatrix}. \tag{13}$$

In addition to these, Yılmaz and Taşkara constructed the following equations with respect to the Padovan and Perrin numbers Yılmaz (2015); Yılmaz and Taşkara (2013):

$$\begin{cases} P_{m-3}P_{n-3} + P_{m-1}P_{n-2} + P_{m-2}P_{n-1} = P_{m+n-1}, & \text{(14a)} \\ P_{m-3}R_{n-3} + P_{m-1}R_{n-2} + P_{m-2}R_{n-1} = R_{m+n-1}, & \text{(14b)} \\ R_{m-3}R_{n-3} + R_{m-1}R_{n-2} + R_{m-2}R_{n-1} = 4P_{m+n-5} + 4P_{m+n-8} & \text{(14c)} \\ \qquad + P_{m+n-11}, & \\ R_{m-3}R_{n-3} + R_{m-1}R_{n-2} + R_{m-2}R_{n-1} = 2R_{m+n-3} + R_{m+n-6}. & \text{(14d)} \end{cases}$$

For more detailed information with respect to the matrix representations and matrix sequences of Padovan and Perrin numbers, we refer to the studies dos Santos Mangueira et al. (2020); Yılmaz (2015); Yılmaz and Taşkara (2013).

Also, Sokhuma examined the Padovan Q-matrix, and gave some relations in Sokhuma (2013a, b). Sokhuma presented the following equations for Padovan and Perrin numbers with initial conditions $P_0 = 0$, $P_1 = 0$, $P_2 = 1$, the Padovan matrix is identical to the matrix given in Eq. (13) and Q^n is written as for all $n \geq 3$ Sokhuma (2013a):

$$Q^n = \begin{bmatrix} P_{n-1} & P_{n+1} & P_n \\ P_n & P_{n+2} & P_{n+1} \\ P_{n+1} & P_{n+3} & P_{n+2} \end{bmatrix}. \tag{15}$$

For all $m, n \in \mathbb{Z}^+$ such that $m < n$, Sokhuma Sokhuma (2013a, b) obtained the following equations with the help of the matrix (15):

$$\begin{cases} P_n = P_{m-1}P_{n-m} + P_{m+1}P_{n-m+1} + P_m P_{n-m+2}, & \text{(16a)} \\ P_n = P_m P_{n-m-1} + P_{m+2}P_{n-m} + P_{m+1}P_{n-m+1}, & \text{(16b)} \\ R_n = P_{m-1}R_{n-m} + P_{m+1}R_{n-m+1} + P_m R_{n-m+2}. & \text{(16c)} \end{cases}$$

Moreover, Seenukul et al. Seenukul et al. (2015) determined the new 3×3 matrices with the initial conditions $P_0 = 0$, $P_1 = 0$, $P_2 = 1$ and studied (16a), (16b) with the help of these new matrices.

Also, Sompong et al. Sompong et al. (2017) established new 3×3 matrices that have similar properties with the Padovan Q matrix with the conditions $P_0 = 0$, $P_1 = 0$, $P_2 = 1$. For all $n, m \in \mathbb{Z}^+$ such that $m \leq n$, the following relations were studied Sompong et al. (2017):

$$\begin{cases} P_{2n} = P_{2m-1}P_{2(n-m)} + P_{2m}P_{2(n-m)+2} + P_{2m+1}P_{2(n-m)+1}, & \text{(17a)} \\ P_{2n} = P_{2m}P_{2(n-m)-1} + P_{2m+1}P_{2(n-m)+1} + P_{2m+2}P_{2(n-m)}, & \text{(17b)} \\ P_{2n+1} = P_{2m-1}P_{2(n-m)+1} + P_{2m}P_{2(n-m)+3} + P_{2m+1}P_{2(n-m)+2}, & \text{(17c)} \\ P_{2n+1} = P_{2m}P_{2(n-m)} + P_{2m+1}P_{2(n-m)+2} + P_{2m+2}P_{2(n-m)+1}, & \text{(17d)} \end{cases}$$

by using

$$Q_1^a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \left(Q_1^a\right)^n = \begin{bmatrix} P_{2n-1} & P_{2n} & P_{2n+1} \\ P_{2n+1} & P_{2n+2} & P_{2n+3} \\ P_{2n} & P_{2n+1} & P_{2n+2} \end{bmatrix}.$$

Then, Khompungson et al. Khompungson et al. (2019) investigated the following equations with respect to the Padovan and Perrin numbers with the initial conditions $P_0 = 0, P_1 = 0, P_2 = 1$. For all $n, m \in \mathbb{Z}^+$ such that $m < n$, the following relations were presented Khompungson et al. (2019):

$$\begin{cases} P_{n+1} = P_{m-1}P_{n-m+1} + P_m P_{n-m+3} + P_{m+1}P_{n-m+2}, & \text{(18a)} \\ R_{n+1} = P_{m-1}R_{n-m+1} + P_m R_{n-m+3} + P_{m+1}R_{n-m+2}, & \text{(18b)} \\ P_{n+2} = P_m P_{n-m+1} + P_{m+1}P_{n-m+3} + P_{m+2}P_{n-m+2}, & \text{(18c)} \\ R_{n+2} = P_m R_{n-m+1} + P_{m+1}R_{n-m+3} + P_{m+2}R_{n-m+2}, & \text{(18d)} \\ R_{2n} = P_{2m-1}R_{2(n-m)} + P_{2m}R_{2(n-m)+2} + P_{2m+1}R_{2(n-m)+1}, & \text{(18e)} \\ R_{2n+1} = P_{2m-1}R_{2(n-m)+1} + P_{2m}R_{2(n-m)+3} + P_{2m+1}R_{2(n-m)+2}, & \text{(18f)} \\ R_{2n+1} = P_{2m}R_{2(n-m)} + P_{2m+1}R_{2(n-m)+2} + P_{2m+2}R_{2(n-m)+1}, & \text{(18g)} \\ P_{2n+2} = P_{2m}P_{2(n-m)+1} + P_{2m+1}P_{2(n-m)+3} + P_{2m+2}P_{2(n-m)+2}, & \text{(18h)} \\ R_{2n+2} = P_{2m}R_{2(n-m)+1} + P_{2m+1}R_{2(n-m)+3} + P_{2m+2}R_{2(n-m)+2}. & \text{(18i)} \end{cases}$$

To examine for detailed information associated with the Padovan and Perrin numbers, we can refer to the studies Cerda-Morales (2017a); Dişkaya and Menken (2019a, b); Günay (2019); Günay and Taşkara (2019); İşbilir and Gürses (2022); Kalman (1982); Shannon and Horadam (1972); Shannon et al. (2006b); Soykan (2020a, b, 2023); Sloane (1964); Sokhuma (2013a, b); Shannon et al. (2006a); Stewart (1996); Taşcı (2018); Waddill (1991); Waddill and Sacks (1967); Stewart (2004); Lucas (1878); Perrin (1899); Khompungson et al. (2019); Sompong et al. (2017); Seenukul et al. (2015); Yılmaz and Taşkara (2013); Yılmaz (2015).

3 Padovan and Perrin hyperbolic spinors

In this section, we investigate and examine new number systems bringing together the hyperbolic spinors and one of the most popular third-order special recurrence numbers Padovan and Perrin numbers with the help of the split Padovan and Perrin quaternions. Moreover, we give some algebraic properties and equalities concerning conjugations. Then, we construct some equations such as recurrence relation, Binet formula, generating function, exponential generating function, Poisson generating function, summation formulas, and matrix formulas. Also, we give some special equalities, which include relations between the Padovan and Perrin hyperbolic spinors and Padovan and Perrin numbers. Then we obtain the determinant equalities for calculating the terms of these sequences.

Definition 1 Let $\breve{P}_n$ and $\breve{R}_n$ be the n^{th} Padovan split quaternion and n^{th} Perrin split quaternion, respectively. The set of n^{th} Padovan and Perrin split quaternion are denoted by $\breve{\mathbb{P}}$ and $\breve{\mathbb{R}}$, respectively. We can construct the following transformations with the help of the correspondence between the split quaternions and hyperbolic spinors as follows for all $n \geq 0$:

$$\begin{aligned} f &: \breve{\mathbb{P}} \rightarrow \mathbb{S} \\ \breve{P}_n &\rightarrow f\left(P_n + P_{n+1}i + P_{n+2}j + P_{n+3}k\right) = \begin{bmatrix} P_n + P_{n+3}j \\ -P_{n+1} + P_{n+2}j \end{bmatrix} \equiv \psi_n \end{aligned} \tag{19}$$

and

$$\begin{aligned} f &: \breve{\mathbb{R}} \rightarrow \mathbb{S} \\ \breve{R}_n &\rightarrow f\left(R_n + R_{n+1}i + R_{n+2}j + R_{n+3}k\right) = \begin{bmatrix} R_n + R_{n+3}j \\ -R_{n+1} + R_{n+2}j \end{bmatrix} \equiv \phi_n, \end{aligned} \tag{20}$$

where the split quaternionic units i, j, k satisfy the rules that are given in Eq. (2). Since these transformations are linear and one-to-one but not onto, these new type sequences are called Padovan and Perrin hyperbolic spinor sequences, respectively. These hyperbolic spinor sequences are linear recurrence sequences and are constructed by using this transformation.

Now, we give some algebraic properties with respect to the Padovan and Perrin hyperbolic spinor sequences, such as addition/subtraction and multiplication by a scalar, respectively. For the sake of brevity, we give algebraic properties only for the Padovan hyperbolic spinor since similar properties can be written easily by substituting P to R for Perrin hyperbolic spinors. Let us consider $\psi_n, \psi_m \in \mathbb{S}$ for all $n, m \geq 0$:

- **Addition/Subtraction:**

$$\begin{aligned} \psi_n \pm \psi_m &= \begin{bmatrix} P_n + P_{n+3}j \\ -P_{n+1} + P_{n+2}j \end{bmatrix} \pm \begin{bmatrix} P_m + P_{n+3}j \\ -P_{m+1} + P_{m+2}j \end{bmatrix} \\ &= \begin{bmatrix} P_n \pm P_m + \left(P_{n+3} \pm P_{m+3}\right)j \\ -\left(P_{n+1} \pm P_{m+1}\right) + \left(P_{n+2} \pm P_{m+2}\right)j \end{bmatrix}. \end{aligned}$$

- **Multiplication by a scalar:**

$$\lambda\psi_n = \begin{bmatrix} \lambda P_n + \lambda P_{n+3}j \\ -\lambda P_{n+1} + \lambda P_{n+2}j \end{bmatrix}, \quad \lambda \in \mathbb{R}.$$

It should be noted that throughout this paper, we use the following explanations in order to the sake of brevity:

- $\breve{P}_n$ is the n^{th} Padovan split quaternion, $\breve{R}_n$ is n^{th} Perrin split quaternion, $\breve{\mathbb{P}}$ is the set of Padovan split quaternions, $\breve{\mathbb{R}}$ is the set of Perrin split quaternion, ψ_n is the n^{th} Padovan hyperbolic spinor, ϕ_n is the n^{th} Perrin hyperbolic spinor.

In order to provide a more concise and straightforward presentation for the readers, the explanations of these notations will not be repeated in each theorem.

Theorem 1 [**Recurrence relation**] *For all $n \geq 0$, the following recurrence relations are given for the Padovan and Perrin hyperbolic spinor sequences, respectively:*

$$\psi_{n+3} = \psi_{n+1} + \psi_n \tag{21}$$

and

$$\phi_{n+3} = \phi_{n+1} + \phi_n. \tag{22}$$

Proof By using the Eqs. (3) and (19), we completed the proof of Eq. (21) as follows:

$$\begin{aligned}
\psi_{n+1}+\psi_n &= \begin{bmatrix} P_{n+1}+P_{n+4}j \\ -P_{n+2}+P_{n+3}j \end{bmatrix} + \begin{bmatrix} P_n+P_{n+3}j \\ -P_{n+1}+P_{n+2}j \end{bmatrix} \\
&= \begin{bmatrix} P_{n+1}+P_{n+4}j+P_n+P_{n+3}j \\ -P_{n+2}+P_{n+3}j-P_{n+1}+P_{n+2}j \end{bmatrix} \\
&= \begin{bmatrix} P_{n+1}+P_n+(P_{n+4}+P_{n+3})\,j \\ -(P_{n+2}+P_{n+1})+(P_{n+3}+P_{n+2})\,j \end{bmatrix} \\
&= \begin{bmatrix} P_{n+3}+P_{n+6}j \\ -P_{n+4}+P_{n+5}j \end{bmatrix} \\
&= \psi_{n+3}.
\end{aligned}$$

The Eq. (22) can be proved by using the Eqs. (4) and (20). □

The following initial values are written for Padovan and Perrin hyperbolic spinors, respectively:

$$\psi_0 = \begin{bmatrix} 1+2j \\ -1+j \end{bmatrix}, \quad \psi_1 = \begin{bmatrix} 1+2j \\ -1+2j \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} 1+3j \\ -2+2j \end{bmatrix}, \tag{23}$$

and

$$\phi_0 = \begin{bmatrix} 3+3j \\ 2j \end{bmatrix}, \quad \phi_1 = \begin{bmatrix} 2j \\ -2+3j \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} 2+5j \\ -3+2j \end{bmatrix}.$$

Now, let us construct the Maple 12 code to calculate the Padovan and Perrin hyperbolic spinors:

- *Calculating the n^{th} Padovan hyperbolic spinor with the help of the Maple 12.*

```
> restart: with(LinearAlgebra):with(linalg):
> P(n):
> P:= proc(n)
> if n = 0 then return 1:
> elif n = 1 then return 1:
> elif n = 2 then return 1:
> elif n = 3 then return 2:
> else return P(n − 2) + P(n − 3)
> end if;
> end proc;
> PH(n):
> PH:= proc(n)
> if n = 0 then return Vector([P(0) + P(3)∗j, −P(1) + P(2)∗j]):
> elif n = 1 then return Vector([P(1) + P(4)∗j, −P(2) + P(3)∗j]):
> elif n = 2 then return Vector([P(2) + P(5)∗j, −P(3) + P(4)∗j]):
> elif n = 3 then return Vector([P(0) + P(1) + (P(3) + P(4))∗j,
              −P(1) − P(2) + (P(2) + P(3))∗j]):
> else return PH(n − 2) + PH(n − 3)
> end if;
> end proc;
```

*The notations P(n) and PH(n) denote the n^{th} Padovan number and Padovan hyperbolic spinor, respectively.

- *Calculating the n^{th} Perrin hyperbolic spinor with the help of the Maple 12.*

```
> restart: with(LinearAlgebra):with(linalg):
> R(n):
> R:= proc(n)
> if n = 0 then return 3:
> elif n = 1 then return 0:
> elif n = 2 then return 2:
> elif n = 3 then return 3:
> else return R(n − 2) + R(n − 3)
> end if;
> end proc;
> RH(n):
> RH := proc(n)
> if n = 0 then return Vector([R(0) + R(3)∗j, −R(1) + R(2)∗j]):
> elif n = 1 then return Vector([R(1) + R(4)∗j, −R(2) + R(3)∗j]):
> elif n = 2 then return Vector([R(2) + R(5)∗j, −R(3) + R(4)∗j]):
> elif n = 3 then return Vector([R(0) + R(1) + (R(3) + R(4))∗j,
              −R(1) − R(2) + (R(2) + R(3))∗j]):
> else return RH(n − 2) + RH(n − 3)
> end if;
> end proc;
```

*The notations R(n) and RH(n) denote the n^{th} Perrin number and Perrin hyperbolic spinor, respectively.

We study only nonnegative subscripted Padovan and Perrin hyperbolic spinors throughout this paper. The definition of the negative subscripted Padovan and Perrin hyperbolic spinors is straightforward by $n \to -n$, but for giving a short introduction, the following definition and recurrence relations can be written for negative subscripted Padovan and Perrin hyperbolic spinors:

Definition 2 Let $\breve{P}_{-n}$ and $\breve{R}_{-n}$ be the $-n^{th}$ negative subscripted Padovan split quaternion and $-n^{th}$ negative subscripted Perrin split quaternion, respectively. The set of $-n^{th}$ Padovan and Perrin split quaternion are denoted by $\breve{\mathbb{P}}$ and $\breve{\mathbb{R}}$, respectively. We can construct the following transformations with the help of the correspondence between the split quaternions and hyperbolic spinors as follows:

$$f : \breve{\mathbb{P}} \to \mathbb{S}$$

$$\breve{P}_{-n} \to f\left(P_{-n} + P_{-n+1}i + P_{-n+2}j + P_{-n+3}k\right) = \begin{bmatrix} P_{-n} + P_{-n+3}j \\ -P_{-n+1} + P_{-n+2}j \end{bmatrix} \equiv \psi_{-n}$$

and

$$f : \breve{\mathbb{R}} \to \mathbb{S}$$

$$\breve{R}_{-n} \to f\left(R_{-n} + R_{-n+1}i + R_{-n+2}j + R_{-n+3}k\right) = \begin{bmatrix} R_{-n} + R_{-n+3}j \\ -R_{-n+1} + R_{-n+2}j \end{bmatrix} \equiv \phi_{-n},$$

where the split quaternionic units i, j, k are satisfied by the rules which are given in the Eq. (2). Since these transformations are linear and one-to-one but not onto, these new type sequences which are called Padovan and Perrin hyperbolic spinor sequences, respectively. These are linear recurrence sequences and are constructed by using this

In addition to these, the following recurrence relations hold for Padovan and Perrin hyperbolic spinors with negative subscripts:

$$\psi_{-n} = \psi_{-(n-3)} - \psi_{-(n-1)}$$

and

$$\phi_{-n} = \phi_{-(n-3)} - \phi_{-(n-1)}.$$

The other equations and properties can be obtained similarly for negative subscripted Padovan and Perrin hyperbolic spinors for the sake of brevity.

Definition 3 Let the conjugate of the n^{th} Padovan and Perrin split quaternion be denoted by $\check{P}_n^* = P_n - P_{n+1}i - P_{n+2}j - P_{n+3}k$ and $\check{R}_n^* = R_n - R_{n+1}i - R_{n+2}j - R_{n+3}k$. The following expressions can be written:

- The n^{th} Padovan hyperbolic spinor ψ_n^* and n^{th} Perrin hyperbolic spinor ϕ_n^* corresponding to the conjugate of the n^{th} Padovan split quaternion and n^{th} Perrin split quaternion are expressed by, respectively:

$$f\left(\check{P}_n^*\right) = f\left(P_n - P_{n+1}i - P_{n+2}j - P_{n+3}k\right) = \begin{bmatrix} P_n - P_{n+3}j \\ P_{n+1} - P_{n+2}j \end{bmatrix} \equiv \psi_n^*,$$
$$f\left(\check{R}_n^*\right) = f\left(R_n - R_{n+1}i - R_{n+2}j - R_{n+3}k\right) = \begin{bmatrix} R_n - R_{n+3}j \\ R_{n+1} - R_{n+2}j \end{bmatrix} \equiv \phi_n^*.$$

- Also, the matrix $\mathcal{C} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is given. The ordinary hyperbolic conjugate of n^{th} Padovan and Perrin hyperbolic spinors ψ_n and ϕ_n are written as follows, respectively:

$$\overline{\psi}_n = \begin{bmatrix} P_n - P_{n+3}j \\ -P_{n+1} - P_{n+2}j \end{bmatrix} \quad \text{and} \quad \overline{\phi}_n = \begin{bmatrix} R_n - R_{n+3}j \\ -R_{n+1} - R_{n+2}j \end{bmatrix}.$$

- Hyperbolic conjugate of Padovan and Perrin hyperbolic spinor $\tilde{\psi}_n = j\mathcal{C}\overline{\psi}_n$ and $\tilde{\phi}_n = j\mathcal{C}\overline{\phi}_n$ of n^{th} Padovan and Perrin hyperbolic spinor ψ_n and ϕ_n are expressed as follows, respectively:

$$\tilde{\psi}_n = \begin{bmatrix} -P_{n+2} - P_{n+1}j \\ P_{n+3} - P_n j \end{bmatrix} \quad \text{and} \quad \tilde{\phi}_n = \begin{bmatrix} -R_{n+2} - R_{n+1}j \\ R_{n+3} - R_n j \end{bmatrix},$$

  where by using the study of Cartan (1966).
- Additionally, the hyperbolic mate of n^{th} Padovan and Perrin hyperbolic spinor $\check{\psi}_n = -\mathcal{C}\overline{\psi}_n$ and $\check{\phi}_n = -\mathcal{C}\overline{\phi}_n$ are given as respectively:

$$\check{\psi}_n = \begin{bmatrix} P_{n+1} + P_{n+2}j \\ P_n - P_{n+3}j \end{bmatrix} \quad \text{and} \quad \check{\phi}_n = \begin{bmatrix} R_{n+1} + R_{n+2}j \\ R_n - R_{n+3}j \end{bmatrix},$$

  where by using the study of Torres del Castillo and Barrales Torres del Castillo and Barrales (2004).

The following Theorem 2-Theorem 12 is given without proofs, since the proofs are clear, by using the matrix $\mathcal{C}$ and the conjugation properties of Padovan and Perrin hyperbolic spinors.

Theorem 2 *The following equations are satisfied:*

$$(a)\ \overline{\psi}_n = \mathcal{C}\check{\psi}_n, \quad (c)\ \check{\psi}_n = -j\tilde{\psi}_n, \quad (e)\ \overline{\psi}_n = -j\mathcal{C}\tilde{\psi}_n,$$
$$(b)\ \overline{\phi}_n = \mathcal{C}\check{\phi}_n, \quad (d)\ \check{\phi}_n = -j\tilde{\phi}_n, \quad (f)\ \overline{\phi}_n = -j\mathcal{C}\tilde{\phi}_n.$$

Theorem 3 *The following equations are satisfied:*

$$(a)\ \psi_n + \psi_n^* = \begin{bmatrix} 2P_n \\ 0 \end{bmatrix}, \quad (c)\ \psi_n - \psi_n^* = 2\begin{bmatrix} P_{n+3}j \\ -P_{n+1} + P_{n+2}j \end{bmatrix},$$
$$(b)\ \phi_n + \phi_n^* = \begin{bmatrix} 2R_n \\ 0 \end{bmatrix}, \quad (d)\ \phi_n - \phi_n^* = 2\begin{bmatrix} R_{n+3}j \\ -R_{n+1} + R_{n+2}j \end{bmatrix}.$$

Theorem 4 *The following equations are satisfied:*

$$(a)\ \psi_n + \overline{\psi}_n = 2\begin{bmatrix} P_n \\ -P_{n+1} \end{bmatrix}, \quad (c)\ \psi_n - \overline{\psi}_n = 2j\begin{bmatrix} P_{n+3} \\ P_{n+2} \end{bmatrix},$$
$$(b)\ \phi_n + \overline{\phi}_n = 2\begin{bmatrix} R_n \\ -R_{n+1} \end{bmatrix}, \quad (d)\ \phi_n - \overline{\phi}_n = 2j\begin{bmatrix} R_{n+3} \\ R_{n+2} \end{bmatrix}.$$

Theorem 5 *The following equations are satisfied:*

$$(a)\ \psi_n + \tilde{\psi}_n = \begin{bmatrix} -P_{n-1} + P_n j \\ P_n + P_{n-1}j \end{bmatrix},$$
$$(b)\ \phi_n + \tilde{\phi}_n = \begin{bmatrix} -R_{n-1} + R_n j \\ R_n + R_{n-1}j \end{bmatrix},$$
$$(c)\ \psi_n - \tilde{\psi}_n = \begin{bmatrix} P_n + P_{n+2} + (P_{n+3} + P_{n+1})\, j \\ -P_{n+1} - P_{n+3} + (P_{n+2} + P_n)\, j \end{bmatrix},$$
$$(d)\ \phi_n - \tilde{\phi}_n = \begin{bmatrix} R_n + R_{n+2} + (R_{n+3} + R_{n+1})\, j \\ -R_{n+1} - R_{n+3} + (R_{n+2} + R_n)\, j \end{bmatrix}.$$

Theorem 6 *The following equations are satisfied:*

$$(a)\ \psi_n + \check{\psi}_n = \begin{bmatrix} P_{n+3} + P_{n+5}j \\ -P_{n+1} + P_n + (P_{n+2} - P_{n+3})\, j \end{bmatrix},$$
$$(b)\ \phi_n + \check{\phi}_n = \begin{bmatrix} R_{n+3} + R_{n+5}j \\ -R_{n+1} + R_n + (R_{n+2} - R_{n+3})\, j \end{bmatrix},$$
$$(c)\ \psi_n - \check{\psi}_n = \begin{bmatrix} P_n - P_{n+1} + (P_{n+3} - P_{n+2})\, j \\ -P_{n+3} + P_{n+5}j \end{bmatrix},$$
$$(d)\ \phi_n - \check{\phi}_n = \begin{bmatrix} R_n - R_{n+1} + (R_{n+3} - R_{n+2})\, j \\ -R_{n+3} + R_{n+5}j \end{bmatrix}.$$

Theorem 7 *The following equations are satisfied:*

$$(a)\ \psi_n^* + \overline{\psi}_n = 2\begin{bmatrix} P_n - P_{n+3}j \\ -P_{n+2}j \end{bmatrix}, \quad (c)\ \psi_n^* - \overline{\psi}_n = 2\begin{bmatrix} 0 \\ P_{n+1} \end{bmatrix},$$
$$(b)\ \phi_n^* + \overline{\phi}_n = 2\begin{bmatrix} R_n - R_{n+3}j \\ -R_{n+2}j \end{bmatrix}, \quad (d)\ \phi_n^* - \overline{\phi}_n = 2\begin{bmatrix} 0 \\ R_{n+1} \end{bmatrix}.$$

Theorem 8 *The following equations are satisfied:*

$$(a)\ \psi_n^* + \tilde{\psi}_n = \begin{bmatrix} -P_{n-1} - (P_{n+3} + P_{n+1})\, j \\ P_{n+1} + P_{n+3} - (P_{n+2} + P_n)\, j \end{bmatrix},$$
$$(b)\ \phi_n^* + \tilde{\phi}_n = \begin{bmatrix} -R_{n-1} - (R_{n+3} + R_{n+1})\, j \\ R_{n+1} + R_{n+3} - (R_{n+2} + R_n)\, j \end{bmatrix},$$

$$(c)\ \psi_n^* - \tilde{\psi}_n = \begin{bmatrix} P_n + P_{n+2} - P_n j \\ P_{n+1} - P_{n+3} - P_{n-1} j \end{bmatrix},$$
$$(d)\ \phi_n^* - \tilde{\phi}_n = \begin{bmatrix} R_n + R_{n+2} - R_n j \\ R_{n+1} - R_{n+3} - R_{n-1} j \end{bmatrix}.$$

Theorem 9 *The following equations are satisfied:*

$$(a)\ \psi_n^* + \breve{\psi}_n = \begin{bmatrix} P_{n+3} + (-P_{n+3} + P_{n+2})\, j \\ P_{n+3} - P_{n+5} j \end{bmatrix},$$
$$(b)\ \phi_n^* + \breve{\phi}_n = \begin{bmatrix} R_{n+3} + (-R_{n+3} + R_{n+2})\, j \\ R_{n+3} - R_{n+5} j \end{bmatrix},$$
$$(c)\ \psi_n^* - \breve{\psi}_n = \begin{bmatrix} P_n - P_{n+1} - P_{n+5} j \\ P_{n+1} - P_n + (-P_{n+2} + P_{n+3})\, j \end{bmatrix},$$
$$(d)\ \phi_n^* - \breve{\phi}_n = \begin{bmatrix} R_n - R_{n+1} - R_{n+5} j \\ R_{n+1} - R_n + (-R_{n+2} + R_{n+3})\, j \end{bmatrix}.$$

Theorem 10 *The following equations are satisfied:*

$$(a)\ \overline{\psi}_n + \tilde{\psi}_n = \begin{bmatrix} -P_{n-1} - (P_{n+3} + P_{n+1})\, j \\ P_n - (P_{n+2} + P_n)\, j \end{bmatrix},$$
$$(b)\ \overline{\phi}_n + \tilde{\phi}_n = \begin{bmatrix} -R_{n-1} - (R_{n+3} + R_{n+1})\, j \\ R_n - (R_{n+2} + R_n)\, j \end{bmatrix},$$
$$(c)\ \overline{\psi}_n - \tilde{\psi}_n = \begin{bmatrix} P_n + P_{n+2} - P_n j \\ -P_{n+1} - P_{n+3} - P_{n-1} j \end{bmatrix},$$
$$(d)\ \overline{\phi}_n - \tilde{\phi}_n = \begin{bmatrix} R_n + R_{n+2} - R_n j \\ -R_{n+1} - R_{n+3} - R_{n-1} j \end{bmatrix}.$$

Theorem 11 *The following equations are satisfied:*

$$(a)\ \tilde{\psi}_n + \breve{\psi}_n = \begin{bmatrix} P_{n+1} - P_{n+2} + (P_{n+2} - P_{n+1})\, j \\ P_{n+3} + P_n - (P_n + P_{n+3})\, j \end{bmatrix},$$
$$(b)\ \tilde{\phi}_n + \breve{\phi}_n = \begin{bmatrix} R_{n+1} - R_{n+2} + (R_{n+2} - R_{n+1})\, j \\ R_{n+3} + R_n - (R_n + R_{n+3})\, j \end{bmatrix},$$
$$(c)\ \tilde{\psi}_n - \breve{\psi}_n = \begin{bmatrix} -P_{n+4} - P_{n+4} j \\ P_{n+1} + P_{n+1} j \end{bmatrix},$$
$$(d)\ \tilde{\phi}_n - \breve{\phi}_n = \begin{bmatrix} -R_{n+4} - R_{n+4} j \\ R_{n+1} + R_{n+1} j \end{bmatrix},$$

Theorem 12 *The following equations are satisfied:*

$$(a)\ \overline{\psi}_n + \breve{\psi}_n = \begin{bmatrix} P_{n+3} + (P_{n+2} - P_{n+3})\, j \\ P_n - P_{n+1} - P_{n+5} j \end{bmatrix},$$
$$(b)\ \overline{\phi}_n + \breve{\phi}_n = \begin{bmatrix} R_{n+3} + (R_{n+2} - R_{n+3})\, j \\ R_n - R_{n+1} - R_{n+5} j \end{bmatrix},$$
$$(c)\ \overline{\psi}_n - \breve{\psi}_n = \begin{bmatrix} P_n - P_{n+1} - P_{n+5} j \\ -P_{n+3} + (P_{n+3} - P_{n+2})\, j \end{bmatrix},$$
$$(d)\ \overline{\phi}_n - \breve{\phi}_n = \begin{bmatrix} R_n - R_{n+1} - R_{n+5} j \\ -R_{n+3} + (R_{n+3} - R_{n+2})\, j \end{bmatrix}.$$

Theorem 13 [**Norm**] *The norm of the n^{th} Padovan and Perrin split quaternions $N\left(\breve{P}_n\right) = \breve{P}_n \breve{P}_n^*$ and $N\left(\breve{R}_n\right) = \breve{R}_n \breve{R}_n^*$ are equal to the norm of the associated Padovan and Perrin* *hyperbolic spinors, respectively:*

$$N(\check{P}_n) = \overline{\psi}_n^t \psi_n \quad and \quad N(\check{R}_n) = \overline{\phi}_n^t \phi_n.$$

By using Theorem 2, we can express the norm of the Padovan and Perrin hyperbolic spinors as follows, respectively:

$$N(\psi_n) = \overline{\psi}_n^t \psi_n = \check{\psi}_n^t C^t \psi_n \quad and \quad N(\phi_n) = \overline{\phi}_n^t \phi_n = \check{\phi}_n^t C^t \phi_n.$$

Theorem 14 [**Generating Function**] *For all $n \geq 0$, the following generating functions hold for the Padovan and Perrin hyperbolic spinors as follows, respectively:*

$$\sum_{n=0}^{\infty} \psi_n = \frac{1}{1-x^2-x^3}\begin{bmatrix} 1+x+(2+2x+x^2)j \\ -1-x-x^2+(1+2x+x^2)j \end{bmatrix}, \tag{24}$$

$$\sum_{n=0}^{\infty} \phi_n = \frac{1}{1-x^2-x^3}\begin{bmatrix} 3-x^2+(3+2x+2x^2)j \\ -2x-3x^2+(2+3x)j \end{bmatrix}. \tag{25}$$

Proof Suppose that the following equation is the generating function for Padovan hyperbolic spinors:

$$\sum_{n=0}^{\infty} \psi_n x^n = \psi_0 + \psi_1 x + \psi_2 x^2 + \ldots + \psi_n x^n + \ldots \tag{26}$$

Then, multiplying the Eq. (26) by x^2 and x^3 and by using the Eq. (21), we get:

$$\sum_{n=0}^{\infty} \psi_n - x^2 \sum_{n=0}^{\infty} \psi_n - x^3 \sum_{n=0}^{\infty} \psi_n = \psi_0 + \psi_1 x + (\psi_2 - \psi_0)\, x^2.$$

With the help of the initial values of Padovan hyperbolic spinor sequence ψ_0, ψ_1 and ψ_2 given in the Eq. (23), then we attain:

$$\begin{aligned} \sum_{n=0}^{\infty} \psi_n - x^2 \sum_{n=0}^{\infty} \psi_n - x^3 \sum_{n=0}^{\infty} \psi_n &= \begin{bmatrix} 1+2j \\ -1+j \end{bmatrix} + \begin{bmatrix} 1+2j \\ -1+2j \end{bmatrix} x + \begin{bmatrix} j \\ -1+j \end{bmatrix} x^2 \\ &= \begin{bmatrix} 1+x+(2+2x+x^2)j \\ -1-x-x^2+(1+2x+x^2)j \end{bmatrix}. \end{aligned}$$

Therefore, we obtain:

$$\sum_{n=0}^{\infty} \psi_n = \frac{1}{1-x^2-x^3}\begin{bmatrix} 1+x+(2+2x+x^2)j \\ -1-x-x^2+(1+2x+x^2)j \end{bmatrix}.$$

Then, Eq. (25) can be proved by using the same manner. □

Theorem 15 [**Binet Formula**] *For all $n \geq 0$, the following Binet formulas are written for Padovan and Perrin hyperbolic spinors, respectively:*

$$\psi_n = A\sigma_1\alpha^n + B\sigma_2\beta^n + C\sigma_3\gamma^n, \tag{27}$$

$$\phi_n = A\alpha^n + B\beta^n + C\gamma^n, \tag{28}$$

where

$$A = \begin{bmatrix} 1+\alpha^3 j \\ \alpha(-1+\alpha j) \end{bmatrix}, \quad B = \begin{bmatrix} 1+\beta^3 j \\ \beta(-1+\beta j) \end{bmatrix}, \quad C = \begin{bmatrix} 1+\gamma^3 j \\ \gamma(-1+\gamma j) \end{bmatrix}.$$

Proof By the Eq. (19) and Binet formula of the Padovan numbers given in the Eq. (10), we write:

$$\begin{aligned}\psi_n =& \begin{bmatrix} \sigma_1\alpha^n + \sigma_2\beta^n + \sigma_3\gamma^n + \left(\sigma_1\alpha^{n+3} + \sigma_2\beta^{n+3} + \sigma_3\gamma^{n+3}\right) j \\ -\left(\sigma_1\alpha^{n+1} + \sigma_2\beta^{n+1} + \sigma_3\gamma^{n+1}\right) + \left(\sigma_1\alpha^{n+2} + \sigma_2\beta^{n+2} + \sigma_3\gamma^{n+2}\right) j \end{bmatrix} \\ =& \begin{bmatrix} 1+\alpha^3 j \\ \alpha\left(-1+\alpha j\right) \end{bmatrix} \sigma_1\alpha^n + \begin{bmatrix} 1+\beta^3 j \\ \beta\left(-1+\beta j\right) \end{bmatrix} \sigma_2\beta^n + \begin{bmatrix} 1+\gamma^3 j \\ \gamma\left(-1+\gamma j\right) \end{bmatrix} \sigma_3\gamma^n \\ =& A\sigma_1\alpha^n + B\sigma_2\beta^n + C\sigma_3\gamma^n.\end{aligned}$$

Therefore, we completed the proof of the Binet formula for Padovan numbers. Also, by using a similar way, Eq. (28) can be proved easily. □

Theorem 16 [**Exponential Generating Function**] *The exponential generating functions are obtained for Padovan and Perrin hyperbolic spinors, respectively:*

$$\sum_{n=0}^{\infty} \psi_n \frac{y^n}{n!} = A\sigma_1 e^{\alpha y} + B\sigma_2 e^{\beta y} + C\sigma_3 e^{\gamma y}, \tag{29}$$

$$\sum_{n=0}^{\infty} \phi_n \frac{y^n}{n!} = Ae^{\alpha y} + Be^{\beta y} + Ce^{\gamma y}. \tag{30}$$

Proof With the help of the Eq. (27), we have the followings:

$$\begin{aligned}\sum_{n=0}^{\infty} \psi_n \frac{y^n}{n!} &= \sum_{n=0}^{\infty} \left(A\sigma_1\alpha^n + B\sigma_2\beta^n + C\sigma_3\gamma^n\right) \frac{y^n}{n!} \\ &= A\sigma_1 \sum_{n=0}^{\infty} \alpha^n \frac{y^n}{n!} + B\sigma_2 \sum_{n=0}^{\infty} \beta^n \frac{y^n}{n!} + C\sigma_3 \sum_{n=0}^{\infty} \gamma^n \frac{y^n}{n!} \\ &= A\sigma_1 \sum_{n=0}^{\infty} \frac{(\alpha y)^n}{n!} + B\sigma_2 \sum_{n=0}^{\infty} \frac{(\beta y)^n}{n!} + C\sigma_3 \sum_{n=0}^{\infty} \frac{(\gamma y)^n}{n!} \\ &= A\sigma_1 e^{\alpha y} + B\sigma_2 e^{\beta y} + C\sigma_3 e^{\gamma y}.\end{aligned}$$

Then, Eq. (30) can be demonstrated easily. □

Theorem 17 [**Poisson Generating Function**] *The following Poisson generating functions hold for Padovan and Perrin hyperbolic spinors, respectively:*

$$e^{-y} \sum_{n=0}^{\infty} \psi_n \frac{y^n}{n!} = e^{-y} A\sigma_1 e^{\alpha y} + e^{-y} B\sigma_2 e^{\beta y} + e^{-y} C\sigma_3 e^{\gamma y}, \tag{31}$$

$$e^{-y} \sum_{n=0}^{\infty} \phi_n \frac{y^n}{n!} = e^{-y} Ae^{\alpha y} + e^{-y} Be^{\beta y} + e^{-y} Ce^{\gamma y}. \tag{32}$$

Proof By using Eqs. (29) and (30), we get the desired results, since the Poisson generating function is expressed as multiplying the exponential generating function by e^{-y} (cf. also Şentürk et al. (2020)). □

Thanks to the Soykan Soykan (2020b, 2023), we get the following summation formulas for Padovan and Perrin hyperbolic spinors without proofs.

Theorem 18 *For all $m, n \geq 0$, the following summation formulas are satisfied for Padovan and Perrin hyperbolic spinors, respectively:*

$$(a)\begin{cases}\star \sum_{n=0}^{m} \psi_n = \psi_{m+5} - \psi_4 = \begin{bmatrix} P_{m+5} + P_{m+8} j \\ -P_{m+6} + P_{m+7} j \end{bmatrix} - \begin{bmatrix} 2+5j \\ -3+4j \end{bmatrix}, \\ \star \sum_{n=0}^{m} \phi_n = \phi_{m+5} - \phi_4 = \begin{bmatrix} R_{m+5} + R_{m+8} j \\ -R_{m+6} + R_{m+7} j \end{bmatrix} - \begin{bmatrix} 2+7j \\ -5+5j \end{bmatrix}, \end{cases}$$

$$(b)\begin{cases}\star \sum_{n=0}^{m} \psi_{2n} = \psi_{2m+3} - \psi_1 = \begin{bmatrix} P_{2m+3} + P_{2m+6} j \\ -P_{2m+4} + P_{2m+5} j \end{bmatrix} - \begin{bmatrix} 1+2j \\ -1+2j \end{bmatrix}, \\ \star \sum_{n=0}^{m} \phi_{2n} = \phi_{2m+3} - \phi_1 = \begin{bmatrix} R_{2m+3} + R_{2m+6} j \\ -R_{2m+4} + R_{2m+5} j \end{bmatrix} - \begin{bmatrix} 2j \\ -2+3j \end{bmatrix}, \end{cases}$$

$$(c)\begin{cases}\star \sum_{n=0}^{m} \psi_{2n+1} = \psi_{2m+4} - \psi_2 = \begin{bmatrix} P_{2m+2} + P_{2m+5} j \\ -P_{2m+3} + P_{2m+4} j \end{bmatrix} - \begin{bmatrix} 1+3j \\ -2+2j \end{bmatrix}, \\ \star \sum_{n=0}^{m} \phi_{2n+1} = \phi_{2m+4} - \phi_2 = \begin{bmatrix} R_{2m+4} + R_{2m+7} j \\ -R_{2m+5} + R_{2m+6} j \end{bmatrix} - \begin{bmatrix} 2+5j \\ -3+2j \end{bmatrix}. \end{cases}$$

Theorem 19 *The following relations between Padovan and Perrin hyperbolic spinors are satisfied:*

(a) $\phi_n = 3\psi_{n-5} + 2\psi_{n-4}$,
(b) $\psi_{n-1} = \frac{1}{23}(\phi_{n-3} + 8\phi_{n-2} + 10\phi_{n-1})$.

Proof (a) By using the Eqs. (11), (19), and (20), we get the followings:

$$\begin{aligned} 3\psi_{n-5} + 2\psi_{n-4} &= 3\begin{bmatrix} P_{n-5} + P_{n-2} j \\ -P_{n-4} + P_{n-3} j \end{bmatrix} + 2\begin{bmatrix} P_{n-4} + P_{n-1} j \\ -P_{n-3} + P_{n-2} j \end{bmatrix} \\ &= \begin{bmatrix} 3P_{n-5} + 2P_{n-4} + (3P_{n-2} + 2P_{n-1})\, j \\ -3P_{n-4} - 2P_{n-3} + (3P_{n-3} + 2P_{n-2})\, j \end{bmatrix} \\ &= \begin{bmatrix} R_n + R_{n+3} j \\ -R_{n+1} + R_{n+2} j \end{bmatrix} \\ &= \phi_n. \end{aligned}$$

(b) This part can be proved by using the Eqs. (12), (19), and (20). □

Theorem 20 *The relations between the Padovan and Perrin hyperbolic spinors are given as follows:*

(a) $\psi^t_{m-3}\psi_{n-3} + \psi^t_{m-1}\psi_{n-2} + \psi^t_{m-2}\psi_{n-1} = P_{m+n-1} + P_{m+n+1} + P_{m+n+3} + P_{m+n+5}$,
(b) $\phi^t_{m-3}\phi_{n-3} + \phi^t_{m-1}\phi_{n-2} + \phi^t_{m-2}\phi_{n-1} = 4P_{m+n+4} + 4P_{m+n-1} + 2P_{m+n-5} + P_{m+n-9}$ P_{m+n-11},
(c) $\phi^t_{m-3}\phi_{n-3} + \phi^t_{m-1}\phi_{n-2} + \phi^t_{m-2}\phi_{n-1} = 2R_{m+n+6} + R_{m+n-1} + R_{m+n-6}$,
(d) $\psi^t_{m-3}\phi_{n-3} + \psi^t_{m-1}\phi_{n-2} + \psi^t_{m-2}\phi_{n-1} = R_{m+n-1} + R_{m+n+1} + R_{m+n+3} + R_{m+n+5}$.

Proof (a) By using the Eqs. (14a) and (19), we can get the followings:

$$\begin{aligned}
&\psi^t_{m-3}\psi_{n-3} + \psi^t_{m-1}\psi_{n-2} + \psi^t_{m-2}\psi_{n-1} \\
&= \begin{bmatrix} P_{m-3} + P_m j & -P_{m-2} + P_{m-1} j \end{bmatrix} \begin{bmatrix} P_{n-3} + P_n j \\ -P_{n-2} + P_{n-1} j \end{bmatrix} \\
&\quad + \begin{bmatrix} P_{m-1} + P_{m+2} j & -P_m + P_{m+1} j \end{bmatrix} \begin{bmatrix} P_{n-2} + P_{n+1} j \\ -P_{n-1} + P_n j \end{bmatrix} \\
&\quad + \begin{bmatrix} P_{m-2} + P_{m+1} j & -P_{m-1} + P_m j \end{bmatrix} \begin{bmatrix} P_{n-1} + P_{n+2} j \\ -P_n + P_{n+1} j \end{bmatrix} \\
&= (P_{m-3}P_{n-3} + P_{m-1}P_{n-2} + P_{m-2}P_{n-1}) + (P_{m-3}P_n + P_{m-1}P_{n+1} + P_{m-2}P_{n+2})\, j \\
&\quad + (P_m P_{n-3} + P_{m+2}P_{n-2} + P_{m+1}P_{n-1})\, j + (P_m P_n + P_{m+2}P_{n+1} + P_{m+1}P_{n+2}) \\
&\quad + (P_{m-2}P_{n-2} + P_m P_{n-1} + P_{m-1}P_n) - (P_{m-2}P_{n-1} + P_m P_n + P_{m-1}P_{n+1})\, j \\
&\quad - (P_{m-1}P_{n-2} + P_{m+1}P_{n-1} + P_m P_n)\, j + (P_{m-1}P_{n-1} + P_{m+1}P_n + P_m P_{n+1}) \\
&= P_{m+n-1} + P_{m+n+1} + P_{m+n+3} + P_{m+n+5}.
\end{aligned}$$

The other parts can be demonstrated similarly. □

Theorem 21 *For $m < n$, the relations between the Padovan and Perrin hyperbolic spinors are given as follows:*

(a) $\psi^t_{m-1}\psi_{n-m} + \psi^t_{m+1}\psi_{n-m+1} + \psi^t_m\psi_{n-m+2} = P_n + P_{n+2} + P_{n+4} + P_{n+6}$,
(b) $\psi^t_m\psi_{n-m-1} + \psi^t_{m+2}\psi_{n-m} + \psi^t_{m+1}\phi_{n-m+1} = P_n + P_{n+2} + P_{n+4} + P_{n+6}$,
(c) $\psi^t_{m-1}\phi_{n-m} + \psi^t_{m+1}\phi_{n-m+1} + \psi^t_m\phi_{n-m+2} = R_n + R_{n+2} + R_{n+4} + R_{n+6}$.

Proof With the help of Eqs. (14b)–(14d) and (19), the proof is completed. □

Theorem 22 *For $m \le n$, the relations between the Padovan and Perrin hyperbolic spinors are given as follows:*

(a) $\left.\begin{array}{l} \star\, \psi^t_{2m-1}\psi_{2(n-m)} + \psi^t_{2m}\psi_{2(n-m)+2} + \psi^t_{2m+1}\psi_{2(n-m)+1} \\ \star\, \psi^t_{2m}\psi_{2(n-m)-1} + \psi^t_{2m+1}\psi_{2(n-m)+1} + \psi^t_{2m+2}\psi_{2(n-m)} \end{array}\right\} = \begin{array}{l} P_{2n} + P_{2n+2} \\ + P_{2n+4} + P_{2n+6}, \end{array}$

(b) $\left.\begin{array}{l} \star\, \psi^t_{2m-1}\psi_{2(n-m)+1} + \psi^t_{2m}\psi_{2(n-m)+3} + \psi^t_{2m+1}\psi_{2(n-m)+2} \\ \star\, \psi^t_{2m}\psi_{2(n-m)} + \psi^t_{2m+1}\psi_{2(n-m)+2} + \psi^t_{2m+2}\psi_{2(n-m)+1} \end{array}\right\} = P_{2n+4} + P_{2n+9}.$

Proof By means of Eqs. (16a)–(16c) and (19), the proof is finished. □

Theorem 23 *For $m < n$, the relations between the Padovan and Perrin hyperbolic spinors are given as follows:*

(a) $\psi^t_{m-1}\psi_{n-m+1} + \psi^t_m\psi_{n-m+3} + \psi^t_{m+1}\psi_{n-m+2} = P_{n+1} + P_{n+3} + P_{n+5} + P_{n+7}$,
(b) $\psi^t_{m-1}\phi_{n-m+1} + \psi^t_m\phi_{n-m+3} + \psi^t_{m+1}\phi_{n-m+2} = R_{n+1} + R_{n+3} + R_{n+5} + R_{n+7}$,
(c) $\psi^t_m\psi_{n-m+1} + \psi^t_{m+1}\psi_{n-m+3} + \psi^t_{m+2}\psi_{n-m+2} = P_{n+2} + P_{n+4} + P_{n+6} + P_{n+8}$,
(d) $\psi^t_m\phi_{n-m+1} + \psi^t_{m+1}\phi_{n-m+3} + \psi^t_{m+2}\phi_{n-m+2} = R_{n+2} + R_{n+4} + R_{n+6} + R_{n+8}$.
(e) $\psi^t_{2\,m-1}\phi_{2(n-m)} + \psi^t_{2\,m}\phi_{2(n-m)+2} + \psi^t_{2\,m+1}\phi_{2(n-m)+1} = R_{2n} + R_{2n+2} + R_{2n+4}$
R_{2n+6},
(f) $\left.\begin{array}{l} \star\, \psi^t_{2m-1}\phi_{2(n-m)+1} + \psi^t_{2m}\phi_{2(n-m)+3} + \psi^t_{2m+1}\phi_{2(n-m)+2} \\ \star\, \psi^t_{2m}\phi_{2(n-m)} + \psi^t_{2m+1}\phi_{2(n-m)+2} + \psi^t_{2m+2}\phi_{2(n-m)+1} \end{array}\right\} = R_{2n+4} + R_{2n+9},$
(g) $\psi^t_{2\,m}\psi_{2(n-m)+1} + \psi^t_{2\,m+1}\psi_{2(n-m)+3} + \psi^t_{2\,m+2}\psi_{2(n-m)+2} = P_{2n+2} + P_{2n+4} + P_{2n+6}$
P_{2n+8},

(h) $\psi^t_{2m}\phi_{2(n-m)+1}+\psi^t_{2m+1}\phi_{2(n-m)+3}+\psi^t_{2m+2}\phi_{2(n-m)+2} = R_{2n+2}+R_{2n+4}+R_{2n+6}$ R_{2n+8}.

Proof By using Eqs. (18a)–(18i) and (19), the proof is demonstrated. □

Theorem 24 *The following matrix equations are obtained for Padovan and Perrin hyperbolic spinors, respectively:*

(a) $\begin{bmatrix}\psi_n\\ \psi_{n+1}\\ \psi_{n+2}\end{bmatrix} = \begin{bmatrix}0&1&0\\0&0&1\\1&1&0\end{bmatrix}^n \begin{bmatrix}\psi_0\\ \psi_1\\ \psi_2\end{bmatrix}$,

(b) $\begin{bmatrix}\phi_n\\ \phi_{n+1}\\ \phi_{n+2}\end{bmatrix} = \begin{bmatrix}0&1&0\\0&0&1\\1&1&0\end{bmatrix}^n \begin{bmatrix}\phi_0\\ \phi_1\\ \phi_2\end{bmatrix}$.

Proof By using mathematical induction, we can complete the proof. For the sake of brevity, we omit them. □

Theorem 25 *The following determinant equations are satisfied for Padovan and Perrin hyperbolic spinors, respectively:*

$$\psi_n = \begin{vmatrix} \psi_0 & -1 & 0 & 0 & 0 & & 0 & 0\\ \psi_1 & 0 & -1 & 0 & 0 & & 0 & 0\\ \psi_2 & 0 & 0 & -1 & 0 & & 0 & 0\\ 0 & 1 & 1 & 0 & -1 & & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 0 \end{vmatrix}_{(n+1)\times(n+1)}, \tag{33}$$

$$\phi_n = \begin{vmatrix} \phi_0 & -1 & 0 & 0 & 0 & & 0 & 0\\ \phi_1 & 0 & -1 & 0 & 0 & & 0 & 0\\ \phi_2 & 0 & 0 & -1 & 0 & & 0 & 0\\ 0 & 1 & 1 & 0 & -1 & & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 0 \end{vmatrix}_{(n+1)\times(n+1)}. \tag{34}$$

Proof For the sake of the brevity, we skip this proof. By using the recurrence relation of Padovan and Perrin hyperbolic spinor sequences Eqs. (21) and (22), and the study of Kızılateş et al. Kızılateş et al. (2019) (see Theorem 5 on page 5). □

4 New further notions: (s, t)-Padovan and (s, t)-Perrin hyperbolic spinors

In this section, we provide a short introduction to some realizable new arguments for where to go from here. These notions include the Padovan and Perrin hyperbolic spinors according to the given values $s = 1$ and $t = 1$.

Definition 4 Let $\breve{\mathscr{P}}_n$ and $\breve{\mathscr{R}}_n$ be the n^{th} (s,t)-Padovan split quaternion and n^{th} (s,t)-Perrin split quaternion. The set of n^{th} (s,t)-Padovan and (s,t)-Perrin split quaternion are denoted by $\widetilde{\mathfrak{P}}$ and $\widetilde{\mathfrak{R}}$, respectively. We can construct the following transformations with the help of the correspondence between the split quaternions and hyperbolic spinors as follows:

$$f:\widetilde{\mathfrak{P}}\to\mathbb{S}$$
$$\breve{\mathscr{P}}_n\to f\left(\mathscr{P}_n+\mathscr{P}_{n+1}i+\mathscr{P}_{n+2}j+\mathscr{P}_{n+3}k\right)=\begin{bmatrix}\mathscr{P}_n+\mathscr{P}_{n+3}j\\-\mathscr{P}_{n+1}+\mathscr{P}_{n+2}j\end{bmatrix}\equiv\ddot{\psi}_n$$

and

$$f:\widetilde{\mathfrak{R}}\to\mathbb{S}$$
$$\breve{P}_n\to f\left(\mathscr{R}_n+\mathscr{R}_{n+1}i+\mathscr{R}_{n+2}j+\mathscr{R}_{n+3}k\right)=\begin{bmatrix}\mathscr{R}_n+\mathscr{R}_{n+3}j\\-\mathscr{R}_{n+1}+\mathscr{R}_{n+2}j\end{bmatrix}\equiv\ddot{\phi}_n,$$

where the split quaternionic units i,j,k are satisfied by the rules that are given in Eq. (2). Since these transformations are linear and one-to-one but not onto, these new type sequences are called (s,t)-Padovan and (s,t)-Perrin hyperbolic spinor sequences, respectively. These sequences are linear recurrence sequences and are constructed by using this transformation.

Theorem 26 [**Recurrence Relation**] *For all $n\geq 0$, the following recurrence relations hold for the (s,t)-Padovan and (s,t)-Perrin hyperbolic spinor sequences, respectively:*

$$\ddot{\psi}_{n+3}=s\ddot{\psi}_{n+1}+t\ddot{\psi}_n$$

and

$$\ddot{\phi}_{n+3}=s\ddot{\phi}_{n+1}+t\ddot{\phi}_n.$$

The other equations given in the previous section can be constructed easily for the (s,t)-Padovan and (s,t)-Perrin hyperbolic spinors, as well.

5 Conclusion and future studies

In this paper, we determined Padovan and Perrin hyperbolic spinors. Then, we obtained some algebraic properties and equalities with respect to the conjugations. Also, we gave some equations such as; recurrence relation, Binet formula, generating function, exponential generating function, Poisson generating function, summation formulas, and matrix formulas. Moreover, we established some special equations that present relations between Padovan and Perrin hyperbolic spinors and Padovan and Perrin numbers. Additionally, we got the determinant equalities for calculating the terms of these sequences. Also, we gave a short introduction with respect to the (s,t)-Padovan and (s,t)-Perrin hyperbolic spinors.

In the near future study, we intend to examine generalized Tribonacci hyperbolic spinors.

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