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Symmetry breaking and Goldstone bosons in holographic strongly coupled field theories

Relativistic and non-relativistic examples

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The original contributions of this thesis are based on the following papers.

[I] Riccardo Argurio, A.M., Andrea Mezzalira, Daniel Naegels, "Note on holographic nonrelativistic Goldstone bosons", Phys. Rev. D **92** (2015) no.6, 066009, Jul 2015, [arXiv:1507.00211 [hep-th]].

[II] Riccardo Argurio, A.M., Andrea Mezzalira, Daniele Musso, "Analytic pseudo-Goldstone bosons", JHEP **1603** (2016) 012, Dec 2015 [arXiv:1512.03750 [hep-th]].

[III] Riccardo Argurio, Gaston Giribet, A.M., Daniel Naegels, J. Anibal Sierra-Garcia, "Holographic Ward identities for symmetry breaking in two dimensions", JHEP **1704** (2017) 007, Dec 2016 [arXiv:1612.00771 [hep-th]].

[IV] Riccardo Argurio, Jelle Hartong, A.M., Daniel Naegels, "Symmetry breaking in holographic theories with Lifshitz scaling", Sep 2017 [arXiv:1709.08383 [hep-th]].

The following paper is also part of the work published by the author during the PhD.

[V] Eduardo Conde and A.M., "Lorentz Constraints on Massive Three-Point Amplitudes", JHEP **1609** (2016) 041, Jan 2016, [arXiv:1601.08113 [hep-th]].

It is not included in this thesis, since it already constitutes part of the material of the following published lecture notes.

[VI] A.M., "The 4D on-shell three-point amplitude in spinor-helicity formalism and BCFW recursion relations" (lecture notes), Proceedings of the *XII Modave Summer School in Mathematical Physics*, in *Proceeding of Science*, PoS(Modave2016)002, May 2017 [arXiv:1705.09678 [hep-th]].

Abstract

In this thesis various holographic models are treated, which describe theories of fields where an internal symmetry is broken, either in relativistic contexts, or in case of violation of the Lorentz invariance.

The first chapter opens with the revision of the notion of symmetry breaking in pure relativistic field theory. The case of spontaneous breaking and the Goldstone theorem are discussed, as well as the case of explicit breaking, where precise Ward identities between conserved current correlators and scalar operators loaded under such current are derived in a completely general way.

We then consider two examples of non-relativistic field theories, which will be reproduced by holographic models: a model in which the invariance of boosts is broken by the presence of a chemical potential, and a model of Lifshitz's invariant theory. We show the non-relativistic realization of Ward's identities for the symmetry breaking.

In the second chapter we briefly introduce the correspondence gravitation / gauge theory and we revise the central tool of this thesis, the holographic renormalization.

In the third chapter, we show how to generate field theories with symmetry breaking by coupling a scalar field to a gauge field, and holographically deriving the Ward identities predicted by the field theory arguments, first in the Relativistic case. We also obtain an analytic expression for the scalar two-point function, where we know how to find the massless boson of Goldstone and the mass of linear mass in the explicit breaking parameter Of the Goldstone pseudo-boson, respectively in the purely spontaneous case and in the case of an explicit small break.

We also consider the two-dimensional case on the edge, where we find that Coleman's theorem is eluded in the wide limit of N, and Ward's identities are not affected.

For non-relativistic cases, we first consider a non-abelian model in which the Lorentz invariance is broken: this situation makes it possible to observe so-called B bosons which exhibit a quadratic dispersion relation and do not respect Not the law of a single Goldstone mode for each broken generator.

Finally, we study in detail the holographic renormalization and the two-point functions for a conserved current and various scalar operators in a space-time of Lifshitz. We also find the Ward identities of symmetry breaking in their non-relativistic realization.

KEY WORDS: Symmetry breaking, Goldstone bosons, Holography, Gauge/Gravity duality, Lifshitz.

Résumé

Dans cette thèse divers modèles holographiques sont traités, qui décrivent des théories des champs où une symétrie interne est brisée, soit dans des contextes relativistes, soit en cas de violation de l'invariance de Lorentz.

Le premier chapitre s'ouvre sur la révision de la notion de brisure de symétrie en théorie des champs relativiste. La cas de brisure spontanée et le théorème de Goldstone sont discutés, ainsi que le cas de brisure explicite, où des identités de Ward précises, entre correlateurs du courant conservé et de l'opérateur scalaire chargé sous tel courant, sont derivées de façon complètement générale.

En suite on considère deux exemples de théories des champs non-relativistes, qui seront reproduits par les modèles holographiques : un modèle où l'invariance de boosts est brisée par la présence d'un potentiel chimique, et un modèle de théorie invariante de Lifshitz. On montre la réalisation non-relativiste des identités de Ward pour la brisure de symétrie.

Dans le deuxième chapitre on introduit brièvement la correspondance gravitation/théorie de jauge et on revise l'outil central de cette thèse, la renormalisation holographique.

Dans le troisième chapitre, on montre comment générer des théories des champs avec brisure de symétrie en couplant un champ scalaire à un champ de jauge, et on derive holographiquement les identités de Ward prédites par les arguments de théorie de champ, d'abord dans le cas relativiste. On obtient aussi une expression analytique pour la fonction à deux points scalaire, où on sait retrouver le pôle sans masse du boson de Goldstone et le pôle massif, de masse linéaire dans le paramètre de brisure explicite, du pseudo-boson de Goldstone, cela respectivement dans le cas purement spontané et dans le cas de petite brisure explicite petite par rapport à la brisure spontanée.

On considère aussi le cas à deux dimensions sur le bord, où l'on retrouve que le théorème de Coleman est éludé dans la limite de large N, et les identités de Ward ne sont pas affectées.

Pour les cas non-relativistes, on considère d'abord un modèle non-abelien où l'invariance de Lorentz est brisée : cette situation permet d'observer des bosons dit de type B, qui présentent une relation de dispersion quadratique et ne respectent pas la loi d'un seul mode de Goldstone pour chaque générateur brisé.

Enfin, on étudie en detail la renormalisation holographique et les fonctions à deux points pour un courant conservé et divers opérateurs scalaires dans un espace-temps de Lifshitz. On retrouve pour ce cas également les identités de Ward de brisure de symétrie dans leur réalisation non-relativiste.

MOTS CLEFS : Brisure de symétrie, Bosons de Goldstons, Holographie, dualités gravitation/théorie de jauge, Lifshitz.

Introduction

Spontaneous symmetry breaking, with its associated Goldstone massless modes, is an ubiquitous phenomenon in physics, especially in particle physics and condensed matter physics. Emblematically, its first observation was at the intersection of the two fields [1]. It finds its theoretical grounds in the Goldstone theorem [2], which is a very well defined mathematical statement, at least in the relativistic framework where it has been first proven [3].

If relativistic invariance is an essential ingredient in particle high energy physics, it is likely disregarded in condensed matter systems, where Goldstone theorem has known many successful applications in describing phase transitions (superconductivity, Bose-Einstein condensation, etc). So, the extension of Goldstone theorem to non-relativistic field theories [4–7] has drawn attention since the early stages of its conception. In the absence of Lorentz invariance, the physics of Goldstone bosons (GB) gets richer, and its mathematical foundations have to be revised and extended.

For instance, the one-to-one matching between broken symmetry generators and Goldstone massless modes breaks down in non-relativistic settings [8]. Moreover, the dispersion relations of gap-less modes can have more diversified forms than the relativistic one $\omega = |\vec{k}|$. We can have Goldstone bosons with linear dispersion relation, $\omega = c_s|\vec{k}|$, but with velocity now non-necessarily equal to the speed of light $c \equiv 1$. But we can also have Goldstone modes which present dispersion relations with quadratic or higher-order low-momentum behavior. In particular, in non-relativistic theories with spontaneously broken *non-abelian* global symmetries, the commutator of two broken charges may get a non-vanishing expectation value. In such situation, the two corresponding broken generators give rise to a single Goldstone mode, which presents in turn a quadratic dispersion relation, $\omega \propto |\vec{k}|^2$ [9–11]. The second 'would-be massless' mode typically gets a mass proportional to the Lorentz breaking parameter [12].

In more recent years, another input for theoretical physicists comes from condensed matter systems: a large variety of experimental devices presents 'strange' behaviors, which can be ascribed to strong coupling effects. By strange it is meant that are not predicted by present-day available theoretical models (weakly-coupled Fermi-liquid theory), which are based on perturbation theory, and so fail at strong coupling.

Of course, whatever is based on symmetries is non perturbative. So, the physics of symmetry breaking, spontaneous or (slightly) explicit, is particularly relevant for strongly coupled systems, where it can constitute some of the (few) things we are able to say about. To this extent, a renewed interest in such an old and broadly discussed topic, enlarged towards these unconventional non-relativistic frameworks, may be justified.

At the same time, new techniques, that be capable to deal with strong coupling, are called for. Holography, which is a quite recent and active field of research, is a promising candidate. Since it has been conjectured, exactly twenty years ago [13], as a duality between compactifications of string theory to Anti de Sitter spacetime (AdS), in the classical limit, and supersymmetric conformal field theories on the boundary of AdS, in the 't Hooft limit (large N, strong coupling), the holographic principle has been extended to a much larger class of dualities. In such dualities the classical limit of the gravitational theory in the bulk corresponds to the strong coupling limit of the quantum field theory on the boundary.

So, holography potentially constitutes a powerful tool to investigate the strong coupling regime of field theory. Indeed, the classical regime of gravity, which is generically well understood and under control, can be used to gain insights over the physics of field theory at strong coupling (confinement, bound states, quantum phase transitions, etc).

Non-relativistic behaviors are observed in many of the above-mentioned condensed matter experiments (high temperature superconductors, cold atoms at unitarity, etc). So, the ambition to apply holography to condensed matter has raised the challenge of extending the correspondence to non-relativistic backgrounds [14–19].

A particular class of non-relativistic field theories, which is dealt with in this dissertation, is given by Lifshitz invariant field theories. Lifshitz symmetry [20] is a non-relativistic, anisotropic version of scaling invariance. In condensed matter theory, Lifshitz scaling is relevant to describe the quantum critical point of certain materials, such as MnP magnets [21], cuprate superconductors, non-Fermi liquids. The Lifshitz critical point is subject matter of various theoretical models [22, 23]. From the pure field-theoretical point of view, the subject of symmetry breaking in Lifshitz field theories has been addressed, and various examples of Lifshitz invariant low-energy effective field theories for Goldstone bosons have been realized [24–27].

In this thesis, we study holography as a powerful tool to generate strongly coupled field theoretical environments, where to robustly probe our understanding of the physics of symmetry breaking. At the same time, we provide the opportunity to test the holographic conjecture in a variety of non-canonical contexts, such as broken Lorentz invariance, Lifshitz space-times, low dimensions.

The objects that we aim to compute holographically are two-point correlation functions of conserved currents and of scalar operators, charged under those currents. Such two-point functions contain the spectrum of the field theory, so in particular the Goldstone pole, and its dispersion relation, which, in case of broken Lorentz invariance, can exhibit deviations from the relativistic behavior. Since we are interested in two-point functions of scalars and conserved currents, gravity is kept non-dynamical throughout all this work. The metric is chosen each time as a fixed background, making sure that back-reaction of other fields on the metric be negligible as far as our computations are concerned.

Furthermore, spontaneous or explicit symmetry breaking entails specific relations (Ward identities) among correlation functions of a conserved current and the scalar operator which is charged under it. These Ward identities, which are derived in full generality in field theory, can be retrieved in holography through a pure boundary analysis, constituting thus an independent check of the correspondence. Such boundary analysis typically requires the treatment of bulk terms that diverge at the boundary, which can be consistently removed by a procedure called holographic renormalization [28–30].

The holographic renormalization procedure depends sensitively on the number of spacetime dimensions, on the scaling dimensions of the considered field, on the background metric, etc. So, it can in some occasions present subtleties, which are extensively discussed in this work, whenever encountered. Furthermore, our interest in non-relativistic field theories pushes us to study holography for Lifshitz bulk geometry, which is a still not so well established correspondence in the zoo of gauge/gravity dualities.

So, along the way, we will derive the Ward identities for symmetry breaking in various holographic setups, recovering the field theoretical expectations. In some favorable cases, we will also be able to obtain explicit expressions for the scalar two-point function, displaying the Goldstone (or pseudo-Goldstone) boson pole, together with the complete spectrum of the theory, which exhibits some typical features of strongly coupled theories (linear confinement).

Outline

Let us go through the contents of this thesis in an orderly fashion. In Part I we discuss some preliminary material, which consists of well-established subjects, but still presented through a partially original point of view. This first part prepares the ground for Part II, where the original contributions of this thesis are reported, based on various published works of the author [I, II, III, IV].

As far as Part I is concerned, in Chapter 1 we start with reviewing, in a pure field theoretical approach, the physics of symmetry breaking. In Section 1.1 we recall the Noether theorem and the relation between symmetry and conserved currents, we introduce the spontaneous symmetry breaking mechanism, and we reproduce the original non-perturbative proof of Goldstone theorem [3]. In this section we are close to the approach of Weinberg's textbook [31]. In Section 1.1.2 we discuss the particular case of 1+1 dimension and the no-Goldstone theorem by Coleman [32], commenting in particular on its quantum nature [33], and on the peculiarities of the large N limit [34].

Then, in Section 1.2, we allow for explicit breaking on top of spontaneous one, and we derive in maximal generality (without specializing to any specific theory) the corresponding Ward identities, which in the purely spontaneous case reveal the presence of a massless mode, and in the case of (little) explicit breaking imply the notorious GMOR relation [35] for the (little) mass of the pseudo-Goldstone boson. Our derivation is standard, but partially novel in its formulation, which follows the one presented in [II]. For these initial sections, we also found a helpful tool in [36].

Then, in Section 1.3, we abandon Lorentz invariance, and we discuss the issues of the generalization of Goldstone theorem to non-relativistic field theories. We considerably owe to [7, 37], however we give a different, original derivation of non-relativistic dispersion relations, starting from the symmetry breaking Ward identity. In Section 1.3.1 we consider a field theory model [38, 39], equipped with a non-abelian global symmetry, where Lorentz invariance is explicitly broken by a chemical potential, and the so-called type B Goldstone boson arises. Finally, in Section 1.3.2, we consider another example of non-relativistic field theories, Lifshitz invariant field theories, and we discuss correlators and Ward identities in this context.

The second chapter is dedicated to the holographic correspondence [13, 40], and in particular to the technique of holographic renormalization [28–30], which is the main computational tool of this thesis. After giving a sketchy review of the holographic principle, close in spirit to [41], in Section 2.1 we introduce the renormalization procedure by a simple, but rather complete example, presented in a quite personal way.

Then, we are ready to move to the second part, where the original contributions of this thesis are collected [I, II, III, IV]. We present there various holographic setups that originate field theories with symmetry breaking: in Chapter 3 we have some relativistic examples, while the non-relativistic examples are left for Chapter 4 and 5.

In Section 3.1 the Ward identities for concomitant spontaneous and explicit symmetry breaking, shown in Section 1.2, are retrieved holographically [II]. An analytic expression for the boundary scalar two-point function is provided [II], exhibiting the pseudo-Goldstone poles, with the proper GMOR linear dependence of the mass on the explicit breaking parameter. We then move to the $1+1$ dimensional case in Section 3.2, where we show that in holography, as far as Ward identities are concerned, the Coleman theorem breaks down [III], due to the classical/large-N nature of our boundary analysis, which does not capture the quantum effects that are responsible for no spontaneous symmetry breaking in two dimensions.

As first non-relativistic example, in Chapter 4 we consider a holographic setup that produces a type B Goldstone boson, retrieving holographically [I] the non-relativistic Ward identities of Section 1.3.1.

Finally, in Chapter 5, holographic renormalization and two-point functions for conserved currents and scalar operators on Lifshitz background are studied in detail, concluding with the derivation, once more, of the non-relativistic Ward identities for symmetry breaking.

Contents

Notations and conventions

We briefly establish here the main notations and conventions that are used throughout this manuscript. First of all, we work in natural units $c = 1 = \hbar$, and in mostly-plus Lorentzian signature. Then, for the majority of prescriptions regarding analytic continuation and $\pm i$ factors in quantum field theory prescriptions, we employ the same conventions as Weinberg in his textbooks [31, 42]. Einstein convention about summation over repeated indices is also understood, unless otherwise specified.

In the holographic sections, we use roman indices m, n, r, s for the $(d+1)$-dimensional bulk coordinates, saving greek letters μ, ν, ρ, σ as indices of the flat Minkowski space on the boundary. When, for non-relativistic examples, a distinction between temporal and spatial components is required, we make use of roman characters i, j, k, l for the spatial components. The notation $x^\mu = (t, \vec{x})$ is also widely employed. So, for instance, for the AdS$_{d+1}$ metric in the Poincaré patch, we will write

$$\mathrm{d}s^2 = \mathrm{d}x_m \mathrm{d}x^m = r^{-2}\left(\mathrm{d}r^2 + \mathrm{d}x_\mu \mathrm{d}x^\mu\right) = r^{-2}\left(\mathrm{d}r^2 - \mathrm{d}t^2 + \mathrm{d}x_i \mathrm{d}x_i\right) .$$

Moreover, in the holographic parts we will specify the space-time dimensions in the integration measure, in order to allow to distinguish bulk terms from boundary terms. Otherwise, by simply $\mathrm{d}x$ we will indicate integration over all space-time coordinates, and by $\mathrm{d}\vec{x}$ integration over all spatial coordinates.

Finally our conventions for Fourier transformations to momentum space are

$$f(x) = \int \mathrm{d}k\, e^{ikx} f(k) = \int \mathrm{d}\omega \mathrm{d}\vec{k}\, e^{-i\omega t + i k_i x_i} f\left(\omega, \vec{k}\right) ,$$

so that we schematically have $\partial_t \to -i\omega$, and $\partial_i \to +ik_i$.

Part I

Preliminary material

1 Symmetry breaking in quantum field theory

In this chapter we want to review some very well known field-theoretical results, re-deriving them in a perspective that will make them easier to be compared to the corresponding results obtained in Part II through holographic techniques.

So we will put on the table the main physical ingredients of this dissertation, that is internal symmetries and Noether conserved currents, spontaneous breaking and Goldstone theorem, explicit breaking and Ward identities.

1.1 Spontaneous breaking and Goldstone theorem

We consider an action functional S on a d-dimensional space-time, depending on some field content $\{X_A\}$, and derivatives. The fields X_A represent all the real scalar degrees of freedom occurring in the action. We assume that such action is invariant under some internal (*i.e.* spacetime independent) continuous symmetry:

$$\delta_\alpha S\big[\{X_A, \partial_\mu X_A\}\big] = 0 \,, \qquad \text{for } \delta_\alpha X_A = T^a_{AB} X_B \alpha^a \,, \tag{1.1}$$

where repeated indices imply a summation, and α^a is the infinitesimal parameter associated to the transformation performed by the generator T^a of the considered symmetry group, in the representation under which the field X_A transforms.

Then we can apply this variation on the Lagrangian density, obtaining

$$\begin{aligned}\delta_\alpha S = \int dx\ \delta_\alpha \mathscr{L} &= \int dx\ \left[\frac{\delta \mathscr{L}}{\delta X_A}\delta_\alpha X_A + \frac{\delta \mathscr{L}}{\delta \partial_\mu X_A}\delta_\alpha \partial_\mu X_A\right] \\ &= \int dx\ \left[\left(\frac{\delta \mathscr{L}}{\delta X_A} - \partial_\mu \frac{\delta \mathscr{L}}{\delta \partial_\mu X_A}\right)\delta_\alpha X_A + \partial_\mu\left(\frac{\delta \mathscr{L}}{\delta \partial_\mu X_A}\delta_\alpha X_A\right)\right] \\ &= \int dx\ \partial_\mu\left(\frac{\delta \mathscr{L}}{\delta \partial_\mu X_A} T^a_{AB} X_B\right)\alpha^a\ ,\end{aligned} \tag{1.2}$$

where we have integrated by part, and used the equations of motion for the classical fields X_A. We can define a current

$$J^{a\mu} = -\frac{\delta \mathscr{L}}{\delta \partial_\mu X_A} T^a_{AB} X_B \tag{1.3}$$

for each generator of the symmetry group. Then, the invariance of the action under the considered symmetry (1.1) implies that the integrated expression (1.2) has to vanish. If we assume that the spatial part of the current vanishes at large enough distances, which is always true for sufficiently causal theories, then we have

$$\begin{aligned}0 = \int dx\ \partial_\mu J^{a\mu} &= \int_0^T dt \int_{\mathbb{R}_{d-1}} d\vec{x}\ \left(-\partial_t J^a_t + \partial_i J^a_i\right) = \\ &= -\int_{\mathbb{R}_{d-1}} d\vec{x}\ J^a_t(t,\vec{x})\Big|_0^T = Q^a(0) - Q^a(T)\ ,\end{aligned} \tag{1.4}$$

where

$$Q^a = \int d\vec{x}\ J^a_t(t,\vec{x}) = \frac{\delta \mathscr{L}}{\delta \partial_t X_A} T^a_{AB} X_B \equiv \int d\vec{x}\ \Pi_A T^a_{AB} X_B\ , \tag{1.5}$$

with Π_A being the canonical conjugate momenta of the field X_A.

We have thus obtained the central result of Noether theorem: for any given symmetry of the action a quantity conserved in time arises, the conserved charge. However, the fact that the integral (1.4) vanishes does not imply that the current (1.3) is locally conserved. It can still be $\partial_\mu J^{a\mu} = \partial_\mu \Omega^\mu$, with Ω^μ vanishing at infinity. Then we can redefine the current as $\tilde{J}^{a\mu} = J^{a\mu} - \Omega^\mu$, so that $\partial_\mu \tilde{J}^{a\mu} = 0$. Furthermore, the current is still defined up to en equivalence class defined by

$$J^{a\mu} \longrightarrow J^{a\mu} + \partial_\nu K^{[\mu\nu]}\ ,$$

where the divergence of any anti-symmetric tensor is annihilated by construction under a further action of ∂_μ. So, the conserved current is actually the part which is not trivially divergence-less[1] Anyway, the integration (1.4) ensures that the conserved charge (1.5) is unambiguously defined.

Let us now consider the very common case where the action is invariant under a certain symmetry, whereas the ground state is not. Indeed, when a system possesses a certain continuous global invariance, the space of vacua likely does as well: the lowest energy state is degenerate, and gives a continuum of possible vacua, related to each other by transformations

[1] In the language of differential geometry the current is a closed form which is not exact.

of the symmetry group.[2] Then, the ground state of the system is just one specific vacuum out of the continuum of possible ones, which thus breaks the invariance. This phenomenon is called *spontaneous* symmetry breaking, as opposed to *explicit* symmetry breaking, where instead the symmetry is broken at the level of the action and the current is not conserved anymore.

The discussion here is completely general, but if we wish, we can just think, as an example, of the paradigmatic toy-model of a complex scalar with Goldstone Mexican-hat potential [2]. In that case the action is invariant under a global U(1), and the potential yields a continuum of vacua, arranged on a circle in the phase space, covered by the U(1) transformations. A specific choice of vacuum on the ground circle will break the U(1) invariance.

We are now ready to see the physical consequences of such situation, which are condensed in the statement of the celebrated Goldstone theorem [2]: for each spontaneously broken generator of a continuous global internal symmetry, a zero-energy mode appears in the spectrum. This statement strictly holds in the context of relativistic invariance and internal symmetries, and so will the proof we are going to report here after. Nonetheless, the statement can be relaxed and generalized to the non relativistic case, as we will see in the following. Goldstone theorem can also be applied, in some cases, to the spontaneous breaking of space-time symmetries [43–46], which however will not be considered in this dissertation.

1.1.1 General proof of relativistic Goldstone theorem

We now outline the general proof (not relying on perturbation theory) of Goldstone theorem, first given in [3], and presented in more detail in chapter 19 of Weinberg's textbook [31]. Our assumptions are: (1) the invariance under a global continuous symmetry, which gives a conserved current and the associated conserved charge; (2) the existence of a vacuum expectation value (vev) which is not invariant under that symmetry, thus spontaneously breaking the symmetry. Translational invariance of the vacuum, and Lorentz invariance are also ingredients that we will make use of in the proof.

Using the defining expressions for conserved current (1.3) and charge (1.5), and the canonical commutation relation for quantum fields, that is

$$\left[X_A(t,\vec{x}),\Pi_A(t,\vec{y})\right] = i\delta(\vec{x}-\vec{y}) \,, \tag{1.6}$$

we can restate the spontaneous symmetry breaking condition as follows

$$\left\langle\left[Q^a, X_A(0)\right]\right\rangle_0 = -i\, T^a_{AB}\left\langle X_B(0)\right\rangle_0 \neq 0 \,, \qquad \text{for some } A \,. \tag{1.7}$$

[2]The idea holds equivalently for discrete symmetries, which in turns gives a discrete sets of vacua. However, the follow-up of the discussion and Goldstone theorem are only valid for continuous symmetries (see for instance the example of the real scalar field in Goldstone's original paper [2]).

Notice that this definition is equivalent to saying that the vacuum be not invariant under the symmetry generator; if it was the case, then $Q^a|0\rangle = 0$, and consequently the commutator (1.7) would vanish as well. The symmetry breaking vev $\langle X_A\rangle$ is also called order parameter.

We now take the vacuum expectation value of the commutator between the four-current and the scalar field, and we insert the decomposition of the identity as sum over the spectrum of energy-momentum eigenstates $|n\rangle \equiv \big|\omega(\vec{k}_n), \vec{k}_n\big\rangle$,[3] that is $P^\mu|n\rangle = k_n^\mu|n\rangle$. At the same time, we translate the fields to the origin, that is

$$e^{-iP_\mu x^\mu} J^\mu(x) e^{+iP_\mu x^\mu} = J^\mu(0)\ ,$$

and the same for X_A, and we use the translational invariance of the vacuum, $e^{iP\cdot x}|0\rangle = 0$, finally obtaining

$$\begin{aligned} \big\langle [J^\mu(x), X_A(y)]\big\rangle_0 &= \\ &= \sum_n \Big(e^{ik_n(x-y)} \langle 0|J^\mu(0)|n\rangle\langle n|X_A(0)|0\rangle - e^{-ik_n(x-y)}\langle 0|X_A(0)|n\rangle\langle n|J^\mu(0)|0\rangle\Big) \\ &= \int \mathrm{d}k \sum_n \delta\big(k-k_n\big)\Big(e^{ik(x-y)} \langle 0|J^\mu(0)|n\rangle\langle n|X_A(0)|0\rangle - e^{-ik(x-y)}\langle 0|X_A(0)|n\rangle\langle n|J^\mu(0)|0\rangle\Big)\ . \end{aligned} \tag{1.8}$$

The expression in the parenthesis carries a Lorentz vector index, so Lorentz invariance dictates that as a function of momentum it must be proportional to k^μ. Thus we can define

$$\sum_n \delta\big(k-k_n\big)\langle 0|J^\mu(0)|n\rangle\langle n|X_A(0)|0\rangle = i\,(2\pi)^{-d+1}\, k^\mu \theta\big(k^0\big)\, \rho_A\big(-k^2\big)\ , \tag{1.9}$$

$$\sum_n \delta\big(k-k_n\big)\langle 0|X_A(0)|n\rangle\langle n|J^\mu(0)|0\rangle = i\,(2\pi)^{-d+1}\, k^\mu \theta\big(k^0\big)\, \tilde{\rho}_A\big(-k^2\big)\ , \tag{1.10}$$

where the Heaviside theta step function is there to ensure k_n be the d-momentum of a physical state. If both J^μ and X_A are hermitian operators, then the two expressions are just complex conjugates, and $\tilde{\rho}_A \equiv -\rho_A^*$. However, even without assuming this, we can write

$$\begin{aligned} \big\langle [J^\mu(x), X_A(y)]\big\rangle_0 &= \\ &= i\,(2\pi)^{-d+1} \int \mathrm{d}k\, \theta\big(k^0\big)\, k^\mu \Big[e^{ik(x-y)} \rho_A\big(-k^2\big) - e^{-ik(x-y)}\tilde{\rho}_A\big(-k^2\big)\Big] \\ &= (2\pi)^{-d+1} \frac{\partial}{\partial x_\mu} \int \mathrm{d}k\, \theta\big(k^0\big) \Big[e^{ik(x-y)} \rho_A\big(-k^2\big) + e^{-ik(x-y)}\tilde{\rho}_A\big(-k^2\big)\Big] \\ &= \frac{\partial}{\partial x_\mu} \int \mathrm{d}m^2 \left[\rho_A\big(m^2\big)\Delta^+(x-y) + \tilde{\rho}_A\big(m^2\big)\Delta^+(y-x)\right]\ , \end{aligned} \tag{1.11}$$

where we have introduced the standard positive frequency scalar distribution,

$$\Delta^+(x-y) = (2\pi)^{-d+1} \int \mathrm{d}k\, \theta\big(k^0\big)\, \delta\big(k^2+m^2\big)\, e^{ik(x-y)}\ . \tag{1.12}$$

[3] In our notation the sum over n is a shorthand for an integration over spatial momenta. Indeed, the orthonormality condition reads $\big\langle \omega(\vec{k}_n), \vec{k}_n \big| \omega(\vec{k}_m), \vec{k}_m\big\rangle = \delta(\vec{k}_n - \vec{k}_m)$, so that $\sum_n |n\rangle\langle n| = \int \mathrm{d}\vec{k}_n\, \big|\omega(\vec{k}_n), \vec{k}_n\big\rangle\big\langle \omega(\vec{k}_n), \vec{k}_n\big|$.

We remind the reader that for space-like separations we have $\Delta^+(x-y) \equiv \Delta^+(y-x)$, so that the canonical commutator for a complex scalar field ϕ,

$$\langle [\phi(x), \phi^\dagger(y)] \rangle_0 = \Delta(x-y) \equiv \Delta^+(x-y) - \Delta^+(y-x) \tag{1.13}$$

vanishes at space-like separations, respecting causality. In the same way, the commutator (1.11) has to vanish at space-like separations. This implies $\rho_A(m^2) = -\tilde{\rho}_A(m^2)$, so giving

$$\langle [J^\mu(x), X_A(y)] \rangle_0 = \frac{\partial}{\partial x_\mu} \int \mathrm{d}m^2 \, \rho_A(m^2) \, \Delta(x-y) \, . \tag{1.14}$$

Now we can use the current conservation, and, recalling $\Delta(x-y)$ is a solution of the Klein-Gordon equation, we can write

$$0 = \frac{\partial}{\partial x^\mu} \langle [J^\mu(x), X_A(y)] \rangle_0 = \Box_x \int \mathrm{d}m^2 \, \rho_A(m^2) \, \Delta(x-y) = \int \mathrm{d}m^2 \, m^2 \, \rho_A(m^2) \, \Delta(x-y) \, .$$

For time-like or light-like separations, this yields the condition

$$m^2 \rho_A(m^2) = 0 \, , \tag{1.15}$$

which implies that ρ_A must vanish for any value of m^2, except $m^2 = 0$. From the definition of ρ_A (1.9), it is clear that this latter possibility occurs only if there is a state $|\bar{n}\rangle$ in the spectrum that has zero mass, *i.e.*: $-k_{\bar{n}}^2 = 0$.

We now have to see that in case of spontaneously broken symmetry the function ρ_A actually cannot vanish everywhere, implying the existence of this massless mode. Let us take the expression (1.14) for the temporal component, and let us integrate it and evaluate it at equal times $x^0 = y^0 = t$, in order to obtain the symmetry breaking condition (1.7). We find

$$\begin{aligned} -i T_{AB} \langle X_B \rangle_0 &= \int \mathrm{d}m^2 \rho_A(m^2) \int \mathrm{d}\vec{x} \int \frac{\mathrm{d}k \, (i k^0)}{(2\pi)^{d-1}} \, \theta(k^0) \delta(k^2 + m^2) \left(e^{i \vec{k} \cdot \vec{x}} + e^{-i \vec{k} \cdot \vec{x}} \right) \\ &= i \int \mathrm{d}m^2 \, \rho_A(m^2) \int \mathrm{d}\vec{x} \int \frac{\mathrm{d}\vec{k}}{(2\pi)^{d-1}} \, e^{i \vec{k} \cdot \vec{x}} = i \int \mathrm{d}m^2 \, \rho_A(m^2) \, , \end{aligned}$$

where we have used the distributional identity

$$\theta(k^0) \, \delta(k^2 + m^2) = \frac{1}{2k^0} \, \delta\left(k^0 - \sqrt{\vec{k}^2 + m^2}\right) . \tag{1.16}$$

We then conclude

$$\rho_A(m^2) = -T_{AB} \langle X_B(0) \rangle_0 \, \delta(m^2) \, ,$$

as announced, implying the existence of a massless state in the spectrum. Now, a delta function contribution can only be generated by a single-particle pole. Furthermore, in order to contribute to ρ_A, such single-particle state has to couple both to the conserved current and to the scalar operator that gets a vev, otherwise either $\langle 0 | J^\mu(0) | n \rangle$ or $\langle n | X_A(0) | 0 \rangle$ would respectively vanish. The latter condition implies that our single-particle state must be a scalar (spin 0) particle; the former entails that it inherits the parity and internal quantum numbers of J^0. This precise gentleman is the notorious Nambu-Goldstone boson.

The proof outlined here has been carried out in generic $d+1$ space-time dimensions. However, by an argument first raised by Coleman [32], there is an obstruction to spontaneous symmetry breaking in $(1+1)$-dimensions. Actually, nothing really breaks down in the steps of the proof of Goldstone theorem, it is rather the possibility for a scalar operator to get a stable vacuum expectation value that is ruled out in two dimensions, so invalidating the very initial point of the reasoning (1.7). In next section we present Coleman's mathematical argument and discuss its physical implications.

1.1.2 Lowering to 1+1 dimensions: Coleman theorem

The key feature, on which Coleman's argument is founded, is the fact that in one spatial dimension the scalar two-point correlator (1.12) has a logarithmic behavior in position space, so having an infrared divergence for massless scalar fields:

$$\begin{aligned}\left\langle \phi(x)\phi^\dagger(y)\right\rangle_0 = \Delta^+_{d=1}(x-y) = (2\pi)^{-1}\int \mathrm{d}^2 p\, \theta\left(k^0\right)\delta\left(k^2+\mu^2\right) e^{ik(x-y)} = \\ = \frac{1}{2\pi}\int \frac{\mathrm{d}k^1\; e^{ik(x-y)}}{2\sqrt{(k^1)^2+\mu^2}} = -\frac{1}{4\pi}\,\ln\left(-\mu^2(x-y)\right),\end{aligned} \tag{1.17}$$

where μ^2 is indeed an infrared regulator to escape the divergence of zero mass.

Rather than reproduce Coleman's proof [32], which is quite mathematical, we will follow the lines of Ma and Ranjaraman [33], which present a more physical picture of what happens in a two dimensional quantum field theory with an internal continuous symmetry. Moreover, this picture will allow to better understand some situations that seem to evade Coleman's argument, in particular the strict large N limit of $\mathrm{SU}(N)$ field theories, such as the Thirring model [47], or the holographic dual theory of Section 3.2.

We consider a U(1)-invariant Lagrangian for a complex scalar field, with an arbitrary (U(1)-invariant, and bounded from below) potential, and we write the field in the polar representation, $\phi = \frac{1}{\sqrt{2}}\rho\, e^{i\vartheta}$:

$$\mathscr{L} = -\partial^\mu\phi^*\partial_\mu\phi - V(\phi^*\phi) = -\tfrac{1}{2}\partial^\mu\rho\partial_\mu\rho - \tfrac{1}{2}\rho^2\partial^\mu\vartheta\partial_\mu\vartheta - V\left(\rho^2\right). \tag{1.18}$$

This Lagrangian can be regarded of course as a low-energy effective one of a more general and involved theory. The vacuum expectation value for this scalar field is dictated by the minimum of the potential in phase space. If we want to have spontaneous symmetry breaking, we need the minimum to be away from the origin $\rho = 0$. We take then $\rho^2 = |v|^2$, with $v \in \mathbb{C}$. Since the potential is invariant under the global U(1), our minimum in $|v|^2$ represents a continuity of possible vacua for ϕ, varying the phase of v. In higher dimensions, the selection of a specific vacuum, that is of a specific phase, yields the spontaneous breaking. In $(1+1)$-dimensions, the IR divergences of the scalar propagator bring large quantum fluctuations of the phase field ϑ, that would spread away any classically selected phase. Thus, the classical possibility of picking a vacuum out of the continuum is ruled out at the quantum level.

Let us see this fact explicitly. By making $\rho = v$ non-dynamical, we are left with the following Lagrangian for ϑ,

$$\mathscr{L} = -\tfrac{1}{2} v^2 \partial^\mu \vartheta \partial_\mu \vartheta \ . \tag{1.19}$$

We then consider the expansion of the scalar field ϑ in its annihilation and creation components, $\vartheta = \vartheta^+ + \vartheta^-$, with

$$\vartheta^+(t,x) = \int \frac{\mathrm{d}k\, e^{-i\omega t + ikx}}{v\sqrt{4\pi\omega}}\, a_k \ , \quad \vartheta^-(t,x) = \int \frac{\mathrm{d}k\, e^{i\omega t - ikx}}{v\sqrt{4\pi\omega}}\, a_k^\dagger \ , \tag{1.20}$$

where (k^0, k^1) have been replaced by (ω, k), and we have an extra v-factor because of the form of the kinetic term in the Lagrangian (1.19). Then, using the standard commutations relations for annihilation/creation operators, we straightforwardly have

$$\left\langle \left[\vartheta^+(x), \vartheta^-(0)\right] \right\rangle_0 = \int \frac{\mathrm{d}k\, e^{ikx}}{4\pi v^2 \sqrt{k^2}} = \frac{1}{v^2} \lim_{\mu^2 \to 0} \Delta^+_{d=1}(x) \ , \tag{1.21}$$

which, from (1.17), is diverging to positive infinity. So, we can evaluate

$$\left\langle e^{i\vartheta(x)} \right\rangle_0 = \left\langle e^{i\left(\vartheta^-(x)+\vartheta^+(x)\right)} \right\rangle_0 = \left\langle e^{i\vartheta^-(x)} e^{i\vartheta^+(x)} e^{-\frac{1}{2}\left[\vartheta^+(x),\vartheta^-(x)\right]} \right\rangle_0 = e^{-\frac{1}{2v^2}\Delta^+_{d=1}(0)} = 0 \ ,$$

where we have used the Baker–Campbell–Hausdorff formula for exponentials of operators, and the annihilation property of the vacuum. Thus, we have shown that the expectation value of $e^{i\vartheta(x)}$ is wiped out by the diverging quantum fluctuations, and so $\langle \cos\vartheta(x) \rangle_0$ and $\langle \sin\vartheta(x) \rangle_0$ vanish as well. No preferred direction in the phase plane survives the quantum smearing in two dimensions.

However, if we had an arbitrary small, but non zero explicit breaking term (such for instance a small mass for ϑ), it would act precisely as the regulator μ^2 in (1.17) and make the commutator (1.21) finite. The concomitant occurrence of spontaneous and explicit breaking, with explicitly breaking parameter hierarchically suppressed with respect to the spontaneously breaking vev, gives rise to so-called pseudo-Goldstone bosons. Pseudo-Goldstone bosons are would-be Goldstone particles that have a mass which is hierarchically lower with respect to the rest of the spectrum, thanks to the mentioned hierarchy between explicit and spontaneous breaking.

This last scenario will be discussed in depth in next section, but, before moving to that, let us discuss a class of model that seems to evade Coleman theorem, that is SU(N)-invariant theories in the large N limit. Of course we are interested in large N theories in view of the holographic realizations of symmetry breaking of Part II, where pure boundary computations correspond to the strict large N limit.

So now we consider the SU(N) extension of Thirring model [47], proposed by Gross and Neveu [48], with our focus on understanding the peculiar features of the infinite N limit for spontaneous symmetry breaking in two dimensions, rather than on the details of the model. This is a theory of N-component massless fermions with quartic interaction in two dimensions, which has a smooth, weak-interacting limit for $N \to \infty$, so that it can be solved in a $1/N$ expansion. On top of the SU(N) invariance, the system enjoys a U(1) chiral symmetry, which normally would prevent the fermions from dynamically getting a mass. However, as pointed out by Gross and Neveu, the theory allows for a minimum of the potential away from the origin in the phase plane, fermions fields acquire mass, and a massless boson arises: so, everything suggests that spontaneous symmetry breaking is occurring in two dimensions.

Nonetheless, Witten showed [34] that the physical fermions are actually not charged under the chiral symmetry, explaining why they are not protected from having a mass, and that no long-range order is present, so no symmetry breaking and no contradiction to Coleman theorem. Indeed, the low-energy effective Lagrangian for Gross-Neveu-Thirring model is of the form [34]

$$\mathscr{L} = -\frac{N}{4\pi}\partial^{\mu}\vartheta\partial_{\mu}\vartheta \,, \tag{1.22}$$

quite analogous to (1.19). Then we have

$$\left\langle e^{-i\vartheta(x)}e^{+i\vartheta(0)}\right\rangle_0 = e^{\frac{\pi}{N}\left(\Delta^+_{d=1}(x)-\Delta^+_{d=1}(0)\right)} \sim x^{-\frac{1}{N}} \,, \tag{1.23}$$

for large x. So the long-range order is suppressed for $x \to \infty$. In a spontaneously broken theory, this expectation value would be finite (proportional to the vev), even for infinite separations, as in (1.7). For unbroken symmetry, instead, we would have an exponential fall-off, at the rate of the lightest excitation in the spectrum: $\left\langle e^{-i\vartheta(x)}e^{+i\vartheta(0)}\right\rangle_0 \sim e^{-mx^2}$. So, this two-dimensional large N case is in-between: the long range order vanishes, but with a power law, so less quickly than in an usual unbroken phase.

This was the main result of Witten's paper [34], but what he did not stress in that work, since it was not yet relevant at that time, is the fact that in the strict infinite N limit the long-range order is restored. And indeed, in the holographic setup of Section 3.2, from a pure boundary analysis (which is at strictly infinite N), we retrieve the Ward identities of spontaneous symmetry breaking, as in any higher dimensions.

In next section we derive in full generality such Ward identities, which are a non-perturbative feature of symmetry breaking in quantum field theory. Furthermore, as already announced, we will consider the concomitant occurrence of spontaneous and explicit breaking, and analyze the physical predictions that can be extracted just from Ward identities, independently of any specific theory.

1.2 Explicit breaking and Ward identities

We consider here an action that is invariant under an internal global continuous symmetry, as in (1.1), and we add a term that breaks the symmetry explicitly:

$$S_{\rm tot} = S_{\rm inv} + S_m \,, \quad \text{with } S_m = \frac{1}{2} m \int {\rm d}x \, \left(\mathcal{O}[X] + \mathcal{O}^*[X]\right) = m \int {\rm d}x \, {\rm Re}\mathcal{O}[X] \,, \tag{1.24}$$

where $\mathcal{O}_\phi$ is a scalar operator of scaling dimension Δ, which is charged under the symmetry, and does not depend on derivatives of the fields. For the sake of simplicity, and for an easier comparison with the results of the holographic model of Section 3.1, we choose as global symmetry an abelian U(1), and we assume

$$\delta_\alpha \mathcal{O} = i\alpha\mathcal{O} \,, \quad \Rightarrow \begin{cases} \delta_\alpha {\rm Re}\mathcal{O} = -\alpha {\rm Im}\mathcal{O} \,, \\ \delta_\alpha {\rm Im}\mathcal{O} = +\alpha {\rm Re}\mathcal{O} \,. \end{cases} \tag{1.25}$$

Since $\delta_\alpha S_{\rm inv} = 0$, as we have seen in Section 1.1, for $m = 0$ we have a conserved current J^μ. But in presence of the explicit breaking term, the equations of motion get modified accordingly,

$$\partial_\mu \frac{\delta \mathcal{L}_{\rm inv}}{\delta \partial_\mu X_A} - \frac{\delta \mathcal{L}_{\rm inv}}{\delta X_A} - m \frac{\delta {\rm Re}\mathcal{O}}{\delta X_A} = 0 \,,$$

so that in the second line of (1.2) we now have

$$0 = \delta_\alpha S_{\rm inv} = \int {\rm d}x \, \left[-m \frac{\delta {\rm Re}\mathcal{O}}{\delta X_A} \delta_\alpha X_A + \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu X_A} \delta_\alpha X_A \right) \right] = -\int {\rm d}x \, \alpha \left(\partial_\mu J^\mu - m {\rm Im}\mathcal{O} \right) \tag{1.26}$$

where we have used the trivial identity $\frac{\delta \mathcal{O}}{\delta X_A} \delta_\alpha X_A \equiv \delta_\alpha \mathcal{O}$. We see that the current is no longer conserved, instead we have the operator identity

$$\partial_\mu J^\mu = m {\rm Im}\mathcal{O} \,. \tag{1.27}$$

We now want to show that such identity holds at the quantum level, and derive the Ward identities corresponding to the breaking of an internal symmetry. For that we employ the path integral formulation, and we use the invariance of the measure under the symmetry transformation.

The path integral and n-point time-ordered correlation functions are defined as follows:

$$\mathcal{Z} = \int \mathcal{D}[X] \, e^{iS_{\rm tot}[X]} \,, \qquad \left\langle \mathcal{F}[X](x_1, \ldots, x_n) \right\rangle_0 = \int \mathcal{D}[X] \, e^{iS_{\rm tot}[X]} \, \mathcal{F}[X](x_1, \ldots, x_n) \,. \tag{1.28}$$

If there are no quantum anomalies (violation of the symmetry at the quantum level), the invariance of the measure $\mathcal{D}[X]$ under field redefinitions assures

$$\int \mathcal{D}[X] \, \delta_\alpha \left(e^{iS_{\rm tot}[X]} \mathcal{F} \right) = 0 \,, \qquad \forall \, \mathcal{F} \,. \tag{1.29}$$

This is then true for the path integral itself, allowing to confirm the operator identity (1.27) at the quantum level. Indeed, from (1.29) we have

$$0 = \int \mathscr{D}[X]\, \delta_\alpha e^{iS_{\rm tot}} = \int \mathscr{D}[X]\, e^{iS_{\rm tot}}\, i\,\delta_\alpha S_{\rm tot}\,. \tag{1.30}$$

In addition, in order to compute correlation functions involving the divergence of the current, which classically vanishes, we need to make the global symmetry local. In this way the variation of the invariant action is not zero anymore, rather $\delta_\alpha S_{\rm inv} = -\int {\rm d}x\ J^\mu \partial_\mu \alpha$, and consequently

$$\delta_\alpha S_{\rm tot} = \delta_\alpha S_{\rm inv} + \delta_\alpha S_m = -\int {\rm d}x\, \left(J^\mu \partial_\mu \alpha + \alpha\, m {\rm Im}\mathscr{O} \right)\,, \tag{1.31}$$

A perfectly equivalent approach is to add to the path integral sources for both the current and Im$\mathscr{O}$, whose variations compensate the variation of the total action (1.31). The source for the current then has to be a vector field A_μ transforming under U(1) gauge transformations, $\delta_\alpha A_\mu = \partial_\mu \alpha$; the source for Im$\mathscr{O}$ has to be a scalar field m transforming under the U(1) as $\delta_\alpha m = m\alpha$. Thus we have

$$\int \mathscr{D}[X]\, e^{iS_{\rm inv} + iS_{\rm m} + i\int A_\mu J^\mu + i\int m {\rm Im}\mathscr{O}}\,. \tag{1.32}$$

Now we can choose either to put the total variation (1.31) into the vanishing expression (1.30), or to vary the sourced path integral (1.32) with respect to the sources only, and require it to be invariant (since transformations of the sources can be absorbed by transformations of the fields),

$$\begin{aligned} 0 &= \int \mathscr{D}[X]\, e^{iS_{\rm inv} + iS_{\rm m} + i\int A_\mu J^\mu + i\int m {\rm Im}\mathscr{O}}\, i \int {\rm d}^d x\, \left(J^\mu \delta_\alpha A_\mu + {\rm Im}\mathscr{O} \delta_\alpha m \right) = \\ &= \int {\rm d}^d x\, \langle J^\mu \partial_\mu \alpha + \alpha\, m {\rm Im}\mathscr{O} \rangle_0\,. \end{aligned}$$

In both ways, after integration by parts, we obtain the quantum version of the operator identity (1.27):

$$\langle \partial_\mu J^\mu \rangle_0 = m \langle {\rm Im}\mathscr{O} \rangle_0\,. \tag{1.33}$$

Analogously, we can start with the variation of the path integral for the operator Im$\mathscr{O}$,

$$\begin{aligned} 0 &= \int \mathscr{D}[X]\, \delta_\alpha \left(e^{iS_{\rm tot}} {\rm Im}\mathscr{O}(x') \right) = \int \mathscr{D}[X]\, e^{iS_{\rm tot}} \left(i\,(\delta_\alpha S_{\rm tot}) {\rm Im}\mathscr{O}(x') + \delta_\alpha {\rm Im}\mathscr{O}(x') \right) = \\ &= \int \mathscr{D}[X]\, e^{iS_{\rm tot}} \left[\alpha(x') {\rm Re}\mathscr{O}(x') + i \int {\rm d}x\, \alpha(x) \left(\partial_\mu J^\mu(x) - m {\rm Im}\mathscr{O}(x) \right) {\rm Im}\mathscr{O}(x') \right]\,, \end{aligned}$$

thus obtaining the following Ward identity for symmetry breaking:

$$\langle \partial_\mu J^\mu(x) {\rm Im}\mathscr{O}(0) \rangle_0 = m \langle {\rm Im}\mathscr{O}(x) {\rm Im}\mathscr{O}(0) \rangle_0 + i \langle {\rm Re}\mathscr{O} \rangle_0\, \delta(x)\,, \tag{1.34}$$

We see that, if spontaneous breaking occurs as well, by the real part of the scalar operator taking a non-zero vacuum expectation value, $\langle \mathrm{Re}\mathscr{O}\rangle_0 = v$, we have a contact term, which survives in the $m \to 0$ limit,

$$\left\langle \partial_\mu J^\mu(x)\mathrm{Im}\mathscr{O}(0)\right\rangle_0 = i\,v\,\delta(x)\ , \tag{1.35}$$

that is even when the current is perfectly conserved at the classical level. Then, Lorentz invariance imposes, Fourier transforming to momentum space,

$$\left\langle J^\mu(k)\mathrm{Im}\mathscr{O}(-k)\right\rangle_0 = i\,v\,\frac{k^\mu}{k^2}\ , \tag{1.36}$$

manifesting the presence of a massless pole in the spectrum, signature of the Goldstone boson. Requiring continuity for m going to zero in the Ward identity (1.34), the same massless pole has to appear in the two-point scalar correlator $\langle \mathrm{Im}\mathscr{O}\mathrm{Im}\mathscr{O}\rangle_0$ as well, although its explicit computation requires an analysis of the (IR) dynamics, and so knowledge over the specific theory.

When $m \neq 0$, instead, thanks to the Ward identity (1.34) and to the operator identity (1.27), we have that the following two-point functions all depend on a single non-trivial function $f(\Box)$:

$$\left\langle \mathrm{Im}\mathscr{O}\mathrm{Im}\mathscr{O}\right\rangle_0 = -i\,f(\Box)\ , \tag{1.37}$$

$$\left\langle \partial_\mu J^\mu \mathrm{Im}\mathscr{O}\right\rangle_0 = -i\,m\,f(\Box) + i\,v\ , \tag{1.38}$$

$$\left\langle \partial_\mu J^\mu \partial_\nu J^\nu\right\rangle_0 = -i\,m^2 f(\Box) + i\,m\,v\ , \tag{1.39}$$

where we have kept the delta function implicit. The last correlator is just a consequence of the operator identity, and when $v = 0$ (pure explicit breaking case) then also the second correlator is a trivial consequence of the operator identity.

When both $v \neq 0$ and $m \neq 0$, we see that the scalar-current correlator (1.38) exhibits both features: a term related to (1.37) and a constant term. On the other hand, since the symmetry is broken explicitly, we do not expect a massless mode in the spectrum contributing to this set of correlators. However, in the case where $m \ll v^{\frac{d-\Delta}{\Delta}}$ (small amount of explicit breaking compared to spontaneous breaking), we expect to find a light particle, whose mass is sensibly lower with respect to the rest of the spectrum and goes to zero in the $m \to 0$ limit. We call such particle pseudo-Goldstone boson, precisely in consequence of this fact that for $m \to 0$ it ‘becomes’ the Goldstone boson. Furthermore, the leading-order contribution to the square mass of the pseudo-Goldstone for small explicit breaking is linear in $\frac{m}{v}$, as we will show now in full generality, requiring continuity in the $m \to 0$ limit. This linear relation for the square mass of the pseudo-Goldstone boson is renown as GMOR (Gell-Mann–Oakes–Renner) relation [35].

The GMOR relation was first found in the context of the effective description of pions as pseudo-Goldstone bosons of the approximate chiral symmetry of quantum chromodynamics, where the role of explicit breaking parameter is played by the masses of the quarks. Standard derivations of GMOR for the pion can be found in the literature, for instance in [49, 50], and also [51] for a derivation based on the effective action. Here we derive it in a more general fashion, as it was done in [II].

In momentum space we can write

$$\langle \mathrm{Im}\mathcal{O}\mathrm{Im}\mathcal{O}\rangle_0 = -i\,f\big(k^2\big)\,, \qquad i k_\mu \langle J^\mu \mathrm{Im}\mathcal{O}\rangle_0 = -i\,m\,f\big(k^2\big) + i\,v\,. \tag{1.40}$$

Note that the dimensions are $[f] = 2\Delta - d$, $[v] = \Delta$ and $[m] = d - \Delta$. Using Lorentz invariance, which imposes $\langle J^\mu \mathrm{Im}\mathcal{O}\rangle_0 = k^\mu g\big(k^2\big)$, the second relation in (1.40) leads to

$$\langle J^\mu \mathrm{Im}\mathcal{O}\rangle_0 = -\frac{k^\mu}{k^2}\Big(m\,f\big(k^2\big) - v\Big)\,. \tag{1.41}$$

We immediately see that there cannot be a massless excitation in the $\langle \mathrm{Im}\mathcal{O}\mathrm{Im}\mathcal{O}\rangle_0$ channel, otherwise there would be a double pole in $\langle J^\mu \mathrm{Im}\mathcal{O}\rangle_0$. Moreover, the massless pole in the above correlator should be spurious, which requires $f\big(k^2\big)$ to satisfy

$$m\,f(0) - v = 0\,. \tag{1.42}$$

So far, everything is valid for any values of m and v. When m^2 and k^2 are both small with respect to v, we can approximate f by a pole of mass M, corresponding to the pseudo-Goldstone mode,

$$f\big(k^2\big) \;\simeq\; \frac{\mu}{k^2+M^2} - \frac{\mu}{M^2} + \frac{v}{m}\,, \tag{1.43}$$

where we have implemented the condition (1.42), and the residue μ is a dynamical quantity of dimension $2\Delta - d + 2$.

We now require that in the $m \to 0$ limit $f\big(k^2\big)$ go over smoothly to what we expect in the pure spontaneously broken case. Namely, we expect μ to be (roughly) constant in the limit, as of course v, while $M^2 \to 0$, so that

$$f\big(k^2\big) \xrightarrow{m\to 0} \frac{\mu}{k^2}\,, \tag{1.44}$$

up to possibly an additive finite constant. From (1.43) we see that this is possible only if there is a relation between all the constants such that

$$M^2 = \frac{\mu}{v}\,m\,. \tag{1.45}$$

This is the generalized form of the GMOR relation [35], which indeed states that, at first order, the square mass of the pseudo-Goldstone boson scales linearly with the small parameter which breaks explicitly the symmetry, as announced. The two other constants entering the expression are both of the order of the dynamical scale generating the vev, *i.e.* the spontaneous breaking of the symmetry. We remark that, since μ has to be positive because of unitarity, then the signs (and more generally the phases) of m and v have to be correlated in order to avoid tachyonic pseudo-Goldstone bosons. This can be understood by the fact that the small explicit breaking removes the degeneracy of the vacua, and thus the phase of the vev v is no longer arbitrary but has to be aligned with the true vacuum selected by m.

We conclude making the link with the usual form in which GMOR relation is stated, which is in terms of the residue of the full current-current correlator,

$$\langle J^\mu J^\nu \rangle_0 = -\frac{i\mu_J}{k^2+M^2} k^\mu k^\nu + \dots \,, \tag{1.46}$$

where the residue μ_J is actually given by the square of the decay constant of the pion, f_π^2. Then we can use the GMOR value of the pole (1.45) into (1.43), and rewrite the longitudinal correlator (1.39) as

$$k_\mu k_\nu \langle J^\mu J^\nu \rangle_0 = \frac{imvk^2}{k^2+\mu\frac{m}{v}} \,. \tag{1.47}$$

The comparison of these two expressions, (1.46) and (1.47), at the location of the pseudo-GB pole (1.45) yields

$$\mu_J = \frac{v^2}{\mu} \,, \tag{1.48}$$

and so

$$M^2 = \frac{v}{\mu_J}\, m \equiv \frac{v}{f_\pi^2}\, m \,,$$

which is the desired usual form of GMOR relation [35], though completely equivalent to (1.45).

Here we have kept both d and Δ arbitrary, and the relation is valid in all generality. In Section 3.1 we will retrieve this relation in a particular holographic model for $d=3$ and $\Delta=2$, determining as well the value of the residue μ, which is specific to the considered model.

1.3 Goldstone theorem in non-relativistic field theories

In our derivation of Goldstone theorem, we have used Lorentz invariance in defining the spectral density (1.9-1.10), as well as in any occurrence of the Lorentz-invariant distributions $\Delta^+(x)$ and $\Delta(x)$. So, in lack of Lorentz invariance, that proof does not apply. However, the central statement, that spontaneously broken symmetries imply zero-energy modes in the spectrum, holds, in a softer formulation and with some modifications, also for non-relativistic field theories, as we will see.

On the other hand, though in Section 1.1 we have presented Noether theorem as well in a relativistic formalism, it relies only on the action principle and so it holds independently of the symmetries of space-time. However, the key assumption in order to have a conserved charge is that the spatial divergence of the current vanishes at spatial infinity. Actually, to ensure time-independence of the vev in the spontaneous breaking condition (1.7), it is sufficient that

$$\int \mathrm{d}\vec{x}\, \left[\partial_i J_i^a(\vec{x}), X_A\right] = 0 \,, \tag{1.49}$$

so that $\partial_t [Q^a, X_A] = 0$. In relativistic theories, which are intrinsically causal, this is guaranteed as long as the operator X_A is localized to a finite domain of space-time, and Goldstone theorem inescapably follows[4]. On the contrary in non-relativistic theories, which can be non-causal, long-range interactions can in some cases make the boundary term in (1.49) not vanish. It can be shown that (1.49) holds as long as the range of interactions is finite [5, 6], otherwise the applicability of condition (1.49) has to be checked case by case. In the following, we assume (1.49) as necessary prerequisite for spontaneous breaking to occur, otherwise the charge commutator and vev itself are not time-independent, and Goldstone theorem cannot be discussed at all.

We can then take the symmetry breaking commutator (1.7), and rewrite it through the spectral decomposition of the unity as in (1.8), obtaining

$$\begin{aligned}
&\langle [Q^a, X_A(0)] \rangle_0 = \\
&= \sum_n \int \mathrm{d}\vec{x} \left(e^{ik_n x} \langle J_t^a(0) | n \rangle \langle n | X_A(0) \rangle_0 - e^{-ik_n x} \langle X_A(0) | n \rangle \langle n | J_t^a(0) \rangle_0 \right) \\
&= (2\pi)^d \sum_n \delta(\vec{k}_n) \left(e^{-i\omega(\vec{k}_n)t} \langle J_t^a(0) | n \rangle \langle n | X_A(0) \rangle_0 - e^{i\omega(\vec{k}_n)t} \langle X_A(0) | n \rangle \langle n | J_t^a(0) \rangle_0 \right) .
\end{aligned} \tag{1.50}$$

We see that, for the left-hand side to be time-independent, the right-hand side must vanish except for states with vanishing energy at zero momentum, that is: $\omega(\vec{k}_n \equiv 0) = 0$ [7]. Furthermore, since the left-hand side is non-zero because of the spontaneous breaking condition (1.7), *at least one* of such states must exist and contribute to the right-hand side. So, even without using any Lorentz invariance, we can state that, when an internal continuous symmetry is spontaneously broken, there exists at least one mode, whose energy goes to zero for spatial momentum going to zero, and which couples to both the conserved current and the scalar operator that does not commute with the current.

Let us now point out the crucial differences between the general statement of Goldstone theorem, which we have just outlined here, and the relativistic one. First, in its more general version Goldstone theorem is a low-energy statement: for $\vec{k} \to 0$ there is a vanishing energy mode, but it is not assured to survive at high energy, and in any case not with the same properties. In the relativistic case, the dispersion relation (*i.e.* the dependence of the energy of an excitation on the momentum) is fixed to $\omega_{\vec{k}} = |\vec{k}|$, at low as at high energy, and the Goldstone mode is an actual massless particle.

In a non-relativistic framework a richer range of dispersion relations are recorded. Let us assume broken Lorentz invariance, but still preserved invariance under spatial rotations. Then, in such case the Fourier transform of the old same correlator (1.8) has a less constrained structure than in the Lorentz invariant case, which we can express in a formally Lorentz

[4]Goldstone theorem predicts a massless particle, but this does not mean that such particle be physically observable. As high-energy physicists we all have in mind the example of gauge theories and Brout-Englert-Higgs mechanism, where the Goldstone boson does not appear in the observable spectrum, being 'hidden' in the longitudinal degree of freedom of the newly massive gauge boson.

covariant way using a time-like vector $b^\mu = (1, \vec{0})$:

$$\int \mathrm{d}x\, e^{ikx} \langle [J^\mu(x), X_A(0)] \rangle_0 = k^\mu \rho_1(k^2, b\cdot k) + b^\mu \rho_2(k^2, b\cdot k) \,. \tag{1.51}$$

In the relativistic case, only the term proportional to k^μ was present (1.9-1.10). Then the current conservation leads to the non-relativistic equivalent [7, 52] of condition (1.15), which constrains (1.51) to take the form

$$k^\mu \delta(k^2)\, \widetilde{\rho}_1(b\cdot k) + (k^2 b^\mu - (b\cdot k) k^\mu)\, \widetilde{\rho}_2(k^2, b\cdot k) + b^\mu\, \delta(b\cdot k)\, \widetilde{\rho}_3(k^2) + b^\mu\, C_4 \delta(k^0) \delta(\vec{k}) \,, \tag{1.52}$$

where C_4 is just a constant, and the $\widetilde{\rho}$'s are functions only of the explicitly indicated arguments. We can see that the introduction of the second term in (1.51) brings quite more possible contributions, which in the early stages had been advocated as signals of a breakdown of Goldstone theorem in the non relativistic framework [4, 52].

However, the last two terms correspond to states with zero energy, and zero energy and zero spatial momentum, respectively. The latter is a spurious isolated energy-momentum eigenstates with eigenvalue $k^\mu = 0$, whose contribution can be excluded if forces have finite (exponentially suppressed) range [6, 8] (see also the discussion around eq. (32) of Brauner's review [37]).

The former can be one of the degenerate ground states of the continuum of possible vacua, which are not actual excitations of the spectrum with energy going to zero *in the limit* $|\vec{k}| \to 0$. The possibility of such term contributing to (1.51) is an artifact of working at infinite volume, and it is ruled out by restricting to a finite volume. In a finite volume, indeed, the integration in (1.50) does not give exactly a delta, rather a function peaked around zero, but which allows for values of $|\vec{k}|$ slightly different from zero. So, at finite volume, the states contributing to the non-vanishing commutator (1.50) are excitations with gapless dispersion relation, rather than states with zero energy, so that contributions from ground states or other spurious states of intrinsically zero energy are ruled out. Taking the limit of infinite volume, and $|\vec{k}| \to 0$, it can be explicitly shown [6] that such states keep on not contributing to (1.50).

So, the actual, admissible contributions come from the first two terms in (1.52). The first one is the analogous of the unique term of the relativistic case, with the exception that there $\widetilde{\rho}_1$ is forced by Lorentz symmetry to be just a constant. The second term has a completely transverse structure, so $\widetilde{\rho}_2$ is left completely arbitrary. This gives rise to the possibility of non-linear dispersion relations for non-relativistic Goldstone modes[5], as we will show now, by an argument based on the symmetry breaking Ward identity.

In previous section we have seen that spontaneous symmetry breaking entails a contact term in the Ward identity (1.35). Such Ward identity holds in absence as in presence of Lorentz

[5] As a curiosity, we point out that such term was excluded by Guralnik, Hagen, and Kibble in their review [7], by a not convincing argument relying on the divergence of the spatial current at infinity. The legitimacy of this term has been restored by Nielsen and Chadha, who indicated it as responsible for non-linear dispersion relations [8], even though without giving any proof, and without confuting Guralnik-Hagen-Kibble's argument, and not even mentioning their work.

invariance, as long as the invariance of the measure and the continuity equation $\partial_\mu J^\mu = 0$ are assured.

In the relativistic case, the Ward identity (1.35) implied the presence of the Goldstone massless pole (1.36), with the completely fixed relativistic dispersion relation, $\omega(|\vec{k}|) = |\vec{k}|$, which is linear with proportionality constant given by the speed of light $c \equiv 1$. Here, with an argument inspired by [9] but somehow original, we use the Ward identity to derive the more general structure of the dispersion relation of non-relativistic GBs.

The non-vanishing contact term proportional to the vev (1.35) implies a pole at dispersion relation $\omega_{\vec{k}}$, with $\omega_{\vec{k}} \to 0$ as $|\vec{k}| \to 0$, in the correlator

$$\langle J^\mu(k) X_A(-k)\rangle_0 = \frac{i v}{-k^0 + \omega_{\vec{k}}} \left(n^\mu A(|\vec{k}|) + b^\mu B(|\vec{k}|)\right) , \tag{1.53}$$

where we have allowed for a completely general non-relativistic structure (which yet preserves spatial rotational invariance), with $n^\mu = \left(1, \vec{k}/|\vec{k}|\right)$, 'normalized' light-like vector, and b^μ the same as in (1.51). In this way A and B have the same mass dimensions. Then the Ward identity becomes an equation for A and B, giving the dispersion relation. Namely,

$$\begin{aligned} i v = k_\mu \langle J^\mu X_A\rangle_0 &= \frac{i v}{-k^0 + \omega_{\vec{k}}} \left(\left(-k^0 + |\vec{k}|\right) A - k^0 B\right) = \\ &= i v\, (A+B) + i v\, \frac{\left(-\omega_{\vec{k}} + |\vec{k}|\right) A - \omega_{\vec{k}} B}{-k^0 + \omega_{\vec{k}}} \, . \end{aligned} \tag{1.54}$$

The realization of the Ward identity demands the term in $A+B$ to be a constant for $|\vec{k}| \to 0$, and the term diverging at the location of the Goldstone pole to vanish. We can write

$$\begin{aligned} A(|\vec{k}|) &= a_0 + a_1 |\vec{k}| + \ldots ; \\ B(|\vec{k}|) &= b_0 + \ldots . \end{aligned}$$

In this way, the first condition that we extract from (1.54), in order to recover the Ward identity, is $a_0 + b_0 = 1$. Then, the other condition is that the second term must vanish, giving

$$\left(-\omega_{\vec{k}} + |\vec{k}|\right) A - \omega_{\vec{k}} B = 0 \, .$$

It is easy to check that for $b_0 \neq 1$ a linear dispersion relation occurs at low momentum

$$\omega_{\vec{k}} = \left(1 - b_0\right) |\vec{k}| \, , \tag{1.55}$$

with the relativistic one being a special case, at $b_0 = 0$. If instead $b_0 = 1$ (and $a_1 \neq 0$), we have

$$\omega_{\vec{k}} = a_1 \vec{k}^2 \, , \tag{1.56}$$

that is a quadratic dispersion relation. We will see in the following that such quadratic dispersion relation can be connected to a non-zero vacuum expectation value of a charge density, event that clearly can only occur when Lorentz invariance is violated.

As a final remark, we point out that throughout this discussion we have assumed the dispersion relation to be analytic in the momentum, as a consequence of the analyticity of the Fourier transform. This is guaranteed if there are no long-range interactions [8]. However, Goldstone modes can occur also in the presence of long-range interactions (as long as the divergence of the spatial current is assured to vanish at infinity), and can exhibit non-analytic dispersion relation, for instance with fractional powers of momentum (see Section 6 of Brauner's review [37] for concrete examples).

Another main issue of the non-relativistic extension of Goldstone theorem concerns the number of Goldstone bosons in relation to the number of broken generators. In the relativistic case we have (at least[6]) one Goldstone particle for each broken generator. Already in early times Nielsen and Chadha [8] showed that, in lack of Lorentz invariance, the possibility of dispersion relations proportional to even powers of momentum demands a modification of the counting rule, resulting in the inequality

$$n_I + 2n_{II} \geq n_{BG} \,, \tag{1.57}$$

where n_{BG} is the number of broken generators, and n_I and n_{II} are the numbers of type I and type II Goldstone modes, with type I having odd-power, and type II even-power dispersion relations. Of course, relativistic Goldstone boson are all necessarily of type I, giving the inequality $n_{GB} \geq n_{BG}$, with n_{GB} being the number of Goldstone bosons.

More than twenty-five years later [39] the missing counting was connected to non-vanishing vacuum expectation value for the charge operators. More precisely, the statement of [39] was that if

$$\left\langle [Q^a, Q^b] \right\rangle_0 = 0 \tag{1.58}$$

for all broken generators, then $n_{GB} \geq n_{BG}$, as in the relativistic case. Of course, Lorentz invariance excludes the possibility of a non-zero vev for a charge, that is defined in term of the *temporal* component of a current. Actually, the original statement of [39] was given as an equality, which does not necessary follows from their argument, and also their reasoning presented some inaccuracies. Anyway, the statement, in the form of an inequality, is correct, and the proof can be refined, as you can see for instance in section 5.3 of Brauner's review [37].

The connection between charge density taking a vev and quadratic dispersion relation was already pointed-out by means of low-energy effective Lagrangian for non-relativistic Goldstones [53]. However, a precise counting rule has been established only relatively recently,

[6]The non-perturbative proof of Goldstone theorem presented here does not rule out the possibility that Goldstone modes may be more than broken generators. However, in all known example the strict equality holds, and proofs of Goldstone theorem based on perturbation theory and effective potential (see section 19.2 of Weinberg's book [31]) show that there is a one-to-one correspondence between number of Goldstone modes and number of independent broken generators. Furthermore, the non-relativistic counting of (1.59) reduces to the equality between numbers of broken generators and Goldstone bosons for the relativistic sub-case rank$\rho = 0$.

again thanks to a low-energy effective Lagrangian approach with some assumed properties, in a series of papers [9–11, 54, 55], whose conclusions summarize as follows:

$$n_A = n_{BG} - \mathrm{rank}\rho\ , \quad n_B = \frac{1}{2}\mathrm{rank}\rho\ , \quad n_A + n_B = n_{GB}\ , \quad n_A + 2n_B = n_{BG}\ , \tag{1.59}$$

where n_A and n_B are the number of type A and type B Goldstones respectively, and

$$i\rho_{ab} = \langle [Q_a, J^0_b] \rangle_0\ . \tag{1.60}$$

The distinction of type A and B, rather than type I and II, is based on the counting rather than on dispersion relation: a type A GB corresponds to a unique broken generator, whereas a type B GB corresponds to a pair of canonically conjugated broken generators. Nonetheless, type A GBs typically have linear dispersion relation, while type B GBs have quadratic dispersion relation, even if there are exceptions, namely when the breaking of spacetime symmetries is involved (see sec. VI.A of [55]).

In next section we will briefly analyze a basic field theory model where Lorentz invariance is explicitly broken by a chemical potential, and both type A and type B Goldstone bosons are encountered.

1.3.1 Relativistic-like theories with a chemical potential

The field theoretical model that we discuss here is inspired to the one presented in [38, 39], which shares in turn the same symmetries and the same pattern of symmetry breaking as the holographic model that we consider in Chapter 4. In the weak coupling limit, we can compute the two-point correlators at tree level, derive the Ward identities in absence of Lorentz invariance, but also extract the dispersion relations and the spectrum.

The model consists in theory of a complex scalar doublet Φ of a global U(2) symmetry, where a non-dynamical abelian gauge field $B_\mu = (\mu, \vec{0})$ introduces a non-zero chemical potential μ, which explicitly breaks Lorentz invariance in the otherwise relativistic action

$$\begin{aligned} S &= -\int \mathrm{d}^d x\ \left[(\partial_\mu + iB_\mu)\Phi^\dagger (\partial^\mu - iB^\mu)\Phi + M^2\Phi^\dagger\Phi + \lambda\left(\Phi^\dagger\Phi\right)^2 \right] \\ &= \int \mathrm{d}^d x\ \left[(\partial_0 + i\mu)\Phi^\dagger (\partial_0 - i\mu)\Phi - \partial_i\Phi^\dagger\partial_i\Phi - M^2\Phi^\dagger\Phi - \lambda\left(\Phi^\dagger\Phi\right)^2 \right]\ . \end{aligned} \tag{1.61}$$

Using (1.3), we can compute the four conserved currents,

$$J^a_\nu = i\Phi^\dagger T^a \partial_\nu \Phi - i\partial_\nu \Phi^\dagger T^a \Phi + 2B_\nu \Phi^\dagger T^a \Phi\ , \quad \text{with } a = \{0,1,2,3\}\ , \tag{1.62}$$

where T^a are the generators of U(2) in the fundamental representation, namely

$$T^a = \frac{1}{2}\tau^a\ , \qquad \text{with } \tau^0 = \mathbb{1}_2\ , \quad \tau^1 = \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}\ , \quad \tau^2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}\ , \quad \tau^3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{1.63}$$

It can be easily checked that, for any value of μ (which is real), the currents (1.62) indeed satisfy the U(2) current algebra, given by $\left[T^a, T^b\right] = i f^{abc} T^c$, with $f^{abc} = 0$ if any of the indices is 0, and $f^{abc} \equiv \epsilon^{abc}$ otherwise, where ϵ^{abc} is the usual completely antisymmetric three-dimensional Levi-Civita symbol.

When $\mu^2 > M^2$ (i.e. always if $M^2 < 0$) the theory settles in a vacuum where the global symmetry is broken down to a U(1), so we have three broken generators. For definiteness, we choose a real vev for the bottom component,

$$\Phi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}, \qquad \text{with } v^2 = \frac{\mu^2 - M^2}{2\lambda}, \tag{1.64}$$

so that the component $\frac{1}{2}(\mathbb{1} + \tau_3)$ is conserved, whereas the other three generators, $\frac{1}{2}(\mathbb{1} - \tau^3)$, τ^1, τ^2, are broken. We can then consider small fluctuations around the vacuum, in their real and imaginary parts,

$$\Phi = \begin{pmatrix} \sigma + i\theta \\ v + \rho + i\pi \end{pmatrix}, \tag{1.65}$$

and work out the quadratic part of the Lagrangian, which reads

$$\begin{aligned}\mathscr{L}_{\text{quad}} = &-\partial_\mu\sigma\partial^\mu\sigma - \partial_\mu\theta\partial^\mu\theta - \partial_\mu\rho\partial^\mu\rho - \partial_\mu\pi\partial^\mu\pi + \\ &- 2\mu\left(\sigma\partial_t\theta - \theta\partial_t\sigma + \rho\partial_t\pi - \pi\partial_t\rho\right) - 4\lambda v^2\rho^2 .\end{aligned} \tag{1.66}$$

This Lagrangian seems to have only one massive particle, and so three Goldstone bosons for the three broken generators. However, we will see that it is not the case, as soon as we inspect the low-momentum eigenvalues of the mass-matrices, which in momentum space ($\partial_t \to -i\omega$) are given by

$$\begin{pmatrix} \sigma & \theta \end{pmatrix} \begin{pmatrix} \omega^2 - \vec{k}^2 & 2i\mu\omega \\ -2i\mu\omega & \omega^2 - \vec{k}^2 \end{pmatrix} \begin{pmatrix} \sigma \\ \theta \end{pmatrix} + \begin{pmatrix} \rho & \pi \end{pmatrix} \begin{pmatrix} \omega^2 - \vec{k}^2 - 4\lambda v^2 & 2i\mu\omega \\ -2i\mu\omega & \omega^2 - \vec{k}^2 \end{pmatrix} \begin{pmatrix} \rho \\ \pi \end{pmatrix} . \tag{1.67}$$

The low-momentum zeros of the determinant of the above matrices yield the dispersion relations.

For the ρ-π sector, we have

$$\omega_0^2 \sim 4\lambda v^2 + 4\mu^2 , \qquad \omega_3^2 \sim \frac{\lambda v^2}{\lambda v^2 + \mu^2}\vec{k}^2 . \tag{1.68}$$

This is the ‘relativistic-like’ sector: indeed we have a Higgs-like massive particle, and a Goldstone boson with linear dispersion relation. However, the constant of proportionality depends on the chemical potential and it is always different from one, except for $\mu \equiv 0$, when Lorentz invariance is restored.

The mass-matrix of the σ-θ sector, instead, yields the following dispersion relations

$$\omega_1 \sim 2\mu , \qquad \omega_2 \sim \frac{\vec{k}^2}{2\mu} . \tag{1.69}$$

Remarkably, a massive mode arises even in this sector, leaving a unique Goldstone boson for a pair of broken generators. Furthermore, this latter exhibits a quadratic dispersion relation. Notice that the mass of the massive partner is proportional to the Lorentz breaking parameter [12], so it can be kept hierarchically small by lowering the chemical potential.

Let us see that we are precisely in a situation where a charge density has a non-zero vacuum expectation value, so that the commutator of two broken generators consequently has. From the expression of the currents (1.62), and using the definition of conserved charge (1.5), we have

$$\langle [Q^1, Q^2] \rangle_0 = \int \mathrm{d}\vec{x} \int \mathrm{d}\vec{x}' \, \langle [J_t^1(x), J_t^2(x')] \rangle_0 = i \int \mathrm{d}\vec{x} \, \langle J_t^3(x)] \rangle_0 \neq 0 \,, \tag{1.70}$$

since

$$\langle J_t^3 \rangle_0 = -\mu \Phi_0^\dagger \tau^3 \Phi_0 = -\mu v^2 \,. \tag{1.71}$$

We notice that, on the contrary, $\langle J_t^1 \rangle_0 \equiv 0$ and $\langle J_t^2 \rangle_0 \equiv 0$, so that (1.70) is the only charge commutator having a non-vanishing vev. We conclude the analysis of this field theory model by deriving the Ward identity for the vev of J_t^3 from the tree-level propagators.

1.3.1.1 Non-relativistic propagators and Ward identities

From the mass-matrices (1.67) we can read off the tree-level propagators. We write down here those of the σ-θ sector, which are those that we will make use of,

$$\begin{aligned}
-i\langle \sigma(-k)\sigma(k) \rangle_0 &= \frac{\omega^2 - \vec{k}^2}{(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)} \,, \\
-i\langle \sigma(-k)\theta(k) \rangle_0 &= \frac{2i\mu\omega}{(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)} = i\langle \theta(-k)\sigma(k) \rangle_0 \,, \\
-i\langle \theta(-k)\theta(k) \rangle_0 &= \frac{\omega^2 - \vec{k}^2}{(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)} \,,
\end{aligned} \tag{1.72}$$

where the location of the poles are given by the dispersion relations that make the determinant of the respective mass matrix vanish, and whose low-momentum form are those showed in (1.68-1.69).

In the broken symmetry vacuum, the currents can also be expanded at the linear order in the fluctuated fields, obtaining:

$$\begin{aligned}
J_\nu^0 &= -v\partial_\nu \pi + 2vB_\nu \rho = -J_\nu^3 \,, \\
J_\nu^1 &= -v\partial_\nu \theta + 2vB_\nu \sigma \,, \\
J_\nu^2 &= -v\partial_\nu \sigma - 2vB_\nu \theta \,.
\end{aligned} \tag{1.73}$$

With this expressions we can compute the current-current correlator $\langle \partial_\nu J^{1\nu} J_t^2 \rangle_0$ in momentum space

$$-ik^{\nu}\langle J^1_{\nu}(-k)J^2_t(k)\rangle_0 = v^2\Big[\big(\omega^2-\vec{k}^2\big)\Big((-i\omega)\langle\theta\sigma\rangle_0+2\mu\langle\theta\theta\rangle_0\Big)+2i\omega\mu\Big(2\mu\langle\sigma\theta\rangle_0-i\omega\langle\sigma\sigma\rangle_0\Big)\Big]$$
$$= 2i\mu v^2\,\frac{\omega^2-\vec{k}^2-4\mu^2\omega^2}{\big(\omega^2-\omega_1^2\big)\big(\omega^2-\omega_2^2\big)} \equiv -2if^{123}\langle J^3_t\rangle_0\,, \tag{1.74}$$

where the simplification in the last line comes from the fact that the numerator is precisely the determinant of the mass-matrix (1.67) which vanishes at the location of the poles, and we have used (1.71) as well.

So we have retrieved the Ward identity from the dispersion relations. We have in a sense reversed the logic of (1.54), where we have shown that is possible to derive the dispersion relations from the Ward identities. Of course, the kind of current-current Ward identity of (1.74) can be obtained in full generality from the path integral invariance, as it was done in Section 1.2 for the current-scalar Ward identity (1.34). We have just to start from the the path integral of the charge density, and repeat the same steps.

In Section 4 we obtain holographically the analogous of the Ward identity (1.74), as well as all the other Ward identities, for a holographic model that enjoys the same symmetry breaking pattern as the one presented here.

1.3.2 Goldstone bosons in Lifshitz invariant field theories

In this section we want to very briefly introduce Lifshitz invariant field theories, and discuss in particular the characteristic features of the elements that we are concerned with: symmetry breaking, Goldstone dispersion relations, Ward identities. We start by deriving correlators between currents and order parameters in low-energy effective field theories of Goldstone bosons. We then discuss how the qualitative features of these correlators can be extracted from the Ward identities.

Lifshitz invariance is an anisotropic form of scaling invariance: it rescales differently space and time. Namely,

$$\begin{aligned} t &\longrightarrow \lambda^z\, t\,,\\ x_i &\longrightarrow \lambda\, x_i\,, \end{aligned} \tag{1.75}$$

where Lifshitz scaling parameter z is also called the dynamical critical exponent. For $z=1$ we have the relativistic scaling invariance, whereas with any $z\neq 1$ Lorentz symmetry is violated. Thus, one of the theoretical interests in Lifshitz invariant field theories is driven by the will to find an analogous of conformal invariance for non-relativistic QFTs.

Consider the following low-energy effective action for a Goldstone boson in a field theory which is invariant under Lifshitz scaling (1.75) [20], as well as under time reversal,

$$S=\int \mathrm{d}^d x\;\frac{1}{2}\Big(\partial_t\pi\partial_t\pi-(-1)^z\xi\pi\nabla^{2z}\pi\Big)\,, \tag{1.76}$$

where $\nabla^2 = \partial_i \partial_i$, and the sign in front of the second term is chosen such that the dispersion relation reads $\omega^2 = \xi k^{2z}$, so that we can set $\xi \geq 0$. The relativistic case is $z = 1$, $\xi = 1$. It can be more reassuring to think of z as an integer, but it can really take any value (here we will mainly consider $z \geq 1$).

Assuming ξ does not scale, the scaling dimensions are the following:

$$[\partial_t] = z\,, \quad [\partial_i] = 1\,, \quad [\pi] = \frac{d-1-z}{2}\,. \tag{1.77}$$

Note that for $d \leq z+1$ the scalar field has vanishing or negative scale dimension, which makes its fluctuations long range, rendering the effective action ill-defined. This will turn out to be a problematic case as well in the corresponding holographic setup of Chapter 5, where the 'BF bound' for temporal component of the vector field will be given precisely by $d = z+1$ (see eq. (5.46) and the discussion below).

The propagator for π, that we can extract from the action (1.76) is the following, in Fourier space,

$$\left\langle \pi(k)\pi(-k)\right\rangle_0 = \frac{i}{\omega^2 - \xi |\vec{k}|^{2z}}\,. \tag{1.78}$$

It has the correct scaling dimension, since $2[\pi] - (d-1+z) = -2z = -2[\omega]$.

The action (1.76) enjoys a shift symmetry $\pi \to \pi + v\alpha$, with v the vev and α the parameter of the transformation. This is indeed expected for a Goldstone boson. In order to find the current that generates this symmetry (which is broken by the vev v), we promote α to a space-time dependent function, and define

$$\delta_\alpha S = \int \mathrm{d}^d x \left(\partial_t \alpha J_t - \partial_i \alpha J_i\right)\,. \tag{1.79}$$

We then obtain

$$J_t = v \partial_t \pi\,, \qquad J_i = (-1)^{z-1} \xi v \nabla^{2z-2} \partial_i \pi\,. \tag{1.80}$$

They are linear in the Goldstone field, as it should be for currents of a broken symmetry (at the lowest order). The conservation law is

$$-\partial_t J_t + \partial_i J_i = -v\left(\partial_t^2 + (-1)^z \xi \nabla^{2z}\right)\pi = 0\,, \tag{1.81}$$

from the equation of motion. Note that it reads exactly as in the relativistic case, although the dimensions of the currents are here changed:

$$[J_t] = d-1\,, \qquad [J_i] = d+z-2\,. \tag{1.82}$$

We can now see how the conservation law appears in two-point functions, that is in the Ward identities. Recall that here the operator breaking the symmetry is π itself, with

$\langle\delta_\alpha\pi\rangle_0 = v$. Using (1.78), we have

$$\langle J_t(k)\pi(-k)\rangle_0 = -i v \omega \,\langle\pi\pi\rangle_0 = \frac{v\omega}{\omega^2 - \xi|\vec{k}|^{2z}} \,, \tag{1.83}$$

$$\langle J_i(k)\pi(-k)\rangle_0 = i\xi v\, k_i|\vec{k}|^{2z-2}\langle\pi\pi\rangle_0 = -\frac{v\xi k_i|\vec{k}|^{2z-2}}{\omega^2 - \xi|\vec{k}|^{2z}} \,, \tag{1.84}$$

so that

$$i\omega\langle J_t\pi\rangle_0 + i k_i\langle J_i\pi\rangle_0 = \frac{i v\omega^2}{\omega^2 - \xi|\vec{k}|^{2z}} - \frac{i v\xi|\vec{k}|^{2z}}{\omega^2 - \xi|\vec{k}|^{2z}} = i v \,. \tag{1.85}$$

This is again the Ward identity (1.35),

$$-\partial_t\langle J_t\pi\rangle_0 + \partial_i\langle J_i\pi\rangle_0 = i\langle\delta_\alpha\pi\rangle_0 \,, \tag{1.86}$$

keeping implicit the delta function of the contact term.

We can briefly consider also the case of a theory which has Lifshitz scaling but not time reversal symmetry. The low energy action is then[7]

$$S = \int \mathrm{d}^d x\; \frac{1}{2}\left(i\psi^*\partial_t\psi - (-i)^z\zeta\,\psi\nabla^z\psi\right) \,. \tag{1.87}$$

Again, it is easier to consider z even, but it can be generic. Now the dimension of the Goldstone field is

$$[\psi] = \frac{d-1}{2} \,. \tag{1.88}$$

Note that it is always positive as long as $d > 1$. Its propagator is

$$\langle\psi\psi^*\rangle_0 = \frac{i}{\omega - \zeta|\vec{k}|^z} \,, \tag{1.89}$$

We can again derive the currents,

$$J_t = -i v\psi \,, \qquad J_i = -(-i)^z v\zeta\nabla^{z-2}\partial_i\psi \,. \tag{1.90}$$

and verify that the Ward identities are realized again:

$$i\omega\langle J_t\psi\rangle_0 + i k_i\langle J_i\psi\rangle_0 = \frac{i v\omega}{\omega - \zeta|\vec{k}|^z} - \frac{i v\zeta|\vec{k}|^z}{\omega - \zeta|\vec{k}|^z} = i v \,. \tag{1.91}$$

1.3.2.1 From Ward identities to the Goldstone boson

Having seen how the Ward identities are realized in a specific example of a low energy effective theory, we now reverse the logic and start from the Ward identities in order to find the

[7] The linearity in time derivatives forces us to take a complex scalar field, otherwise the action would be a total derivative. However, only one massless physical degree of freedom arises, the Goldstone boson. We assume boundary terms that make the action real.

Goldstone boson. We have

$$-\partial_t \langle J_t \mathcal{O} \rangle_0 + \partial_i \langle J_i \mathcal{O} \rangle_0 = i \langle \mathcal{O} \rangle_0 \,, \tag{1.92}$$

for some operator which transforms under the symmetry generated by the currents, and which has a vev that breaks the symmetry.

Using the invariance under spatial rotations, we parametrize the correlators in Fourier space as follows,

$$\langle J_t \mathcal{O} \rangle_0 = f\left(\omega, |\vec{k}|\right) \,, \qquad \langle J_i \mathcal{O} \rangle_0 = -k_i\, g\left(\omega, |\vec{k}|\right) \,. \tag{1.93}$$

Note that $[f] = \Delta - z$ and $[g] = \Delta - 2$, where Δ is the dimension of the operator $\mathcal{O}$. The Ward identity then implies

$$\omega f - \vec{k}^2 g = \langle \mathcal{O} \rangle_0 \,. \tag{1.94}$$

Obviously, assuming $\langle \mathcal{O} \rangle_0$ finite and non zero when $\omega, |\vec{k}| \to 0$, then either f or g, or both, have to blow up, signaling the presence of a massless particle in the spectrum, the Goldstone boson.

Let us be more precise. Take first $|\vec{k}| \to 0$ with $\omega \neq 0$. Then, assuming g finite in this limit, we have $f \to \frac{\langle \mathcal{O} \rangle_0}{\omega}$. Similarly, when $\omega \to 0$ at $|\vec{k}| \neq 0$, we have $g \to -\frac{\langle \mathcal{O} \rangle_0}{\vec{k}^2}$. We can then rewrite

$$f = \frac{\langle \mathcal{O} \rangle_0}{\omega} \tilde{f} \,, \qquad g = \frac{\langle \mathcal{O} \rangle_0}{|\vec{k}|^2} \left(\tilde{f} - 1\right) \,. \tag{1.95}$$

where $\tilde{f}$ is a dimensionless function of ω and $|\vec{k}|$.

There are two trivial ways to satisfy the Ward identity, which is setting either $\tilde{f} = 1$ or $\tilde{f} = 0$. These two choices do not corresponding to propagating degrees of freedom: they correspond to the two previously taken limits, and yields the degenerate dispersion relations $k^2 \equiv 0$ (independently of ω), and $\omega \equiv 0$ (independently of $\vec{k}^2$), respectively. We want rather $\omega \to 0$ *as* $|\vec{k}| \to 0$, giving a proper dispersion relation $\omega = \omega(|\vec{k}|)$. Thus we consider the relevant case where $\tilde{f}$ is a non-trivial function.

If we require that the low energy theory has Lifshitz scaling, then $\tilde{f}$ must be a function of the ratio $x = \frac{|\vec{k}|^z}{\omega}$. If we also impose time reversal symmetry, then it must be a function of x^2. The conditions on the $\omega \to 0$ and $|\vec{k}| \to 0$ limits translate into

$$\tilde{f}(x = 0) = 1 \,, \qquad \tilde{f}(x = \infty) = 0 \,. \tag{1.96}$$

We can readily find simple functions that satisfy the above requirements, and reproduce the single-particle correlators obtained previously.

First, we let time reversal be broken, and we take

$$\tilde{f} = \frac{1}{1 - \zeta x} = \frac{\omega}{\omega - \zeta |\vec{k}|^z} \,. \tag{1.97}$$

In this way, using (1.93) and (1.95), we have

$$\langle J_t \mathcal{O} \rangle_0 = \frac{\langle \mathcal{O} \rangle_0}{\omega} \frac{\omega}{\omega - \zeta |\vec{k}|^z} = \frac{\langle \mathcal{O} \rangle_0}{\omega - \zeta |\vec{k}|^z} \,, \tag{1.98}$$

$$\langle J_i \mathcal{O} \rangle_0 = k_i \frac{\langle \mathcal{O} \rangle_0}{|\vec{k}|^2} \left(1 - \frac{\omega}{\omega - \zeta |\vec{k}|^z} \right) = -\frac{\zeta k_i |\vec{k}|^{z-2} \langle \mathcal{O} \rangle_0}{\omega - \zeta |\vec{k}|^z} \,, \tag{1.99}$$

recovering the same form as in (1.91) (with the identification $\langle \mathcal{O} \rangle_0 \equiv v$).

Imposing now time reversal invariance, we can take

$$\widetilde{f}_T = \frac{1}{1 - \xi x^2} = \frac{\omega^2}{\omega^2 - \xi |\vec{k}|^{2z}} \,, \tag{1.100}$$

so that

$$\langle J_t \mathcal{O} \rangle_0 = \frac{\langle \mathcal{O} \rangle_0}{\omega} \frac{\omega^2}{\omega^2 - \xi |\vec{k}|^{2z}} = \frac{\omega \langle \mathcal{O} \rangle_0}{\omega^2 - \xi |\vec{k}|^{2z}} \,, \tag{1.101}$$

$$\langle J_i \mathcal{O} \rangle_0 = k_i \frac{\langle \mathcal{O} \rangle_0}{|\vec{k}|^2} \left(1 - \frac{\omega^2}{\omega^2 - \xi |\vec{k}|^{2z}} \right) = -\frac{\xi k_i |\vec{k}|^{2z-2} \langle \mathcal{O} \rangle_0}{\omega^2 - \xi |\vec{k}|^{2z}} \,, \tag{1.102}$$

which are the same as (1.83)–(1.84) (again with the identification $\langle \mathcal{O} \rangle_0 \equiv v$).

Note that in both cases, more complicated functions can be taken. However, all these functions will involve a denominator with a polynomial in x (or x^2). Near the roots of this polynomial, the functions will all be very close to those that we have taken.

Along this first chapter, we have seen in a general fashion and in various simple examples how the Ward identities for symmetry breaking are a strongly general feature, which applies to non-relativistic field theories in the same way as they do for relativistic ones. Moreover, we have pointed out how contact terms in the Ward identities are connected to the presence of massless Goldstone modes in the spectrum, and we have explored what information they can unveil over the dispersion relations of such gapless modes.

In part II, we will see how to retrieve the non-perturbative structure of Ward identities for symmetry breaking in holography. Before, let us introduce, in the next Chapter, the holographic correspondence, and the main computational tool that we will adopt in part II, holographic renormalization.

2 Holography from a boundary-oriented perspective

The duality between quantum, strongly-coupled field theories and classical, weekly-coupled gravitational theories has been originally conjectured [13] and affirmed between asymptotically Anti de Sitter (AdS) space-times and conformal field theories in flat space. Yet the holographic principle seems to hold in a much broader variety of different cases, as for instance in the case, relevant for us, of non-relativistic bulk geometries. As Jared Kaplan affirms in his lecture notes on AdS/CFT correspondence [56]: "AdS/CFT is many things to many people". The gauge/gravity duality is then even more things for at least as many people, so let us say what it is to us.[1]

Let us start from the original concrete realization of AdS/CFT [13]. Maldacena considered N coincident D3-branes in type IIB string theory, where the low-energy limit provides two alternative descriptions. On the one hand, in perturbative string theory at low-energy, where only massless excitations are important, massless close strings have irrelevant interactions and decouple, while massless open strings ending on the D3-branes have dimensionless coupling and survive. The resulting theory of interacting $SU(N)$ gauge fields living on the four-dimensional (flat) world-volume of the N D3-branes is $\mathcal{N}=4$ $SU(N)$ super-Yang-Mills (SYM), with Yang-Mills coupling g_{YM} given in terms of the string coupling g_s, by the identification $g_{\text{YM}}^2 \sim g_s$. On the other hand, D3-branes are black brane solutions of supergravity (low-energy classical limit of string theory). In this picture the low-energy limit selects the near-horizon region, and the near-horizon geometry of D3-branes in 10-dimensional type IIB supergravity is given by $\text{AdS}_5 \times S^5$, where S^5 is the five-sphere and the AdS radius L is given by $L^4 \sim g_s N l_s^4$, where l_s is the string length.

So, we have taken the same limit of a single theory, and obtained two apparently very different theories, which nonetheless should be equivalent: $\mathcal{N}=4$ $SU(N)$ super-Yang-Mills

[1] Giving a review on holography is far beyond the purpose of this dissertation. For complete and pedagogical introductions we refer to the copious existing material, as for instance [41, 57–62].

on flat space on one side, and supergravity on $AdS_5 \times S^5$ on the other. Let us first remark that the symmetries match on the two sides: the group of isometries of AdS_5 coincide with the conformal group in one less dimension, SO(4,2) (and this is true for any dimensions), the symmetry of the five-sphere, SO(6), matches the SU(4) R-symmetry of $\mathcal{N} = 4$ SYM, and supersymmetries are the same on both sides.

However, the two approximations are valid in opposite regimes, which is the essential property of a duality. Indeed, the perturbative description holds when perturbation theory applies, that is for $g_s N \ll 1$. On the other hand supergravity, *i.e.* the classical approximation, is reliable if the quantum corrections are negligible, so $g_s \to 0$, and if the AdS curvature radius L is larger than the string length l_s, which is true for $g_s N \gg 1$, which implies $N \to \infty$.

Let us summarize by introducing the 't Hooft coupling λ [63], so that

$$\frac{L^4}{l_s^4} \sim g_s N \sim g_{\text{YM}}^2 N \equiv \lambda \,. \tag{2.1}$$

If in SU(N) Yang-Mills theory one takes N to be large, but g_{YM} small enough so that $\lambda \ll 1$, perturbation theory applies, and 't Hooft showed [63] that at large N quantum loop diagrams organize according to the topology, with decreasing powers of N as the genus of the diagram increases. In this way, planar diagrams are dominant in the large N limit (which for this reason is also called planar limit).

In conclusion, if we can commute the low-energy limit with the fact of moving continuously between the weak-coupling regime $\lambda \ll 1$ and the strong-coupling regime $\lambda \gg 1$, then the duality is established. This last assumption is assumed to be true, but it is very hard to be proved or disproved, precisely for the fact that in the regime where one side of the duality is controllable, the other is hardly tractable, and vice-versa. At the same time, such duality is a very powerful and promising tool, precisely for the fact that when one of the side is obscure, the other is generically under control.

Anyway, many non-trivial checks of the duality have been performed, in particular on supersymmetry-protected quantities, which can be computed on both sides [64], but also by a large variety of constructions, which suggests that the duality holds beyond its original AdS/CFT framework. Actually, in the perspective of our discussion, we precisely wish it to extend to the broadest possible class of theories (namely non-relativistic space-time, non-conformal field theories, and so on). On the other hand, since we are mainly interested in using holography to study quantum field theories at strong coupling (in this sense, a boundary-oriented perspective), we are happy with the classical approximation for bulk gravity (which for our purposes will actually be even non-dynamical).

Letting the quantum completion of our classical gravity be string theory as well as whatever other theory of quantum gravity, for the aims of this thesis we assume

$$\frac{L^4}{L_{Pl}^4} \gg 1 \,, \tag{2.2}$$

meaning that the size L of our bulk space-time is large with respect to the Planck length L_{Pl}, that is the scale where quantum gravitational effects become relevant, so that the curvature is weak and the classical description of gravity holds. In turns, this implies that the dual QFT contains a large number of degrees of freedom.

In this range of validity, a weaker/broader holographic principle can be formulated, which is embodied by the following statement [40]: the on-shell partition function of a classical gravitational theory in a given $(d+1)$-dimensional space-time $\mathscr{M}$ is equivalent to the (off-shell) generating functional for correlation functions of a gauge field theory with no gravity on the boundary $\partial\mathscr{M}$. In formulæ:

$$\begin{aligned} e^{iS_{\text{cl}}^{\text{grav}}} &\equiv \exp\left[i\int_{\mathscr{M}} \mathrm{d}^{d+1}x\,\sqrt{-g}\;\mathscr{L}_{\text{cl}}[\varphi]\Big|_{\text{on-shell}}\right] = \\ &= \left\langle \exp\left[i\int_{\partial\mathscr{M}} \mathrm{d}^{d}x\,\sqrt{-\hat{g}}\;\varphi_0\mathscr{O}_{\varphi}\right]\right\rangle_0 \equiv \mathscr{Z}_{\text{QFT}}, \end{aligned} \tag{2.3}$$

where g is the determinant of the bulk metric, $\hat{g}$ the one of the corresponding induced metric on the boundary, φ is a generic classical bulk field, φ_0 is its asymptotic boundary value, and $\mathscr{O}_{\varphi}$ is the boundary operator defined in this way as dual to the bulk field φ. Notice that the left-hand side of this equation explicitly relies on a weak-coupling local Lagrangian formulation, whereas the right-hand side is the generating functional for correlation function in a general QFT, which is valid regardless of the strength of the coupling.

The actual meaning of the left-hand side will be made clear in the following, but it is already evident from the right-hand side, that equation (2.3) furnishes a prescription for computing QFT n-point correlation functions. Defining $\mathscr{W}_{\text{QFT}} = \ln\mathscr{Z}_{\text{QFT}}$, we have the following formula for *connected* n-point correlation functions

$$\left\langle \mathscr{O}_{A_1}(x_1)\cdots\mathscr{O}_{A_n}(x_n)\right\rangle_0 \equiv (-i)^n \left.\frac{\delta^n\,\mathscr{W}_{\text{QFT}}[\varphi_0]}{\delta\varphi_0^{A_1}(x_1)\cdots\delta\varphi_0^{A_n}(x_n)}\right|_{\varphi_0=0} \tag{2.4}$$

$$= (-i)^n \left.\frac{\delta^n\, iS_{\text{cl}}^{\text{grav}}}{\delta\varphi_0^{A_1}(x_1)\cdots\delta\varphi_0^{A_n}(x_n)}\right|_{\varphi_0=0} \tag{2.5}$$

where we have allowed for various boundary operators $\mathscr{O}_A$, dual to various bulk fields φ^A.

Furthermore, we remark that the boundary dual operator has not to be a scalar, it can be any sort of composite QFT expressions, however, since it couples to the source φ_0 in the generating functional $\mathscr{Z}_{\text{QFT}}$, it must have the same symmetry properties and quantum numbers as the corresponding bulk field φ. In particular, for a conserved current J^μ we have

$$\exp\left[i\int_{\partial\mathscr{M}} \mathrm{d}^{d}x\,\sqrt{-\hat{g}}\;A_{0\mu}J^{\mu}\right]. \tag{2.6}$$

As already discussed in Section 1.2 around eq. (1.31), the gauge invariance of the measure in the generating functional entails that the source of a conserved current undergoes gauge transformations. Thus, in holography this implies that the dual field of the conserved current of a global symmetry on the boundary is given by an actual gauge field in the bulk.

Let us now focus on the left-hand side of the foundational equation (2.3): the bulk action should be put on-shell by means of the bulk equations of motion, and the sources for dual operators identified. Then, suitable boundary conditions, and the capability to solve the equations of motion, allow to express the boundary action in terms of the sources only. However, the operation of reduction on the boundary may produce diverging terms. Such divergences can be removed in a consistent way, identifying at the same time the sources for dual operators, by a procedure called holographic renormalization, by analogy with its field theoretical counterpart. Next section is devoted to illustrate such procedure in detail.

2.1 Holographic renormalization

The idea of holographic renormalization was first proposed in [65, 66], and then developed in a systematic way in [28–30]. Here we illustrate it through the minimal example of a real scalar field φ of mass m_φ on an AdS_{d+1} fixed background, following an approach similar to [67]. Several more examples of holographic renormalization will follow throughout the holographic models of Part II.

We thus consider the following bulk action,

$$S = -\frac{1}{2}\int \mathrm{d}^{d+1}x\ \sqrt{-g}\ \left(g^{mn}\partial_m\varphi\partial_n\varphi + m_\varphi^2\varphi^2\right), \tag{2.7}$$

where Latin letters indicate indices in the $(d+1)$-dimensional bulk, while Greek letters indicate indices in the d-dimensional boundary, which are contracted in the usual way with mostly-plus Minkowski metric[2], and where the AdS metric in the Poincaré patch is defined by

$$\mathrm{d}s^2 = \frac{1}{r^2}\left(\mathrm{d}r^2 + \mathrm{d}x^\mu \mathrm{d}x_\mu\right), \qquad \text{so that} \quad g^{rr} = r^2\,, \quad g^{\mu\nu} = r^2\eta^{\mu\nu}\,, \tag{2.8}$$

where we have set the AdS radius $L \equiv 1$. We call r holographic coordinate or radial coordinate, and the boundary is located at $r = 0$.

The variational principle gives the following equation of motion

$$0 = -\frac{1}{\sqrt{-g}}\partial_m\left(\sqrt{-g}\,\frac{\delta\mathscr{L}}{\delta\partial_m\varphi}\right) - \frac{\delta\mathscr{L}}{\delta\varphi} = r^{d+1}\partial_r\left(r^{-d+1}\partial_r\varphi\right) + r^2\Box\varphi - m_\varphi^2\varphi\,, \tag{2.9}$$

where with the box we indicate the d'Alembertian in Minkowski space, $\Box = \partial_\mu\partial^\mu$. Exact analytic solutions to this equation are given in terms of Bessel functions, as we will see in Section 2.1.1.

[2] We choose to adopt from the beginning the 'physical' Lorentzian signature, rather than the Euclidean one, even if the holographic renormalization and the computation of correlation functions require careful treatments [68, 69], related to the causal $i\epsilon$ prescription for (time-ordered, retarded, advanced) correlation functions in QFT. However, for the setups presented in this work, which involve no temperature and no time-evolution, the actual Lorentzian computation would be uselessly complicated and equivalent to the Euclidean one with a final suitable analytic continuation to real time (Wick rotation), and indeed it is what we will be implicitly doing all the time.

For the moment, we just need the asymptotic behavior, which can be easily determined by solving the equation order by order in the near-boundary expansion, obtaining [40, 70]

$$\varphi = r^{d-\Delta}\left(\varphi_0 + r^2\varphi_1 + \dots\right) + r^{\Delta}\left(\tilde{\varphi}_0 + r^2\tilde{\varphi}_1 + \dots\right), \tag{2.10}$$

$$\text{with } \Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m_\varphi^2}\,. \tag{2.11}$$

This power expansion holds whenever $2\Delta - d$ is different from zero, and is not an even integer, so for $\Delta \neq \frac{d}{2} + n$, with n non-negative integer. Otherwise, the two parts of (2.10) mix and logarithmic terms arise, as it can be checked from the equation of motion (2.9) itself. The presence of logarithms does not represent any insurmountable obstacle, nonetheless we avoid them for the moment in order to keep the discussion simpler, and we postpone their treatment to Section 2.1.2.

The value $\Delta = \frac{d}{2}$ is a minimal value for Δ, and it corresponds to the minimal value that the scalar mass can get for the argument of the square root in the definition of Δ to be non-negative, that is

$$m_\varphi^2 = -\frac{d^2}{4}\,. \tag{2.12}$$

This defines the BF (Breitenlohner-Freedman) bound for the AdS scalar mass [71]. Masses below this value correspond to tachyons in AdS. Nonetheless, AdS squared mass can get negative values, with still positive energy, since AdS curvature compensates with a positive contribution to the energy.

If we identify the asymptotic leading term φ_0 as the source for the dual boundary operator $\mathcal{O}_\varphi$ (2.3), then Δ (that is the asymptotic exponent of the sub-leading mode $\tilde{\varphi}_0$) turns out to be the scaling dimension of $\mathcal{O}_\varphi$, so that we will call $\tilde{\varphi}_0$ the 'response'. Hence, the BF bound constitutes a lower bound for the dimension of the boundary operator: $\Delta \geq \frac{d}{2}$. Actually, in a conformal field theory there is a lower bound for the dimension of scalar operators, given by unitarity [72]:

$$\Delta > \frac{d}{2} - 1\,. \tag{2.13}$$

However, the BF bound is above the unitarity bound, and there are no reasons from the boundary point of view for the dimension of $\mathcal{O}_\varphi$ not to be allowed to go down to the unitarity bound. And indeed values of the scaling dimension between the unitarity bound and the BF bound can be accessed by making $\tilde{\varphi}_0$ the source, so that the the dual scalar operator is of dimension $d - \Delta$ (*i.e.* φ_0 is the response).

Breitenlohner and Freedman themselves [71], demanding finiteness of the AdS action and positive definiteness of the energy, actually showed that for $m_\varphi^2 > -\frac{d^2}{4} + 1$, that is $\Delta > \frac{d}{2} + 1$, only one choice of boundary condition (φ_0 is the source) is acceptable. Instead, for

$$-\frac{d^2}{4} < m_\varphi^2 < -\frac{d^2}{4} + 1 \tag{2.14}$$

two alternative AdS-invariant quantizations are admissible. For φ_0 as source, we have an operator of dimensions Δ, with $\frac{d}{2} < \Delta < \frac{d}{2}+1$; for $\widetilde{\varphi}_0$ as source, we have an operator of dimensions $\widetilde{\Delta} \equiv d-\Delta$, with $\frac{d}{2}-1 < \widetilde{\Delta} < \frac{d}{2}$, filling in this way the window of dimensions between the BF bound and the unitarity bound [70, 73–75].

We will see in detail how to set these different boundary conditions, and how holographic renormalization, besides removing divergences, actually allows to fix which are the sources, and consequently select the quantization.

Now, let us go back to the action (2.7), and try to reduce it to the boundary, in order to be able to apply our holographic principle (2.3). We first integrate by parts, and use the equation of motion to put the action on shell:

$$\begin{aligned} S &= -\frac{1}{2}\int \mathrm{d}^{d+1}x \, \left[\partial_r\left(r^{-d+1}\varphi\partial_r\varphi\right) - \varphi\partial_r\left(r^{-d+1}\partial_r\varphi\right) - r^{-d+1}\varphi\Box\varphi + r^{-d-1}m_\varphi^2\varphi^2\right] \\ &= -\frac{1}{2}\int \mathrm{d}^{d+1}x \quad \partial_r\left(r^{-d+1}\varphi\partial_r\varphi\right). \end{aligned} \tag{2.15}$$

We are left with a total derivative in the radial coordinate, so the action can be reduced to a boundary term[3], and expanded according to the asymptotic expression (2.10),

$$\begin{aligned} S_{\text{reg}} &= \frac{1}{2}\int_{r=\epsilon} \mathrm{d}^d x \;\, r^{-d}\,\varphi\, r\partial_r\varphi \;= \\ &= \frac{1}{2}\int_{r=\epsilon} \mathrm{d}^d x \;\left(r^{d-2\Delta}\,(d-\Delta)\,\varphi_0 + r^{d-2\Delta+2}\,2(d-\Delta+1)\,\varphi_1 + \,\dots\, + d\,\widetilde{\varphi}_0\right)\varphi_0\,, \end{aligned} \tag{2.16}$$

where we used a small ϵ as regulator for the potentially divergent terms, and the dots represent sub-leading terms of the order $O\big(r^{d-2\Delta+4}\big)$. We can see that the leading term is always divergent, the minimal value of Δ being $\frac{d}{2}$, that is at the BF bound, where the divergence is logarithmic, as we have already pointed out.

Let us focus first on the most dangerous term. We want to remove it by a boundary counterterm, which should not drastically modify our bulk theory. Then, we demand the counterterm to be *local* and invariant under all the symmetries induced on the boundary by the bulk action (gauge invariance, space-time symmetries and internal symmetries). For the specific case of the action (2.7), which possesses Poincaré symmetry and a global $\mathbb{Z}_2$ invariance, the only terms we can write with the field content of the bulk action, *i.e.* the real scalar φ, are a kinetic term or a mass term. We will see that the former actually leads to alternative quantization, but we consider first the latter, namely

$$\begin{aligned} S_{\text{ct}}^{(0)} &= -\frac{1}{2}\int \mathrm{d}^d x \;\sqrt{-\hat{g}}\;\varphi^2 = \\ &= -\frac{1}{2}\int \mathrm{d}^d x \;\left(r^{d-2\Delta}\,\varphi_0 + r^{d-2\Delta+2}\,2\varphi_1 + \,\dots\, + 2\,\widetilde{\varphi}_0\right)\varphi_0\,, \end{aligned} \tag{2.17}$$

where $\hat{g}_{\mu\nu}$ is the d-dimensional metric induced on the boundary by the bulk $(d+1)$-dimensional

[3] The on-shell action reduces to a purely boundary term because the considered action is quadratic. With cubic interactions, for instance, bulk terms would survive. However, as far as *two-point* functions are concerned, the quadratic part of the action is all we need.

metric g_{mn}, which in the present case is just the flat Minkowski metric rescaled by r^2: $\hat{g}_{\mu\nu} = r^{-2}\eta_{\mu\nu}$.

This counterterm serves to cancel the leading divergence in the regularized action (2.16), but leaves sub-leading divergences whenever $\Delta \geq \frac{d}{2}+1$,

$$S_{\text{reg}} + (d-\Delta)\, S_{\text{ct}}^{(0)} = \frac{1}{2}\int \mathrm{d}^d x\ \left(r^{d-2\Delta+2}\, 2\varphi_1 + \ldots + (2\Delta-d)\,\widetilde{\varphi}_0\right)\varphi_0\,. \tag{2.18}$$

However, the coefficients of sub-leading terms φ_n can always be related to the coefficient of the leading term φ_0, by means of the equation of motion. This traces the road to write the additional counterterms we may need.

For instance for the term proportional to φ_1, which by the equations of motion is given by

$$\varphi_1 = \frac{1}{2(2\Delta-d-2)}\,\Box\varphi_0\,, \tag{2.19}$$

we have

$$\begin{aligned} S_{\text{ct}}^{(1)} &= \frac{1}{2}\int \mathrm{d}^d x\ \sqrt{-\hat{g}}\ \hat{g}^{\mu\nu}\varphi\partial_\mu\partial_\nu\varphi = \frac{1}{2}\int \mathrm{d}^d x\ \left(r^{d-2\Delta+2}\,\varphi_0\Box\varphi_0 + \ldots\right) = \\ &= \frac{1}{2}\,(2\Delta-d-2)\int \mathrm{d}^d x\ \left(r^{d-2\Delta+2}\, 2\varphi_0\varphi_1 + \ldots\right)\,, \end{aligned} \tag{2.20}$$

which, dressed with the suitable numerical factor, is indeed able to cancel the leading surviving divergence in (2.18). The dots here represent sub-leading terms of order $O\big(r^{d-2\Delta+4}\big)$. They can still be divergent, but again we can use the equation of motion, which actually give, for any n,

$$\varphi_n = \frac{1}{2n(2\Delta-d-2n)}\,\Box\varphi_{n-1} = \frac{\Gamma\left[\Delta-\frac{d}{2}-n\right]}{4^n n!\,\Gamma\left[\Delta-\frac{d}{2}\right]}\,\Box^n\varphi_0\,, \tag{2.21}$$

as it can be easily verified by induction. Thus, the sub-leading divergences can be removed, order by order, through a counterterm $S_{\text{ct}}^{(n)}$, similar to (2.20), but with n powers of d'Alembert operator, until no divergent terms are left.

However, for the purpose of computing correlators and Ward identities, we do not even need to do that. Indeed, with the exception of the special case where the scalar mass saturates the BF bound[4], the term proportional to $\varphi_0\widetilde{\varphi}_0$ (*i.e.* to the two independent leading modes of the solution (2.10), which will be identified with the source and the vev of the dual operator) is always finite in the leading counterterm $S_{\text{ct}}^{(0)}$ (2.17), and always vanishing in the rest of sub-leading counterterms $S_{\text{ct}}^{(n)}$[5]. Thus, precisely this term, bi-linear in the leading and sub-leading,

[4] At the BF bound the presence of a logarithm in the leading term affects the definition of the source, forcing the introduction of a scale a so partially violating conformal invariance [76, 77]. In Section 3.2 we will treat the case of a Maxwell vector field in AdS_3, which is analogous to that of a scalar at the BF bound.

[5] This fact has been shown more systematically using a different renormalization technique based on dimensional regularization [78].

yields the main part of the renormalized action

$$S_{\text{ren}} = \left(\Delta - \frac{d}{2}\right) \int \mathrm{d}^d x \;\, \varphi_0 \widetilde{\varphi}_0 \, . \tag{2.22}$$

When logarithmic divergences are present, this renormalized action gets modified by scheme-dependent finite pieces, as we will see in Section 2.1.2.

Now, we affirm that, in this renormalization scheme (2.17), φ_0 is the source for the dual boundary operator $\mathcal{O}_\varphi$, which has dimension Δ, with $\Delta > \frac{d}{2}$ by the definition (2.11).[6]

In the path integral formulation, the source is a field that is coupled to a given operator of the theory, and whose variation should be set to zero in order to preserve the variational principle (invariance of the action). In this way, the source is dual to the associated operator, and allows to generate the correlation functions for that operator.

So, in order to verify that φ_0 is actually the source, we have to check the variational principle on the renormalized action (2.22). The correct variation of the renormalized action is not the naïve variation of expression (2.22), is rather the variation of the bulk action (2.7), put on-shell and reduced to the boundary, and then renormalized by the variation of employed counterterms, which removes divergent pieces. Basically we have to repeat the renormalization procedure on the variation δS, so in practice the variation of the renormalized action is rather the renormalization of the varied action.

For the on-shell variation of the bulk action we have

$$\begin{aligned} \delta S &= -\int \mathrm{d}^{d+1}x \;\sqrt{-g}\; \left(g^{mn}\partial_m\varphi\partial_n\delta\varphi + m_\varphi^2\varphi\delta\varphi\right) = \int_{r=\epsilon} \mathrm{d}^d x \;\, r^{-d}\,\delta\varphi\, r\partial_r\varphi = \\ &= \int_{r=\epsilon} \mathrm{d}^d x \,\left[(d-\Delta) r^{d-2\Delta}\,\varphi_0\delta\varphi_0 + \ldots + \Delta\,\widetilde{\varphi}_0\delta\varphi_0 + (d-\Delta)\varphi_0\delta\widetilde{\varphi}_0\right] . \end{aligned} \tag{2.23}$$

On the other hand, the variation of counterterm (2.17) yields

$$\delta S_{\text{ct}}^{(0)} = -\int \mathrm{d}^d x \;\sqrt{-\hat{g}}\;\varphi\delta\varphi = -\int \mathrm{d}^d x \;\left[r^{d-2\Delta}\,\varphi_0\delta\varphi_0 + \ldots + \widetilde{\varphi}_0\delta\varphi_0 + \varphi_0\delta\widetilde{\varphi}_0\right] ,$$

which, with the same numerical factor as in (2.18), removes the leading divergence. Possible additional divergences can be removed by the variation of the respective counterterms, which yet do not affect the finite terms. Thus, we finally have

$$\delta S_{\text{ren}} = \delta S + (d-\Delta)\,\delta S_{\text{ct}}^{(0)} = (2\Delta - d)\int \mathrm{d}^d x \;\,\widetilde{\varphi}_0\,\delta\varphi_0 \, , \tag{2.24}$$

where only the variation of ϕ_0 appears. The variational principle for the boundary action imposes $\delta\varphi_0 = 0$, so that φ_0 is correctly identified as the source for the dual operator.

Finally, we remark that, once φ_0 is established as the source, the response $\widetilde{\varphi}_0$ can be related to the vacuum expectation value of the dual operator $\mathcal{O}_\varphi$. Indeed, applying (2.5) on the

[6] In the so-called alternative quantization scheme, the renormalized action will be changed by a sign (see eq. (3.94)) and the source would be φ, sourcing a boundary operator of dimension $\widetilde{\Delta} = d - \Delta$, with $-\frac{d}{2} < \widetilde{\Delta} < \frac{d}{2}$, as we will see in the following.

renormalized action (2.22),

$$\langle \mathcal{O}_\varphi \rangle_0 = \frac{\delta S_{\rm ren}}{\delta \varphi_0}\bigg|_{\varphi_0=0} = (2\Delta - d)\, \widetilde{\varphi}_0\big|_{\varphi_0=0} \,. \tag{2.25}$$

For these reason the response is usually referred to as the 'vev'. A fixed (non-vanishing with the source) value for the coefficient of the sub-leading mode, then returns a non-vanishing vacuum expectation value for the dual operator in the boundary theory. This is relevant when we want to describe spontaneous symmetry breaking in the boundary field theory.

We will now show that a different counterterm can cancel the same divergence as (2.17), yet giving the opposite quantization, where $\widetilde{\varphi}_0$ is the source and φ_0 the response. We recall that we are in the window (2.14), where double quantization is possible, so $\frac{d}{2} < \Delta < \frac{d}{2} + 1$, and for this range of values of Δ the leading divergence is the unique divergence in the on-shell action (2.16) [74, 75].

We consider the following boundary term,

$$\begin{aligned} \widetilde{S}^{(0)}_{\rm ct} &= -\frac{1}{2}\int \mathrm{d}^d x\, \sqrt{-\hat{g}}\, \left(r\partial_r \varphi\right)^2 = \\ &= -\frac{d-\Delta}{2}\int \mathrm{d}^d x\, \left((d-\Delta)\, r^{d-2\Delta}\varphi_0 + 2\Delta \widetilde{\varphi}_0\right)\varphi_0 \,. \end{aligned} \tag{2.26}$$

We point out that this counterterm is made out of the square of the conjugate momentum of φ in the holographic coordinate, and it is actually related to counterterm (2.17), which is a square in φ, by a Legendre transform[7], namely

$$\int \mathrm{d}^d x\, \varphi \Pi_\varphi - (d-\Delta)\, S^{(0)}_{\rm ct} = \frac{1}{d-\Delta}\, \widetilde{S}^{(0)}_{\rm ct} \,, \tag{2.27}$$

where $\Pi_\varphi \equiv r\partial_r \varphi$. This expression actually furnishes the suitable numerical factor for (2.26) to cancel the leading divergence in the regularized action (2.16), yielding

$$\widetilde{S}_{\rm ren} = S_{\rm reg} + \frac{1}{d-\Delta}\, \widetilde{S}^{(0)}_{\rm ct} = -\left(\Delta - \frac{d}{2}\right)\int \mathrm{d}^d x\, \varphi_0 \widetilde{\varphi}_0 = \left(\widetilde{\Delta} - \frac{d}{2}\right)\int \mathrm{d}^d x\, \varphi_0 \widetilde{\varphi}_0 \,. \tag{2.28}$$

So, the renormalized action for a real scalar in alternative quantization is identical to the one in standard quantization (2.22), except for the opposite sign, which again corresponds to replacing Δ by $\widetilde{\Delta} = d - \Delta$.

It is straightforward now to check the variational principle on this last renormalized action. From the variation of the bulk action (2.23) and the variation of counterterm (2.26), we obtain

$$\delta \widetilde{S}_{\rm ren} = \delta S + \frac{1}{d-\Delta}\, \delta \widetilde{S}^{(0)}_{\rm ct} = (d-2\Delta)\int \mathrm{d}^d x\, \varphi_0\, \delta \widetilde{\varphi}_0 \,, \tag{2.29}$$

which indeed requires to set $\delta\widetilde{\varphi}_0$ to zero, so that now $\widetilde{\varphi}_0$ is the source.

We are now ready to use the holographic principle (2.3) to explicitly compute two-point

[7] The fact that the holographic generating functional in alternative quantization is related to the holographic generating functional in standard quantization by a Legendre transform was first suggested in [70].

correlation functions for the dual operator $\mathscr{O}_\varphi$ of dimension Δ (in standard quantization), and for $\widetilde{\mathscr{O}}_\varphi$ of dimension $\widetilde{\Delta}$ (in alternative quantization).

2.1.1 Two-point functions from the renormalized action

The two-point correlation function is by definition a *non-local* function. Let us define, for conciseness,

$$\left\langle \mathscr{O}_\varphi(x)\mathscr{O}_\varphi(x')\right\rangle_0 \equiv -i f_\varphi(x-x') , \tag{2.30}$$

and then express the generating functional (2.4) at quadratic order in the sources,

$$\mathscr{W}_{\text{QFT}}[\varphi_0] = \frac{i}{2}\int \mathrm{d}y\mathrm{d}y'\, \varphi_0(y) f_\varphi(y-y')\varphi_0(y') ,$$

so that the formula (2.4) on this last expression correctly reproduces (2.30). This suggests that, in the standardly quantized renormalized action (2.22), the response $\widetilde{\varphi}_0$ should be related to the source by a non-local function, namely

$$(2\Delta - d)\, \widetilde{\varphi}_0(x) = f_\varphi(x-x')\varphi_0(x') . \tag{2.31}$$

This fact is achieved by imposing 'boundary'[8] conditions in the bulk. Indeed, with the renormalization procedure we have fixed boundary conditions on the boundary of AdS. In order to have a unique solution to the second-order equation (2.9), we need to fix another boundary condition, on the opposite side, that is in the deep bulk $r = \infty$. Precisely this latter will relate the source and the response of the near-boundary expansion, as we will see explicitly in a while for the example presented here.

Before, let us apply the formula (2.5) to the renormalized action in standard quantization (2.22), and use (2.31) to obtain

$$\left\langle \mathscr{O}_\varphi(x)\mathscr{O}_\varphi(x')\right\rangle_0 = -i\, \frac{\delta^2 S_{\text{ren}}}{\delta\varphi_0\delta\varphi_0}\bigg|_{\varphi_0=0} = -i\,(2\Delta-d)\,\frac{\delta\widetilde{\varphi}_0(x')}{\delta\varphi_0(x)} = -i f_\varphi(x-x') , \tag{2.32}$$

again consistently with our renaming (2.30).

If we are in alternative quantization, from (2.28) we get

$$\left\langle \widetilde{\mathscr{O}}_\varphi(x)\widetilde{\mathscr{O}}_\varphi(x')\right\rangle_0 = -i\, \frac{\delta^2 \widetilde{S}_{\text{ren}}}{\delta\widetilde{\varphi}_0\delta\widetilde{\varphi}_0}\bigg|_{\widetilde{\varphi}_0=0} = +i\,(2\Delta-d)\,\frac{\delta\varphi_0(x')}{\delta\widetilde{\varphi}_0(x)} = -i \widetilde{f}_\varphi(x-x') , \tag{2.33}$$

where we have defined

$$-\,(2\Delta - d)\,\varphi_0(x) = \widetilde{f}_\varphi(x-x')\widetilde{\varphi}_0(x') . \tag{2.34}$$

[8] We apologize for the unfortunate recurrence of the word boundary, in this case to design the conditions at the extremes of a boundary value differential problem, so that we have *boundary* boundary conditions and *bulk* boundary conditions

Now we have to find a full solution to the equation of motion, in order to explicitly determine the non-local functions $f_\varphi, \widetilde{f}_\varphi$. Consider the equation of motion (2.9) in momentum space,

$$r^{d+1}\partial_r\left(r^{-d+1}\partial_r\varphi_k\right) - k^2\varphi_k + \Delta(d-\Delta)\varphi_k = 0\ , \tag{2.35}$$

where φ_k is the boundary Fourier transform of the field, defined by

$$\varphi(r,x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{ikx}\,\varphi_k(r)\ . \tag{2.36}$$

Rescaling $\varphi_k(r) \equiv r^{\frac{d}{2}}\,\bar{\varphi}(k,r)$, and then performing the change of variable $\rho = kr$,[9] this equation reduces to the modified Bessel equation

$$\rho^2\,\bar{\varphi}''(\rho) + \rho\,\bar{\varphi}'(\rho) - \left(\rho^2 + \left(\Delta - \tfrac{d}{2}\right)^2\right)\bar{\varphi}(\rho) = 0\ , \tag{2.37}$$

whose solution is given by

$$\bar{\varphi}(\rho) = C_1\,\mathsf{I}[\nu;\rho] + C_2\,\mathsf{K}[\nu;\rho]\ , \tag{2.38}$$

where we have defined $\nu = \Delta - \frac{d}{2}$.

As announced at the beginning of this section, we have to impose boundary conditions in the interior in order to get rid of one integration constant, the other one being solved by boundary conditions at the boundary. As you can check in your favorite reference on special functions, the asymptotic behavior of the modified Bessel function of the first kind $\mathsf{I}[\nu;\rho]$ for $\rho \to \infty$ is $\mathsf{I} \sim \rho^{-\frac{1}{2}}e^{\rho} + e^{-\rho}$, while the asymptotic behavior of the modified Bessel function of the second kind $\mathsf{K}[\nu;\rho]$ is $\mathsf{K} \sim \rho^{-\frac{1}{2}}e^{-\rho}$. We prefer our solution not to explode in the deep bulk, and rather be small, if we want bulk gravitational effects be negligible. If we assume ρ real and positive (that means, as priorly discussed in footnote 9, k real and positive), the function K vanishes at $r = \infty$.

So, we choose $C_1 = 0$, and we finally obtain

$$\varphi_k = C_2\,r^{\frac{d}{2}}\,\mathsf{K}[\Delta - \tfrac{d}{2};\,kr]\ . \tag{2.39}$$

For $r \to 0$, this solution correctly presents

$$\varphi_k = r^{d-\Delta}\,\frac{C_2}{2}\,\Gamma\left[\Delta - \tfrac{d}{2}\right]\left(\frac{k}{2}\right)^{\frac{d}{2}-\Delta} + \ldots + r^{\Delta}\,\frac{C_2}{2}\,\Gamma\left[\tfrac{d}{2} - \Delta\right]\left(\frac{k}{2}\right)^{\Delta-\frac{d}{2}} + \ldots\ ,$$

[9] Here by k we rather mean $\sqrt{k^2}$, which will assume to be real when we impose the bulk boundary condition here after. This corresponds to $k^2 > 0$, rather than the other way round, as we would expect in our mostly-plus signature. However, the spectrum of the final correlator will be on the right side, *i.e.* for $k^2 < 0$. Actually it is a quite general feature of holography that the domain imposed by bulk regularity requirements and the region where the spectrum is located be complementary.

which, by comparison with the boundary expansion (2.10), allows to determine

$$\varphi_0 = \frac{C_2}{2}\,\Gamma\Big[\Delta - \tfrac{d}{2}\Big]\left(\frac{k}{2}\right)^{\frac{d}{2}-\Delta}, \qquad \widetilde{\varphi}_0 = \frac{C_2}{2}\,\Gamma\Big[\tfrac{d}{2} - \Delta\Big]\left(\frac{k}{2}\right)^{\Delta-\frac{d}{2}}, \tag{2.40}$$

where, with a little abuse of notation, we indicate the Fourier transforms in the same way as their counterparts in position space. The reader can notice that so far we have not specified anything about which is the source and which is the response. The solution of the equation of motion is one and unique, giving the asymptotic expansion (2.10). Everything about quantization, sources and responses is contained in the renormalization procedure, which selects either the prescription (2.32) or (2.33).

Since we are in momentum space, the relation (2.31) becomes an algebraic relation, $(2\Delta - d)\widetilde{\varphi}_0 = f_\varphi(k)\varphi_0$, so that in standard quantization, from (2.32), we have

$$\big\langle \mathscr{O}_\varphi(k)\mathscr{O}_\varphi(-k)\big\rangle_0 = -i\,f_\varphi(k) = -i\,(2\Delta-d)\,\frac{\widetilde{\varphi}_0}{\varphi_0} = -i\,(2\Delta-d)\,\frac{\Gamma\big[\frac{d}{2}-\Delta\big]}{\Gamma\big[\Delta-\frac{d}{2}\big]}\left(\frac{k}{2}\right)^{2\Delta-d}.$$

In order to obtain the correlator in position space, we have to Fourier transform. Using the formula

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d}\,k^n = \frac{2^n}{\pi^{\frac{d}{2}}}\,\frac{\Gamma\big[\frac{d+n}{2}\big]}{\Gamma\big[-\frac{n}{2}\big]}\,\frac{1}{|x|^{d+n}},$$

we obtain

$$\big\langle \mathscr{O}_\varphi(x)\mathscr{O}_\varphi(0)\big\rangle_0 = -i\,\frac{2\Delta-d}{\pi^{\frac{d}{2}}}\,\frac{\Gamma[\Delta]}{\Gamma\big[\Delta-\frac{d}{2}\big]}\,\frac{1}{|x|^{2\Delta}}, \tag{2.41}$$

which is precisely the expression in conformal field theory for the two-point correlator of a primary operator of scaling dimension Δ, thus confirming our holographic dictionary for Δ.

On the other hand, in alternative quantization we obtain from (2.33)

$$\begin{aligned}\big\langle \widetilde{\mathscr{O}}_\varphi(k)\widetilde{\mathscr{O}}_\varphi(-k)\big\rangle_0 &= i\,(2\Delta-d)\,\frac{\varphi_0}{\widetilde{\varphi}_0} = i\,(2\Delta-d)\,\frac{\Gamma\big[\Delta-\frac{d}{2}\big]}{\Gamma\big[\frac{d}{2}-\Delta\big]}\left(\frac{k}{2}\right)^{-2\Delta+d}\\ &\equiv -i\,\big(2\widetilde{\Delta}-d\big)\,\frac{\Gamma\big[\frac{d}{2}-\widetilde{\Delta}\big]}{\Gamma\big[\widetilde{\Delta}-\frac{d}{2}\big]}\left(\frac{k}{2}\right)^{2\widetilde{\Delta}-d},\end{aligned}$$

and Fourier transforming we get

$$\big\langle \mathscr{O}_\varphi(x)\mathscr{O}_\varphi(0)\big\rangle_0 = -i\,\frac{2\widetilde{\Delta}-d}{\pi^{\frac{d}{2}}}\,\frac{\Gamma[\widetilde{\Delta}]}{\Gamma\big[\widetilde{\Delta}-\frac{d}{2}\big]}\,\frac{1}{|x|^{2\widetilde{\Delta}}}. \tag{2.42}$$

So, the expressions of the two-point correlator in alternative quantization are identical, both in momentum and position space, to those in standard quantization, provided that we switch Δ with $\widetilde{\Delta}$. This latter indeed represents the scaling dimension of the dual operator in alternative quantization.

2.1.2 Holographic renormalization in presence of logarithms: scheme dependence and anomalies

Let us conclude this crash course on holographic renormalization by briefly considering a case where logarithms arise. Logarithmic terms are a bit more annoying from the technical point of view, but they can be treated in a standard way without any conceptual issue. For the equation of motion (2.9), logarithmic terms appear in the expansion (2.10) whenever $2\Delta - d$ is an even integer (zero included). Let us distinguish two different situations: when we are at the BF bound, and so $\Delta = d - \Delta = \frac{d}{2}$, and when we are away from it.

In the former situation, the logarithmic term is the leading, and it is the source in standard quantization, meaning with this that a mass-like counterterm (2.17) is employed to renormalize the boundary action; by means of the corresponding Legendre transformed counterterm, the source can be switched to the term without logarithm, but in both cases an operator of dimension $\frac{d}{2}$ is described. In the latter situation, instead, the source and response are not affected by logarithmic terms, which on the contrary appear in the sub-leading divergences.

In both situations, the renormalization works in the same way as described above, and counterterms have the same form, except that they may need to be dressed with logarithms where necessary, in order to remove logarithmic divergences. The only crucial consequence of logarithmic counterterms is that they introduce scheme dependent terms in the renormalized action (the coefficient of such terms can be modified by arbitrary finite counterterms).

Let us consider again our real scalar field in AdS, for the case $2\Delta - d = 2$ (other cases are not conceptually different), and so

$$\varphi = r^{\frac{d}{2}-1}\left(\varphi_0 + r^2 \ln r\, \varphi_1 + \ldots\right) + r^{\frac{d}{2}+1}\left(\widetilde{\varphi}_0 + \ldots\right) . \tag{2.43}$$

With this asymptotic expansion, the on-shell action (2.16) becomes

$$^{(\log)}S_{\text{reg}} = \frac{1}{2}\int_{r=\epsilon} \mathrm{d}^d x \,\left[r^{-2}\left(\frac{d}{2}-1\right)\varphi_0 + \ln r\; d\,\varphi_1 + d\,\widetilde{\varphi}_0 + \varphi_1\right]\varphi_0 \,. \tag{2.44}$$

We can notice that, besides the log-divergent term, an additional finite term is present. The equation of motion (2.9) relates φ_1 to the source, as in (2.19), which though now reads

$$\varphi_1 = -\frac{1}{2}\Box\varphi_0 \,. \tag{2.45}$$

Thus, a counterterm of the form (2.20), but dressed with an additional logarithmic divergent factor,

$$\frac{d}{2}\ln(r)\, S^{(1)}_{\text{ct}} = \frac{d}{4}\int \mathrm{d}^d x\, \left(\ln r\, \varphi_0 \Box\varphi_0\right) = -\frac{d}{2}\int \mathrm{d}^d x\; \ln r\, \varphi_0\varphi_1 \,, \tag{2.46}$$

is able to remove the logarithmic divergence in the the regularized action (2.44). However, we are still allowed to add (finite) counterterms of the form of $S^{(1)}_{\text{ct}}$ (2.20) (that is, without logarithmic factor). Such counterterms, which are related to the holographic *matter* conformal anomaly [28, 79, 80], would modify the coefficient of the finite term proportional to $\varphi_0\varphi_1$ in (2.44), which can be even removed: such term depends therefore on the chosen renormalization scheme.

On the other hand, there is no way to touch the term proportional to $\varphi_0\widetilde{\varphi}_0$ by means of finite counterterms: the coefficient of such term is fixed once and for all by the leading counterterm $S_{\rm ct}^{(0)}$ (2.17). Then, we can eventually state again that the final (scheme independent) form of the renormalized action is fully determined by the leading counterterm. We finally have

$$^{(\log)}S_{\rm ren} = \frac{1}{2}\int \mathrm{d}^d x\ \varphi_0\left(\widetilde{\varphi}_0 + \ln\Lambda\Box\varphi_0\right)\ , \tag{2.47}$$

where $\ln\Lambda$ is an arbitrary constant, which we can adapt by choosing a certain renormalization scheme, and whose logarithmic form will be justified in a second. Let us first notice that the expressions for the one-point function (2.25) gets shifted by the scheme-dependent piece of the renormalized action (2.47).

We analyze now the explicit solution, which, for the Fourier transform, can be read off from (2.39), replacing the current value for Δ:

$$\varphi_k = C_0\, r^{\frac{d}{2}}\ \mathsf{K}[1; kr]\ , \tag{2.48}$$

with C_0 a constant. The expansions near $r=0$ gives

$$\varphi_k \sim r^{\frac{d}{2}-1}\,\frac{C_0}{k} + r^{\frac{d}{2}+1}\ln r\ C_0\frac{k}{2} + r^{\frac{d}{2}+1}\ C_0\frac{k}{2}\left(\ln\frac{k}{2} - \frac{1}{2} + \gamma_{EM}\right)\ , \tag{2.49}$$

where γ_{EM} is the Euler-Mascheroni constant, and from where we read the following expressions for the coefficients:

$$\varphi_0 = \frac{C_0}{k}\ , \tag{2.50}$$

$$\widetilde{\varphi}_0 = C_0\frac{k}{2}\left(\ln\frac{k}{2} - \frac{1}{2} + \gamma_{EM}\right)\ , \tag{2.51}$$

$$\varphi_1 = C_0\frac{k}{2} \equiv \frac{1}{2}k^2\varphi_0\ , \tag{2.52}$$

where in the last line we have pointed out that the analytic expression for φ_1 correctly reproduces (2.45).

Finally using the formula (2.5) on the renormalized action (2.47), we obtain the two-point function

$$\left\langle \mathcal{O}_\varphi(k)\mathcal{O}_\varphi(-k)\right\rangle_0 = -i\,\frac{k^2}{4}\left(\ln\frac{k^2}{\Lambda^2} - 1 + 2\gamma_{EM} - \ln 4\right)\ . \tag{2.53}$$

We see here that the meaning of Λ: it is a scale, introduced by the conformal anomaly, which makes the argument of the logarithm dimensionless.

We conclude this chapter by admitting that here we have presented holographic renormalization in a very operational way. We do not pretend to be exhaustive about the subject, and we point out that a more formal and general approach exists, based on Hamiltonian formalism and renormalization group flows [81–86] (see [87] for a pedagogical review). However, the hope is that this introduction will be adequate to guide the reader through the various applications of holographic renormalization of Part II.

Part II

Symmetry breaking in holographic field theories

3 Symmetry breaking in relativistic holographic setups

In this chapter a minimal holographic model is presented, describing a quantum field theory where a continuous global symmetry is broken. We consider at the same time the case of an operator taking a vacuum expectation value, which breaks the symmetry spontaneously, and the case of a charged operator that breaks it explicitly.

By the procedure of holographic renormalization illustrated in Section 2.1, we retrieve the Ward identity (1.34), for concomitant spontaneous and explicit breaking. For some specific values of space-time dimensions and dimension of the dual scalar operator, we provide an explicit analytic solution for the two-point scalar correlator, exhibiting the pseudo-Goldstone pole, which correctly reproduce the GMOR relation (1.45).

In Section 3.2 we go down to the two-dimensional case, and investigate the fate of Coleman theorem in holography, where the large N limit allows for an evasion of the theorem, as discussed in Section 1.1.2. The derivation of the symmetry breaking Ward identity, however, will encompass some subtleties related to the holographic renormalization of a vector field in AdS_3.

3.1 Spontaneous and explicit breaking in a holographic-dual relativistic CFT

As introduced in Chapter 2, eq. (2.6), a global conserved current in the boundary theory is dual to a local gauge field in the bulk. It is then natural to add a scalar that couples to the gauge field in the bulk, in order to have a scalar operator that is charged under the global symmetry on the boundary. The most basic bulk action that we can write with these two fundamental ingredients[1] is then

$$S = \int d^{d+1}x \, \sqrt{-g} \left[-\frac{1}{4} F^{mn} F_{mn} - D_m \phi^* D^m \phi - m_\phi^2 \phi^* \phi \right] , \tag{3.1}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$, $D_m \phi = \partial_m \phi - i A_m \phi$, and g_{mn} is the AdS_{d+1} metric (2.8). For our quest of maximal simplicity, we choose an abelian U(1). Non-abelian symmetries will be treated in the non-relativistic example of Chapter 4. Here, they would add complications without increase in generality.

The AdS metric is chosen to be a fixed background here and gravity is kept non-dynamical, since we are not going to compute correlation functions which involve the stress-energy tensor. Moreover, we are allowed to neglect the back-reaction of the other fields on the metric because we will carry out a near-boundary analysis, and the back-reaction would emerge at higher order in the near-boundary expansions with respect to the computations that we will perform.

In the introductory section on holographic renormalization, we have pointed out that a vacuum expectation value for the boundary operator corresponds to a fixed profile for the response-mode (2.25). On the other hand, a fixed value for the source-mode would correspond to a deformation of the boundary QFT by an operator of dimension Δ (in standard quantization; $d - \Delta$ in alternative quantization). If the operator is charged under the symmetry, as in our case, due to the coupling to the gauge field, then such deformation breaks the symmetry explicitly [70, 90, 91]. So, giving a background profile to our complex scalar will allow to trigger spontaneous and explicit breaking of the boundary global symmetry.

Let us then consider the equation of motion for the complex scalar alone, which are coming from the variation of the action (3.1) with respect to ϕ, setting $A_m \equiv 0$. It reads

$$r^{d+1} \partial_r \left(r^{-d+1} \partial_r \phi \right) + r^2 \Box \phi - m_\phi^2 \phi = 0 , \tag{3.2}$$

which is the same equation as for the real scalar (2.9). Thus the asymptotic solution is

$$\phi = r^{\frac{d}{2} - \nu} \left(\phi_0 + \dots \right) + r^{\frac{d}{2} + \nu} \left(\widetilde{\phi}_0 + \dots \right) , \quad \text{with } \nu = \sqrt{\frac{d^2}{4} + m_\varphi^2} . \tag{3.3}$$

So, in order to arouse the desired spontaneous and explicit breaking on the boundary, we should give background values to the leading and the sub-leading, namely

$$\phi_B = m \, r^{\frac{d}{2} - \nu} + v \, r^{\frac{d}{2} + \nu} . \tag{3.4}$$

[1] For the reader familiar with the literature of the holographic superconductor, a disclaimer is here in order. The action (3.1) is identical to the one of the very first holographic superconductor [88, 89]. However, besides the fact that we will not switch on a background for the vector, since we do not want to break Lorentz invariance having a chemical potential, our purposes are completely different here. In the holographic superconductor the aim is to describe a system at finite temperature and to furnish a dynamical mechanism which generates a critical scale where an order parameter appears and a phase transition occurs. In our case, we want just to mimic an ordinary relativistic field theory at zero temperature, which enjoys a global symmetry, which in turns is broken (spontaneously and/or explicitly) by a scalar operator. So, in a sense, we are always in the broken phase.

In standard quantization, the leading piece (source) is triggering explicit breaking, whereas the sub-leading one (vev) is triggering spontaneous breaking. Of course, in alternative quantization source and vev are switched, and so are explicit and spontaneous breaking accordingly.

The Ward identity structure for symmetry breaking in the boundary field theory emerges through the precise holographic renormalization procedure, which therefore constitutes our first task. Let us stay in the window $0 < \nu < 1$, where no scheme dependence driven by logarithmic terms arise in the holographic renormalization procedure, and in addition it is the window where alternative quantization is possible (2.14). This restriction will not decrease the generality of our conclusions.

We then apply holographic renormalization to the action (3.1), for fluctuations above the background (3.4), and we divide also the fluctuations of the complex scalar into real and imaginary part,

$$\phi = \frac{1}{\sqrt{2}}\left(\phi_B + \rho + i\pi\right) , \tag{3.5}$$

where ϕ_B is assumed to be real for simplicity (and, as we have already remarked in Section 1.2, on page 22, also for consistency). The rescaling pre-factor $\sqrt{2}$ with respect to the generic shape of the scalar profile (3.4) is designed to match the field theory derivations of Section 1.2.

We partially fix the gauge freedom by setting ourselves in the radial gauge $A_r = 0$, leaving unbroken the boundary gauge invariance only. In such situation, the equation of motion for A_r gives a constraint on the other fields, namely

$$r^2\partial_r\partial_\mu A^\mu - (\phi_B + \rho)\partial_r\pi + \pi\partial_r(\phi_B + \rho) = 0 . \tag{3.6}$$

The equation of motion for A_μ, instead, reads

$$r^{d-1}\partial_r\left(r^{-d+3}\partial_r A_\mu\right) + r^2\left(\Box A_\mu - \partial_\mu\partial_\nu A^\nu\right) + \left(\phi_B + \rho\right)\partial_\mu\pi - \pi\partial_\mu\rho - (\phi_B + \rho)^2 A_\mu = 0 , \tag{3.7}$$

from which we can extract the asymptotic behavior

$$A^\mu = a_0^\mu + r^2 a_1^\mu \ldots + r^{d-2}\left(\tilde{a}_0^\mu + \ldots\right) . \tag{3.8}$$

We can notice that for $d = 2$ the leading and the sub-leading are of the same order (as for the scalar at the BF bound), and so a logarithmic term appears: this case will be precisely the subject of next section, Section 3.2, where we also discuss the issues related to Goldstone theorem in two-dimensions. For $d = 4$, a logarithm occurs as well, since a_1^μ and $\tilde{a}_0^\mu$ are of same order, but this is less problematic since the source a_0^μ is not affected[2]. However, in this section, we will set $d = 3$, so that we do not have to renormalize the vector, and we have no scheme dependence for either the vector and the scalar. Also, for this value of dimensions, we are able to provide an explicit analytic solution, in Section 3.1.1. Furthermore, we remark that for our chosen window of values for ν ($0 < \nu < 1$), the scalar background intervenes in the equation

[2] For an example of holographic renormalization of the gauge field in $d = 4$, we point the reader to [92], where it is treated in a similar context.

of motion for A_μ (3.7) at the order $d-2\nu > d-2$, thus not affecting the sub-leading of A_μ. Otherwise, we could not have avoided considering a background for the vector as well.

So, we stay at $d=3$, and in addition, since we are interested in two-point functions at most, we will consider the renormalized action up to quadratic order in the fluctuations. In such case, we can consider the linearized equations of motion for the fluctuated field over ϕ_B, which read

$$r^4\partial_r\left(r^{-2}\partial_r\rho\right)+r^2\left(\Box-m_\phi^2\right)\rho=0\,, \tag{3.9}$$

$$r^4\partial_r\left(r^{-2}\partial_r\pi\right)+r^2\left(\Box-m_\phi^2\right)\pi-r^2\phi_B\partial_\mu A^\mu=0\,, \tag{3.10}$$

$$r^2\partial_r^2A_\mu+r^2\left(\Box A_\mu-\partial_\mu\partial_\nu A^\nu\right)+\phi_B\partial_\mu\pi-\phi_B^2A_\mu=0\,, \tag{3.11}$$

$$r^2\partial_r\partial_\mu A^\mu-\phi_B\partial_r\pi+\pi\partial_r\phi_B=0\,. \tag{3.12}$$

Then, the on-shell action at quadratic order in the fluctuations reduces to the following boundary term

$$S_{\text{reg}}=\int_{r=\epsilon}\mathrm{d}^3x\,\left[\frac{1}{r^2}(\partial_r\phi_B)\rho+\frac{1}{2}A^\mu\partial_rA_\mu+\frac{1}{2r^2}\left(\rho\partial_r\rho+\pi\partial_r\pi\right)\right]\,. \tag{3.13}$$

At this point, we note that the quadratic terms are exactly the same that would arise in a configuration with vanishing backgrounds. The presence of a non-trivial background must then show up when expanding the fluctuations near the boundary as powers of r. There is however one more substitution that we can make, that makes the dependence on the background manifest even before expanding the fluctuations. We can indeed use the equation of motion coming from the variation with respect to A_r (3.12), which at linear order in the fluctuations rewrites

$$r^2\partial_r\Box L-\phi_B\partial_r\pi+\partial_r\phi_B\pi=0\,, \tag{3.14}$$

where we have also introduced the splitting of the gauge field in its irreducible components,

$$A_\mu=T_\mu+\partial_\mu L\,,\qquad\text{with }\;\partial_\mu T^\mu=0\,. \tag{3.15}$$

Noting that the second term of (3.13) has a longitudinal part that can be rewritten, after integration by parts, as $\frac{1}{2}L\partial_r\Box L$, then the regularized action for the longitudinal part and the scalars becomes

$$S_{\text{reg}}=\frac{1}{2}\int\mathrm{d}^3x\,\left[T^\mu\partial_rT_\mu\;+\frac{2}{r^2}\rho\partial_r\phi_B+\frac{1}{r^2}L\left(\pi\partial_r\phi_B-\phi_B\partial_r\pi\right)+\frac{1}{r^2}\left(\rho\partial_r\rho+\pi\partial_r\pi\right)\right]\,. \tag{3.16}$$

Using the splitting (3.15), we now rewrite the equation of motion for the imaginary part of the scalar (3.10),

$$r^4\partial_r\left(r^{-2}\partial_r\pi\right)+r^2\left(\Box-m_\phi^2\right)\pi-r^2\phi_B\Box L=0\,, \tag{3.17}$$

and the equation of motion for the vector field (3.11),

$$r^2\partial_r^2 L + \phi_B \pi - \phi_B^2 L = 0\,, \tag{3.18}$$

$$r^2\partial_r^2 T_\mu + r^2 \Box T_\mu - \phi_B^2 T_\mu = 0\,, \tag{3.19}$$

where we see that the equation for the transverse part of the vector decouples, exactly as for the real part of the scalar (3.9). Then, from the equations of motion we can derive the following asymptotic expansions for the fluctuated fields:

$$\begin{aligned} \rho &= r^{\frac{3}{2}-\nu}\rho_0 + r^{\frac{3}{2}+\nu}\widetilde{\rho}_0 + \dots\,, \qquad & T^\mu &= t_0^\mu + r\,\widetilde{t}_0^\mu + \dots\,, \\ \pi &= r^{\frac{3}{2}-\nu}\pi_0 + r^{\frac{3}{2}+\nu}\widetilde{\pi}_0 + \dots\,, \qquad & L &= l_0 + r\,\widetilde{l}_0 + \dots\,. \end{aligned} \tag{3.20}$$

The regularized action (3.16) then becomes

$$\begin{aligned} S_{\text{reg}} = \frac{1}{2}\int d^3x\, \Big[& t_0\cdot\widetilde{t}_0 + (3-2\nu)\, m\big(r^{-2\nu}\rho_0 + \widetilde{\rho}_0\big) + (3+2\nu)\, v\rho_0 + \\ & + \Big(\frac{3}{2}-\nu\Big) r^{-2\nu}\left(\rho_0^2 + \pi_0^2\right) + 3\left(\rho_0\widetilde{\rho}_0 + \pi_0\widetilde{\pi}_0\right) + 2\nu\, l_0\big(v\pi_0 - m\widetilde{\pi}_0\big)\Big]\,. \end{aligned} \tag{3.21}$$

The counterterm needed to cancel the divergences in standard quantization is analogous to the one for the real scalar (2.17), and reads

$$S_{\text{ct}} = -\Big(\frac{3}{2}-\nu\Big)\int d^3x\, \sqrt{-\hat{g}}\,\phi^*\phi = -\frac{1}{2}\Big(\frac{3}{2}-\nu\Big)\int d^3x\, r^{-3}\left(2\phi_B\rho + \rho^2 + \pi^2\right)\,. \tag{3.22}$$

Note that we neglect the constant term, as it would only be relevant with dynamical gravity. After adding the counterterm (3.22) to the regularized action (3.21), we obtain the renormalized action

$$S_{\text{ren}} = 2\nu\int d^3x\, \Big[v\rho_0 + \frac{1}{2}\widetilde{\rho}_0\rho_0 + \frac{1}{2}\widetilde{\pi}_0\big(\pi_0 - m l_0\big) + \frac{1}{2} v l_0\pi_0 \Big]\,, \tag{3.23}$$

where we have dropped the term for the transverse part of the vector, since it is completely decoupled from the scalar sector.

We can then notice that there are two kinds of terms in the quadratic renormalized action: those which are bilinears of a source and a response of the fluctuations, and those which involve only sources. The latter are all proportional to the non-trivial scalar background that we have introduced, thus they would not be there for trivial profile $v = 0 = m$. In practice, the background profiles make the scalar and vector sectors 'talk' to each others.

However, terms of the second kind are also hidden into terms of the first kind, because of gauge invariance. Indeed, gauge transformations that preserve our gauge choice $A_r = 0$ yield

$$\delta L = \alpha, \qquad \delta\phi = i\alpha\phi\,. \tag{3.24}$$

The first transformation above tells that α should be considered of the same order as the fluctuations L and ρ, π. It then implies that the gauge variations of ρ, π have actually terms of first and second order

$$\delta\rho = -\alpha\pi\ , \qquad \delta\pi = \alpha\phi_B + \alpha\rho\ . \tag{3.25}$$

On the coefficients of the near-boundary expansions (3.20), the transformations read

$$\begin{aligned} \delta l_0 &= \alpha\ , \qquad \delta\rho_0 = -\alpha\pi_0\ , \qquad \delta\pi_0 = \alpha m + \alpha\rho_0\ , \\ \delta\widetilde{l}_0 &= 0\ , \qquad \delta\widetilde{\rho}_0 = -\alpha\widetilde{\pi}_0\ , \qquad \delta\widetilde{\pi}_0 = \alpha v + \alpha\widetilde{\rho}_0\ . \end{aligned} \tag{3.26}$$

With the transformations given above, one can check that all the actions $S_{\rm reg}$, $S_{\rm ct}$, and $S_{\rm ren}$ are gauge invariant. We note that gauge invariance requires the cancellation between the variations of the linear and quadratic parts of the actions, and we have of course neglected orders higher than quadratic[3] (*i.e.* in the variations of the quadratic part of the action, only the terms of first order in the variations of ρ,π are considered).

We want to the derive the holographic correlators, assuming that the terms coupling the sources to the operators are

$$\int d^3x\ \left(\rho_0 {\rm Re}\mathcal{O}_\phi + \pi_0 {\rm Im}\mathcal{O}_\phi - l_0\partial_\mu J^\mu\right)\ , \tag{3.27}$$

where the last term comes from integration by parts. We also assume the usual holographic prescription in its Wick-rotated, Lorentzian version (2.5). In this way, from the renormalized action (3.23) we immediately have that ${\rm Re}\mathcal{O}_\phi$ has a non-zero vev, namely

$$\langle {\rm Re}\mathcal{O}_\phi\rangle_0 = \frac{\delta i S_{\rm ren}}{\delta i\rho_0} = v\ . \tag{3.28}$$

From (3.23), it is also manifest that the ρ-sector decouples from the L,π-sector. We will focus on this latter, since it is the sector where we expect the (pseudo) Goldstone boson to appear.

In order to solve for $\widetilde{\pi}_0$ in terms of the sources π_0 and l_0, one should assure that the deep bulk (IR) boundary conditions preserve gauge invariance, hence they have to be imposed on gauge invariant combinations. At linear order, the gauge invariant combinations are $\pi_0 - ml_0$ and $\widetilde{\pi}_0 - vl_0$. As a consequence, one can express the sub-leading mode of π in terms of the sources as

$$\widetilde{\pi}_0 = vl_0 + f(\Box)(\pi_0 - ml_0)\ . \tag{3.29}$$

The renormalized action for this sector can be rewritten accordingly

$$S_{\rm ren} = -\int d^3x\ \left[-\frac{1}{2}(\pi_0 - ml_0)f(\Box)(\pi_0 - ml_0) - vl_0\pi_0 + \frac{1}{2}mvl_0l_0\right]\ . \tag{3.30}$$

We observe that we have a term that is linear in m, which encodes the operator identities that

[3]It is possible also to parametrize the complex scalar in terms of its modulus and phase as in [92]; the latter parametrization, being well-adapted to gauge transformations (which consist in shifts of the phase), features manifest gauge invariance without mixing among different orders in the fluctuations. However, this brings disadvantages in the renormalization procedure: indeed, given that the phase has to be non-dimensional, the would-be Goldstone boson mixes non-trivial with the scalar background ϕ_B.

are present when the symmetry is explicitly broken. Then we have a term linear in v, which embodies the Ward identities when the symmetry is spontaneously broken. Eventually we have a term which is linear both in m and v and is necessary in order to recover the proper Ward identities in the case of concomitant spontaneous and explicit breaking.

Indeed, again using the prescription (2.5) for deriving two-point functions, we obtain the following relations among correlators of the longitudinal sector:

$$\langle \text{Im}\mathcal{O}_\phi \text{Im}\mathcal{O}_\phi \rangle_0 = -\frac{\delta^2 i S_{\text{ren}}}{\delta\pi_0 \delta\pi_0} = -i f(\Box) , \tag{3.31}$$

$$\langle \partial_\mu J^\mu \text{Im}\mathcal{O}_\phi \rangle_0 = \frac{\delta^2 i S_{\text{ren}}}{\delta l_0 \delta\pi_0} = -i m f(\Box) + i v , \tag{3.32}$$

$$\langle \partial_\mu J^\mu \partial_\nu J^\nu \rangle_0 = -\frac{\delta^2 i S_{\text{ren}}}{\delta l_0 \delta l_0} = -i m^2 f(\Box) + i m v . \tag{3.33}$$

These are exactly the equations (1.37)–(1.39), obtained in Section 1.2 from QFT arguments, where they have been used to derive the GMOR relation. Now we proceed to compute holographically the non-trivial function $f(\Box)$, and show that it reproduces all the physics that one expects on general grounds.

3.1.1 Analytical study of the fluctuations and two-point correlators

In this section we study the bulk equations of motion for the fluctuations, in order to extract exact expressions for the correlators. We will be able to find an analytic solution for $m_\phi^2 = -2$ (and so $\nu = \frac{1}{2}$, $\Delta = 2$), and thus to verify explicitly that the non-local function defined in (3.29) satisfies the non-trivial conditions discussed in Section 1.2, in particular eq. (1.42).

We start back from the equations of motion for the fluctuations, and we consider the system of equations involving the longitudinal component of the vector and the imaginary part of the scalar, that is equations (3.14), (3.17), and (3.18). We rewrite them here for the special case $m_\phi^2 = -2$.

$$r^2 \partial_r^2 L + \phi_B \pi - \phi_B^2 L = 0 , \tag{3.34}$$

$$r^2 \partial_r \Box L - \phi_B \partial_r \pi + \partial_r \phi_B \pi = 0 , \tag{3.35}$$

$$r^4 \partial_r \left(r^{-2} \partial_r \pi \right) + r^2 \left(\Box - m_\phi^2 \right) \pi - r^2 \phi_B \Box L = 0 . \tag{3.36}$$

We can extract π from the first equation,

$$\pi = \phi_B L - \frac{r^2}{\phi_B} \partial_r^2 L . \tag{3.37}$$

Note that gauge transformations, at linear order, are given by $\delta L = \alpha$ and $\delta\pi = \phi_B \alpha$, where α

does not depend on r because of the gauge fixing condition $A_r = 0$. We then see that both $\partial_r L$ and $\pi - \phi_B L$ are gauge invariant quantities.

We then plug (3.37) into (3.35), and we obtain a second order differential equation for $\partial_r L \equiv L' \equiv M$ (as expected from gauge invariance), which reads

$$r^2 M'' + 2r\, M' - 2r^2 \frac{\phi_B'}{\phi_B} M' + r^2 \Box M - \phi_B^2 M = 0\,. \tag{3.38}$$

The system of three equations is therefore reduced to a single second order ODE.

For the current value of $\nu = \frac{1}{2}$, the scalar profile is $\phi_B = mr + \upsilon r^2$, so that the equation (3.38) yields, in Fourier transformed version,

$$M'' - \frac{2\upsilon}{m + \upsilon r} M' - k^2 M - (m + \upsilon r)^2 M = 0\,, \tag{3.39}$$

and, by the simple change of variable $y = r + \dfrac{m}{\upsilon}$, it reduces to

$$M'' - \frac{2}{y} M' - \left(k^2 + \upsilon^2 y^2\right) M = 0\,, \tag{3.40}$$

which can be recast as a general confluent hypergeometric equation. Its solutions are given in terms of the Tricomi's confluent hypergeometric function $\mathsf{U}[a,b;x]$ and the generalized Laguerre polynomial $\mathsf{L}[a,b;x]$:

$$M(y) = \exp\left[-\frac{\upsilon y^2}{2}\right]\left(C_1\, \mathsf{U}\Big[\frac{k^2-\upsilon}{4\upsilon}, -\frac{1}{2};\, \upsilon y^2\Big] + C_2\, \mathsf{L}\Big[\frac{\upsilon - k^2}{4\upsilon}, -\frac{3}{2};\, \upsilon y^2\Big]\right)\,. \tag{3.41}$$

In the deep bulk ($y \to \infty$), we have $e^{-\frac{\upsilon y^2}{2}}\mathsf{U} \sim e^{-\frac{\upsilon y^2}{2}}$ whereas $e^{-\frac{\upsilon y^2}{2}}\mathsf{L} \sim e^{+\frac{\upsilon y^2}{2}}$. Since $\partial_r L$ is gauge-invariant, we are allowed to impose IR boundary conditions on it, and we choose bulk normalizability of the solution setting $C_2 \equiv 0$. We thus obtain

$$M(y) = C_1\, e^{-\frac{\upsilon}{2} y^2}\, \mathsf{U}\Big[\frac{k^2-\upsilon}{4\upsilon}, -\frac{1}{2};\, \upsilon y^2\Big]\,. \tag{3.42}$$

Note that this solution has a very fast decrease towards the interior of the bulk, confirming that back-reaction would only affect mildly the correlators that we will extract from it.

In this way we have obtained an exact analytical solution for the derivative of the gauge field, but we still have to derive a solution for π, in order to compute the scalar correlator (3.31). If we consider the near-boundary expansion for the fluctuations

$$\pi = r\,\pi_0 + r^2\,\tilde{\pi}_0 + \dots\,, \tag{3.43}$$

$$L = l_0 + r\,\tilde{l}_0 + r^2\, l_1 + r^3 \tilde{l}_1 + \dots\,, \tag{3.44}$$

then we need to know the expressions for π_0 and $\tilde{\pi}_0$ in order to compute the scalar correlator.

Indeed, from (3.29) and (3.31), we see that

$$\langle \mathrm{Im}\mathcal{O}_\phi \mathrm{Im}\mathcal{O}_\phi \rangle_0 = -i\frac{\delta\tilde{\pi}_0}{\delta\pi_0} = -i f(k^2)\,. \tag{3.45}$$

In other words, the correlator is essentially extracted from (3.29), that we rewrite here as

$$\tilde{\pi}_0 - v\, l_0 = f(k^2)\left(\pi_0 - m l_0\right)\,. \tag{3.46}$$

Using equation (3.37), we can express the gauge invariant combinations appearing in eq. (3.46) in terms of L,

$$\pi - \phi_B L = -\frac{r^2}{\phi_B} L''(r)\,. \tag{3.47}$$

Order by order near the boundary, through the expansions (3.43)-(3.44), we obtain

$$\pi_0 - m l_0 = -\frac{1}{m} 2 l_1\,; \tag{3.48}$$

$$\tilde{\pi}_0 - v\, l_0 = m\tilde{l}_0 + \frac{v}{m^2} 2 l_1 - \frac{1}{m} 6\tilde{l}_1\,. \tag{3.49}$$

We can then realize that $\tilde{l}_0$, l_1 and $\tilde{l}_1$ can be associated to M, M' and M'' evaluated at $r=0$, or equivalently at $y = \frac{m}{v}$, as follows:

$$\begin{aligned} M\big(x/\sqrt{v}\big) &= L'(0) = \tilde{l}_0\,,\\ M'\big(x/\sqrt{v}\big) &= L''(0) = 2 l_1\,,\\ M''\big(x/\sqrt{v}\big) &= L'''(0) = 6\tilde{l}_1\,, \end{aligned} \qquad \text{with } x \equiv \frac{m}{\sqrt{v}}\,.$$

Thus we can establish the expression for f in terms of B and its derivatives,

$$f(k^2) = \frac{\tilde{\pi}_0 - v\, l_0}{\pi_0 - m l_0} = -\frac{v}{m} + \frac{M''\big(x/\sqrt{v}\big) - v x^2 M\big(x/\sqrt{v}\big)}{M'\big(x/\sqrt{v}\big)}\,. \tag{3.50}$$

We can then use (3.42) to express the correlator (3.45) in terms of Tricomi functions, finally obtaining

$$\langle \mathrm{Im}\mathcal{O}_\phi \mathrm{Im}\mathcal{O}_\phi \rangle_0 = i\,\frac{x(k^2-v)\Big(4v\,\mathrm{U}\Big[\frac{k^2+3v}{4v},\frac{1}{2};x^2\Big] + (k^2+3v)\,\mathrm{U}\Big[\frac{k^2+7v}{4v},\frac{3}{2};x^2\Big]\Big)}{2\sqrt{v}\Big(2v\,\mathrm{U}\Big[\frac{k^2-v}{4v},-\frac{1}{2};x^2\Big] + (k^2-v)\,\mathrm{U}\Big[\frac{k^2+3v}{4v},\frac{1}{2};x^2\Big]\Big)}\,. \tag{3.51}$$

Let us show now how this expression reproduces all the physical features required by the field theory analysis of Section 1.2. First of all, in the limit of zero momenta, $f(k^2)$ as given in (3.50) satisfies the relation (1.42), i.e. $f(0) = \frac{v}{m}$. This can be easily seen by using (3.40) in order to obtain

$$f(k^2) = \frac{v}{m} + k^2\frac{M\big(x/\sqrt{v}\big)}{M'\big(x/\sqrt{v}\big)}\,. \tag{3.52}$$

Moreover, we can graphically find the poles of the propagator by plotting the correlator

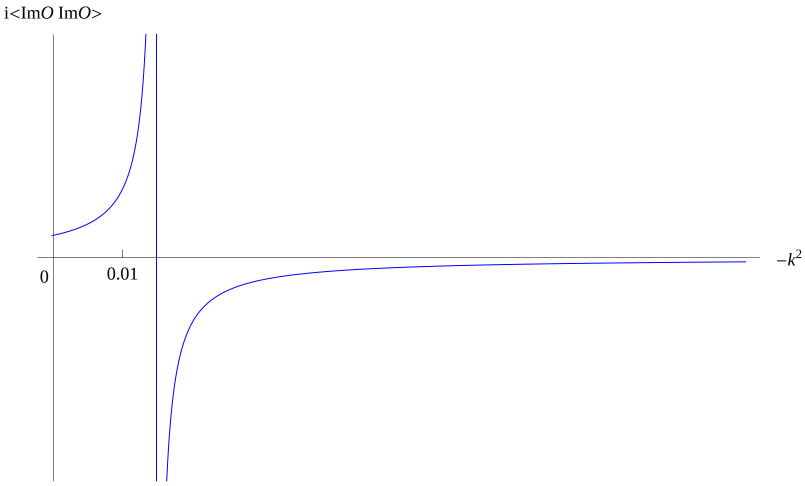

Figure 3.1. The lightest pole (PGB) in $\langle \text{Im}\mathcal{O}_\phi \text{Im}\mathcal{O}_\phi \rangle_0$, for $v = 1$ and $x = 0.01$, is displayed at a value of the order of x.

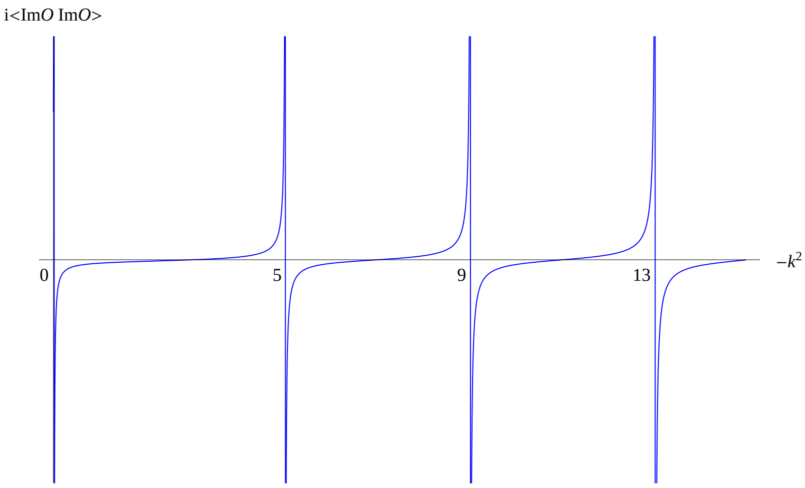

Figure 3.2. The first poles of the spectrum in $\langle \text{Im}\mathcal{O}_\phi \text{Im}\mathcal{O}_\phi \rangle_0$, for $v=1$ and $x=0.01$. They exhibit a gap of the order of $5v$ with respect to the lightest pole (PGB).

for specific values of the ratio $x = \frac{m}{\sqrt{v}}$. For instance, with $x = 0.01$, that is spontaneous breaking dominating on explicit breaking, we find a first pole close to zero (see fig. 3.1), and then a complete spectrum of higher massive poles with a gap considerably bigger than the mass of the first pole (see fig. 3.2). This is the hallmark of a pseudo-Goldstone boson. Furthermore, the gapped spectrum presents an interesting feature that we will show analytically for the purely spontaneous case in the next section: the poles are separated by a regular gap in squared mass (except for the first higher pole after the PGB, which exhibits a slightly bigger gap from the rest of the spectrum). This is reminiscent of linear confinement.[4]

Finally, we are able to find analytically the GMOR linear relation (1.45). Indeed, one finds that the numerator of expression (3.51) is just a constant in the limits $k^2 \to 0$ and $x \to 0$ (taken

[4] Indeed a phenomenological model like [93], that is designed in order to achieve linear confinement, also ends up having confluent hypergeometric equations for the bulk fluctuations.

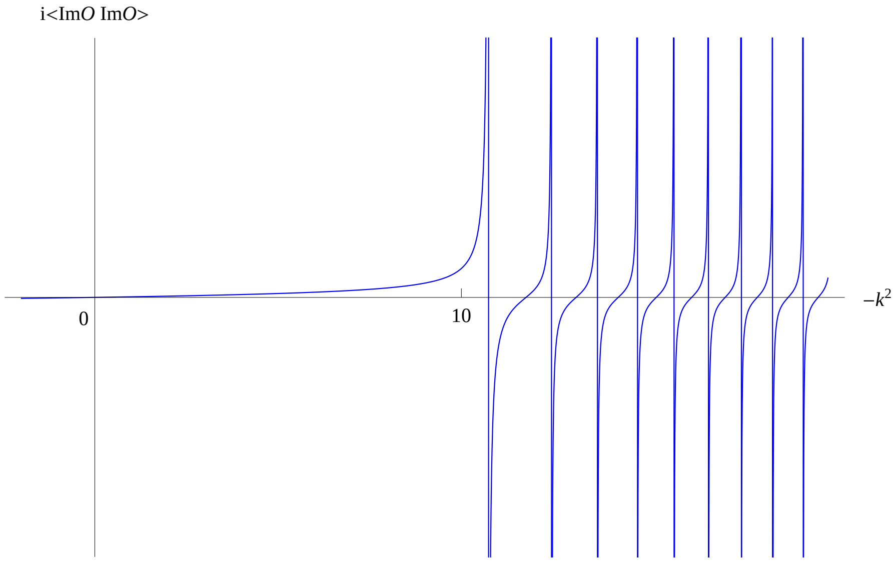

Figure 3.3. The low $|k^2|$ portion of the spectrum of $\langle \mathrm{Im}\mathcal{O}_\phi \mathrm{Im}\mathcal{O}_\phi \rangle_0$, for $v = 0.1$ and $x = 10$; the first pole is of the order of $m^2 = 100v = 10$.

in this precise order). If one then takes the denominator and expands it to the first order in x, and afterwards to the first order in k^2, it vanishes for

$$k^2 = -2\sqrt{v}\,\frac{\Gamma\left[\frac{5}{4}\right]}{\Gamma\left[\frac{3}{4}\right]}\,m\,, \tag{3.53}$$

where Γ is the Euler function. So we have found the explicit value of the residue μ appearing in (1.45) for the specific model at hand, namely

$$\mu = \frac{vM^2}{m} = 2\,v^{3/2}\,\frac{\Gamma\left[\frac{5}{4}\right]}{\Gamma\left[\frac{3}{4}\right]}\,. \tag{3.54}$$

It is also possible to find deviations from the linear GMOR behavior at the desired order in $\frac{m}{\sqrt{v}}$, as it is depicted in the appendix B of ref. [II].

Let us underscore that expression (3.50) is valid not only for small m, but actually for any m. We can then take $x \gg 1$ and find that, as expected, the first pole gets larger and larger with m, and it is pushed towards the rest of the spectrum, as it can be seen in fig. 3.3. Actually, the ratio between the first gap and the subsequent ones increases with x, so that if one keeps the first pole fixed, the following poles will be increasingly dense just after it. This is the signal that a cut is emerging in the $x \to \infty$ limit, i.e. in the purely explicit case.

In the next subsections we make further comments on the sub-cases of purely spontaneous and purely explicit breaking.

3.1.1.1 Purely spontaneous case

For purely spontaneous breaking, i.e. $m = 0$, the equation (3.39) becomes

$$M'' - \frac{2}{r} M' - k^2 M - v^2 r^2 M = 0 \, . \tag{3.55}$$

This is the same as (3.40), but directly in the r variable. Its solution is given in (3.42), where now B is a function of r

$$M(r) = C_1 \, e^{-\frac{v}{2} r^2} \, \mathrm{U}\Big[\frac{k^2 - v}{4v}, -\frac{1}{2}; v r^2\Big] \, . \tag{3.56}$$

In this case, we see from eq. (3.26) that, at linear order in the fluctuations, π_0 is gauge invariant by itself, so

$$f(k^2) = \frac{\widetilde{\pi}_0 - v l_0}{\pi_0} \, . \tag{3.57}$$

Then, using the equations of motion (3.34) for $\phi_B = v r^2$, we obtain

$$\langle \mathrm{Im}\mathscr{O}_\phi \mathrm{Im}\mathscr{O}_\phi \rangle_0 = -i \, f(k^2) = -i \, \frac{M'''(0)}{2M''(0)} = \frac{8 i \, v^{\frac{3}{2}}}{k^2} \, \frac{\Gamma\Big[\frac{k^2+5v}{4v}\Big]}{\Gamma\Big[\frac{k^2-v}{4v}\Big]} \, , \tag{3.58}$$

which, in the limit of small momenta, actually exhibits a massless pole, signature of the expected Goldstone boson,

$$\langle \mathrm{Im}\mathscr{O}_\phi \mathrm{Im}\mathscr{O}_\phi \rangle_0 \sim -2 i \, v^{\frac{3}{2}} \, \frac{\Gamma\big[\frac{5}{4}\big]}{\Gamma\big[\frac{3}{4}\big]} \, \frac{1}{k^2} \, .$$

As a double check, we can recover the same result of eq. (3.58) by taking the limit $m \to 0$ of expression (3.51). Furthermore, we are able to find explicitly the position of the gapped poles of the spectrum. Indeed, Gamma functions have no zeros, and they have poles at non-positive integer numbers. Therefore from (3.58) we infer the following spectrum

$$m_{n+1}^2 = (5 + 4n) \, v \, ,$$

with n being a non-negative integer. As anticipated in the previous section (see fig. 3.2), this spectrum presents the feature of equally gapped poles, except for the first massive one, whose gap from zero is bigger than the others by one unit in v.

3.1.1.2 Purely explicit case

For $v = 0$ the equation reduces to

$$M'' - \big(k^2 + m^2\big) \, M = 0 \, . \tag{3.59}$$

Note that the limit of vanishing scalar profile is trivially achieved putting $m = 0$ in the above equation, and in its solutions. The solutions are

$$B = C_+ e^{\sqrt{k^2+m^2}z} + C_- e^{-\sqrt{k^2+m^2}z} \,. \tag{3.60}$$

Bulk normalizability imposes $C_+ = 0$. The gauge invariant combination is

$$\pi - r\,mL = -\frac{r}{m} M' = C_- \frac{\sqrt{k^2+m^2}}{m} \left(r - r^2 \sqrt{k^2+m^2} + \dots \right) , \tag{3.61}$$

where we have used the constraint (3.37), with $\phi_B \equiv mr$. From this we read

$$\pi_0 - m l_0 = C_- \frac{\sqrt{k^2+m^2}}{m} \,, \qquad \tilde{\pi}_0 - m\tilde{l}_0 = -C_- \frac{k^2+m^2}{m} \,. \tag{3.62}$$

Now we can extract $\tilde{l}_0$ directly as the constant term of M, that is $\tilde{l}_0 = C_-$. This yields

$$\tilde{\pi}_0 = -C_- \frac{k^2}{m} = -\frac{k^2}{\sqrt{k^2+m^2}} \left(\pi_0 - m l_0 \right) \,. \tag{3.63}$$

Finally, the correlator is given by

$$\langle \mathrm{Im}\mathcal{O}_\phi \mathrm{Im}\mathcal{O}_\phi \rangle_0 = -i \frac{\delta \tilde{\pi}_0}{\delta \pi_0} = \frac{i\,k^2}{\sqrt{k^2+m^2}} \,. \tag{3.64}$$

It presents a cut starting after a gap given by m^2. This is what is expected from the $m/\sqrt{v} \to \infty$ limit of the correlator in the general case (3.51). Note that for $m = 0$ we obtain the conformal result $\langle \mathrm{Im}\mathcal{O}_\phi \mathrm{Im}\mathcal{O}_\phi \rangle_0 = i k$, with a cut as well, but without any gap.

It is important also to notice that (3.64) goes as k^2 for small k, which is necessary to ensure that the correlator $\langle J^\mu \mathrm{Im}\mathcal{O}_\phi \rangle_0 = -i m \frac{k^\mu}{k^2} \langle \mathrm{Im}\mathcal{O}_\phi \mathrm{Im}\mathcal{O}_\phi \rangle_0$ (1.41) does not have a spurious massless pole.

3.2 The $1+1$ dimensional case: Coleman fate in holography

In the previous section we have seen how the procedure of holographic renormalization brings to light the Ward identities structure of a holographic-dual field theory where a non-trivial scalar background profile triggers the breaking of a global symmetry. In this section, we want to consider the the special case of two boundary dimensions, where the arguments of Coleman theorem [32], as we have seen in Section 1.1.2, put some obstractions to spontaneous symmetry breaking.

However, as we will see, the procedure of holographic renormalization for a gauge vector field presents some subtleties in 2+1 bulk dimensions. Most of them are related to the fact that a vector in AdS_3 has analogous properties as a scalar at the BF bound [71]. We then start the study of the peculiarities of holographic renormalization in two dimensions with a preliminary discussion of a free gauge field in AdS_3, before coupling it to matter and analyzing the physics of symmetry breaking.

3.2.1 Maxwell gauge field in AdS_3

Let us consider the following bulk action for a free Abelian gauge field in AdS_3:[5]

$$S = \int \mathrm{d}^3x \, \sqrt{-g} \left(-\frac{1}{4} F^{mn} F_{mn} \right) , \tag{3.65}$$

where F_{mn} is the usual Maxwell field strength, and g_{mn} is the AdS_3 metric in the Poincaré patch (2.8). We choose the radial gauge $A_r = 0$, and we divide the remainder into transversal and longitudinal components,

$$A_\mu = T_\mu + \partial_\mu L \, , \quad \text{with } \partial_\mu T^\mu = 0 \, , \tag{3.66}$$

so that the action becomes

$$S = -\int \mathrm{d}^3x \; \frac{r}{2} \Big[-\partial_r L \Box \partial_r L + \partial_r T^\mu \partial_r T_\mu - T^\mu \Box T_\mu \Big] \, . \tag{3.67}$$

The variation of this action with respect to A_r, L, and T^μ respectively leads to the following equations of motion:

$$\Box \partial_r L = 0 \, , \tag{3.68}$$

$$r\partial_r (r\partial_r L) = 0 \, , \tag{3.69}$$

$$r\partial_r (r\partial_r T_\mu) + r^2 \Box T_\mu = 0 \, . \tag{3.70}$$

The first equation allows to drop from the action (3.67) the term involving the longitudinal component, while from the last two equations we derive the leading asymptotic behaviors of the fields,

$$L = \ln r \, \tilde{l}_0 + l_0 + \dots \, , \qquad T^\mu = \ln r \, \tilde{t}_0^\mu + t_0^\mu + \dots \, . \tag{3.71}$$

We notice the presence of logarithmic terms, as predicted in previous section. The constant terms, contrary to higher dimensions, are not the leading one here, and we expect logarithmic divergences in the on-shell action. Moreover, as discussed in Section 2.1.2, logarithms entail ambiguities for the constant terms in the renormalized action (scheme dependence).

[5] Being in three dimensions, one could include a Chern-Simons term for the vector (see for instance [94] for a careful discussion in a similar perspective). Since our aim is to stay as close as possible to the higher dimensional case of the previous section, we will take here the minimalistic approach and set it to zero. This choice is of course protected by parity.

In particular, in this case, given that the logarithm is one of the leading modes, the ambiguity will affect the definition of source and vev, analogously to what happens for scalars at the BF bound. Let us notice, in this respect, that we could reasonably introduce a scale in order to make the argument of the logarithm dimensionless. This could in turn be reabsorbed by a redefinition of the constant piece, giving us a hint of the fact that selecting the constant term as source will entail fixing a scale.

Let us put the action (3.67) on-shell, by integrating by parts and using the equation of motion for T_μ. The action reduces to the following boundary term:

$$S_{\text{reg}} = \frac{1}{2}\int_{r=\epsilon} \mathrm{d}^2x \;\, T^\mu r\partial_r T_\mu = \frac{1}{2}\int_{r=\epsilon} \mathrm{d}^2x \;\left(\ln r\, \tilde{t}_0 + t_0\right)\cdot \tilde{t}_0 \,, \tag{3.72}$$

which indeed displays a logarithmic divergence, which needs to be renormalized. If we want to write a counterterm that removes this divergence and is gauge invariant, we may build it out of the field strength, but we soon realize that we then have to make it non local. This turns out to be equivalent to a mass term, which is absolutely local, but gauge invariant only on-shell, by equation (3.68). Indeed,[6]

$$\begin{aligned} S^{(0)T}_{\text{ct}} &= -\frac{1}{4}\int \mathrm{d}^2x \;\frac{\sqrt{-\hat{g}}}{\ln r\, \hat{g}^{\rho\sigma}\partial_\rho\partial_\sigma}\; \hat{g}^{\kappa\mu}\hat{g}^{\lambda\nu}F_{\kappa\lambda}F_{\mu\nu} = \\ &= \frac{1}{2}\int \mathrm{d}^2x \;\frac{1}{\ln r}\; T_\mu T^\mu = \frac{1}{2}\int \mathrm{d}^2x \;\frac{\sqrt{-\hat{g}}}{\ln r}\; \hat{g}^{\mu\nu}A_\mu A_\nu = \\ &= \frac{1}{2}\int_{r=\epsilon} \mathrm{d}^2x \;\left(\ln r\, \tilde{t}_0 + 2t_0\right)\cdot \tilde{t}_0 \,, \end{aligned} \tag{3.73}$$

where $\hat{g}_{\mu\nu}$ is the induced metric on the two-dimensional boundary, and the identity in the second line holds indeed thanks to the constraint (3.68).

With such counterterm the renormalized action, $S_{\text{ren}} = S_{\text{reg}} - S_{\text{ct}}$, reads

$$S_{\text{ren}} = -\frac{1}{2}\int_{r=\epsilon} \mathrm{d}^2x \;\, \tilde{t}_0\cdot t_0 \,. \tag{3.74}$$

After eq.s (3.71) we have mentioned an ambiguity related to the constant term in the asymptotic expansion of the vector, which due to the presence of the logarithmic term is not the leading piece as in higher dimensions. So one may wonder if the constant term is still the source or not. The source can be established by checking the variational principle, as we have done in Section 2.1, given that the source is defined as the mode that has to be fixed in order to satisfy the variational principle.

We then take the variation of the bulk action (3.67) with respect to the fields and we put it on-shell, obtaining

$$\delta S_{\text{on-shell}} = \int_{r=\epsilon} \mathrm{d}^2x \;\left[\,\delta T^\mu\, r\partial_r T_\mu - \delta L\Box r\partial_r L\right] = \int_{r=\epsilon} \mathrm{d}^2x \;\, \tilde{t}_0\cdot\left(\ln r\, \delta\tilde{t}_0 + \delta t_0\right).$$

[6] A counterterm with a $1/\ln r$ pre-factor is typically needed for scalars at the BF bound, see e.g. [30].

We vary the counterterm (3.73) as well, and we eventually get the variation of the renormalized action,

$$\delta S_{\text{ren}} = \delta S_{\text{on-shell}} - \delta S_{\text{ct}} = -\int \mathrm{d}^2x \ \ t_0 \cdot \delta \tilde{t}_0 \,. \tag{3.75}$$

Hence we see that we have to fix $\tilde{t}_0^\mu$ in order to satisfy the variational principle, and so the source for the operator dual to A^μ is the coefficient of the logarithm.

We stress that the counterterm we have introduced for two boundary dimensions, which has the form of a mass term, does not have an equivalent in any higher dimensions. [7] We wonder if we can introduce a Legendre transformed version of such counterterm, as we have done for the scalar field in (2.27), in order to obtain an "alternative quantization" for the vector, and move the source to the constant term.

The answer is given by the following counterterm,

$$\begin{aligned} S^{(2)}_{\text{ct}} &= -\frac{1}{2}\int \mathrm{d}^2x \ \sqrt{-\hat{g}}\ r^2 \ln r\, \hat{g}^{\mu\nu} F_{r\mu} F_{r\nu} \\ &= -\frac{1}{2}\int_{r=\epsilon} \mathrm{d}^2x \ \ln r \left(\tilde{t}_0 \cdot \tilde{t}_0 - \tilde{l}_0 \Box \tilde{l}_0\right) = -\frac{1}{2}\int_{r=\epsilon} \mathrm{d}^2x \ \ln r\, \tilde{t}_0 \cdot \tilde{t}_0 \,, \end{aligned} \tag{3.76}$$

where in the last step we have used the equation of motion (3.68). We notice that, contrary to (3.73), this counterterm is manifestly gauge invariant and local, and it is in the form of a square of the conjugate momentum of A_μ. It removes the divergence in (3.67), and yields

$$S'_{\text{ren}} = S_{\text{reg}} + \tilde{S}_{\text{ct}} = \frac{1}{2}\int_{r=\epsilon} \mathrm{d}^2x \ \ \tilde{t}_0 \cdot t_0 \,, \tag{3.77}$$

which has opposite sign with respect to (3.74).

The variational principle for this last renormalized action gives

$$\delta S'_{\text{ren}} = \delta S_{\text{on-shell}} + \delta S^{(2)}_{\text{ct}} = \int \mathrm{d}^2x \ \ \tilde{t}_0 \cdot \delta t_0 \,.$$

As announced, the source in this alternative renormalization scheme is the constant term.

The case of a vector in AdS_3 was first treated in [77], where it is briefly discussed along with higher dimensions. In that work was shown that from the point of view of bulk normalizability only the boundary conditions corresponding to what we called standard quantization (logarithmic term as source) are admissible. However, in [96] different boundary conditions were considered, actually all intermediate mixed boundary conditions interpolating between standard and alternative quantizations, corresponding to double-trace deformations. The 'standard' boundary condition indicated by [77] was argued to yield a pure gauge field as dual boundary operator, whereas the 'mixed' boundary conditions chosen by [96] are meant to give a conserved current with a marginal double-trace deformation. The 'alternative' boundary

[7] This might be reminiscent of the Schwinger model (see e.g. [95] for a modern exposition), where the photon mass is generated by exactly the same non-local term. Note however that here we are dealing with a non-local counterterm, due to a non-local UV divergent term, while in two-dimensional QED the loop-generated mass of the photon is finite.

condition given by the counterterm (3.76) then would correspond to a conserved current with no double-trace deformation.

In both papers [77, 96], the dual formulation (special to three dimension) of the (transverse part of the) vector in terms of a massless scalar field was used, that is

$$\partial_l \vartheta = \sqrt{-g}\, g^{mr} g^{ns} \varepsilon_{lmn} \partial_r A_s\,, \tag{3.78}$$

and it was noted that boundary conditions for the vector corresponds to boundary conditions on the scalar. So, as it is straightforward to see, the usual 'mass' counterterm for as scalar in standard quantization (see eq. (2.17)) corresponds to the non-local counterterm (3.73), whereas the scalar counterterm in alternative quantization (see eq. (2.26)) corresponds to (3.76).

Anyway, from the point of view of holographic renormalization, we have no reason at this stage to prefer one boundary condition with respect to the other, even though the counterterm (3.73) has that non-local form, whereas the counterterm (3.76) has absolutely no issues. However, in the next section, we will see that coupling the vector to a charged scalar demands the choice of the alternative boundary condition, in order to describe a conserved current and retrieve the correct Ward identities.

3.2.2 Holographic renormalization with a charged scalar

We consider now the holographic model of Section 3.1 for $d = 2$. The action (3.1) then becomes

$$S = \int \mathrm{d}^3x\, \sqrt{-g}\, \left[-\frac{1}{4} F^{mn} F_{mn} - (D_m\phi)^* D^m \phi - m_\phi^2 \phi^* \phi \right] . \tag{3.79}$$

From the equation of motion for the free complex scalar in AdS_3, the exponents of the leading and sub-leading boundary modes in two dimensions are

$$\Delta_\pm = 1 \pm \nu\,, \quad \text{with } \nu = \sqrt{1 + m_\phi^2}\,. \tag{3.80}$$

Then, as in the previous section, we fluctuate the complex scalar around a fixed background,

$$\phi = \frac{\phi_B + \rho + i\pi}{\sqrt{2}}\,, \quad \text{with } \phi_B = m\, r^{1-\nu} + v\, r^{1+\nu}\,, \tag{3.81}$$

in order to induce symmetry breaking in the boundary field theory. We take m and v to be real for definiteness. As we know, in standard quantization, the sub-leading piece (proportional to v) triggers a vev for the real part of the dual boundary operator, and so leads to spontaneous symmetry breaking of the global U(1), whereas the leading piece (proportional to m) corresponds to explicit breaking of the symmetry. In alternative quantization their roles are inverted. For the moment we keep both of them different from zero.

Again, we fix the radial gauge $A_r = 0$ and we conveniently split the gauge field into transverse and longitudinal components as in (3.66). We then derive from the variation of the action the following linearized equations of motion for the fluctuated fields:

$$\Box r^2 \partial_r L - \left(\phi_B \partial_r \pi - \pi \partial_r \phi_B\right) = 0 \,, \tag{3.82}$$

$$r^2 \partial_r^2 T_\mu + r \partial_r T_\mu + r^2 \Box T_\mu - \phi_B^2 T_\mu = 0 \,, \tag{3.83}$$

$$r^2 \partial_r^2 L + r \partial_r L - \phi_B^2 L + \phi_B \pi = 0 \quad , \tag{3.84}$$

$$r^2 \partial_r^2 \rho - r \partial_r \rho - m^2 \rho + r^2 \Box \rho = 0 \quad , \tag{3.85}$$

$$r^2 \partial_r^2 \pi - r \partial_r \pi - m^2 \pi + r^2 \Box \pi - r^2 \phi_B \Box L = 0 \,. \tag{3.86}$$

As we have seen in the previous section, in three dimensions the vector field is at the BF bound, and indeed we have the following asymptotic expansions near the boundary:

$$T^\mu = \ln r \, \tilde{t}_0^\mu + t_0^\mu + \dots \,, \qquad L = \ln r \, \tilde{l}_0 + l_0 + \dots \,. \tag{3.87}$$

The asymptotic expansion of the two scalar components depends on the value of the bulk mass. As in the previous section, let us set ourselves in the window between the BF bound ($m^2 = -1$) and the “massless bound” ($m^2 = 0$), and exclude the two extremal values, which would need to be treated separately since they entail logarithms. For all values in this window, $0 < \nu < 1$, the scalar asymptotic expansions are logarithm-free. We thus have the following expansions,

$$\begin{aligned} \rho &= r^{1-\nu}\left(\rho_0 + r^2 \tilde{\rho}_0 + \dots\right) + r^{1+\nu}\left(\tilde{\rho}_0 + r^2 \tilde{\rho}_1 + \dots\right) \,, \\ \pi &= r^{1-\nu}\left(\pi_0 + r^2 \tilde{\pi}_0 + \dots\right) + r^{1+\nu}\left(\tilde{\pi}_0 + r^2 \tilde{\pi}_1 + \dots\right) \,. \end{aligned} \tag{3.88}$$

We can now, integrating by parts and using the equations of motion, reduce the action to a boundary term, which reads

$$S_{\text{reg}} = \int_{r=\epsilon} \mathrm{d}^2 x \; \frac{1}{2}\left[T^\mu r \partial_r T_\mu - \Box L r \partial_r L + \frac{1}{r}\left(\pi \partial_r \pi + \rho \partial_r \rho + 2\rho \partial_r \phi_B\right)\right] , \tag{3.89}$$

where we have neglected the terms at the zeroth order in the fluctuations[8]. By using the asymptotic expansions we obtain

$$\begin{aligned} S_{\text{reg}} = \int_{r=\epsilon} \mathrm{d}^2 x \, \frac{1}{2}\Big[&\left(\ln r \, \tilde{t}_0 + t_0\right) \cdot \tilde{t}_0 - \left(\ln r \, \tilde{l}_0 + l_0\right) \Box \tilde{l}_0 + \\ &+ \rho_0 \Big((1-\nu)\left(\rho_0 + 2m\right) r^{-2\nu} + 2\tilde{\rho}_0 \Big) + 2m(1-\nu)\tilde{\rho}_0 + 2\nu(1+\nu)\rho_0 + \\ &+ \pi_0 \Big((1-\nu)\pi_0 \, r^{-2\nu} + 2\tilde{\pi}_0 \Big) \Big] \,. \end{aligned} \tag{3.90}$$

[8] The zeroth order terms would be relevant if we where interested in the zero point energy, but not in our discussion.

We see that the divergent pieces of the scalar sector can be removed by the usual counterterm (in which we subtract the background value)

$$\begin{aligned} S_{\rm ct}^{(m)} &= (1-\nu)\int_{r=\epsilon} \mathrm{d}^2x\ \sqrt{-\hat{g}}\ \left(\phi^*\phi - \frac{\phi_B^2}{2}\right) \\ &= \frac{1}{2}(1-\nu)\int_{r=\epsilon} \mathrm{d}^2x\ \sqrt{-\hat{g}}\ \left[\rho^2 + 2\phi_B\rho + \pi^2\right] , \end{aligned} \tag{3.91}$$

leaving only the logarithmic divergences of the vector sector:

$$\begin{aligned} S_{\rm reg} - S_{\rm ct}^{(m)} = \frac{1}{2}\int_{r=\epsilon} \mathrm{d}^2x\ \Big[&\left(\ln r\, \tilde{t}_0 + t_0\right)\cdot \tilde{t}_0 - \left(\ln r\, \tilde{l}_0 + l_0\right)\Box \tilde{l}_0 + \\ &+ 2\nu\left(\rho_0\tilde{\rho}_0 + 2v\,\rho_0 + \pi_0\tilde{\pi}_0\right)\Big] . \end{aligned} \tag{3.92}$$

We would like to remove also these divergences, and then express the renormalized action in terms of the sources only. To do so, we need to identify which are the sources. We could be tempted to substitute the term $\Box L$ in (3.92), as we did in Section 3.1, by means of the constraint (3.82) to, which in $d = 2$ reads

$$\Box \tilde{l}_0 = 2\nu\left(m\tilde{\pi}_0 - v\,\pi_0\right) . \tag{3.93}$$

However, until we fix the source of the longitudinal component, we cannot use this constraint. Indeed, since we want the renormalized action to be eventually expressed in terms of the sources only, if $\tilde{l}_0$ is the source, then we should not remove it from the action. On the contrary, if the source is l_0, then we can use (3.93) with no worries.

So, we do not use the constraint for the moment, first of all we decide which of the two leading terms we want to be the source. In the previous section, for the transverse part, we have seen that we have the two choices (and even more, intermediate choices), one of the two (the logarithmic term as source) however leading to a pure gauge field, rather than a conserved current, as dual boundary operator. This fact is manifest by considering the longitudinal component, which here does not disappear from the on-shell action.

The longitudinal part of the vector shifts under gauge transformations, $\delta_\alpha L = \alpha$, which in the radial gauge $A_r = 0$ are constant in r. It is then the constant term in L that shifts, in two dimensions as in any higher boundary dimensions. In other words, it is the constant part of A_μ that has gauge transformations, and so should be the source of a boundary conserved current. On the contrary, the coefficient of the logarithm is gauge invariant, and so it would source an operator which enjoys gauge transformations: a gauge field.

Thus, if we want to describe a global conserved current, we should choose l_0 as the source. The discussion of the previous section about the vector alone in AdS$_3$ suggests that alternative quantization should be performed.

Let us then take the counterterm (3.76) and add it to (3.92)[9]. We obtain

$$\begin{aligned} S_{\text{ren}} &= S_{\text{reg}} - S_{\text{ct}}^{(m)} + S_{\text{ct}}^{(2)} = \\ &= \frac{1}{2}\int \mathrm{d}^2x \, \Big[t_0 \cdot \tilde{t}_0 - l_0 \Box \tilde{l}_0 + 2\nu \Big(\rho_0\tilde{\rho}_0 + 2v\,\rho_0 + \pi_0\tilde{\pi}_0\Big)\Big] \,. \end{aligned} \tag{3.94}$$

Let us check the variational principle, to verify that l_0 is indeed the source in this scheme. We then compute first the variation of the action (3.79), putting it on-shell:

$$\begin{aligned} \delta S_{\text{on-shell}} &= \int_{r=\epsilon} \mathrm{d}^2x \, \Big[\delta T^\mu r\partial_r T_\mu - \delta L \Box r \partial_r L + \frac{1}{r}\left(\delta\pi\partial_r\pi + \delta\rho\partial_r\rho + \delta\rho\partial_r\phi_B\right)\Big] \\ &= \int_{r=\epsilon} \mathrm{d}^2x \, \Big[\tilde{t}_0 \cdot \left(\ln r\,\delta\tilde{t}_0 + \delta t_0\right) - \Box\tilde{l}_0\left(\ln r\,\delta\tilde{l}_0 + \delta l_0\right) + \\ &\qquad + (1-\nu)\pi_0\left(r^{-2\nu}\delta\pi_0 + \delta\tilde{\pi}_0\right) + (1+\nu)\tilde{\pi}_0\delta\pi_0 + \\ &\qquad + (1-\nu)\left(\rho_0 + m\right)\left(r^{-2\nu}\delta\rho_0 + \delta\tilde{\rho}_0\right) + (1+\nu)\left(\tilde{\rho}_0 + v\right)\delta\rho_0 \Big] \,. \end{aligned} \tag{3.95}$$

As already remarked, it is crucial here not to use the constraint (3.93), which relates the coefficient of the logarithm to the source and vev of the fluctuating scalar π. The equations of motion can be used to express vevs in term of sources, but since we do not know yet whether $\tilde{l}_0$ will be a source or not, we have to remain off-shell to check the variational principle. To use this constraint to remove $\tilde{l}_0$ would actually mean to imply already that $\tilde{l}_0$ is not a source, that is precisely what we are trying to prove.

We go on varying the counterterm for the scalar divergences (3.91),

$$\begin{aligned} \delta S_{\text{ct}}^{(m)} = (1-\nu)\int_{r=\epsilon} \mathrm{d}^2x \, \Big[\Big(r^{-2\nu}\left(\rho_0 + m\right) + \left(\tilde{\rho}_0 + v\right)\Big)\delta\rho_0 + \left(\rho_0 + m\right)\delta\tilde{\rho}_0 + \\ + \Big(r^{-2\nu}\pi_0 + \tilde{\pi}_0\Big)\delta\pi_0 + \pi_0\delta\tilde{\pi}_0 \Big] \,, \end{aligned} \tag{3.96}$$

and the one for the vector divergences (3.76),

$$\delta S_{\text{ct}}^{(2)} = -\int_{r=\epsilon} \mathrm{d}^2x \, \ln r \left(\delta\tilde{t}_0 \cdot \tilde{t}_0 - \delta\tilde{l}_0 \Box \tilde{l}_0\right) \,. \tag{3.97}$$

We finally obtain the variation of the renormalized action (3.94),

$$\begin{aligned} \delta S_{\text{ren}} &= \delta S_{\text{on-shell}} - \delta S_{\text{ct}}^{(m)} + \delta S_{\text{ct}}^{(2)} = \\ &= \int \mathrm{d}^2x \left[\tilde{t}_0 \cdot \delta t_0 - \Box\tilde{l}_0\,\delta l_0 + 2\nu\Big(\left(\tilde{\rho}_0 + v\right)\delta\rho_0 + \tilde{\pi}_0\delta\pi_0\Big)\right] \,, \end{aligned} \tag{3.98}$$

where we see that the sources of the gauge field are the constant terms, as desired. Of course the scalar sources are those expected in standard quantization, which is customary with the standard counterterm (3.91).

Furthermore, now that we have established that $\tilde{l}_0$ is not the source, we can use the

[9]We point out that now the term in $\Box\tilde{l}_0$ in (3.76) does not vanish anymore on-shell.

constraint (3.93) in the renormalized action (3.94) without any issues, obtaining

$$S_{\text{ren}} = \frac{1}{2}\int d^2x \left[t_0 \cdot \widetilde{t}_0 + 2\nu \left(\rho_0 \widetilde{\rho}_0 + 2\nu\, \rho_0 + \pi_0 \widetilde{\pi}_0 - \left(m \widetilde{\pi}_0 - \nu \pi_0 \right) l_0 \right) \right], \tag{3.99}$$

This renormalized action is completely identical to those in higher space-time dimensions (see (3.23) for the three-dimensional one), and gives the suitable Ward identities for a pseudo-Goldstone boson [II].

The counterterm (3.76), however, has an explicit $\ln r$ factor, leaving the possibility of adding an additional *finite* counterterm with identical structure and arbitrary pre-factor. This is indeed what was analyzed in [96] in the dual frame (3.78), with the interpretation of a double-trace current-current deformation appearing, and consequent non-trivial RG flow. Here we note that such a finite counterterm would spoil the identification of t_0^μ and l_0 as sources, shifting them by an arbitrary amount linear in t_0^μ and $\widetilde{l}_0$ respectively. This corresponds to introducing a scale in the asymptotic expansion (3.71) in order to make the argument of the logarithm dimensionless. So, in other words, our choice of renormalization scheme (3.76) is equivalent to fixing a scale, which in turn corresponds to a specific choice of the parameter characterizing the boundary conditions of [96], the one related to the 'trivial' double-trace deformation (*i.e.* no deformation). In the following, we take the point of view that the ambiguity in the $\ln r$ has been fixed, and we have taken t_0^μ and l_0 to be our sources. This prescription allows us to find the expected Ward identities, which nevertheless would not be affected by double-trace deformations[10].

One last comment we should make about the choice of alternative quantization for the vector field is that, if one holds t_0^μ fixed and lets $\widetilde{t}_0^\mu$ loose, then according to [77] the fluctuations are not normalizable. This seems the price to pay to describe in the boundary theory a proper conserved current, whose existence we have no reason to exclude for a two-dimensional CFT. In addition, let us say that, at least for scalars, bulk non-normalizability is usually connected to boundary operators with dimension below the unitarity bound, whereas in the present case we do not see which problematic scenario this non-normalizability would correspond to in the dual theory. On the contrary, we will show that everything works as smoothly as in higher dimensions *precisely* when we choose the alternative quantization for the vector field.

Thus, before moving to the usual holographic derivation of Ward identities for symmetry breaking and giving some analytic evidence for holographic Goldstone bosons in two dimensions, let us first briefly discuss the 'standard quantization' for the vector, that is when we allow the coefficient of the logarithm to be the source, and show that in such case the holographic renormalization in presence of a scalar background presents some difficulties which cannot be overcome.

[10] On the contrary, it will not be possible to derive the correct Ward identities for symmetry breaking in the 'ordinary' quantization scheme (which as already said *does not* describe a conserved current), as we will see in Section 3.2.2.1.

3.2.2.1 Ordinary quantization for the vector

Let us go back to the boundary action (3.92), where the scalar divergences are removed whereas the vector divergences are still present, and try to find a renormalization scheme for the vector that select the coefficient of the logarithm as the source. A mass-like counterterm like the one that we have used for the transverse part (3.73) will not help in presence of a scalar background, since it does not take care of the divergence of the longitudinal component, which now does not vanish on-shell. We propose instead the following gauge invariant, local counterterm:

$$\begin{aligned} S_{\rm ct}^{(0)} &= -\int_{r=\epsilon} \mathrm{d}^2x \, \frac{\sqrt{-\hat{g}}\;\hat{g}^{\mu\nu}}{\ln r\;\phi_B^2}\,(D_\mu\phi)^* D_\nu\phi \\ &= -\frac{1}{2}\int_{r=\epsilon} \mathrm{d}^2x \,\left[\left(\ln r\,\tilde{t}_0 + 2t_0\right)\cdot\tilde{t}_0 - \left(\ln r\,\tilde{l}_0 + 2l_0 - \frac{2}{m}\pi_0\right)\Box\tilde{l}_0\right]. \end{aligned} \tag{3.100}$$

We stress that such counterterm cannot be written in case of vanishing scalar background, and it is anyway singular at $m=0$, that is the purely spontaneous case, already suggesting that we are not doing the proper way to describe a theory with symmetry breaking. However, it actually removes the logarithmic divergences, yielding

$$\begin{aligned} \hat{S}_{\rm ren} &= S_{\rm reg} - S_{\rm ct}^{(m)} + S_{\rm ct}^{(0)} = \\ &= \frac{1}{2}\int \mathrm{d}^2x\,\left[-t_0\cdot\tilde{t}_0 + l_0\Box\tilde{l}_0 + 2v\left(\rho_0\tilde{\rho}_0 + 2v\,\rho_0 + \pi_0\tilde{\pi}_0\right)\right]. \end{aligned} \tag{3.101}$$

Let us check by the variational principle that with this renormalization scheme the coefficients of the logarithmic terms are indeed the sources. Taking (3.95) and (3.96), and adding the variation of the counterterm (3.100), we obtain

$$\delta\hat{S}_{\rm ren} = \int \mathrm{d}^2x\left[-t_0\cdot\delta\tilde{t}_0 - \frac{\Box}{m}(\pi_0 - m l_0)\delta\tilde{l}_0 + \left(2v\tilde{\pi}_0 - \frac{\Box}{m}\tilde{l}_0\right)\delta\pi_0 + 2v\left(\tilde{\rho}_0 + v\right)\delta\rho_0\right].$$

We see that in this way the variational principle is well defined (even if still singular in $m\to 0$), and in particular $\tilde{l}_0$ should be considered as the source[11].

If $\tilde{l}_0$ is a source, then we have to re-interpret the constraint (3.93) as an expression for $\tilde{\pi}_0$ in terms of the sources $\tilde{l}_0$, π_0. With this substitution the renormalized action (3.101) becomes

$$\hat{S}_{\rm ren} = \frac{1}{2}\int \mathrm{d}^2x\,\left[-\tilde{t}_0\cdot t_0 - (\pi_0 - m l_0)\frac{\Box}{m}\tilde{l}_0 + 2v\left(\rho_0\tilde{\rho}_0 + 2v\rho_0 + \frac{v}{m}{\pi_0}^2\right)\right]. \tag{3.102}$$

Again we see that all the terms involving the source of the imaginary part of the dual scalar operator explode for $m=0$. No theory of spontaneous breaking can be extracted out of this, which is consistent with the fact that the response of the longitudinal component of the vector, l_0, is gauge-dependent in this quantization. Moreover, again in the purely spontaneous

[11] Notice that the mass parameter m acts here as a regulator in the case of the purely spontaneous case. The structure of counterterms looks different in the case one fixes $m=0$ ab initio.

case $m=0$, the constraint (3.93) becomes

$$\Box\widetilde{l}_0 = -2\nu\, v\pi_0\,, \tag{3.103}$$

which strengthens the idea that $\widetilde{l}_0$ cannot be the source of the conserved current. Indeed, given that π_0 is the source for the imaginary part of the scalar, $\widetilde{l}_0$ cannot be another source, giving another signal of the sickness of this renormalization scheme as we remove the regulator m.

So, the issues of the counterterm (3.100), and of the corresponding renormalized action (3.101), at $m=0$ corroborate the idea that the alternative quantization scheme is not only the proper one to describe a conserved current and symmetry breaking, but it is also the unique consistent way of holographically renormalizing the three-dimensional action (3.79) in presence of a non-trivial scalar background.

We may then wonder if the alternative quantization counterterm (3.76) can be obtained from the standard quantization counterterm (3.100) through a Legendre transform, as it is the case for a scalar (2.27) [70]. The answer is of course positive, and is given by the counterterm

$$\begin{aligned} S^{(1)}_{\rm ct} &= \frac{-i}{\sqrt{2}}\int_{r=\epsilon} \mathrm{d}^2x\ \frac{\sqrt{-\hat{g}}}{\phi_B}\ \hat{g}^{\mu\nu} r\partial_r A_\mu\Big(D_\nu\phi-(D_\nu\phi)^*\Big) = \\ &= -\int_{r=\epsilon}\mathrm{d}^2x\ \Big[\Big(\ln r\,\widetilde{t}_0+t_0\Big)\cdot\widetilde{t}_0-\Big(\ln r\,\widetilde{l}_0-\frac{1}{m}\big(\pi_0-ml_0\big)\Big)\Box\widetilde{l}_0\Big]\,, \end{aligned} \tag{3.104}$$

and it is straightforward to check that indeed $S^{(1)}_{\rm ct}-S^{(0)}_{\rm ct}=S^{(2)}_{\rm ct}$, and the same for the respective variations. However, we see that also this boundary term $S^{(1)}_{\rm ct}$ is singular for $m=0$ like $S^{(0)}_{\rm ct}$, whereas the Legendre transformed one $S^{(2)}_{\rm ct}$ is independent of m.

Note that this could be interpreted to mean that the limits $r\to0$ (high UV) and $m\to0$ (purely spontaneous breaking) do not commute at large N. In order to make explicit this non-commutativity of the two limits, we can write

$$S^{(1)}_{\rm ct}-S^{(0)}_{\rm ct}=S^{(2)}_{\rm ct}-\frac{1}{2}\int_{r=\epsilon}\mathrm{d}^2x\ \frac{1}{\ln r}\left[2\Box l_0\,\frac{\pi_0+\widetilde{\pi}_0 r^{2\nu}}{m+vr^{2\nu}}-\Box\pi_0\,\frac{\pi_0+2\widetilde{\pi}_0 r^{2\nu}}{m^2+2mvr^{2\nu}+v^2r^{4\nu}}\right].$$

We see immediately that if we take first the limit $r\to0$ we correctly get the counterterm $S^{(2)}_{\rm ct}$ (which is independent of m and so the subsequent $m\to0$ limit is ineffective); whereas, if we take first the limit $m\to0$, we have no singularities thanks to $v\neq0$, but we have surviving divergences in r when we take the $r\to0$ limit afterwards, namely:

$$\Big[S^{(1)}_{\rm ct}-S^{(0)}_{\rm ct}\Big]_{m=0}=S^{(2)}_{\rm ct}+\frac{1}{2}\int_{r=\epsilon}\mathrm{d}^2x\ \left[\frac{2}{v}\pi_0\Box l_0\frac{r^{-2\nu}}{\ln r}-\frac{1}{v^2}\Big(\pi_0\frac{r^{-4\nu}}{\ln r}+2\widetilde{\pi}_0\frac{r^{-2\nu}}{\ln r}\Big)\Box\pi_0\right].$$

One could be tempted to interpret this as a signal of the impossibility of taking $v\neq0$ and $m=0$ at the same time, i.e. no spontaneous symmetry breaking is allowed. However the counterterm (3.76) is perfectly legitimate in its own right, and it is the only one which can still be written also when the scalar background completely vanishes ($m=0$ *and* $v=0$), and actually even when the scalar is not there at all, as in Section 3.2.1. On the contrary, the standard quantization counterterm (3.100) and the one necessary to Legendre transform (3.104) cannot be written in absence of the scalar background.

The main understanding that we have gained from the analysis of this section is that the 'alternative quantization' counterterm (3.76) provides the only sensible way of renormalizing a vector field in AdS_3 in presence of a non-trivial scalar back-ground, in order to describe a conserved current as dual boundary operator and consistently retrieve Ward identities for symmetry breaking [III]. Let us then compute now those Ward identities embodying spontaneous (and explicit) symmetry breaking, in two dimensions exactly as in higher dimensions, and so confirming the announced holographic evasion of Coleman theorem.

3.2.3 Symmetry breaking and Goldstone boson in AdS_3/CFT_2

We start back from the renormalized action in alternative quantization (3.94), which we have extensively argued to be the proper one to describe a global conserved current and symmetry breaking in the dual boundary field theory. Then showing that the Ward identities are realized is straightforward, and proceeds exactly as in Section 3.1. First we rewrite the action as

$$S_{\text{ren}} = \frac{1}{2}\int_{r=\epsilon} \mathrm{d}^2x \left[t_0\cdot\tilde{t}_0 + 2\nu\Big(\rho_0\tilde{\rho}_0 + v\left(2\rho_0 + 2\pi_0 l_0 - m l_0 l_0\right) + \left(\pi_0 - m l_0\right)\left(\tilde{\pi}_0 - v l_0\right)\Big)\right] .$$

Then we remark that the equations of motions and gauge invariance dictate the relations between vevs and sources to take the following form:

$$\tilde{t}_0^\mu = f_t(\Box)\, t_0^\mu\,, \quad \tilde{\rho}_0 = f_\rho(\Box)\rho_0\,, \quad \tilde{\pi}_0 - v l_0 = f_\pi(\Box)\left(\pi_0 - m l_0\right), \tag{3.105}$$

where the f's are typically non-local functions, obtained by solving the equations of motion with appropriate IR boundary conditions in the deep bulk.

Replacing in the action yields the generating functional for one- and two-point functions, depending explicitly on sources only:

$$\begin{aligned} S_{\text{ren}} = \frac{1}{2}\int_{r=\epsilon} \mathrm{d}^2x \;\Big[\, & t_0\cdot f_t(\Box)\, t_0 + 2\nu\Big(v\left(2\rho_0 + 2\pi_0 l_0 - m\, l_0 l_0\right) + \\ & + \rho_0 f_\rho(\Box)\rho_0 + \left(\pi_0 - m l_0\right) f_\pi(\Box)\left(\pi_0 - m l_0\right)\Big)\Big] . \end{aligned} \tag{3.106}$$

Given the dictionary (2.5), we get, for the correlators that are most relevant to the Ward identities,

$$\begin{aligned} \left\langle \mathrm{Im}\mathcal{O}(x)\,\mathrm{Im}\mathcal{O}(x')\right\rangle_0 &= -i\,2\nu\, f_\pi(\Box)\,\delta(x-x')\,, \\ \left\langle \partial_\mu j^\mu(x)\,\mathrm{Im}\mathcal{O}(x')\right\rangle_0 &= -i\,2\nu\left(m f_\pi(\Box) - v\right)\delta(x-x')\,. \end{aligned} \tag{3.107}$$

As in Section 3.1, we can obtain directly the Goldstone boson pole in the purely spontaneous case from the above relations. In momentum space, relativistic invariance and the Ward identity force the mixed correlator to be

$$\left\langle j_\mu(k)\,\mathrm{Im}\mathscr{O}(-k)\right\rangle_0 = v\,\frac{k_\mu}{k^2}\,, \tag{3.108}$$

displaying the expected massless pole. Furthermore when turning on m, one can argue that f_π has to have a pole with a mass square proportional to m. Hence also f_π has a massless pole in the $m=0$ limit. We will not repeat here all these steps, because they are explained in detail in Section 3.1 and they are clearly independent of the dimension of space-time.

The Coleman theorem kicks in only after one considers (perturbative) quantum corrections due to the massless particle. Clearly holography does not capture such quantum corrections, which we then assume to be suppressed by the large N limit, as in the example of Thirring model discussed in Section 1.1.2.

Hence, the main result of [III] has been reproduced, namely to show what is the correct prescription for the boundary conditions and for the renormalization of the vector in order to obtain the expected Ward identities in the two-dimensional boundary field theory. This has been derived through a physically intuitive operational method, based on locality, gauge-invariance and the variational principle. The same conclusions (counterterms built out of canonical momenta are necessary to correctly renormalize a Maxwell gauge field in AdS_3) can be obtained in a more formal way through Hamiltonian formalism, as discussed in Section 3.2 of [97].

In the following section, in order to cover all possibilities (namely, all scalar operator dimensions between 0 and 2), we will briefly perform alternative quantization also in the scalar sector. Moreover, this will allow us to work out an analytic expression for f_π for a specific value of the dimension of the dual boundary operator.

3.2.3.1 Scalar correlator in alternative quantization

Here, as we did for the vector field, the goal is to move the sources to the subleading terms for the scalar as well. That is, we are interested in considering $\widetilde{\rho}_0, \widetilde{\pi}_0$ as the sources. As explained in Section 2.1, in order to change the boundary conditions we should consider a Legendre transformation of the scalar counterterm (3.91), that is

$$\begin{aligned}\widetilde{S}^{(m)}_{\rm ct} &= \int_{r=\epsilon} \mathrm{d}^2x\,\sqrt{-\hat{g}}\left(\phi^* r\partial_r\phi + \phi^* r\partial_r\phi - \phi_B r\partial_r\phi_B\right) \\ &= \int_{r=\epsilon} \mathrm{d}^2x\left[(1-v)\Big(\rho_0\big(\rho_0+2m\big)+\pi_0\pi_0\Big)r^{-2v} + 2\big(v\rho_0 + m\widetilde{\rho}_0 + \rho_0\widetilde{\rho}_0 + \pi_0\widetilde{\pi}_0\big)\right].\end{aligned} \tag{3.109}$$

Then the following combination is free from scalar divergences,

$$S_{\rm reg} + S^{(m)}_{\rm ct} - \widetilde{S}^{(m)}_{\rm ct} = \frac{1}{2}\int_{r=\epsilon}\mathrm{d}^2x\;\left[\big(\ln r\,\widetilde{t}_0 + t_0\big)\cdot\widetilde{t}_0 - \big(\ln r\,\widetilde{l}_0 + l_0\big)\Box\widetilde{l}_0 - 2v\big(\rho_0\widetilde{\rho}_0 + 2m\,\widetilde{\rho}_0 + \pi_0\widetilde{\pi}_0\big)\right],$$

and v and m have opposite meanings with respect to (3.92). We can verify by the variational principle that indeed the sources and vevs are switched. If we take the expression (3.95) and subtract the variation of the present countenterm, we obtain

$$\delta S_{\text{on-shell}} + \delta S^{(m)}_{\text{ct}} - \delta \widetilde{S}^{(m)}_{\text{ct}} = \int_{r=\epsilon} \mathrm{d}^2x \left[\widetilde{t}_0 \cdot \left(\ln r \delta \widetilde{t}_0 + \delta t_0\right) - \left(\ln r\, \delta \widetilde{l}_0 + \delta l_0\right) \Box \widetilde{l}_0 + \right.$$
$$\left. -2\nu \Big((\rho_0 + m)\delta\widetilde{\rho}_0 + \pi_0 \delta\widetilde{\pi}_0 \Big) \right],$$

as desired.

Then we use the counterterm (3.76) to remove the vector divergences as well, and we get the renormalized action where both the vector and the scalar are in the alternative quantization:

$$\widetilde{S}_{\text{ren}} = \frac{1}{2} \int \mathrm{d}^2x \left[t_0 \cdot \widetilde{t}_0 - 2\nu \Big(\rho_0 \widetilde{\rho}_0 + 2m\widetilde{\rho}_0 + \pi_0 \widetilde{\pi}_0 + \left(m\widetilde{\pi}_0 - v\pi_0\right) l_0 \Big) \right]. \tag{3.110}$$

We remark that, since now the purely spontaneous breaking occurs for $v = 0$, the two counterterms (3.100, 3.104) are now well behaved for the purely spontaneous case, whereas they are singular for the purely explicit one. Since we do not expect any obstruction for explicit symmetry breaking specific to two dimensions, this confirms once more that the ordinary quantization for the vector should be excluded.

If we now express the vevs in terms of the gauge-invariant sources in the following way:

$$\widetilde{t}^{\mu}_0 = f_t(\Box)\, t^{\mu}_0 , \quad \rho_0 = \widetilde{f}_\rho(\Box)\widetilde{\rho}_0 , \quad \pi_0 - m l_0 = \widetilde{f}_\pi(\Box)\left(\widetilde{\pi}_0 - v l_0\right), \tag{3.111}$$

we can rewrite the renormalized action uniquely in terms of the sources,

$$\widetilde{S}_{\text{ren}} = \frac{1}{2} \int \mathrm{d}^2x \left[t_0 \cdot f_t(\Box)\, t_0 - 2\nu \Big(m\left(2\widetilde{\rho}_0 + 2\widetilde{\pi}_0 l_0 - v\, l_0 l_0\right) + \right. \tag{3.112}$$
$$\left. + \widetilde{\rho}_0 \widetilde{f}_\rho(\Box)\widetilde{\rho}_0 + \left(\widetilde{\pi}_0 - v l_0\right)\widetilde{f}_\pi(\Box)\left(\widetilde{\pi}_0 - v l_0\right) \Big) \right].$$

From this renormalized action we can retrieve Ward identities that are completely equivalent to those in (3.107), with inverted roles for v and m (and ν going into $-\nu$).

To conclude the discussion, we would like to provide an explicit expression for the two-point correlator of $\mathrm{Im}\widetilde{\mathcal{O}}$, where the massless Goldstone pole should be found. For $v = 0$, that in alternative quantization corresponds to purely spontaneous breaking, and $\nu = 1/2$, corresponding to the dimension of the boundary operator equal to 1/2, the equation of motion (3.84) becomes

$$M''(r) - \left(k^2 + m^2 r^{-1}\right) M(r) = 0\,, \tag{3.113}$$

where $M = r\partial_r L$. This equation can be analytically solved, and, if we impose boundary conditions such that the solution is not exploding in the deep bulk, we obtain the following well-behaved function

$$M(r) = C\, r\, e^{-\sqrt{k^2}\, r}\, \mathrm{U}\left[1 + \frac{m^2}{2\sqrt{k^2}}, 2; 2\sqrt{k^2}\, r\right], \tag{3.114}$$

where $\mathrm{U}[a,b;x]$ is the Tricomi's hypergeometric function.

From the constraint (3.82) we get

$$\widetilde{\pi}_0 = -\frac{1}{m}\, k^2 M\big|_{r^0} \,, \tag{3.115}$$

where $M|_{r^0}$ is the constant term in the small r expansion. Similarly, from the equation of motion (3.84) we can express the gauge invariant combination involving π_0 in the following way:

$$\pi_0 - m l_0 = -\frac{1}{m}\, M'\big|_{r^0} \,. \tag{3.116}$$

Then we can derive the final expression for the correlator

$$\begin{aligned}\left\langle \mathrm{Im}\widetilde{\mathcal{O}}(k)\,\mathrm{Im}\widetilde{\mathcal{O}}(-k)\right\rangle_0 &= i\,\widetilde{f}(k^2) = i\,\frac{\pi_0 - m l_0}{\widetilde{\pi}} \\ &= -\frac{i}{k^2}\left[\sqrt{k^2} - m^2\left(2\gamma_{EM} + \ln\!\left(2\sqrt{k^2}\right) + \psi_{(2)}\left[1 + \frac{m^2}{2\sqrt{k^2}}\right]\right)\right] \,,\end{aligned} \tag{3.117}$$

where γ_{EM} is the Euler-Mascheroni constant, and $\psi_{(2)}[x]$ is the di-gamma function. Using the expansion $\psi_{(2)}[1+x] \simeq \ln(x) + 1/(2x) + \mathcal{O}(1/x^2)$ for large x, one verifies that both the linear term $\sim |k|$ and the logarithmic term $\sim \ln|k|$ in the numerator of the equation above cancel in the $k \to 0$ limit. In this way, the low energy behavior of this correlator exhibits the expected Goldstone massless pole, namely

$$\left\langle \mathrm{Im}\widetilde{\mathcal{O}}\,\mathrm{Im}\widetilde{\mathcal{O}}\right\rangle_0 \approx i\,\frac{2m^2}{k^2}\left(\gamma_{EM} + \ln m\right) . \tag{3.118}$$

We have thus explicitly confirmed the presence of the Goldstone boson, in addition to the deduction (3.108) based on Ward identities.

4 Holographic non-relativistic type B Godstone boson

In this chapter we will see the first of two examples of non-relativistic holographic setups. Such example is formulated in completely Lorentz covariant formalism, and Lorentz boost invariance is explicitly broken by giving a background profile to the temporal component of the bulk gauge field (in this way preserving invariance under spatial rotations). Taking a non-abelian gauge group, we will produce a suitable setup for the type B Goldstone boson to appear.

In Section 1.3, we have seen that the occurrence of Goldstone bosons with quadratic dispersion relation in non-relativistic field theories can be related to the presence of pairs of broken generators whose commutator has a non-trivial vacuum expectation value, which would be instead forbidden in case of Lorentz invariance. Such Goldstone bosons with quadratic dispersion relation (type B GBs) are usually accompanied by a massive partner whose mass is proportional to the amount of Lorentz breaking. Eventually, there can still be massless modes with linear dispersion relations (type A GBs), as in the relativistic case, but with a model dependent velocity, likely different from $c = 1$.

In this chapter we will explore in a holographic setup what can be learned on these different kinds of light modes. In particular, we will focus on what can be extracted just by specifying how the symmetries are broken. The holographic model is supposed to represent a field theory, with a large number of degrees of freedom and at strong coupling, which displays a pattern of symmetry breaking allowing for the various types of Goldstone bosons to be present. We will study a model with the minimal requirements in order to expect all these light modes, namely a U(2) global symmetry on the boundary, which will be spontaneously broken, and a source for the time-like component of the abelian U(1) current, *i.e.* a chemical potential, which explicitly breaks boost invariance (but preserves, besides rotations, the global U(2) as well).

Such model was previously considered in [98], where the system was set at finite temperature and numerical techniques were essentially employed to produce the results (see also [99] for a different holographic model). We will rather consider the model at zero temperature, in analogy with the original field theory model [38, 39], reviewed in Section 1.3.1, and perform a purely boundary analysis through analytic techniques.

4.1 Renormalized action for a holographic type B Goldstone model

Let us outline the model discussed in [98]. This is a typically strongly coupled theory represented by its holographic bulk dual. We will assume that the theory has a non-trivial UV conformal fixed point. In this UV CFT, we will focus on the conserved currents J_i^a which form a U(2) algebra, and on a relevant operator $\mathcal{O}_\Phi$ which is a doublet of U(2). In the holographic dual, this means that we need to have a bulk theory in an asymptotically AdS spacetime, which includes dynamical U(2) gauge fields and a complex doublet scalar with negative squared mass. Since we are not interested in computations involving the stress energy tensor, we set gravity in the bulk not to be dynamical. Finally, in order to enjoy the simplifications discussed in Section 3.1, we choose four bulk dimensions (and hence three dimensions for the boundary theory), assuming that our results would not be qualitatively modified in higher dimensions.

Then we consider the following action,

$$S = \int d^4x\, \sqrt{-g}\left[-\frac{1}{4}F^a{}_{mn}F^{amn} - D_m\Phi^\dagger D^m\Phi - m_\Phi^2\Phi^\dagger\Phi\right] , \tag{4.1}$$

where $F^a_{mn} = \partial_m A^a_n - \partial_n A^a_m + f^{abc}A^b_m A^c_n$, $D_m\Phi = \partial_m\Phi - iA^a_m T^a\Phi$, and the field

$$\Phi \equiv \begin{pmatrix} \psi \\ \phi \end{pmatrix} \tag{4.2}$$

is a complex scalar doublet. The two-by-two matrices $T^a = \frac{1}{2}\tau^a$, with $a = \{0,1,2,3\}$, are the generators of U(2) in the fundamental representation, where the τ^a are given by the identity and the Pauli matrices, as defined in (1.63). They satisfy $[T^a, T^b] = i f^{abc}T^c$, with $f^{abc} = 0$ if any of the indices is 0, and $f^{abc} \equiv \epsilon^{abc}$ otherwise, where ϵ^{abc} is the completely antisymmetric three-dimensional Levi-Civita symbol.

The metric is fixed to be the AdS one, as defined in (2.8), again with no gravitational dynamics, given that we are not interested in computing correlators involving the stress-energy tensor. Moreover, we will not consider any back-reaction on the metric. This is motivated by the fact that we will in any case restrict our attention to the near-boundary region, where back-reaction effects can be shown to be sub-leading and actually completely irrelevant to our considerations. Finally, note that since this is pure AdS, on the field theory side it corresponds to considering zero temperature, in contrast with the analysis of [98] where a non-zero temperature was always required.

Let us first consider the field configuration defining the background, and thus the vacuum of the dual theory. First of all, as anticipated, the theory is non-relativistic because boost invariance is broken by the presence of a chemical potential, $\mathscr{L}_{\text{QFT}} \supset \mu J_t^0$. This means that there should be a non-trivial source for J_t^0, namely A_t^0 should have a leading mode turned on. This source breaks Lorentz invariance, but preserves all of U(2) since the U(1) generator commutes with the whole algebra. In the vacuum of the theory, we expect that non-trivial dynamics generates vev's for the operator $\mathscr{O}_\Phi$, thus breaking U(2) to U(1). For simplicity we will take this U(1) to be the one generated by $T^0 + T^3$. The unbroken symmetries allow for vev's to be generated also for J_t^0 and J_t^3. The latter is crucial for obtaining a non-vanishing vev for commutators of charges, and thus for the appearance of type B Goldstone bosons. In the bulk, we will thus have sub-leading modes for the profiles of A_t^0, A_t^3 and for Φ in its bottom component ϕ.

From now on, we fix the dimension of $\mathscr{O}_\Phi$ to be two, which implies $m_\Phi^2 = -2$ (in units of the AdS radius). The background profiles are thus the following:

$$\phi\Big|_B = \phi_B r^2, \qquad A_t^0\Big|_B = \mu + A_B^0 r\,, \qquad A_t^3\Big|_B = A_B^3 r\,, \tag{4.3}$$

where we take ϕ_B, μ, A_B^0 and A_B^3 to be real constants. These profiles satisfy the free equations of motion. Normally, in the deep bulk (and in particular in presence of a horizon/temperature) the equations of motion would relate the sub-leading to the leading mode. However, as a far as our *near-boundary* analysis is concerned, the back-reaction of the above profiles on the metric can be safely neglected,[1] and the leading and sub-leading modes can be chosen independently. In particular, with $A_B^{0,3}$ independent of μ, this latter could be even set to zero. However this would mean that some physical mechanism should generate spontaneously a Lorentz breaking vev such as $\langle J_t^3\rangle_0$. So, in the following we will rather assume that $\mu \neq 0$ even if it will eventually not appear in any of the results.

We now proceed to fluctuate the fields over the background. The aim is to obtain the on-shell action up to quadratic order, since we are interested in two- and one-point functions. As we know, at this order the on-shell action reduces to a boundary term. In the following, we fix as usual the radial gauge $A_r^a = 0$. We write then the equations of motion at linear order in the fluctuations above the backgrounds, which we label in the same way as the respective fields themselves. The constraint coming from the variation with respect to A_r^a reads:

$$\begin{aligned} &r\partial_r\partial_\mu A_\mu^a - r A_B^3 f^{3ab}\left(A_t^b - r\partial_r A_t^b\right) - i\phi_B (T^a)_{21}\big(2\psi - r\partial_r\psi\big) + \\ &\quad + i\phi_B (T^a)_{12}\left(2\psi^* - r\partial_r\psi^*\right) - i\phi_B (T^a)_{22}\left(2(\phi-\phi^*) - r\partial_r(\phi-\phi^*)\right) = 0\,. \end{aligned} \tag{4.4}$$

The equations of motion for the vector field fluctuations, separated in spatial and temporal

[1] The back-reaction would intervene at $\mathscr{O}(r^4)$ in ϕ and at $\mathscr{O}(r^2)$ in $A_t^{0,3}$, hence at orders that do not contribute to the rest of our computations.

components, are

$$
\begin{aligned}
& r^2\partial_r^2 A_i^a + r^2\partial_j^2 A_i^a - r^2\partial_i\partial^\mu A_\mu^a - r^4\phi_B^2 A_i^b\{T^a,T^b\}_{22} + \\
&\quad + 2r^3 f^{ab3} A_B^3 \partial_t A_i^b - r^3 f^{ab3} A_B^3 \partial_i A_t^b - r^4 \left(A_B^3\right)^2 f^{ab3} f^{bc3} A_i^c + \\
&\quad + i r^2 \phi_B \left[(T^a)_{12}\partial_i\psi^* - (T^a)_{21}\partial_i\psi - (T^a)_{22}\partial_i(\phi-\phi^*)\right] = 0\,;
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
& r^2\partial_r^2 A_t^a + r^2\partial_i^2 A_t^a - r^2\partial_t\partial_i A_i^a + r^3 f^{ab3} A_B^3 \partial_i A_i^b - r^4\phi_B^2 A_t^b\{T^a,T^b\}_{22} + \\
&\quad + i r^2\phi_B\left[(T^a)_{12}\partial_t\psi^* - (T^a)_{21}\partial_t\psi - (T^a)_{22}\partial_t(\phi-\phi^*)\right] + \\
&\quad - r^2\left(\mu + A_B^0 r\right)\phi_B\left[(T^a)_{21}\psi + (T^a)_{12}\psi^* + (T^a)_{22}(\phi+\phi^*)\right] + \\
&\quad - r^3 A_B^3\phi_B\left[\{T^a,T^3\}_{21}\psi + \{T^a,T^3\}_{12}\psi^* + \{T^a,T^3\}_{22}(\phi+\phi^*)\right] = 0\,.
\end{aligned}
\tag{4.6}
$$

Finally, the equations for the scalar fluctuations are

$$
\begin{aligned}
& r^2\partial_r^2\psi - 2r\partial_r\psi + r^2\partial_i^2\psi + 2\psi - i r^4\phi_B (T^a)_{12}\partial_i A_i^a + \\
&\quad + i r^2\left(\mu + A_B^0 r + A_B^3 r\right)\partial_t\psi + \frac{1}{4} r^2\left(\mu + A_B^0 r + A_B^3 r\right)^2\psi + \\
&\quad + r^4\left(\mu + A_B^0 r\right)\phi_B (T^a)_{12} A_t^a = 0\,;
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
& r^2\partial_r^2\phi - 2r\partial_r\phi + r^2\partial_i^2\phi + 2\phi - i r^4\phi_B (T^a)_{22}\partial_i A_i^a + \\
&\quad + i r^2\left(\mu + A_B^0 r - A_B^3 r\right)\partial_t\phi + \frac{1}{4} r^2\left(\mu + A_B^0 r - A_B^3 r\right)^2\phi + \\
&\quad + r^4\left(\mu + A_B^0 r\right)\phi_B (T^a)_{22} A_t^a + r^5 A_B^3 \phi_B \{T^a,T^3\}_{22} A_t^a = 0\,.
\end{aligned}
\tag{4.8}
$$

The last two equations should be supplemented by their complex conjugates.

The on-shell action is obtained expanding (4.1) up to quadratic order in the fluctuations, and then substituting the equations of motion. At the regularizing surface $r=\epsilon$, we obtain

$$
\begin{aligned}
S_{\text{reg}} = -\int_{r=\epsilon} \mathrm{d}^3x\, \Big[& A_B^0 A_t^0 + A_B^3 A_t^3 + \frac{1}{2} A_t^a \partial_r A_t^a - \frac{1}{2} A_i^a\partial_r A_i^a + \\
& - \frac{2}{r}\phi_B(\phi+\phi^*) - \frac{1}{2} r^{-2}\left(\psi^*\partial_r\psi + \psi\partial_r\psi^* + \phi^*\partial_r\phi + \phi\partial_r\phi^*\right)\Big]\,.
\end{aligned}
\tag{4.9}
$$

Now, we can also use the equation (4.4), which only has first order derivatives in r. However, we also have to split the vector into its irreducible components. We choose here to split into transverse and longitudinal parts with respect to the spatial coordinates i, while keeping A_t^a as the temporal component[2], namely

$$
A_i^a = T_i^a + \partial_i L^a, \qquad \partial_i T_i^a = 0\,. \tag{4.10}
$$

[2] In Appendix A of [I], a different splitting is discussed, where the transverse and longitudinal parts are taken with respect to the space-time coordinates μ, as in the relativistic case (3.15). Even if the final results are the same, namely the Ward identities can be correctly recovered by both ways, we find more logical to show the computation here in this manifestly non-relativistic splitting.

We eventually arrive at

$$S_{\rm reg} = -\int_{r=\epsilon} \mathrm{d}^3x \, \Big\{ A_B^0 A_t^0 + A_B^3 A_t^3 - \frac{2}{r}\phi_B(\phi+\phi^*) + \frac{1}{2}A_t^a \partial_r A_t^a - \frac{1}{2}T_i^a \partial_r T_i^a + \\ + \frac{1}{2}L^a \partial_r \partial_t A_t^a + \frac{1}{2}A_B^3 f^{ab3} L^a\big(A_t^b - r\partial_r A_t^b\big) + \\ + \frac{i}{2r}\phi_B L^a\Big[(T^a)_{21}\big(2\psi - r\partial_r\psi\big) - (T^a)_{12}\big(2\psi^* - r\partial_r\psi^*\big) + \\ + (T^a)_{22}\big(2(\phi-\phi^*) - r\partial_r(\phi-\phi^*)\big)\Big] + \\ - \frac{1}{2r^2}\left(\psi^*\partial_r\psi + \psi\partial_r\psi^* + \phi^*\partial_r\phi + \phi\partial_r\phi^*\right)\Big\} \,. \tag{4.11}$$

We now consider the following near-boundary expansion of the fluctuating fields, which can be straightforwardly derived from the equation of motion (4.5–4.8):

$$\begin{aligned} A_t^a &= a_0^a + r\,\tilde{a}_0^a + \dots \\ T_i^a &= t_{i0}^a + r\,\tilde{t}_{i0}^a + \dots \\ L^a &= l_0^a + r\,\tilde{l}_0^a + \dots \\ \psi &= r\,\psi_0 + r^2\,\tilde{\psi}_0 + \dots \\ \phi &= r\,\phi_0 + r^2\,\tilde{\phi}_0 + \dots \end{aligned} \tag{4.12}$$

We remark that all non-tilded modes will be the sources, while all tilded modes will be the responses. The fact that the latter occur at next-to-leading order in r is a specific feature of three boundary dimensions and scalar mass $m_\Phi^2 = -2$, as we have already remarked in the pseudo-Goldstone model of Section 3.1.

The divergent terms that one finds in $S_{\rm reg}$ are taken care of by adding counter-terms, which are independent of the presence of the profile. Again, due to the involved dimensions, no logarithms are present, and hence no finite counter-terms arise and no scheme dependence (see Section 2.1.2). However, as it should be clear in the following, scheme dependent terms would not affect the part of the renormalized action that we are interested in. Moreover, the only divergent terms are those involving scalars, so that the only needed counterterm is

$$S_{\rm ct} = \int \mathrm{d}^3x\, \sqrt{-\hat{g}}\; \Phi^\dagger\Phi = \int \mathrm{d}^3x \left[\frac{1}{r}\left(\psi_0^*\psi_0 + \phi_0^*\phi_0\right) + \psi_0^*\tilde{\psi}_0 + \psi_0\tilde{\psi}_0^* + \phi_0^*\tilde{\phi}_0 + \phi_0\tilde{\phi}_0^*\right] .$$

Eventually, we obtain the following renormalized action,

$$S_{\rm ren} = -\int \mathrm{d}^3x \,\Big[A_B^0 a_0^0 + A_B^3 a_0^3 - 2\phi_B\big(\phi_0 + \phi_0^*\big) + \\ - \frac{1}{2}t_{i0}^a \tilde{t}_{i0}^a + \frac{1}{2}\big(a_0^a - \partial_t l_0^a\big)\tilde{a}_0^a + \frac{1}{2}A_B^3 f^{ab3} l_0^a a_0^b + \\ + \frac{i}{2}\phi_B l_0^a\Big((T^a)_{21}\psi_0 - (T^a)_{12}\psi_0^* + (T^a)_{22}\big(\phi_0 - \phi_0^*\big)\Big) + \\ - \frac{1}{2}\Big(\psi_0^*\tilde{\psi}_0 + \psi_0\tilde{\psi}_0^* + \phi_0^*\tilde{\phi}_0 + \phi_0\tilde{\phi}_0^*\Big)\Big] \,. \tag{4.13}$$

The first three terms of the above functional, linear in the fluctuations, just give the vev's of the corresponding dual operators J^0_t, J^3_t and Re$\mathcal{O}_\phi$. The other terms, quadratic in the fluctuations, contain in principle all the information about two-point functions. However, this information is encoded in the way the sub-leading (tilded) modes with depend on the (untilded) sources. Such dependence, as we know, is fixed through the bulk boundary conditions and is typically non-local, but this requires solving the full equations for the fluctuations inside the bulk, which in this case are quite involved. Instead, in the following we will show what can be extracted from the renormalized action (4.13) without solving the bulk equations.

4.2 Holographic Ward identities for symmetry breaking in presence of charge density

A brief inspection of S_{ren} (4.13) shows that again, as discussed in Section 3.1 after eq. (3.23), there are two kinds of quadratic terms: those which are bilinears of a source and a vev, and those which involve only sources, and latter would vanish for trivial profiles.

However, terms of the second kind are also hidden into terms of the first kind, because of gauge invariance. Indeed, if we consider the bulk gauge invariance of the action (4.1), we have $\delta\Phi = i\alpha\Phi$ and $\delta A_\mu = \partial_\mu\alpha + i[\alpha, A_\mu]$, where, after the gauge fixing $A^a_r = 0$, the gauge parameter α does not depend anymore on r. It is then easy to see how the residual gauge transformations act on each mode in the expansions (4.12). Recalling further that we are only interested in the quadratic part of the action, we neglect terms in the gauge variations which are bilinear in the gauge parameter and the mode of the fluctuation. We are then left with:

$$\begin{aligned}
\delta\psi_0 &= 0\,, & \delta\tilde{\psi}_0 &= i\alpha^a (T^a)_{12}\phi_B\,,\\
\delta\phi_0 &= 0\,, & \delta\tilde{\phi}_0 &= i\alpha^a (T^a)_{22}\phi_B\,,\\
\delta a^a_0 &= \partial_t\alpha^a\,, & \delta\tilde{a}^a_0 &= -f^{ab3}\alpha^b A^3_B\,,\\
\delta t^a_{i0} &= 0\,, & \delta\tilde{t}^a_{i0} &= 0\,,\\
\delta l^a_0 &= \alpha^a\,, & \delta\tilde{l}^a_0 &= 0\,.
\end{aligned} \tag{4.14}$$

By solving the bulk equations of motion for the fluctuations, the sub-leading modes would be expressed in terms of non-local functions of the sources. However, in order to solve the equations, we would have to impose boundary conditions in the bulk and, in order to preserve the gauge symmetry, we should take care of imposing them on gauge-invariant combinations of the fields. So, we have to consider only the gauge-invariant combinations of both sources and vev's, which are respectively

$$a^a_0 - \partial_t l^a_0\,, \tag{4.15}$$

for the sources, and

$$\widetilde{\psi}_0 - i\phi_B (T^a)_{12} l_0^a \,, \qquad \widetilde{\phi}_0 - i\phi_B (T^a)_{22} l_0^a \,, \qquad \widetilde{a}_0^a + A_B^3 f^{ab3} l_0^b \,, \tag{4.16}$$

for the vev's.

We are now able to define the following relations between vev's and sources:

$$\begin{aligned}
\widetilde{\psi}_0 =& i\phi_B (T^a)_{12} l_0^a + f_\psi(\partial)\psi_0 + g_\psi^a(\partial)\big(a_0^a - \partial_t l_0^a\big) \,, \\
\widetilde{\phi}_0 =& i\phi_B (T^a)_{22} l_0^a + f_\phi(\partial)\phi_0 + g_\phi^a(\partial)\big(a_0^a - \partial_t l_0^a\big) \,, \\
\widetilde{a}_0^a =& - A_B^3 f^{ab3} l_0^b + f_t^{ab}(\partial)\big(a_0^b - \partial_t l_0^b\big) + \\
& + g_1^a(\partial)\psi_0 + g_1^a(\partial)^* \psi_0^* + g_2^a(\partial)\phi_0 + g_2^a(\partial)^* \phi_0^* \,, \\
\widetilde{t}_{i0}^a =& f_T^{ab}(\partial)\, t_{i0}^b \,,
\end{aligned}$$

where all the functions of ∂ collectively indicate expressions that are typically non-local in space and/or time derivatives, and that cannot be determined without explicitly solving the equations in the bulk. However, we will see that in some combinations of the correlators the dependence on these unknown functions drops out.

First we rewrite S_{ren} (4.13) using the expressions above, so that it becomes an expression in terms of the sources only:

$$\begin{aligned}
S_{\text{ren}} = \int \mathrm{d}^3x \Big\{ & - A_B^0 a_0^0 - A_B^3 a_0^3 + 2\phi_B\big(\phi_0 + \phi_0^*\big) + \frac{1}{2} t_{i0}^a f_T^{ab}(\partial)\, t_{i0}^b + \\
& - \frac{1}{2}\big(a_0^a - \partial_t l_0^a\big) f_t^{ab}(\partial)\big(a_0^b - \partial_t l_0^b\big) + \frac{1}{2} A_B^3 f^{ab3}\big(2a_0^a - \partial_t l_0^a\big) l_0^b + \\
& + \frac{1}{2}\psi_0^*\big(f_\psi(\partial) + f_\psi(\partial)^*\big)\psi_0 + \frac{1}{2}\phi_0^*\big(f_\phi(\partial) + f_\phi(\partial)^*\big)\phi_0 + \\
& - i\phi_B l_0^a \Big[(T^a)_{21}\psi_0 - (T^a)_{12}\psi_0^* + (T^a)_{22}\big(\phi_0 - \phi_0^*\big)\Big] + \\
& + \frac{1}{2}\Big[\psi_0^*\big(g_\psi^a(\partial) + g_1^a(\partial)^*\big) + \psi_0\big(g_\psi^a(\partial)^* + g_1^a(\partial)\big)\Big]\big(a_0^a - \partial_t l_0^a\big) + \\
& + \frac{1}{2}\Big[\phi_0^*\big(g_\phi^a(\partial) + g_2^a(\partial)^*\big) + \phi_0\big(g_\phi^a(\partial)^* + g_2^a(\partial)\big)\Big]\big(a_0^a - \partial_t l_0^a\big) \Big\} \,.
\end{aligned} \tag{4.17}$$

This is the generating functional for the one- and two-point functions in our theory.[3] The precise relations between sources of operators in the boundary theory and modes of bulk fluctuations are the following. For the scalar operators we have

$$\int_{\partial AdS} \mathrm{d}^3x \,\Big(\psi_0 \mathcal{O}_\psi + \phi_0 \mathcal{O}_\phi + c.c.\Big) \,, \tag{4.18}$$

while for the currents

$$\int_{\partial AdS} \mathrm{d}^3x \,\Big(- a_0^a J_t^a + t_{i0}^a J_i^{Ta} - l_0^a \partial_i J_i^a\Big) \,, \tag{4.19}$$

[3] Note that the scheme-ambiguity, which would arise from possible logarithmic terms in higher dimensions, would be contained in the possibility to redefine the non-local functions in the above expression. However, the Ward identity structure (4.20–4.23) is independent of such non-local functions.

so that t^a_{i0} sources the purely transverse part of J^a_i, while l^a_0 its longitudinal piece.

Some two-point functions will be entirely determined by their non-local part, for instance those with two transverse currents or two scalar operators, and we will have nothing to say about them, since we do not solve the bulk equations. On the other hand, we see from the final expression of our generating functional (4.17) that some other two-point functions might be directly determined by our analysis. It should be the case for two-point functions involving the temporal and longitudinal components of the currents, both among themselves or mixed with scalar operators. Indeed, local constant terms involving the sources of these operators appear in (4.17).

Let us list here a number of such correlators:

$$
\begin{aligned}
\langle J^a_t(x) J^b_t(y)\rangle_0 &= -i\frac{\delta^2 S_{\text{ren}}}{\delta a^a_0(x)\delta a^b_0(y)} = i f^{ab}_t(\partial)\delta(x-y)\,,\\
\langle \partial_i J^b_i(x) J^a_t(y)\rangle_0 &= -i\frac{\delta^2 S_{\text{ren}}}{\delta l^b_0(x)\delta a^a_0(y)} = i\left(f^{ab}_t(\partial)\partial_t - A^3_B f^{ab3}\right)\delta(x-y)\,,\\
\langle \partial_i J^a_i(x) \partial_j J^b_j(y)\rangle_0 &= -i\frac{\delta^2 S_{\text{ren}}}{\delta l^a_0(x)\delta l^b_0(y)} = -i\left(f^{ab}_t(\partial)\partial_t^2 - A^3_B f^{ab3}\partial_t\right)\delta(x-y)\,,\\
\langle J^a_t(x)\mathcal{O}_\psi(y)\rangle_0 &= +i\frac{\delta^2 S_{\text{ren}}}{\delta a^a_0(x)\delta \psi_0(y)} = \frac{i}{2}\left(g^a_\psi(\partial)^* + g^a_1(\partial)\right)\delta(x-y)\,,\\
\langle \partial_i J^a_i(x)\mathcal{O}_\psi(y)\rangle_0 &= +i\frac{\delta^2 S_{\text{ren}}}{\delta l^a_0(x)\delta \psi_0(y)} = \\
&= \frac{i}{2}\left[\left(g^a_\psi(\partial)^* + g^a_1(\partial)\right)\partial_t + 2i\phi_B (T^a)_{21}\right]\delta(x-y)\,,\\
\langle J^a_t(x)\mathcal{O}_\phi(y)\rangle_0 &= +i\frac{\delta^2 S_{\text{ren}}}{\delta a^a_0(x)\delta \phi_0(y)} = \frac{i}{2}\left(g^a_\phi(\partial)^* + g^a_2(\partial)\right)\delta(x-y)\,,\\
\langle \partial_i J^a_i(x)\mathcal{O}_\phi(y)\rangle_0 &= +i\frac{\delta^2 S_{\text{ren}}}{\delta l^a_0(x)\delta \phi_0(y)} = \\
&= \frac{i}{2}\left[\left(g^a_\phi(\partial)^* + g^a_2(\partial)\right)\partial_t + 2i\phi_B (T^a)_{22}\right]\delta(x-y)\,.
\end{aligned}
$$

Then, we immediately see that some combinations are given entirely by the constant terms, or trivially vanish:

$$-\langle \partial_t J^a_t(x) J^b_t(y)\rangle_0 + \langle \partial_i J^a_i(x) J^b_t(y)\rangle_0 = -i A^3_B f^{ab3}\,\delta(x-y)\,, \tag{4.20}$$

$$-\langle \partial_t J^a_t(x) \partial_i J^b_i(y)\rangle_0 + \langle \partial_i J^a_i(x) \partial_i J^b_i(y)\rangle_0 = 0\,, \tag{4.21}$$

$$-\langle \partial_t J^a_t(x)\mathcal{O}_\psi(y)\rangle_0 + \langle \partial_i J^a_i(x)\mathcal{O}_\psi(y)\rangle_0 = -\phi_B (T^a)_{21}\,\delta(x-y)\,, \tag{4.22}$$

$$-\langle \partial_t J^a_t(x)\mathcal{O}_\phi(y)\rangle_0 + \langle \partial_i J^a_i(x)\mathcal{O}_\phi(y)\rangle_0 = -\phi_B (T^a)_{22}\,\delta(x-y)\,. \tag{4.23}$$

These are of course nothing else than the Ward identities relating the two-point functions of currents associated to broken generators, to the vev's of the operators that break the symmetry. In particular, the relations (4.22) and (4.23) are the usual identities for spontaneous symmetry breaking, which are identical to the relativistic one (3.32).

Of more interest is the identity (4.20), analogous to the (1.74), derived in the field theoretical model of Section 1.3.1.1, which is non-trivial due to the fact that we allow the temporal component of a current (J^3_μ here) to have a non-zero vev. We assume that this Lorentz violating vev is permitted by the presence of a chemical potential, though in our holographic set-up this is not technically necessary (indeed μ does not explicitly appear anywhere in the above expressions). In addition, note that the identity (4.21) is consistently trivial since the spatial (longitudinal) components of the same current cannot get a vev in our setup, otherwise the invariance under spatial rotations would be violated.

The above Ward identities imply the presence of Goldstone bosons, i.e. of gapless modes in the spectrum. More precisely, we see that in order to satisfy the identities (4.20)–(4.23), the Fourier transformed correlators $\langle J^a_t J^b_t\rangle_0(\omega, k_i)$, $\langle J^a_t \mathcal{O}_\psi\rangle_0(\omega, k_i)$, and similar ones, must be singular when the energy ω and the momentum k_i go to zero. Indeed, in Fourier space (4.20) reads

$$-i\omega\langle J^a_t J^b_t\rangle_0 + i k_i\langle J^a_i J^b_t\rangle_0 = -i A^3_B f^{ab3}\,. \tag{4.24}$$

We thus deduce the presence of massless poles in all of these correlators. This additional Ward identity, which defines type B Goldstone bosons [11], requires the dispersion relation to be quadratic [9], with the following argument.

In the Lorentz invariant case, the dispersion relation of a Goldstone boson is trivially determined from a Ward identity similar to (4.22) and (4.23), giving

$$\langle J_i \mathcal{O}\rangle_0 \propto \frac{k_i}{\omega^2 - k_i^2}\,. \tag{4.25}$$

In lack of Lorentz invariance, we have to consider two different situations, one in which time-reversal invariance is preserved and one in which it is broken. When it is preserved, for small values of ω and k_i we can admit:

$$\langle J_t \mathcal{O}\rangle_0 \simeq \frac{\widetilde{T}\omega}{\omega^2 - c_s^2 k_i^2}\,, \qquad \langle J_i \mathcal{O}\rangle_0 \simeq \frac{\widetilde{U} k_i}{\omega^2 - c_s^2 k_i^2}\,, \tag{4.26}$$

while when it is broken, we can have

$$\langle J_t \mathcal{O}\rangle_0 \simeq \frac{\bar{T}}{M\omega - k_i^2}\,, \qquad \langle J_i \mathcal{O}\rangle_0 \simeq \frac{\widetilde{U} k_i}{M\omega - k_i^2}\,, \tag{4.27}$$

where $\bar{T}$, $\widetilde{T}$, $\widetilde{U}$, c_s and M are constants. Note now that (4.24) breaks time-reversal invariance for the currents J^1_i and J^2_i, and actually requires $\bar{T} \neq 0$ as it was proved in [9]. This implies that (4.20) and (4.22) lead to quadratic dispersion relations, $\omega \simeq \frac{k_i^2}{M}$ with $M \equiv \bar{T}/\phi_B$. On the other hand for the Goldstone boson contributing to (4.23), which is of type A [11], the time-reversal invariance is still preserved and the dispersion relation is linear (assuming $\widetilde{U}$ has a finite limit for vanishing momentum), but with velocity depending on the ratio $\widetilde{U}/\widetilde{T} \equiv c_s^2$.

We stress once more that in order to obtain quantitative results on all these dispersion relations, we should determine the non-local functions that we have left unspecified in (4.17). The poles in these functions would give us the dispersion relations of the massless modes, together with all the rest of the massive spectrum. In order to do that, we would need first to have a background which is reliable down to the deep bulk. Performing the back-reaction, also on the metric, would then be necessary. One should be warned though that in the present zero-temperature set-up, that would most inevitably lead to a singular geometry. That should however not prevent us from imposing boundary conditions in the form of boundedness of the fluctuations. Any other stratagem to avoid the singularity would introduce a new scale to the problem, as for instance a finite temperature. That has been carried out by numerical methods in [98], where however the back-reaction is not studied, thus limiting the analysis to situations where the temperature and the chemical potential are roughly at the same scale.

Without specifying the unknown non-local functions $f_t^{ab}(\partial)$, $f_\psi^a(\partial)$, $f_\phi^a(\partial)$, etc, we cannot go further and, for instance, find the exact expression for $\bar{T}$, $\widetilde{T}$, $\widetilde{U}$ for all the massless (and light) excitations. In the present model this is of course in principle possible, but would imply solving the equations of motion for the fluctuations in the bulk. This in turn would necessitate to find the back-reacted geometry. The point in this chapter was to exploit up to its limits the technique of holographic renormalization, i.e. to extract the maximal information about the system purely from boundary considerations.

5 Conserved currents and symmetry breaking in Lifshitz holography

In this chapter we will review holographic renormalization and two-point functions for conserved current and scalar operators on Lifshitz invariant backgrounds, so setting the premises to study symmetry breaking and Ward identities in Lifshitz holography.

Our goal is to discuss symmetry breaking and Goldstone bosons in theories which enjoy Lifshitz scaling invariance, rather than relativistic or even conformal invariance. Low-energy effective theories for such Goldstone bosons were considered for instance in [24–27]. If one is interested in seeing how such Goldstone bosons arise in strongly coupled theories, the natural answer is to turn to large N field theories and their holographic description. Holography for theories with a Lifshitz scaling has been initiated in [16, 17] (see [19] for a recent review). Some other relevant references are [18, 100–108].

We will be concerned with the breaking of a global symmetry which commutes with the space-time symmetries of the theory. This entails as usual the presence of a conserved current (i.e. a charge density and a spatial current related by a conservation equation), independent from the energy-momentum complex. Holographically, this means that we will have to deal with a bulk massless vector in a fixed Lifshitz background. In particular, the non-relativistic form of the latter is typically generated with the help of a massive bulk vector which acquires a profile. We will not consider the dynamics neither of the metric nor of this massive vector field, but we will make use of the background massive vector field in order to write more general bulk actions for the scalar and the massless vector.

Since in order to discuss symmetry breaking, we need to consider the coupled system of a vector and a scalar, we will need to first make sure we understand how Lifshitz holography works for a massless vector and a scalar separately, and what are the different options in the holographic description. Indeed, having less constraints with respect to, i.e., conformal symmetry, we expect a larger diversity of field theory outcomes from holography, depending on choices that one makes in the bulk theory.

We have briefly discussed basic examples of Lifshitz invariant field theories in Section 1.3.2. We recall here the defining transformations (1.75) of Lifshitz scaling invariance,

$$\begin{aligned} t &\longrightarrow \lambda^z t\,, \\ x_i &\longrightarrow \lambda x_i\,. \end{aligned}$$

Whenever $z \neq 1$, such scaling transformations are not compatible with Lorentz symmetry, and they constitute non-relativistic scale transformations. We will always assume that $z \geq 1$. If we add to Lifshitz scaling (1.75) time and space translations as well as spatial rotations, we obtain what is called the Lifshitz symmetry group.

A $d+2$-dimensional Lifshitz invariant metric with $SO(d)$ rotational symmetries is unique and takes the following form

$$ds^2 = \frac{dr^2}{r^2} - \frac{dt^2}{r^{2z}} + \frac{\sum_i dx_i^2}{r^2}\,. \tag{5.1}$$

It can be shown that such a metric for $z \neq 1$ is a solution of a suitably chosen theory of Einstein gravity with a negative cosmological constant and a massive vector field [17] that we will denote as B and that is given by

$$B = \beta \frac{\mathrm{d}t}{r^z}\,, \quad \text{with } \beta \equiv \sqrt{\frac{2(z-1)}{z}}\,. \tag{5.2}$$

For $z = 1$ we recover the standard AdS metric in Poincaré coordinates that has relativistic scale invariance. The spatial coordinate r will be our holographic coordinate that is zero at the boundary and that tends to infinity in the bulk.

In the following we will consider the metric (5.1) and the massive vector (5.2) as fixed, non-dynamical background quantities, and we will study the holographic renormalization and boundary two-point functions of operators dual to fields on a fixed Lifshitz bulk geometry. We start reviewing the properties of a scalar field on a Lifshitz background.

5.1 Two-point functions for Lifshitz scalars

The basic setup of a free scalar field in a Lifshitz spacetime has of course been considered previously in the literature [16, 17, 19]. This simple case in fact already displays some peculiar features with respect to the AdS version, namely the appearance of imaginary poles in the two-point function. These diffusive poles at zero temperature are a characteristic feature in holographic Lifshitz theories, and they will appear also in the vector sector as we will show in the next section.

Let us consider the $(d+1)$-dimensional bulk Klein-Gordon action for a complex scalar field

$$S = \int \mathrm{d}^{d+1}x\ \sqrt{-g}\left(-g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - m^2\phi^*\phi\right), \tag{5.3}$$

where g is the Lifshitz metric, as defined by (5.1). We allow for arbitrary values of the bulk mass m, but for simplicity, we do not consider higher order potential terms.

From the variation of this action, we obtain the equation of motion

$$r\partial_r(r\partial_r\phi) - (d+z-1)\, r\partial_r\phi - r^{2z}\partial_t^2\phi + r^2\partial_i^2\phi - m^2\phi = 0 \,, \tag{5.4}$$

where ∂_i^2 implies a sum over the $d-1$ values of i. This equation has two independent solutions, with the following near boundary asymptotics ($r=0$):

$$\phi = r^{\frac{\tilde{d}}{2}-\nu}\left(\phi_0+\ldots\right) + r^{\frac{\tilde{d}}{2}+\nu}\left(\tilde{\phi}_0+\ldots\right), \qquad \text{with} \begin{cases} \tilde{d} = d+z-1\,, \\ \nu = \sqrt{\dfrac{\tilde{d}^2}{4}+m^2}\,. \end{cases} \tag{5.5}$$

When $\nu = 0$ we have a Lifshitz version of the BF (Breitenlohner-Freedman) bound [71] on the bulk mass of the scalar, with an 'effective' dimension $\tilde{d}$, obtained by adding the dynamical critical exponent z to the number of spatial boundary dimensions $d-1$ (that is, counting the time dimension z times). From now on, in order to avoid technical difficulties related to the renormalization procedure, we will take $0<\nu<1$.

Using the equation of motion (5.4) we can put the action (5.3) on-shell, and reduce it to a boundary term,

$$\begin{aligned} S_{\text{on-shell}} &= \frac{1}{2}\int \mathrm{d}^{d+1}x\ \partial_r\Big[-r^{-\tilde{d}}\big(\phi^*\, r\partial_r\phi + \phi\, r\partial_r\phi^*\big)\Big] \\ &= \int_{r=\epsilon} \mathrm{d}^d x\ \left[\left(\frac{\tilde{d}}{2}-\nu\right) r^{-2\nu}\,\phi_0^*\phi_0 + \frac{\tilde{d}}{2}\Big(\phi_0\tilde{\phi}_0^* + \phi_0^*\tilde{\phi}_0\Big)\right], \end{aligned} \tag{5.6}$$

where in the second line we have used the asymptotic expansion (5.5), and, since we have a term diverging at the boundary $r=0$, kept a small ϵ as a regulator. We can renormalize such a divergence through a standard mass counterterm

$$S_{\text{ct}} = \left(\frac{\tilde{d}}{2}-\nu\right)\int_{r=\epsilon} \mathrm{d}^d x\ \sqrt{-\hat{g}}\ \phi^*\phi\,, \tag{5.7}$$

where $\hat{g}$ is the induced metric on the boundary, thus obtaining

$$S_{\text{ren}} = S_{\text{on-shell}} - S_{\text{ct}} = \nu\int \mathrm{d}^d x\ \left(\phi_0\tilde{\phi}_0^* + \phi_0^*\tilde{\phi}_0\right). \tag{5.8}$$

A full solution to the equation of motion (5.4) would relate the subleading $\tilde{\phi}_0$ to the leading ϕ_0 through a non-local function, *i.e.* $\tilde{\phi}_0 = f_\phi(\partial_t,\partial_i^2)\,\phi_0$, which we assume real. Then we can write the formula for the two-point correlator of the dual boundary operator $\mathcal{O}_\phi$:

$$\langle \mathcal{O}_\phi(x)\mathcal{O}_\phi^*(x')\rangle_0 = -i\,\frac{\delta^2 S_{\text{ren}}}{\delta\phi_0(x)\delta\phi_0^*(x')} = -i\,2\nu\, f_\phi(\partial_t,\partial_i^2)\,\delta(x-x')\,. \tag{5.9}$$

The equation of motion (5.4) can be analytically solved for $z = 2$. Writing

$$\phi(r,t,x) = e^{-i\omega t + i\vec{k}\cdot\vec{x}}\phi(r,\omega,k_i) \tag{5.10}$$

we find for the Fourier modes

$$r\partial_r(r\partial_r\phi) - (d+1)\,r\partial_r\phi - \left(m^2 + k_i^2 r^2 - \omega^2 r^4\right)\phi = 0\,. \tag{5.11}$$

The solution is given in terms of confluent hypergeometric functions, of the first kind, M, and of the second kind, U:

$$\phi(r,\omega,k_i^2) = e^{\frac{i}{2}\omega r^2} r^{\frac{d+1}{2}+\nu}\left(C_1\,\mathsf{U}\left[a,1+\nu;-i\omega r^2\right] + C_2\,\mathsf{M}\left[a,1+\nu;-i\omega r^2\right]\right), \tag{5.12}$$

$$\text{with }\; a = \frac{1}{2}\left(1+\nu+\frac{ik_i^2}{2\omega}\right). \tag{5.13}$$

In order for the solution to be of in-falling type[1] at the Lifshitz horizon ($r\to\infty$), we need to set $C_2 = 0$. Indeed the function $\mathsf{M}\left[a,1+\nu;-i\omega r^2\right]$ for large argument goes as $e^{-i\omega r^2}$, thus introducing an outgoing wave. Note that the Fourier mode in (5.12), when C_2 is set to zero, decays in the deep bulk for $\mathrm{Im}\omega > 0$. Wick-rotating to Euclidean signature, this corresponds to requiring regularity in the deep bulk.

From the expansion of $\mathsf{U}[a,b;x]$ around $x=0$,

$$\mathsf{U}[a,b;x] = \frac{\Gamma[1-b]}{\Gamma[a-b+1]} + \cdots + x^{1-b}\frac{\Gamma[b-1]}{\Gamma[a]} + \ldots\,, \tag{5.14}$$

we can read off the coefficient of the leading term, going as $r^{\frac{d+1}{2}-\nu}$, which is ϕ_0, and the coefficient of the subleading term, going as $r^{\frac{d+1}{2}+\nu}$, which is $\widetilde{\phi}_0$. Then from the formula (5.9) we obtain the two-point function of the dual scalar operator with Lifshitz scaling $z = 2$:

$$\begin{aligned}\left\langle \mathcal{O}_\phi(k)\mathcal{O}_\phi^*(-k)\right\rangle_0 &= -i\,2\nu\, f_\phi(\omega,k_i^2) = -i\,2\nu\,\frac{\widetilde{\phi}_0}{\phi_0}\\ &= 2i(-i\omega)^\nu\,\frac{\Gamma[1-\nu]}{\Gamma[\nu]}\,\frac{\Gamma\left[\frac{1}{2}\left(1+\nu+\frac{ik_i^2}{2\omega}\right)\right]}{\Gamma\left[\frac{1}{2}\left(1-\nu+\frac{ik_i^2}{2\omega}\right)\right]}\,.\end{aligned} \tag{5.15}$$

We first remark the branch-cut starting at $\omega = 0$. Then, the Euler gamma function Γ has poles at negative integer values of its argument and $1/\Gamma$ is an entire function, so we find that the poles of the correlator are given by

$$\omega = \frac{-ik^2}{2(2n+1+\nu)}\,, \quad \text{with } n\in\mathbb{N}. \tag{5.16}$$

[1] Note that we use here the prescription of [109]. A more rigorous derivation of the analytic structure of the time-ordered Minkowski correlator should be recovered by accurately performing real-time holography [68, 69].

We thus see that these poles all lie on the negative imaginary axis.[2] In fact the locations of the poles accumulate as one approaches the origin along the negative imaginary axis. One may expect this behavior to be a consequence of the singularity at $r=\infty$ where for $z>1$ Lifshitz spacetimes display diverging tidal forces [16]. In other words Lifshitz spacetimes with $z>1$ are, contrary to AdS, geodesically incomplete. In the next section, we will use "hard wall" boundary conditions to cut-out the singularity at $r=\infty$, in order to see what happens to the imaginary poles. For other considerations on the appearance of imaginary poles in Lifshitz holography, together with a field theoretic perspective on the issue, see [101, 110, 111].

5.1.1 The hard wall solution for Lifshitz Klein-Gordon scalar

We want to see how the two-point function, and its poles, change if we break the Lifshitz scale invariance setting a hard wall in the IR, i.e. deep in the bulk. We consider exactly the same bulk action (5.3) as in the previous section, so we have the same solution of the equation of motion (5.12) for $z=2$. But now we impose a different IR regularity condition, namely that the field vanishes[3] in the bulk on a surface at $r=\mu^{-1}$. This yields

$$\phi^{HW}(\mu^{-1},\omega,k_i^2)\equiv 0 \quad \Leftrightarrow \quad C_2=-C_1\,\frac{\mathsf{U}\left[a,1+\nu;-\frac{i\omega}{\mu^2}\right]}{\mathsf{M}\left[a,1+\nu;-\frac{i\omega}{\mu^2}\right]}\,, \tag{5.17}$$

with the expression for a already given in (5.13), so that the hard wall solution reads

$$\begin{aligned}\phi^{HW}(\mu^{-1},\omega,k_i^2)\ &= \\ &= C_1\,e^{\frac{i}{2}\omega r^2}\,r^{\frac{d+1}{2}+\nu}\left(\mathsf{U}\left[a,1+\nu;-i\omega r^2\right]-\frac{\mathsf{U}\left[a,1+\nu;-\frac{i\omega}{\mu^2}\right]}{\mathsf{M}\left[a,1+\nu;-\frac{i\omega}{\mu^2}\right]}\,\mathsf{M}\left[a,1+\nu;-i\omega r^2\right]\right),\end{aligned} \tag{5.18}$$

and $\phi^{HW}\equiv 0$ for $r>\mu^{-1}$.

In order to determine the two-point correlator, we consider the terms $\propto r^{\frac{d+1}{2}-\nu}$ and $\propto r^{\frac{d+1}{2}+\nu}$ in the series expansion of the solution ϕ^{HW} around $r=0$. Using the near boundary expansion (5.14) for the confluent hypergeometric function of the second kind $\mathsf{U}\left[a,b;x\right]$, also

[2]We notice that the imaginary poles are symmetric under $\omega\to-\omega^*$, that is the equivalent condition in the complex ω plane to time-reversal invariance, which is indeed a symmetry of Lifshitz Klein-Gordon equation (5.11). The branch-cut in (5.15) may also be taken to be symmetric under $\omega\to-\omega^*$, i.e. lying along the upper imaginary axis.

[3]This IR boundary condition might seem closer in spirit to the ones used after a Euclidean rotation. However as we have noted, IR vanishing and in-falling boundary conditions coincide when going to the upper half ω plane. See also [106] for a similar situation.

known as the Tricomi function, and recalling that $\mathsf{M}[a,b;x]=1+O(x)$ for small x, we find

$$\begin{aligned}\phi_0^{HW} &= C_1\,(-i\omega)^{-\nu}\,\frac{\Gamma[\nu]}{\Gamma[a]}\,,\\ \tilde{\phi}_0^{HW} &= C_1\left(\frac{\Gamma[-\nu]}{\Gamma[a-\nu]}-\frac{\mathsf{U}\left[a,1+\nu;-\mu^{-2}i\omega\right]}{\mathsf{M}\left[a,1+\nu;-\mu^{-2}i\omega\right]}\right).\end{aligned} \tag{5.19}$$

The correlator of the dual scalar operator, following formula (5.9), is then given by

$$\begin{aligned}\langle O_\phi O_\phi^*\rangle_0^{HW} = -i\,2\nu\,\frac{\tilde{\phi}_0^{HW}}{\phi_0^{HW}} &= -i\,2\nu\,(-i\omega)^{\nu}\,\frac{\Gamma[a]}{\Gamma[\nu]}\left(\frac{\Gamma[-\nu]}{\Gamma[a-\nu]}-\frac{\mathsf{U}\left[a,1+\nu;-\mu^{-2}i\omega\right]}{\mathsf{M}\left[a,1+\nu;-\mu^{-2}i\omega\right]}\right)\\ &= i\,2\nu\,\mu^{2\nu}\,\frac{\mathsf{M}\left[a-\nu,1-\nu;-\mu^{-2}i\omega\right]}{\mathsf{M}\left[a,1+\nu;-\mu^{-2}i\omega\right]}\,,\end{aligned} \tag{5.20}$$

where in the second line we have used the expression of the Tricomi function as a combination of confluent hypergeometric functions, that is

$$\mathsf{U}[a,b;x] = \frac{\Gamma[1-b]}{\Gamma[a+1-b]}\,\mathsf{M}[a,b;x] + x^{1-b}\,\frac{\Gamma[b-1]}{\Gamma[a]}\,\mathsf{M}[a+1-b,2-b;x]\,.$$

Notice that this simplification precisely removes the branch cut on the real axis which was present in the pure Lifshitz correlator (5.15).

Note that a depends on ω so that the frequency dependence of $\mathsf{M}[a,b;x]$ is through both a and x. Since the confluent hypergeometric function has no poles, the poles of the hard-wall correlator are determined by the zeros of M. Since zeros of analytic functions lie isolated we obtain a discrete spectrum as shown in the numerical plot in Fig. 5.1, for the case of poles that lie on the real-ω axis. The zeros of M do not have a simple analytic expression and are hard to find even numerically. The numerical plot in Fig. 5.2 shows that the correlator (5.20) has no poles that lie exactly on the imaginary axis, as in the pure Lifshitz case (5.16), but it should have poles that lie off the real and imaginary axes, in the complex plane, that in the $\mu\to 0$ limit (no hard wall) align on the imaginary axis, at the locations of (5.16).

Indeed, the limit of the correlator (5.20) for $\mu\to 0$ correctly gives the expression for the scalar correlator in the absence of the hard wall (5.15), provided we approach the origin from the half-plane where $\mathrm{Im}\,\omega>0$ (in agreement with the choice we made below (5.13)). In fact, from the first line of (5.20), one can see that the correlator of the hard wall scalar is written in terms of the correlator of the pure Klein-Gordon scalar, plus a piece proportional to U/M. In the limit $\mu\to 0$, so large third argument, we have $\mathsf{U}[a,b;x]\propto x^a$, while $\mathsf{M}[a,b;x]\propto e^x$, with $x\equiv -\mu^{-2}i\omega$. So, for $\mathrm{Im}\,\omega>0$, the U/M piece is exponentially (and non-analytically) suppressed as we push the hard wall into the deep bulk, eventually recovering the pure Klein-Gordon expression in the strict $\mu=0$ limit.

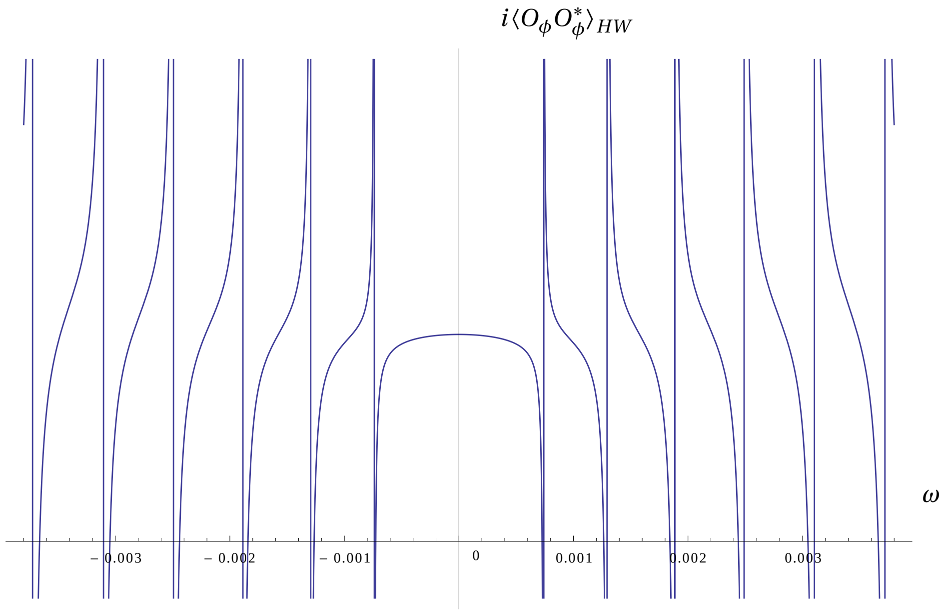

Figure 5.1. Plot of the hard-wall correlator for *real* values of ω, at fixed $\mu = 0.01$, $\nu = \frac{1}{4}$, $k_i^2 = 0.001$.

5.1.2 Two-point function for a general scalar

We discuss here a modification of the Klein-Gordon scalar bulk action (5.3), by allowing couplings to the background massive vector field (5.2). This will allow for a time-reversal breaking kinetic term inspired by [107], which yields a more general model for a non-relativistic probe scalar, including a case with enhanced $z = 2$ Schrödinger symmetry.

The background massive vector is given by equation (5.2). Note that since $B_\mu B^\mu = -\beta^2$ we can define the metric orthogonal to B_μ as

$$\gamma_{\mu\nu} = g_{\mu\nu} + \beta^{-2} B_\mu B_\nu \, . \tag{5.21}$$

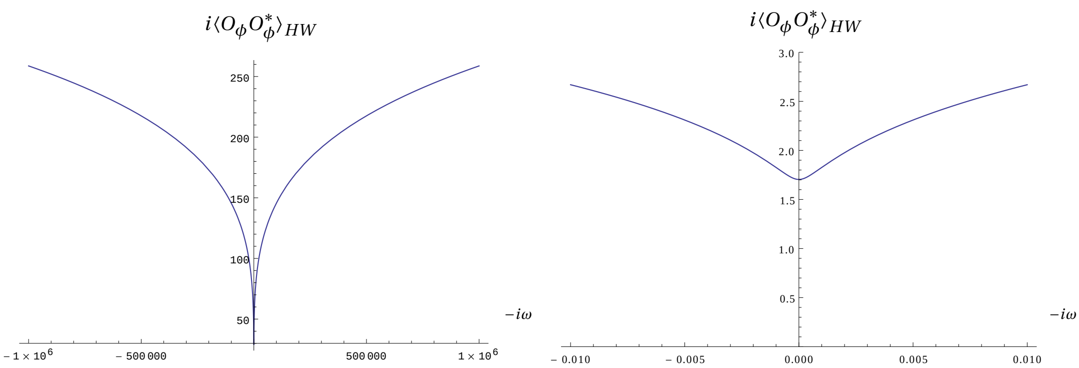

Figure 5.2. Plot of the hard-wall correlator for purely *imaginary* values of ω, at fixed $\mu = 0.01$, $\nu = \frac{1}{4}$, $k_i^2 = 0.001$. The second plot is a zoom close to the origin of the first plot.

We will denote by $\gamma^{\mu\nu}$ the above projector with its indices raised with $g^{\mu\nu}$.

Taking into account the fact that B_μ, being massive, has no gauge invariance, the most general bulk action for a complex scalar is the following:

$$S = \int \mathrm{d}^{d+1}x \,\sqrt{-g}\,\Big[\,\frac{1}{c^2\beta^2}\,B^\mu\partial_\mu\phi^*\,B^\nu\partial_\nu\phi - \frac{ih}{2\beta}B^\mu\big(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\big) + \\ -\gamma^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - m^2\phi^*\phi\Big]\,, \tag{5.22}$$

where c and h are real numbers. Note that in order to have the second term, with the h coefficient, we need a complex scalar for otherwise any term linear in $B^\mu\partial_\mu$ would be a total derivative. The contribution of the second kinetic term with coefficient h to the kinetic energy is positive for positive frequency modes provided we take h to be positive. We will therefore assume throughout that $h \geq 0$. We recover the Lifshitz Klein-Gordon case of the previous two subsections for $h = 0$ and $c = 1$, whereas the Schrödinger invariant case is recovered for $c = \infty$ as we will see later.

Since B_μ has no radial component, the terms involving B_μ in (5.22) do not contribute to the on-shell boundary action, which is thus the same as in (5.6). Thus, in our case, with $0 < \nu < 1$, the counterterm (5.7) removes the only divergence, so that the renormalized action is again given by (5.8). Therefore, the two-point function is given by (5.9), where the unknown non-local function is now obtained by solving the equation of motion resulting from the variation of the generalized action (5.22), which is

$$r\partial_r(r\partial_r\phi) - \tilde{d}\,r\partial_r\phi + r^2\partial_i^2\phi - c^{-2}r^{2z}\partial_t^2\phi + ih\,r^z\partial_t\phi - m^2\phi = 0\,. \tag{5.23}$$

In order to find analytical results, once again we specialize to $z = 2$. The above equation, after Fourier transforming, is identical to the Lifshitz Klein-Gordon equation (5.11), with ω replaced by ω/c and k_i^2 replaced by $\tilde{k}_i^2 = k_i^2 + h\omega$:

$$r\partial_r(r\partial_r\phi) - (d+1)\,r\partial_r\phi + \frac{\omega^2}{c^2}\,r^4\phi - \tilde{k}_i^2\,r^2\phi - m^2\phi = 0\,. \tag{5.24}$$

For $c \to \infty$ this equation is identical to that of a complex scalar on a Schrödinger spacetime with $z = 2$ [15] and hence possesses $z = 2$ Schrödinger invariance (see also [107]).[4] The solution, again picking the in-falling part as $r \to \infty$, is given by

$$\phi(r,\omega,k_i^2) = C\,e^{\frac{i\omega}{2c}r^2}\,r^{\frac{d+1}{2}+\nu}\,\mathrm{U}\Big[\,\tilde{a}, 1+\nu; -\frac{i\omega}{c}\,r^2\Big]\,, \tag{5.25}$$

$$\text{with}\;\; \tilde{a} = \frac{1}{2}\left(1 + \nu + \frac{ic\tilde{k}_i^2}{2\omega}\right). \tag{5.26}$$

From the expansion of this solution near $r = 0$, we again extract the source and the VEV

[4]We note however that this is no longer true for other values of z, i.e. a complex scalar field on a Schrödinger spacetime with $z \neq 2$ satisfies a different equation than we would find here for $z \neq 2$ and $c \to \infty$.

for ϕ, from which we then derive an expression similar to (5.15). This results in the following boundary two-point correlator of the operator dual to our more general scalar:

$$\langle \mathcal{O}_\phi(k)\mathcal{O}^*_\phi(-k)\rangle_0 = 2i\Big(-\frac{i\omega}{c}\Big)^\nu \frac{\Gamma[1-\nu]}{\Gamma[\nu]} \frac{\Gamma\Big[\frac{1}{2}\Big(1+\nu+\frac{ic}{2\omega}(k_i^2-h\omega)\Big)\Big]}{\Gamma\Big[\frac{1}{2}\Big(1-\nu+\frac{ic}{2\omega}(k_i^2-h\omega)\Big)\Big]}\,. \tag{5.27}$$

This expression matches the pure Klein-Gordon correlator (5.15) for $h=0$ and $c=1$. Let us however see how the analytic structure is changed by $h>0$, and what happens in the Schrödinger limit $c\to\infty$.

The poles of the correlator are situated where the argument of the Gamma function in the numerator is a negative integer. In the complex ω-plane, these are located at

$$\omega = \frac{i\,k_i^2}{\frac{2}{c}\big(2n+1+\nu\big)-ih}\,, \quad \text{with } n\in\mathbb{N}. \tag{5.28}$$

We can see immediately that for $h=0$ and $c=1$ we retrieve the same poles as in the Lifshitz Klein-Gordon case (5.16). For $c\gg 1$, instead, the imaginary part of the poles for small n is very small, and so they lie close to the negative real axis (since h is taken to be positive). Note that time-reversal invariance is violated, as a consequence of the time-reversal breaking term proportional to h in the equation of motion (5.23). In the strict $c\to\infty$ limit the structure of the zeros and poles of the correlator changes in a non-trivial manner as we will show next.

We can assume that in the Schrödinger $c\to\infty$ limit, $\frac{ic\tilde{k}^2}{4i\omega}\to\infty$, so that we can use Stirling's formula, valid for $x\to\infty$,

$$\Gamma(x)\simeq e^{(x-\frac{1}{2})\log x - x+\frac{1}{2}\log 2\pi}\,, \tag{5.29}$$

which implies

$$\frac{\Gamma(x+a)}{\Gamma(x+b)}\simeq x^{a-b}\,. \tag{5.30}$$

We thus get,

$$\langle O_\phi(k)O^*_\phi(-k)\rangle_0 = 2i\frac{\Gamma[1-\nu]}{\Gamma[\nu]}\left(\frac{k_i^2-h\omega}{4}\right)^\nu\,, \tag{5.31}$$

which agrees with [15]. Note that for $c=\infty$ the equation (5.24) reduces to a Bessel equation, as for a relativistic scalar in AdS. Hence the correlator is similar to the one in the conformal case, where however the Lorentz invariant $q^2=k^2-\omega^2$ is replaced by the Lifshitz covariant combination $k^2-h\omega$. There is a branch cut which we choose to be placed at those points for which the argument of the fractional power $0<\nu<1$ is negative, that is for real and positive $\omega\geq\frac{k^2}{h}$.

Before concluding our discussion of complex scalar probes on Lifshitz spacetimes, we consider one final case: $c=\infty$ and $z=4$. We saw that for $c=\infty$ and $z=2$ we obtain the case of a Schrödinger scalar. In the same limit but with $z=4$ we do not see a symmetry enhancement but we do find an example where we have analytic control of the solution. Since this is rare for values of z different from 2 we pause here to study this special case in some detail.

Consider equation (5.23) with $c \to \infty$ and $z = 4$. We obtain, for the Fourier transformed field,

$$r\partial_r(r\partial_r\phi) - \tilde{d}\, r\partial_r\phi - \left(m^2 + k_i^2 r^2 - h\omega\, r^4\right)\phi = 0\,, \tag{5.32}$$

which is again an equation of the confluent hypergeometric kind. Imposing the usual in-falling boundary conditions, we obtain the following solution:

$$\phi(r,\omega,k) = C\, e^{i\frac{1}{2}\sqrt{h\omega}\,r^2}\, r^{\frac{d+3}{2}+\nu}\, \mathsf{U}\left[\,\hat{a}, 1+\nu; -i\sqrt{h\omega}\,r^2\right]\,, \tag{5.33}$$

$$\text{with}\;\; \hat{a} = \frac{1}{2}\left(1+\nu+\frac{i k_i^2}{2\sqrt{h\omega}}\right). \tag{5.34}$$

Repeating the procedure that allowed us to derive the correlator (5.15), we can write the two-point function for this $z = 4$ scalar as:

$$\left\langle \mathscr{O}_\phi(k)\mathscr{O}_\phi^*(-k)\right\rangle = 2i\left(-i\sqrt{h\omega}\right)^\nu \frac{\Gamma[1-\nu]}{\Gamma[\nu]}\, \frac{\Gamma\left[\frac{1}{2}\left(1+\nu+\frac{ik^2}{2\sqrt{h\omega}}\right)\right]}{\Gamma\left[\frac{1}{2}\left(1-\nu+\frac{ik^2}{2\sqrt{h\omega}}\right)\right]}\,. \tag{5.35}$$

This correlator exhibits a branch cut, as well as poles on the negative real axis located at

$$\omega = -\frac{k^4}{4h(2n+1+\nu)^2}\,, \quad \text{with } n \in \mathbb{N}\,. \tag{5.36}$$

It is not straightforward to come with an interpretation of the modes associated with these poles, besides the obvious fact that the dispersion relations respect the $z = 4$ Schrödinger scaling, and that they violate time-reversal invariance, as expected.

5.2 Lifshitz holography for a massless vector

We now discuss conserved currents in a Lifshitz theory, that holographically correspond to massless vectors in the bulk. This is a set up that has received surprisingly little attention (see [112] for a rather different set up, at finite density). As we will see, the correlators that we will be able to compute analytically display very similar features to the case of the Klein-Gordon scalar.

We consider from the outset the most general bulk action for a massless vector in a fixed Lifshitz background,

$$S = -\frac{1}{4}\int \mathrm{d}^{d+1}x\, \sqrt{-g}\, \gamma^{\mu\nu}\left(\gamma^{\rho\sigma} - \frac{2\kappa}{\beta^2}B^\rho B^\sigma\right)F_{\mu\rho}F_{\nu\sigma}\,, \tag{5.37}$$

where B^μ and $\gamma^{\mu\nu}$ are defined in (5.2) and (5.21), and $\kappa \geq 0$ is a parameter which generalizes the action for the massless vector (by putting κ to one, we recover the Maxwell action for a free massless vector).

Using bulk gauge invariance $\delta A_\mu = \partial_\mu \alpha$, we can fix the radial component to vanish: $A_r = 0$. Hence, the residual gauge transformations will only depend on the boundary coordinates t, x_i. The equations of motion obtained by varying (5.37) with respect to A_r, A_t, and A_i respectively are

$$\begin{aligned}
&\kappa\, r^{2z}\partial_r\partial_t A_t - r^2\,\partial_r\partial_i A_i = 0\,,\\
&r^{d-z}\partial_r\big(r^{-d+z+2}\partial_r A_t\big) + r^2\big(\partial_i\partial_i A_t - \partial_t\partial_i A_i\big) = 0\,,\\
&r^{d+z-2}\partial_r\big(r^{-d-z+4}\partial_r A_i\big) - \kappa\, r^{2z}\big(\partial_t^2 A_i - \partial_i\partial_t A_t\big) + r^2\big(\partial_j\partial_j A_i - \partial_i\partial_j A_j\big) = 0\,.
\end{aligned}$$

We will split the vector A_i into a transverse and a longitudinal part[5]:

$$A_i = T_i + \partial_i L\,, \quad \text{with } \partial_i T_i = 0\,. \tag{5.38}$$

Under the residual gauge transformations, we have $\delta T_i = 0$, $\delta L = \alpha$, and $\delta A_t = \partial_t \alpha$. The splitting leads to the following four equations:

$$\kappa\, r^{2z}\partial_r\partial_t A_t - r^2\,\partial_r\partial_i\partial_i L = 0\,, \tag{5.39}$$

$$r^{d-z}\partial_r\big(r^{-d+z+2}\partial_r A_t\big) + r^2\,\partial_i^2\big(A_t - \partial_t L\big) = 0\,, \tag{5.40}$$

$$r^{d+z-2}\partial_r\big(r^{-d-z+4}\partial_r\partial_i L\big) + \kappa\, r^{2z}\,\partial_i\partial_t\big(A_t - \partial_t L\big) = 0\,, \tag{5.41}$$

$$r^{d+z-2}\partial_r\big(r^{-d-z+4}\partial_r T_i\big) - \kappa\, r^{2z}\,\partial_t^2 T_i + r^2\,\partial_j^2 T_i = 0\,. \tag{5.42}$$

Every term in the equations above is gauge invariant.

Starting from (5.37), we now compute the renormalized action. The first step is to reduce the bulk action to a boundary action, using the equations of motion:

$$S = \int \mathrm{d}^{d+1}x\ r^{-d-z}\frac{1}{2}\Big[\kappa\, r^{2+2z}F_{rt}F_{rt} - r^4 F_{ri}F_{ri}\Big]\Big|_{\text{on-shell}} \tag{5.43}$$

$$= -\int_{r=\epsilon}\mathrm{d}^d x\ \frac{1}{2}\Big[\kappa\, r^{-d+z+2}A_t\partial_r A_t - r^{-d-z+4}A_i\partial_r A_i\Big]\,. \tag{5.44}$$

We now use the split (5.38) and the constraint (5.39) to obtain

$$S_{\text{reg}} = -\frac{1}{2}\int_{r=\epsilon}\mathrm{d}^d x\,\Big[\kappa\, r^{-d+z+2}\big(A_t - \partial_t L\big)\partial_r A_t - r^{-d-z+4}T_i\partial_r T_i\Big]\,. \tag{5.45}$$

All terms in the expression above are manifestly gauge invariant.

The next step is to see if there are any divergent terms, and if so, to add the appropriate counterterms to cancel the divergences. This procedure depends on the specific values of d and z, nonetheless we are going to sketch it assuming those values are generic enough.

Let us first consider the timelike/longitudinal part. The purely radial solutions of the

[5]Note that insisting on a transverse splitting over all boundary coordinates would be very inconvenient here (contrarily to the case of the previous chapter, where it is eventually equivalent), since $\partial_m A^m = -r^{2z}\partial_t A_t + r^2\partial_i A_i$, i.e. the r-dependence does not factorize.

equations of motion (5.39)–(5.41) are respectively $A_t \propto 1,\, r^{d-z-1}$ and $L \propto 1,\, r^{d+z-3}$. Hence, from the equations of motion and assuming $z \geq 1$, we expand A_t and L as

$$A_t = a_0 + a_1 r^2 + \ldots + \tilde{a}_0 r^{d-z-1} + \ldots \tag{5.46}$$

$$L = l_0 + l_1 r^{2z} + \ldots + \tilde{l}_0 r^{d+z-3} + \ldots \tag{5.47}$$

where all coefficients depend on the boundary coordinates t, x_i and where we included the first order corrections with the coefficients a_1 and l_1 as well. Under the residual gauge transformations we have that $\delta a_0 = \partial_t \alpha$ and $\delta l_0 = \alpha$. To avoid complications (*i.e.* logarithmic terms and/or alternative quantization, as for instance in [III]), we assume $d - z - 1 > 0$ (equivalently $\tilde{d} > 2z$), which since $z \geq 1$ also implies $d + z - 3 > 0$. Let us notice that the constraint (5.39) imposes

$$\partial_t a_1 = \frac{z}{\kappa} \partial_i^2 l_1 \,, \qquad \kappa\,(d-z-1)\partial_t \tilde{a}_0 = (d+z-3)\partial_i^2 \tilde{l}_0 \,. \tag{5.48}$$

and that (5.40) and (5.41) imply

$$a_1 = \frac{\partial_i^2 \left(a_0 - \partial_t l_0\right)}{2(d-z-3)} \,, \qquad \partial_i l_1 = \frac{\kappa}{z} \frac{\partial_t \partial_i \left(a_0 - \partial_t l_0\right)}{2(d-z-3)} \,. \tag{5.49}$$

The regularized action is

$$S_{\text{reg}}^{(t/L)} = -\kappa \int_{r=\epsilon} \mathrm{d}^d x \left[r^{-d+z+3} \big(a_0 - \partial_t l_0\big)\, a_1 + \ldots + \frac{1}{2}(d-z-1)\big(a_0 - \partial_t l_0\big)\, \tilde{a}_0 \right] . \tag{5.50}$$

where the superscript t/L means that we ignore the second T^i dependent term in (5.45). The first term is divergent if $d - z > 3$. In that case, the dots represent other possibly (less) divergent terms. We note that if $d - z = 3$, there would have been log-divergences that would have to be taken care of. The last term is finite and is the only one involving $\tilde{a}_0$ as all other terms with $\tilde{a}_0$ vanish as $r \to 0$.

If the first term diverges, we have to add a proper local, gauge invariant counterterm, which in this case is

$$S_{\text{ct}}^{(t/L)} = -\int_{r=\epsilon} \mathrm{d}^d x \, \sqrt{-\hat{g}} \left[\frac{\kappa}{2(d-z-3)} F_{ti} F^{ti} \right] , \tag{5.51}$$

where $\hat{g}$ is the induced metric on the boundary. Using the expression for a_1 given in equation (5.49) we obtain the 'renormalized' action for this sector:

$$\begin{aligned} S_{\text{ren}}^{t/L} = S_{\text{reg}}^{t/L} - S_{\text{ct}}^{t/L} = -\kappa \int_{r=\epsilon} d^d x \, \bigg[&\frac{1}{2}(d-z-1)\big(a_0 - \partial_t l_0\big)\tilde{a}_0 + \\ &+ \frac{r^{-d+z+3}}{2(d-z-3)} \partial_t T_i \partial_t T_i \bigg] . \end{aligned} \tag{5.52}$$

We observe that our counterterm succeeds in suppressing the divergent terms in the time-like/longitudinal sector without adding finite terms, but that it leads to an additional term in the transverse sector. This term is divergent, but, as we will see in a moment, it will be canceled by the counterterm in the transverse sector.

So, let us concentrate on the transverse sector. The boundary behaviors of the solutions of the equation of motion (5.42) are $T_i \propto 1,\, r^{d+z-3}$. By using (5.42), T_i can be expanded near the boundary as

$$T_i = t_0 + r^2\, t_1 + \,\dots\, + r^{2z}\, t_z + \,\dots\, + r^{d+z-3}\, \widetilde{t}_0 + \,\dots\,, \tag{5.53}$$

where we are omitting the i-index on the coefficients and where we have

$$t_1 = \frac{1}{2(d+z-5)}\partial_i^2\, t_0\,, \qquad t_z = -\frac{\kappa}{2z(d-z-3)}\partial_t^2\, t_0\,. \tag{5.54}$$

Again, as a matter of simplicity (no logarithmic terms), we will consider $d+z-3 \neq 0, 2, 2z$. The boundary action is then

$$S^T_{\text{on-shell}} = \int_{r=\epsilon} d^d x \left[r^{-d-z+5}\, t_0 t_1 + \,\dots\, + z\, r^{-d+z+3}\, t_0 t_z + \,\dots\, + \frac{1}{2}(d+z-3)\, t_0 \widetilde{t}_0 \right], \tag{5.55}$$

where the superscript T means that we ignore the first time/longitudinal term in (5.45).

If the first term in (5.55) diverges, it can be canceled by the following counterterm

$$S^T_{\text{ct}} = \int_{r=\epsilon} d^d x \,\sqrt{-\hat{g}} \left[-\frac{1}{4}\frac{1}{d+z-5}\, F_{ij}F^{ij} \right], \tag{5.56}$$

which does not generate any finite term containing $\widetilde{t}_0$. Further, if the second term in (5.55) diverges, this part will be canceled by the second (transverse) term in (5.52). Therefore, by combining the counterterms of the two sectors, we obtain the fully renormalized action

$$\begin{aligned} S_{\text{ren}} &= S_{\text{on-shell}} - S^{t/L}_{\text{ct}} - S^T_{\text{ct}} = \\ &= \int d^d x \left[\frac{1}{2}(d+z-3)\, t_0 \widetilde{t}_0 - \frac{1}{2}\kappa(d-z-1)\big(a_0 - \partial_t l_0\big)\, \widetilde{a}_0 \right]. \end{aligned} \tag{5.57}$$

Note that as soon as $z \neq 1$ the two counterterms (5.51) and (5.56) are of the general form that we used in the action (5.37).

Solving the equations of motion for the fluctuations with some bulk boundary conditions would allow us to express $\widetilde{a}_0$ and $\widetilde{t}_0$ in terms of the gauge invariant combinations of the sources, $(a_0 - \partial_t l_0)$ and t_0 respectively, through some non-local function. The sources for the currents are

$$S_{\text{sources}} = \int d^d x \{-a_0 J_t - l_0 \partial_i J_i + t_{i0} J^T_i\}\,. \tag{5.58}$$

This results into

$$\langle J_t J_t \rangle = i\,\kappa\,(d-z-1)\, \frac{\delta \widetilde{a}_0}{\delta a_0}\,, \tag{5.59}$$

$$\langle J^T_i J^T_j \rangle = -i\,(d+z-3)\left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial_k^2}\right)\frac{\delta \widetilde{t}_0}{\delta t_0}\,, \tag{5.60}$$

and the correlators involving $\partial_i J_i$ being proportional to (5.59). Note that due to invariance under the residual gauge transformations there is an operator identity:

$$\frac{\delta S_{\text{ren}}}{\delta l_0} = \partial_t \frac{\delta S_{\text{ren}}}{\delta a_0} \quad \Leftrightarrow \quad \partial_i J_i = \partial_t J_t \, , \tag{5.61}$$

i.e. the current $(-J_t, J_i)$ is conserved.

5.2.1 Two-point function for a Lifshitz vector

Since we have performed the holographic renormalization of the most general bulk action for a massless vector field and derived the formulæ (5.59,5.60) for the two-point functions of the charge densities and currents, we may try, for specific cases, to get analytic expressions for these correlators. They turn out to be very similar to the scalar case.

Again, when $z = 2$ the equations for the fluctuations can be solved analytically. The conditions from the previous subsection for avoiding logarithmic terms imply that d has to be even. Let us start with the transverse part. The Fourier transformed version of equation (5.42) reads (we write $T_i \equiv T$)

$$r^2 \partial_r^2 T - (d-2)\, r \partial_r T + r^4 \tilde{\omega}^2 T - r^2 k_i^2 T = 0 \, . \tag{5.62}$$

where $\tilde{\omega}^2 = \kappa \omega^2$. Again we have an equation of the confluent hypergeometric form, whose solution, that is regular in the interior, is

$$T = C_T \, e^{\frac{i}{2}\tilde{\omega} r^2} \mathrm{U}\left[\frac{i k_i^2}{4\tilde{\omega}} - \frac{d-3}{4}, -\frac{d-3}{2}; -i\tilde{\omega} r^2 \right] . \tag{5.63}$$

Using the expansion of $\mathrm{U}[a,b;x]$ for small small x (5.14), from (5.60) we obtain the correlator

$$\langle J_i^T J_j^T \rangle_0 = 2i \left(\delta_{ij} - \frac{k_i k_j}{k_l^2} \right) (-i\tilde{\omega})^{\frac{d-1}{2}} \frac{\Gamma\left[-\frac{d-3}{2}\right] \Gamma\left[\frac{i k_i^2}{4\tilde{\omega}} + \frac{d+1}{4}\right]}{\Gamma\left[\frac{d-1}{2}\right] \Gamma\left[\frac{i k_i^2}{4\tilde{\omega}} - \frac{d-3}{4}\right]} \, , \tag{5.64}$$

for d even. The scaling dimension is the correct one: taking into account that $[\tilde{\omega}] = 2$, the correlator has dimension $d-1$, which is $2[J_i] - (d+1)$ given that $[J_i] = d$. We notice that it is similar to the scalar one (5.15), and it also has analogous imaginary poles, located at

$$\sqrt{\kappa}\, \omega = \frac{-i\, k_i^2}{4n + d + 1} \, . \tag{5.65}$$

Let us also investigate the $k \to 0$ and $\omega \to 0$ limits of the correlator (5.64). For $k \to 0$ and fixed ω, all the Γ-functions approach constants and the propagator is just proportional to a transverse projector times $\omega^{(d-1)/2}$. For $\omega \to 0$ (so, $\tilde{\omega} \to 0$) at fixed k, we have to employ

Stirling's formula (see (5.30)), leading to

$$\langle J_i^T J_j^T \rangle_0 \simeq 2i\left(\delta_{ij} - \frac{k_i k_j}{k_l^2}\right)\left(\frac{k_l^2}{4}\right)^{\frac{d-1}{2}} \frac{\Gamma\left(-\frac{d-3}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} . \tag{5.66}$$

Up to a prefactor both limits are fixed by scaling considerations and the fact that the correlator is proportional to a projector.

We can also consider the temporal/longitudinal sector. Let us start by establishing an ordinary differential equation for the temporal part. If we call

$$\dot{A} \equiv \partial_t A_t , \qquad \dot{L} \equiv \partial_i \partial_i L , \tag{5.67}$$

then the equations (5.39), (5.40) and (5.41) become

$$\kappa\, r^{-d+z+2} \partial_r \dot{A} - r^{-d-z+4} \partial_r \dot{L} = 0 , \tag{5.68}$$
$$\partial_r\left(r^{-d+z+2} \partial_r \dot{A}\right) + r^{-d+z+2}\left(\partial_i \partial_i \dot{A} - \partial_t^2 \dot{L}\right) = 0 , \tag{5.69}$$
$$\partial_r\left(r^{-d-z+4} \partial_r \dot{L}\right) + \kappa\, r^{-d+z+2}\left(\partial_i \partial_i \dot{A} - \partial_t^2 \dot{L}\right) = 0 , \tag{5.70}$$

the third being obviously redundant. The first equation gives

$$\partial_r \dot{L} = \kappa\, r^{2z-2} \partial_r \dot{A} , \tag{5.71}$$

so that we eventually have an equation for $\dot{A}$ only:

$$\partial_r\left(r^{d-z-2} \partial_r\left(r^{-d+z+2} \partial_r \dot{A}\right)\right) + \partial_i^2 \partial_r \dot{A} - \kappa\, r^{2z-2} \partial_t^2 \partial_r \dot{A} = 0 . \tag{5.72}$$

This is a second order equation for $\dot{A}' \equiv \partial_r \dot{A}$:

$$r^2 \partial_r^2 \dot{A}' - (d-z-2)\, r \partial_r \dot{A}' + (d-z-2)\dot{A}' - \kappa\, r^{2z} \partial_t^2 \dot{A}' + r^2 \partial_i \partial_i \dot{A}' = 0 . \tag{5.73}$$

Performing a Fourier transform and setting $z = 2$, the later equation becomes

$$r^2 \partial_r^2 \dot{A}' - (d-4)\, r \partial_r \dot{A}' + (d-4)\, \dot{A}' + r^4 \tilde{\omega}^2 \dot{A}' - r^2 k_i^2 \dot{A}' = 0 , \tag{5.74}$$

which is again of the confluent hypergeometric form. Thus, we obtain

$$\dot{A}' = C_A\, e^{\frac{i}{2}\tilde{\omega} r^2}\, r\, \mathrm{U}\left[\frac{i k_i^2}{4\tilde{\omega}} - \frac{d-7}{4}, -\frac{d-7}{2}; -i\tilde{\omega} r^2\right] . \tag{5.75}$$

One must be careful with the expansions, because leading and subleading orders can get inverted. This happens for example for $d = 4$, $1 - b = -\frac{1}{2}$, where the $\tilde{a}_0$ term is more leading than a_1, with both being subleading to the constant term a_0. However their expansions do not mix and one can still extract the coefficients.

The two leading modes are identified as follows:

$$2\omega\, a_1 = iC_A\,\frac{d-7}{2}\,\frac{\Gamma\left[\frac{d-7}{2}\right]}{\Gamma\left[\frac{ik_i^2}{4\widetilde{\omega}}+\frac{d-3}{4}\right]}\,,\qquad (d-3)\omega\,\widetilde{a}_0 = iC_A\,(-i\widetilde{\omega})^{\frac{d-5}{2}}\,\frac{\Gamma\left[-\frac{d-5}{2}\right]}{\Gamma\left[\frac{ik_i^2}{4\widetilde{\omega}}-\frac{d-7}{4}\right]}$$

so that, using (5.49), we get

$$(d-3)\widetilde{a}_0 = -\frac{2}{(d-5)(d-7)}\,k_i^2(-i\widetilde{\omega})^{\frac{d-5}{2}}\,\frac{\Gamma\left[-\frac{d-5}{2}\right]\Gamma\left[\frac{ik_i^2}{4\widetilde{\omega}}+\frac{d-3}{4}\right]}{\Gamma\left[\frac{d-7}{2}\right]\Gamma\left[\frac{ik_i^2}{4\widetilde{\omega}}-\frac{d-7}{4}\right]}\left(a_0+i\omega l_0\right)\,. \tag{5.76}$$

Eventually, from (5.59),

$$\left\langle J_t(k)J_t(-k)\right\rangle_0 = -\frac{i}{2}\,\kappa\,k_i^2(-i\widetilde{\omega})^{\frac{d-5}{2}}\,\frac{\Gamma\left[-\frac{d-5}{2}\right]\Gamma\left[\frac{ik_i^2}{4\widetilde{\omega}}+\frac{d-3}{4}\right]}{\Gamma\left[\frac{d-3}{2}\right]\Gamma\left[\frac{ik_i^2}{4\widetilde{\omega}}-\frac{d-7}{4}\right]}\,. \tag{5.77}$$

One expects the above correlator to have dimension $2[J_t]-(d+1)=d-3$, which it has. It also displays poles for

$$\sqrt{\kappa}\,\omega = \frac{-i\,k_i^2}{4n+d-3}\,. \tag{5.78}$$

In the $\omega\to 0$ limit, the correlator (5.77) becomes

$$\left\langle J_t(k)J_t(-k)\right\rangle_0 = -2i\kappa\left(\frac{k_i^2}{4}\right)^{\frac{d-3}{2}}\,\frac{\Gamma\left[-\frac{d-5}{2}\right]}{\Gamma\left[\frac{d-3}{2}\right]}\,, \tag{5.79}$$

which is finite. It is to be noted that this is in line with the relativistic case, where the correlators of a conserved current are finite for $\omega\to 0$ and fixed k.

We have thus shown that a general massless vector in a Lifshitz background presents two-point correlators completely analogous to the ones produced by a Klein-Gordon scalar in the same background. They display the same analytic structure, comprising a cut ending at the origin and an accumulation of poles on the upper imaginary axis. It is also worth noting that for the vector, this is the most general result, since we do not have the possibility (without introducing other field content) to generalize the bulk action with Schrödinger-like terms.

5.3 Lifshitz holography for symmetry breaking

We are finally ready to study the physics of symmetry breaking for a Lifshitz invariant theory in a holographic setup. Our main goal is to retrieve the correct non-relativistic Ward identities for symmetry breaking.

To this end we will consider the most general U(1)-invariant action for a complex scalar field coupled to a massless vector field on a non-dynamical Lifshitz background. This amounts to combining the actions for the most general scalar (5.22) and for the most general massless vector (5.37), and replacing ordinary derivatives with covariant derivatives, leading to:

$$S=\int \mathrm{d}^{d+1}x\ \sqrt{-g}\,\Bigg[-\frac{1}{4}\gamma^{\mu\nu}\Big(\gamma^{\rho\sigma}-\frac{2\kappa}{\beta^2}B^\rho B^\sigma\Big)F_{\mu\rho}F_{\nu\sigma}+ \\ -\gamma^{\mu\nu}D_\mu\phi^* D_\nu\phi+\frac{1}{c^2\beta^2}B^\mu D_\mu\phi^* B^\nu D_\nu\phi+ \\ -\frac{ih}{2\beta}B^\mu\Big(\phi^* D_\mu\phi-\phi D_\mu\phi^*\Big)-m^2\phi^*\phi\Bigg]\,, \tag{5.80}$$

where $\gamma_{\mu\nu}$ and B_μ are defined in (5.21,5.2), and

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,, \\ D_\mu=\partial_\mu-iA_\mu\,.$$

We will partially fix the gauge freedom by imposing radial gauge, i.e. $A_r=0$, like we did in Section 5.2. The equations of motion, obtained by varying the action (5.80) with respect to A_r, A_i, A_t, and ϕ^* respectively, are

$$r^2\partial_i\partial_r A_i-\kappa r^{2z}\partial_t\partial_r A_t+i\left(\phi^*\partial_r\phi-\phi\partial_r\phi^*\right)=0\,, \tag{5.81}$$

$$r\partial_r\big(r\partial_r A_i\big)-\big(\tilde{d}-2\big)r\partial_r A_i+r^2\Big(\partial_j^2A_i-\partial_i\partial_jA_j\Big)-\kappa r^{2z}\left(\partial_t^2A_i-\partial_i\partial_tA_t\right)+ \\ -i\left(\phi^*\partial_i\phi-\phi\partial_i\phi^*\right)-2\phi^*\phi A_i=0\,, \tag{5.82}$$

$$r\partial_r\big(r\partial_r A_t\big)-\big(\tilde{d}-2z\big)r\partial_r A_t+r^2\left(\partial_i^2A_t-\partial_t\partial_iA_i\right)+ \\ -\frac{i}{\kappa c^2}\left(\phi^*\partial_t\phi-\phi\partial_t\phi^*\right)-\frac{2}{\kappa c^2}\phi^*\phi A_t-\frac{h}{\kappa}r^{-z}\phi^*\phi=0\,, \tag{5.83}$$

$$r\partial_r\big(r\partial_r\phi\big)-\tilde{d}\,r\partial_r\phi-m^2\phi+r^2\left(\partial_i^2\phi-2iA_i\partial_i\phi-i(\partial_iA_i)\phi-A_i^2\phi\right)+ \\ -\frac{r^{2z}}{c^2}\left(\partial_t^2\phi-2iA_t\partial_t\phi-i(\partial_tA_t)\phi-A_t^2\phi\right)+ihr^z\big(\partial_t\phi-iA_t\phi\big)=0\,. \tag{5.84}$$

In order to trigger the breaking of the U(1) global symmetry on the boundary, we assign a background profile ϕ_B to the scalar, which can be taken to be real by virtue of the symmetry. In order not to break boundary spacetime symmetries, we will construct a set of (t,x^i)-independent solutions (ϕ_B, A_{Bt}, A_{Bi}) of the equations of motion where $A_{Bi}=0$ and $\phi_B\in\mathbb{R}$. As we will see a posteriori, we have to consider $A_{Bt}\neq 0$ otherwise the scalar background has to be zero. With these specific requirements for the solutions, equations (5.81) and (5.82) are trivially satisfied and (5.83), (5.84) provide respectively:

$$r\partial_r\big(r\partial_r A_{tB}\big)-\big(\tilde{d}-2z\big)r\partial_r A_{tB}-\frac{2}{\kappa c^2}\phi_B^2A_{tB}-\frac{h}{\kappa}r^{-z}\phi_B^2=0\,, \tag{5.85}$$

$$r\partial_r\big(r\partial_r\phi_B\big)-\tilde{d}\,r\partial_r\phi_B-m^2\phi_B+\frac{r^{2z}}{c^2}A_{tB}^2\phi_B+hr^zA_{tB}\phi_B=0\,. \tag{5.86}$$

We see that, whenever $h\neq 0$, on the one hand, if we want to set $A_{tB}\equiv 0$, then the first of the two equations is not satisfied, save for a trivial scalar background $\phi_B \equiv 0$. On the other hand, if A_{tB} has a constant term, *i.e.* a chemical potential (see eq. (4.3)), again the scalar profile must vanish for eq. (5.85) to be satisfied. Let us then set $A_{tB} = 0$ in (5.86), so that we have the usual near-boundary solution

$$\phi_B = w\, r^{\frac{\tilde{d}}{2}-\nu} + v\, r^{\frac{\tilde{d}}{2}+\nu} \,. \tag{5.87}$$

where $\tilde{d}$ and ν are the same as defined in Section 5.1, eq. (5.5). We choose $\nu \in]0,1[$, so that we will not have logarithms in the boundary action[6], but in this way we also guarantee that the term $r^z A_{tB}\phi_B$ in (5.86) does not affect the profile, since $\frac{\tilde{d}}{2} - \nu + z > \frac{\tilde{d}}{2} + \nu$ is always true in our range of ν.

Then, the profile (5.87), through the term $r^{-z}\phi_B^2$ in (5.85), sources a background profile for the temporal component of the vector, namely

$$A_{tB} = A_B\, r^{\tilde{d}-z-2\nu} + \ldots \qquad \text{with } \; A_B = \frac{h}{\kappa} w^2 \,. \tag{5.88}$$

We will see that such background profile is subleading enough in order to disappear from the boundary action if $d > 3$, but for the moment we will keep track of it. Finally, let us stress that for the full Lifshitz case $h = 0$, the problematic terms in the equations (5.85-5.86) are not there, and A_{tB} can be set to zero without any issue.

We will now treat the fields as small fluctuations over these background profiles, expanding the action up to quadratic term in the fluctuations. For the scalar fluctuations we also split into real and imaginary part, so

$$\phi = \frac{\phi_B + \rho + i\pi}{\sqrt{2}} \,. \tag{5.89}$$

For the moment we keep both the leading mode with coefficient w and the subleading mode with coefficient v of the scalar profile, corresponding respectively to explicit and spontaneous breaking in ordinary quantization, and vice versa in alternative quantization. We can switch off the appropriate explicit breaking term when studying the case of spontaneous symmetry breaking. The fluctuations around the time component of the background gauge field will again be denoted by A_t. We hope that this will not lead to any confusion. Finally, we split the spatial component of the gauge field into a transverse and a longitudinal part,

$$A_i = T_i + \partial_i L \,, \quad \text{with } \; \partial_i T_i = 0 \,.$$

Since we are interested in an action that be quadratic in the fluctuations, we keep equations motions that are linear in the fluctuations, which read (by also using the equations for

[6] Also $\nu > 1$ would be fine, but we want to stay in the window where alternative quantization is possible.

the background profiles):

$$r^2\partial_i^2\, r\partial_r L - \kappa\, r^{2z}\partial_t\, r\partial_r A_t - \phi_B\, r\partial_r\pi + \pi\, r\partial_r\phi_B = 0 \tag{5.90}$$

$$r\partial_r(r\partial_r T_i) - (\tilde{d}-2)\, r\partial_r T_i - \kappa\, r^{2z}\partial_t^2 T_i + r^2\partial_j^2 T_i - \phi_B^2 T_i = 0 \tag{5.91}$$

$$r\partial_r(r\partial_r L) - (\tilde{d}-2)\, r\partial_r L - \phi_B^2 L + \kappa\, r^{2z}\partial_t\big(A_t - \partial_t L\big) + \phi_B\pi = 0 \tag{5.92}$$

$$\begin{aligned} r\partial_r(r\partial_r A_t) - (\tilde{d}-2z)\, r\partial_r A_t - \kappa^{-1}c^{-2}\phi_B^2 A_t - 2\kappa^{-1}c^{-2}\rho\phi_B A_{tB} +& \\ + r^2\partial_j^2\big(A_t - \partial_t L\big) + \kappa^{-1}c^{-2}\phi_B\partial_t\pi - \kappa^{-1}h\, r^{-z}\phi_B\rho = 0& \end{aligned} \tag{5.93}$$

$$\begin{aligned} r\partial_r(r\partial_r\rho) - \tilde{d}\, r\partial_r\rho - \Big(m^2 - r^2\partial_j^2 + c^{-2}r^{2z}\partial_t^2 - c^{-2}r^{2z}A_{tB}^2 - h\, r^z A_{tB}\Big)\rho +& \\ + h\, r^z\big(A_t\phi_B - \partial_t\pi\big) = 0& \end{aligned} \tag{5.94}$$

$$\begin{aligned} r\partial_r(r\partial_r\pi) - \tilde{d}\, r\partial_r\pi - \Big(m^2 - r^2\partial_j^2 + c^{-2}r^{2z}\partial_t^2 - c^{-2}r^{2z}A_{tB}^2 - h\, r^z A_{tB}\Big)\pi +& \\ + h\, r^z\partial_t\rho - \phi_B\big(r^2\partial_j^2 L - c^{-2}r^{2z}\partial_t A_t\big) + 2c^{-2}r^{2z}A_{tB}\partial_t\rho = 0& \end{aligned} \tag{5.95}$$

We see that the equation for T_i decouples, whereas we get a system of coupled equations for ρ, π, L, and A_t. If $h = 0$, in which case we take $A_{Bt} = 0$, the real part ρ decouples as well.

Using these equations of motion the part of the action that is quadratic in the fluctuations can be put on-shell and reduced to a boundary term:

$$\begin{aligned} S_{\text{on-shell}} = \frac{1}{2}\int_{r=\epsilon} \mathrm{d}^d x\, r^{-\tilde{d}}\Big[r^2 T_i\, r\partial_r T_i - r^2 L\, r\partial_r\partial_j^2 L - \kappa\, r^{2z}A_t\, r\partial_r A_t +& \\ - 2\kappa\, r^{2z}A_t r\partial_r A_{tB} + 2\rho\, r\partial_r\phi_B + \rho\, r\partial_r\rho + \pi\, r\partial_r\pi\Big]& \end{aligned} \tag{5.96}$$

where the term containing A_{Bt} actually vanishes in the near-boundary expansion of the fields by virtue of the assumption $z > 2\nu$.

We now have to study the divergent pieces in the regularized action that need to be renormalized. Since we have already done this for the scalar and the vector when they are decoupled, we need only focus on the effects on the procedure of the presence of the background profiles Φ_B and A_{Bt}.

5.3.1 Holographic renormalization and Ward identities

We start by considering the components of the gauge field. From equations (5.91–5.93) we can see that the generic backgrounds (5.87–5.88) impact the asymptotic expansions by the following terms:

$$T_i = t_{i0} + r^{\alpha_T}\, t_{i1} + \ldots + t_{i(z)}r^{2z} + \ldots + r^{\tilde{d}-2}\big(\tilde{t}_{i0} + \ldots\big)\,, \tag{5.97}$$

$$L = l_0 + r^{\alpha_L}\, l_1 + \ldots + r^{\tilde{d}-2}\big(\tilde{l}_0 + \ldots\big)\,, \tag{5.98}$$

$$A_t = a_0 + r^{\alpha_A}\, a_1 + \ldots + r^{\tilde{d}-2z}\big(\tilde{a}_0 + \ldots\big)\,. \tag{5.99}$$

where the first term appearing in the expansion for A_t above is present only when $h \neq 0$ (it is due to the last term in (5.93)).

It is straightforward to see that none of these background-dependent coefficients will survive in (5.96), provided we stick to the (simplifying) assumptions $\nu < 1$ and $z > 2\nu$. Therefore, the renormalization of the vector sector goes through exactly in the same way as for the free vector discussed in the previous section. Namely, there will be counterterms in the transverse sector if $\widetilde{d} > 4$ (i.e. $d + z - 5 > 0$), and in the timelike/longitudinal one if $\widetilde{d} > 2 + 2z$ (i.e. $d - z - 3 > 0$). In any case, such counterterms (5.51) and (5.56) do not affect the finite part of the action.

To complete the renormalization of the action (5.96), now we only need to remove the scalar divergences. Note that the expansions for ρ and π are the same as in (5.5), even in presence of the background. The scalar divergences are then cured by the following standard counterterm, which includes the effect of the background profile (but removing the zeroth order term)

$$
\begin{aligned}
S^{\phi}_{\mathrm{ct}} &= \frac{\widetilde{d}}{2}\,(1-\nu)\int_{r=\epsilon} \mathrm{d}^d x\,\sqrt{-\hat{g}}\,\left(\phi^*\phi - \frac{\phi_B^2}{2}\right) \\
&= \frac{\widetilde{d}}{4}(1-\nu)\int_{r=\epsilon} \mathrm{d}^d x\,\sqrt{-\hat{g}}\,\left(\rho^2 + 2\phi_B\rho + \pi^2\right).
\end{aligned}
\tag{5.100}
$$

The final renormalized action would thus be[7]

$$
\begin{aligned}
S_{\mathrm{ren}} = \frac{1}{2}\int \mathrm{d}^d x\,\Big[&(\widetilde{d}-2)\,t_{i0}\widetilde{t}_{i0} - (\widetilde{d}-2)\,l_0\partial_i^2\widetilde{l}_0 - \kappa\,(\widetilde{d}-2z)\,a_0\widetilde{a}_0 + \\
&+ 2\nu\left(\rho_0\widetilde{\rho}_0 + 2\nu\,\rho_0 + \pi_0\widetilde{\pi}_0\right)\Big],
\end{aligned}
\tag{5.101}
$$

which, using the constraint (5.90), which reads

$$
(\widetilde{d}-2)\,\partial_i^2\widetilde{l}_0 - \kappa\,(\widetilde{d}-2z)\,\partial_t\widetilde{a}_0 + 2\nu\left(\nu\pi_0 - w\widetilde{\pi}_0\right) = 0\,, \tag{5.102}
$$

can be eventually rewritten as

$$
\begin{aligned}
S_{\mathrm{ren}} = \frac{1}{2}\int \mathrm{d}^d x\,\Big[&(\widetilde{d}-2)\,t_{i0}\widetilde{t}_{i0} - \kappa\,(\widetilde{d}-2z)\left(a_0 - \partial_t l_0\right)\widetilde{a}_0 + \\
&+ 2\nu\left(\rho_0\widetilde{\rho}_0 + \left(\pi_0 - w\,l_0\right)\left(\widetilde{\pi}_0 - \nu\,l_0\right) + 2\nu\left(\rho_0 + \pi_0 l_0 - \tfrac{1}{2}w\,l_0 l_0\right)\right)\Big].
\end{aligned}
\tag{5.103}
$$

From this renormalized action it is straightforward to recognize the Ward identities for symmetry breaking, it is sufficient to express the action only in term of the gauge invariant combinations of the sources. Recall the non-trivial gauge transformations are

$$
\delta l_0 = \alpha\,, \qquad \delta a_0 = \partial_t\alpha\,, \qquad \delta\pi_0 = w\alpha\,, \qquad \delta\widetilde{\pi}_0 = \nu\alpha\,. \tag{5.104}
$$

[7] We keep general d and z, since, once the divergences are taken care of, the form of the renormalized action is the same, provided that there are no logarithms (so no scheme dependent pieces).

Then, through the following identifications,

$$\begin{aligned} \widetilde{t}_{i0} &= f_t(\Box)\, t_{i0} \,, \\ \widetilde{a}_0 &= f_a(\Box)\big(a_0 - \partial_t l_0\big) + g_a(\Box)\big(\pi_0 - w l_0\big) + h\, h_a(\Box)\rho_0 \,, \\ \widetilde{\pi}_0 - v l_0 &= f_\pi(\Box)\big(\pi_0 - w l_0\big) + g_\pi(\Box)\big(a_0 - \partial_t l_0\big) + h\, h_\pi(\Box)\rho_0 \,, \\ \widetilde{\rho}_0 &= f_\rho(\Box)\,\rho_0 + h\, g_\rho(\Box)\big(a_0 - \partial_t l_0\big) + h\, h_\rho(\Box)\big(\pi_0 - w l_0\big) \,, \end{aligned} \tag{5.105}$$

we obtain indeed

$$\begin{aligned} S_{\text{ren}} = \frac{1}{2}\int \mathrm{d}^d x \Big[&(\widetilde{d}-2)\, t_{i0} f_t(\Box)\, t_{i0} - \kappa\,(\widetilde{d}-2z)\big(a_0 - \partial_t l_0\big) f_a(\Box)\big(a_0 - \partial_t l_0\big) + \\ &+ \big(\pi_0 - w l_0\big)\Big(2\nu\, g_\pi(\Box) - \kappa\,(\widetilde{d}-2z)\, g_a(\Box)\Big)\big(a_0 - \partial_t l_0\big) + \\ &+ h\rho_0 \Big(2\nu\, g_\rho(\Box) - \kappa\,(\widetilde{d}-2z)\, h_a(\Box)\Big)\big(a_0 - \partial_t l_0\big) + \\ &+ 2\nu\Big(\rho_0 f_\rho(\Box)\,\rho_0 + 2v\,\big(\rho_0 + \pi_0 l_0 - \tfrac{1}{2} w\, l_0 l_0\big)\Big) \\ &+ 2h\nu\,\rho_0\Big(h_\pi(\Box) + h_\rho(\Box)\Big)\big(\pi_0 - w l_0\big) + \\ &+ 2\nu\,\big(\pi_0 - w l_0\big) f_\pi(\Box)\big(\pi_0 - w l_0\big)\Big)\Big] \,. \end{aligned} \tag{5.106}$$

We point out that the presence of $h \neq 0$, which couples ρ to π, A_t, L, has forced us to introduce three additional non-local functions that are not present at $h = 0$ (see [I] for an analogous example). However, since there is no explicit dependence on h in the on-shell action (5.96), the terms that are bilinear in π_0 and $\big(a_0 - \partial_t l_0\big)$ are the same both for $h \neq 0$ and $h = 0$. So, the Ward identities for symmetry breaking are smoothly recovered in both cases. Indeed, we have

$$\begin{aligned} \big\langle \partial_i J_i(x)\, \mathrm{Im}\mathcal{O}(0)\big\rangle_0 &= i\,\frac{\delta^2 S_{\text{ren}}}{\delta l_0 \delta \pi_0} \\ &= i\Big[2\,\nu\big(v - w\, f_\pi(\Box)\big) + \Big(\nu\, g_\pi(\Box) - \frac{\kappa}{2}(\widetilde{d}-2z)\, g_a(\Box)\Big)\partial_t\Big]\delta(x) \,, \end{aligned} \tag{5.107}$$

$$\big\langle J_t(x)\, \mathrm{Im}\mathcal{O}(0)\big\rangle_0 = i\,\frac{\delta^2 S_{\text{ren}}}{\delta a_0 \delta \pi_0} = i\Big(\nu\, g_\pi(\Box) - \frac{\kappa}{2}(\widetilde{d}-2z)\, g_a(\Box)\Big)\delta(x) \,; \tag{5.108}$$

and consequently

$$-\big\langle \partial_t J_t(x)\, \mathrm{Im}\mathcal{O}(0)\big\rangle_0 + \big\langle \partial_i J_i(x)\, \mathrm{Im}\mathcal{O}(0)\big\rangle_0 = i\, 2\,\nu\big(v - w\, f_\pi(\Box)\big)\delta(x) \,, \tag{5.109}$$

which is, as announced, the usual Ward identity for concomitant spontaneous and explicit symmetry breaking, completely identical to the one derived in the relativistic holographic models of Chapter 3 (see eq. (3.32) and eq. (3.107)).

Main results

In this dissertation we have accomplished a quite complete journey, deriving Ward identities and analytic expressions for two-point functions in holography, from the paradigmatic relativistic case to some non-relativistic examples. We summarize here the main achievements collected on the road.

In Part I we reviewed the basic material needed for the understanding of the second part, where the original contributions of this thesis are contained. In the first chapter we discussed, from a field theory perspective, in relativistic as in non-relativistic contexts, various features of the physics of symmetry breaking, which then were recovered in the holographic setups of Part II. Then, in Chapter 2, we briefly introduced the holographic correspondence and the prescription for computing field theory correlation functions, and we presented the procedure of holographic renormalization through a paradigmatic example, discussing some peculiarities as alternative quantization and scheme dependence.

Then, in the second part, various holographic realization of quantum field theories with symmetry breaking were presented, both in relativistic and non-relativistic frameworks. In Section 3.1, the Ward identity structure and symmetry breaking pattern were neatly embodied in a simple and prototypic bulk toy-model for an abelian U(1) symmetry breaking. The precise relations among correlators are dictated by the field theory arguments of Section 1.2, which pinpoint the Ward identity structure independently of the strength of the coupling, and are realized in the AdS/CFT model thanks to holographic renormalization. The holographic derivation relies just on an asymptotic near-boundary analysis, therefore, as far as Ward identities are concerned, only UV knowledge is necessary, and we could indeed perform the analysis before actually solving the model and discussing its IR properties.

In turn, in order to access quantitative data such as masses and residues, the IR properties are crucial and hence solving for bulk fluctuations becomes necessary. We thus explicitly studied the toy-model which allows for complete analytic control of its solutions and the dual correlators. Our analytic correlator exhibited the Goldstone boson massless pole (with a numeric residue specific to the theory) in the purely spontaneous case, and a hierarchically lighter massive pole in the case where a little explicit breaking intervenes. In this latter case, the squared mass of the pseudo-Goldstone reproduces the expected GMOR relation, linear in the explicit breaking parameter.

Thus holography, already in such a basic realization, is able to reproduce general quantum field theoretical expectations and allows explicit quantitative computations. Even if analytic control was possible only for specific space-time dimensions and scaling dimension of the dual scalar operator, we expect the results of this analytic study to hold at the qualitative level for higher space-time dimensions and generic scaling dimensions.

The lower dimensional case was discussed in Section 3.2, where we verified from the holographic point of view the fact that in the strict large N limit spontaneous symmetry breaking can occur in two dimensions. Indeed, considering the AdS_3/CFT_2 version of the model of Section 3.1, we retrieved the same Ward identities as they appear in higher dimensions. Nevertheless, the way to get this result involves subtleties and peculiarities which are specific to two dimensions, and can be regarded as a premonition of the fact that spontaneous breaking is ruled out as soon as one moves away from strict infinite N. The most crucial subtlety is that the gauge field has to be renormalized in a sort of 'alternative quantization', in order to have it properly sourcing a conserved current and yielding the correct Ward identities for the breaking of a global symmetry on the boundary.

In a fashion inspired by [113], quantum corrections can be taken into account, by computing a bulk tadpole correction to the scalar profile. This may reproduce the infrared divergence which is responsible for preventing the formation of a vacuum expectation value, that is the core mechanism of Coleman's no-Goldstone theorem. Such a quantum effect, by the holographic correspondence, is equivalent to a $1/N$ correction in the boundary theory.

However, if the existence of Goldstone bosons in two dimensions is under threat, that is absolutely not the case for pseudo-Goldstone bosons. Indeed, in 1+1 dimensions, the large quantum fluctuations of the phase prevent the selection of a specific ground state out of the continuum of possible vacua; however, if we added an arbitrarily small (but finite) explicit breaking, this would select a particular ground state, and act as a regulator for the infra-red divergence, making such vacuum stable under quantum fluctuations. Hence, for explicit breaking parametrically smaller than the spontaneous one, we expect (even at finite N) a mode that is hierarchically lighter than the rest of the spectrum, and whose mass undergoes the usual GMOR relation. So, there are no obstruction for pseudo-Goldstone bosons in two dimensions, and we provided a holographic model for them.

Then, Chapter 4 and 5 were dedicated to the non-relativistic examples. In Section 4 we adopted a non-abelian version of the relativistic model of Section 3.1, with a global U(2) symmetry, and we broke explicitly Lorentz invariance by giving a background profile to the temporal component of the gauge field. In this way we studied the consequences of the spontaneous breaking of U(2) to U(1), in a strongly coupled field theory where a chemical potential breaks explicitly boost invariance, in a holographic dual field theory with the same symmetry breaking pattern of the toy-model of Section 1.3.1.

Performing the holographic renormalization of this setup, we showed holographically how the Ward identities associated to the broken symmetries are reproduced in a non-relativistic framework. So the system must have Goldstone excitations associated to the three broken generators. However, because of broken Lorentz symmetry, not only the scalar operator acquired a symmetry breaking vev's, but also the temporal components of one of the three conserved currents. We were thus exactly in the situation where type B Goldstone bosons arise, *i.e.* when the commutator of two broken charges has a non-vanishing vev, and only one massless excitation is associated to these two broken generators. The third broken generator is not involved in commutators with non-trivial VEVs, and hence gives rise to a type A Goldstone boson.

From the field theoretical analysis of Section 1.3, we expect the type B GB to have quadratic dispersion relation and the type A GB to have linear dispersion relation, but with a velocity smaller than the speed of light c. Moreover, we expect the type B Goldstone boson to be accompanied by an almost Goldstone boson, *i.e.* a light mode whose mass is related to the coefficient of the quadratic dispersion relation of its partner [12]. Unfortunately, due to littler symmetry and a much more involved system of equations of motion, contrary to Section 3.1, we were not able to find analytic expressions where such dispersion and mass relations would appear.

Finally, in Chapter 5, we studied conserved currents and charged scalar operators in Lifshitz invariant space-times. We first discussed a free scalar field, computing explicit expressions for the two-point function for various bulk actions and different values of the critical exponent. A peculiar feature of the Lifshitz Klein-Gordon scalar correlator for $z = 2$ is the presence of purely imaginary poles, which seems related to the tidal singularity at the center of Lifshitz geometry. We tried to cure this by cutting out the singularity with hard-wall boundary conditions, which have analogous consequences as those for AdS space-time (appearance of a discrete spectrum). The imaginary poles seem to be shifted away from the imaginary axis, in the complex plan, though surviving.

Then we performed in detail the holographic renormalization for the most general gauge vector field on Lifshitz background, and we computed the two-point functions, which turn out to be completely similar to the scalar ones. We finally put together the pieces, coupling a charged scalar to a gauge field, and giving a background profile to the scalar field in order to trigger symmetry breaking on the boundary. After renormalization, we obtain again the proper non-relativistic realization of Ward identities for symmetry breaking.

The study of these various holographic setups allowed a clearer understanding of the physics of symmetry breaking in quantum field theory, especially in the less well-defined non-relativistic framework. At the same time, it gave the occasion to explore the holographic correspondence in a variety of canonical and less canonical examples. The implementation of holographic renormalization to those various examples has encountered some specificities and subtleties, which have been extensively discussed, engendering a deeper understanding of the technique.

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