CONFORMAL COMPACTIFICATIONS OF SPACE-TIME AND MOMENTUM-SPACE
Possible Observable Consequences

PAOLO BUDINICH
Interdisciplinary Laboratory for Natural and Humanistic Sciences and International Centre for Theoretical Physics, Strada Costiera, 11, 34014 Trieste, Italy.
budinich@tsmi19.sissa.it

(Received: November 19, 1993)

Abstract. It is conjectured that space-time and momentum space are both conformally compactified and represented by homogeneous spaces of the conformal group generated from two different subgroups transformed in each other by conformal inversion. It is proposed that this hypothesis may be possibly supported by the recently discovered large scale correlation of galaxies in pencil beam surveys, as far as space-time is concerned, while the hydrogen atom in stationary states might represent a support to the conjecture of momentum-space compactification, connected to the space-time one by conformal inversion. Further possible consequences of the hypothesis are briefly outlined.

1. Introduction

Historically the main, often implicit, axiom of physical sciences is represented by the postulated geometrical properties of space and time where physical phenomena occur. As an example newtonian space-time is constituted by $\mathbb{R}^3$-space where euclidean geometry holds and by an oriented straight line $\mathbb{R}^1$ representing time. This space-time is still appropriate for the description of non-relativistic local motions.

The discovery of electromagnetic phenomena and of Maxwell's equations with their Poincaré covariance induced to the axiomatic introduction of Minkowski space-time apt for the local description of relativistic phenomena. This space-time may also be defined group theoretically as an homogeneous space constituted by the Poincaré group divided by its Lorentz subgroup. In this way the axiomatic role is shifted from space-time to its isometry group.

For the description of global phenomena dealt by cosmology Robertson-Walker space-time is often postulated, this also characterized by the symmetry implied by cosmological principle.

Several motivations [1] (of which the main is Maxwell's equations conformal covariance [2]) suggest the hypothesis that the conformal symmetry is a fundamental one for physics of massless systems. If we adopt it, then space-time should be conceived as the compact homogeneous space $\bar{M}$ diffeomorphic to $(S^3 \times S^1)/Z_2$ generated by the conformal group G as follows:

$$\bar{M} = \frac{G}{H_I} = \frac{S^3 \times S^1}{Z_2} \tag{1.1}$$

where the subgroup $H_I = L \times D \bowtie K$ of G is constituted by L = Lorentz-, D = dilatations- and K = special conformal-transformations. In this case the particular space-time mentioned above should be considered as densely imbedded in $\bar{M}$. As a consequence of the embedding, the flat space (as Minkowski) result conformally flat with conformal factors depending on the modality of the embedding [3].

If space-time is represented by the compact manifold $\bar{M}$ given in (1.1) then one should expect that any field theory in it should be free from infrared divergences, but not from ultraviolet ones [4], an unsatisfactory result for a conformal world where dilatation invariance should hold.

An opposite result, that is regularization of ultraviolet divergences but not of infrared ones, could be expected in case of compactified momentum-space.

A suggestion of momentum space compactification may be found in spinor theory. In fact the E. Cartan definition of a simple spinor Ψ associated with the pseudoeuclidean space $V = \mathbb{R}^{m+1,m-1}$ is represented by [5]:

$$p_a \gamma^a \Psi = 0 \quad a = 1, 2, \ldots, 2m \tag{1.2}$$

where p_a are the components of $p \in V$ and γ^a are the generators of the Clifford algebra $Cl(m+1, m-1)$. Notoriously for $\Psi \neq 0$ we have $p_a p^a = 0$, therefore $p \in V$ lies in the projective light cone $\bar{P}$ of V which is compact. In fact:

$$\bar{P} = \frac{S^m \times S^{m-2}}{Z_2} \tag{1.3}$$

The fact that $\bar{P}$ is a good candidate for compactified momentum space derives from the observation that from (1.2) one may easily get several of the fundamental equations of physics like, for $m = 2$, Weyl equation for massless neutrinos:

$$p_\mu \gamma^\mu (1 \pm \gamma_5) \Psi_\pm = 0, \quad \mu = 1, 2, 3, 4$$

and Maxwell equations:

$$p_\mu F_\pm{}^{\mu\nu} = 0, \tag{1.4}$$

and, for $m = 3$, the twistor equations

$$p_a\Gamma^a(1 \pm \Gamma_7)\Pi_\pm = 0 \quad a = 1,2,3,4,5,6 \tag{1.5}$$

and so on [6]. All of them in compactified momentum space represented by eq.(1.3)..

Going back to compactified space-time $\bar{M}$ given by eq.(1.1), one may transform it with conformal inversion I, one of the global transformations of the conformal group. The result is:

$$I\bar{M}I^{-1} = \bar{P} = \frac{G}{H_{II}} = \frac{S^3 \times S^1}{Z_2} \tag{1.6}$$

where $H_{II} = L \times D \bowtie T$ obtained from H_I above by substituting special conformal transformations K with Poincar'e translations T.

Observe that $\bar{P}$ may be also considered as a compactification of $P = \mathbb{R}^{3,1}$, however if in space-time $M = \mathbb{R}^{3,1}$ the coordinates of a point x have the dimension of a length in $P = \mathbb{R}^{3,1}$, the coordinates of a point p have the dimension of the inverse of a length, since: $Ix_\mu = \pm x_\mu/x^2$.

The action of G in $\bar{M}$ unambiguously defines the action of G in $\bar{P}$, and in conformally flat momentum space it might be identified under certain conditions with the standard one [7].

Adopting the suggestion of spinor geometry, we will then adopt the hypothesis that $\bar{P}$ represents compactified momentum space. Furthermore, if we admit the validity of global conformal covariance in nature, since $\bar{P}$ is obtained from $\bar{M}$ as a consequence of conformal inversion: one of the transformations of G, then we will suppose that both space-time $M = \mathbb{R}^{3,1}$ and momentum space $P = \mathbb{R}^{3,1}$ are simultaneously compactified in $\bar{M}$ given by and $\bar{P}$ given by (1.2) respectively, connected by I, conformal inversion.

As a consequence then in a fully conformal theory both infrared and ultraviolet divergences should both be absent, as expected in a conformal world.

It is well known that if one adopts $\bar{M}$ given in (1.1) as space-time, then its dual momentum space may be conceived as an infinite lattice, whose points are labelled by the indices of spherical harmonics [4]. If momentum space is also compactified, then this lattice will have to be mapped in $\bar{P}$ and result therefore finite. From $\bar{P}$ in turn a finite lattice will result in $\bar{M}$. It could be expected that on these two finite lattices, labelled by the quantum number of the spherical functions on $S^3 \times S^1$, Hopf groups may act[1]. This will be further analyzed elsewhere.

In the following we will try to explore if our hypothesis of simultaneous compactification of space-time and momentum space, correlated by eq.(1.6) may find support from some observable phenomena.

[1] This possibility was suggested to me by S.Majid, to whom I am grateful.

2. Observable Consequences

2.1. Space-compactification and the Structure of the Universe

In cosmological applications compactified space-time $\bar{M}$ given in eq.(1.1) is often substituted by [4]:

$$M_{R.W} = S^3 \times I\!R^1 \tag{2.1}$$

where $I\!R^1$ is interpreted as infinite covering of S^1. It is one of the possible compact-space realizations of Robertson-Walker space-time with metric:

$$ds^2 = -dt^2 + R^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 - \sin^2\theta d\phi^2)], \tag{2.2}$$

which may be also obtained by imposing the cosmological principle stating: "the homogeneity and isotropy of space in the universe"[8].

Several cosmological models consider the universe evolution in Robertson-Walker space-time. In a previous paper [9] we took as an example the inflationary model which considers a scalar field ϕ: the "inflaton field", for which the equation of motion $M_{R.W.}$ is:

$$\{R^{-3}\frac{\partial}{\partial t}R^3\frac{\partial}{\partial t} + \frac{1}{R^2}[\Delta(S^3) - 1]\}\phi = 0 \tag{2.3}$$

where $\Delta(S^3)$ is the Laplace-Beltrami operator in S^3, with elementary solutions:

$$\phi_{n\ell m} = f_n(t)Y_{n\ell m}(\chi,\theta,\phi) \tag{2.4}$$

and the spherical functions $Y_{n\ell m}(\chi,\theta,\phi)$ are defined by

$$\Delta(S^3)Y_{n\ell m} = -n(n+2)Y_{n\ell m}. \tag{2.5}$$

Then the component T_{00} of the energy momentum tensor $T_{\mu\nu}$ interpreted as energy-density $\aleph$ has, for eigenmode, the form:

$$\aleph_{n\ell m} = K(t)[Y_{n\ell m}(\chi,\theta,\phi)]^2, \tag{2.6}$$

which represents then a possible eigenvibration of a closed universe compatible with the cosmological principle as already anticipated by E. Schroedinger [10].

Recent observations up to 1400 Mpc in the North-South galactic poles directions have revealed striking correlations of galaxies showing 10 peaks spaced by 128 Mpc [11]. They may be described by eq. (2.6) by identifying energy density with matter density. Assuming the value 3000 Mpc for the inverse $H^{-1} = R\pi/2$ of the Hubble constant, the observations may be rather well represented with just one eigenmode $Y_{n,0,0}$, for which

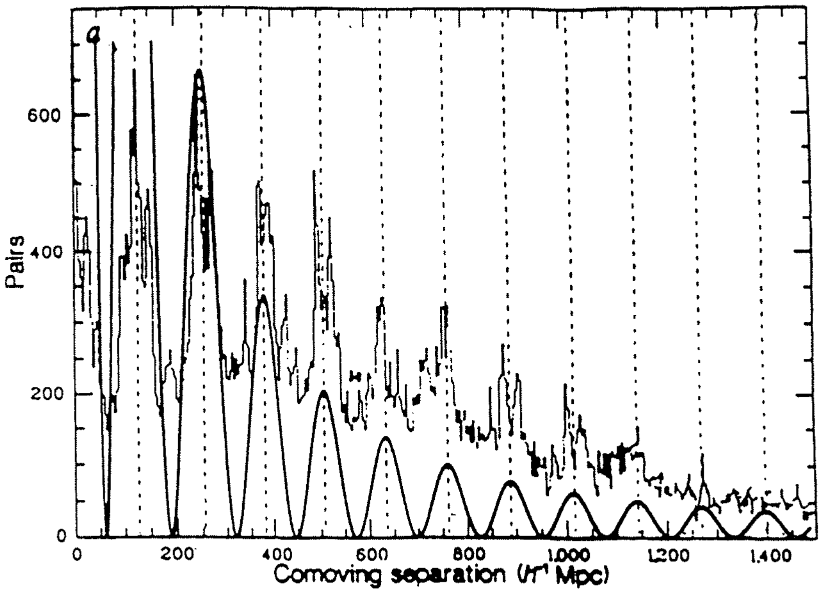

Fig. 1. Observational result on large-scale distribution of galaxies in the direction of south and north galactic poles reproduced from ref. [11], fig. 2a) (broken line). The continuous curve represents the theoretical energy density $\rho(x, x_0)$ given by eq.(2.8). It is normalized to the third peak of the observational data.

$$\aleph_n := \aleph_{n,00}(\chi, t) = K(t)\frac{\sin^2(n+1)\chi}{\sin^2\chi} \tag{2.7}$$

and the unknown time-dependent factor may be eliminated by considering the relative density. For $n = 46$ one has:

$$\rho(\chi_0\chi) = \frac{\aleph_{46}(\chi, t)}{\aleph_{46}(\chi_0 t)} = \frac{\sin^2 47\chi \sin^2\chi_0}{\sin^2\chi \sin^2 47\chi_0}. \tag{2.8}$$

The result is reported from reference [9] in fig. 1.

More observations are being performed confirming the previous ones with some variations in other directions [12]. They seem to be rather well reproducible with the single eigenmode $Y_{45,1,0}$ as it will be reported in a subsequent paper.

Should the present trend be confirmed from further observations and computations, one might conclude that they may constitute a confirmation of the role of $\bar{M}$ in the universe (or at least of the S^3 part of it).

2.2. Momentum-space Compactification and the H-atom

In the frame of the hypothesis proposed in the introduction the compactification $\bar{M}$ of space-time should imply the compactification $\bar{P}$ of momentum space and the two should be correlated by conformal inversion I, as shown in eq.(1.6).

It is well known that I maps every point x_μ of Minkowski space-time $\mathbb{R}^{3,1}$ in x_μ/x^2, say:

$$I : x_\mu \rightarrow \frac{x_\mu}{x^2}$$

and therefore for space-like x_μ, every point inside a sphere S^2 of radius one centered in the origin is mapped by I to a point outside it; one could call "small" and "large" the space inside and outside S^2 respectively however these words have no meaning in a conformal world. In order to give them meaning one should give a dimensional radius to S^2 but then conformal covariance would be broken. The situation is different with the dimensionless scalar product $x_\mu k^\mu$ where k^μ is a point of the wave-number space dual to x_μ; then, in fact:

$$I : x_\mu k^\mu \rightarrow \frac{x_\mu k^\mu}{x^2 k^2},$$

and if one substitutes k_μ with momentum $p_\mu = \hbar k_\mu$, then S^2 has radius $\hbar^2$ and the words "large" and "small" refer to classical and quantum system. Therefore if we consider the universe considered in section 2.1 as a "large" classical system we should, by conformal inversion, obtain a system in

$$P_{R.W.} = S^3 \times \mathbb{R} \tag{2.9}$$

where S^3 represents compactified momentum space (3-dimensional) and R energy and search for quantum systems whose equation of motion is represented by eq.(2.5) in momentum space. Such systems indeed exist and they represent the most common element in the universe: the H-atom.

In fact let us represent the sphere S^3 in (2.9) by:

$$S^3 : \pi_1^2 + \pi_2^2 + \pi_3^2 + \pi_5^2 = 1,$$

then, as shown by V.Fock [13] eq.(2.5) on S^3 is equivalent to the integral equation

$$\phi(\pi) = \frac{\lambda}{2\pi^2} \int \frac{\phi(\pi')}{(\pi - \pi')^2} d\Omega'$$

where $\pi = \{\pi_1\pi_2\pi_3\pi_5\}$ is a point on S^3 and which, for $\lambda = n+1$ is satisfied by the spherical harmonics $\phi = Y_{n\ell m}$. If we stereographically project S^3 on $\mathbb{R}^3$-momentum space through

$$p_j = \frac{p_0\pi_j}{1+\pi_5}, \qquad j = 1,2,3,$$

where $p_0 = mc$ is a unit of momentum (m is the mass of the electron) we easily obtain, after setting

$$n+1 = \lambda = \frac{e^2}{\hbar c}\sqrt{\frac{mc^2}{-2E}}, \qquad n = 0,1,2...$$

and

$$\Psi_{n\ell m}(\mathbf{p}) = K(p,p_0)Y_{n\ell m}(\chi,\theta,\phi),$$

with $K(p,p_0)$ a normalization factor which renders $\Psi_{n\ell m}$ orthonormal in $\mathbb{R}^3$, the Schrödinger equation of the H-atom in momentum space:

$$\frac{1}{2m}p^2\Psi_{n\ell m}(\mathbf{p}) + \frac{e^2}{2\pi\hbar^2}\int\frac{\Psi_{n\ell m}(\mathbf{p}')}{|\mathbf{p}-\mathbf{p}'|^2}d^3p = E_n\Psi_{n\ell m}$$

where

$$E_n = \frac{me^4}{2\hbar(n+1)^2}.$$

It may be shown that the introduction of p_0 is equivalent to giving a radius p_0 to S^3.

It is known that the whole spectrum of the H-atom may be obtained by acting on $\Psi_{n\ell m}$ with the algebra of SO(4,2) realized in momentum space [14].

Therefore the eigenfunction $Y_{n\ell m}(\chi,\theta,\phi)$, possible candidates for the interpretation of the large-scale distribution of matter in the universe are proportional to the Fourier transforms of the H-atom eigenfunctions in stationary states.

Obviously this correspondence is not unique. In fact there might be more systems both in space-time and, correspondingly, in momentum space, whose geometrical structure (and dynamics) is determined by that of $\bar{M}$ and correspondingly of $\bar{P}$. One of them is represented by planetary systems. But more may exist. Each might be characterized by the possible radius of S^3 like the two considered here (R for the universe and $p_0 = mc$ for the H-atom).

3. Conclusion

Should the hypothesis formulated in this paper be confirmed by further observations and computations then one might draw from it several further interesting consequences. One of them could be that if the eigenvibrations $Y_{n\ell m}(\chi, \theta\phi)$ are apt to represent the observed large scale structure of the universe, then they would mean that the cosmological principle is broken, and not just locally by matter condensations as we well know, but also by the large scale structure of space represented in our particular model by the eigenvibrations of the inflaton field ϕ.

It would be an example of spontaneous symmetry breaking, of the group $SO(4)$ in this case, which is instead the isometry group of S^3 in $M_{R.W.}$ given by (2.1), where the equations of motion (2.3) and (2.5) are formulated. One should then distinguish between the "homogeneous space" on one side, which is determined by the group of symmetry, in this case $SO(4)$, dictated by the symmetry implied by the Cosmological principle, where the equations of motions are written and the "physical space" on the other side obtained from the solutions of equations of motion written in that homogeneous space, which then may break spontaneously the postulated symmetry (in a similar way as the Kepler orbits spontaneously break the $SO(3)$ symmetry of the equations of motions). The cosmological principle then applies to the "homogeneous space" but not to the "physical one". Furthermore, while the "homogeneous space" is finite and continuous for the transitive action of the group, the "physical space" might be both finite and discrete.

The concept of a finite space is not new: it is contemplated in the cosmology of a closed universe where space is represented by S^3 with radius R equal $\sim 10^{10}$ light years, say. In such a universe the concept of distance has an upper limit: πR beyond which the mere concept of distance looses its meaning. Such a closed universe implies the existence of a minimal momentum $\sim \hbar R^{-1}$, below which the mere concept of momentum is meaningless. This, by itself originates the conception of physical momentum space as an infinite lattice.

We have seen that spinor geometry and conformal inversion induces to conjecture that homogeneous momentum space should also be compact, represented by a sphere S^3 of radius M, say, But then the above lattice will also have to be finite; not only but the existence of a maximal momentum $\sim \pi M$ will imply the existence of a minimal distance $\sim \hbar M^{-1}$ in physical ordinary space, below which the mere concept of distance is meaningless, and the physical ordinary space will also have to be represented by a finite lattice, dual to the one of physical momentum space.

Concluding, we may affirm that the postulated conformal symmetry axiomatically defines the "homogeneous, compact space" where the postulated group acts continuously and transitively. The "physical space" instead may not only spontaneously break that symmetry, but also might naturally result discrete and finite, on which then the mere concepts of infinity and infinitesimal are absent.

It is to be underlined that this last property derives not only from the identification of the "physical space" as originated from some functional of the solutions of some field equation on the compact "homogeneous space", but also from any single valued function on it. This should then in principle be applicable also to general relativity where space-time, conceived as a field (obviously not scalar), should, in our language, be classified as a "physical space" to which all the considerations exposed above (after extending them to tensor fields) should apply.

Observe that this possible group-theoretical genesis of finite, dual lattices in ordinary and momentum "physical spaces" differs both from the one pursued, after the introduction of a fundamental length, in non local field theories some decades ago, and from the one adopted for the numerical solution of Q.C.D. field equations.

These ideas will be further analyzed elsewhere.

Acknowledgements

The author wishes to thank L. Dąbrowski, W. Heidenreich, S. Majid, P. Nurowski, R. Rączka and A. Trautman for helpful discussions and suggestions.

References

[1] Dirac P.A.M., *Ann Math* **37** (1939) 429; M. Flato and Sternheimer D., *C.R. Acad. Sc.* (Paris) **263** (1963) A936; Zeeman E.C., *J. Math. Phys.* **5** (1964) 490; Mack G. and Todorov I.T., *Phys. Rev.* **D8** (1973) 1764; Penrose R. and Rindler W., *Spinors and Spacetime* (Cambridge Univ. Press, Cambridge, (1984).
[2] Cunningham E., *Proc. Londorn Math. Soc.* **8** (1909) 77; Bateman H., Proc. London Math. Soc. **8** (1909) 223
[3] Barut A.O., Budinich P., Niederle J. and Rączka R., in preparation.
[4] Heidenreich W.F., *Invariant causal propagators in conformal space,* to appear in the Proceedings of the Adanved Research Workshop, San Antonio, Texas, 1993.
[5] Cartan E., *Lecons sur la théorie des spineurs* (Herman, Paris 1937).
Budinich P. and Trautman A., *the spinorial chessboard* (Trieste Notes in *Physics, Springer*, New York 1988).
[6] Budinich P., *Found. of Physics* **23** (1993) 949.
[7] Budinich P. and Rączka R., *Found. of Physics* **23** (1993) 599.
[8] Weinberg S., *Gravitation and Cosmology*, pg. 409 (Wiley, New York 1972).
[9] Budinich P. and Rączka R., *Found. of Physics* **23** (1993) 225 and *Europhys. Lett.* **23** (4) (1993) 295.
Buninich P., *Ordinary - and momentum-space conformal compactifications,* to apperar in the Proceedings of the Advanced Research Workshop, San Antonio, Texas, 1993.
[10] Schrödinger E., *Physica VI* (1939) 272.
[11] Broadhurst T.J., Ellis R.S., Koo D.C. and Szalay A.S., Nature (London) 343 (1990) 726.
[12] Szalay A.S., Broadhurst T.J., Ellman N., Koo D.C. and Ellis R.S., *Proc. Nat. Acad. Sci. U.S.A.* **90** (1993) 4853.
[13] Fock V., Zeitschrift *f. Physik*, **98** (1935) 145.
[14] Barut A.O. and Bronzin G.L., *J. Math. Phys.* **15** (1974) 1000.