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On the Status of Newtonian Gravitational Radiation

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Abstract
We discuss the status of gravitational radiation in Newtonian theories. In order to do so, we (i) consider various options for interpreting the Poisson equation as encoding propagating solutions, (ii) reflect on the extent to which limit considerations from general relativity can shed light on the Poisson equation's conceptual status, and (iii) discuss various senses in which the Poisson equation counts as a (non-)dynamical equation.

1 Introduction

In recent years, there has been a heightening of interest in *Newton-Cartan theory* (NCT): a geometrised version of Newtonian gravity, in which gravitational effects, as in general relativity (GR), are understood to be a manifestation of spacetime curvature. This interpretation of NCT is evident from the form of the field equation of the theory, the *geometrised Poisson equation*,

$$R_{ab} = 4\pi t_{ab}\rho. \quad (1)$$

Here, R_{ab} is the Ricci tensor associated with a derivative operator which is compatible with a spatial 'metric' h^{ab} of signature (0, 1, 1, 1) and a temporal 'metric' t_{ab} of signature (1, 0, 0, 0), and $\rho = T^{ab}t_{ab}$ is a matter density scalar field (T^{ab} is the stress-energy tensor). The coupling of geometry to matter in (1) is manifest, as it is for Einstein's equation of GR.[1]

It is increasingly well-appreciated that the parallels between NCT and GR run deep: in both theories is it possible to prove geodesic theorems [34], singularity

[1] Here is not the place for a detailed explication of the theoretical underpinnings of NCT; see [23, ch. 4] for the *locus classicus*.

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theorems [35], and recovery theorems [31]; moreover, in both theories is it possible to define conformal structures [8], and in both theories is the notion of gravitational energy problematic [9, 11]. With these parallels in mind, in this article we direct our attention towards the following question: to what extent do gravitational *waves* exist in NCT? Here there is a puzzle, for opinions in the extant literature diverge radically. On the one hand, Hansen *et al.* write that

> ... the equations of motion of [NCT, (1)] do not admit gravitational wave solutions: they are true relativistic phenomena. [19, p. 6]

On the other hand, Dewar and Weatherall, after having constructed a non-trivial Newtonian analogue of the Weyl tensor, conclude that

> This suggests that we should take homogeneous solutions to the Poisson equation (1) to include, among other things, the Newtonian analog of gravitational waves [9, p. 574]

Who is correct here? The purpose of the present article is to resolve this issue. While not all of the aforementioned parallels between GR and NCT are necessarily surprising given that GR reduces to NCT,[2] the existence of gravitational waves in NCT—so far always considered a characteristic trait of the full general relativistic regime—would indeed come as a revelation. Central to settling this interpretational issue is the following question: to what extent is the matter-free geometrised Poisson equation of NCT (or, for our purposes of similarly high interest, the homogeneous Newton-Poisson equation[3]) in fact a *dynamical* equation? Alternatively: to what extent are the solutions of this equation to be interpreted as *propagating* solutions? It is these questions which we seek to answer in this article; we proceed as follows. In Sect. 2, we consider options for interpreting elliptic equations (such as the Poisson equation (1)) as encoding propagating solutions. In Sect. 3, we consider the extent to which considering (1) as arising from a non-relativistic limit of GR can shed light on its conceptual and dynamical status. Section 4 discusses various senses in which the Poisson equation may or may not count as a dynamical equation—arguably with repercussions for whether its solution can genuinely count as *propagating* waves after all. Section 5 concludes.

[2] A rigorous formulation of this formal reduction is due to [13], but it was only very recently that it was given a compelling physical interpretation by [15]. Recent reconsiderations of the Newtonian limit of GR in the physics literature have been presented by [19–21]; these investigations have allowed authors to write down for the first time an action principle for an extended version of NCT (albeit not for NCT in the traditional presentation of e.g. [23, ch. 4]).

[3] In light of the Trautman geometrisation and recovery theorems [23, ch. 4], every solution to the homogeneous geometrised Poisson equation has a corresponding solution to the homogeneous Newton-Poisson equation, and *vice versa* (at least assuming the canonical additional curvature conditions—see [23, ch. 4]). For convenience we will in fact often work with the Newton-Poisson equation in the ensuing—although our points will apply to both equations.

2 The Poisson Equation

Dewar and Weatherall recognise that (1) is elliptic, and thus cannot straightforwardly represent propagating solutions in the same sense as do hyperbolic equations.[4] Nevertheless, they claim that elliptic equations can still represent propagating solutions, in two (weaker) senses. The first is that the solutions of elliptic equations may nevertheless be 'wavelike'; the second is that solutions of elliptic equations can be considered to be propagating waves insofar as they are obtained via the Newtonian limit of GR wave solutions. In the following two subsections, we consider these arguments in turn.

2.1 Wavelike Solutions to Elliptic Equations

Dewar and Weatherall's first argument that there can be gravitational wave solutions of the geometrised Poisson equation (1) (and also, given Trautman geometrisation and recovery, to the Newton-Poisson equation) begins with the claim that the solutions of an elliptic equation

> may [nevertheless] be "wavy", in the sense of exhibiting some periodicity: e.g., $\varphi = e^x e^y \sin\left(\sqrt{2}z\right)$ is a homogeneous solution to Poisson's equation that is sinusoidal in the z direction [9, p. 574].

Dewar and Weatherall take the alleged 'waviness' of such solutions as indicating a sense in which these solutions should still count as propagating. In our view, there are several issues with this. First, it is not clear why a solution should count as wavelike just because it features oscillatory functions, such as (in the above case) a sine function of one (local) spatial coordinate. Generally, accepted forms of waves are after all functions of the form $f(z \pm v(x, y, z, t)t)$ when expressed in an appropriately adapted coordinate system,[5] together with some constraints on the form of f and v.

Secondly, even if the (spatial) solutions to the Poisson equation are stacked onto each other *over time* with the appropriate boundary conditions imposed, they seem at first sight only to be associated with *standing* waves: the time component seems to only (if at all) oscillate independently of the spatial component—giving rise to standing wave solutions such as

$$\Psi(x, y, z, t) = \sin(\omega t)e^x e^y \sin\left(\sqrt{2}z\right).$$

[4] This sentiment strikes us as the orthodox view. Just very recently, for example, James writes that

> If we wish to be informed about *dynamics*, an elliptic system will not do. Given that the study of dynamics *just is* the study of processes evolving in time, it is to be expected that the mathematical tools designed for this purpose distinguish timelike directions, as hyperbolic PDEs do [22, p. 27].

Later in this article, we will evaluate critically some of James' claims in this regard.

[5] We do not necessarily want to hereby say that this is the most general form of an admissible wave function. One might, for instance, maintain that an argument of the form $f\left(z^3 \pm v(x, y, z, t)t^3\right)$ just as well gives rise to wavelike phenomena. We omit in this paper a discussion of these more general functions.

More generally, an evolution of solutions to the vacuum gravitational Poisson equation over time can be given by a continuous piecewise function, each piece of which is a product of a temporal factor and a solution to the homogeneous Poisson equation (cf. [9, p. 574] and [33]), that is

$$\Psi(x, y, z, t) = \sum_a f_a(t)\varphi_a,$$

where $f_a(t)$ and φ_a are time-dependent functions and oscillatory solutions to the Poisson equation, respectively.[6]

Thus considered, there is, in fact, a straightforward—albeit unorthodox—way of making spatial oscillatory functions 'move' over time: consider two spatial solutions oscillatory in the z-direction, with approximately the same (spatial) periodicity and amplitude, but one solution shifted by a small distance in z direction with respect to the other. A moving 'wave' can then be obtained by letting time functions appropriately weigh 'neighbouring' spatially oscillatory solutions such that neighbouring oscillatory solutions subsequently blend into one another. If this *ad hoc* construction of a moving wave is after all convincing, we have indeed found a sense in which NCT allows for *propagating* waves. That is, we can arbitrarily approximate travelling waves

$$\Psi'(x, y, z, t) = \sin(z - z_0(t))$$

through the superposition of standing wave solutions

$$\Psi(x, y, z, t) = \int daA(a, t)\sin(z - z_0'(a)),$$

where $A(a, t)$ is an appropriately-chosen modulating amplitude (essentially allowing for the blending of the oscillatory function $\sin(z - z_0'(a))$ into $\sin(z - z_0'(a'))$ over time), and a is a variable marking individual oscillatory functions, in particular their characteristic shift $z_0'(a)$ and their aforementioned time-varying amplitudes $A(a, t)$).

It is important to stress that these wavelike potentials are not gauge artefacts, but empirically relevant insofar as the force field associated to the potential inherits its propagating nature; for instance, for the potential $\Psi(x, y, z, t)$ above, the corresponding force field amounts to

$$\vec{F}(x, y, z, t) := -\vec{\nabla}\Psi(x, y, z, t) = \int daA(a, t)\begin{pmatrix} \sin(z - z_0'(a)) \\ \sin(z - z_0'(a)) \\ -\cos(z - z_0'(a)) \end{pmatrix}.$$

[6] The superposition principle we are exploiting here for the vacuum Newton Poisson equation holds also for its NCT analogue: as [23, p. 269] shows, $R_{bc} = t_b t_c \nabla_a \nabla^a \phi$, where R_{bc} is the Ricci tensor of the geometrised NCT connection, and ∇ is a flat connection of standard Newtonian gravity. This implies linearity of the NCT Ricci tensor R^{ϕ}_{ab} relative to the corresponding Newtonian potential ϕ (as linked to one another via the recovery theorem): $R^{\phi+\psi}_{bc} = t_b t_c \nabla_a \nabla^a (\phi + \psi) = t_b t_c (\nabla_a \nabla^a \phi + \nabla_a \nabla^a \psi) = R^{\phi}_{bc} + R^{\psi}_{bc}$. Our thanks to Neil Dewar for discussion on this point.

This force field can be directly measured through point particle probes. Transitioning back to the geometric picture of Newton-Cartan gravity via the Trautman recovery theorem, such propagating changes in the potential/Newtonian force field correspond to moving 'ripples' of spacetime (which are just as well measurable through point particle probes).[7]

Nevertheless, a straightforward reification of these *individual* standing waves making up the propagating wave, *à la* a naïve realist, seems to first of all speak against such a construction: there is no *single* wave moving here, but rather a conspiratorial evolution of individual standing waves. However, progress can be made by asking the following question: independently of the mathematical *means of representation* of a certain state of affairs, what is *effectively* expressed by the mathematical expressions under consideration? And here it seems without doubt that the right temporal modulation of spatially oscillatory functions can codify a moving pattern—even though, within our mathematical description, this looks like a conspiratorial set-up of standing-waves.[8] So, as long as we think of the mathematical language at play as first of all a toolkit, without premature attempts at *direct* reification, a set of appropriately modulated 'standing' waves can just as well be used to describe signal transport as an ordinary plane wave package.[9]

Still, a propagating solution that is in any physically meaningful sense associable with information/signal transfer must, one might think, be linked to energy transfer. But standing waves are exactly those kind of waves that do *not* allow for energy transfer—why should some concerted movement of several standing waves be linked to energy transfer, then? Now, in response to this, one might (1) attack the view that information/signal transfer is necessarily associated with energy transfer as such, or one might (2) associate some notion of energy to the piecewise constructed propagating solution at an emergent level. In support of (1), one could say that it is only under the presupposition of a possibly-problematic energy-transfer account of causality that we arrive at this view; and that a physical notion of wave propagation is possible without such an energy-transfer account. In fact, Dewar and Weatherall express in the very same paper their concern about the status of gravitational energy in NCT. Thus, they should indeed, it seems, regard it as being problematic to attribute energy transfer to gravitational waves in NCT; they might then seek to extend this separation of the notions of information/signal transfer and energy transfer to all Newtonian contexts. One potential resource to which Dewar and Weatherall could appeal here is the work of Dürr on gravitational waves in GR: they exist, and can propagate to mediate information transfer, but they do not carry *energy* (cf. [10]).

[7] It is worth flagging that, in the geometrised framework, these 'ripples' are not spatial, given that NCT is spatially flat [23, ch. 4].

[8] See [9, fn. 48] for some related metaphysical discussion.

[9] By analogy with the Goodman paradox [17], one might regard the piecewise functions at play in our example as trading exclusively in terms of non-projectable predicates, each defined at one time only (cf. [28])—while still allowing for talk of the matters at hand in terms of different, projectable predicates: the diachronic description of the wave. Less linguistically, and more ontologically speaking, one might simply say that 'being a propagating effect' or a 'propagating wave' are functional concepts which find physical realisation in many ways.

What about option (2), given that a functionalist stance on signal propagation is arguably a natural inspiration for taking the relevant notion of energy at play to be a higher level functionalist concept as well?[10] We accepted a long time ago that a notion of energy at one level of description does not require a notion of energy at a lower level of description. This is because notions of energy are associated with notions of time-translation invariance which itself is a symmetry which can be realised at one level of description even if it is not realised at more or less fundamental levels of description. While we do not see ourselves in a position to calculate such a putative higher-level notion of energy explicitly in the current case, there is no clear sense in which this option is ruled out either; the objection can at least not be that there is *no* way to associate the piecewise-constructed wave functions to a sensible notion of energy transfer at a coarse-grained level.

Furthermore, it is worth pointing out that the current proposal for information-carrying solutions to the (purely spatial) Poisson equation(s) *and* an additional(!) temporal direction need not clash with any claim to the effect that elliptic equations are not informative in a preferred direction (cf. [22]). The temporal direction is not part of the elliptic geometric Poisson equation to begin with; rather, the information-propagating aspect is simply gained by suitable 'stacking' of solutions to the Poisson equation at each instant.

2.2 Newtonian Limits of Relativistic Waves

The second sense in which elliptic equations such as the Newton-Poisson equation can still, for Dewar and Weatherall, represent propagating solutions is articulated by considering the Newtonian limit of GR. In particular, they write that

> we may think of Newtonian Weyl curvature as characterizing relativistic Weyl curvature–including gravitational waves–in the limit where the metric light cones flatten–and thus, the propagation velocity diverges. [Footnote suppressed.] In this limit, one would expect gravitational degrees of freedom to propagate instantaneously, in precisely the way described by an eliptic [*sic*] equation. [9, p. 575]

There are two possible interpretations of the solutions to the Poisson equation (1)[11]—as representing either (i) no propagation, or (ii) instantaneous propagation—and, indeed, authors often vacillate between them.[12] Dewar and Weatherall,

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[10] Cf. the functionalist take on gravitational stress-energy in GR presented in [27].

[11] The geometric Poisson equation (1) encodes information in 3-dimensional spacelike hypersurfaces; the question is to what extent such equations can (at least indirectly—see below) model evolution in the additional (temporal) direction.

[12] For example, at one point in their response to Dewar and Weatherall, Dürr and Read write of NCT that

> Its defining partial differential equations are elliptic. Hence, information about variations in a region propagates instantaneously. [11, p. 1002, fn. 28]
>
> Later, however, they write that
>
> NCT's Poisson Equation is elliptic. Hence its solutions can't propagate. In that sense, there is of course no gravitational radiation [11, p. 1003].

however, seek to break the symmetry in the Newtonian case with respect to gravitational radiation in that theory by considering the Newtonian limit of GR. To this, however, we would reply in the spirit of [11] as follows: the only way in which Dewar and Weatherall justify an interpretation of the Poisson equation as giving rise to gravitational waves is by reference to GR: but they thus do not establish that Newtonian gravity or NCT, *considered on their own terms*, in any clear and comprehensible manner features anything earning the name of a gravitational wave.

To make this point clear, note that the three-dimensional Newton-Poisson equation can be obtained from all sorts of differential equations, not just from hyperbolic partial differential equations, and in particular not just from the four-dimensional wave equations. For instance, it might just as well be obtained from a four-dimensional diffusion/heat equation: one simply need take the $\kappa \to 0$ limit of

$$\kappa \frac{\partial \psi}{\partial t} = \nabla^2 \psi. \tag{2}$$

Without specific *theory-relative* information undergirding Dewar and Weatherall's interpretation of the Poisson equation (1), there is no reason then why certain solutions should be read as idealised propagation processes akin to those described by a wave equation. Even if one upholds the posit that instantaneously-propagated physical effects are ultimately idealised descriptions of a finite propagation speed, nothing in NCT itself reveals whether the solutions to homogeneous equations describe propagation to begin with!

So, there are issues with using inter-theory relations to justify the existence of instantaneously-propagating gravitational waves in NCT. To bring these problems out further, consider the following reasoning, which would go through by analogy with Dewar and Weatherall's own: is it the case that a point charge in NCT is (at times) a classical black hole—just because the former can be obtained from the latter in the course of the reduction of Schwarzschild spacetime to an NCT spacetime? Or: is the Newtonian mass a relativistic mass just because it comes out of the limit of a relativistic mass?[13]

How might one respond on behalf of Dewar and Weatherall here? The following seems to us a reasonable line to take: although it is true that theories do, at least in part, 'wear their interpretations on their sleeves' (in, e.g., fixing their own senses of what is observable: see [30]), it is nevertheless the case that there are certain aspects of the interpretation of a theory which are determined only by pragmatic and contextual considerations regarding the application of that theory currently being made. For example, a simple harmonic oscillator equation might represent a block on a Hookean spring in one context, and an LC-circuit in another [16, §3.1]. Likewise, when we are discussing the Poisson equation as above, this may represent gravitational degrees of freedom in one context (in particular: when one is considering it *qua* limit of the field equations of GR), but not in another context (e.g.: when one is considering it *qua* limit of the diffusion equation). Thus, Dewar and Weatherall's

[13] For a rigorous discussion of mass in the Newtonian limit of GR, see [5].

claims here are more defensible, if one backs away from a specific central conceit of the ‘semantic approach’ to scientific theories (cf. [32] for some classic discussion)—that models of a given theory may be interpreted *ab initio* as representing possible worlds, without further input contextual considerations—and moves towards a more ‘pragmatic’ approach to theory interpretation.[14] If one does so, then one is able to say that, since Dewar and Weatherall’s discussion is situated entirely in the context of considerations of gravitational physics, it is indeed appropriate to interpret terms in (1) in a way which is informed by limiting relations from other gravitational theories.

To summarise this section, then: Dewar and Weatherall maintain that solutions to the homogeneous Poisson equation in NCT can represent propagating gravitational degrees of freedom, and provide two arguments for this. First: it is possible to understand solutions to an elliptic equation as representing propagation. We have seen that Dewar and Weatherall’s own argument for this (in terms of oscillatory functions) is too fast—but that there does remain room to defend this claim, by appealing to piecewise-composed standing waves, together with a functionalist interpretation of these objects. (In addition, we have argued that concerns regarding the capacity of such objects to mediate energy transfer can be overcome.) Second: Dewar and Weatherall argue that one can interpret solutions of the homogeneous Poisson equation, when extended by an additional time direction, as mediating energy transfer by considering the Newtonian limit of GR—to which we respond that one has to be very careful with this style of reasoning, for it has the capacity to over-generate (as we will also see again in the next section); in our view, in making these claims, one has to be explicit about the theoretical context currently under consideration.

3 Limit Considerations on the Poisson Equation

In this section, we demonstrate that the central dynamical equation of NCT, the Poisson equation (1), is obtained by taking the Newtonian limit of the Hamiltonian constraint of 3+1 GR, at least when restriction is made to asymptotically flat such spacetimes. Insofar as constraints (one might argue) cannot encode dynamics, one might conclude that Dewar and Weatherall’s argumentation given in the previous section, to the effect that the NCT Poisson equation should be interpreted by considering the Newtonian limit of GR, fails on its own terms as a means of establishing

[14] In this regard, we agree with many of the morals laid out in [7]. For more background on a ‘pragmatic’ approach to theory interpretation, see [36]. Note that our concerns about *ab initio* interpretation here are different from those expressed by the so-called ‘motivationalist’ about symmetries (see [24, 25, 29, 30]), for whom symmetry-related models of a given theory may not be taken from the outset to represent the same physical state of affairs, but rather for whom such interpretation is only legitimate after a ‘metaphysically perspicuous characterisation’ of the ontology of those models has been secured. In particular, our cause for hesitance here raises its head at an earlier stage: it is not legitimate to interpret models of a given theory as representing particular worlds *at all*, absent appreciation for the wider context in which that theory is currently being used.

a dynamics of propagating gravitational waves in NCT. The general question for the dynamical status of the Poisson equation will be addressed in the next section.

To begin, recall that, in the 3+1 decomposition of the Einstein equation, the Hamiltonian constraint—part of the elliptic content of the Einstein equation—reads [18, p. 60]

$$R_\Sigma + K^2 - K_{ij}K^{ij} = 16\pi E. \tag{3}$$

Here, R_Σ is the Ricci scalar associated with the intrinsic curvature of the hypersurface Σ, K_{ij} is the extrinsic curvature of Σ, and $K := \gamma^{ij}K_{ij}$, where γ^{ij} is the induced Euclidean 3-metric on Σ. Now, using the scalar Gauss relation [18, p. 35], we can rewrite the left-hand side of (3) in terms of the curvature of the Lorentzian 4-metric g_{ab}, to obtain

$$R + 2R_{ab}n^a n^b = 16\pi E, \tag{4}$$

where n^a is a unit vector normal to Σ. We are now in a position to consider the Newtonian limit of this equation; we do so using Ehlers' 'frame theory' (see [13]). In the frame theory limit,[15] one finds that

$$g_{ab} \to t_{ab}, \tag{5}$$

$$g^{ab} \to h^{ab}. \tag{6}$$

Moreover, the derivative operator compatible with g_{ab} maps to a (not necessarily unique) derivative operator compatible with both h^{ab} and t_{ab}. Replacing the geometrical objects in (4) with their frame theory limits (see [23, p. 258]), one then finds

$$h^{ab}R_{ab} + 2R_{ab}n^a n^b = 16\pi E. \tag{7}$$

From hereon, we assume temporal orientability, in the sense that $t_{ab} = t_a t_b$ [23, p. 251]: this assumption is innocuous, in light of the fact that we are considering the Newtonian limits of general relativistic spacetimes which admit of a 3+1 decomposition. In the frame theory limit, n^a becomes a timelike vector, in the sense that $n^a t_a = 1$ [5, p. 104]. Now, (i) assuming that one is dealing with asymptotically flat 3+1 solutions of general relativity, we recover $R^{ab}{}_{cd} = 0$ [12, p. A121], which implies spatial flatness, and so $h^{ab}R_{ab} = 0$,[16] (ii) that $E \doteq \rho = T^{ab}t_a t_b$, and (iii) that $T^{ab} = \rho n^a n^b + p^{ab}$ where p^{ab} is a smooth, symmetric field that is spacelike in both indices ($t_a p^{ab} = 0$) [23, p. 266], we have

$$n^a n^b R_{ab} = 8\pi \rho n^a n^b t_a t_b, \tag{8}$$

[15] There are delicate issues regarding both the existence and uniqueness of frame theory limits: see [5, ch. 6] for discussion.

[16] This follows straightforwardly: $h^{ab}R_{ab} = h^{ab}R^c{}_{abc} = R^{cb}{}_{bc} = 0$, where the last step follows from $R^{ab}{}_{cd} = 0$. As an alternative (but more restricted) approach here, one could use the results for static spacetimes presented at [5, p. 104].

from which it follows, after absorbing a factor of two into the energy density, that

$$R_{ab} = 4\pi\rho t_a t_b + \Xi_{ab}, \tag{9}$$

where Ξ_{ab} is a term such that $n^a n^b \Xi_{ab} = 0$. Contracting (9) with h^{ab}, we find that $h^{ab}\Xi_{ab} = 0$; moreover, contracting (9) with a term of the form $\sigma^a \eta^b$, where σ^a is a spacelike vector and η^b is a timelike vector, we also find that $\sigma^a \eta^b \Xi_{ab} = 0$ (cf. [23, p. 279]; the result deployed here assumes only $R^{ab}{}_{cd} = 0$, which follows already from the fact that we are considering the Newtonian limits of asymptotically flat 3+1 GR spacetimes); thus, Ξ_{ab} vanishes when projected onto spacelike, timelike, and mixed directions: and, so, it must vanish identically. In turn, then, we obtain

$$R_{ab} = 4\pi\rho t_a t_b, \tag{10}$$

which is the geometrised Newton-Cartan Poisson equation. Thus, the central dynamical equation of NCT is obtained by taking the Newtonian limit of the Hamiltonian constraint in 3+1 GR, at least when one restricts to asymptotically flat spacetimes.

What is the philosophical upshot of this result? Insofar as one thinks that constraints cannot be dynamical, and that significant aspects of the interpretation of one's physics (including whether a given object is to be regarded as being a constraint or not) are preserved across limiting relations between theories (a view which we have already seen that Dewar and Weatherall endorse), this would seem to imply that, for Dewar and Weatherall, (1) cannot be regarded as being dynamical after all. We explore this line of reasoning in detail in the following section.

4 The Dynamical Status of the Poisson Equation

Another take on the question of whether the Poisson equation (1) gives rise to gravitational wave solutions proceeds via reflection on the 'dynamical' status of that equation. In GR, gravitational waves are solutions to the field equations. If the Poisson equation turned out, in some sense to be specified, to be less 'dynamical' than the field equations of GR, then there would again be reason to think that the putative gravitational waves in NCT are unsatisfactory, as compared with relativistic gravitational waves.

To make progress in exploring this line of reasoning, we must first cash out salient senses of 'dynamical'. One standard such sense derives from the kinematics/dynamics distinction, which as such is a widely familiar element of physical theorising (see [6] for some recent discussion). In turn, this distinction can be disambiguated in at least three ways:[17]

[17] All three of these understandings are at least implicit in how [6] presents the kinematics/dynamics distinction. Arguably, Curiel's own view is closest to the second of the three options presented here.

Modal reading	Kinematical structure is comprised of concrete relations[18] between physical quantities that are by stipulation invariant across all possible solutions of a physical theory; dynamical structure is specified by the dynamical equations of the theory under consideration, and need not be invariant across all possible solutions of the theory.
Semantic reading	Kinematical structure is comprised of those relations between physical quantities in a given physical theory which hold in virtue of meaning.
Essentialist reading	Kinematical structure is comprised of those relations between physical quantities which hold in a given physical theory by virtue of the very essence of those quantities.

Arguably, the modal and essentialist readings of the kinematics/dynamics distinction are closely related: on the modern post-Finean view, essentialist claims are automatically metaphysically necessary, but not *vice versa* (see [14]). It is also worth stressing that the kinematics/dynamics distinction applies across the entire spectrum of physical theories.[19] For what follows in this article, it is not necessary to pick one reading over another.

The kinematics/dynamics distinction comes apart from two other distinctions related to a notion of 'dynamics': diachronic/synchronic (does the equation express evolution in time?), and that regarding hyperbolic/elliptic equations.[20] On the first:

[18] As opposed to placeholder relations that feature quantities still in need of further specification: In the context of Newtonian mechanics, $\dot{x} = v$ is a concrete relation, whereas $F = m\ddot{x}$ is a placeholder relation (as the concrete form of F still needs to be fixed further in terms of the basic dynamical variables x, v and certain constants, as for instance done through choosing $F := \frac{Gm_1m_2}{x^2}$ in certain scenarios—cf. [6, p. 5]).

[19] Although this point can only be sketched here, let us quickly stress that even two theories at first sight not well-linked to a kinematics/dynamics distinction in fact turn out to be so:

1. According to [2], the kinematics/dynamics distinction gets blurred in general relativity: inertial structure—normally associated to kinematical structure—now becomes a matter of dynamics as well. However, note that the field equations together with the matter field equations of motion are part of the dynamical structure, and it is now only the underlying manifold structure together with once-and-for-all restrictions on which fields are permitted (for instance, just tensor/spinor fields, as arguably the case in GR) that forms the kinematical structure. Brown's point should thus rather be taken as illustrating that what is kinematical, and what is dynamical, is a theory-relative matter (inertial structure is kinematical in Newtonian gravity but dynamical in GR).
2. Thermodynamics only makes explicit statements about equilibrium states and transitions between equilibrium states; it thus at most seems only to feature trivially the kinematics/dynamics distinction (there is simply no dynamical structure, one might claim). However: the kinematics/dynamics distinction is not one of time-invariant versus time-evolving quantities (see below), but rather one which distinguishes what is invariant about a system/picks out a system/essentially characterises a system from the viewpoint of that theory (see above), and what is said about these systems over and above. In fact, the so-called 'minus first law' of thermodynamics, according to which "An isolated system in an arbitrary initial state within a finite fixed volume will spontaneously attain a unique state of equilibrium" [3, p. 528], is exactly what picks out a thermodynamic system in the first place. The first law of thermodynamics, by contrast, is a condition a thermodynamic system could or could not fulfil: it is therefore dynamical.

[20] For simplicity, we set aside parabolic differential equations in this article.

a kinematical constraint need not be synchronic: consider, for example, the second homogenous Maxwell equation presented in Table 1, which is obviously diachronic. At the same time, the homogeneous Maxwell equations are best understood as being kinematical constraints, as they are constitutive of the electric and magnetic fields in the first place—in particular, these equations have the same concrete form independently of the specific scenario of interest [6, p. 5]. Moreover, a dynamical constraint need not be diachronic: consider, for example, the first inhomogeneous Maxwell equation in Table 1, which is obviously synchronic. At the same time, the inhomogeneous Maxwell equations are dynamical equations: these involve placeholders which depend on the specific scenario of interest—they do not have the same *concrete* form in all scenarios; more precisely, in the case of the inhomogeneous Maxwell equations, the density ρ and current $\mathbf{J}$ need to be determined externally,[21] and their concrete forms differs from scenario to scenario [6, p. 5]. Hence, the kinematics/dynamics distinction is orthogonal to the diachronic/synchronic distinction.

On the second: note that elliptic equations (sometimes loosely described as 'non-dynamical', as they do not model evolution, in the sense that, unlike hyperbolic equations, they do not have characteristics—see [4, 22] for some recent philosophical discussion) can be regarded as being both dynamical constraints (as in the case of the first inhomogeneous Maxwell equation in Table 1), and as kinematical constraints (as in the case of the first homogeneous Maxwell equation in Table 1). In addition, the diachronic/synchronic distinction can be teased apart from the hyperbolic/elliptic distinction, in several different ways (thus all three distinctions are indeed independent). First: elliptic equations may still be used to model temporally evolving systems, if judicious use of boundary conditions is made: this is one of the lessons drawn in Sect. 2 of the present paper. Second: elliptic equations have Cauchy problems, if not *well-posed* Cauchy problems: in this sense, they can be used to model temporal evolution (cf. [26]). Third: although, as [22, p. 19] notes,

> no particular set of directions is privileged by elliptic PDEs. We therefore have no reason to regard the informative directions of an elliptic PDE as temporal.

it is nonetheless the case that, as James also registers, elliptic equations satisfy well-posedness conditions when supplemented with Dirichlet or Neumann boundary conditions [22, p. 25]. In this sense, although nothing in the structure of an elliptic equation *privileges* any particular direction as temporal, it is still the case that such equations can be used (at least locally) to *model* temporal evolution. On the other hand, hyperbolic differential equations may or may not be used to model temporal evolution, insofar as one or more of the variables therein is interpreted as being temporal.

The upshot of the foregoing is this: we have three different distinctions which could be used to cash out a relevant sense of 'dynamical': kinematics/dynamics, diachronic/synchronic, and hyperbolic/elliptic; moreover, the first can be disambiguated in (at least) three further ways. How, then, do the Poisson equation and the field

21 There is a sense in which one can think of ρ and $\mathbf{J}$ as boundary conditions.

equations fare relative to these three senses of 'dynamics'? There are (at least) two senses in which the Poisson equation is less dynamical than the field equations:

- The Poisson equation is elliptic; the field equations are mixed hyperbolic-elliptic.
- The Poisson equation is synchronic, in the sense that none of the variables therein are typically interpreted as being related to temporal evolution; the field equations are diachronic, i.e., demonstrably have temporal components.[22]

There is one sense in which the Poisson question is as dynamical as the field equations:

- The Poisson equation and the field equations are both dynamical in the sense of the kinematics/dynamics distinction (both equations are placeholder relations as they leave open the energy-matter content for further specification). That is, in Newtonian theories the actual physical system of interest is constrained/semantically identified/essentially characterised independently of this equation.

There is a fair amount of subtlety regarding this final point. Suppose that one does indeed interpret the Hamiltonian constraint of 3+1 GR, as its name suggests, as a kinematical constraint—as per e.g. [18, p. 64].[23] In relation to our work in Sect. 3, the question then arises as to whether an object counting as a constraint can keep this status across limiting relations. If so, then, since the Poisson equation (1) can be derived as the limit of the Hamiltonian constraint in asymptotically flat 3+1 GR, one might think that it, too, should be regarded as being a kinematical constraint: thereby rendering the solutions of the Poisson equation less dynamical than the solutions of the (full) field equations of GR. Although this line of reasoning is certainly worthy of mention, we are, ultimately, not convinced by it, for several reasons: (i) whether an object should be regarded as (e.g.) a gravitational degree of freedom (as in e.g. the case of the Newtonian Weyl tensor) is a matter of physical interpretation—but it is not obvious that whether or not an object/equation should be regarded as being a *constraint* is a matter of physical interpretation in the same way (at the very least, more needs to be said on why these cases are analogous); (ii) this limiting relation was established only for a subsector of GR—so any interpretation of the Poisson equation procured via consideration of limiting relations from that subsector should, indeed, be relativised to that subsector (cf. the contextual considerations raised in Sect. 2). But most importantly (iii): in spite of its name, the Hamiltonian constraint is *not* a kinematical constraint in the sense upon which we have focused in this section, for it does not express a concrete relation: the energy E differs from dynamical solution to dynamical solution. Since it is not a kinematical constraint in this sense, one need not worry about whether the Poisson equation of Newtonian

[22] Of course, however, as we have already discussed both in Sect. 2 and above, there are more sophisticated senses in which the Poisson equation may be understood to be related to temporal evolution.

[23] Note, indeed, that Gourgoulhon diverges from Curiel on which of the Maxwell equations are to be classified as kinematical constraints: this difference tracks the distinction discussed in this paragraph.

Table 1 Maxwell's equations in their 3-vector formulation

Homogeneous equations	Inhomogeneous equations
$\vec{\nabla} \cdot \vec{B} = 0$	$\vec{\nabla} \cdot \vec{E} = \rho$
$\vec{\nabla} \cdot \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$	$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J}$

theories is a kinematical constraint or not *on the grounds of* limit considerations: for even if one *does* take it that this status is preserved on taking the Newtonian limit of GR, the status of the object of which it is a limit—the Hamiltonian constraint—is dynamical rather than kinematical anyway. If these points are correct, then it is reasonable to continue to place the Newtonian Poisson equation on the dynamics side of the kinematics/dynamics divide, and our third point above—that there is parity between the Poisson equation of Newtonian theories and the field equartions of GR—stands.

Our above investigations have had the fruitful upshot of clarifying different possible senses of 'dynamical'; using this, we see that there are clear senses in which the Poisson equation is dynamical, as well as other senses in which it is non-dynamical. Of course, whether one takes this to mean that the solutions of the Poisson equation cannot be dynamical, and therefore that there can be no gravitational wave propagation in Newtonian theories, will depend upon how one weighs these factors: one who places no store by the first of the above two points would be entitled to argue that there *can* still be dynamical gravitational wave solutions in Newtonian theories; one who places some store by them might demur. Our purpose in this paper is not so much to argue conclusively one way or the other on this matter, but rather to emphasise that anyone making a firm claim in this regard—including Dewar and Weatherall—must negotiate explicitly the above-articulated distinctions.[24]

5 Conclusions

Our goal in this paper has been to assess the status of gravitational radiation in theories such as NCT. In Sect. 2, we saw that there are ways of understanding the (elliptic) Poisson equation (1) as representing propagating solutions. In Sect. 3, we saw that this equation can be obtained via the Newtonian limit of asymptotically flat 3+1 GR; insofar as one thinks that whether an object counts as a constraint (and therefore, in one sense, non-dynamical) is preserved across mathematical limits, one might worry that, in fact, the Poisson equation cannot represent dynamical solutions of propagating gravitational waves—although we went on to argue in Sect. 4

[24] It bears mentioning that there is also a sense in which the *Hamiltonian* formulation of GR is less dynamical than e.g. NCT: for the Hamiltonian of GR—the generator of time translations—is a full constraint equation, including the Hamiltonian constraint (it is this conundrum of a vanishing Hamiltonian which has by now become (in)famous as the 'problem of time'). According to [1, p. 14], this issue cannot arise in NCT due to its formulation in terms of a preferred foliation. So, if it were possible to formulate the theory in Hamiltonian form, the Hamiltonian would not simply end up as a mere sum of constraints.

that this argument is not convincing, for a number of reasons. In Sect. 4, we also disambiguated more systematically three senses in which an equation might count as dynamical, related to three distinctions: dynamics/kinematics, diachronic/synchronic, and elliptic/hyperbolic. On the third: one might ask why we should place an *ab initio* embargo on using elliptic equations as a decisive part in modelling dynamical situations: while admittedly non-standard, we have seen that there does not seem to be any reason in principle why elliptic equations cannot be used to this end. That, then, leaves only the diachronic/synchronic distinction: but, again, we have witnessed certain senses in which elliptic equations *can* form a natural part of modeling dynamical evolution.

The upshot, then, is this: when considered *qua* limit of the field equations of GR, the Poisson equation of e.g. NCT inherits a gravitational interpretation; in this context, solutions of this equation *can* be used to model gravitational waves, gravitation radiation, and pure gravitational degrees of freedom when stacked upon one another over time and made subject to appropriate boundary conditions. This vindicates, in substantially more detail than hitherto achieved, the claims made by Dewar and Weatherall to this effect.

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