THÈSE DE DOCTORAT
SORBONNE UNIVERSITÉ

Spécialité : Physique

École doctorale n°564: Physique en Île-de-France

réalisée

au Laboratoire de Physique Théorique et Hautes Energies

sous la direction de Marc Bellon

présentée par

Enrico Russo

pour obtenir le grade de :

DOCTEUR SORBONNE UNIVERSITÉ

Sujet de la thèse :

Ward Schwinger Dyson and resurgence

soutenue le 18/12/2020

devant le jury composé de :

Mme.	Sylvie Paycha	Examinatrice
M.	Dominique Manchon	Rapporteur
Mme.	Karen Yeats	Rapportrice
M.	Christian Brouder	Examinateur
M.	Loïc Foissy	Examinateur
M.	Marc Bellon	Directeur de thèse

Contents

I – Introduction

It is almost 90 years from what is recognised as the first attempt to formulate a quantum field theory [25] and still we crack our heads around it. It is the most tested theory we have at our disposal, with predictions that agree up to 0.12 part per billion [48], but still we do not have a clear interpretation of it. It can not be cast in axiomatic form, at least not with unanimous agreement (yet) but the attempt to formalise it has led to the development of vast area of mathematics. Consider for instance the case of non commutative geometry: absolutely rooted in Von Neumann theories of quantum mechanics it has changed the perception of functional spaces, algebra spectra, fractals, with even cosmological perspectives [41, 39]. Quantum field theory poses problems of different kind:

- combinatorial : what are the most appropriate algebraic objects we can deploy to describe how objects of the theory combine together?
- analytical : what is the structure of singularities involved and what are the functional objects appropriate with that?
- numerical : what kind of numbers does the theory produce? What are the relationships between them?

and many of them are worth a lifetime of research. In this thesis I will focus on a solution scheme to Schwinger Dyson equations I put forward with my supervisor Marc Bellon. A *fil rouge* connects the work of Broadhurst and Kreimer [14] in which the authors initiated the study of Schwinger Dyson equations with Hopf algebraic techniques; the work of Kreimer and Yeats [37] in which they generalised the case to a non-linear kind of Schwinger Dyson equations; the work of Bellon and Schaposnik [2] where the Wess-Zumino case was fully solved; and the work of Bellon and Clavier [5] where resurgent techniques have been introduced. This thesis is thus the latest stone in a long road. During my PhD we have developed tools to tackle non-linear Schwinger Dyson equations where also the vertices appear as full renormalised quantities. We have done that for the case of a massless scalar field model with cubic interactions in its critical dimension. We have characterised the solution with tools from Resurgent analysis with the aim to extract a non perturbative description. Our method was able to replicate calculations available in literature of the beta function and raised an interesting problem: in the linear Yukawa and in the Wess-Zumino case, it is possible to resum the different trans-series contributions; in the fully interactive case this is hindered by the appearance of non-entire powers and the degeneracy of some trans-series contributions. Like a bust of Janus this thesis has two faces: Schwinger Dyson equations and Resurgence theory, the hard stone is Renormalisation.

Renormalisation plays a central role in understanding quantum field theory, and currently renormalisation techniques are spreading in different areas from stochastic analysis [17] to information theory [38]. In quantum field theory the introduction of renormalisation traces back to Petermann and Stueckelberg [52], was crucial to the development of quantum electro dynamics, but it was with the work of Wilson [55], linking statistical mechanics with high energy physics, that it was recognised as a universal approach to physical problems. In Wilson's point of view there is no such thing as the "absolute charge" or the "absolute mass" of a particle. One could be tempted to define them as the outcome of a measurement performed if the system was isolated; but quantum field theory teaches us that the notion of isolated system depends on the scale at which we are probing it. These properties thus emerge from interactions. No wonder that when we write a Lagrangian to represent these interactions, the parameters we use to describe them are subject to change according to a reference scale. Sometimes there are physical cut-off: for example we might want to describe a physical system modelled by a lattice at its critical point; Wilson showed us that a quantum field theory provides a very good description of the correlations occurring in this system and in this context a natural candidate for regulator would be the lattice spacing. He showed that quantum field theories provide effective descriptions of physical models, and that there is a notion of flow we can assign to these theories: the farther we are from the energy scales of a microscopical feature of the system, e.g., the lattice spacing, the more accurate is the quantum field theory description. In common jargon, theories flow from ultraviolet point to infrared fixed points. For good measure, all the parameters appearing in the Lagrangian can change: this means that different kinds of new interactions may come and go depending on the energy scale: there will be "relevant", "irrelevant" and "marginal" couplings according to their behaviour across distances. The problem, though, remains for high energy physics for which there is no underlying lattice, if we do not turn to discretisation of space-time itself; a possible cut-off is in fact the Plank scale where a quantum gravity seems necessary, but the problem of "running coupling constant" is already there from inter-atomic distances of few $10^{-10}m$, way down to Plank length of $10^{-35}m$, across 20 orders of magnitudes; so maybe renormalisation is not a bunch of alchemical rules we have to use while we are waiting to understand a deeper theory, but rather a fundamental piece in our understanding of physical theories.

Technically we can distinguish two main steps in treating a quantum field theory model: the introduction of regulators and the change of couplings parameters. Many different frameworks have been put forward to introduce regulators: in old literature it was a common habit to just introduce a cut-off, usually called Λ, representing, for instance, the energy reachable by experimental set-up; around the seventies, notably [10] and [53], it was realised that a complex space-time dimension defined in a neighbourhood of the usual real one, might serve well as a regulator. This approach goes by the name of dimensional regularisation or *DimReg*. Unfortunately, despite the value of the Gaussian integral

$$\int e^{-k|x|^2} d^\delta x = \left(\frac{\pi}{k}\right)^{\delta/2}$$

that can be defined for $\delta \in \mathbb{C}$, there is no "traditional" geometric candidate for space-time manifold where such a theory could be defined; allow me to refer, though, to [40] where the author proposes Feynman motivic sheaves, and [22] based on the interplay between non commutative geometry and mixed Hodge structure; a third notable way to regularise is the zeta function regularisation method, particularly used in models of quantum field theory on curved space-time manifolds [49, 31], where different Dirichlet series are used to assign regularised values to traces and determinants of self-adjoint operators; in this thesis I will work in a still older regularisation, the so-called analytical regularisation that basically avoids the use of explicit regulators *tout-court*, because finite values for observables are obtained by analytic continuation.

The second aspect of a renormalisation procedure is the introduction of anomalous dimensions. In general these are functions of the coupling parameters and the energy scale reference. They encode the variation of the observables of the theory across different scales and, their value represents the deviation from classical dimensional analysis. This interpretation is particularly neat for classically conformal field theories. The functions that describe how coupling constants change across energy scales are called β-functions. In physics literature, it is common practise to classify different models according to the sign of the first terms of the β-function series development. For models at their critical dimensions the first non trivial term is the one proportional to the square of the relevant coupling constant. If this term is positive the theory might have a Landau pole: the interaction is weakly coupled at low energies but the value of the constant increases at high energies; but further terms in the development are required to understand whether a Wilson-Fisher point or perhaps an anomaly might appear. Whereas if this term is negative the theory is called asymptotically free: the deeper we go in energy scales, the weaker the interaction becomes. A notorious example of a possible Landau pole is the case of quantum electro dynamics, but its pole is disputably so far away in energies that it might outdistance Plank length; while a pivotal example of asymptotically free theory is quantum chromo-dynamics. A very nice and intuitive explanation of asymptotic freedom could be found in [45, 33] where they make the analogy with (dia/para) electric/magnetic properties of materials and (dia/para) colour properties of the vacuum: in a sense an asymptotically free theory is one where the vacuum state behaves as a dielectric.

In general the strenuous calculations of Feynman integrals seem unavoidable. These integrals are individually divergent or contain sub-divergent components. Renormalisation unties the intricacy of sub-divergent graphs nested one into another one. The procedure to hierarchically extract divergent contributions is called BPHZ by the name of those who invented it [9] or refined it [32, 57]. Almost 20 years ago, Connes and Kreimer [20, 21] showed that BPHZ procedure can be described in algebraic terms and the appropriate object to consider are Hopf algebras. A Hopf algebra is a bialgebra endowed with an antihomomorphism called antipode. For the Feynman Hopf algebra the elements of the underlying set are the admissible Feynman diagrams of the theory; the product is just the disjoint union of graphs, and the co-product is defined as $\Delta(G) = 1 \otimes G + G \otimes 1 + \sum_{\gamma \in Sub(G)} \gamma \otimes G/\gamma$ where in $Sub(G)$ we consider only one particle irreducible (1PI) divergent sub-diagrams. The antipode assigns to a given graph a linear combination of product of graphs where the sub-divergences have been extracted. In *DimReg* BPHZ can be seen as the Birkhoff factorisation of the Rota-Baxter algebra of Laurent polynomials with the projection along the polar part acting as Rota-Baxter operator [27]. The fascinating fact, for me, is that for a given Hopf algebra $\mathcal{H}$ there is an affine group scheme, i.e., a covariant functor F from the category of commutative algebras to the category of groups such that $F(A) = Hom_{\mathcal{A}}(\mathcal{H}, A)$ for every $A \in Obj(\mathcal{A})$; the affine group scheme corresponding to the Hopf algebra of Feynman graphs is called the group of diffeographisms; the complex points of this group scheme act as diffeomorphisms on the coupling constants of the theory! This algebraic approach combines thus combinatorics of the diagrams together with analytical properties of coupling constants.

More or less at the same time, in some seminal papers, Connes with Marcolli in [22] Kontsevich in [34] and Cartier in [19] proposed the existence of a bigger group, a "Cosmic Galois group" in the words of Cartier, that would contain the renormalisation one. Such a group would bring together two facts: Feynman integrals are periods of algebraic varieties and a lot of them are evaluated in terms of multiple zeta values.

A period is given once the data of an algebraic variety X and a pairing morphism between a meromorphic form ω and a cycle Δ are given. For Feynman integrals the algebraic variety is the zero locus of the graph polynomial and the pairing is the integration of Green functions in the domain where all Schwinger parameters are positive; in principle this might require a number of blow ups. This suggested that the Cosmic Galois group would appear as a motivic group, as it would serve as intermediary between graph algebraic varieties and their cohomology (periods).

Moreover, the link between multiple zeta values and Feynman integrals even led to the notorious Broadhurst and Kreimer conjecture about multiple zeta values [13]. At the same time Bloch, Kreimer and Esnault showed in [8] that it was possible to associate mixed Tate motives to certain graph polynomials. The periods over Mixed Tate motives are multiple zeta values, so the Motivic Galois group was thought to be the one acting on Mixed Tate motives.

All these results were coherent with a conjecture put forward by Kontsevich, claiming that the number of points of X_G, the algebraic variety of zero loci of graph polynomial, in a finite field $\mathbb{F}_q$, $X_G(\mathbb{F}_q)$ should be a (quasi)-polynomial in q. However, Brown showed in [16] together with Schnetz a counterexample in which $X_G(\mathbb{F}_q)$ is actually a modular form. They showed that in the family of Feynman diagrams of ϕ^4 theory there was a hidden K_3 surface. A little curiosity: apparently André Weil had suggested to look at K_3 surface as a counter example for the Hodge conjecture. This result declared the impossibility to describe all Feynman integrals as mixed Tate motives, but did not rule out the existence of a Cosmic Galois Group.

A motivic Galois group would act on Feynman amplitudes like a Galois group acts on its base field. The evaluation map from the ring of periods $\mathcal{P} \to \mathbb{R}$ factorises through a motivic ring of periods $\mathcal{P}^M$ [44]. It is not known whether the first map is injective, but if it was so, we would be able to prove that π and all $\zeta(2n+1)$ are transcendental numbers and algebraically independent over the rational field; that is because in the abstract ring of periods they are so. To my knowledge, no explicit motive whose periods are modular form has been constructed. Nevertheless, if we exclude the class of theories for which this counterexamples can be built, it is possible to explicitly construct a Group whose dual would give a co-action on amplitudes [15]. Having this group at our disposal means to be able to predict whether a Feynman amplitude appearing at high order loops can be ruled out from the calculations: this would occur if the co-action broke it down into not allowed terms at the motivc level. In other words, in the space of motivic Feynman integrals we can pick an element; look at its co-orbits; and if we land on an element that is not allowed, then so it is the element we started from.

Resurgence theory can shed some light both in renormalisation and non perturbative

physics. First of all, resurgence well adapts to differential system with singularities with prescribed monodromic data, i.e, a Riemann Hilbert problem. A Riemann Hilbert problem appears also in renormalisation: in dimensional regularisation, if we consider the fibre product with the multiplicative group representing a local change of scale, a Cosmic Galois group would classify flat equisingular[1] connections [22]. To my knowledge this has not been explored in literature yet. Then, the connections between modular forms, q-series and multiple zeta values seal another relevant link with resurgence theory. The most simple example is the fact that the space of modular forms is spanned by two Eisentein series $\mathbb{G}_4$ and $\mathbb{G}_6$ and a generic Eisenstein series satisfies $\mathbb{G}_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n\geq 1} \sigma_{k-1}(n) q^n$ where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the divisor sum. The last term is a so called q-series. Modular forms are particularly symmetric holomorphic functions of the unit disk that are analytically continuable in the origin; q-series on the other hand evaluate to multiple zeta values in the limit of $q \to 1$. Multiple zeta values appear thus as intermediaries in a problem of connecting analytic extensions of series defined in the unit disk. In the work of Écalle they appear in the expression of resurgent invariants as functions of the parameters of the problem [29] [12]. At last, if Schwinger Dyson equations are a doorway to non perturbative physics, then resurgence seems to be the key. As I reminded above, perturbation theory is basically the only tool that seems to work for a generic quantum field theory model. It is very difficult to compute physical properties beyond the regime of perturbative calculations.
Particular cases, like topological field theories or conformal field theories, might be sufficiently constrained, but in general the non perturbative physics seems simply unreachable. Resurgence is the appropriate tool to extract information from the, generically divergent, perturbation series [1, 46, 26, 43]; nonetheless it requires the knowledge of the whole asymptotic series and factorial growth of Feynman diagrams makes it a hard task. Here we propose a radically different point of view: the Green functions of a model satisfy Schwinger Dyson equations so they must encode the data described by the asymptotic series. They are functional equation, but if they are written with fully renormalised Green functions they imply differential equations for the anomalous dimensions. The knowledge of anomalous dimensions, in turns, allows to integrate the renormalisation group equation and ultimately reconstruct correlation functions. This frame seems thus a promising and coherent approach to non perturbative physics.

This thesis will be divided as follows: in the first chapter I report different languages in which Schwinger Dyson equations can be written. No new result is presented but rather hints and suggestions to the underlying themes of this thesis; the second chapter retraces the study of linear Schwinger Dyson equations done in the famous paper [14] and presents its resurgent analysis in close fashion with the recent publication [11], but providing a simple groundwork for the following results; as a third and forth chapter I attach my works done with Bellon that are currently under peer revision.

[1] the equisingular property expresses the energy scale independence of the counterterms

II – Schwinger Dyson equations

II.1 Introduction

As goddesses from mythology Schwinger Dyson equations might appear in many shapes and aspects. They express the relations between different n-point functions of a given quantum field theory model. In this chapter, entirely based on existing literature, I will show some of these different languages used to spell them, following more or less the historical development. We will see Schwinger Dyson examples in the language of

- traditional operator theory
- path integral
- functional equations
- combinatorial equations

I will try to provide a unifying picture across these different languages.

II.2 Lagrangian approach to Feynman rules

Let us consider a free scalar field theory. The Hilbert space correspondent to it will just be the bosonic Fock space. Suppose we could deform our theory by an interaction term (arguably small), and that the Hilbert space does not change. Let us call $\phi(\mathbf{x}, t)$ the interactive field operator. Let its interactions be described by the following Lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi\left(\Box + m^2\right)\phi + \mathcal{L}_{int}[\phi],$$

and let us suppose also that $\phi(\mathbf{x}, t)$ satisfies its classical Euler-Lagrangian equations

$$\left(\Box + m^2\right)\phi - \mathcal{L}'_{int}[\phi] = 0 \tag{II.1}$$

just like the free theory would do. To preserve causality structure and uncertainty principle let $\phi(\mathbf{x}, t)$ satisfy the following commutations relations

$$[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = 0, \tag{II.2}$$

$$[\phi(\mathbf{x}, t), \partial_t\phi(\mathbf{x}, t)] = i\hbar\delta^3(\mathbf{x} - \mathbf{x}'). \tag{II.3}$$

Under these conditions, the Schwinger Dyson equations are

$$\left(\Box_x + m^2\right)\langle\phi_x\phi_1\dots\phi_n\rangle = \langle\mathcal{L}'_{int}[\phi_x]\phi_1\dots\phi_n\rangle - i\hbar\sum_j \delta^4(x-x_j)\langle\phi_1\dots\phi_{j-1}\phi_{j+1}\dots\phi_n\rangle. \tag{II.4}$$

First, observe that these are equations for correlation functions and not for the operators. Then, notice that in the limit $\hbar \to 0$ the last term vanishes, and equation (II.3) becomes $[\phi(\mathbf{x},t), \partial_t\phi(\mathbf{x},t)] = 0$, the commutation relation of a classical field theory. This means that they can be seen as a quantum deformation of the equations of motion. Remark, also, that this formulation is Lorentz invariant since equation (II.2) ensures that for a given foliation of space-time no faster then light signal is allowed. Finally equation (II.4) remain valid no matter the specific kind of interactions.
To prove it, let $|\Omega\rangle$ be the vacuum state of the interacting theory. We just need to take a look at the time-ordered product:

$$T\{\phi(x),\phi(x')\} = \phi(x)\phi(x')\theta(t-t') + \phi(x')\phi(x)\theta(t'-t),$$

where $\theta(t-t')$ is the Heaviside step function, and consider its time derivative

$$\partial_t\langle T\{\phi(x)\phi(x')\}\rangle_\Omega = \langle T\{\partial_t\phi(x)\phi(x')\}\rangle_\Omega + \langle[\phi(x)\phi(x')]\rangle_\Omega\,\delta(t-t'),$$

where we have used that $\delta(t-t') = \partial_t\theta(t-t')$. A second derivation brings

$$\begin{aligned}\partial_t^2\langle T\{\phi(x)\phi(x')\}\rangle_\Omega &= \langle T\{\partial_t^2\phi(x)\phi(x')\}\rangle_\Omega + \delta(t-t')\langle[\partial_t\phi(x)\phi(x')]\rangle_\Omega\\ &= \langle T\{\partial_t^2\phi(x)\phi(x')\}\rangle_\Omega - i\hbar\delta^4(x-x').\end{aligned}$$

So when we are considering the full operator $\Box + m^2$ we have

$$\left(\Box + m^2\right)\langle T\{\phi(x)\phi(x')\}\rangle_\Omega = \langle T\{(\Box+m^2)\phi(x)\phi(x')\}\rangle_\Omega - i\hbar\delta^4(x-x'),$$

and if we impose the Euler-Lagrangian equations (II.1) for ϕ become

$$\left(\Box + m^2\right)\langle T\{\phi(x)\phi(x')\}\rangle_\Omega = \langle T\{\mathcal{L}'_{int}[\phi](x)\phi(x')\}\rangle_\Omega - i\hbar\delta^4(x-x'),$$

that is equation (II.4) in case of the two-point function. The generalisation to n-point function is straightforward.

No reference to perturbation theory was invoked, so already one of the main point arises here: they are non perturbative relations. Nonetheless we can solve them perturbatively and start to see some diagrams. You will see, though, that this formalism includes disconnected Green functions and this makes the attempt to write Schwinger Dyson equations with only fully renormalised terms harder. Let me show this with an example, and for homogeneity with the following chapters, consider a massless scalar field with cubic interaction. Let its action be simply

$$S = \int -\frac{1}{2}\phi(\Box+m^2)\phi - \frac{g}{3!}\phi^3.$$

Let $K_{x,y}$ be the Green function satisfying

$$\Box_x K_{xy} = -i\delta_{xy}$$

obtained considering the weak solution to the Euler-Lagrangian equation of the free theory. Notice the identity

$$\langle \phi_x \phi_y \rangle = \int dz\, \delta_{zx} \langle \phi_x \phi_y \rangle = i \int dz\, \Box_x K_{xy} \langle \phi_x \phi_y \rangle = i \int dz\, K_{xy} \Box_x \langle \phi_x \phi_y \rangle$$

Schwinger Dyson equations (II.4) can be used to express the last term on the right hand side. This gives

$$\langle \phi_x \phi_y \rangle = i \int dz\, K_{xz} \left(\frac{g}{2} \langle \phi_z^2 \phi_y \rangle - i \delta_{zy} \right);$$

we can repeat the process and have

$$\begin{aligned} \langle \phi_x \phi_y \rangle &= K_{xy} - \frac{g}{2} \int dz\, dw\, K_{xz} K_{yw} \Box_w \langle \phi_z^2 \phi_w \rangle \\ &= K_{xy} - \frac{g^2}{4} \int dz\, dw\, K_{xz} K_{yw} \Box_w \langle \phi_z^2 \phi_w^2 \rangle + ig \int dz\, K_{xz} K_{zy} \langle \phi_z \rangle . \end{aligned}$$

This procedure might continue indefinitely but, as said above, it is not particularly convenient to represent connected full quantities. We can, though, calculate $\langle \phi_x \phi_y \rangle$ at any order in g that we want. For example we can show the result at second order: the same algorithm applied to $\langle \phi_x \rangle$ brings

$$\langle \phi_x \rangle = \int dz\, \delta_{xz} \langle \phi_z \rangle = i \int dz\, K_{xz} \Box_z \langle \phi_z \rangle = i \frac{g}{2} \int dz\, K_{xz} \langle \phi_z^2 \rangle$$

that at second order is just

$$\langle \phi_x \rangle = i \frac{g}{2} \int dz\, K_{xz} K_{zz} + O(g^2);$$

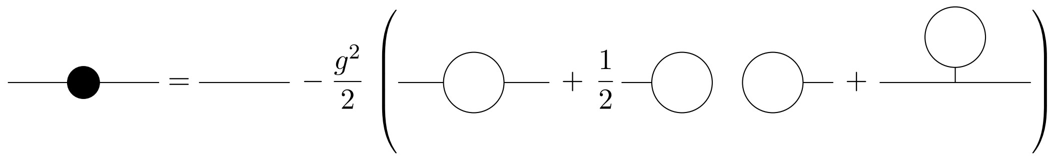
and for $\langle \phi_z^2 \phi_w^2 \rangle$ gives

$$\langle \phi_z^2 \phi_w^2 \rangle = 2K_{zw}^2 + K_{zz} K_{ww} + O(g).$$

If we put all together we have

$$\langle \phi_x \phi_y \rangle = K_{xy} - \frac{g^2}{2} \int dz\, dw\, \left(K_{xz} K_{zw}^2 K_{wy} + \frac{1}{2} K_{xz} K_{zz} K_{ww} K_{wy} + K_{xz} K_{zw} K_{ww} K_{zy} \right) + O(g^3)$$

The last equation has the following diagrammatic representation:



As expected, the second term in parenthesis is manifestly disconnected.
This formalism has highlighted one crucial aspect: the non perturbative nature of Schwinger Dyson equations. The presence of disconnected diagrams make the renormalisation process more difficult. Extra counter-terms would be required. Let us move on to a path integral representation.

II.3 Path integral

Euler Lagrangian equation are derived requiring the action to be stationary with respect to a variation of the field $\delta\varphi(x) = \varepsilon(x)$. Schwinger Dyson can be obtained in the same way but now requiring the partition function to be stationary to such a variation. They will appear in the shape of a differential equation for the partition function. This formalism has the benefit to mimic the classical variational approach and to provide with the same technique the Ward-Slavnov identities.
Let φ denote a generic field, either scalar or fermionic, and let its partition function

$$Z[J] = \int \mathcal{D}\varphi \, exp \, i \left(S[\varphi] + \int J\varphi \right)$$

be a functional on the classical source J. We can introduce the standard notation

$$\langle O \rangle := \int \mathcal{D}\varphi \, O e^{iS}$$

With this notation

$$Z[J] = \langle e^{i \int J\varphi} \rangle \tag{II.5}$$

and

$$\langle \varphi \rangle = -i \frac{1}{Z(0)} \frac{\delta Z}{\delta J}$$

where the evaluation at $J = 0$ is understood. It is well known that we don't have a rigorously defined measure theory for which this expression makes sense; for the difficulties related to the definition of infinite dimensional integral I recommend reading [18]. Nevertheless admit that we could really integrate on all possible configurations of the field φ; the measure $\mathcal{D}\varphi$ would be invariant under such variation, i.e., its Jacobian would be simply one. We could thus consider

$$\delta Z = \langle e^{i \int J\varphi} \left(\frac{\delta S}{\delta \varphi} + J \right) \delta\varphi \rangle = 0, \tag{II.6}$$

at first order in the variations. If we specify the Lagrangian as

$$\mathcal{L} = -\frac{1}{2} \varphi D \varphi + \mathcal{L}_{int}$$

where D might represent any kinetic operator, we have

$$\frac{\delta S}{\delta \varphi} = -D\varphi + \mathcal{L}'_{int},$$

so equation (II.6) can be rewritten as

$$\langle e^{i \int J\varphi} \left(-D\varphi + \mathcal{L}'_{int} + J \right) \delta\varphi \rangle \, .$$

Now observe two facts: the operator D can get out of the integral; the formal expression $\mathcal{L}'_{int}$ can be thought as a functional of the fields as well as an operator acting on functionals as

$$\langle e^{i \int J\varphi} \mathcal{L}'_{int}(\varphi) \rangle = \mathcal{L}'_{int} \left(\frac{\delta}{\delta J} \right) Z[J].$$

So equation (II.6) can be rewritten as

$$-iD\frac{\delta Z}{\delta J} = \left\{\mathcal{L}'_{int}\left(\frac{\delta}{\delta J}\right) + J\right\} Z[J]. \tag{II.7}$$

This equation generates the same perturbation expansion of the previous section if we develop equation (II.5) in powers of the source J.
Again, no mention to specific interactions nor perturbative expansion was needed to derive equation (II.7). These relations are thus purely non perturbative. Let us see, in fact, how the same techniques bring to Ward identities.

Consider now the case of a spinor field ψ. The partition function is

$$Z[\eta, \bar{\eta}] = \langle e^{i\int \eta\psi + \bar{\eta}\bar{\psi}}\rangle .$$

Consider now a group action on the fields described by the variation

$$\begin{aligned} \delta\psi(x) &= -i\varepsilon(x)\psi(x) \\ \delta\bar{\psi}(x) &= i\varepsilon(x)\bar{\psi}(x) \end{aligned}$$

characterised by a function ε, without a gauge field that compensates their transformation. The n-point functions are stable with respect to this variation. Consider thus the case of a 2-point function

$$\langle\psi(x)\bar{\psi}(y)\rangle = -\frac{\delta^2 Z}{\delta\eta(x)\delta\bar{\eta}(y)}.$$

We have

$$\delta\langle\psi(x)\bar{\psi}(y)\rangle = \langle\left(\int dz\,\frac{\delta S}{\delta\psi(z)}\delta\psi(z) + \frac{\delta S}{\delta\bar{\psi}(z)}\delta\bar{\psi}(z)\right)\psi(x)\bar{\psi}(y) + \delta\psi(x)\bar{\psi}(y) + \psi(x)\delta\bar{\psi}(y)\rangle = 0$$

or equivalently to first order in ε

$$\langle\left(\int dz\,\bar{\psi}(z)\partial\!\!\!/\varepsilon(z)\right)\psi(x)\bar{\psi}(y) - i\varepsilon(x)\psi(x)\bar{\psi}(y) + i\varepsilon(y)\psi(x)\bar{\psi}(y)\rangle = 0$$

Since this must be true for a general $\varepsilon(z)$ we have

$$\partial_\mu\langle j^\mu(z)\psi(x)\bar{\psi}(y)\rangle = -\delta_{zx}\langle\psi(x)\bar{\psi}(y)\rangle + \delta_{zy}\langle\psi(x)\bar{\psi}(y)\rangle \tag{II.8}$$

where $j^\mu(z) = \bar{\psi}(z)\gamma^\mu\psi(z)$ is the notorious quantum electro dynamics current. In Fourier space this relation is

$$iq_\mu M^\mu(q, p_1, p_2) = M_0(q + p_1, p_2) - M_0(p_1, p_2 - q)$$

This procedure can be generalised for any correlation function appearing in the theory. Even if we add new fields ψ_i and we consider the variations

$$\delta\psi_i(x) = -iQ_i\varepsilon(x)\psi(x),$$

equation (II.8) becomes

$$\partial_\mu\langle j^\mu(z)\psi(x_1)\ldots\bar{\psi}(x_n)\rangle = \sum_k i\delta_{zx_k}Q_k\langle\psi(x_1)\ldots\bar{\psi}(x_n)\rangle .$$

that in Fourier space becomes

$$iq_\mu M^{\mu\sigma_1\ldots\sigma_n}(q, r_1\ldots r_n, p_1\ldots p_m) = \sum sign_i Q_i M^{\sigma_1\ldots\sigma_n}(r_1\ldots r_n, p_1\ldots p_i - q\ldots p_m)$$

where $sign_i$ specifies the momentum of p_i.

The Ward identities prevent the occurrence of anomalies and are valid at any order in perturbation theory. They are essential tools for the study of systems of Schwinger Dyson equations. We have not yet solved the case for quantum electro dynamics where one would see them in action, but our aim is to do that in the future.

II.4 Schwinger Dyson equations from Effective action

Here I present a functional approach to the construction of Schwinger Dyson equations. A key message that this approach suggests is that whenever we introduce an equation also for the vertex then necessarily we are facing an infinite tower of equations. In general, unless some particular limit is evoked or in the presence of symmetries, studies of Schwinger Dyson equations are compelled to truncate somewhere along this tower. We follow here [24].

II.4.1 Quantum and classical action

Once again, consider

$$Z[J] = \exp\{iW[J]\} = \int \mathcal{D}\varphi \exp\{i(S + J_i\varphi_i)\}.$$

for any field theory model φ where J_i describes its classical external source. In this section the path integral is just a convenient representation. $Z[J]$ generates all the Feynman diagrams of the theory and its logarithm $W[J]$ generates all connected Green's functions. I adopt here the standard notation $J_i\varphi^i = \int d^4x\, J(x)\varphi(x)$. The Legendre transform of the $W[J]$ function is called quantum action $\Gamma[\varphi]$. As usual we need to define a conjugate variable $J \leftrightarrow \varphi$, so let φ be

$$\varphi^i := \frac{\delta W[J]}{\delta J_i};$$

the quantum action is defined as

$$\Gamma[\varphi] = W[J] - J_i\varphi^i.$$

The connection between quantum and classical action is given by

$$\frac{\delta\Gamma}{\delta\varphi^i} = \Lambda\frac{\delta S}{\delta\varphi^i}. \tag{II.9}$$

The operator Λ is defined

$$\Lambda :=: \exp\left\{\frac{i}{\hbar}\sum_{n=2}^{\infty}\frac{(-i\hbar)^n}{n!}G_{i_1i_2\cdots i_n}\frac{\delta^n}{\delta\varphi_{i_1}\delta\varphi_{i_2}\cdots\delta\varphi_{i_n}}\right\}: \tag{II.10}$$

where the colons represent the order in which the functional derivatives are put at the rightmost position so that $G^{i_1 \cdots i_n}$ are not touched. The latter are connected Green functions defined as

$$G^{i_1 \cdots i_n} = \frac{\delta^n W}{\delta J_{i_1} \cdots \delta J_{i_n}}$$

Equation (II.9) generates relations for connected Green's functions from classical equations of motion, but the quantum action Γ is not known, so we need to fix the so called quantum equations of motion

$$\frac{\delta \Gamma}{\delta \varphi^i} = -J_i. \tag{II.11}$$

as they mimic classical ones in presence of a source. So

$$-J_i = \frac{\delta \Gamma}{\delta \varphi^i} =: \exp\left\{ \frac{i}{\hbar} \sum_{n=2}^{\infty} \frac{(-i\hbar)^n}{n!} G^{i_1 i_2 \ldots i_n} \frac{\delta^n}{\delta \varphi_1^i \delta \varphi_2^i \ldots \delta \varphi_n^i} \right\} : \frac{\delta S}{\delta \varphi^i}$$

expresses how classical equations of motions are changed by establishing relations between Green functions. From quantum equations of motion (II.11) we have

$$\frac{\delta}{\delta J^j} \frac{\delta \Gamma}{\delta \varphi^i} = -\delta_{ij}.$$

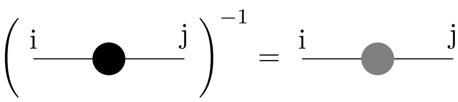
and since

$$\frac{\delta \varphi^k}{\delta J_l} = \frac{\delta}{\delta J_l} \frac{\delta W}{\delta J_k} = \frac{\delta^2 W}{\delta J_l \delta J_k} = G^{lk}$$

we have the relation

$$\frac{\delta^2 W}{\delta J_l \delta J^k} \frac{\delta^2 \Gamma}{\delta \varphi_k \delta \varphi_i} = -\delta^{il}.$$

This equation has a pictorial representation as



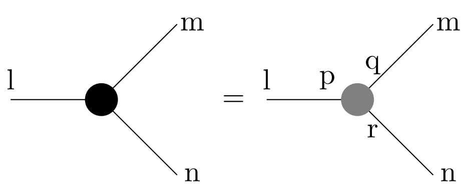
or prosaically that the inverse of a full Green's functions is a full 2-vertex. If we act twice with the functional derivative $\frac{\delta}{\delta J_i}$ on the quantum equations of motion we have:

$$\frac{\delta^3 W}{\delta J_l \delta J_m \delta J_n} = \frac{\delta^2 W}{\delta J_l \delta J_p} \frac{\delta^2 W}{\delta J_m \delta J_q} \frac{\delta^2 W}{\delta J_n \delta J_r} \frac{\delta^3 \Gamma}{\delta \varphi^p \delta \varphi^q \delta \varphi^r}$$

or equivalently

$$G^{lmn} = G^{lp} G^{mq} G^{nr} \Gamma_{pqr}, \qquad \Gamma_{pqr} = \frac{\delta^3 \Gamma}{\delta \varphi^p \delta \varphi^q \delta \varphi^r}.$$

We can again depict it with graphs



(II.12)

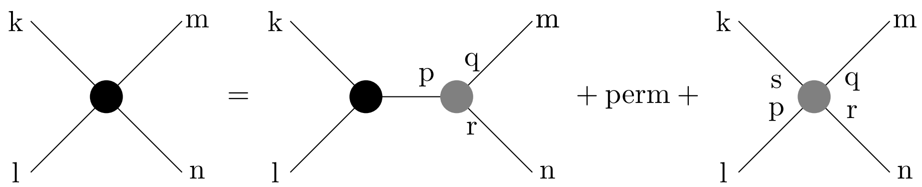
And we can go on and on with this method. As a last example let us show the relations for the 4-point function:

$$
\begin{aligned}
\frac{\delta^4 W}{\delta J_k \delta J_l \delta J_m \delta J_n} = & \frac{\delta^3 W}{\delta J_k \delta J_l \delta J_p} \frac{\delta^2 W}{\delta J_m \delta J_q} \frac{\delta^2 W}{\delta J_n \delta J_r} \frac{\delta^3 \Gamma}{\delta \varphi^p \delta \varphi^q \delta \varphi^r} \\
& + \frac{\delta^2 W}{\delta J_l \delta J_q} \frac{\delta^3 W}{\delta J_k \delta J_m \delta J_q} \frac{\delta^2 W}{\delta J_n \delta J_r} \frac{\delta^3 \Gamma}{\delta \varphi^p \delta \varphi^q \delta \varphi^r} \\
& + \frac{\delta^2 W}{\delta J_l \delta J_q} \frac{\delta^2 W}{\delta J_m \delta J_q} \frac{\delta^3 W}{\delta J_k \delta J_n \delta J_r} \frac{\delta^3 \Gamma}{\delta \varphi^p \delta \varphi^q \delta \varphi^r} \\
& + \frac{\delta^2 W}{\delta J_l \delta J_q} \frac{\delta^2 W}{\delta J_m \delta J_q} \frac{\delta^2 W}{\delta J_n \delta J_r} \frac{\delta^2 W}{\delta J_k \delta J_s} \frac{\delta^4 \Gamma}{\delta \varphi^p \delta \varphi^q \delta \varphi^r \delta \varphi^s}
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
G^{klmn} = & G^{klp} G^{mq} G^{nr} \Gamma_{pqr} + G^{lp} G^{kmq} G^{nr} \Gamma_{pqr} + \\
& + G^{lp} G^{mq} G^{knr} \Gamma_{pqr} + G^{lp} G^{mq} G^{nr} G^{ks} \Gamma_{pqrs}.
\end{aligned}
$$

This has a graphical representation as



where we could use equation (II.12) to re-express the black blob of the first three terms on the right hand side with grey blobs. This would characterise the full 4-points functions only in terms of lower or equal order vertices.

II.4.2 The ϕ^3 model

The new results of this thesis concern the Schwinger Dyson equations for a massless scalar field theory with cubic interaction. Let me derive in this formalism the equations for the generic cubic interaction described by $g d_{ijk} \varphi^i \varphi^j \varphi^k$, where for example d_{ijk} is totally symmetric.
Consider again the classical action

$$
S = -\frac{1}{2} \varphi^i (\partial^2 + m^2)_{ij} \varphi^j - \frac{g}{3!} d_{ijk} \varphi^i \varphi^j \varphi^k = -\frac{1}{2} \varphi^i K^{-1}_{ij} \varphi^j - \frac{g}{3!} d_{ijk} \varphi^i \varphi^j \varphi^k
$$

where the Latin indices represent both space-time variables and internal ones, and K^{-1}_{ij} is the Green function of the free theory. With such classical action, polynomial in the fields, there are just a finite number of functional derivatives different from zero. This implies a truncation for the operator Λ in equation (II.9). For our case, since

$$
\frac{\delta S}{\delta \varphi^l} = -K^{-1}_{li} \varphi^i - \frac{g}{2} d_{lij} \varphi^i \varphi^j,
$$

only the first non trivial term will survive in the expansion from equation (II.10). So from equation (**??**) we have

$$
\begin{aligned}
J_l = \Lambda\frac{\delta S}{\delta\varphi^l} =& K^{-1}_{li}\varphi^i + \frac{g}{2}d_{lij}\varphi^i\varphi^j + \\
&+ \frac{i}{\hbar}\left(\sum_{\substack{\mathbf{n}\\|\mathbf{n}|\geq 2}} \frac{(-i\hbar)^{|\mathbf{n}|}}{|\mathbf{n}|!}G^{i_\mathbf{n}}\frac{\delta^{|\mathbf{n}|}}{\delta\varphi^{i_\mathbf{n}}}\right)\left(K^{-1}_{li}\varphi^i + \frac{g}{2}d_{lij}\varphi^i\varphi^j\right) + \\
&+ -\frac{1}{\hbar^2}\left(\sum_{\substack{\mathbf{n},\mathbf{m}\\|\mathbf{n}|,|\mathbf{m}\geq 2}} \frac{(-i\hbar)^{|\mathbf{n}+\mathbf{m}|}}{|\mathbf{n}+\mathbf{m}|!}G^{i_\mathbf{n}}G^{i_\mathbf{m}}\frac{\delta^{|\mathbf{n}+\mathbf{m}|}}{\delta\varphi^{i_{\mathbf{n}+\mathbf{m}}}}\right)\left(K^{-1}_{li}\varphi^i + \frac{g}{2}d_{lij}\varphi^i\varphi^j\right) + \dots
\end{aligned}
$$

where I have used the multi-index notation and $|\mathbf{n}| = n$ the length of the index. We can simplify this equation in the more appealing form

$$J_l = K^{-1}_{li}\varphi^i + \frac{g}{2}d_{lij}\varphi^i\varphi^j + \frac{i\hbar}{2}g d_{lij}G^{ij}.$$

Now if we derive both sides by $\frac{\delta}{\delta J_m}$, and evaluate it to 0,

$$\delta^m_l = K^{-1}_{li}G^{im} + g d_{lij}\varphi^j G^{im} + \frac{i\hbar}{2}g d_{lij}G^{ijm},$$

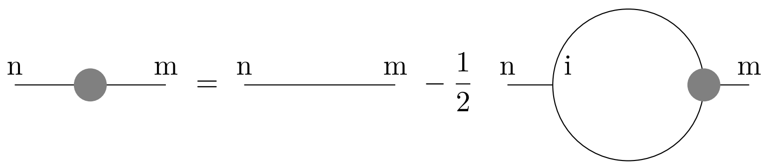
and multiply by K^{nl}, we obtain

$$K^{nm} = G^{nm} + g d_{lij}K^{nl}G^{im}\varphi^j + \frac{i\hbar}{2}gK^{nl}d_{lij}G^{ijm}$$

or better

$$G^{nm} = K^{nm} - g d_{lij}K^{nl}G^{im}\varphi^j - \frac{i\hbar}{2}gK^{nl}d_{lij}G^{ijm}.$$

This is the Schwinger Dyson equation for ϕ^3 model. It has the following pictorial representation



The purpose of this section was to show a derivation of the Schwinger Dyson equation in a functional setting and in particular for the case of the ϕ^3 model that we will study in depth in the next chapters. In this way, the reader should not be surprised by the starting point of our introduction of the Ward Schwinger Dyson method. More importantly here I would like to make a suggestion: in differential Galois theory [50, 51], for a linear system of differential equations the Stokes matrix S_ℓ associated to a singular line ℓ is the unique unipotent element of the Galois group G associated to the system; it is the exponential of a unique, nilpotent infinitesimal Stokes matrix belonging to the Lie algebra of G. Écalle [28] was able to characterise this Stokes automorphism as the exponential of *alien derivatives* Δ: a derivation of the convolution algebra of germs of holomorphic functions in the Borel plane,

usually represented the universal covering of pointed disk, i.e., by the Riemann surface of the logarithm. In general, one could write $M^\bullet \Delta_\bullet$, where $M^\bullet$ is the so called mould (a character in the Hopf algebra of words) and $\Delta_\bullet$ are the alien derivatives taken at the singularity indexed by $\bullet$.
I find particularly inspiring the idea to consider the operator Λ appearing in (II.9) and write it in the appealing form

$$\Lambda = e^{\frac{i}{\hbar} G^\bullet \delta_{\varphi^\bullet}}.$$

This would act on the classical equations of motion uncovering the monodromic properties that are then encoded in the quantum action. Such an interpretation would need a bridge equation that links the resurgent structure to the differential one, of the kind found for differential equation. In this case, though, it would be a functional relation. In the context of differential equations, the bridge equations link the alien derivatives at a given singular point with ordinary ones with respect to extra parameters representing the boundary conditions of the problem. In the functional context the analogous of these parameters would be the values of the fields in extremal points. In quantum field theory this would mean the asymptotic states where the free theory is defined.

II.5 Combinatorial Schwinger Dyson

In the previous sections I have shown how Schwinger Dyson appear in a quantum field theory context. As I pointed out in the introduction, Connes and Kreimer realised in [20, 21] that renormalisation in quantum field theory could be described in algebraic terms introducing an Hopf algebra. The two algebras deployed are usually the polynomial algebra over the set of rooted trees, and the Hopf algebra of graphs.

Schwinger Dyson equations are fixed point equations representing a sub-Hopf algebra condition. The solution to a Schwinger Dyson equation is in general a formal series that contains all possible elements of the Hopf algebra that can be generated by an iterative insertion of particular elements singled out by the co-product. In the case of trees, these elements will be new roots; in the case of Feynman graphs they will be primitively divergent graphs. The compatibility of the insertion map and the co-product can be expressed by demanding it to be a co-cycle in the Hochschild cohomology associated to the Hopf algebra.

These combinatorial equations have been classified by Foissy [30] and their solution extensively studied. As Yeats clearly points out in [56] to a given combinatorial equation in the Hopf algebra, one can associate an analytical equation by virtue of the Mellin transform. Despite that, these equations can not be written in terms of fully renormalised objects because of the occurrence of overlapping divergences. To solve this problem we have introduced the Ward Schwinger Dyson methods that we will discuss in the last two chapters. Our method departs a bit from this formulation in the language of Hopf algebra but they are an essential step to understand the current context of the theory. Let me thus introduce here the main features.

II.5.1 Hopf algebras

Let me start with the definition.

Let $\mathbb{K}$ be a field of characteristic zero. An Hopf algebra H is the data of an associative, co-associative, commutative, unital and co-unital bialgebra $(\tilde{H}, m, u, \Delta, \varepsilon)$ together with a anti-homomorphism S called the antipode, such that

$$m \circ (id \otimes S) \circ \Delta = m \circ (S \otimes id) \circ \Delta = u \circ \varepsilon.$$

Equivalently, given an associative, co-associative, commutative, unital and co-unital bialgebra $(\tilde{H}, m, u, \Delta, \varepsilon)$, that admits a grading

$$\tilde{H} = \bigoplus_{n \geq 0} \tilde{H}_n,$$

and that is connected

$$H_0 = \mathbb{K},$$

the map S can be uniquely canonically constructed as the convolution inverse of characters with values in a unital algebra A.

The characters are morphisms of unital algebras $Hom(\tilde{H}, A)$: for $\phi \in Hom(\tilde{H}, A)$

$$\phi \circ u = u_A \qquad \phi \circ m = m_A \circ (\phi \otimes \phi).$$

Let $\phi, \psi \in Hom(\tilde{H}, A)$, their convolution product is defined as

$$\phi * \psi := m_A \circ (\phi \otimes \psi) \circ \Delta.$$

With the data of this product characters form a group. The convolution inverse of $\phi \in Hom(\tilde{H}, A)$ will be

$$\phi^{*-1} = \phi \circ S$$

with S the antipode.

II.5.2 Hochschild cohomology

To study the Hopf algebra H we can introduce the Hochschild chain complex [7] and study only the first cohomology group.

Let construct the cochain complex as follows:

- the zero-cochains are the functionals $Hom(H, \mathbb{K})$
- the one-cocycles $Z^1(H)$ are the maps $L \in End(H)$ subject to the condition

$$\Delta \circ L = (id \otimes L) \circ \Delta + L \otimes \mathbb{1}$$

- the differential δ is defined as

$$\delta : Hom(H, \mathbb{K}) \to Z^1(H)$$
$$\delta : \alpha \mapsto \delta\alpha := (id \otimes \alpha) \circ \Delta - u \circ \alpha$$

- the coboundaries are defined as $B^1(H) := \delta(Hom(H, \mathbb{K}))$
- the first cohomology group will be $H^1(H) = Z^1(H)/B^1(H)$

The Hochschild cohomology is a very powerful tool: if a grafting map is given, then the requirement of being a one-cocyle induces a choice on coproducts. I refer also to the beautiful work [36].

Now for the rest of the section let us focus on the Hopf algebra of trees for two reasons: it suffices to describe the nesting of subdivergences for the case we will study in the next chapter; we can build simple examples that show how Schwinger Dyson equations work.

II.5.3 Hopf algebra of trees

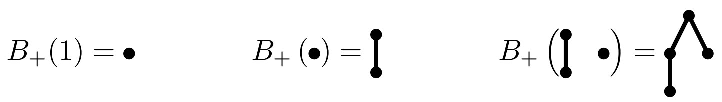
The algebra of trees is the polynomial algebra over the set of rooted trees. The product is simply the disjoint union of trees and a monomial is generally called a forest. The grading is given by the number of vertices. Call an admissible cut the set of vertices of a given rooted tree T with the property that no vertex is an ancestor of another. An admissible cut c splits the tree in two parts: the pruned part $P_c(T)$, i.e, is the forest of trees rooted at the vertices of c, and the rooted part $R_c(T)$, i.e., the tree resulting from removing $P_c(T)$ from T. The coproduct for the tree Hopf algebra is defined as

$$\Delta(T) = \sum_{c \in Adm(T)} P_c(T) \otimes R_c(T)$$

for any tree $T \in H$. Now, $\varnothing, T \in Adm(T)$ so in the coproduct the terms $T \otimes 1 + 1 \otimes T$ will appear. When these are the only terms of the coproduct, the tree T is said primitive, whereas when $\Delta T = T \otimes T$ the tree T is said to be group-like element. The counit is defined simply as $\varepsilon(1) = 1$ and $\varepsilon(T) = 0$ for any tree not in H_0.

Let us consider some examples of trees:

Define the grafting operator $B_+ \in End(H)$ attaching a root to all the trees of a forest. This means



It satisfies the cocyle property

$$\Delta \circ B_+ = (id \otimes B_+) \circ \Delta + B_+ \otimes \mathbb{1}$$

and the fact that $B_+(\mathbb{1}) = \bullet$ implies that the first cohomology group $H^1(H)$ is non trivial. In general to obtain this condition it might be necessary to take the quotient of the Hopf algebra with respect to its ideals [54].

All the tools are there to define a combinatorial Schwinger Dyson equation as the fixed point equation

$$X(x) = 1 \pm \sum_{n \geq 1} x^n B_+^n(P_n(X(x)))$$

where $X \in H$ and P_n are rational functions. In physical applications usually there exists a particular monomial, called the invariant charge Q, for which $P_n(X) = Q^n X$. The case of a system of m Schwinger Dyson equations it is given by

$$Q = \prod_{k=1}^{m} X_k^{s_k}$$

where $s_k \in \mathbb{Z}$. If an energy scale L is introduced one defines the beta function of the theory as being

$$\beta := \partial_L|_{L=0} \log Q.$$

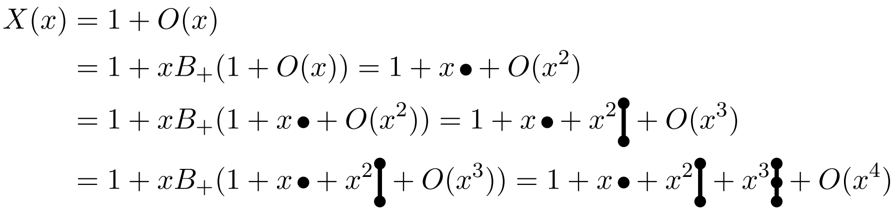
We will see this definition in action in the following chapters. If we are looking at a single equation the definition of invariant charge imposes the form

$$X(x) = 1 + sign(s) \sum_{n \geq 1} x^n B_+^n (X^{1+ns}(x))$$

Let us see some examples of this equation as in [56]. Example I

$$X(x) = 1 + xB_+(X(x)).$$

The solution is found by iteration

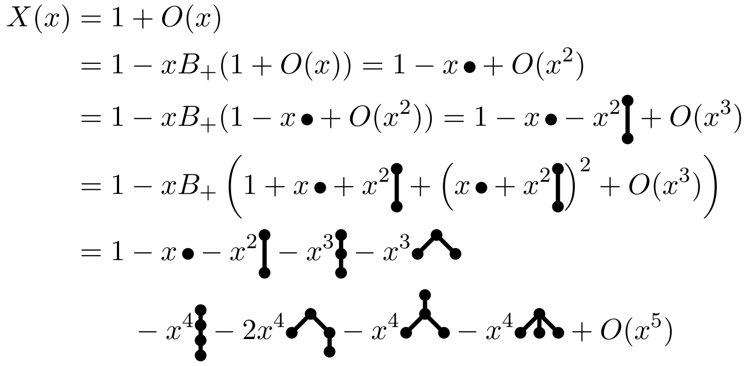


we can easily guess that this equation generates all the trees with n vertices and no branches.

Example II

$$X(x) = 1 - xB_+(X^{-1}(x)).$$

Again let us expand



In this case the coefficients represent the number of different embeddings of the tree in the plane. In [42] the authors used this example to decode the solution to the Yukawa model in terms of chord diagrams. To understand the link with quantum field theory models we have to translate the combinatorial Schwinger Dyson equation into an analytical one. The map is done with the introduction of Feynman rules. For a detailed description I refer to [47]. The Feynman rules associate to combinatorial objects a Feynman integral. These integrals, in general, will be divergent. In analytical regularisation a single integral is substituted by a family of integrals parametrised by an extra variable defined in a neighbourhood of the origin. The integral is then evaluated in terms of its Mellin transform. This procedure is very natural in the context of integrals of fully renormalised Green functions of massless theories. The reason for that is that renormalisation corrections come from log divergences. So the series correction becomes a pseudodifferential operator and the log terms shift the power in the free propagator.

Given a combinatorial Dyson-Schwinger of the form

$$X(x) = 1 + \mathrm{sgn}(s) \sum_{k \geq 1} x^k B_+^k (X^{1+ks}(x))$$

its analytic version is

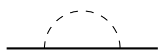
$$G(x, L) = 1 + \mathrm{sgn}(s) \sum_{k \geq 1} x^k G\left(x, \frac{d}{d(-\rho)}\right)^{1+sk} (e^{-L\rho} - 1) F_k(\rho)\Big|_{\rho=0}$$

where $L := \log p^2/\mu^2$, with p^2 the incoming momentum and μ^2 the reference energy scale. F_k is the Mellin transform of a given Feynman integral.

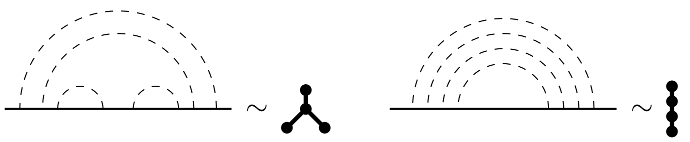
In the next chapter we will report the solution to the "linear" case of Schwinger Dyson equations as it was found in [14], and a different method that allows an easy resurgent analysis. A single combinatorial equation represent both the Yukawa model in four dimensions and scalar field model with cubic interactions in six dimensions. It is, thus, the right moment to show that this problem can be spelled as a combinatorial Schwinger Dyson equation.

II.5.4 Linear Schwinger Dyson equation in combinatorial language

In a theory where the basic interaction is described by a 3-vertex, if only one field gets renormalised, all the graphs can be constructed from a single building block



and the structure of nesting can be described by trees. For example



As you can see these trees are the same generated by the last equation. The combinatorial Schwinger Dyson equation that describes both Yukawa and ϕ^3 is thus

$$X(x) = \mathbb{I} - xB_+\left(\frac{1}{X(x)}\right),$$

and its analytic version is

$$G(x, L) = 1 - \frac{x}{p^2} \int d^4k \frac{k \cdot p}{k^2 G(x, \log k^2)(k+p)^2} - \cdots \Big|_{p^2=\mu^2}$$

Note that only one subtraction is necessary because the structure of sub-divergences is encoded in the combinatorial equation.

Before ending this chapter let me just mention the result found in [39]. Following Manin's suggestion to halting problem via Hopf algebra methods [38], the authors study Schwinger

Dyson solutions in the non commutative algebra of flow charts. They also generalise their results to the case of Schwinger Dyson equations for operads and even properads. It has been indeed known from [7] that behind Hopf algebras lies an operadic viewpoint. We believe that the methods we have introduced to solve the full vertices cases has an operadic interpretation. Unluckily we do not know it.

III – The Linear case

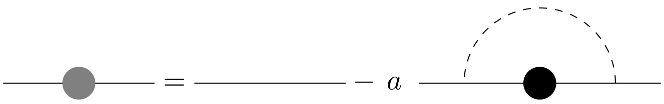
In this chapter I discuss two methods to solve an example of "linear" Schwinger Dyson equation. First, I consider the method introduced by Broadhurst and Kreimer in the seminal work [14], then I present the analysis using the same techniques that brought to the introduction of Ward Schwinger Dyson method explained in the next chapters. It will be thus a good playground to introduce notations and the resurgent analysis in a simpler case.

In their work Broadhurst and Kreimer settled down a non perturbative approach for quantum field theory: consider the Schwinger Dyson equation of the model of interest; demand that the full Green functions appearing in it are renormalised; extract a differential equation for the anomalous dimension; generate all subsequent coefficient integrating renormalisation group equation; re-sum the series.
The models studied by Broadhurst and Kreimer have two virtues: no vertex correction is required; they are both described by a "linear " Schwinger Dyson equation, i.e., Schwinger Dyson equations where only one full Green function is fully renormalised. The first virtue implies that there is no need for a Schwinger Dyson equation for the vertex. This cuts down the infinite tower of differential equations into just one for the anomalous dimension. In particular, both are first order equations, but the Yukawa case is a degree 2 non-linear differential equation, and the scalar case a degree 4 non-linear differential equation. A first attempt to generalise this condition can be found in [35] where the authors introduced tools to include overlapping divergences in the Hopf algebra of graphs; however no explicit solution for a Schwinger Dyson equation is provided. The second condition was relaxed first in [37] and then concrete cases were constructed in [6, 3]. In the first, the authors proposed to study the Wess Zumino model for which, due to supersymmetry, the vertex does not get renormalised; in the second one the non linear Schwinger Dyson equation is an approximation. In [4] Bellon and Clavier extended the study of the Wess Zumino model including a resurgence analysis of the problem. My work with Bellon is the completion of this extension to the case where also the vertices need renormalisation and the next chapters will be dedicated to that.

The principle advantage of the second methods is to factor the evaluation of integrals and the full Green function by Mellin transform: the integral can be calculated in analytic regularisation and the Green function becomes a pseudodifferential operator that acts on it. In the recent publication [11] the authors were able to characterise the solution to Yukawa model with resurgent analysis. They were able to extract the values of Stokes constants and the trans-series contributions to its solution. Here with our methods we are able to reproduce their results and also show the first trans-series contributions for the scalar case.

III.1 The problem

Both models will be described by a single Schwinger Dyson equation:



(III.1)

where $a = (g/4\pi)^2$. In the case of Yukawa model the solid line represents the massless spinor field ψ and the dashed line the massless scalar field σ; their interaction is described by the Lagrangian

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}(\partial\sigma)^2 - g\bar{\psi}\sigma\psi.$$

For the massless scalar field model the solid line represents the massless charged field ϕ and the dashed line again the massless neutral scalar field σ; their interaction is now described by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}|\partial\phi|^2 - \frac{1}{2}(\partial\sigma)^2 - g\phi^*\sigma\phi.$$

Note that only the full Green function associated to the solid line gets fully renormalised. We will describe the full propagator as

$$P(p^2) = \frac{1}{p\!\!\!/}G(p^2),$$

in Yukawa model, and as

$$P(p^2) = \frac{1}{p^2}G(p^2),$$

in the scalar model. The formal series G will satisfy a renormalisation group equation

$$\partial_L G = (\gamma + \beta a \partial_a)\, G$$

where in this case $\beta = 2\gamma$ as no vertex correction is included, and $L = \log p^2$. We can give a general solution to this equation as a series in L with coefficients that depend on a:

$$G(a, L) = \sum_{n\geq 0} \gamma_n \frac{L^n}{n!}. \qquad \text{(III.2)}$$

In such a way the first coefficient can be interpreted as the anomalous dimension, usually indicated simply as γ. This recovers the definition of the anomalous dimension as

$$\gamma = \partial_L|_{L=0}G.$$

III.2 Method I: Yukawa model

We can translate equation (III.1) into

$$G(p^2)p\!\!\!/ = p\!\!\!/ - \frac{2a}{\pi^2}\int d^4l\, \frac{1}{G(l^2)l\!\!\!/(p+l)^2} - \text{subtractions}$$

where the subtractions depend on the initial condition of the renormalisation group equation. This equation is equivalent to

$$G(p^2) = 1 - \frac{2a}{\pi^2}\int \frac{d^4 l}{p^2}\frac{1}{G(l^2)l^2}\frac{p\cdot l}{(p+l)^2} - \text{subtractions} \tag{III.3}$$

where we have used $\not{p}^2 = p^2$. We have

$$\frac{p\cdot l}{(p+l)^2} = \frac{1}{2} - \frac{p^2+l^2}{2(q+l)^2}$$

and we can perform the angular integration considering

$$p^2 l^2 \left\langle \frac{1}{(p+l)^2} \right\rangle_{d=4} = \min(p^2, l^2). \tag{III.4}$$

Finally, recall

$$vol_{S^d} = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}.$$

We can split the terms in equation (III.3) considering the two intervals, when $l^2 \geq p^2$ and when $l^2 \leq p^2$, and have

$$G(p^2) = 1 + a\left[\int_p^{+\infty} dl \frac{1}{lG(l^2)} + \frac{1}{p^4}\int_0^p dl \frac{l^3}{G(l^2)}\right] - \text{subtractions}$$

because apart from common factors, the integrand

$$\int dl \left(1 - \min(p^2, l^2)\frac{p^2+l^2}{(p+l)^2}\right) = \int_0^{+\infty} dl\, 1 - \int_0^p dl\, \frac{p^2+l^2}{p^2} - \int_p^{+\infty} dl\, \frac{p^2+l^2}{l^2}$$

splits into terms that cancel the first. These terms themselves might still be divergent; that is why we still keep the subtractions. As we will discuss in next chapters we could even be fine with divergent integral, as long as there exists a finite step procedure to extract a certain anomalous dimension. These expression might be, in short, just formal. In any case, we can fix these subtraction terms if we interpret G as a series of corrections due to renormalisation and asking $G(\mu^2) = 1$ at the reference point. We obtain

$$G(p^2) = 1 + a\left[\frac{1}{p^4}\int_0^p dl \frac{l^3}{G(l^2)} - \frac{1}{\mu^4}\int_0^{\mu^2} dl \frac{l^3}{G(l^2)} - \int_{\mu^2}^p dl \frac{1}{lG(l^2)}\right].$$

We can introduce an auxiliary function

$$F(x) = \frac{a}{2}\int_0^x dy \frac{1}{yG(y)}\left(\frac{y}{x}\right)^2.$$

and rewrite our equation as

$$\begin{aligned} G(p^2) &= 1 + \frac{a}{2}\left[\frac{1}{p^4}\int_0^{p^2} dy \frac{y}{G(y)} - \frac{1}{\mu^4}\int_0^{\mu^2} dy \frac{y}{G(y)} - \int_{\mu^2}^{p^2} dy \frac{1}{yG(y)}\right] \\ &= 1 - \frac{a}{2}\int_{\mu^2}^{p^2} dy \frac{1}{yG(y)} + F(p^2) - F(\mu^2). \end{aligned} \tag{III.5}$$

Note that, for $x = p^2$,

$$\frac{d}{dx}F(x) = \frac{a}{2}\frac{1}{xG(x)} - \frac{a}{x^3}\int_0^x dy \frac{y}{G(y)}$$

so we can rewrite equation (III.5) as

$$\frac{d}{dx}G(x) = -\frac{a}{x^3}\int_0^x dy \frac{y}{G(y)}$$

or better

$$x^3 \frac{d}{dx}G(x) = -a\int_0^x dy \frac{y}{G(y)}.$$

This is an integro-differential equation. It suffices, though, to take the derivative with respect to x to both sides to have

$$3x^2 \frac{d}{dx}G(x) + x^3 \frac{d^2}{dx^2}G(x) = -\frac{ax}{G(x)}. \tag{III.6}$$

Introduce

$$D := x\frac{d}{dx}$$

and note that

$$\left[D, \frac{d}{dx}\right] = -\frac{d}{dx}.$$

We can rewrite equation (III.6) in the more appealing form

$$G(x)D(D+2)G(x) = -a. \tag{III.7}$$

III.2.1 Parametric solution

Rewrite the equation with dimensionless terms: let

$$z := \left(\frac{q^2}{\mu^2}\right)^2 = \frac{x^2}{\mu^4}$$

and

$$\tilde{G}(z) := \sqrt{2/a}\, zG(\mu^2\sqrt{z}).$$

The boundary condition is thus

$$\tilde{G}(1) = \sqrt{2/a}$$

With this change of variable

$$D = x\frac{d}{dx} = 2z\frac{d}{dz},$$

thus (III.7) becomes

$$\frac{a}{2}\frac{\tilde{G}}{z}2z\frac{d}{dz}\left(2z\frac{d}{dz}+2\right)\frac{\tilde{G}}{z} = -a,$$

or better

$$2\tilde{G}\frac{d}{dz}\left(z\frac{d}{dz}+1\right)\frac{\tilde{G}}{z} = -1.$$

Note at last that

$$\left(z\frac{d}{dz}+1\right)\frac{\tilde{G}}{z} = z\left(\frac{\tilde{G}'(z)}{z}-\frac{\tilde{G}(z)}{z^2}\right)+\frac{\tilde{G}(z)}{z} = \tilde{G}'(z)$$

Then (III.7) is simply

$$2\tilde{G}(z)\tilde{G}''(z) = -1. \tag{III.8}$$

Start by multiplying both sides by $\tilde{G}'(z)$ to have

$$2\tilde{G}'(z)\tilde{G}''(z) = -\frac{\tilde{G}'(z)}{\tilde{G}(z)}$$

that amounts to

$$\left[\tilde{G}'(z)\right]^2 = c - \log \tilde{G}(z)$$

with c a constant. This equation can be cast in the form of a system by the introduction of a parameter $p := \tilde{G}'(z)$, and then another differential equation for it. So first,

$$\tilde{G} = \sqrt{\frac{2}{a}}\exp(p_0^2 - p^2) \tag{III.9}$$

where we have used the initial condition $\tilde{G}(1) = \sqrt{2/a}$ and the parameter p at $z = 1$: $p_0 := \tilde{G}(1)$; then introduce $\tilde{\alpha}(p) = z/\tilde{G}$ that satisfies:

$$\frac{d\tilde{\alpha}(p)}{dp} = \frac{1}{\tilde{G}}\frac{dz}{dp} - \frac{z}{\tilde{G}^2}\frac{d\tilde{G}}{dp} = -2 + 2p\tilde{\alpha}(p). \tag{III.10}$$

This equation was obtained by inverting the functional dependence of z and $\tilde{G}$ in terms of p:

$$\begin{cases} \frac{d\tilde{G}}{dp} &= -2p\tilde{G}, \\ \frac{d\tilde{G}}{dp} &= \frac{dz}{dp}p \end{cases}$$

where we have used the definition of p for the last equality. These two impose

$$\frac{dz}{dp} = -2\tilde{G}.$$

that explains equation (III.10). The latter can be written as

$$\tilde{\alpha} = \frac{1}{p} + \frac{1}{2p}\frac{d\tilde{\alpha}}{dp},$$

and its solution has the form

$$\tilde{\alpha}(p) = e^{p^2}\sqrt{\pi}\left(\lambda - \mathrm{erf}(p)\right)$$

with erf the error function defined by

$$\mathrm{erf}(p) = \frac{2}{\sqrt{\pi}}\int_0^p ds\,\exp(-s^2)$$

and λ a constant. To determine its values notice that $\tilde{\alpha}$ has to be regular at infinity. Indeed, at $p \longrightarrow \infty$, the intermediate result (III.9) gives $\tilde{G} \to 0$, and therefore $z = 0$ since we can assume that G is vanishing nowhere, from the hypothesis of analyticity of the propagator. Then, writing

$$\tilde{\alpha}(p) = \sqrt{\frac{a}{2}}\frac{1}{G}$$

we have

$$\tilde{\alpha} \sim_{p\to\infty} \sqrt{\frac{a(q=0)}{2}}\frac{1}{G(0)},$$

which is finite. Since $\text{erf}(p) \sim_{p\to\infty} 1$, then $\lambda = 1$. Finally we have

$$\tilde{\alpha}(p) = e^{p^2}\sqrt{\pi}\text{erfc}(p) = 2\int_p^{+\infty} ds \exp(p^2 - s^2)$$

with erfc the complementary error function defined by

$$\text{erfc}(p) = 1 - \text{erf}(p) = \frac{2}{\sqrt{\pi}}\int_p^{+\infty} ds \exp(-s^2).$$

In [14] this result was obtain from an expansion in powers of $1/p$ of $\tilde{\alpha}$.

Let us go back to the solution of equation (III.8). Recall $z = (q^2/\mu^2)^2$, so

$$\tilde{\alpha}(p) = \frac{z}{\tilde{G}} = \sqrt{\pi}\exp(p^2)\text{erfc}(p)$$

gives

$$z = \left(\frac{q^2}{\mu^2}\right)^2 = \sqrt{\pi}\exp(p^2)\text{erfc}(p)\tilde{G} = \sqrt{\frac{2\pi}{a}}e^{p_0^2}\text{erfc}(p). \tag{III.11}$$

Recall also that p_0 is defined as the value of p at $z = 1$, so

$$\tilde{\alpha}(p_0) = \frac{1}{\tilde{G}(1)} = \sqrt{\frac{a}{2}} = \sqrt{\pi}e^{p_0^2}\text{erfc}(p_0),$$

that implies

$$e^{p_0^2} = \sqrt{\frac{a}{2\pi}}\frac{1}{\text{erfc}(p_0)}.$$

Thus equation (III.11) gives us

$$q^2 = \mu^2\sqrt{\frac{\text{erfc}(p)}{\text{erfc}(p_0)}}.$$

We are left with

$$G(q^2) = \sqrt{\frac{a}{2\pi}}\frac{e^{-p^2}}{\text{erfc}(p)}$$
$$q^2 = \mu^2\sqrt{\frac{\text{erfc}(p)}{\text{erfc}(p_0)}}.$$

This result can be cast in an equation for the anomalous dimension. Recall

$$\tilde{\gamma} := q^2 \frac{dG}{dq^2}\bigg|_{q^2=\mu^2},$$

the definition of the parameter p allow us to write

$$p_0 = \tilde{G}'(1) = \sqrt{\frac{2}{a}} \left(G + \frac{1}{2} q^2 \frac{dG}{dq^2} \right)\bigg|_{q^2=\mu^2},$$

that takes the form of

$$p_0 = \frac{2+\tilde{\gamma}}{\sqrt{2a}}.$$

From that and from equation (III.10) we have the equation

$$2\tilde{\gamma} = -a - \tilde{\gamma}^2 + 2a\tilde{\gamma}\frac{d}{da}\tilde{\gamma}.$$

To pay a tribute to the spirit of [14], note that the asymptotic series for $\tilde{\alpha}$

$$\tilde{\alpha}(p) \sim \frac{1}{p} + \frac{1}{p}\sum_{n=1}^{+\infty} \frac{(2n-1)!!}{(-2p^2)^n}$$

allows efficient computation to high orders of perturbation theory.

III.3 Method I: Charged scalar model

Let us now consider the case of the scalar field ϕ. We can now repeat the analysis but now the analogue for equation (III.4) is

$$p^2 l^2 \left\langle \frac{1}{(p+l)^2} \right\rangle_{d=6} = \mathfrak{m} - \frac{\mathfrak{m}^3}{3p^2 l^2}$$

where

$$\mathfrak{m} := min(p^2, l^2).$$

This implies that the Schwinger Dyson equations (III.5) take the form

$$G(p^2) = 1 - \frac{a}{6}\int_{\mu^2}^{p^2} dy \frac{1}{yG(y)} + F(p^2) - F(\mu^2)$$

but now the F function is more complicated, namely

$$F(x) = \frac{a}{6}\int_0^x dy \frac{1}{yG(y)} \left(\frac{y^3}{x^3} - 3\frac{y^2}{x^2} + 3\frac{y}{x} \right).$$

Note that

$$\begin{aligned}
\frac{d}{dx}F &= \frac{a}{6}\frac{1}{xG(x)} + \frac{a}{6}\int_0^x dy \frac{1}{yG(y)} \frac{d}{dx}\left(\frac{y^3}{x^3} - 3\frac{y^2}{x^2} + 3\frac{y}{x} \right) \\
&= \frac{a}{6}\frac{1}{xG(x)} - \frac{a}{2x^4}\int_0^x dy \frac{1}{G(y)} \left(y^2 - 2xy + x^2 \right) \\
&= \frac{a}{6}\frac{1}{xG(x)} - \frac{a}{2x^4}\int_0^x dy \frac{(x-y)^2}{G(y)},
\end{aligned}$$

so now the differential equation that G satisfies will be

$$x^4 \frac{d}{dx} G(x) = -\frac{a}{2} \int_0^x dy \frac{(x-y)^2}{G(y)},$$

or equivalently

$$GD(D+1)(D+2)(D+3)G = -a. \tag{III.12}$$

To write the associated equation for the anomalous dimension, let us write the last equation in the following way:

$$x^3 x \frac{d}{dx} G = -\frac{a}{2} \int_0^1 dz\, x^3 \frac{(1-z)^2}{G(xz)}$$

where we have performed just a change of variables in the integral. Thus cancelling the factor x^3 it is easier to write down

$$\gamma := \partial_L|_{L=0} G = -\frac{a}{2} \int_0^1 dz\, x^3 \frac{(1-z)^2}{G(\mu^2 z)}.$$

III.3.1 Parametric solution

Like before, introduce a variable to rephrase the problem in dimensionless quantities. Let

$$y := q^2/\mu^2,$$

and

$$\tilde{G}(y) := \sqrt{6/a} y G(\mu^2 y),$$

with the boundary condition

$$\tilde{G}(1) = \sqrt{6/a}.$$

Now D is simply

$$D = y \frac{d}{dy},$$

and equation (III.12) becomes

$$\frac{\tilde{G}}{6} \frac{d}{dy} (D+1)(D+2)(D+3) \frac{\tilde{G}}{y} = -1. \tag{III.13}$$

For $u, w \in \mathbb{R}$

$$[D+u, D+w] = 0,$$

for $\epsilon = \pm 1$

$$(D+u) y^\epsilon = (u+\epsilon) y^\epsilon,$$

and it satisfies a slightly modifies Leibniz rule

$$(D+u)(f \cdot g) = (D+u) f \cdot g + f (D+u) g - u f \cdot g.$$

So we can switch operators and calculate

$$(D+1) \frac{\tilde{G}}{y} = \frac{d}{dy} \tilde{G},$$

then

$$(D+3)(D+2)\frac{d}{dy}\tilde{G} = (D+3)\left(2\frac{d}{dy}\tilde{G} + y\frac{d^2}{dy^2}\tilde{G}\right)$$
$$= 6\frac{d}{dy}\tilde{G} + (D+5)y\frac{d^2}{dy^2}\tilde{G}$$

so we rewrite equation (III.13) as

$$\tilde{G}\tilde{G}'' + \frac{\tilde{G}}{6}\frac{d}{dy}(D+5)y\tilde{G}'' = -1. \qquad \text{(III.14)}$$

Again introduce a parameter

$$p := \tilde{G}'$$

and two functions

$$\begin{aligned} \alpha(p) :=& y/\tilde{G} \\ \beta(p) :=& -\tilde{G}\tilde{G}''. \end{aligned} \qquad \text{(III.15)}$$

The definition of these two functions is equivalent to the equation

$$\alpha(p) = \int_p^\infty \frac{ds}{\beta(s)} e^{-\int_p^s \frac{tdt}{\beta(t)}}. \qquad \text{(III.16)}$$

To see this, it suffices to prove that α satisfies the differential equation

$$\frac{d\alpha}{dp} - \frac{p}{\beta}\alpha = -\frac{1}{\beta} \qquad \text{(III.17)}$$

and solution (III.16) is obtained by the method of integrating factor and requiring the solution to be regular at infinity. From definition (III.15)

$$\frac{d\alpha}{dp} = \frac{dy}{dp}\frac{1}{\tilde{G}} - \frac{y}{\tilde{G}^2}\frac{d\tilde{G}}{dp} = \frac{dy}{dp}\frac{\tilde{G}''}{\tilde{G}\tilde{G}''} - \frac{\alpha\tilde{G}''}{\tilde{G}\tilde{G}''}\frac{d\tilde{G}}{dp},$$

together with

$$\frac{d\tilde{G}}{dp} = \frac{d\tilde{G}}{dy}\frac{dy}{dp} = p\frac{dy}{dp},$$

and

$$\tilde{G}''\frac{dy}{dp} = \frac{d\tilde{G}'}{dy}\frac{dy}{dp} = \frac{d\tilde{G}'}{dp} = 1,$$

equation (III.17) is obtained.

The differential equation for β can be written from equation (III.14) by first noticing that

$$y\tilde{G}'' = -\alpha\beta,$$

then

$$\frac{d}{dy} = \tilde{G}''\frac{d}{dp},$$

and

$$D = y\frac{d}{dy} = y\frac{d^2\tilde{G}}{dy^2}\frac{d}{dp} = -\alpha\beta\frac{d}{dp}.$$

So we have

$$\beta = 1 + \frac{\beta}{6}\frac{d}{dp}\left(5 - \alpha\beta\frac{d}{dp}\right)\alpha\beta.$$

So we have the differential system

$$\begin{cases} \frac{d\alpha}{dp} - \frac{p}{\beta}\alpha = -\frac{1}{\beta} \\ \beta = 1 + \frac{\beta}{6}\frac{d}{dp}\left(5 - \alpha\beta\frac{d}{dp}\right)\alpha\beta. \end{cases}$$

that can be solved iteratively yielding the two series

$$\alpha(p) \sim \frac{1}{p} + \frac{6}{p}\sum_{n>0}\frac{A_n}{(-6p^2)^n}$$

$$\beta(p) \sim 1 + \sum_{n>0}\frac{B_n}{(-6p^2)^n}$$

where $A_1 = 1$ and $B_1 = 5$.
From the asymptotic series of these two functions, the series for G can be reconstructed and thus also the series for γ:

$$\gamma \sim 6\sum_{n>0} A_n \left(\frac{-a}{(6\gamma+6)^2}\right)^n$$

This analysis by Broadhurst and Kreimer was a milestone. Their method allowed hundreds of terms of the series for the anomalous dimension to be calculated. However, non linear equations are not expressible as simply as differential equations; another method has to be deployed. I will present here a second method which is valid for non linear equations and that allows a resurgent approach.

III.4 Method II: Yukawa model

Let us go back to equation (III.1) and rewrite it as

$$\not{p}G^{-1}(p^2) = \not{p} - a\int_{\mathbb{R}^4} du\,\frac{1}{u^2}\frac{G((u+p)^2)}{\not{p}+\not{u}}. \tag{III.18}$$

Recall from equation (III.2) that we have the following formal series

$$G(a, L) = \sum_{n\geq 0}\gamma_n\frac{L^n}{n!},$$

and there is an associated pseudodifferential operator given by the same coefficients

$$G(a, \partial_x) = \sum_{n\geq 0}\gamma_n\frac{\partial_x^n}{n!},$$

where instead of the variable $L = \log(p^2/\mu^2)$, we have the operator ∂_x. The connection is immediate since

$$L^n = \partial_x^n e^{xL},$$

where the evaluation at zero is understood. With this tool at hand we can rewrite equation (III.18)

$$\not{p} G^{-1}(p^2) = \not{p} - aG_x \int_{\mathbb{R}^4} du\, \frac{\not{u} + \not{p}}{u^2((u+p)^2)^{1-x}}.$$

We will meet many times from now on this kind of integral so let me solve it in this case in which the gamma matrices makes the computation a little subtle. To calculate this integral let us introduce the Schwinger parameters:

$$\int_{\mathbb{R}^4} du\, \frac{\not{u} + \not{p}}{u^2((u+p)^2)^{1-x}} = \frac{1}{\Gamma(1)\Gamma(1-x)} \int_{\mathbb{R}^2_+} dt_1\, dt_2\, t_1^{-x} \int_{\mathbb{R}^4} du\, e^{-t_1(u+p)^2 - t_2 u^2} (\not{u} + \not{p}) \quad \text{(III.19)}$$

and complete the square

$$t_1(u+p)^2 + t_2 u^2 = t_{12}(u+p)^2 - t_2 p^2 - 2t_2 u \cdot p$$

where $t_{12} := t_1 + t_2$. We can change the variable of the integral letting $w = u + p$ obtaining

$$\int_{\mathbb{R}^4} du\, e^{-t_1(u+p)^2 - t_2 u^2} (\not{u} + \not{p}) = e^{-t_2 p^2} \int_{\mathbb{R}^4} dw\, \not{w} e^{-t_{12} w^2 + 2t_2 w \cdot p}.$$

This is a case of Gaussian integral

$$\int_{\mathbb{R}^4} dw\, w_\nu\, e^{-t_{12} w^2 + 2t_2 w \cdot p} = \frac{\partial_{p^\nu}}{2t_2} \left(\int_{\mathbb{R}^4} dw\, e^{-t_{12} w^2 + 2t_2 w \cdot p} \right) = \frac{\partial_{p^\nu}}{2t_2} \left(\frac{2\pi}{2t_{12}} \right)^{4/2} e^{\frac{t_2^2}{t_{12}} p^2}.$$

So we can rewrite equation (III.19) as

$$\int_{\mathbb{R}^4} du\, \frac{\not{u} + \not{p}}{u^2((u+p)^2)^{1-x}} = \frac{\pi^2}{\Gamma(1)\Gamma(1-x)} \not{p} \int_{\mathbb{R}^2_+} dt_1\, dt_2\, \frac{t_1^{-x} t_2}{t_{12}^3} e^{-\frac{t_1 t_2}{t_{12}} p^2}$$

The last integral is an homogeneous integral: since all $t_i \geq 0$ we can insert the identity $1 = \int_0^\infty d\lambda\, \delta(\lambda - \sum_{i \in T} t_i)$ where the set T can be chosen arbitrarily. Another change of variable $\tau_i = \lambda t_i$ leads to

$$\int_{\mathbb{R}^2_+} d\tau_1 d\tau_2 \frac{\tau_1^{-x} \tau_2}{\tau_{12}^3} \delta(1 - \sum_{i \in T} \tau_i) \int_0^\infty \lambda^{-x} e^{-\lambda \frac{\tau_1 \tau_2}{\tau_{12}} p^2} = \Gamma(-x)(p^2)^x \int_{\mathbb{R}^2_+} d\tau_1 d\tau_2 \frac{\tau_2^{1+x}}{\tau_{12}^3} \delta(1 - \sum_{i \in T} \tau_i).$$

Now if we choose $T = \{1, 2\}$ the last integral can be evaluated as a Euler Beta function

$$B(\alpha, \beta) = \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1},$$

and this leads to the result

$$\int_{\mathbb{R}^4} du\, \frac{\not{u} + \not{p}}{u^2((u+p)^2)^{1-x}} = \pi^2 \not{p} (p^2)^x \frac{\Gamma(-x)\Gamma(1)\Gamma(2+x)}{\Gamma(1)\Gamma(1-x)\Gamma(3+x)}.$$

This function is clearly divergent in a neighbourhood of the origin, but to have a regular expression it suffices to take the derivative with respect to L. In some sense, the point is that this kind of expressions are not simply divergent, they have more information. It is by taking (perhaps a finite number of) derivatives that we have a finite expression. An analogy could be done with the physically meaning of energy levels compared to their absolute value, or in more pertinent context, to the Bogoliubov approach to renormalisation. So let us go back to the Schwinger Dyson equation (III.18) and rewrite it as

$$G^{-1} = 1 + aG_x(p^2)^x \frac{\Gamma(-x)}{\Gamma(1-x)} \frac{1}{2+x}. \tag{III.20}$$

Recall that if G is the series

$$G = 1 + \gamma L + \gamma_2 \frac{L^2}{2} + \dots,$$

then its inverse will be

$$G^{-1} = 1 - \gamma L + (2\gamma^2 - \gamma_2)\frac{L^2}{2} + \dots \tag{III.21}$$

so if we hit both terms of equation (III.20) with ∂_L we have an equation for the anomalous dimension

$$\gamma(a) = aG_x(a)\left(\frac{1}{2+x}\right). \tag{III.22}$$

We can introduce now a family of functions F indexed by $k \in \mathbb{Z}$ as follows:

$$F_k := G_x\left(\frac{x}{k+x}\right)$$

They encode the behaviour of the Mellin transform around integer poles. Note that from the definition we have

$$\begin{aligned} F_k &= -G_x \sum_{m\geq 0} \left(\frac{-x}{k}\right)^{m+1} \\ &= -\sum_{\substack{n\geq 0\\ m\geq 1}} \gamma_n \frac{m!}{n!}\left(\frac{-1}{k}\right)^m \delta_{n,m} \\ &= -\sum_{n\geq 1}\left(\frac{-1}{k}\right)^n \gamma_n. \end{aligned}$$

Due to the renormalisation equation that G satisfies, if we indicate simply with ∂ the derivation of formal power series that shifts the coefficient by one, we have an equation

$$\partial_L G_x\left(\frac{x}{k+x}\right) = \gamma - kF_k,$$

or equivalently

$$\gamma(1 + 2a\partial_a)F_k = \gamma - kF_k.$$

To rewrite equation (III.22) in term of this functions F_k it suffices to observe

$$\frac{1}{2+x} = \frac{1}{2}\left(1 - \frac{x}{2+x}\right)$$

so we can write the system

$$\gamma = \frac{a}{2}(1 - F_2) \tag{III.23}$$
$$2F_2 = \gamma - \gamma(1 + 2a\partial_a)F_2$$

because $G_x 1 = \gamma_0 = 1$ since it satisfies a first order equation and we can fix the boundary conditions so that $G(1) = 1$.

This system is sufficient to generate the asymptotic series for the gamma function. To ease the resurgent computations let us change variable. Recall that for this model $\beta = 2\gamma$ and from equation (III.23) we know that

$$\beta(a) = a + O(a^2)$$

With a slight abuse of notation we want a $\beta(r)$ that starts as

$$\beta(r) = \frac{1}{r} + O(1/r^2)$$

so in this case the change of variable is very simple: $a = \frac{1}{r}$. Note that

$$r\partial_r = -a\partial_a$$

so our equations become

$$\gamma = \frac{1}{2r}(1 - F_2) \tag{III.24}$$
$$2F_2 = \gamma - \gamma(1 - 2r\partial_r)F_2 \tag{III.25}$$

We can solve it iteratively with a computer. The result matches with [14] but here I will just report the first terms as they will be useful later. Let me denote with [0] the zeroth order in a trans-series expansion.

$$\gamma[0] = \frac{1}{2r} - \frac{1}{8r^2} + \frac{1}{8r^3} + \dots \tag{III.26}$$
$$F_2[0] = \frac{1}{4r} - \frac{1}{4r^2} + \frac{27}{64r^3} + \dots \tag{III.27}$$

III.4.1 Trans-series solution

Consider the following trans-series expansion

$$\gamma = \sum_n e^{2nr} r^{\tau_n} c_n (1 + g_n/r + \dots) \tag{III.28}$$
$$F_2 = \sum_n e^{2nr} r^{\varphi_n} c'_n (1 + f_n/r + \dots)$$

and recall that $\tau_0 = \varphi_0 = -1$ and the [0]-coefficients are given by equations (III.26) and (III.27).

The first equation (III.24) is algebraic so we could substitute it in (III.25), but for conformity with the next cases let us keep it in this form. Before inserting the trans-series expansion observe that equation (III.25) features a peculiar property called resonance: at the order (1) the highest degree term r^{φ_1} appears on both sides with the same coefficient, so it vanishes; this imposes an equation for φ_1 where also the subdominant term of γ, the term g_0/r, contributes. In general the equation will be resonant whenever $(2 - 2c_0 2n) = 0$ where the first term comes from the left hand side of the equation, and the factor of two comes from the renormalisation differential operator. Note that we changed the variable in such a way that $c_0 = 1/2$, so resonance will occur only for the case $n = 1$. Whenever the equation is not resonant we can throw away all subdominant terms, in particular only the highest terms of $2\gamma r\partial_r F_2 = \beta r\partial_r F_2$ will contribute.

Let us calculate these contributions: from equation (III.24) we have

$$e^{2r} r^{\tau_1} c_1 = -e^{2r} r^{\varphi_1 - 1} \frac{c_1'}{2},$$

so

$$\tau_1 = \varphi_1 - 1 \tag{III.29}$$

$$c_1 = -\frac{c_1'}{2}. \tag{III.30}$$

To tackle equation (III.25) note that when two trans-series A, B are multiplied

$$(AB)[n] = \sum_k A[k]B[n-k],$$

so

$$\begin{aligned} 2e^{2r} r^{\varphi_1} c_1' = e^{2r} r^{\tau_1} c_1 - e^{2r} \Big(& r^{\tau_0 + \varphi_1} c_1' \left(c_0 + \frac{g}{r} \right) \left(1 - 2r \left(2 + \frac{\varphi_1}{r} \right) \right) \\ & + r^{\tau_1 + \varphi_0} c_0' c_1 \left(1 - 2r \frac{\varphi_0}{r} \right) \Big). \end{aligned}$$

We can simplify the right hand side to

$$\begin{aligned} 2e^{2r} r^{\varphi_1} c_1' = e^{2r} r^{\tau_1} c_1 - e^{2r} \Big(& r^{\tau_0 + \varphi_1} c_0 c_1' \left(-4r + 1 - 2\varphi_1 - 4g_0 + O(1/r) \right) \\ & + r^{\tau_1 + \varphi_0} c_1 c_0' \left(1 - 2\varphi_0 + O(1/r) \right) \Big). \end{aligned}$$

The first term in parenthesis matches the left hand side, as previously foretold. Recalling equations (III.29) and (III.30) we have

$$0 = r^{\varphi_1 - 1} c_1 (1 + 1 - 2\varphi_1 - 4g_0)$$

because the other terms are subdominant. This implies

$$\varphi_1 = 1 - 2g_0 = 1 + \frac{1}{2} = \frac{3}{2}$$

so from equation (III.29)

$$\tau_1 = \frac{1}{2}.$$

The coefficient c_1 remains a free parameter of the problem. This is not surprising since these equations are first order differential equations.

Now let us proceed with higher terms. Now resonance will not appear anymore but a factor $(2 - 2c_0 2n)^{-1}$ will appear on the right hand side. Only the highest terms can be considered so I will not write the rest. At order two we have again

$$\tau_2 = \varphi_2 - 1 \tag{III.31}$$

$$c_2 = -\frac{c_2'}{2}. \tag{III.32}$$

Without resonance the fact that $\tau_2 < \varphi_2$ allows to neglect the first term of the right hand side. We obtain

$$e^{4r} r^{\varphi_2} c_2' = (2 - 2c_0 4)^{-1} e^{4r} r^{\tau_1 + \varphi_1 + 1} 4 c_1 c_1'$$

that with equations (III.29), (III.30), (III.31) and (III.32) leads to

$$\tau_2 = 2\tau_1 + 1$$
$$c_2 = -2c_1^2.$$

For the third order term as always

$$\tau_3 = \varphi_3 - 1$$
$$c_3 = \frac{c_3'}{2}$$

and from the second equation, again without resonance as $(2 - 2c_0 6) \neq 0$,

$$e^{6r} r^{\varphi_3} c_3' = (2 - 2c_0 6)^{-1} e^{6r} \left(r^{\tau_1 + \varphi_2 + 1} c_1 c_2' 2(2 \cdot 2) + r^{\tau_2 + \varphi_1 + 1} c_2 c_1' 2 \cdot 2 \right).$$

Inserting all the previous values, we obtain

$$\tau_3 = \varphi_3 - 1 = 3\varphi_3 - 1 = 3\tau_1 + 2$$
$$c_3 = (2 - 2c_0 6)^{-1} 4(c_1 c_2 + 2c_2 c_1) = 6c_1^3$$

We are ready to guess the behaviour of the series:

$$\tau_n = \varphi_n - 1 = n\tau_1 + n - 1$$

$$c_n = (2 - 2c_0 2n)^{-1} 4 \sum_{k=1}^{n-1} k c_k c_{n-k} \tag{III.33}$$

These result match with those found in [11]. Both the latter and [5] found a way to re-sum the series: define the generating function

$$S(x) := \sum_{n \geq 1} c_n x^n,$$

then equation (III.33) induces the differential equation for S

$$\frac{S}{x} - S' = 2SS'$$

or

$$\frac{1}{x} = 2S' + \frac{S'}{S}. \tag{III.34}$$

Note that both terms of this equations are the derivative of a log: let

$$\chi = Se^{2S}$$

we have

$$\log \chi = \log S + 2S,$$

and

$$\partial_x \log \chi = 2S' + \frac{S'}{S}.$$

We can thus integrate equation (III.34) to

$$2\eta x = 2Se^{2S}. \tag{III.35}$$

where $\eta \in \mathbb{R}$ is an integration constant. Its value can be obtained by differentiating again equation (III.35) together with the conditions

$$S(0) = 0$$
$$S'(0) = c_1.$$

This leads to $\eta = c_1$. So equation (III.35) can be inverted with the Lambert function $W(x)$

$$2S = W(2c_1 x).$$

We can go back to equation (III.28) and consider the lowest term in non perturbative sector

$$\gamma = \sum_n e^{2nr} r^{n\tau_1+n-1} c_n + \ldots = \frac{1}{2r} W(2c_1 r^{\tau_1+1} e^{2r}) + O(1/r^2).$$

This is defined for $|2c_1 r^{\tau_1+1} e^{2r}| < e^{-1}$. For a similar application of trans-series resummations, but in the context of differential equations, I refer to [23].

Let us go now to the linear case for the scalar field.

III.5 Method II : Charged scalar model

For the scalar model, equation (III.1) can be translated into

$$p^2 G^{-1} = p^2 - a \int_{\mathbb{R}^6} du \, \frac{1}{u^2} \frac{G((u+p)^2)}{(u+p)^2}. \tag{III.36}$$

Now the integral to solve is a little simpler. The result is

$$\int_{\mathbb{R}^6} du \, \frac{1}{u^2((u+p)^2)^{1-x}} = (p^2)^{1+x} \frac{\Gamma(-1-x)\Gamma(2)\Gamma(2+x)}{\Gamma(4+x)\Gamma(1)\Gamma(1-x)}$$

so equation (III.36) can be written as

$$G^{-1} = 1 - a \, G_x \, (p^2)^x \frac{\Gamma(-1-x)\Gamma(2+x)}{\Gamma(1-x)\Gamma(4+x)}.$$

Notice that now simple derivative is not enough to make this expression regular. In fact we need to apply $\partial_L^2 + \partial_L$ to have

$$(\partial_L^2 + \partial_L)(p^2)^x = (x^2 + x)(p^2)^x = x(x+1)(p^2)^x,$$

and shift the arguments of Gamma function enough. With equation (III.21) this means

$$(2\gamma^2 - \gamma_2) - \gamma = -a\, G_x \frac{1}{(2+x)(3+x)}. \tag{III.37}$$

Recall that the renormalisation group equation for G induces a relation on its coefficients given by

$$\gamma_{n+1} = \gamma(1 + 2a\partial_a)\gamma_n,$$

thus we can finally write an equation for the anomalous dimension γ as

$$\gamma = \gamma(1 - 2a\partial_a)\gamma + a\, G_x \frac{1}{(2+x)(3+x)}.$$

The next step, as for the Yukawa case, is the introduction of the functions F_k. For this case observe

$$\begin{aligned}
\frac{1}{(2+x)(3+x)} &= \frac{1}{2+x} - \frac{1}{3+x} \\
&= \frac{1}{2}\left(1 - \frac{x}{2+x}\right) - \frac{1}{3}\left(1 - \frac{x}{3+x}\right) \\
&= \frac{1}{6} - \frac{x}{2(2+x)} + \frac{x}{3(3+x)}
\end{aligned}$$

So equation (III.37) can be put in the form of a system

$$\begin{aligned}
\gamma &= \gamma(1 - 2a\partial_a)\gamma + \frac{a}{6} - \frac{a}{2}F_2 + \frac{a}{3}F_3 \\
2F_2 &= \gamma - \gamma(1 - 2a\partial_a)F_2 \\
3F_3 &= \gamma - \gamma(1 - 2a\partial_a)F_3.
\end{aligned}$$

We can again change variable looking at the beta function. The beta function is always $\beta = 2\gamma$

$$\beta(a) = \frac{a}{3} + O(a^2)$$

so now the change of variable is $a = 3/r$. Finally,

$$\gamma = \gamma(1 + 2r\partial_r)\gamma + \frac{1}{2r}\left(1 - 3F_2 + 2F_3\right) \tag{III.38}$$

$$2F_2 = \gamma - \gamma(1 - 2r\partial_r)F_2 \tag{III.39}$$

$$3F_3 = \gamma - \gamma(1 - 2r\partial_r)F_3. \tag{III.40}$$

With this system we can again generate a series that matches the result of [14]. A computer can provide hundreds of terms in few seconds; here just report the first few terms

$$\begin{aligned}
\gamma[0] &= \frac{1}{2r} - \frac{11}{24r^2} + \frac{47}{36r^3} + \dots \\
F_2[0] &= \frac{1}{4r} - \frac{5}{12r^2} + \frac{775}{576r^3} + \dots \\
F_3[0] &= \frac{1}{6r} - \frac{17}{72r^2} + \frac{17}{24r^3} + \dots
\end{aligned} \tag{III.41}$$

III.5.1 Trans-series solution

The trans-series analysis is still incomplete: these are partial results waiting for a clearer picture.

Two features distinguish this case with the previous one: the equation for γ is no more algebraic; there is also an equation for F_3. All three equations have resonances at different trans-series order. This hinders in a non trivial way the all-order expansion. Let me just report the values of the dominant powers of the first three trans-orders, and explain how to extract them.

Let us adopt the same notation as for the Yukawa case §III.4.1, and let us start with the first contributions to the trans-series. Equation (III.38) is resonant at this order so let us start with equations (III.39) and (III.40)

$$2F_2[1] = \gamma[1] + \left(\gamma(1-2r\partial_r)F_2\right)[1] \tag{III.42}$$
$$3F_3[1] = \gamma[1] + \left(\gamma(1-2r\partial_r)F_3\right)[1]. \tag{III.43}$$

The term

$$\left(\gamma(1-2r\partial_r)F_2\right)[1] = \gamma[0](1-2r\partial_r)F_2[1] + \gamma[1](1-2r\partial_r)F_2[0]$$

and

$$\partial_r F_2[1] \sim F_2[1]$$

so equations (III.42) and (III.43) reduce to

$$F_2[1] \sim (2-2c_0)^{-1}\gamma[1]$$
$$F_3[1] \sim (3-2c_0)^{-1}\gamma[1].$$

If we now consider equation (III.38), the first term on the right hand side of equation (III.38) gives

$$\left(\gamma(1+2r\partial_r)\gamma\right)[1] \sim \gamma[1] + \frac{\tau_1+2g}{r}\gamma[1],$$

where g is the second term in the expansion (III.41) and τ_1 is the dominant exponent for order [1]. To see that, notice

$$\left(\gamma(1+2r\partial_r)\gamma\right)[1] = \gamma[0](1+2r\partial_r)\gamma[1] + \gamma[1](1+2r\partial_r)\gamma[0]$$

and

$$r\partial_r\gamma[1] \sim \gamma[1] + \frac{\tau_1}{r}\gamma[1],$$

where we have consider also the subleading term due to the resonance. The g factor comes from the product $\gamma[0]2r\partial_r\gamma[1]$. Now, the left hand side of equation (III.38) is cancelled and we remain with

$$0 = \frac{\tau_1+2g}{r}\gamma[1] + \frac{2F_3[1]-3F_2[1]}{2r}$$

that implies

$$\tau_1 + 2g + \frac{1}{2} - \frac{3}{2} = 0,$$

so

$$\tau_1 = \frac{23}{12}.$$

Since $F_2[1] \sim F_3[1] \sim \gamma[1]$, we can consider c_1, the parameter associated to $\gamma[1]$, to be the free trans-series parameter.

Three are the differential equations for the scalar model, so three should be the free parameters. Equations (III.38), (III.39) and (III.40) feature resonances at the first three orders: $\gamma[1]$ at first order, $F_2[2]$ at second order and $F_3[3]$ at third one. At trans-series order [2] equations (III.38) and (III.40) will not have resonance, but equation (III.39) will do. For terms proportional to e^{2r}, there are two contributions: one from the new trans-series parameter c_2, and another one from c_1^2. We would expect for the dominant powers to appear analogously combined: if at order one we have $r^{\alpha}c_1$, at order n, $r^{n\alpha}c_1^n$. Notice, though, that the right hand side of equation (III.40) will now have several terms and between them, some will have greater powers. In practice,

$$\left(\gamma(1-2r\partial_r)F_3\right)[2] = \gamma[0](1-2r\partial_r)F_3[2] + \gamma[1](1-2r\partial_r)F_3[1] + \gamma[2](1-2r\partial_r)F_3[0],$$

and now

$$\gamma[1]r\partial_r F_3[1] \sim r^{1+\tau_1+\phi_1} = r^{2\tau_1+1}$$

one degree higher than naively combinatorially expected. For the part proportional to the new trans-series parameter c_2 all but one of these terms are subdominant so

$$\begin{aligned} \phi_2 &= \tau_2 \\ c_2'' &= c_2(3-2\cdot 2c_0)^{-1} \end{aligned} \tag{III.44}$$

where the 3 comes from the term $3F_3$, a factor of 2 from the operator $2r\partial_r$ and the last one comes from $\gamma[0]r\partial_r F_3[2]$. In a similar fashion we study equation (III.38) and we have

$$\gamma[2] \sim (1-2\cdot 2c_0)^{-1}\left(2r\big(\gamma[1]\big)^2 - \frac{3}{2r}F_2[2] + \frac{1}{r}F_3[2]\right)$$

that with equation (III.44) implies

$$\tau_2 = \varphi_2 - 1 \tag{III.45}$$

$$c_2 = (1-2\cdot 2c_0)^{-1}\Big(-\frac{3}{2}\Big)c_2' \tag{III.46}$$

Now the resonant equation (III.39): the highest degree term

$$\gamma[0](1-2r\partial_r)F_2[2] \sim -2c_0 2F_2[2]$$

cancels the left hand side, so we need to go to the next order

$$\gamma[0](1-2r\partial_r)F_3[2] \sim -2F_3[2] + \frac{c_0(1-\varphi_2)-4g}{r}F_2[2]$$

where g is the second term in $\gamma[0]$. Using equations (III.45) and (III.46), we have

$$(1-2c_0 2)^{-1}\Big(-\frac{3}{2}\Big) = c_0(1-\varphi_2) - 4g$$

thus

$$\varphi_2 = \frac{5}{6} \Rightarrow t_2 = -\frac{1}{6}.$$

At third trans-series order the resonant equation is (III.40). This resonance allows us to calculate the dominant power. As for the previous order the terms coming from trans-series parameters c_1 and c_2 will have a shift in the highest power. In short, we find

$$\tau_3 = \frac{11}{4}.$$

The particular values of these powers suggest us that there might be logarithmic terms in the trans-series. We are currently working on a possible resummation of these trans-series contributions as for the Yukawa case, and we look forward to understand better the trans-series structure in the forthcoming future.

Ward–Schwinger–Dyson equations in ϕ^3_6 Quantum Field Theory

Marc P. Bellon, Enrico I. Russo

Sorbonne Université, CNRS, Laboratoire de Physique Théorique et Hautes Energies, LPTHE, 75005 Paris, France

Abstract

We develop a system of equations for the propagators and three point functions of the ϕ^3 quantum field theory in six dimensions. Inspired from a refinement by Ward on the Schwinger–Dyson equations, the main characteristics of this system are to be formulated purely in terms of renormalized quantities and to give solutions satisfying renormalisation group equations. These properties were difficult to get together, due to the overlapping divergences in the propagator. The renormalisation group equations are an integral part of any efficient resolution scheme of this system and will be instrumental in the study of the resurgent properties of the solutions.

It is our belief that this method can be generalized to the case of gauge fields, shedding some light on their quantum properties.

Mathematics Subjects Classification: 81T10, 81T15, 81Q40.
Keywords: Renormalisation, Schwinger-Dyson equation

Introduction

In this work we introduce a system of generalized Schwinger–Dyson equations for a massless ϕ^3 model in 6 dimensions.

Usual works on Schwinger–Dyson equations miss one of their fundamental interest for us, the possibility to solve them purely in terms of renormalized Green functions. This possibility was first encountered in the case of a linear Schwinger–Dyson equation in [1], developed in [2] and found a powerful illustration in [3], from which a whole series of refinements have emerged, see e.g., [4, 5, 6].

The Schwinger–Dyson equations start from the implication of field equations on the field correlators and involve at first a bare vertex, but in many cases can be converted to ones depending only on dressed vertices. However, this is not possible for the propagator corrections, which are written asymmetrically, with a bare vertex on one side and a dressed vertex on the other side. In the language of renormalisation Hopf algebras, the resulting combinatorial Schwinger–Dyson equation is not based on a Hopf algebra cocycle. It has been proposed to put suitable combinatorial factors to ensure this cocycle property [7], but this does not solve all problems. First of all, as has been remarked by K. Yeats in her thesis [8], this cannot solve the problem in theories where different particles can run in the one loop corrections for a single propagator, as in QCD. Furthermore, even in cases where such a formulation is valid at the combinatorial level, the dependence of the combinatorial factors on the internal structure of the graphs precludes its transformation into an analytic Schwinger–Dyson equation, where one simply use the values up to some order of the vertex functions in the right hand side of the equation. The problem is directly linked to the presence of overlapping divergences. The Schwinger–Dyson equations build complex diagrams by assembling simpler parts, but in the presence of overlapping divergences, the obtained diagram have alternative ways of being dismantled. Since each of the ways to disassemble a diagram imply a possible chain of counterterms, the counterterms associated to the constituents are not sufficient for a full preparation of the diagram: subdivergences remain.

The solution we adopt has quite a long history, since we found its first expression in a paper by J.C. Ward [9], completed in [10]. The most complete exposition in these early times appears in a conference report of Symanzik [11] and some further application of these ideas appear for example in [12] and especially [13], where the case of QCD is worked out. However the complexity of the proposed solution did not really allow for applications. An important further advantage of these methods is that they are naturally compatible with the Ward or Slavnov–Taylor identities expressing gauge invariance, when no consistent truncation of the usual Schwinger–Dyson equations can be found with this property. This will not appear in this paper, where we limit ourselves to scalar interactions, but we hope to go back to this question in future works.

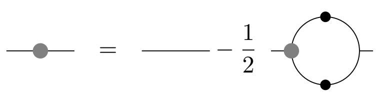
We have two additional ingredients in our proposal. First, since the general analytic dependence of three point functions on the kinematic invariants rapidly becomes cumbersome, we will reduce to the use of a single scale version, which has the same type of functional dependence as the propagator. We will sketch a systematic computation of the corrections to this first approximation. The second one is that renormalization group equations are a direct consequence of our Ward–Schwinger–Dyson equations. The proof we give depends only on the fact that we deal with a massless theory and apply also to the Wess–Zumino model considered in previous works. It has the double advantage to simplify the proof with respect to the one used in [3] and to make it independent on previous works on renormalisability. This avoids to make a detour by the corresponding combinatorial Schwinger–Dyson equations and their renormalisation. Since the renormalization group equations were an essential part in our analysis of the asymptotic properties of the perturbative solutions in the case of the Wess–Zumino model, we surmise that the same kind of analysis can be done for the singularities of the Borel transform as in [14] and that, as in [5], higher order terms in the Ward–Schwinger–Dyson equations only bring higher order corrections to the properties of these singularities.

The rest of the paper is structured as follows: in section §1 we will present the Schwinger–Dyson equations we will be studying and introduce the problem of overlapping divergences that until now has precluded the use of Schwinger–Dyson equations in the presence of vertex corrections; in section §1.3 following [15, 16] we will present a deformation of Feynman rules that resembles the IR rearrangement used in QCD to have single-scale vertices, paving the way to treat them as solutions of renormalisation equation; section §2 is a curious intermezzo that tangles Schwinger–Dyson equations and renormalization group equations providing us with a recipe for the β-function; and in section §4.3 and §5 we will provide results for the various anomalous dimensions and thus reconstructing up to order a^2 the 2-point and 3-point functions; then we conclude.

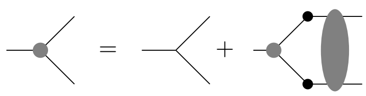
1 The Ward–Schwinger–Dyson equations

After the introduction of Hopf algebra methods to deal with the combinatorics of renormalisation, it has been recognized that Schwinger–Dyson equations can be formulated in terms of 1-cocycles B_+ in Hochschild cohomology [17, 18]. In the case of the Hopf algebra of (decorated) trees, $B_+(F)$ is the tree made by putting the roots of the trees in the forest F as direct descendants of a new root (which in the decorated case, will have the decoration associated to B_+). One obvious question is whether the solution of the said Schwinger–Dyson equations satisfies renormalization group equations. An important stepping stone is to know whether the series coefficients of the solution would generate a sub Hopf algebra of the Hopf algebra of graphs. The possible forms of such Schwinger–Dyson equations were investigated in [19] and one of the possible class of Schwinger–Dyson equations seems to correspond to the situation in quantum field theory. They indeed have variables which can be interpreted as propagators or vertex functions and the different B_+ act on products of powers of the variables, with the exponents corresponding to the number of vertices and propagators in a diagram.

However, Feynman rules for the evaluation do not apply to rooted trees but to Feynman diagrams which do not always have a nice translation in a tree structure, due to the overlapping divergences. To make things more concrete, we start from the Schwinger–Dyson equations for our model ϕ^3_6 :

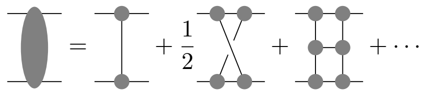


(1)



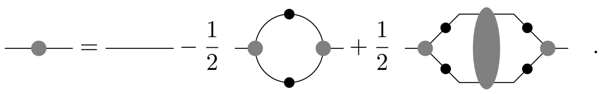
(2)

where the black and gray elements denote respectively dressed propagators and vertices. The big oval denotes a four particle kernel, the Bethe–Salpeter kernel, which can be expanded as a sum of two-particle irreducible graphs. In the following equation, the internal lines denote the full propagator, without an additional dot for the sake of readability:

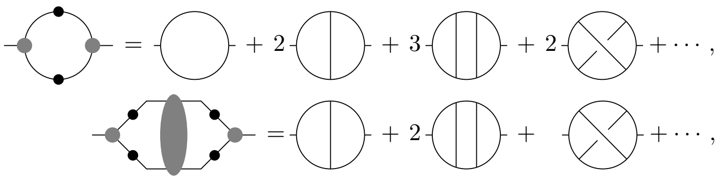
 (3)

The factor before the second diagram is a symmetry factor linked to the invariance of this diagram through the exchange of its two outputs. Since we will only make explicit computations at rather low order, the first term will be generally sufficient. Combining the three previous equations, one may obtain solutions as series of one-particle irreducible graphs, but it is not the path we will follow.

The asymmetry in the first equation might puzzle at first glance. Indeed, it is not possible to write an equation with a single diagram and a constant combinatorial factor. To restore its symmetry we use the expression in (2) at the cost of introducing a second object:

 (4)

Each term in this equation produces multiple counting as can be seen already from the first few terms of their expansion:

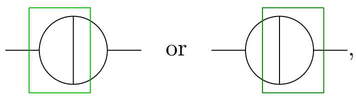


but the combination of the two restore the proper count.

1.1 The problem of overlapping divergences

A Feynman diagram Γ could have divergent evaluation despite a negative superficial degree of divergence $\omega(\Gamma)$. This happens whenever a sub-graph $\gamma \subset \Gamma$ is divergent, thus with $\omega(\gamma) \geq 0$. It was the main contribution of Bogolubov [20] to show that a recursive subtraction of counterterms could suppress such divergences. Alternatively, in the context of Schwinger–Dyson equations, we make use of renormalised vertices.

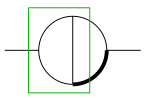
But how do we generalize this approach to the case of overlapping divergences? Take as an example

 (5)

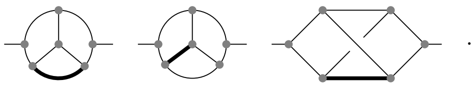
with its two possible subdivergence structures, which overlap. We would prefer a simple tree structure, clearly separating all primitive contributions in an expressions while avoiding double counting. If we look at equation (4), the first correction is fine, with a simple loop integration if we replace all vertices with their renormalized value, but the second one has visibly overlapping subdivergences, and the compensations between the two show that each one should be more difficult to evaluate than it seems.

The rather old solution, first proposed by Ward [9] is to derive with respect to the external momentum. The remark that the derivative of a propagator has better ultraviolet behavior is quite old and is at the base of the Bogolubov method [20]. Once the subdivergences are properly dealt with, a sufficient number of derivations makes the diagram convergent and its renormalized evaluation is obtained through integration of this finite result. The integration constants correspond exactly to the parameters of the Lagrangian and their arbitrariness corresponds to the choice of renormalisation scheme.

For us, it will be sufficient that the inclusion of the derivative of a propagator in a three point function makes it superficially convergent. If we mark the propagator with a derivation by a heavy line, one see that now the following diagram has only one possible subdivergence:

 (6)

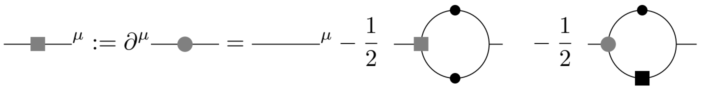
The highlighted line splits the subdivergences, in the sense that there are those coming before it, and those coming after it in the flow of momentum through the diagram. For the derivative of the two-point function, there are many possible primitive diagrams beyond the simple one loop one. Here are some examples



(7)

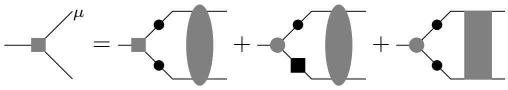
It remains however to know how these diagrams contribute to the derivative of the two-point function.

The first step is to derive the equation (1) for the two-point function:



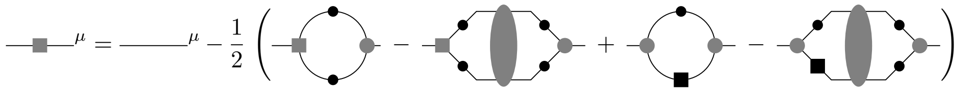
(8)

This equation involves the derivative of the vertex, for which one obtain an equation by deriving the equation (2):



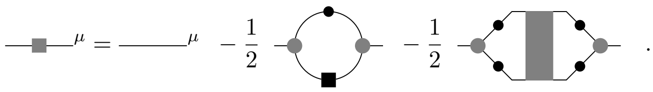
(9)

where the squares represent the derived objects and the round ones the original ones. We can now use equation (2) to re-express the right bare vertices in the derived propagator (8) and obtain



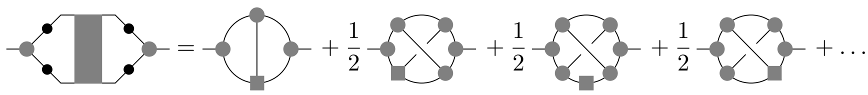
(10)

which ultimately, by use of equation (9) in the first term of the parenthesis, simplifies to:



(11)

If we expand the last term by deriving the expression for the four-point kernel (3) we have



(12)

Whenever we have a derivative of a vertex, we should make use of equation (9) to obtain a more explicit value, with the remaining parts involving derivatives of the vertex or the 4-point kernel having to be recursively expanded. The final objects we want to evaluate should have the derivative only on one propagator. We could in this way obtain certain of the diagrams in equation (7) with determined weights. However, since the first contributions beyond the one-loop one appear at three loop order, we will not need them in this work, but should keep in mind that the full solution will require an infinite set of primitives.

This concludes the determination of the equations that we will use. As in many cases in quantum field theory, it is not clear whether this derivation can be made rigorous, since we should at least put some regularization. In a sense, this is not really important if we can show that the solutions of these equations have the properties of the Green functions of a quantum field theory.

1.2 Renormalised propagators

As renormalised quantities, all our n-point functions depend on the coupling constant g and a renormalisation momentum scale μ. The two point function P has more natural variables: $a := g^2/\left(2\sqrt{\pi}\right)^D$ since all the graphs involved have an even number of vertices, and $L := \log(p^2/\mu^2)$ since the general solution to a Callan-Symanzik equation is a series in g with coefficients that are polynomials in L (see for instance [21], chap 5). We can represent $P(a, L)$ as

$$P(a, L) = \frac{1}{p^2} G(a, L) := \frac{1}{p^2} \sum_{n=0} \frac{\gamma_n(a)}{n!} L^n. \tag{13}$$

The free propagator $1/p^2$ has been factored out and the series $G(a,L)$ contains all the corrections due to the renormalisation process. In this representation, the first two coefficients of the series in L have special significance: $\gamma_0 = 1$, reflecting the renormalisation condition that the propagator is unchanged at the renormalisation point $L = 0$, and $\gamma_1 = \gamma$ the anomalous dimension of the field.

Noting that $L^n = \partial_x^n e^{xL}|_{x=0}$, we can describe the series $G(a,L)$ as a pseudodifferential operator obtained by replacing L by ∂_x in the series, acting on e^{xL}:

$$G(a,L) = G(a,\partial_x)\left(p^2/\mu^2\right)^x\Big|_{x=0}. \tag{14}$$

From now on, we will not indicate the evaluation at $x = 0$ but it remains implied.

The derivation $\mathcal{D}$ with respect to L will be important for us:

$$\mathcal{D}G(a,L) = \sum_{n=0} \frac{\gamma_{n+1}(a)}{n!} L^n. \tag{15}$$

In the context of the corresponding pseudodifferential operators, the iterations of $\mathcal{D}$ have a simple expression

$$G(a,\partial_x)\,(x^m \cdot f) = (\mathcal{D}^m G)(a,\partial_x)\, f \tag{16}$$

as we can easily show by direct inspection

$$G(a,\partial_x)\,(x^m \cdot f) = \sum_{n=0} \frac{\gamma_n(a)}{n!}\,\partial_x^n\,(x^m \cdot f) = \sum_{n=m} \frac{\gamma_n(a)}{(n-m)!}\,\partial_x^{n-m} f \tag{17}$$

using that $\partial_x^k x^m = m!\delta_{km}$ when evaluated at 0 and the binomial expansion of the higher derivatives of a product.

The Ward-Schwinger-Dyson equations depend on other quantities for which we will also need convenient expression. First we have the relation between the 2-point function (the propagator) and the corresponding vertex function (the 1PI two-point function)

$$P(a,L) \cdot \Gamma_2(a,L) = 1, \tag{18}$$

where the 1 comes from the fact that we are working in Euclidean space. The first few terms of $\Gamma_2(a,L)$ are given by

$$\Gamma_2(a,L) = p^2\left(1 - \gamma_1 L + \left(2\gamma_1^2 - \gamma_2\right)L^2/2 + O(L^3)\right). \tag{19}$$

We will need the derivatives with respect to p_ν of both P and Γ_2. This derivation will be abbreviated to ∂^ν in many cases. We have

$$K^\nu(a,L) := \frac{\partial}{\partial p_\nu} P(a,L) = \frac{2p^\nu}{(p^2)^2}\left(\mathcal{D}G(a,L) - G(a,L)\right) \tag{20}$$

since $\partial^\nu L = 2p^\nu/p^2$. This can be expressed in terms of the γ_n:

$$K^\nu(a,L) = \frac{2p^\nu}{(p^2)^2}\sum_{n=0}\frac{\gamma_{n+1}(a) - \gamma_n(a)}{n!} L^n. \tag{21}$$

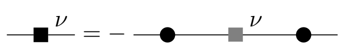
Likewise we define the derivative of the Γ_2 function,

$$C^\nu(a,L) := \partial^\nu \Gamma_2(a,L) = 2p^\nu\left(\mathcal{D}G^{-1}(a,L) + G^{-1}(a,L)\right) \tag{22}$$

with its coefficients obtained by adding two subsequent coefficients in the development of G^{-1}. The derivation of equation (18) gives a relation between K^ν and C^ν

$$K^\nu(a,L) = -P(a,L) \cdot C^\nu(a,L) \cdot P(a,L) \tag{23}$$

which graphically becomes



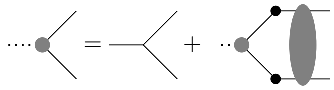
(24)

1.3 Infrared rearrangement

We still need a way to treat the renormalised vertex function Γ_3. The full three point function depends on all the possible scalar products of the external momenta minus the delta function condition, so in our case 3 variables. Their functional dependence on these kinematic invariants are unknown at higher loop order, but even the known cases do not easily allow for the evaluation of diagrams including vertices with the full dependence on all invariants.

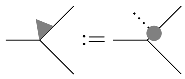
A guide for a solution is that the renormalisation functions only depend on the three point function evaluated at a multiple of the renormalisation condition. We will therefore try to only use this special case of the three-point function, which will have the same kind of functional dependence as the two-point one. This will use the trick called infrared rearrangement, where one transform a diagram in one with the same subdivergences, but with simpler evaluation. A nice characteristics of this six-dimensional theory is that we do not introduce infrared divergences in the process.

In the definition of the vertex in equation (2) we consider the case where one incoming momentum is set to zero. In Euclidean space, the null momentum condition is equivalent to the one of a zero norm for the momentum and impose that the invariants associated to the two other inputs of the vertex diagram are equal. A dotted line signals the zero momentum input.


(25)

In fact, it will be the only equation for the vertex that we will solve. It is well defined since in 6 dimensions, the product of two propagators with the same momentum is not infrared divergent.

However in this very equation, the vertices included in the four-point kernel have generic configurations of the inputs. We therefore want to reexpress all diagrams using only this single scale approximation of the vertex, by a process known as infrared rearrangement. This is made by joining two of the external legs of the vertex on one of its inputs while letting an other one of the inputs without any connection to the rest of the diagram and therefore with a zero momentum entry. Graphically, this is represented by two lines connecting to a single point of the boundary of a blob, while an interrupted dotted lines represent the unused third input of the vertex. This can be further simplified by using the following convention

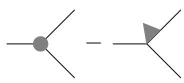

(26)

to denote the one scale object. Like the correction to the propagator, it can be characterised by a formal series, again in $\log(p^2/\mu^2)$:

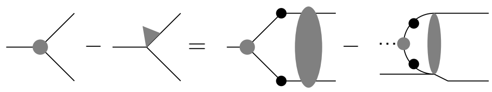
$$\Gamma_3(p;a) := gY(a,L) = g\sum_{n\geq 0} \upsilon_n(a)\frac{L^n}{n!} \tag{27}$$

with again the convention that the first term of the series υ_0 is set to 1, ensuring the renormalisation condition when $p^2 = \mu^2$.

Substituting the one scale object in the diagrams allows for their simple evaluation, since only logarithms of the momenta in some propagators are introduced, but introduces errors which must be controlled and computed. We therefore have to compute the following difference

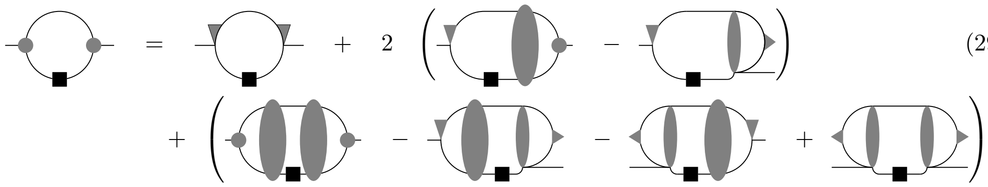


with both terms evaluated using respectively equations (2) and (25). The bare vertex is independent of all momenta and thus the same in both equations so that we obtain:


(28)

The two terms in the right hand side present the same subdivergences, so that substitution of this difference in a diagram will produce a primitive composite diagram. This will in particular mean that this correction will produce contributions subleading in powers of the log at a given order in the coupling. This is akin to the angle scale separation introduced in [15], which was an inspiration for this part of our work. A particular care should be paid to the last term in equation (28), since it seems to involve a elementary four-particle vertex, when the expansion from equation (3) is inserted in it. The two branches must merge into a single input outside of every expansion of the blobs. This means that we will have diagrams which are no longer strictly diagrams of the ϕ^3 theory.

We can now go back to equation (4) and insert equation (28) in it to obtain


(29)

where all propagators are supposed to be full propagators, even if we do not mark them by black circles. We see that the first term will be the dominant one in the expansion in a. The next parenthesis (with a factor of two indicating that you have the drawn diagrams plus their mirror images) will give a correction starting at order a^2 that will be addressed in section 5. The last group start contributing at order a^3 and is neglected in this work. We remark that in all these groupings, subdivergences are compensated between the different members, so that they behave as primitive divergences. We will not try to give an all order ansatz for the generation of all the primitive terms which will be generated by the recursive application of equation (28) in the diagrams. We suppose that they could be generated by a combinatorial Schwinger–Dyson equations for the full three-point functions followed by what should look akin to a renormalization where the counterterms are obtained by moving one of the exterior link of the subdiagram to make it single scale.

At each stage of our approximation scheme, we only deal with single scale versions of the vertices, with a logarithmic dependence on one the momentum entering it. This means that in our diagrams, the vertices do not introduce any new analytic difficulty, since they just add logarithmic factors similar to the ones coming from one of the neighbouring lines. New kind of dressed propagators will appear, like


. (30)

Their description in terms of series or pseudo-differential operators will be given just by the multiplication of the associated series G and Y. Even though we distinguish, at first, the action of these operators by applying them on different dummy variables, in the end a single variable can be associated to the internal line, since the product of exponents of the same p^2/μ^2 can be combined in the exponentiation by the sum of the variables. Take as an example the case of the operator YGY, associated to the third example in equation (30): by taking the Cauchy product of the series and by using the sum $x_{123} = x_1 + x_2 + x_3$

$$Y(a, \partial_{x_1}) G(a, \partial_{x_2}) Y(a, \partial_{x_3}) f(x_{123}) = \sum_{n \geq 0} (\upsilon\gamma\upsilon)_n \frac{\partial^n_{x_{123}}}{n!} f(x_{123}) \,.$$

Furthermore, we will see in the next section that, since each of the factor satisfies the same kind of renormalisation group equation, the product satisfies a renormalisation group equation which allows for the easy evaluation of its higher orders in L, so that, *in fine*, solving our equations written at a given order is no more complex than for preceding studies of Schwinger–Dyson equations.

2 WSD equations and the renormalisation group

Our strategy to solve the model is to use the Ward–Schwinger–Dyson equations to extract the anomalous dimensions of the 2 and 3 point functions and then generate all their higher orders in L by means of the renormalisation group equations. This is specially important for the propagator, since our equation (11) does not directly give the anomalous dimension γ as its coefficient of L, but a sum of the two first coefficients of the propagator. Higher order terms in L will likewise give sums of two terms. It is only because, at a given order in the coupling, the right hand side of the equation is polynomial in L that we can have definite values for all these coefficients. A non zero value for the higher orders in L of the inverse propagator Γ_2 would give an $\exp(-L)$ contribution, which would produce a constant term in Γ_2: such a constant is in contradiction with the renormalisation condition of our massless theory. The fact that a constant term is allowed by our equation can be easily understood, since it is an equation for the derivative of Γ_2.

Deducing the renormalisation group equations from the usual renormalisation procedure is however no longer the option it was in preceding works like [2, 3], since the propagators are deduced from their derivatives and do no longer correspond to the evaluation of a set of Feynman diagrams. However we will see that the renormalization group equations can be seen as consequences of the Schwinger–Dyson equations, in a rather simple way, without any need for special properties of the combinatorial solution of the Schwinger–Dyson equations. In particular, the β-function appears naturally as a combination of different anomalous dimensions and an "effective coupling" can be obtained as a combination of two- and three-point functions.

We will start from a rather naive observation: we consider a formal series $A(a, L)$ over the ring of functions in the variable a, which is supposed to satisfy a renormalisation group like equation with the anomalous dimension γ_A:

$$\mathcal{D}A(a, L) = (\gamma_A(a) + \beta a\partial_a)A(a, L). \tag{31}$$

We have the following lemma: let $\mathcal{P}(\mathcal{D})$ be a polynomial with constant coefficients in the variable $\mathcal{D}$, then if $A(a, L)$ satisfies the preceding equation (31)

$$\mathcal{D} \circ \mathcal{P}(\mathcal{D})A^m(a, L) = (m\gamma_A(a) + \beta a\partial_a)\mathcal{P}(\mathcal{D})A^m(a, L). \tag{32}$$

We see that $\mathcal{P}(\mathcal{D})A^m(a, L)$ satisfies a renormalization group equations with $m\gamma_A$ as anomalous dimension. This happens simply because the coefficients of $\mathcal{P}(\mathcal{D})$ do not depend on a and the $\mathcal{D}$ commutes with $(\gamma_A(a) + \beta a\partial_a)$. The lemma could be even generalised if we consider another formal series $B(a, L)$ that satisfies (31) with an anomalous dimension $\gamma_B(a)$ but with the same β-function. We have

$$\mathcal{D} \circ \mathcal{P}(\mathcal{D})A^m(a, L)B^n(a, L) = (m\gamma_A(a) + n\gamma_B(a) + \beta a\partial_a)\mathcal{P}(\mathcal{D})A^m(a, L)B^n(a, L) \tag{33}$$

and now the anomalous dimension is $m\gamma_A(a) + n\gamma_B(a)$.

We will now apply that lemma to the two series we have used to represent the 2 and the 3 point functions, respectively $G(a, L)$ and $Y(a, L)$. Supposing that they satisfy renormalisation group equations with anomalous dimensions γ and υ up to some order in a and L, we will show that the Ward–Schwinger–Dyson equations allow to extend them to a higher order.

In order to make the formulas less cumbersome, we will write G_x instead of $G(a, \partial_x)$, with the dependency on a understood. We can write the general Schwinger-Dyson equation as a series of primitive graph contributions:

$$C^\nu(a, L) = 2p^\nu \sum_n a^n \sum_{\mathbf{p}\in\mathbf{P}_n} \left(\prod_{i=1}^{I-1} G_{x_i}\right)(\mathcal{D}G_{x_I} - G_{x_I})\Big(\prod_{j=I+1}^{I+V} Y_{x_j}\Big)e^{-\omega_\mathbf{p}L}F_\mathbf{p}\left(\{x_i\}, \mu^2\right) \tag{34}$$

with in the I-th position the special link with a derivative of the propagator (marked by a square in the graphs). The terms on the right hand side are monomials of pseudo-differential operators applied to some characteristic integrals depending on the chosen model. These integrals are similar to those usually found in the literature, but they are evaluated in fixed dimension and appear with arbitrary decorations of the propagators and not the usual ones, integers eventually shifted by multiples of the dimensional parameter. They are generically divergent for $\omega_\mathbf{p} = \sum x_i$ equal to 0, where the evaluation finally takes place, but this divergence is a simple pole which is compensated by taking at least one derivative with respect to L. This is coherent with the renormalization approach, since the value of the left-hand part for L null can be fixed at will.

Our aim is to express the derivative with respect to L of equation (34). First of all, we can get rid of the special rôle of x_I by using equation (16) to rewrite $(\mathcal{D}G_{x_I} - G_{x_I}) = G_{x_I}(x_I - 1)$ and include the term $(x_I - 1)$, as well as a factor $\omega_\mathbf{p}$, to suppress the divergence, into $F_\mathbf{p}$ to define $\tilde{F}_\mathbf{p}$. Furthermore let us denote

$$\mathcal{O}_\mathbf{x} := \prod_{i=1}^{I} G_{x_i} \prod_{j=I+1}^{I+V} Y_{x_j} \tag{35}$$

and rewrite (34) as

$$\mathcal{D}C^\nu(a, L) = 2p^\nu \sum_{n\geq 1} a^n \sum_{\mathbf{p}\in\mathbf{P}_n} \mathcal{O}_\mathbf{x}\, e^{\sum x_i L}\tilde{F}_\mathbf{p}\left(\mathbf{x}, \mu^2\right). \tag{36}$$

or equivalently

$$\mathcal{D}^2 G^{-1} + \mathcal{D} G^{-1} = \sum_{n\geq 1} a^n \sum_{\mathbf{p}\in\mathbf{P}_n} \mathcal{O}_{\mathbf{x}}\, e^{\sum x_i L} \tilde{F}_{\mathbf{p}}\left(\mathbf{x}, \mu^2\right). \tag{37}$$

If we now take $\mathcal{D}$ on both sides, we have

$$\mathcal{D}^3 G^{-1} + \mathcal{D}^2 G^{-1} = \sum_n a^n \sum_{\mathbf{p}\in\mathbf{P}_n} \mathcal{O}_{\mathbf{x}} \left(\sum x_i\right) e^{\sum x_i L} \tilde{F}_{\mathbf{p}}\left(\mathbf{x}, \mu^2\right) \tag{38}$$

$$= \sum_n a^n \sum_{\mathbf{p}\in\mathbf{P}_n} \sum_k \mathcal{O}_{\mathbf{x}\backslash x_k} \mathcal{D}\mathcal{O}_{x_k}\, e^{\sum x_i L} \tilde{F}_{\mathbf{p}}\left(\mathbf{x}, \mu^2\right). \tag{39}$$

Here $\mathcal{O}_{x_k}$ denotes either G or Y according to k and since both of them satisfy a renormalisation group equation with the same β according to our recurrence hypothesis, then

$$\sum_k \mathcal{O}_{\mathbf{x}\backslash k} \mathcal{D}\mathcal{O}_k = \left(-\gamma + (3\gamma + 2\upsilon)n + \beta a \partial_a\right) \mathcal{O}_{\mathbf{x}} \tag{40}$$

since at a given order n we have $I = 3n - 1$ and $V = 2n$. Defining $\beta = 3\gamma + 2\upsilon$ and using that $a\partial_a a^n \mathcal{O}_{\mathbf{x}} = a^n \left(n + a\partial_a\right) \mathcal{O}_{\mathbf{x}}$, we have

$$\mathcal{D}^2 C^{\nu}(a, L) = \left(-\gamma + \beta a \partial_a\right) \mathcal{D} C^{\nu}(a, L). \tag{41}$$

We need to now recognise that C is as an inverse propagator, so that the minus sign in front of γ in the preceding equation is just the case $m = -1$ of equation (32). A similar computation for the three point function would give similarly, using that in this case, at order n, we have that $I = 3n$ and $V = 2n + 1$,

$$\mathcal{D}^2 Y(a, L) = \left(\upsilon + \beta a \partial_a\right) \mathcal{D} Y(a, L). \tag{42}$$

It seems that we are missing the case where the Green functions are taken at $L = 0$, but since the renormalisation conditions imply that the function at $L = 0$ are simply constants independent on a, this part of the renormalisation group equations are used to define γ and υ. It is the very fact that the Schwinger–Dyson equations can only define the derivative with respect to L of the Green functions, due to the divergence in the constant part, which introduces the possibility of a breaking of the scale invariance of the theory through the introduction of the anomalous dimensions γ and υ, which will produce the β-function. Higher point functions, which cannot be primitively divergent, do not entail the introduction of new renormalisation group functions.

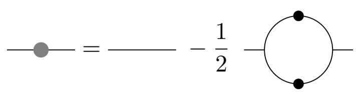
We therefore have accomplished our aim of proving that knowing the renormalization group equation up to a given order in the coupling a allows to get them on the following order and therefore to arbitrary order through the recurrence principle.

The derivation in this section should allow to consider the Schwinger–Dyson equations as an alternative approach to the renormalisation of a theory. The only regularisation we use is a simple consequence of our need to invoke propagators with arbitrary exponents in order to be able to consider the logarithmic corrections to the propagators and vertices, in the spirit of analytic regularisation [22, 23, 24], with the simplification that we only consider it for primitively divergent diagrams with a single pole in the neighborhood of 0. The consideration of a scale invariant, massless theory simplified the argument, but we are confident that the same approach could be used in more general cases.

3 Consistency check: no vertex correction

Before delving in the computations for our model, let us indulge first in a case already studied by one of the author in [4, 25], where no vertex correction is needed. It will be useful to set up notations and to verify that we can recover usual results.

We consider a complex field with an interaction Lagrangian density proportional to $\phi^3 + \phi^{*3}$. In this case we do not have vertex corrections at the one loop approximation and it is consistent to only consider the Schwinger–Dyson equation for the propagator. We start from the equation

 (43)

that in integral form reads

$$\Gamma_2(a, L) = p^2 - \frac{g^2}{2} \int_{\mathbb{R}^6} \frac{du}{(2\pi)^6} P(a, \log\, u^2/\mu^2) P(a, \log\,(p+u)^2/\mu^2). \tag{44}$$

This enforces an equation for the $G^{-1}(a, L)$ series:

$$G^{-1}(a, L) = 1 - \frac{a}{2} G_x G_y \, e^{(x+y)L} \frac{\Gamma(-1-x-y)\Gamma(2+x)\Gamma(2+y)}{\Gamma(1-x)\Gamma(1-y)\Gamma(4+x+y)}. \tag{45}$$

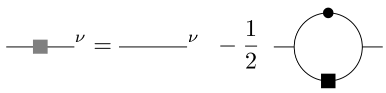
This expression is manifestly divergent in the neighbourhood of the origin due to the pole of $\Gamma(-1-x-y)$ for $x+y=0$. This divergence could be compensated by a mere differentiation with respect to L, but other terms can be added. In particular, we can follow [4] and add a second derivative with respect to L while in [25], a third derivative was used. To simplify notations, ∂_L, the partial derivative with respect to L, will imply an evaluation at $L = 0$. Applying $\partial_L + \partial_L^2$ to equation (45) result in the multiplication of the second hand by $(x+y)+(x+y)^2$ and we get

$$(\partial_L + \partial_L^2) G^{-1}(a, L) = -\frac{a}{2} G_x G_y H(x, y) \tag{46}$$

with $H(x, y)$ regular in the neighbourhood of the origin given by

$$H(x, y) = \frac{\Gamma(1-x-y)\Gamma(2+x)\Gamma(2+y)}{\Gamma(1-x)\Gamma(1-y)\Gamma(4+x+y)}. \tag{47}$$

In our new scheme, we consider an equation for the derivative C^ν, with a divergent second member, so that we still have to do a further differentiation with respect to L to get a finite equation. The left hand side, using the property (22) of C^ν, will also be given as $(\partial_L + \partial_L^2) G^{-1}$. We therefore start from



(48)

or in integral form

$$C^\nu(a, L) = 2p^\nu - \frac{g^2}{2} \int_{\mathbb{R}^6} \frac{du}{(2\pi)^6} \; P(a, \log\, u^2/\mu^2) K^\nu(a, \log\,(p+u)^2/\mu^2). \tag{49}$$

In this particular one loop case, the result can be obtained simply by taking the derivative ∂^ν of the equation (45) to obtain

$$C^\nu(a, L) = 2p^\nu \left(1 - a G_x G_y \, e^{(x+y)L} \frac{\Gamma(-x-y)\Gamma(2-x)\Gamma(2-y)}{\Gamma(1-x)\Gamma(1-y)\Gamma(4+x+y)}\right). \tag{50}$$

We still have a divergence coming from the pole in $\Gamma(-x-y)$, which can be cancelled by a differentiation with respect to L. Using equation (22) to express C^ν, we have

$$\partial_L^2 G^{-1} + \partial_L G^{-1} = -\frac{a}{2} G_x G_y H(x, y) \tag{51}$$

with the same $H(x, y)$ defined in equation (47), so that we obtain the same equations for the propagator as in the previous case. In the case of all other graphs, simple primitive ones described in equation (7) or combinations coming from the corrections to the infrared rearrangement introduced in equation (29), such a simple derivation is not possible and therefore we present a direct computation in appendix A which can be generalised to these other cases. The Mellin transform of the graph with a propagator $p^\nu (p^2)^{y-2}$ is the same function appearing in equation (50), but for a $\Gamma(2-y)$ in the denominator. However the expression of K^ν in terms of G in equation (20) introduces a factor $y-1$ which gives back the preceding result.

To solve equation (46), we must express the derivatives of the inverse propagator in terms of γ with the help of equation (19) and use the renormalisation group equation for $G(a, L)$, which can be spelled out as the following recursion for the coefficients

$$\gamma_{n+1} = \gamma(1 + 3a\partial_a)\gamma_n \tag{52}$$

since $\beta = 3\gamma$ when there is no correction to the vertices. This ultimately brings us to

$$\gamma = \gamma(1 - 3a\partial_a)\gamma + \frac{a}{2} G_x G_y H(x, y). \tag{53}$$

Using the first derivative of $H(x, y)$, we have

$$G_x G_y H(x, y) = \left(\frac{1}{6} - \frac{5}{18}\gamma + \dots\right) \tag{54}$$

and the solution starts as

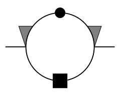
$$\gamma = \frac{1}{12}a - \frac{11}{432}a^2 + \dots \tag{55}$$

4 Computations

It is now time to apply all the machinery to the full ϕ^3 theory and check that it can reproduce known results. This will involve calculating corrections to the propagator and to the vertex.

4.1 Propagator: The Cat

Let's consider the first non trivial term in the full Schwinger-Dyson equation (11):



$$:= \int_{\mathbb{R}^6} \frac{du}{(2\pi)^6} \Gamma_3(-u) P(u) \Gamma_3(u) K^\nu(u + p) \tag{56}$$

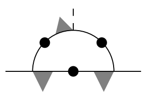
The integral is identical to the one encountered in the previous section and give rise to the same Mellin transform $H(x, y)$, given in equation (47). We therefore obtain a quite similar equation for γ apart for two differences. The second derivative of the propagator now involves a β function which depends on the vertex corrections and the vertex corrections introduce further terms proportional to $\log(u^2/\mu^2)$. We get a contribution for the anomalous dimension

$$-\gamma_2 + 2\gamma^2 - \gamma = -\frac{a}{2} (YGY)_x G_y H(x, y). \tag{57}$$

where we see now the product of series along the line with variable x. It is important to notice that we could not have moved the vertex corrections to the line with a derivative of the propagator, since the trick of obtaining the part with $\mathcal{D}G$ in K^ν simply by a multiplication by y would no longer work. It is interesting to notice that the natural separation of variables allows us to think of the diagram as composed of the two following propagator like objects:

4.2 Vertex: The rising sun

We must also look at the vertex contribution. In equation (25), we look for the lowest order correction and use infrared rearrangement to obtain a one loop diagram quite similar to the one studied in the previous section, only simpler since it does not involve K^ν. We look therefore at



$$:= \int_{\mathbb{R}^6} \frac{du}{(2\pi)^6} P(u)\Gamma_3(u)P(u)\Gamma_3(u+p)P(u+p)\Gamma_3(u+p) \tag{58}$$

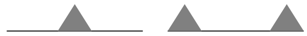
where the dashed line describes a 0-momentum coming in. We will see that the position of the vertex corrections does not modify the value of this diagram at the approximation level we compute in this work, but we argue that it is nevertheless the good one. A first, quite mundane argument is that it allows to reuse a propagator like object already appearing in the previous section. The more serious one is that in the asymptotic analysis we plan to do in the near future [26], this is the form which allows for the clearer derivation. This diagram is divergent by power counting, so we expect a contribution to the anomalous dimension υ. Looking at the first coefficient in L of equation (25), we have

$$\upsilon = -a (GYG)_x (YGY)_y H_2(x, y). \tag{59}$$

with the new Mellin transform $H_2(x, y)$ given by

$$H_2(x, y) = \frac{\Gamma(1 - x - y)\Gamma(1 + x)\Gamma(2 + y)}{\Gamma(2 - x)\Gamma(1 - y)\Gamma(3 + x + y)} \tag{60}$$

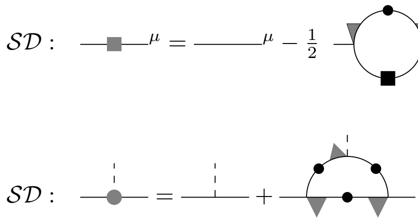
Again in this case we can imagine the diagram made of two composite objects:



where we have omitted the dashed lines.

4.3 First primitive diagram approximation

We are now ready to set up a system of equation that will allow us to solve the theory in the approximation of one primitive diagram for either of the propagator and vertex corrections. We have



$$\begin{cases} \mathcal{RG}: \quad \gamma_{n+1} = (\gamma_1 + \beta a \partial_a)\, \gamma_n \\ \mathcal{SD}/\mathcal{RG}: \quad \beta = 3\gamma + 2\upsilon \end{cases} \tag{61}$$

Using the results of the preceding subsections, this can be converted in the following system of equations:

$$\begin{cases} 2\gamma^2 - \gamma_2 - \gamma = -\frac{a}{2}\, (YGY)_x\, G_y\, H(x, y) \\ \upsilon = -a\, (GYG)_x\, (YGY)_y\; H_2(x, y) \\ \gamma_2 = (\gamma + \beta a \partial_a)\gamma \\ \beta = 3\gamma + 2\upsilon. \end{cases} \tag{62}$$

We do not write explicitly the equations needed to express the higher orders of G or Y, since they would only appear in computations at higher order, which could only be made exact through the introduction of many more primitives than the one we will consider here. At this stage, we simply indicate that it would be much more efficient to directly compute the products GYG and YGY from appropriate version of the renormalisation group equations than to compute explicitly the products. The first step is to expand the right hand sides in the system (62). We limit ourselves to the linear terms in the functions H and H_2:

$$\begin{aligned} (YGY)_x\, (G)_y\; H(x, y) &= (1 + (2\upsilon + \gamma)\partial_x + \dots)\,(1 + \gamma\partial_y + \dots)\, H(x, y) \\ &= \frac{1}{6} - \frac{5}{18}(\upsilon + \gamma) + \dots; \end{aligned} \tag{63}$$

$$\begin{aligned} (GYG)_x\, (YGY)_y\; H_2(x, y) &= (1 + (\upsilon + 2\gamma)\partial_x + \dots)\,(1 + (\gamma + 2\upsilon)\partial_y + \dots)\, H_2(x, y) \\ &= \frac{1}{2} - \frac{3}{4}(\upsilon + \gamma) + \dots. \end{aligned} \tag{64}$$

We can now write the equations in (62) as

$$\begin{cases} \gamma = (\gamma - \beta a \partial_a)\, \gamma + \dfrac{a}{12}\left(1 - \dfrac{5}{3}(\upsilon + \gamma) + \dots\right) \\ \upsilon = -\dfrac{a}{2}\left(1 - \dfrac{3}{2}(\upsilon + \gamma) + \dots\right) \\ \beta = 3\gamma + 2\upsilon. \end{cases} \tag{65}$$

The first two steps of the solution of this system are

$$1st: \qquad \gamma = \frac{a}{12}; \quad \upsilon = -\frac{a}{2}; \quad \beta = -\frac{3}{4}a \tag{66}$$

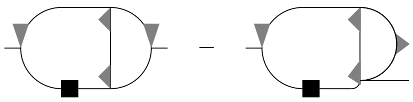
$$2nd: \qquad \gamma = \frac{a}{12} + \frac{55}{432}a^2; \quad \upsilon = -\frac{a}{2} - \frac{5}{16}a^2; \quad \beta = -\frac{3}{4}a - \frac{35}{144}a^2 \tag{67}$$

5 Corrections to the order a^2

The results from the previous paragraph are not complete at order a^2. We have two corrections to consider: one to the anomalous dimension γ and one to the anomalous dimension υ.

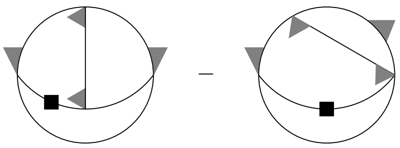
5.1 From the propagator

We first have the second term in (29), which is of minimum order 2:



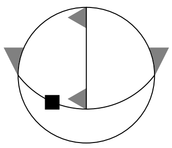
$$\sim a^2 \tag{68}$$

To estimate this contribution let us close the diagrams by connecting the external lines with such a propagator that they become conformal [1] :



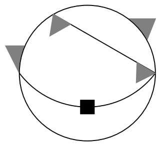
(69)

The propagator with the square is of the form $p^\mu/(p^2)^2$ so the line we added will carry a similar factor q^μ to make the diagram a scalar. To fix notations, the lines 1, 2 and 3 form the triangular subdivergence, 5 is the line carrying the box decoration and 6 is the line added to close the graph. We call $\Gamma_{\tilde{i}}$ the completed graphs and Γ_i the propagator-like ones. Massless vacuum diagrams have a vanishing second Symanzik polynomial and the first one is related to the Symanzik polynomials of the uncompleted graph. Calling the Schwinger parameters t_i, we have



$$\psi_{\tilde{1}} = t_6\psi_1 + \phi_1. \tag{70}$$

and



$$\psi_{\tilde{2}} = t_6\psi_2 + \phi_2. \tag{71}$$

We point out that $\psi_1 = \psi_2$, thus

$$\psi_{\tilde{1}} - \psi_{\tilde{2}} = \phi_1 - \phi_2. \tag{72}$$

so in particular it does not depend on t_6. In the absence of any decoration, for a zero momentum insertion on the line 5, we would calculate

$$\int dt_1 \dots dt_6 t_5 t_6^2 \left(\frac{1}{\psi_{\tilde{1}}^3} - \frac{1}{\psi_{\tilde{2}}^3} \right) \delta_H \tag{73}$$

while with the numerators we have

$$\int dt_1 \dots dt_6 t_5 t_6^3 \left(\frac{C_{\tilde{1}}}{\psi_{\tilde{1}}^4} - \frac{C_{\tilde{2}}}{\psi_{\tilde{2}}^4} \right) \delta_H. \tag{74}$$

[1] In a forthcoming paper there will be a more complete characterisation of the method [27]

The δ_H factors in these integrals represent the restriction of these integrations to a hyperplane necessary to make these scale invariant integrals finite. We will not further precise these factors since these homogeneous integrals are independent on their precise choice. The exact formulas involve Γ factors which become important when considering modified propagators, but at this stage we can ignore them, because they are either taken for an index 1 or 2 and give 1, or there is a compensation between a $\Gamma(D/2)$ factor in the numerator and the same factor in the denominator coming from the exponent of t_6. The computation of the numerators was completely described in a book by Nakanishi [28]: they could be calculated as a minor of a matrix associated to the graph, or by finding subgraphs with exactly one cycle including the lines 5 and 6. We have

$$C_{\tilde{1}} = t_4(t_1 + t_2 + t_3) + t_3 t_2 \tag{75}$$

$$C_{\tilde{2}} = t_4(t_1 + t_2 + t_3) + (t_1 + t_2)t_3 \tag{76}$$

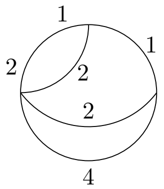
and therefore

$$C_{\tilde{1}} - C_{\tilde{2}} = -t_1 t_3. \tag{77}$$

We can therefore write the parenthesis in equation (74) as:

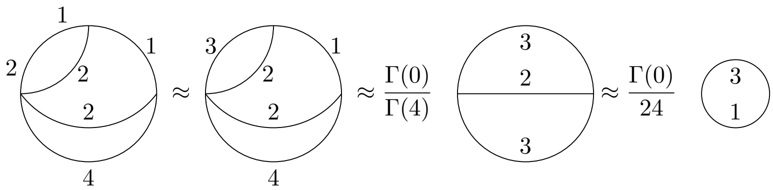
$$\begin{aligned}\frac{C_{\tilde{1}}}{\psi_{\tilde{1}}^4} - \frac{C_{\tilde{2}}}{\psi_{\tilde{2}}^4} &= C_{\tilde{1}}\left(\frac{1}{\psi_{\tilde{1}}^4} - \frac{1}{\psi_{\tilde{2}}^4}\right) - \frac{t_1 t_3}{\psi_{\tilde{2}}^4} \\ &= C_{\tilde{1}}\left(\psi_{\tilde{2}} - \psi_{\tilde{1}}\right)\left(\frac{1}{\psi_{\tilde{1}}^4\psi_{\tilde{2}}} + \frac{1}{\psi_{\tilde{1}}\psi_{\tilde{2}}^4} + \frac{1}{\psi_{\tilde{1}}^2\psi_{\tilde{2}}^3} + \frac{1}{\psi_{\tilde{1}}^3\psi_{\tilde{2}}^2}\right) - \frac{t_1 t_3}{\psi_{\tilde{2}}^4}.\end{aligned} \tag{78}$$

From equation (72) we know that $\left(\psi_{\tilde{2}} - \psi_{\tilde{1}}\right)$ does not depend on t_6 so that the big first term in the preceding equation behaves as $1/t_6^5$ for large t_6, making therefore a convergent contribution in equation (74) which will be absorbed in the renormalisation condition.[2] At order a^2, we are left with the contribution coming from $-t_1 t_3 t_5 t_6^3/\psi_{\tilde{2}}^4$. This integral is the one we would obtain from the graph $\Gamma_{\tilde{2}}$ in $D = 8$ since the exponent of the denominator is 4. The factors in the numerator imply however that it must be taken with propagators with different indices.



(79)

We can proceed by reduction as shown in the appendix B

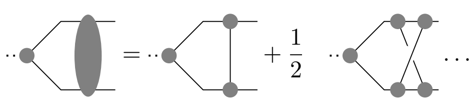


(80)

We consider the last graph as the natural by-product of closing the diagram and therefore evaluate it to 1. The factor $\Gamma(0)$ represents the divergence which will be compensated by taking the derivative with respect to L, so that we remain with a residue $1/24$ which is the period of the graph. Finally, the 1/2 symmetry factor and the doubling associated to the presence of the two symmetric possibilities for the corrected vertex cancel, so that we end up with a contribution $-1/24a^2$ for γ.

5.2 From the vertex

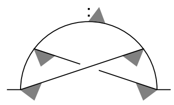
In the non trivial term appearing in equation (25), the 4-point function can be expanded as in equation (3) to give



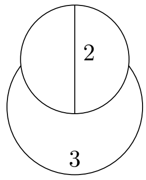
(81)

[2]Since the difference we look at makes a primitive divergence, this global divergence is the only one which has to be checked, but one can also explicitly see that there are no subdivergences associated to the (1, 2, 3) subdiagram.

A correction comes from the second term, in which, according to our approximation scheme, we can shift the decorations of the vertices to obtain

$$\frac{1}{2}$$  (82)

In fact, at the order a^2 we can simply ignore the decorations. We now close the diagram to obtain a vacuum one without any divergent subgraph

$$\frac{1}{2}$$ 

(83)

so that the divergence will come only from the integration over t_6. In fact

$$\frac{1}{2}\int_{\mathbb{R}_+^6} dt_1 \dots dt_6 \frac{t_5 t_6^2}{\tilde{\psi}^3}\delta_H \tag{84}$$

where again

$$\tilde{\psi} = t_6\psi + \phi \tag{85}$$

The divergence takes the form of a pole with a residue linked to the factorisation of the diagram in the additional line 6 and the original diagram without its exterior lines. In the context of fixed dimension analytic regularisation, such poles have been studied in the general case in [5] and the residue is obtained as $1/\Gamma(D/2)$ times the product of the evaluation of the parts. The single loop as well as the remaining two loop graphs with all propagators with index 2 evaluate to 1, so that this residue is just $1/\Gamma(D/2) = 1/2$. With the symmetry factor $1/2$ of this graph, we have therefore a contribution of $a^2/4$ for the anomalous dimension υ.

At the same order, we seem to have an analog of the contribution studied in the preceding subsection, coming from a correction to the infrared rearrangement. However this contribution turns out to be finite, like the term proportional to C_1 in equation (78), so that it does not affect the anomalous dimension υ.

5.3 Comparison with known results

Our results are easily generalisable to include multicomponent fields. The structure of the diagrams remains unchanged and likewise the structure of the Schwinger–Dyson equations. The dependence on a single coupling can be obtained if we have a symmetry group. We would need to include Casimir factors that in the literature are usually denoted by T_i. Starting from an interaction term where ϕ^3 is replaced by $d_{ijk}\phi^i\phi^j\phi^k$, the T_i relate the trace of the i-fold product of tensors d_{ijk} and δ_{ij} (in the case of T_2) or d_{ijk} (in the case of all others T_j, where j is always odd). The first two Casimir T_2 and T_3 are defined by

$$d_{ijk}d_{ljk} = T_2\delta_{il} \qquad d_{ilm}d_{jln}d_{kmn} = T_3 d_{ijk}. \tag{86}$$

In fact, the order is not sufficient to characterize these Casimir factors, starting at order 7, but since they would only appear in vertex corrections with at least three loops or propagator corrections with at least one additional loop, this is of no concern to us in this work. These group factors change our equations in the following way

$$\gamma = (\gamma - \beta a\partial_a)\gamma + \frac{T_2 a}{2}(YGY)_x G_y H(x,y) \tag{87}$$

$$\upsilon = -T_3 a (GYG)_x (YGY)_z H_2(x,z) \tag{88}$$

$$\beta = 3\gamma + 2\upsilon \tag{89}$$

Since the operators contain themselves products of γ and υ we expect products of T_i appearing all over. Our first correction from (68) will intervene with a factor T_2T_3 while the second one from

(82) will come with a factor T_5. With these corrections we have

$$\gamma = \frac{T_2}{12}a + (-11T_2 + 48T_3)\,\frac{T_2}{432}a^2 \tag{90}$$

$$\upsilon = -\frac{T_3}{2}a + \left(-\frac{T_5}{4} + \frac{T_3}{16}\,(T_2 - 6T_3)\right)a^2 \tag{91}$$

$$\beta = (T_2 - 4T_3)\,\frac{a}{4} + \left(-11T_2^2 + 66T_2T_3 - 108T_3^2 - 72T_5\right)\frac{a^2}{144} \tag{92}$$

We can compare it with previous results for the β-function [29, 30]. This comparison is not immediate, since our definition of the β-function is unusual, as can be seen from the way we have written the renormalisation group equations: $\beta(a) := \mu\frac{d}{d\mu}\log a = \frac{2}{g}\mu\frac{d}{d\mu}g$. The coefficient of a is twice the coefficient of g^3 in other works and the one of a^2 is twice the coefficient of g^5. Furthermore, in the work [29], the conventions are such that the terms with odd powers of a have an additional minus sign. With these provisions, we recover the usual β-function, but the T_2T_3 term in γ is off by a factor of 2: this is not completely unexpected, since our renormalisation conditions are different and only the β-function is scheme independent at this order.

Conclusions

In this paper we have established and solved the Ward–Schwinger–Dyson equations for a massless ϕ^3 in 6 dimensions. Our interest in this model is motivated by the presence of vertex corrections, in the simple context of a scalar theory, allowing to go beyond the Wess–Zumino model studied in our previous works. The presence of overlapping divergences was a major block for the expression of Schwinger–Dyson equations solvable in terms of renormalised Green functions. We have deployed a method inspired by Ward [9] that consisted in studying a version of the Schwinger–Dyson equations for the derivative of the 2-point functions with respect to the external momentum. The reduction to analytically simple versions of these equations was achieved by a deformation of the diagrams akin to the infrared rearrangements used in other contexts (section §1.3).

An interesting part of this work is that in section 2, we have shown that the Schwinger–Dyson equations imply the renormalization group equations, by passing the need to deduce them from the usual renormalisation procedure. We could not develop it in this letter, but this could be the base of a proof of the renormalized perturbative series independent on the methods of BPHZ [20, 31, 32] and their more recent avatars [33, 34]. As in the works of Epstein and Glaser [35], fully renormalized lower order results are used to produce the next step in the computation.

Finally, in sections §4.3 and §5 we have given the solution up to the order a^2, fully compatible with known results. It would be possible to proceed to higher orders, but we think that the present computations are sufficient to illustrate the general procedure. We suppose that computations at higher orders would be competitive in complexity with usual methods based on dimensional regularisation, especially for the contributions coming from deeply nested divergences. We believe these methods to be applicable to many other theories and we look forward to apply them to the case of theories with a gauge symmetry.

Acknowledgement

E. Russo would like to thank S. Teber for useful discussions about calculations of Feynman integrals.

A Parametric representations

It is well known that it is possible to rewrite a Feynman integral as a projective integral. The first step is to use the so called "Schwinger trick":

$$x^{-\alpha}\Gamma(\alpha) = \int_{\mathbb{R}_+} dt\, t^{\alpha-1}e^{-tx} \tag{93}$$

on each propagator.

Then we need some definitions. A graph is a collection of vertices and edges, a tree (T) is a graph with no loops and a forest is a disjoint union of trees. A spanning tree (forest) is a tree (forest) which contains all the vertices of the graph. Let $\mathcal{T}$ be the set of spanning trees and $\mathcal{F}_k$ the set of spanning forests with exactly k components. Let I be the set of internal edges e_i, V the set of vertices. We associate to each edge a mass parameter m_i and a power of the propagator α_i. Finally let p_T be the sum of the external momenta entering the subgraph T. In the case of a graph with only one connected component, $h_1(\Gamma) = |I| - |V| + 1$ is the number of loops.

It is then possible to associate to a given graph Γ the two Symanzik polynomials ψ_Γ and ϕ_Γ with the following definitions:

$$\psi_\Gamma(\{t_i\}) := \sum_{T \in \mathcal{T}} \prod_{e_i \notin T} t_i$$

$$\phi_\Gamma\left(\{t_i\}, \{p_i \cdot p_j\}, \{m_i\}\right) := \sum_{(T_1,T_2) \in \mathcal{F}^2} p_{T_1}^2 \Bigg(\prod_{e_i \notin (T_1,T_2)} t_i \Bigg) + \psi_\Gamma(\{t_i\}) \sum_{i=1}^{|I|} t_i m_i^2.$$

The Feynman integral for the graph Γ can then be written as

$$\mathscr{I}\left(\{p_i \cdot p_j\}, \{\alpha_i\}, \{m_i\}, D\right) = \int_{\mathbb{R}_+^{|I|}} \prod_{i \in I} \Big(\frac{dt_i t_i^{\alpha_i - 1}}{\Gamma(\alpha_i)} \Big) \frac{e^{-\phi/\psi}}{\psi^{D/2}}$$

where we have omitted the variables on which ψ and ϕ depend for the sake of readability. From their definition, we can see that the Symanzik polynomials ψ and ϕ are homogeneous in the t_i variables with respective degrees $h_1(\Gamma)$ and $h_1(\Gamma) + 1$. If we insert the equality

$$1 = \int_0^{+\infty} d\lambda \, \delta(\lambda - H(t)) \tag{94}$$

for any non-zero hyperplane equation $H(t) = H^j t_j$, we obtain:

$$\mathscr{I}\left(\{p_i\}, \{\alpha_i\}, \{m_i\}, D\right) = \prod_i^{|I|} \int_{\mathbb{R}_+} \frac{dt_i t_i^{\alpha_i - 1}}{\Gamma(\alpha_i)} \frac{\delta(1 - H(t))}{\psi^{D/2}} \int_0^{+\infty} d\lambda \, \lambda^{\sum \alpha_i - D/2 \, h_1(\Gamma) - 1} e^{-\lambda \phi/\psi}$$

with a small abuse of notation due to the scaling of t_i. In terms of the superficial degree of divergence ω,

$$\omega := h_1 \frac{D}{2} - \sum_i^{|I|} \alpha_i,$$

the integral on λ gives $\Gamma(-\omega)(\phi/\psi)^\omega$. The presence of the hyperplane in the delta function induces a split on the domain and on the volume form which brings us to

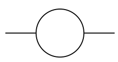
$$\frac{\Gamma(-\omega)}{\prod_i \Gamma(\alpha_i)} \int_{\mathbb{RP}_+^{|I|-1}} \Omega_H \; H^{|I|} \frac{1}{\psi^{D/2}} \left(\frac{\phi}{\psi} \right)^\omega \prod_i t_i^{\alpha_i - 1}$$

written as an integral on the projective space $\mathbb{RP}_+^{|I|-1}$ with the volume form Ω_H given by

$$\Omega_H := \sum_i^{|I|} (-1)^{i-1} \frac{t_i}{H} \quad d\left(\frac{t_1}{H}\right) \wedge \cdots \widehat{d\left(\frac{t_i}{H}\right)} \cdots \wedge d\left(\frac{t_{|I|}}{H}\right).$$

This integral is independent on the particular choice of the hyperplane thanks to its homogeneity, since for a different choice H', $\Omega_{H'} = (H/H')^{|I|} \Omega_H$.

In the case of the one-loop massless diagram ($m_i = 0 \quad \forall i \in I$)



the polynomials are very simple:

$$\psi = t_1 + t_2$$
$$\phi = t_1 t_2 p^2.$$

By the choice of the hyperplane $H(t) = t_2$ we get

$$\begin{aligned}\mathscr{I}(p^2, \alpha_1, \alpha_2, D) &:= (p^2)^\omega \frac{\Gamma(-\omega)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int dt_1 dt_2\, \delta(1-t_2) \frac{t_1^{D/2-\alpha_1-1} t_2^{D/2-\alpha_2-1}}{(t_1+t_2)^{D-\alpha_1-\alpha_2}} = \\ &= (p^2)^\omega \frac{\Gamma(-\omega)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} B(D/2-\alpha_1, D/2-\alpha_2) = \\ &= (p^2)^{D/2-\alpha_1-\alpha_2} \frac{\Gamma(D/2-\alpha_2)\Gamma(D/2-\alpha_1)\Gamma(-\omega)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(D/2+\omega)}\end{aligned}$$

Since we are studying a model in 6 dimensions, in the main text we will use this value.

Let us also report here the explicit calculation mentioned in section §3. Let us consider the integral

$$\int_{\mathbb{R}^6} dq \left(\frac{1}{(p+q)^2} \right)^{1-x} 2q^\nu \left(\frac{1}{q^2} \right)^{2-y}. \tag{95}$$

We can perform the Schwinger trick (93) and have

$$\frac{2}{\Gamma(1-x)\Gamma(2-y)} \int_{\mathbb{R}_+} dt_1 \int_{\mathbb{R}_+} dt_2\, t_1^{-x} t_2^{1-y} \int_{\mathbb{R}^6} dq\, q^\nu e^{-t_1(p+q)^2 - t_2 q^2}. \tag{96}$$

We can complete the square in the exponent

$$t_1(p+q)^2 + t_2 q^2 = t_{12} \left(q + \frac{t_1}{t_{12}} p \right)^2 + \frac{t_1 t_2}{t_{12}} p^2 \tag{97}$$

and then perform a change of variable $u = q + \frac{t_1}{t_{12}} p$ to get

$$\int_{\mathbb{R}^6} dq\, q^\nu e^{-t_1(p+q)^2 - t_2 q^2} = \int_{\mathbb{R}^6} du \left(u^\nu - \frac{t_1}{t_{12}} p^\nu \right) e^{-t_{12} u^2} e^{-\frac{t_1 t_2}{t_{12}} p^2}. \tag{98}$$

In all these formula, we use the abbreviation $t_{12} := t_1 + t_2$. The term proportional to u^ν vanishes since it is the integral of an odd function and the second term is a gaussian integral which gives

$$\int_{\mathbb{R}^6} dq\, q^\nu e^{-t_1(p+q)^2 - t_2 q^2} = -\pi^3 p^\nu \frac{t_1}{t_{12}^4} e^{-\frac{t_1 t_2}{t_{12}} p^2}. \tag{99}$$

Going back to equation (95)

$$\int_{\mathbb{R}^6} dq \left(\frac{1}{(p+q)^2} \right)^{1-x} 2q^\nu \left(\frac{1}{q^2} \right)^{2-y} = \frac{-2\pi^3 p^\nu}{\Gamma(1-x)\Gamma(2-y)} \int_{\mathbb{R}_+} dt_1 \int_{\mathbb{R}_+} dt_2 \frac{t_1^{1-x} t_2^{1-y}}{t_{12}^4} e^{-\frac{t_1 t_2}{t_{12}} p^2} \tag{100}$$

As before, we can insert the equality (94) and choose the hyperplane $H(t) = \lambda\, t_{12}$ to get

$$\int_{\mathbb{R}_+} dt_1 \int_{\mathbb{R}_+} dt_2 \frac{t_1^{1-x} t_2^{1-y}}{t_{12}^4} e^{-\frac{t_1 t_2}{t_{12}} p^2} = \left(p^2\right)^{x+y} \Gamma(-x-y) \int_0^1 dt_1\, t_1^{1+y} (1-t_1)^{1+x} \tag{101}$$

which finally brings us to

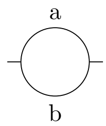
$$\int_{\mathbb{R}^6} dq \left(\frac{1}{(p+q)^2} \right)^{1-x} 2q^\nu \left(\frac{1}{q^2} \right)^{2-y} = -2p^\nu \pi^3 \left(p^2\right)^{x+y} \frac{\Gamma(-x-y)\Gamma(2+x)\Gamma(2+y)}{\Gamma(1-x)\Gamma(2-y)\Gamma(4+x+y)} \tag{102}$$

B Massless loop trick

In this article we are studying a massless model, so the propagators that appear in the whole text are powers of p^2. With the Feynman rules for momentum space a simple loop is the convolution of two propagators. In the massless case though, the Fourier transform back in position space will also be a power of x^2 by homogeneity reasons and the convolution can be evaluated as a multiplication in position space of the Fourier transforms. Apart from some π factors independent on α, we have that

$$\mathcal{F}\left[\left(\frac{1}{p^2} \right)^a \right](x) \propto \frac{\Gamma(D/2-a)}{\Gamma(a)} \left(\frac{1}{x^2} \right)^{D/2-a} \tag{103}$$

so that in the case of a loop



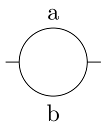
$$\propto \left(\frac{1}{p^2}\right)^a * \left(\frac{1}{p^2}\right)^b = \mathcal{F}^{-1}\left(\mathcal{F}\left(\frac{1}{p^2}\right)^a \cdot \mathcal{F}\left(\frac{1}{p^2}\right)^b\right) \tag{104}$$

$$\propto \frac{\Gamma(D/2-a)\Gamma(D/2-b)}{\Gamma(a)\Gamma(b)}\mathcal{F}^{-1}\left(\frac{1}{x^2}\right)^{D-a-b} \tag{105}$$

$$\propto \frac{\Gamma(D/2-a)\Gamma(D/2-b)\Gamma(a+b-D/2)}{\Gamma(a)\Gamma(b)\Gamma(D-a-b)}\left(\frac{1}{p^2}\right)^{a+b-D/2} \tag{106}$$

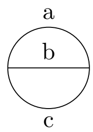
where we have used $\mathcal{F}(\alpha * \beta) = \mathcal{F}(\alpha)\cdot\mathcal{F}(\beta)$. Using repetitively these transformations allow to evaluate the diagram appearing in section §5.1.

In this section, we also have used the following construction: starting from the diagram



(107)

we can add another line of index c



$$\propto \frac{\Gamma(D/2-a)\Gamma(D/2-b)\Gamma(D/2-c)\Gamma(a+b+c-D)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(3D/2-a-b-c)}\left(\frac{1}{p^2}\right)^{a+b+c-D} \tag{108}$$

Now, if we want to consider this graph as a vacuum one, the dependence on the exterior momentum p should disappear. We set therefore $c = D - a - b$, which is also the condition that $\omega = 0$ for the completed graph, making it logarithmically divergent. The preceding formula becomes

$$\frac{\Gamma(D/2-a)\Gamma(D/2-b)\Gamma(a+b-D/2)}{\Gamma(a)\Gamma(b)\Gamma(D-a-b)} \quad \frac{\Gamma(0)}{\Gamma(D/2)} \tag{109}$$

The first factor is the same we would have found if we had transformed the a, b loop. There is however a $\Gamma(0)/\Gamma(D/2)$ additional factor, which is infinite, reflecting the divergence of the whole diagram. This factor can however be interpreted as the integral

$$\int_{\mathbb{R}^D} \frac{d^D u}{(u^2)^{D/2}}$$

the simplest scale invariant integral in D dimensions. This same integral can appear either in x-space, since the product of the three propagators gives the power $D/2$ of x^2 or in p-space, where the combination of any two of the propagators and the last one combine to give $(p^2)^{D/2}$. The scale invariance can be broken by fixing the momentum in any of the propagators while giving the same number, giving another approach to the completion invariance of the residues of propagator graphs.

In this work, we have been interested in the pole structure when one of the propagator of a completed graph becomes scale invariant. Since the whole structure is scale invariant, the complementary of this line becomes also scale invariant, giving two infinite factors. We therefore understand that there should be a pole in the evaluation of the diagram, but it is not so clear how to evaluate the residue of this pole. In a previous work [5], a procedure was devised from the parametric representation, which has the advantage of generalising to poles associated to a propagator with a power larger than $D/2$ by any positive integer, but another approach is possible in this simple case. We consider the diagram in x-space: the scale invariance is broken by fixing the distance between the two vertices while the almost scale invariant link contributes only by its normalisation, $\Gamma(\varepsilon)/\Gamma(D/2-\varepsilon)$. In the limit of vanishing ε, $\Gamma(\varepsilon)$ gives a pole of residue 1 while all other terms have a smooth limit. Differentiating with respect to L compensate the pole, so that we end up with $1/\Gamma(D/2)$ times the residue of the remaining scale invariant diagram. From its scale invariance, it comes that the choice of any two fixed vertices will give the same value for the residue, allowing for a simpler evaluation through suitable choices of these vertices.

The same result on the residue of the pole has been previously obtained in the appendix of [36], by a slightly different derivation. We thank Andrei Kataev for pointing out this reference.

Potential conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] D. J. Broadhurst and D. Kreimer. Exact solutions of Dyson–Schwinger equations for iterated one-loop integrals and propagator-coupling duality. *Nucl. Phys.*, B 600:403–422, 2001. `arXiv:hep-th/0012146`.

[2] Dirk Kreimer and Karen Yeats. An etude in non-linear Dyson-Schwinger equations. *Nucl. Phys. Proc. Suppl.*, 160:116–121, 2006. `arXiv:hep-th/0605096`, `doi:10.1016/j.nuclphysbps.2006.09.036`.

[3] Marc Bellon and Fidel Schaposnik. Renormalization group functions for the Wess-Zumino model: up to 200 loops through Hopf algebras. *Nucl. Phys. B*, 800:517–526, 2008. `arXiv:0801.0727`.

[4] Marc P. Bellon. An efficient method for the solution of Schwinger–Dyson equations for propagators. *Lett. Math. Phys.*, 94:77–86, 2010. `arXiv:1005.0196`, `doi:10.1007/s11005-010-0415-3`.

[5] Marc Bellon and Fidel A. Schaposnik. Higher loop corrections to a Schwinger–Dyson equation. *Lett. Math. Phys.*, 103:881–893, 2013. `arXiv:1205.0022`, `doi:10.1007/s11005-013-0621-x`.

[6] Marc P. Bellon and Pierre J. Clavier. Higher order corrections to the asymptotic perturbative solution of a Schwinger–Dyson equation. *Lett. Math. Phys.*, 104:1–22, 2014. `arXiv:1311.1160`, `doi:10.1007/s11005-014-0686-1`.

[7] Dirk Kreimer. Anatomy of a gauge theory. *Annals Phys.*, 321:2757–2781, 2006. `arXiv:hep-th/0509135`, `doi:10.1016/j.aop.2006.01.004`.

[8] Karen Amanda Yeats. *Growth estimates for Dyson-Schwinger equations*. PhD thesis, Boston University, 2008. `arXiv:0810.2249`.

[9] John Clive Ward. Renormalization theory of the inteactions of nucleons, mesons and photons. *Phys. Rev.*, 84:897–901, 1951. `doi:10.1103/PhysRev.84.897`.

[10] J C Ward. On the renormalization of quantum electrodynamics. *Proceedings of the Physical Society. Section A*, 64(1):54–56, jan 1951. `doi:10.1088/0370-1298/64/1/309`.

[11] Kurt Symanzik. Green's functions method and renormalization of renormalizable quantum field theories. In Borisov Jakšić, editor, *Lectures on high energy physics*, pages 485–517. Gordon and Breach, 1965. Lecture given in Herceg Novi, Montenegro, 1961. URL: `https://cds.cern.ch/record/929298/files/CM-P00056764.pdf`.

[12] R. L. Mills and C. N. Yang. Treatment of overlapping divergences in the photon self-energy function. *Progress of Theoretical Physics Supplement*, 37-38:507–511, 1966. `doi:10.1143/PTPS.37.507`.

[13] M. Baker and Choonkyu Lee. Overlapping-divergence-free skeleton expansion in non-abelian gauge theories. *Phys. Rev. D*, 15:2201–2234, Apr 1977. `doi:10.1103/PhysRevD.15.2201`.

[14] Marc P. Bellon and Pierre J. Clavier. A Schwinger–Dyson equation in the Borel plane: singularities of the solution. *Lett. Math. Phys.*, 105, 2015. `arXiv:1411.7190`, `doi:10.1007/s11005-015-0761-2`.

[15] Francis Brown and Dirk Kreimer. Angles, Scales and Parametric Renormalization. *Lett. Math. Phys.*, 103:933–1007, 2013. `arXiv:1112.1180`, `doi:10.1007/s11005-013-0625-6`.

[16] Francis Brown and Dirk Kreimer. Decomposing Feynman rules. In *Proceedings, 11th DESY Workshop on Elementary Particle Physics: Loops and Legs in Quantum Field Theory: Wernigerode, Germany, April 15-20, 2012*, 2012. [PoSLL2012,049(2012)]. `arXiv:1212.3923`, `doi:10.22323/1.151.0049`.

[17] D. J. Broadhurst and D. Kreimer. Towards cohomology of renormalization: bigrading the combinatorial Hopf algebra of rooted trees. *Comm. Math. Phys.*, 215(1):217–236, Dec 2000. arXiv:hep-th/0001202, doi:10.1007/PL00005540.

[18] Dirk Kreimer. New mathematical structures in renormalizable quantum field theories. *Annals of Physics*, 303(1):179–202, Jan 2003. arXiv:hep-th/0211136, doi:10.1016/S0003-4916(02)00023-4.

[19] Loïc Foissy. General Dyson–Schwinger equations and systems. *Comm. Math. Phys.*, 327(1):151–179, Apr 2014. arXiv:1112.2606, doi:10.1007/s00220-014-1941-0.

[20] N. Bogoliubov and O. Parasiuk. Über die Multiplikation der Kausalfunktionen in der Quantentheorie der Felder. *Acta Mathematica*, 97:227–266, 1957. doi:10.1007/BF02392399.

[21] John C. Collins. *Renormalization: An Introduction to Renormalization, the Renormalization Group and the Operator-Product Expansion*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1984. doi:10.1017/CBO9780511622656.

[22] Eugene R. Speer. Analytic renormalization. *J. Math. Phys.*, 9:1404–1410, 1969.

[23] Eugene R. Speer. On the structure of analytic renormalization. *Commun. Math. Phys.*, 23:23–36, 1971.

[24] Eugene R. Speer. Dimensional and analytic renormalization. In G. Velo and A. S. Wightman, editors, *Renormalization Theory*, NATO Advanced Study Institutes Series, page 25–93. Springer Netherlands, 1976. doi:10.1007/978-94-010-1490-8_2.

[25] Marc P. Bellon. Approximate differential equations for renormalization group functions in models free of vertex divergencies. *Nucl. Phys. B*, 826 [PM]:522–531, 2010. arXiv:0907.2296, doi:10.1016/j.nuclphysb.2009.11.002.

[26] Marc P. Bellon and Enrico I. Russo. Resurgent analysis of Ward–Schwinger–Dyson equations. in preparation, 2020.

[27] Marc P. Bellon. Numerators in parametric representations of Feynman diagrams. in preparation, 2020.

[28] Noboru Nakanishi. *Graph theory and Feynman integrals*, volume 11 of *Mathematics and its applications*. Gordon and Breach, New York, 1971.

[29] J. A. Gracey. Four loop renormalization of ϕ^3 theory in six dimensions. *Phys. Rev.*, D92(2):025012, 2015. arXiv:1506.03357, doi:10.1103/PhysRevD.92.025012.

[30] Yi Pang, Junchen Rong, and Ning Su. ϕ^3 theory with F_4 flavor symmetry in $6-2\epsilon$ dimensions: 3-loop renormalization and conformal bootstrap. *JHEP*, 12:057, 2016. arXiv:1609.03007, doi:10.1007/JHEP12(2016)057.

[31] Klaus Hepp. Proof of the Bogoliubov–Parasiuk theorem on renormalization. *Comm. Math. Phys.*, 2:301–326, 1966. doi:10.1007/BF01773358.

[32] W. Zimmermann. Convergence of Bogoliubov's method of renormalization in momentum space. *Comm. Math. Phys.*, 15:208–234, 1969. doi:10.1007/BF01645676.

[33] Alain Connes and Dirk Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem. I: The Hopf algebra structure of graphs and the main theorem. *Comm. Math. Phys.*, 210:249–273, 2000. arXiv:hep-th/9912092, doi:10.1007/s002200050779.

[34] Alain Connes and Dirk Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem. II: The beta-function, diffeomorphisms and the renormalization group. *Commun. Math. Phys.*, 216:215–241, 2001. arXiv:hep-th/0003188, doi:10.1007/PL00005547.

[35] Henri Epstein and Vladimír Glaser. The role of locality in perturbation theory. *Annales de l'IHP*, 19:211–295, 1973. URL: http://www.numdam.org/article/AIHPA_1973__19_3_211_0.pdf.

[36] S. G. Gorishnii and A. P. Isaev. An approach to the calculation of many-loop massless Feynman integrals. *Theoretical and Mathematical Physics*, 62(3):232–240, 1985. doi:10.1007/BF01018263.

Resurgent analysis of Ward–Schwinger–Dyson equations

Marc P. Bellon[1,2], Enrico I. Russo[1]

[1] *Sorbonne Université, CNRS, Laboratoire de Physique Théorique et Hautes Energies, LPTHE, 75005, Paris, France*

Abstract

Building on our recent derivation of the Ward–Schwinger–Dyson equations for the cubic interaction model, we present here the first steps of their resurgent analysis. In our derivation of the WSD equations, we made sure that they had the properties of compatibility with the renormalisation group equations and independence from a regularisation procedure which was known to allow for the comparable studies in the Wess–Zumino model. The interactions between the transseries terms for the anomalous dimensions of the field and the vertex is at the origin of unexpected features, for which the effect of higher order corrections is not precisely known at this stage: we are only at the beginning of the journey to use resurgent methods to decipher non-pertubative effects in quantum field theory.

Keywords: Renormalization, Schwinger-Dyson equation, Resurgence

1 Introduction

Quantum field theories do not produce convergent perturbative series, while these perturbative series are often the only information accessible up to now through an analytic treatment. Converting these series in numbers and properties of the theory therefore requires some non trivial summation methods, especially in the large coupling regime. A very useful method goes through the definition of a Borel transform that will give the solution through a Laplace integral. However, in many cases, the Borel transform has singularities on the real axis which make the naive Borel–Laplace summation ambiguous: integration on rotated axis lose the reality properties of the original series. In such situations, results with much reduced ambiguities can be obtained through the use of real averages of the different analytic continuations.

It has been known for a long time that a successful resummation of a divergent series requires the knowledge of its asymptotic properties and recent works have tried to quantify the gains which can result from a deeper knowledge of the properties of the Borel transform. We would single out the paper [1] that show how the use conformal maps of the Borel plane allows for the most precise results. However, such gains are only possible if the precise structures of the singularities of the Borel transform are known.

In quantum mechanics, singularities of the Borel transform stem from the presence of non trivial saddle point of the action functional, dubbed instantons, but in renormalisable quantum field theories, new singularities appear, related to the behaviour of diagrams with a maximal number of subdivergences which are called renormalons. Much work has been devoted to the finding of classical field configurations which could explain these singularities, but with limited success even if we must cite [2, 3]. This work will be based on a quite different approach, in the spirit of [4, 5], which made use of the tools of resurgence theory and in particular the alien derivatives, in the study of the solution of a Schwinger–Dyson equation. This work had been prepared by a number of studies [6, 7, 8] which addressed only what was later recognized as the singularities nearest the origin of the Borel transform. These previous studies were however limited to the supersymmetric Wess–Zumino model, first solved perturbatively at high order in [9], or a very special case of the ϕ^3_6 model we study here, where the vertex gets no radiative corrections at one loop [7].

It is therefore very interesting that our recent work [10] gives a system of equations for the determination of the renormalisation group functions of the ϕ^3_6 model, that we called Ward–Schwinger–Dyson equations, most suitable for a resurgent analysis. Indeed, this scheme has the properties which made the analysis in [4] possible, the absence of any explicit regularisation parameter and the invariance of the solution under the renormalisation group. Indeed one may say that our computations tend to transform a leading log approximation for the propagator and the vertex in a computation of the leading terms for the high order terms of the perturbative solution, but with the possibility to go beyond these leading behaviours through the systematic inclusion of corrections.

It is quite old observation by Giorgio Parisi that the renormalisation group equations written for the Borel transform of the propagator imply that a position ρ in the Borel plane, the propagator has a leading correction like $(p^2/\mu^2)^{b_1\rho}$, with b_1 the leading coefficient of the β function [11]. These power corrections give rise to new divergences which seemed impossible to renormalize, in particular the infrared ones. We will instead show that these divergences can be given a precise meaning and are the tools to understand the singularities of the Borel transforms of the anomalous dimensions.

2 The Borel-Laplace resummation method

2.1 General properties

We do not have the presumption here to introduce the topic of Borel-Laplace resummation techniques; there are excellent introductions in the literature [12] or [13]. We report here though some basic properties that we will need in the further development.

The formal Borel transform is defined on formal series as

$$\begin{aligned} \mathcal{B} : (z^{-1}\,\mathbb{C}[[z^{-1}]], \cdot) \quad &\longrightarrow \quad (\mathbb{C}[[\xi]], \star) \\ \tilde{f}(z) = \frac{1}{z}\sum_{n=0}^{+\infty} \frac{c_n}{z^n} \quad &\longrightarrow \quad \hat{f}(\xi) = \sum_{n=0}^{+\infty} \frac{c_n}{n!}\xi^n. \end{aligned}$$

Let $\tilde{f}, \tilde{g}$ in $z^{-1}\,\mathbb{C}[[z^{-1}]]$ be two formal series and $\hat{f}, \hat{g}$ in $\mathbb{C}[[\xi]]$ be their Borel transforms. The following properties hold

$$\mathcal{B}(\tilde{f}.\tilde{g}) = \hat{f} \star \hat{g}; \qquad \mathcal{B}(\partial\tilde{f}) = -\zeta\hat{f}; \qquad \mathcal{B}(z^{-1}\tilde{f}) = \int \hat{f};$$

$$\tilde{f}(z) \in z^{-2}\,\mathbb{C}[[z^{-1}]] \Longrightarrow \mathcal{B}(z\tilde{f}) = \frac{d\hat{f}}{d\zeta};$$

with the derivatives and the integral defined term by term and $\star$ denoting the convolution product of formal series. If $\hat{f}$ and $\hat{g}$ are convergent,

$$\hat{f} \star \hat{g}(\zeta) = \int_0^\zeta \hat{f}(\eta)\hat{g}(\zeta - \eta)d\eta \tag{1}$$

for ζ in the intersection of the convergence domains of $\hat{f}$ and $\hat{g}$. The analytic continuation of the convolution product on a given path can also be expressed through such an integral, but the integration path is in general much more complex than the path on which the analytic continuation is taken.

The definition of the Borel transform can be extended to series with constant terms through the introduction of a unit δ for the convolution product. The Borel transform is extended by mapping the constant function equal to the constant a to $a\delta$ and then to the whole space of formal series $\mathbb{C}[[z^{-1}]]$ by linearity.

A formal series $\tilde{f}(z) = \frac{1}{z}\sum_{n=0}^{+\infty}\frac{a_n}{z^n}$ is 1-Gevrey if

$$\exists A, B > 0 :\ |a_n| \leq AB^n n!\ \ \forall n \in \mathbb{N}.$$

In this case, we write $\tilde{f}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]_1$. In this case and only in this case, its Borel transform has a finite radius of convergence and we denote by $\mathbb{C}\{\zeta\}$ the space of such functions.

The Borel transform can be inverted through the Laplace transform. Let $\theta \in [0, 2\pi[$ and set $\Gamma_\theta := \{Re^{i\theta}, R \in [0, +\infty[\}$. Let $\hat{f} \in \mathbb{C}\{\zeta\}$ be a germ admitting an analytic continuation in an open subset of $\mathbb{C}$ containing Γ_θ and such that

$$\exists c \in \mathbb{R},\ K > 0 : |\hat{f}(\zeta)| \leq Ke^{c|\zeta|} \tag{2}$$

for any ζ in Γ_θ. Then the Laplace transform of $\hat{f}$ in the direction θ is defined as

$$\mathcal{L}^\theta[\hat{f}](z) = \int_0^{e^{i\theta}\infty} \hat{f}(\zeta)e^{-\zeta z}\,d\zeta.$$

When the bound (2) is verified, this expression is finite in the half-plane $\Re(ze^{i\theta}) > c$ and therefore defines an analytic function of z in this domain, which is called a Borel sum of $\tilde{f}$.

For a formal series $\tilde{f}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ with a non-zero radius of convergence, equation (2) is true for all $\theta \in [0, 2\pi[$ and its Borel sum in any direction coincide with the usual sum of the series. For more general Borel summable series, many interesting phenomena can arise, such as the Stokes phenomenon: the singularities of the Borel transform imply differences between the Borel sums defined in directions separated by these singularities and even in cases where the condition (2) is satisfied for all directions, the Borel sum will differ from its analytic continuation in a path around infinity, giving a non trivial monodromy. These problems are at the heart of the renewed interest in summability techniques in particular in the physics community [14, 15, 16, 17].

2.2 Resurgent functions and alien derivatives

Borel summation heavily relies on the possibility of analytically continuing the Borel transform in the whole complex plane minus some set of singularities. These singularities can be studied through alien derivatives. The alien operator Δ_ω extracts the singularity around ω of the Borel transform and translates it to the origin. Some care must be taken when n singularities lie on the segment $[0, \omega]$ since there is no longer a canonical analytic continuation of the Borel transform to the neighborhood of ω: each singularity can be avoided in two different ways resulting in 2^n possible analytic continuations. It is easy to show that in the simple case without singularities between 0 and ω, Δ_ω is a derivation with respect to the convolution product. With adequate weighting of the different paths to ω, this can be made true in the general case with any number of singularities on the path.

Since the alien derivative involves a translation and the ordinary derivative is the multiplication by $-\zeta$, these two derivatives do not commute: we have

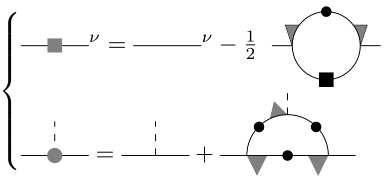
$$[\partial, \Delta_\omega] = \omega \Delta_\omega. \tag{3}$$

Alien derivatives can be considered to act on the formal power series in z^{-1} and one generally keeps the same notation, since confusion is not possible. In this case, we can multiply the operator Δ_ω by a transmonomial to obtain $\dot{\Delta}_\omega \equiv e^{-\omega z}\Delta_\omega$. This modified operator $\dot{\Delta}_\omega$ now commutes with the derivative and since it is a derivation, $\dot{\Delta}_\omega f$ represent a possible deformation of the solution f of a system of differential equations.

When a system of equations is given, alien derivatives can therefore be determined in two stages. One first determines all possible deformations of a solution involving transmonomial factors $e^{-\omega_i z}$, extending the solution to a transseries. This transseries will depend on parameters c_i and the possible alien derivatives are given by bridge equations, which express each alien derivative through the action of some differential operator in the parameters c_i.

3 WSD equations for ϕ_6^3

We introduced the Ward–Schwinger–Dyson scheme (often abbreviated to WSD in the following) in the case of the model ϕ_6^3 in [10]. We here recall only the equations which will be studied. For their origin and possible extensions, the reader is invited to go back to this previous work. They can be considered as variations on the Schwinger Dyson equations written in terms of derivatives of the effective action [18]. The lowest order primitive terms of the Ward-Schwinger-Dyson equations for ϕ_6^3 have the following diagrammatic form



(4)

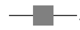
The first equation allows us to determine the 2-point function, while the second one is for the 3-point function. Dotted lines represent a vanishing incoming momentum, the decorations represent the functions we compute:

$$G(a,L) := \sum_{n\geq 0} \frac{1}{n!}\gamma_n(a)L^n \qquad \text{for} \tag{5}$$

$$Y(a,L) := \sum_{n\geq 0} \frac{1}{n!}\upsilon_n(a)L^n \qquad \text{for} \tag{6}$$

and the square appears once and only once in the diagrams for the derivative of the propagator

$$K^\nu(a,L) := \partial^\nu\left(G(a,L)/p^2\right) = \frac{2p^\nu}{(p^2)^2}\sum_{n\geq 0}(\gamma_{n+1}-\gamma_n)\frac{L^n}{n!} \qquad \text{for} \tag{7}$$



In these equations, $a := g^2/(4\pi)^3$ is an equivalent of the fine structure constant which hides irrelevant π factors, g is the coupling constant, $\partial^\nu := \frac{\partial}{\partial p_\nu}$, and $L := \log\left(p^2/\mu^2\right)$ is the logarithmic kinematic variable for a reference energy scale μ^2. These decorations and the functions they denote are always relative to the free propagator. They do not depend on any regularisation parameter and satisfy the renormalisation group equations

$$\partial_L G = (\gamma + \beta\, a\partial_a)\, G, \tag{8}$$

$$\partial_L Y = (\upsilon + \beta\, a\partial_a)\, Y. \tag{9}$$

where $\gamma(a)$ and $\upsilon(a)$ are the anomalous dimensions of the 2-point and 3-point function and $\beta(a)$ is the beta-function of the model. In [10], we established that $\beta = 2\upsilon + 3\gamma$, but we must point out that our convention for the function β differs from the usual ones. Since γ_0 and υ_0 are left undetermined by the equations, we fix them to 1 as a normalisation condition. Then equations (8) and (9) impose the relations $\gamma_1 = \gamma$ and $\upsilon_1 = \upsilon$ which are used to determine the renormalisation group functions γ and υ.

Decorations can be further composed as Cauchy products of the G and Y series. We can assign to each internal line an operator $\mathcal{W}$ given as a product of G and Y, with its anomalous dimension w given as

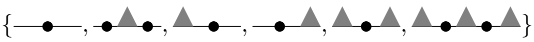
$$w = \#G\,\gamma + \#Y\,\upsilon, \tag{10}$$

and the renormalisation group equation:

$$\partial_L \mathcal{W} = (w + \beta a\partial_a)\,\mathcal{W}. \tag{11}$$

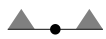
Here is a non-exhaustive list of operators $\mathcal{W}$ that we could consider:

$$\mathcal{W} \in \{\ \ldots\ \} \tag{12}$$



In equations (4), two products appear, 𐤎 (read "Samekh") and 𐤒 (read "Qof"), defined as

$$\text{𐤎}(a,L) := YGY(a,L) = \sum_{n\geq 0}\frac{1}{n!}s_n(a)L^n \qquad \text{for} \tag{13}$$



$$\text{𐤒}(a,L) := GYG(a,L) = \sum_{n\geq 0}\frac{1}{n!}q_n(a)L^n \qquad \text{for} \tag{14}$$

In particular, they satisfy

$$\partial_L \text{𐤎} = (\gamma + 2\upsilon + \beta\, a\partial_a)\,\text{𐤎}, \tag{15}$$

$$\partial_L \text{𐤒} = (2\gamma + \upsilon + \beta\, a\partial_a)\,\text{𐤒}, \tag{16}$$

and we will write $s = 2\upsilon + \gamma$ and $q = 2\gamma + \upsilon$. This formalism was tested in [10] by showing that the renormalisation functions to order a^2 computed with it matched with known results.

For our next computations, it will be convenient to change the variable a to an other one r proportional to its inverse such that

$$r\beta(r) = -1 + o(1/r). \tag{17}$$

The sign here is important because we want to keep this trademark of asymptotic freedom. With this change of variable the other anomalous dimensions become:

$$\gamma(r) = -\frac{T_2}{12\beta_1}\frac{1}{r} + \dots, \tag{18}$$

$$\upsilon(r) = \frac{T_3}{2\beta_1}\frac{1}{r} + \dots. \tag{19}$$

where T_2 and T_3 are the usual notations for the Casimir factors associated to the "color" structure of the interactions (see, e.g., [19] for their precise definitions). With these notations, we have that $\beta_1 = \frac{T_2}{4} - T_3$, which is generically negative.

Our computations make use of the Mellin transform representation of graphs. It means that we deduce their properties after the replacement of all propagators by full propagators from the case where these propagators get an $\exp(xL)$ factor. We therefore obtain a function of n_e complex parameters, meromorphic with poles on linear subspaces.

Indeed, using the relation

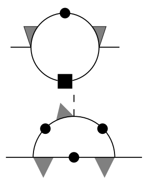
$$L^n = ev_0 \circ \partial_x^n e^{xL}, \tag{20}$$

any function of L can be obtained through the action of an infinite order differential operator on $\exp(xL)$. The effect of the replacement of any propagator in a diagram can be obtained from the action of this differential operator on (one of the parameters of) the Mellin transform, followed by putting a parameter to 0. So, for example, the G series is described by:

$$G_x := G(a, \partial_x) = \sum_{n\geq 0} \gamma_n(a)\frac{\partial_x^n}{n!}. \tag{21}$$

We assign also an operator $\mathcal{O}$ to the whole graph by multiplying the operators associated to all of the internal lines $\mathcal{W}_i$, with a factor $1/r^l$ for a diagram with l loops, so that the evaluation of the graph is just by applying this operator on the Mellin transform of the graph. For example, for the graphs appearing in equations (4)

$$\mathcal{O}_{xy} = \begin{cases} \frac{1}{r} K_x^\nu \text{₸}_y & \text{for} \\ \frac{1}{r} \phi_x \text{₸}_y & \text{for} \end{cases} \tag{22}$$



We do not keep K^ν in our operators, since it can be expressed through G and its derivative with respect to L: K_x^ν can be traded for G_x, with only a multiplication of the Mellin transform by $x-1$. In the one loop diagram we consider here, the propagator K^ν is the only one in a path between the exterior legs of the diagram, so that it could be obtained through the derivation of the whole diagram with respect to the exterior momentum. In any case, we can just write G instead of K^ν. We can assign to $\mathcal{O}$ an anomalous dimension $\gamma_{\mathcal{O}}$:

$$\gamma_{\mathcal{O}} \equiv \#G\,\gamma + \#Y\,\upsilon - l\beta, \tag{23}$$

and write

$$\partial_L \mathcal{O} = (\gamma_{\mathcal{O}} - \beta r \partial_r)\,\mathcal{O}. \tag{24}$$

The β function appears in equation (23) due to the $1/r^l$ factor in the definition of $\mathcal{O}$. It is a combination of the anomalous dimensions γ and υ, in our model

$$\beta = 3\gamma + 2\upsilon, \tag{25}$$

such that for the graphs in equation (4)

$$\gamma_{\mathcal{O}} = \begin{cases} -\gamma & \text{for} \quad \mathcal{O}_\gamma := \frac{1}{r} G_x \text{₸}_y \\ \upsilon & \text{for} \quad \mathcal{O}_\upsilon := \frac{1}{r} \phi_x \text{₸}_y. \end{cases} \tag{26}$$

This ensures that both sides of the WSD equations obey the same renormalisation group equations and is instrumental in the proof we have given in [10] that the solutions of the WSD equations obey renormalisation group equations. The β is then the logarithmic derivative of effective charge $r^{-1}Y^2G^3$, which have important combinatorial properties, see for example [20].

From Ward Schwinger Dyson equations (4) we can extract equations for the anomalous dimensions:

$$\gamma = (\gamma + \beta r \partial_r)\,\gamma - \frac{T_2}{2\beta_1}\mathcal{O}_\gamma H^\gamma \tag{27}$$

$$\upsilon = \frac{T_3}{\beta_1}\mathcal{O}_\upsilon H^\upsilon \tag{28}$$

where H^γ and H^υ are functions associated to the two graphs through Mellin transform.

4 Singularity structure of the Mellin transform

4.1 General properties

The poles of the Mellin transforms H^γ and H^υ give the dominant contributions in the evaluation of the anomalous dimensions. These two functions are given by

$$H^\gamma(x,y) = \frac{\Gamma(1-x-y)\Gamma(2+x)\Gamma(2+y)}{\Gamma(4+x+y)\Gamma(1-x)\Gamma(1-y)}, \tag{29}$$

$$H^\upsilon(x,y) = \frac{\Gamma(1-x-y)\Gamma(1+x)\Gamma(2+y)}{\Gamma(3+x+y)\Gamma(2-x)\Gamma(1-y)}, \tag{30}$$

and they have poles when the argument of one of the Γ function in their numerators is a negative integer or zero. In terms of natural integers n, n', n'', we have therefore poles on the line with equations

$$\begin{cases} 2+x = -n, \\ 2+y = -n', \\ 1-x-y = -n'', \end{cases} \quad \text{and} \quad \begin{cases} 1+x = -n, \\ 2+y = -n', \\ 1-x-y = -n'', \end{cases} \tag{31}$$

respectively for H^γ and H^υ.

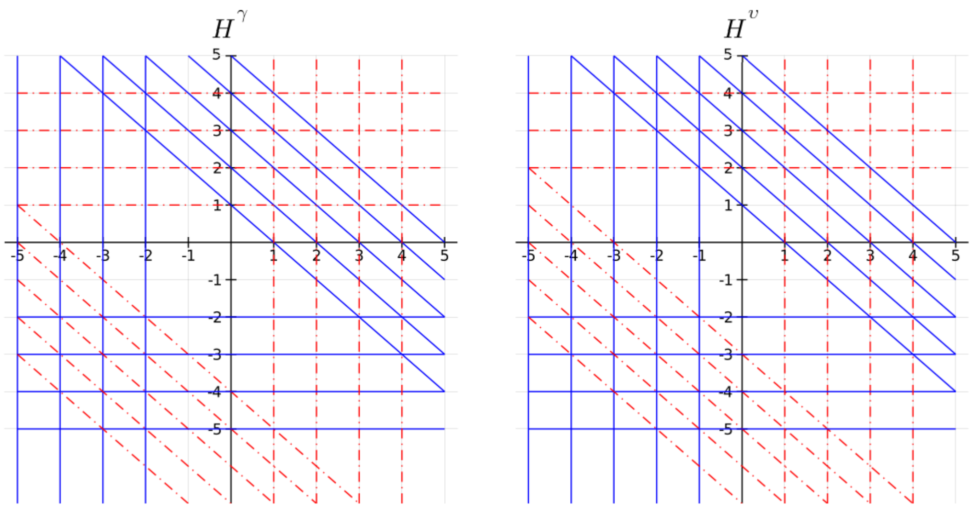


Figure 1: Singularity structure of the characteristic functions H: the blue lines represent the poles, the red ones the zeroes.

From fig.1 we see the structure of singularities. At a generic intersection point the effect coming from poles and zeroes compensate, which means that the residues along the poles have no poles and a finite number of zeroes. The closest singularity to the origin is in both cases due to the blue line $1-x-y=0$. For any H function we write the decomposition

$$H = \sum_k \frac{h_k(x,y)}{k-x-y} + \frac{h'_k(y)}{k+x} + \frac{h''_k(x)}{k+y} + \sum_{n,m} \tilde{h}_{n,m} x^n y^m \tag{32}$$

where the index sum is intentionally left unspecified because it depends on the particular choice of H and a priori there should be four different summations but we did not want to weight the notation too much. This description separates the poles of different kind whose residues are h_k, h'_k, h''_k and the analytic part $\tilde{h}_{n,m}$. Obviously if H is symmetric for $x \leftrightarrow y$ then $h'_k = h''_k$ but in general this might not be the case. We give here their values for small values of k.

	k	h_k	h'_k	h''_k
H^γ	1	$\frac{xy(2+xy)}{4!}$	-	-
	2	$\frac{xy(3+xy)(1-xy)}{5!}$	$\frac{(y-2)(y-1)}{2}$	$\frac{(x-2)(x-1)}{2}$
	3	$\frac{xy(xy-2)^2(xy+4)}{6!2}$	$\frac{(y^2-1)(2-y)(3-y)}{3!}$	$\frac{(x^2-1)(2-x)(3-x)}{3!}$
H^v	1	$\frac{x(1+y)}{3!}$	$\frac{1-y}{2}$	-
	2	$\frac{xy(1+y)(1-x)}{4!}$	$\frac{(1-y^2)(y-2)}{3!}$	$\frac{2-x}{2}$

To this polar parts of the Mellin transform, we associate functions obtained by applying the differential operators associated to propagators to them. They were first considered in [7] and were fundamental tools in [21] and in [8].

4.2 $\mathbf{F}_k$ functions

These functions capture the contribution to anomalous dimensions due to $k + x = 0$ or $k + y = 0$ poles of the characteristic functions H. For $\mathcal{W}$ in {G, Ŧ, Ϙ} and the corresponding w in $\{\gamma, s, q\}$, we define

$$F_k^w := \mathcal{W}_x\Big(\frac{x}{k+x}\Big) = \mathcal{W}_x\Big(\sum_{n\geq 0}\frac{(-1)^n x^{n+1}}{k^{n+1}}\Big) = -\sum_{m\geq 1}\frac{(-1)^m w_m}{k^m}. \tag{33}$$

We could have defined F_k^w as the action of $\mathcal{W}$ on the rational function $\frac{1}{k+x}$, but this would produce terms which have a constant part and this would bring some difficulties with the Borel transforms.

It follows immediately from the definition that

$$\partial_L \mathcal{W}\left(\frac{x}{k+x}\right) = -kF_k^w + w. \tag{34}$$

Indeed, we can write

$$\partial_L \mathcal{W}\sum_{n\geq 0}(-1)^n\frac{x^{n+1}}{k^{n+1}} = (w - \beta r\partial_r)\sum_{n\geq 0}(-1)^n\frac{w_{n+1}}{k^{n+1}} \tag{35}$$

$$= \sum_{n\geq 0}(-1)^n\frac{w_{n+2}}{k^{n+1}} = k\sum_{n\geq 2}(-1)^n\frac{w_n}{k^n} \tag{36}$$

$$= -k(F_k^w - \frac{w}{k}). \tag{37}$$

or the three different kind of F_k, this will give the following equations, which allow to define them:

$$(\gamma - \beta r\partial_r)F_k^\gamma = \gamma - kF_k^\gamma$$
$$(s - \beta r\partial_r)F_k^s = s - kF_k^s$$
$$(q - \beta r\partial_r)F_k^q = q - kF_k^q.$$

There is potentially an infinity of F_k terms which can contribute to the WSD equations, but in our computations, we will need only a finite number of them. The first one in importance is F_1^q but we also have all the F_k with a constant term in the residue of the corresponding pole. These cases can be read in fig. 1 since they correspond to the poles parallel to an axis which are not crossed by zeroes at their intersection with the other axis. In the case of H^γ, since it is symmetric by $x \leftrightarrow y$, it suffices to describe the poles along one direction. These poles come from $\Gamma(2+x)$ and thus start for $x = -2$ and end when a zero is crossed for $y = 0$, produced by the term $1/\Gamma(4+x+y)$, therefore for $x = -4$ and lower. In the end, we have contributions from two values of k, 2 and 3. In the case of H^{υ}, we do not have this symmetry and we must distinguish the two directions: along x, the poles start at $x = -1$, from $\Gamma(1+x)$, and hit a zero at $x = -3$, from $1/\Gamma(3+x+y)$ taken at $y = 0$, leaving the two poles at -1 and -2. Along the y direction, the important poles are determined by $\Gamma(2+y)$ for the start of the poles and get a zero in the residue from the same factor than for the x poles, leaving only one pole for $y = -2$. In the equation for the anomalous dimension γ there will be contributions from F_k^s and F_k^{γ}, while for the one for the anomalous dimension υ there will be contributions from F_k^s and F_k^q.

4.3 $\mathbf{E}_k$ functions

The E_k functions capture the contribution to anomalous dimensions due to $k - x - y = 0$ poles of the characteristic functions H. Differently from $\boldsymbol{F}_k$ functions, they are not associated to a single propagator but to the whole diagram. We define them through the action of the operators $\mathcal{O}_{xy}$:

$$\boldsymbol{E}_k = \mathcal{O}_{xy} \frac{\boldsymbol{h}_k(x,y)}{k-x-y}. \tag{38}$$

In this article we will work with the functions E_k^{γ} and E_k^{υ} given by:

$$E_k^{\gamma} = \frac{G_x \text{Ŧ}_y}{r} \frac{h_k^{\gamma}(x,y)}{k-x-y} \tag{39}$$

$$E_k^{\upsilon} = \frac{\text{Ϙ}_x \text{Ŧ}_y}{r} \frac{h_k^{\upsilon}(x,y)}{k-x-y}. \tag{40}$$

As remarked already in [7], the relation (38) can also be written with exchanged places of derivatives and variables

$$\boldsymbol{E}_k = \frac{\boldsymbol{h}_k(\partial_1, \partial_2)}{k - \partial_1 - \partial_2} \mathcal{O}(L_1, L_2) \tag{41}$$

where ∂_i stands for ∂_{L_i} and as usual, the variables are set to zero after all differentiations are evaluated. Inverting $k - \partial_1 - \partial_2$ is challenging, but when acting on $\mathcal{O}(L_1, L_2$, it can be brought to a form that only refers to the r variable and therefore act similarly on the function E_k. We can also see that an arbitrary power of $\partial_1 + \partial_2$ followed by the evaluation at $L_1 = L_2 = 0$ can be replaced by first evaluating $L_1 = L_2 = L$, applying the same power of ∂_L and finally put $L = 0$. We can then write, with $\mathcal{O}' = \boldsymbol{h}_k(\partial_1, \partial_2)\mathcal{O}$,

$$\frac{1}{k - \partial_1 - \partial_2} \mathcal{O}' = \frac{1}{k} \sum_n \frac{1}{k^n} \partial_L^n \mathcal{O}' = \frac{1}{k} \sum_n \frac{1}{k^n} \left(\gamma_{\mathcal{O}} - \beta r \partial_r\right)^n \mathcal{O}' = \frac{1}{k - \gamma_{\mathcal{O}} + \beta r \partial_r} \mathcal{O}' \tag{42}$$

which ultimately brings the equation

$$\left(k - \gamma_{\mathcal{O}} + \beta r \partial_r\right) \boldsymbol{E}_k = \boldsymbol{h}_k(\partial_1, \partial_2) \mathcal{O}(L_1, L_2). \tag{43}$$

The rather formal definitions (39) and (40) can be converted to the following equations

$$\left(k + \gamma + \beta r \partial_r\right) E_k^{\gamma} = h_k^{\gamma}(\partial_1, \partial_2) \frac{G(L_1) \text{Ŧ}(L_2)}{r} \tag{44}$$

$$\left(k - \upsilon + \beta r \partial_r\right) E_k^{\upsilon} = h_k^{\upsilon}(\partial_1, \partial_2) \frac{\text{Ϙ}(L_1) \text{Ŧ}(L_2)}{r}. \tag{45}$$

One must remark that it is the inclusion of the $1/r$ factor in the definition of these E_k that brings the simplification of the anomalous dimensions $\gamma_{\mathcal{O}}$ to $-\gamma$ and υ.

Using the functions E_k and F_k, the contribution of the diagrams can be written as

$$\mathcal{O}H = \frac{c}{r} + \sum_k E_k + \frac{1}{r} \sum_w \sum_k \mathfrak{z}_w^k F_k^w + R, \tag{46}$$

with c the constant giving the leading term and R collecting all other possible terms. Notice that E_k functions include a factor $1/r$ while the F_k do not. This explains that the two sums do not come with the same $1/r$ factor. The F_k^w come with a factor $\mathfrak{z}_w^k$, which is just a number which is given by

$$\mathfrak{z}_w^k = -\frac{1}{k} h_k^w(0), \tag{47}$$

where $h_k^w(0)$ denotes the constant term of either $h'_k(y)$ or $h''_k(x)$ according to which Mellin variable is associated to the line and the prefactor $-\frac{1}{k}$ comes from the identity

$$\frac{1}{k+x} = \frac{1}{k}\left(1 - \frac{x}{k+x}\right). \tag{48}$$

We could have kept other terms of the residue, but they would not contribute to the exponents.

4.4 R function

Our computations presume that the dominant contributions come from the poles of the Mellin transform and more specifically from the poles near the origin. We therefore need some way of bounding the contributions coming from the remainder of the Mellin transform, once a finite number of poles has been subtracted.

The solution is not easy, since for the terms with exponential factors, all derivatives with respect to L of the propagator corrections are now of the some order. In a first attempt to find the corrections to the asymptotic behaviour of the series for the Wess–Zumino model [8], a solution could be devised by using the conjecturally exact expansion of the Mellin transform as sum of the pole contributions, when using a particular extension of the residues to the whole (x, y) plane. This approach is however limited by the appearance of multizeta values with high depth, which rapidly go beyond the cases with known reductions.

In a following work [4], we could find a much easier solution. It used a transformation of the propagator which can be written as

$$G(L) = \sum_n \frac{1}{n!}\gamma_n L^n \quad \longrightarrow \quad \check{G}(\lambda) = \sum_n \gamma_n \lambda^{-n-1}. \tag{49}$$

With the relation between G and its transform looks like G is the Borel transform of this new function $\check{G}$, but it is not a proper interpretation, since the natural product for G is not a convolution product. Nevertheless, the derivation with respect to L becomes the multiplication by λ for this transform, so that it is easy to convert the renormalisation group equation for G in an equation for $\check{G}$. It is easy to see the equation for a term with an $\exp(kr)$ factor in $\check{G}$ is multiplied by $\lambda - k$.

On the other side, the pairing of G with the Mellin transform is easy to obtain in this form. We just have to make the sum of the products of the x^n terms in H and the λ^{-n-1} terms in $\check{G}$ which is just the residue of $H(x, y)\check{G}(x)$ at $x = 0$. This can be conveniently expressed as a contour integral around the origin. If the dependence of $\check{G}$ on r involves exponentials, it will also have poles for $\lambda = k$ and the contour integral will involve also evaluation of H or its derivatives at $x = k$: this does certainly work out if H has a pole at this point, so that this computation can only be done after subtracting a number of poles of H, but the upshot is that one can obtain in this way the contribution from the remainder of H in a form which contain sufficiently many $1/r$ factor to not have any influence on the exponents we compute.

This construction must necessarily in our case be applied also for the $\mp$ and q products, which must be directly obtained from their respective renormalisation group equations, since the point by point product in the variable L has no easy equivalent in the variable λ. The full development of this formalism is certainly complex, with the necessity of evaluating the Mellin transform not only at the origin, but also, after suitable subtraction, at integer points. Nevertheless, for the sake of our limited ambition in this work, it is possible to consider that the rest function which accounts for all but linear terms in h'_k and h''_k and the regular part of the H function is controlled. We can characterise it as being

$$R = \oint \oint H_{reg} \check{\mathcal{O}} \tag{50}$$

where H_{reg} is the function H with subtracted polar parts. This subtraction unfortunately cannot be put in the form of a canonical projection as would be possible for H a function of a single variable.

5 Trans-series corrections

5.1 Results

From the resurgent point of view the situation is quite intricate, but there are things that we can easily establish. First of all, from the dominant term in $r\beta(r)$ in equation (17), one sees that there is a dominant $(k+\partial_r)F_k$ term in the equations for the F_k and $(k-\partial_r)E_k$ in the ones for the E_k. We therefore see immediately that F_k can be modified by a term proportional to e^{-kr} and E_k by a term proportional to e^{kr}. This is however not sufficient to characterise these terms. The next term in an expansion in $1/r$ cannot be compensated by the derivation of a series in powers of r^{-1}, which have a derivative starting with r^{-2}, so that one has to multiply such terms by a power of r, generically with a non-integer exponent which will be the dominant one in the exponential terms.

The purpose of this section is to show how to compute the values of these dominant exponents for the corrections proportional to e^r and e^{-r} for the anomalous dimension γ and υ. The situation appears more complex than in the Wess–Zumino model, where there is only one E_k and one F_k for each positive integer k and all exponents have been computed in [4]. Talking about the first trans-series corrections means to talk about the closest singularities in the Borel plane through their relations to alien derivatives. In turn, these singularities control the asymptotic behaviour of the perturbative series. Terms that are proportional to e^r are linked to the singularities at -1 in the Borel plane while the e^{-r} terms are linked to the singularities at 1. We will use the notation that $[k]$ indicates the part of a function which has a factor e^{kr}, so that $[0]$ indicates the classical part and we will compute the exponents for the $[1]$ and $[-1]$ parts.

The results of this section are expressed in terms of the three quantities g, u and b appearing in the first orders of the renormalisation group functions as:

$$\gamma[0] = g/r + O(1/r^2) = -\frac{T_2}{12\beta_1}\frac{1}{r} + O(1/r^2) \tag{51}$$

$$\upsilon[0] = u/r + O(1/r^2) = \frac{T_3}{2\beta_1}\frac{1}{r} + O(1/r^2) \tag{52}$$

$$\beta[0]r = (3\gamma[0] + 2\upsilon[0])r = -1 + b/r + O(1/r^2). \tag{53}$$

with

$$\beta_1 = T_2/4 - T_3 \tag{54}$$

$$b = \frac{\beta_2}{\beta_1^2} = \frac{1}{\beta_1^2}\Big(\frac{11}{24}T_2T_3 - \frac{11}{144}T_2^2 - \frac{3}{4}T_3^2 - \frac{1}{2}T_5\Big) \tag{55}$$

The exponents are η, θ and the pair of conjugated numbers $\lambda^\pm$. We have that $\gamma[1]$ is proportional to $e^r r^\eta$, $\upsilon[1]$ to $e^r r^\theta$, while both $\gamma[-1]$ and $\upsilon[-1]$ are dominated by the two terms $e^{-r}r^{\lambda^\pm}$, with a definite relation between the dominant terms in $\gamma[-1]$ and $\upsilon[-1]$. Our results are summarized by

$$\eta = g + b \qquad \theta = b - \frac{2}{3}u \qquad \lambda = -2g - b \pm |3g|\sqrt{1 + \frac{4u}{3g}}. \tag{56}$$

Remarkably

$$\eta - \theta = 1/3 \tag{57}$$

and this difference does not depend on the choice of the ϕ^3 model. We were surprised to find that $\lambda^\pm$ is algebraic and even complex in general. Real asymptotic behaviors can only be obtained by combining two conjugate terms involving λ^+ and λ^-.

Let us show how they are calculated.

5.2 Preparatory steps

We start again from the WSD equations:

$$\gamma = (\gamma + \beta r\partial_r)\gamma - \frac{T_2}{2\beta_1}\mathcal{O}_\gamma H^\gamma \tag{58}$$

$$\upsilon = \frac{T_3}{\beta_1}\mathcal{O}_\upsilon H^\upsilon, \tag{59}$$

and rewrite them using equation (46) as

$$\gamma = (\gamma + \beta r\partial_r)\gamma - \frac{T_2}{2\beta_1}\left(\frac{1}{6r} + \sum_k E_k^\gamma + \frac{1}{r}\left(-\frac{1}{2}(F_2^\gamma + F_2^s) + \frac{1}{3}(F_3^\gamma + F_3^s)\right)\right) \tag{60}$$

$$\upsilon = \frac{T_3}{\beta_1}\left(\frac{1}{2r} + \sum_k E_k^\upsilon + \frac{1}{r}\left(-\frac{1}{2}F_1^q + \frac{1}{6}F_2^q - \frac{1}{2}F_2^s\right)\right) \tag{61}$$

while neglecting the R terms.

For both e^r and e^{-r} trans-series order we can neglect E_k for $k \geq 2$. This occurs because these functions satisfy the equations:

$$(k + \gamma + \beta r\partial_r)\, E_k^\gamma = h_k^\gamma(\partial_1, \partial_2)\frac{G(L_1)\mp(L_2)}{r} \tag{62}$$

$$(k - \upsilon + \beta r\partial_r)\, E_k^\upsilon = h_k^\upsilon(\partial_1, \partial_2)\frac{\Phi(L_1)\mp(L_2)}{r}. \tag{63}$$

and if we denote the descending powers by

$$N^{\underline{n}} = N(N-1)\ldots(N-n+1) \tag{64}$$

we have

$$h_k^\gamma(x, y) = \frac{(-1)^{k-1}}{(k-1)!\Gamma(4+k)}(2+x)^{\underline{k+2}}(2+y)^{\underline{k+2}} \tag{65}$$

$$h_k^\upsilon(x, y) = \frac{(-1)^{k-1}}{(k-1)!\Gamma(3+k)}(1+y)^{\underline{k}}x^{\underline{k}}. \tag{66}$$

The lowest degree monomial will be xy for these h_k except for h_1^υ where it is just x. Higher orders in x and y for h_k extract terms with higher order in L in the propagators and these terms begin also at higher order in $1/r$. Indeed, since we have that $w_{n+1} = (w - \beta r\partial_r)w_n$, w_{n+1} is of one order higher than w_n. Since the equations for the E_k with k larger than two are not resonant, they are of the same order than the term produced by h_k, which is at least two order smaller than γ or υ, unable to modify the exponents. At this stage, we only have to consider E_1^γ and E_1^υ. This simplification might not be true at higher trans-series order. From now on let us forget the subscript 1.

Furthermore, the F_k functions do not contribute to the dominant exponents for trans-series term proportional to e^r, as can be seen from the computation of $F_k[1]$.

$$\Big(\big(k + w - \beta r\partial_r\big)F_k^w\Big)[1] = w[1]. \tag{67}$$

The left hand side can be expanded as

$$\Big(\big(k + w - \beta r\partial_r\big)F_k^w\Big)[1] = \big(k + w - \beta r\partial_r\big)[0]F_k^w[1] + \big(k + w - \beta r\partial_r\big)[1]F_k^w[0]. \tag{68}$$

If we parametrise $F_k^w[1]$ as follows

$$F_k^w[1] = e^r r^{f_k^w} c_k^w(1 + \ldots), \tag{69}$$

we have

$$\partial_r F_k^w[1] = \left(1 + \frac{f_k^w}{r} + \mathcal{O}(r^{-2})\right) F_k^w[1]. \tag{70}$$

Using that $F_k^w[0]$ is at least of order 1 in $1/r$, the dominant term of equation (67) gives that

$$F_k^w[1] \sim \frac{1}{k+1}w[1], \tag{71}$$

which can be specialised to the following dominant behaviours

$$F_k^\gamma[1] \sim \frac{1}{k+1}\gamma[1], \tag{72}$$

$$F_k^s[1] \sim \frac{1}{k+1}s[1] = \frac{1}{k+1}(\gamma[1] + 2\upsilon[1]), \tag{73}$$

$$F_k^q[1] \sim \frac{1}{k+1}q[1] = \frac{1}{k+1}(2\gamma[1] + \upsilon[1]). \tag{74}$$

Since in equations (60) and (61) they appear with a prefactor of r^{-1} they will always be subdominant and for our purpose negligible. The dominant contributions proportional to e^r come from $E[1]$.

The opposite situation occurs for the e^{-r} terms. The $E[-1]$ do not contribute. It suffices to consider the equations

$$\Big((1+\gamma+\beta r\partial_r)E^\gamma\Big)[-1] = \frac{1}{12r}(\gamma s)[-1] \tag{75}$$

$$\Big((1-\upsilon+\beta r\partial_r)E^\upsilon\Big)[-1] = \frac{1}{6r}q[-1], \tag{76}$$

and realise that the left hand side is dominated by $(k+1)E_k[-1]$. This happens because $E_k[0] \in r^{-2}\mathbb{C}[[1/r]]$, since $\mathcal{O}$ contains a factor of r^{-1} and h_k do not contain constant terms; also

$$\beta\, r\partial_r E[-1] \sim E[-1] \tag{77}$$

because the factor -1 coming from e^{-r} compensates with $\beta[0]r = -1 + \ldots$.

This means that $E^\gamma[-1]$ will always be subdominant. Any contribution are suppressed by two factors of r: one from the original equation and one from order [0] series. So even if there was resonance, which in fact occur, the shift is sufficient to neglect these terms. The situation is, a priori, different for $E^\upsilon[-1]$. In fact due to the exceptional right hand side for E^υ where no order [0] series appear, E^υ might contribute. Nevertheless we will see in section §5.4 that $F_1^q[-1]$ is resonant and will dominate over E^υ.

5.3 Exponents of the leading singularities at 1

We take the following form for the functions E^γ:

$$E^\gamma[1] = e^r r^\epsilon c_\epsilon(1+\ldots) \tag{78}$$

$$E^\upsilon[1] = e^r r^{\bar\epsilon} c_{\bar\epsilon}(1+\ldots) \tag{79}$$

In equation (60), the first term on the right hand side has the following leading contribution

$$\big((\gamma+\beta r\partial_r)\gamma\big)[1] \sim -\gamma[1] \tag{80}$$

which generates a factor of 2 with the $\gamma[1]$ of the left hand side. Putting this in equation (61) we obtain the dominant terms

$$\gamma[1] \sim 3gE_1^\gamma[1] \tag{81}$$

$$\upsilon[1] \sim 2uE_1^\upsilon[1] \tag{82}$$

Then ϵ and $\bar\epsilon$ can be obtained from the equations for E^γ and E^υ

$$\Big((1+\gamma+\beta r\partial_r)E^\gamma\Big)[1] \sim -\frac{1}{12r}(\gamma s)[1] \tag{83}$$

$$\Big((1-\upsilon+\beta r\partial_r)E^\upsilon\Big)[1] \sim -\frac{1}{6r}q[1] \tag{84}$$

where we have neglected higher order terms on the right hand side. At this transseries order we have resonance: the highest order terms in the left hand side cancel exactly due to $\beta[0]r = -1 + O(1/r)$. No contribution from $E^\gamma[0]$ or $E^\upsilon[0]$ occurs because $E^\gamma[0] \in r^{-3}\mathbb{C}[[1/r]]$ and $E^\upsilon[0] \in r^{-2}\mathbb{C}[[1/r]]$. We are left with

$$\frac{g+b-\epsilon}{r}E^\gamma[1] \sim 0 \tag{85}$$

$$\frac{-u+b-\bar\epsilon}{r}E^\upsilon[1] \sim -\frac{1}{6r}q[1] \tag{86}$$

In the first one of these equations, the right hand side is negligible, because it contains either $\gamma[0]$ or $s[0]$ which give an additional $1/r$ factor. In the last one, we use $q[1] = 2\gamma[1] + \upsilon[1]$ and the relation (82) to end up with

$$\epsilon = g + b \tag{87}$$

$$\bar\epsilon = -\frac{2}{3}u + b \tag{88}$$

in order to have non trivial solutions for E^γ and E^υ. Then, using the relations (81) and (82) shows that the dominant exponents in $\gamma[1]$ and $\upsilon[1]$ are the ones announced before.

5.4 Exponents of the leading singularities at -1

Here the situation is less straightforward, with different important terms in the equations for γ and υ.

For $\gamma[-1]$, the leading resonance comes from the term $(\gamma+\beta r\partial_r)\gamma$. The F_2 and F_3 terms, even if they are not resonant, give contributions of the order $\gamma[-1]/r$ which cannot be neglected. The first subdominant terms come from $\beta[-1]$ and $\gamma[0]$. In particular equation (60) becomes, when regrouping all $g\gamma[1]$ terms

$$(g+\lambda+b)\gamma[-1]+2g\upsilon[-1]\sim 6g\left(-\frac{1}{2}(F_2^\gamma+F_2^s)[-1]+\frac{1}{3}(F_3^\gamma+F_3^s)[-1]\right). \tag{89}$$

The right hand side can be further simplified using

$$F_k^w[-1]\sim\frac{1}{k-1}w[-1]. \tag{90}$$

We therefore obtain that

$$(g+\lambda+b)\gamma[-1]+2g\upsilon[-1]\sim -4g(\gamma[-1]+\upsilon[-1]), \tag{91}$$

giving

$$(5g+\lambda+b)\gamma[-1]\sim -6g\,\upsilon[-1]. \tag{92}$$

If $\upsilon[1]$ had a smaller exponent than $\gamma[1]$, this would give an equation for λ, but this is not the case.

In the expression for $\upsilon[-1]$, F_1 stands out because it is resonant, so it adumbrates the other F_k. Indeed, in the equation for F_1^q

$$((1+q-\beta r\partial_r))\,F_1^q[-1]=q[-1] \tag{93}$$

the highest terms cancel, so if we call φ the dominant exponent for F_1^q, we have

$$\frac{1}{r}(q_1+b+\varphi)F_1^q[-1]\sim 2\gamma[-1]+\upsilon[-1], \tag{94}$$

with $q_1=u+2g$ the first coefficient in q. The other F_k do not have these cancellations, so that they give contributions smaller by a factor $1/r$ to $\upsilon[-1]$ in equation (61), which at the approximation level we use, gives a further relation:

$$\upsilon[-1]\sim -\frac{u}{r}F_1^q[-1]. \tag{95}$$

Putting together equations (92), (94) and (95), we get an equation for the dominant power λ.

$$q_1+b+\varphi=-u(2\chi+1) \tag{96}$$

where

$$\chi=\frac{-6g}{5g+\lambda+b} \tag{97}$$

is the proportionality factor between $\gamma[-1]$ and $\upsilon[-1]$ from equation (92). Equation (94) further shows that $\varphi=\lambda+1$ and we will use that with our rescaling such the first coefficient of β is 1, we have that $1=-3g-2u$, so that finally λ satisfies

$$\lambda^2+2(2g+b)\lambda+b^2+4gb-5g^2-12ug=0. \tag{98}$$

The solutions are

$$\lambda=-2g-b\pm|3g|\sqrt{1+\frac{4u}{3g}}. \tag{99}$$

The fact that it is algebraic is already strange, but for generic ϕ^3 models the argument of the square root is even negative, thus giving imaginary numbers. For example for the one-component case where $T_2=T_3=T_5=1$, its value is $\lambda=\frac{143}{81}\pm\frac{i\sqrt{7}}{3}$.

6 Conclusion

In this paper we have started the resurgent analysis of the Ward–Schwinger–Dyson equations for the ϕ_6^3 model. While this is a clear illustration of the power of this new approach to quantum field theory to address asymptotic properties of the perturbative series. The exponents we compute are totally inaccessible to the simple minded considerations of specific graphs that allowed to locate renormalon singularities of the Borel transform. The identification of specific classical field configuration which could reproduce such terms through a semiclassical expansion does not seem to provide much added value.

It is clear that the situation is much more intricate than for the Wess–Zumino model studied in [5], where infinite families of possible transseries deformations could be readily obtained. Already at the level we consider here of the nearest singularities of the Borel transform, we have a pair of complex conjugated exponents in the transseries expansion. At the following levels, we would have three different objects of the type F_2, which could mean that three different exponents are possible at level $[-2]$. And we cannot neglect the possibility that higher order corrections to the Ward–Schwinger–Dyson equations have an influence on this whole picture, since some aspects of our computations depend on the precise way we have done the infrared rearrangements. It is therefore our hope that new constraints can be obtained that would allow us to tame this proliferation of new series and use them to study non-perturbative effects for this model through these methods.

It has recently been remarked that in a quite interesting special case, where the field is in a bi-adjoint representation, the first coefficient of the β function vanishes, while the theory remains asymptotically free due to the sign of the second coefficient [22]. In this case, the transseries solution would include powers of $\exp(r^2) = \exp(1/a^2)$, meaning that singularities appear only for the Borel plane dual to a variable $u = r^2$ or some equivalent one. Such a study could reveal new aspects of resurgence studies, but needs a knowledge of the β function at higher loop for us to be able to control the precise transmonomial appearing in the expansion of the renormalisation group function.

We can therefore see that this study is but a small first step in the study of non-pertubative effects in quantum field theory, but it nevertheless presents results not accessible by other methods.

References

[1] Ovidiu Costin and Gerald V. Dunne. Physical resurgent extrapolation. Mar 2020. `arXiv:2003.07451`.

[2] Philip Argyres and Mithat Ünsal. A semiclassical realization of infrared renormalons. *Phys. Rev. Lett.*, 109:121601, Sep 2012. `arXiv:1204.1661`, `doi:10.1103/PhysRevLett.109.121601`.

[3] Masahiro Ashie, Okuto Morikawa, Hiroshi Suzuki, and Hiromasa Takaura. More on the infrared renormalon in $SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$. *Progress of Theoretical and Experimental Physics*, 2020(9):093B02, Sep 2020. `arXiv:2005.07407`, `doi:10.1093/ptep/ptaa096`.

[4] Marc P. Bellon and Pierre J. Clavier. A Schwinger–Dyson equation in the Borel plane: singularities of the solution. *Lett. Math. Phys.*, 105, 2015. `arXiv:1411.7190`, `doi:10.1007/s11005-015-0761-2`.

[5] Marc P. Bellon and Pierre J. Clavier. Alien calculus and a Schwinger–Dyson equation: two-point function with a nonperturbative mass scale. *Lett. Math. Phys.*, 108:391–412, 2017. `arXiv:1612.07813`, `doi:10.1007/s11005-017-1016-1`.

[6] Marc P. Bellon. Approximate differential equations for renormalization group functions in models free of vertex divergencies. *Nucl. Phys. B*, 826 [PM]:522–531, 2010. `arXiv:0907.2296`, `doi:10.1016/j.nuclphysb.2009.11.002`.

[7] Marc P. Bellon. An efficient method for the solution of Schwinger–Dyson equations for propagators. *Lett. Math. Phys.*, 94:77–86, 2010. `arXiv:1005.0196`, `doi:10.1007/s11005-010-0415-3`.

[8] Marc P. Bellon and Pierre J. Clavier. Higher order corrections to the asymptotic perturbative solution of a Schwinger–Dyson equation. *Lett. Math. Phys.*, 104:1–22, 2014. `arXiv:1311.1160`, `doi:10.1007/s11005-014-0686-1`.

[9] Marc Bellon and Fidel Schaposnik. Renormalization group functions for the Wess-Zumino model: up to 200 loops through Hopf algebras. *Nucl. Phys. B*, 800:517–526, 2008. arXiv:0801.0727.

[10] Marc P. Bellon and Enrico I. Russo. Ward–Schwinger–Dyson equations in ϕ_6^3 quantum field theory. 2020. arXiv:2007.15675.

[11] Giorgio Parisi. Xi. the borel transform and the renormalization group. *Physics Reports*, 49(2):215–219, Jan 1979. doi:10.1016/0370-1573(79)90111-X.

[12] Olivier Bouillot. *Invariants Analytiques des Difféomorphismes et MultiZêtas*. PhD thesis, Université Paris-Sud 11, 2011. URL: http://tel.archives-ouvertes.fr/tel-00647909.

[13] David Sauzin. Introduction to 1-summability and resurgence. 2014. arXiv:1405.0356v1.

[14] Inès Aniceto and Ricardo Schiappa. Nonperturbative ambiguities and the reality of resurgent transseries. *Comm. Math. Phys.*, 335:183–245, 2013. arXiv:1308.1115.

[15] Iñês Aniceto, Ricardo Schiappa, and Marcel Vonk. The resurgence of instantons in string theory. *Comm. in Number Theory and Physics*, 6(2):339–496, 2012. doi:10.4310/CNTP.2012.v6.n2.a3.

[16] Gerald V. Dunne and Mithat Ünsal. Uniform WKB, multi-instantons, and resurgent transseries. *Phys. Rev. D*, 89:105009, 2014. doi:10.1103/PhysRevD.89.105009.

[17] Marcos Marino and Tomas Reis. Renormalons in integrable field theories. Sep 2019. arXiv:1909.12134.

[18] Bryce S. DeWitt. *Dynamical Theory of Groups and Fields*. Documents on modern physics. Gordon and Breach, New York, 1965.

[19] J. A. Gracey. Four loop renormalization of ϕ^3 theory in six dimensions. *Phys. Rev.*, D92(2):025012, 2015. arXiv:1506.03357, doi:10.1103/PhysRevD.92.025012.

[20] Guillaume van Baalen, Dirk Kreimer, David Uminsky, and Karen Yeats. The QCD beta-function from global solutions to dyson-schwinger equations. *Annals of Physics*, 325(2):300–324, 2010. arXiv:0906.1754, doi:10.1016/j.aop.2009.10.011.

[21] Marc Bellon and Fidel A. Schaposnik. Higher loop corrections to a Schwinger–Dyson equation. *Lett. Math. Phys.*, 103:881–893, 2013. arXiv:1205.0022, doi:10.1007/s11005-013-0621-x.

[22] J. A. Gracey. Asymptotic freedom from the two loop term of the β-function in a cubic theory. Apr 2020. arXiv:2004.14208.

VI – Conclusion

This thesis is a contribution to the problem of extracting non perturbative information from quantum field theory in the absence of supersymmetry or integrability conditions. Resurgence theory is a highly promising candidate for that. Not only it has been unravelling deep connections between dynamical systems, algebra, and arithmetic; it also connects perturbative with non perturbative physics. The non perturbative part are the instantons: they are functions of the couplings with essential singularities at the origin so no convergent series might describe them; they are interpreted as saddle points of the action around which a path integral should be computed. The perturbative part is the usual series expressed in terms of Feynman integrals. Also this series is doomed to diverge, because of the factorial growth of Feynman graphs. Nevertheless, if we could guess its asymptotics, it could be a priori resummed with Borel techniques: first enhancing convergence with Borel transform, then Laplace-transforming to have a function whose expansion at infinity coincided with the series we started with. This is possible if the line along with we perform the Laplace transformation does not include singularities. Resurgence theory showed us that if there are singularities, especially of they are isolated, the difference on two analytic continuations obtained dodging them is exactly what we would call an instanton. Furthermore, it showed that the behaviour in the neighbourhood of singularity points is captured by another asymptotic series, connected to the one at the origin in a very precise way.

The hard point in applying resurgence to quantum field theory is that we do not have the whole asymptotic series. We can barely calculate its first terms, though sometimes heroically, let alone guess its asymptotics. Schwinger Dyson equations together with renormalisation offer a solution to this dearth: when written in terms of renormalised quantities, the functional equations imply differential equations for the anomalous dimensions. At that point resurgence can strike. Our result is thus to extract non perturbative contributions to anomalous dimensions. This data, in principle, leads to a non perturbative expression for the propagator through the integration of renormalisation equations.

The main result of this work is thus the introduction of the Ward Schwinger Dyson method and its resurgent characterisation.

- The first allows to formulate Schwinger Dyson equations where also the vertices are renormalised, opening the study to full interactive theories;

- The second highlighted the intricate structure of singularities that, for the moment, impedes a complete integration of the Green functions.

While for linear Yukawa and non-linear Wess-Zumino model, the hierarchy of trans-series allows an efficient organisation frame for computations, in the fully non linear and renormalised case, this is hindered due to the irrational value of the powers in the trans-series expansion.

In this thesis I have tried to provide a global picture of the problem at hand, trying to suggest that resurgence has a big role to play in future developments. Our main direction will be the extension of our method for a gauge model. The implementation is not straightforward due to the interplay of Ward identities with our equations, but the shape of vertices are the same. The main obstacle is a proliferation of terms that are computationally demanding and that veil proofs. Another outlook could be towards new shape of vertices like ϕ^4 theory. Finally, the fully non linear Yukawa case might be a very interesting intermediate step and might lead to deep insights in the trans-series hierarchy.

Bibliography

[1] Inès Aniceto and Ricardo Schiappa. Nonperturbative ambiguities and the reality of resurgent transseries. *Comm. Math. Phys.*, 335:183–245, 2013.

[2] Marc Bellon and Fidel Schaposnik. Renormalization group functions for the Wess-Zumino model: up to 200 loops through Hopf algebras. *Nucl. Phys. B*, 800:517–526, 2008.

[3] Marc P. Bellon. An efficient method for the solution of Schwinger–Dyson equations for propagators. *Lett. Math. Phys.*, 94:77–86, 2010.

[4] Marc P. Bellon and Pierre J. Clavier. Higher order corrections to the asymptotic perturbative solution of a Schwinger–Dyson equation. *Lett. Math. Phys.*, 104:1–22, 2014.

[5] Marc P. Bellon and Pierre J. Clavier. Alien calculus and a Schwinger–Dyson equation: two-point function with a nonperturbative mass scale. *Letters in Mathematical Physics*, 108(2):391–412, Oct 2017.

[6] Marc P. Bellon and Fidel A. Schaposnik. Higher loop corrections to a Schwinger–Dyson equation. *Lett. Math. Phys.*, 103:881–893, 2013.

[7] Christoph Bergbauer and Dirk Kreimer. Hopf algebras in renormalization theory: Locality and Dyson-Schwinger equations from Hochschild cohomology. *IRMA Lect. Math. Theor. Phys.*, 10, 07 2005.

[8] Spencer Bloch, Hélène Esnault, and Dirk Kreimer. On Motives Associated to Graph Polynomials. *Communications in Mathematical Physics*, 267(1):181–225, May 2006.

[9] N. N. Bogoliubow and O. S. Parasiuk. Über die Multiplikation der Kausalfunktionen in der Quantentheorie der Felder. *Acta Math.*, 97:227–266, 1957.

[10] Carlos G. Bollini, Juan J. Giambiagi, and Alberto González Domínguez. Analytic regularization and the divergences of quantum field theories. *Nuevo Cimento*, 31:550–561, 1964.

[11] Michael Borinsky and Gerald V. Dunne. Non-perturbative completion of Hopf-algebraic Dyson-Schwinger equations. *Nuclear Physics B*, 957:115096, Aug 2020.

[12] Olivier Bouillot and Jean Ecalle. Invariants of identity-tangent diffeomorphisms: explicit formulae and effective computation, 2014.

[13] David J. Broadhurst and Dirk Kreimer. Association of multiple zeta values with positive knots via feynman diagrams up to 9 loops. *Phys. Lett.*, 393(3):403–412, Feb 1997.

[14] David J. Broadhurst and Dirk Kreimer. Exact solutions of Dyson–Schwinger equations for iterated one-loop integrals and propagator-coupling duality. *Nucl. Phys.*, B 600:403–422, 2001.

[15] Francis Brown. Periods and Feynman amplitudes, 2015.

[16] Francis Brown and Oliver Schnetz. A K3 in ϕ^4. *Duke Mathematical Journal*, 161(10):1817–1862, Jul 2012.

[17] Yvain Bruned, Martin Hairer, and Lorenzo Zambotti. Algebraic renormalisation of regularity structures. *Inventiones mathematicae*, 215(3):1039–1156, Mar 2019.

[18] Pierre Cartier. L'intégrale de chemins de Feynman : d'une vue intuitive à un cadre rigoureux. In *Leçons de mathématiques d'aujourd'hui*, pages pp.27–58. Cassini, 1993.

[19] Pierre Cartier. A mad day's work: from Grothendieck to Connes and Kontsevich. the evolution of concepts of space and symmetry. *Bulletin (New Series) of the American Mathematical Society*, 38, 08 2001.

[20] Alain Connes and Dirk Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem. I: The Hopf algebra structure of graphs and the main theorem. *Comm. Math. Phys.*, 210:249–273, 2000.

[21] Alain Connes and Dirk Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem. II: The beta-function, diffeomorphisms and the renormalization group. *Commun. Math. Phys.*, 216:215–241, 2001.

[22] Alain Connes and Matilde Marcolli. Renormalization and motivic Galois theory. *International Mathematics Research Notices*, 2004(76):4073–4091, 01 2004.

[23] Ovidiu Costin and Rodica Costin. On the formation of singularities of solutions of nonlinear differential systems in antistokes directions. *R. Invent. math.*, (145: 425), 2001.

[24] Bryce S. DeWitt. *Dynamical Theory of Groups and Fields.* Documents on modern physics. Gordon and Breach, New York, 1965.

[25] Paul A. M. Dirac and Niels H. D. Bohr. The quantum theory of the emission and absorption of radiation. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 114(767):243–265, 1927.

[26] Gerald V. Dunne and Mithat Ünsal. Uniform WKB, multi-instantons, and resurgent trans-series. *Phys. Rev. D*, 89:105009, 2014.

[27] Kurusch Ebrahimi-Fard, Li Guo, and Dirk Kreimer. Spitzers identity and the algebraic Birkhoff decomposition in pQFT. *Journal of Physics A: Mathematical and General*, 37(45):11037–11052, oct 2004.

[28] Jean Écalle. *Les fonctions résurgentes, Vol.1.* Pub. Math. Orsay, 1981.

[29] Jean Écalle. Multizetas, perinomal numbers, arithmetical dimorphy, and ari/gari. *Annales de la Faculté des sciences de Toulouse : Mathématiques*, Ser. 6, 13(4):683–708, 2004.

[30] Loïc Foissy. Classification of systems of Dyson-Schwinger equations in the Hopf algebra of decorated rooted trees. *Advances in Mathematics*, 224:2094–2150, 2010.

[31] Steven W. Hawking. Zeta function regularization of path integrals in curved spacetime. *Comm. Math. Phys.*, 55(2):133–148, 1977.

[32] Klaus Hepp. Proof of the Bogoliubov–Parasiuk theorem on renormalization. *Comm. Math. Phys.*, 2:301–326, 1966.

[33] Richard J. Hughes. More comments on asymptotic freedom. *Nuclear Physics B*, 186(2):376 – 396, 1981.

[34] Maxim Kontsevich. Operads and Motives in Deformation Quantization. *Letters in Mathematical Physics*, 48(1):35–72, 1999.

[35] Thomas Krajewski and Raimar Wulkenhaar. On Kreimer's Hopf algebra structure of Feynman graphs. *The European Physical Journal C*, 7(4):697–708, Mar 1999.

[36] Dirk Kreimer. Anatomy of a gauge theory. *Annals Phys.*, 321:2757–2781, 2006.

[37] Dirk Kreimer and Karen Yeats. An Étude in non-linear Dyson–Schwinger equations. *Nuclear Physics B - Proceedings Supplements*, 160:116–121, Oct 2006.

[38] Yuri I. Manin. Renormalization and computation I: motivation and background, 2009.

[39] Yuri I. Manin and Matilde Marcolli. Big bang, blowup, and modular curves: Algebraic geometry in cosmology. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications [electronic only]*, 10, 02 2014.

[40] Matilde Marcolli. Motivic renormalization and singularities, 2009.

[41] Matilde Marcolli and Elena Pierpaoli. Early universe models from noncommutative geometry. *Advances in Theoretical and Mathematical Physics*, 14, 08 2009.

[42] Nicolas Marie and Karen Yeats. A chord diagram expansion coming from some Dyson-Schwinger equations, 2012.

[43] Marcos Mariño and Tomás Reis. Renormalons in integrable field theories. *Journal of High Energy Physics*, 2020(4), Apr 2020.

[44] Don Zagier Maxim Kontsevich. Periods. In *Mathematics unlimited-2001 and beyond*, pages pp.771–808. Springer, 2001.

[45] N. K. Nielsen. Asymptotic freedom as a spin effect. *American Journal of Physics*, 49(12):1171–1178, 1981.

[46] Iñês Aniceto, Ricardo Schiappa, and Marcel Vonk. The resurgence of instantons in string theory. *Comm. in Number Theory and Physics*, 6(2):339–496, 2012.

[47] Erik Panzer. Renormalization, Hopf algebras and Mellin transforms. *Contemporary Mathematics*, page 169–202, 2015.

[48] Richard H. Parker, Chenghui Yu, Weicheng Zhong, Brian Estey, and Holger Müller. Measurement of the fine-structure constant as a test of the standard model. *Science*, 360(6385):191–195, 2018.

[49] Sylvie Paycha. *Regularised integrals, sums and traces. An analytic point of view.* American Mathematical Society, 2012.

[50] Jean Pierre Ramis. Irregular connections, savage π_1, and confluence. In *Proceedings of a Conference at Katata, Japan 1987, Taniguchi Fundation*, 1988.

[51] David Sauzin and Claude Mitschi. *Divergent Series, Summability and Resurgence I, Monodromy and Resurgence.* Springer International Publishing, 2016.

[52] Ernst Carl Gerlach Stueckelberg de Breidenbach and Andreas Petermann. Normalization of constants in the quanta theory. *Helv. Phys. Acta*, 26:499–520, 1953.

[53] Gerardus 't Hooft and Martinus Veltman. Regularization and renormalization of gauge fields. *Nuclear Physics B*, 44(1):189 – 213, 1972.

[54] Walter D. van Suijlekom. Multiplicative renormalization and Hopf algebras. In H. Bass, J. Oesterlé, A. Weinstein, Özgür Ceyhan, Yu. I. Manin, and Matilde Marcolli, editors, *Arithmetic and Geometry Around Quantization*, volume 279 of *Progress in Mathematics*, pages 277–292. Birkhäuser Boston, 2010.

[55] Kenneth G. Wilson. Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture. *Phys. Rev. B*, 4:3174–3183, Nov 1971.

[56] Karen Yeats. Some combinatorial interpretations in perturbative quantum field theory. *Contemp. Math.*, 648:261, 2015.

[57] W. Zimmermann. Convergence of Bogoliubov's method of renormalization in momentum space. *Comm. Math. Phys.*, 15:208–234, 1969.