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General Theory of Constructing Potential with Bound States in the Continuum

Mao Kurino* and Kazuo Takayanagi*

Department of Physics, Sophia University, 7-1 Kioi-cho, Chiyoda-ku, Tokyo 102, Japan
*E-mail: m-kurino-ex5@eagle.sophia.ac.jp (M.K.); k-takaya@sophia.ac.jp (K.T.)

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We present a general theory of potentials that support bound states at positive energies (bound states in the continuum). On the theoretical side, we prove that, for systems described by nonlocal potentials of the form $V(r, r')$, bound states at positive energies are as common as those at negative energies. At the same time, we show that a local potential of the form $V(r)$ rarely supports a positive-energy bound state. On the practical side, we show how to construct a (naturally nonlocal) potential that supports an arbitrary normalizable state at an arbitrary positive energy. We demonstrate our theory with numerical examples both in momentum and coordinate spaces with emphasis on the important role played by nonlocal potentials. Finally, we discuss how to observe bound states at positive energies, and where to search for nonlocal potentials that may support them.

Subject Index A64, D00, D06, D13, D14

1. Introduction

Since the pioneering work of von Neumann and Wigner [1], a lot of attempts have been made both theoretically and experimentally in search of bound states at positive energies, or equivalently bound states in the continuum (BICs) in quantum systems. Theoretically, the BIC has been described as a resonance with vanishing width that occurs only accidentally [2–4]. On the experimental side, however, no clear BIC has been observed experimentally in quantum systems, while many BICs have been found in a broad range of classical and semiclassical systems [5,6].

In this situation, we present a general theory of the potential that supports a BIC via a simple one-body Schrödinger equation.[1] We express a Hermitian potential as $V = V_S + V_B$, which we call SB-decomposition [7], where V_S and V_B operate on the spaces of scattering and bound states, respectively. Then, by examining transformations of V_S and V_B separately, we show how to design a Hermitian potential that describes an arbitrary normalizable state as a BIC at an arbitrary positive energy. Conversely, any Hermitian potential supporting a BIC can be described by the present theory. In other words, we explicitly present a general solution to the inverse problem for an arbitrary BIC.

We develop the theory in momentum space, i.e., we construct a BIC-supporting potential in the form of $\langle k|V|k'\rangle$ where k and k' are the relative momenta for a fixed angular momentum.

[1] In this work, we are concerned with localized potentials with which we are familiar in nuclear and atomic physics. Therefore, the Anderson localization in solid state physics, for instance, is not relevant in the present work though it gives a localized state at a positive energy. This is because it requires a random potential that is not localized but exists all over the system.

By taking its Fourier transform, we obtain a BIC-supporting potential $V(r, r') = \langle r|V|r'\rangle$ in coordinate space that generally depends on r and r' separately, and is therefore nonlocal. In the present work, it is important to distinguish between a familiar "local" potential $V(r)$ and a general "nonlocal" potential $V(r, r')$.[2]

The plan of this paper is as follows. In Section 2, we briefly explain the general theory to design a Hermitian potential V using the SB-decomposition. Then, in Section 3, we present two designs of BIC-supporting potentials. One design requires specification of both V_S and V_B, and is most general but demands sizable calculations. The other uses transformation of V_B only, and is quite simple and numerically feasible without complicated calculations. In Sections 4 and 5, we show numerical results that demonstrate our designs of BIC-supporting potentials in both momentum and coordinate spaces. Here, we shall realize the important role played by the nonlocality of the potential to support a BIC. In Section 6, we discuss the BIC from the viewpoint of familiar local potentials, and prove that, for a local potential $V(r)$ to describe a bound state wave function $\phi(r)$ as a BIC, both $V(r)$ and $\phi(r)$ must be oscillating functions that decay very slowly as $r \to \infty$. Therefore, a local potential can rarely support a BIC. In Sections 7 and 8, we discuss the BIC using nonlocal potentials on the basis of both the Schrödinger equation for a bound state and the structure of the group of all Hermitian potentials with a bound state. Surprisingly enough, the present theory clarifies that, in systems described by Hermitian nonlocal potentials in general, positive-energy bound states (BICs) are as common a phenomenon as negative-energy bound states. To find a BIC, therefore, it is crucial to examine systems with nonlocal potentials. In Section 9, we show how we can observe a BIC, if it exists, on the basis of Levinson's theorem and $\langle k|V_B|k'\rangle$ both theoretically and experimentally. Then, in Section 10, we discuss how and where to search for a BIC in practice. Here, facing the problem of nonlocal potentials, which are not commonplace in many fields of physics, we present possible scenarios to look for a BIC in real physical systems. Finally, in Section 11, we summarize the present work.

2. General design of the potential

We consider a spinless particle of mass m described by the Hamiltonian:

$$H = H_0 + V, \tag{1}$$

where H_0 is the kinetic energy and V is the potential energy of the particle with a fixed force center.

In the following, we briefly explain our method to construct a potential V in three steps (i), (ii), and (iii) in a way that clarifies its relation to the bound and scattering states. Our method is based on the SB-decomposition of a Hermitian potential V, which derives from the generalized optical theorem for the half-on-shell T-matrix [7]. In what follows, we assume a rotationally invariant system for simplicity, and work in the space of a fixed partial wave.

[2] A Hermitian potential is generally a nonlocal potential $V(r, r') = \langle r|V|r'\rangle$ in the coordinate representation. A nonlocal potential $V(r, r')$ enters the Schrödinger equation for $\phi(r)$ as $\int_0^\infty r'^2 \mathrm{d}r'\, V(r, r')\phi(r')$ as in Eq. (35), while a local potential $V(r)$ shows up in the form of $V(r)\phi(r)$ as in Eq. (36). Note that a local potential $V(r)$ can be expressed as a nonlocal potential as $V(r, r') = \dfrac{\delta(r' - r)}{r'r} V(r)$, meaning that a local potential is a special case of a nonlocal potential. As a simple example of the nonlocal potential, one may think of the Fock potential in the Hartree–Fock theory.

(i) $\mathcal{B}$

First, in the whole Hilbert space $\mathcal{H}$, we specify an N-dimensional subspace $\mathcal{B}$ that is to be spanned by N bound states that we are going to fix below. Naturally, its orthogonal complement $\mathcal{S}$ is to be spanned by as-yet-unknown scattering states. With knowledge of the subspace $\mathcal{B}$, we can define the projection operators $P_{\rm B}$ onto $\mathcal{B}$[3] and $P_{\rm S} = 1 - P_{\rm B}$ onto $\mathcal{S}$. Then, we decompose the Hermitian potential V that we are going to construct as

$$V = V_{\rm S} + V_{\rm B}, \tag{2}$$

where we have used abbreviations $VP_{\rm S} \to V_{\rm S}$ and $VP_{\rm B} \to V_{\rm B}$. In the following, Eq. (2) is referred to as the SB-decomposition of V for obvious reasons.

(ii) $V_{\rm S}$

Second, for the fixed subspace $\mathcal{B}$, the generalized optical theorem states the following [7–9]: a function $\langle k'|T|k\rangle$ of two variables k' and k is physically acceptable as a half-on-shell T-matrix,[4] i.e., it has an underlying Hermitian potential V if and only if it satisfies the following conditions (A) and (B):

$$T\Omega_+^\dagger - H_0 P_{\rm B} = \Omega_+ T^\dagger - P_{\rm B} H_0, \tag{A}$$

$$\Omega_+^\dagger T = T^\dagger \Omega_+, \tag{B}$$

where Ω_+ is the Møller wave operator that transforms a plane wave state $|k\rangle$ into its corresponding scattering state $|k\rangle_+ = \Omega_+|k\rangle$ with the outgoing wave boundary condition [10,11].[5] There are many T's that satisfy conditions (A) and (B) for the fixed subspace $\mathcal{B}$. Therefore, we arbitrarily single out a T from them, which in turn defines its corresponding $V_{\rm S}$ as

$$\langle k'|V_{\rm S}|k\rangle = \langle k'|T|k\rangle + \int \frac{p^2 {\rm d}p}{(2\pi)^3}\, \langle k'|T|p\rangle \frac{1}{p^2 - k^2 - i\eta} \langle p|T^\dagger|k\rangle. \tag{3}$$

Then, it is easy to show that $V_{\rm S}$ of Eq. (3) describes our scattering states $\{|k\rangle_+ = \Omega_+|k\rangle\}$ via the following Schrödinger equation:

$$(H_0 + V)|k\rangle_+ = (H_0 + V_{\rm S})|k\rangle_+ = k^2|k\rangle_+. \tag{4}$$

(iii) $V_{\rm B}$

Third, we arbitrarily specify an orthonormal set of bound states $\{|\phi_i\rangle, i = 1, 2, \cdots, N\}$ that span the subspace $\mathcal{B}$, i.e., $\mathcal{B} = {\rm Span}\,\{|\phi_i\rangle\}$. Furthermore, we arbitrarily allot the eigenenergy E_i to $|\phi_i\rangle$. Then, it is straightforward to show that[6]

$$\langle k'|V_{\rm B}|k\rangle = \sum_i (E_i - k'^2)\langle k'|\phi_i\rangle\langle\phi_i|k\rangle \tag{5}$$

[3]Once the N-dimensional subspace $\mathcal{B}$ is fixed, any orthonormal set $\{|\chi_i\rangle, i = 1, 2, \cdots, N\}$ satisfying $\mathcal{B} = {\rm Span}\,\{|\chi_i\rangle\}$ expresses the projection operator $P_{\rm B}$ as $P_{\rm B} = \sum_i |\chi_i\rangle\langle\chi_i|$.

[4]The half-on-shell T-matrix $\langle k'|T|k\rangle$ is defined via the Lippmann–Schwinger equation [10,11], $\langle k'|T|k\rangle = \langle k'|V|k\rangle + \int \frac{p^2 {\rm d}p}{(2\pi)^3}\, \langle k'|V|p\rangle \frac{1}{k^2 - p^2 + i\eta} \langle p|T|k\rangle$. Note that we adopt the usual unit system where $\hbar = 2m = 1$.

[5]Ω_+ is defined in terms of T as $\langle p|\Omega_+|k\rangle = \frac{(2\pi)^3}{pk}\delta(p-k) + \frac{1}{k^2 - p^2 + i\eta}\langle p|T|k\rangle$. Therefore, conditions (A) and (B) compose a coupled set of equations for $\langle p|T|k\rangle$, a function of two variables p and k, to be compatible with the given $P_{\rm B}$.

[6]As in the standard notation, $\langle k|\phi\rangle = \phi(k)$ is the wave function in momentum space.

describes our bound states $\{|\phi_i\rangle\}$ via the Schrödinger equation

$$(H_0 + V)|\phi_i\rangle = (H_0 + V_{\rm B})|\phi_i\rangle = E_i|\phi_i\rangle. \tag{6}$$

Here, conditions (A) and (B) of the generalized optical theorem guarantee that, for $V_{\rm S}$ of Eq. (3) and $V_{\rm B}$ of Eq. (5), V of Eq. (2) is Hermitian [7]. Conversely, any Hermitian potential V can be uniquely SB-decomposed into the form of Eq. (2). To summarize, the above steps (i), (ii), and (iii) compose a general method to design a Hermitian potential V while keeping its eigenstates $\{|k\rangle_+,\ k^2\}$ and $\{|\phi_i\rangle,\ E_i\}$ under control.[7]

We stress that, once the subspace $\mathcal{B}$ is fixed in step (i), the SB-decomposition determines $V_{\rm S}$ and $V_{\rm B}$ quite independently via steps (ii) and (iii), respectively. This is because $V_{\rm S}$ and $V_{\rm B}$ depend on the bound states in different ways; $V_{\rm B}$ of Eq. (5) demands all bound states $\{|\phi_i\rangle\}$ and their energies $\{E_i\}$ for its description. On the other hand, $V_{\rm S}$ of Eq. (3), which expresses $V_{\rm S}$ in terms of T, requires the subspace $\mathcal{B}$ only, i.e., it does not demand further information on bound states for its description.[8] As we shall see, the above feature of the SB-decomposition suits the study of BICs.

3. Potential with a BIC

Using the three steps in Section 2, we explain how to construct a BIC-supporting potential. After giving an explanation of the most general method in Section 3.1, we present a simple and practical method in Section 3.2.

3.1. *General method: Specification of $V_{\rm S}$ and $V_{\rm B}$*

Let us note that the SB-decomposition in Section 2 does not require E_i to be negative in step (iii). This implies that we already have a general design of the potential supporting a BIC; by giving a positive value to E_i in step (iii), our potential V of Eq. (2) describes the bound state $|\phi_i\rangle$ as a BIC. Therefore, via steps (i), (ii), and (iii) in Section 2, we can construct a potential that supports an arbitrary normalizable state $|\phi\rangle$ as a BIC at an arbitrary positive energy. Conversely, because of the uniqueness of the SB-decomposition, any Hermitian potential supporting a BIC can surely be expressed uniquely by Eqs. (2), (3), and (5).

3.2. *Simple method: Transformation of $V_{\rm B}$*

In the general method in Section 3.1, we can arbitrarily specify the wave function and the eigenenergy of the BIC in steps (i) and (iii). In trade for these degrees of freedom, however, we have to look for a solution $\langle k'|T|k\rangle$ to the set of Eqs. (A) and (B) in step (ii), which is sometimes inconvenient. In this section, therefore, we show a simple version of the general method, which does not require an explicit solution to Eqs. (A) and (B). We shall find that the present method is quite useful not only practically but also theoretically.[9]

[7] For convenience, we present the eigenstate $|\phi_i\rangle$ and its eigenenergy E_i in a single pair of braces.

[8] This is because bound states constrain T (and therefore $V_{\rm S}$) through conditions (A) and (B) that utilize the information of bound states via $P_{\rm B}$ only.

[9] We shall make use of the present simple method not only in actual calculations in Section 4 but also in theoretical arguments in Section 7.1.

3.2.1. *Derivation.* Here, we start with an arbitrary Hermitian potential V_0 that supports, for simplicity, a single bound state $|\phi\rangle$ at a negative energy $E = -\gamma_0^2$:

$$(H_0 + V_0)|\phi\rangle = -\gamma_0^2|\phi\rangle. \tag{7}$$

In the following, we derive a potential V with a BIC by establishing a transformation $V_0 \to V$ using the three steps in Section 2, which we shall find quite simple and feasible.

(i) We adopt $|\phi\rangle$ of Eq. (7) as the wave function of the BIC; instead of giving arbitrarily a wave function of the BIC, we give arbitrarily a Hermitian potential V_0 and then adopt its bound state $|\phi\rangle$ as the BIC of the as-yet-unknown potential V that describes $|\phi\rangle$ via

$$(H_0 + V)|\phi\rangle = E|\phi\rangle, \tag{8}$$

with a positive E. The present choice of the BIC means that $\mathcal{B} = \text{Span}\{|\phi\rangle\}$ of V is the same as for V_0. Therefore, their SB-decompositions, $V = V_\text{S} + V_\text{B}$ and $V_0 = V_{0,\text{S}} + V_{0,\text{B}}$, are defined in terms of the same projection operators P_B and P_S.

(ii) Let T and T_0 be the half-on-shell T-matrices of V and V_0, respectively. Because $\mathcal{B}$ of V is the same as for V_0, conditions (A) and (B) for T are the same as for T_0. We can choose, therefore, a trivial solution of Eqs. (A) and (B) without solving them for T, i.e., $T = T_0$, yielding $V_\text{S} = V_{0,\text{S}}$ via Eq. (3). Then, because the SB-decomposition of V_0 gives $V_{0,\text{B}}$ as

$$\langle k'|V_{0,\text{B}}|k\rangle = \left(-\gamma_0^2 - k'^2\right)\langle k'|\phi\rangle\langle\phi|k\rangle, \tag{9}$$

it immediately yields $V_\text{S} = V_{0,\text{S}} = V_0 - V_{0,\text{B}}$ as

$$\langle k'|V_\text{S}|k\rangle = \langle k'|V_0|k\rangle + \left(\gamma_0^2 + k'^2\right)\langle k'|\phi\rangle\langle\phi|k\rangle. \tag{10}$$

(iii) We allot a positive energy $E = K^2$ to $|\phi\rangle$, to which V_0 assigned the negative energy $-\gamma_0^2$ in Eq. (7). Then, Eq. (5) immediately expresses the desired V_B as

$$\begin{aligned}\langle k'|V_\text{B}|k\rangle &= (K^2 - k'^2)\langle k'|\phi\rangle\langle\phi|k\rangle \\ &= \left(-\gamma_0^2 - k'^2\right)\langle k'|\phi\rangle\langle\phi|k\rangle + \left(K^2 + \gamma_0{}^2\right)\langle k'|\phi\rangle\langle\phi|k\rangle \\ &= \langle k'|V_{0,\text{B}}|k\rangle + \langle k'|\Delta V_\text{B}|k\rangle,\end{aligned} \tag{11}$$

where we have defined

$$\Delta V_\text{B} \equiv V_\text{B} - V_{0,\text{B}} = \left(K^2 + \gamma_0^2\right)|\phi\rangle\langle\phi|. \tag{12}$$

Note that ΔV_B of Eq. (12) is an energy-shift operator, i.e., it shifts the eigenenergy $-\gamma_0^2$ of $|\phi\rangle$ to K^2. To be precise, $H_0 + V = H_0 + V_0 + \Delta V_\text{B}$ has the same set of eigenstates $\{|k\rangle_+, |\phi\rangle\}$ as $H_0 + V_0$, but assigns a shifted eigenenergy K^2 to $|\phi\rangle$.

With V_S of Eq. (10) and V_B of Eq. (11), we define V by Eq. (2), which gives

$$\begin{aligned}\langle k'|V|k\rangle &= \langle k'|V_0|k\rangle + \langle k'|\Delta V_\text{B}|k\rangle \\ &= \langle k'|V_0|k\rangle + \left(K^2 + \gamma_0^2\right)\langle k'|\phi\rangle\langle\phi|k\rangle.\end{aligned} \tag{13}$$

It is then easy to confirm the Schrödinger equation (8) with $\langle k'|V|k\rangle$ of Eq. (13), which now reads

$$k^2\phi(k) + \int\frac{p^2\text{d}p}{(2\pi)^3}\,\langle k|V|p\rangle\phi(p) = K^2\phi(k), \tag{14}$$

in the same way as Eq. (6) in the general theory. To summarize, Eq. (13) defines the desired transformation $V_0 \to V = V_0 + \Delta V_\text{B}$, where V describes the bound state $|\phi\rangle$ as a BIC at the

positive energy $E = K^2$.[10] It is straightforward to generalize the above method to the most general case with several bound states, some of which are at positive energies and others at negative energies.

The present transformation $V_0 \to V = V_0 + \Delta V_{\rm B}$ is "minimal" in the sense that it only changes the eigenenergy of $|\phi\rangle$ as $-\gamma_0^2 \to K^2$, while leaving all the eigenstates, $|\phi\rangle$ and $\{|k\rangle_+\}$, untouched. Correspondingly, the SB-decomposition of the transformation is summarized by $V_{0,\rm B} \to V_{\rm B} = V_{0,\rm B} + \Delta V_{\rm B}$ and $V_{\rm S} = V_{0,\rm S}$. We say also that the present method is "simple" because we can immediately write down the right-hand side of Eq. (13) without complicated numerical calculations. So follows the title of the present Section 3.2.

3.2.2. *Comment.* If our goal were just to derive Eq. (13), the easiest way would be the following. By definition, we can express $H_0 + V_0$ and $H_0 + V$ as

$$H_0 + V_0 = \int \frac{k^2 {\rm d}k}{(2\pi)^3}\, k^2 |k\rangle_{++}\langle k| \; - \; \gamma_0^2 |\phi\rangle\langle\phi|, \tag{15}$$

$$H_0 + V = \int \frac{k^2 {\rm d}k}{(2\pi)^3}\, k^2 |k\rangle_{++}\langle k| \; + \; K^2 |\phi\rangle\langle\phi|. \tag{16}$$

Then, the difference between Eqs. (16) and (15) gives

$$V - V_0 = \left(K^2 + \gamma_0^2\right) |\phi\rangle\langle\phi|, \tag{17}$$

which immediately leads to Eq. (13). However, the above derivation does not SB-decompose V of Eq. (13) into $V_{\rm S}$ of Eq. (10) and $V_{\rm B}$ of Eq. (11). It would not allow, therefore, the detailed study of $\langle k'|V|k\rangle$ in the following sections, where we will make full use of the SB-decomposition.

4. Numerical example I

In this section, we explicitly construct a Hermitian potential supporting a BIC. We use the simple method in Section 3.2 rather than the general method in Section 3.1, because the minimal nature of its transformation $V_0 \to V = V_0 + \Delta V_{\rm B}$ is suitable to vary the eigenenergy of a fixed bound state continuously without changing scattering states. In the following, we work in the S-wave channel for simplicity.

4.1. *Step (i): Gaussian potential V_0*

Here, we adopt the following local Gaussian potential for V_0 in step (i) of Section 3.2:[11]

$$V_0(r) = \lambda\, \exp\left(-\frac{r^2}{b^2}\right), \quad \lambda = -30.0\,{\rm fm}^{-2}, \quad b = 0.5\,{\rm fm}, \tag{18}$$

which is given in momentum space as

$$\langle k'|V_0|k\rangle = 4\pi\lambda\,(b\sqrt{\pi})^3\, j_0\left(-i\frac{kk'b^2}{2}\right) \exp\left(-\frac{(k^2+k'^2)b^2}{4}\right), \tag{19}$$

[10] The present method can also be used to change the eigenenergy $-\gamma_0^2$ of $|\phi\rangle$ to another negative value $-\gamma_1{}^2$; we just need to replace K^2 with $-\gamma_1{}^2$ in Eq. (13).

[11] In the calculations below, m stands for the proton–neutron reduced mass, $M_pM_n/(M_p+M_n)$. Then, our unit system, $\hbar = 2m = 1$, gives $1 = \hbar^2/(2m) = 41.47\,{\rm MeV\,fm}^2$. Therefore, $\lambda = -30.0\,{\rm fm}^{-2}$ corresponds to $-30.0\,{\rm fm}^{-2} \times 41.47\,{\rm MeV\,fm}^2 = -1244\,{\rm MeV}$.

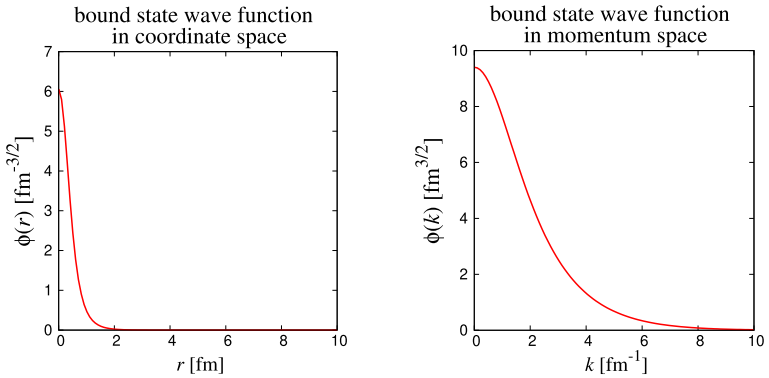

Fig. 1. Wave function $\phi(r)$ of the bound state at $-\gamma_0^2 = -5.373\,\mathrm{fm}^{-2}$ ($= -222.8\,\mathrm{MeV}$) in coordinate space (left panel) and $\phi(k)$ in momentum space (right panel) of the Gaussian potential in Eq. (18). See the text.

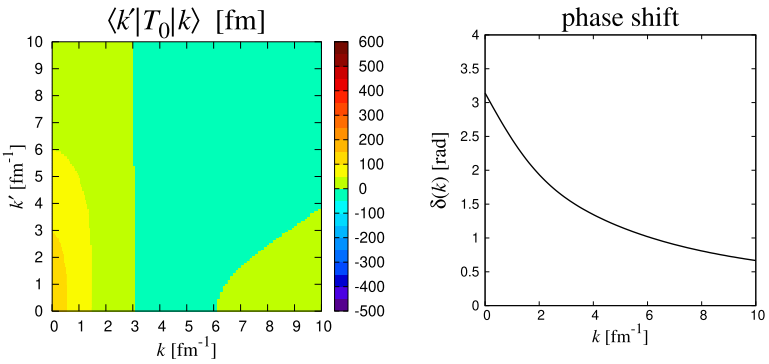

Fig. 2. Half-on-shell T-matrix $\langle k'|T_0|k\rangle$ (left panel, contour map of the real part) and phase shift $\delta(k)$ (right panel) of the Gaussian potential V_0 in Eq. (18). See the text.

where $j_0(x) = \dfrac{\sin x}{x}$ is the spherical Bessel function of zeroth order. It is straightforward to show numerically that the above V_0 supports a single bound state $|\phi\rangle$ at $-\gamma_0^2 = -5.373\,\mathrm{fm}^{-2}$ ($= -222.8\,\mathrm{MeV}$), of which the wave function $\phi(r)$ in coordinate space and $\phi(k)$ in momentum space are respectively shown in the left and right panels in Fig. 1. For later convenience, we also show in Fig. 2 the half-on-shell T-matrix $\langle k'|T_0|k\rangle$ and the phase shift $\delta(k)$ of V_0.

4.2. *Step (ii):* $V_S = V_{0,S}$

Now we make the SB-decomposition of $\langle k'|V_0|k\rangle$ in Eq. (19), to arrive at the results shown in Fig. 3. In the right panel, we display $\langle k'|V_0|k\rangle$ of Eq. (19), which is decomposed into $\langle k'|V_{0,B}|k\rangle$ of Eq. (9) and $\langle k'|V_{0,S}|k\rangle = \langle k'|V_0|k\rangle - \langle k'|V_{0,B}|k\rangle$ shown in the middle and left panels, respectively.[12] Then, following step (ii) in Section 3.2, we define V_S by $\langle k'|V_S|k\rangle = \langle k'|V_{0,S}|k\rangle$.

We now examine Fig. 3 before going to step (iii). First, let us look into $\langle k'|V_{0,B}|k\rangle$ in the middle panel. Then, Eq. (9) explains its behavior as follows: the k-dependence of $\langle k'|V_{0,B}|k\rangle$ is given by $\phi(k)$ in Fig. 1. On the other hand, its k'-dependence is given by $(-\gamma_0^2 - k'^2)\phi(k')$, which is negative and extends to a higher-momentum region than $\phi(k)$. Thus the product

[12] In the general method in Section 3.1, we would have to calculate $\langle k'|V_{0,S}|k\rangle$ via Eq. (3) in terms of $\langle k'|T_0|k\rangle$ in Fig. 2. In the present "simple" method in Section 3.2, on the other hand, we can immediately write down $\langle k'|V_{0,S}|k\rangle$ without the knowledge of $\langle k'|T_0|k\rangle$.

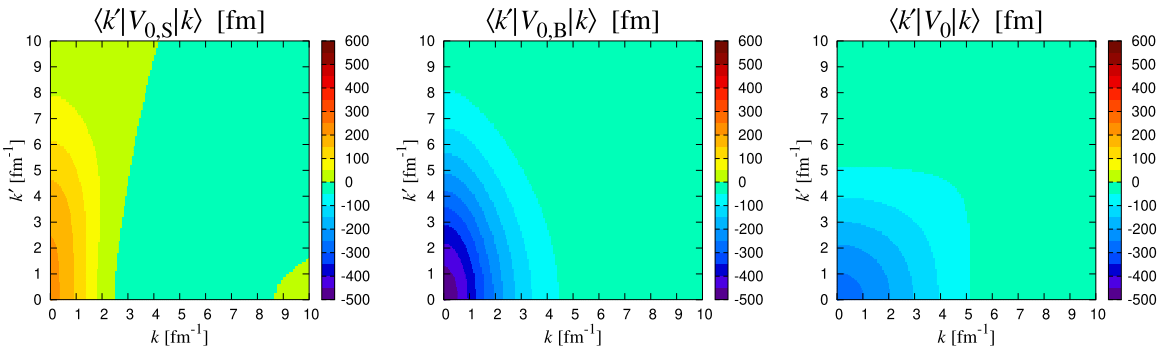

Fig. 3. SB-decomposition $V_{0,\mathrm{S}} + V_{0,\mathrm{B}} = V_0$ of the Gaussian potential V_0 in Eq. (19), which has the bound state at $E = -\gamma_0^2 = -5.373\,\mathrm{fm}^{-2}$ ($= -222.8\,\mathrm{MeV}$). Left: contour map of $\langle k'|V_{0,\mathrm{S}}|k\rangle$. Middle: contour map of $\langle k'|V_{0,\mathrm{B}}|k\rangle$. Right: contour map of $\langle k'|V_0|k\rangle = \langle k'|V_{0,\mathrm{S}}|k\rangle + \langle k'|V_{0,\mathrm{B}}|k\rangle$. See the text.

of these two factors clearly explains $\langle k'|V_{0,\mathrm{B}}|k\rangle$ in the middle panel. Second, let us turn to $\langle k'|V_{0,\mathrm{S}}|k\rangle$ in the left panel of Fig. 3. We can see that, because the range of $\phi(k)$ is approximately $\sim \dfrac{1}{b} = 2.0\,\mathrm{fm}^{-1}$, $\langle k'|V_{0,\mathrm{S}}|k\rangle$ is highly repulsive for $k \lesssim 2.0\,\mathrm{fm}^{-1}$ to guarantee the orthogonality of $\{|k\rangle_+,\ k \lesssim 2.0\,\mathrm{fm}^{-1}\}$ to $|\phi\rangle$. We also realize that, because $V_{0,\mathrm{S}}|\phi\rangle = 0$ by definition, $\langle k'|V_{0,\mathrm{S}}|k\rangle$ becomes slightly negative in the large-k region for each value of k' to guarantee $\langle k'|V_{0,\mathrm{S}}|\phi\rangle = \displaystyle\int \frac{k^2 \mathrm{d}k}{(2\pi)^3}\, \langle k'|V_{0,\mathrm{S}}|k\rangle\, \phi(k) = 0$. The above observations explain the main features of $\langle k'|V_{0,\mathrm{B}}|k\rangle$ and $\langle k'|V_{0,\mathrm{S}}|k\rangle$, and therefore of $\langle k'|V_0|k\rangle$, shown in Fig. 3.

4.3. *Step (iii): Transformation $V_{0,\mathrm{B}} \to V_{\mathrm{B}}$*

Now, we implement step (iii) of Section 3.2, i.e., we transform $\langle k'|V_{0,\mathrm{B}}|k\rangle$ of Eq. (9) into $\langle k'|V_{\mathrm{B}}|k\rangle$ of Eq. (11). For later convenience, we replace K^2 in $\langle k'|V_{\mathrm{B}}|k\rangle$ of Eq. (11) with the energy variable E, to express $\langle k'|V_{\mathrm{B}}|k\rangle$ as a function of E as

$$\langle k'|V_{\mathrm{B}}|k\rangle = (E - k'^2)\langle k'|\phi\rangle\langle\phi|k\rangle. \tag{20}$$

4.4. *Numerical results for $V_{\mathrm{S}} + V_{\mathrm{B}} = V$ in momentum space*

We are now ready to investigate the transformed potential in Eq. (13). For each value of eigenenergy E of the fixed bound state $|\phi\rangle$, $\langle k'|V_{\mathrm{B}}|k\rangle$ of Eq. (20), together with $\langle k'|V_{\mathrm{S}}|k\rangle = \langle k'|V_{0,\mathrm{S}}|k\rangle$ of Eq. (10), makes its corresponding $\langle k'|V_{\mathrm{S}}|k\rangle + \langle k'|V_{\mathrm{B}}|k\rangle = \langle k'|V|k\rangle$, which is expressed in the form of Eq. (13) as

$$\langle k'|V|k\rangle = \langle k'|V_0|k\rangle + \left(E + \gamma_0^2\right)\langle k'|\phi\rangle\langle\phi|k\rangle, \tag{21}$$

where we have replaced K^2 with E as in Eq. (20).[13] Here we examine how the SB-decomposition $\langle k'|V_{0,\mathrm{S}}|k\rangle + \langle k'|V_{0,\mathrm{B}}|k\rangle = \langle k'|V_0|k\rangle$ in Fig. 3 changes as we continuously vary the eigenenergy E. In actual calculations, we have varied $E = -\gamma_0^2 = -5.373\,\mathrm{fm}^{-2}$ for V_0 (Fig. 3) to $-4.0\,\mathrm{fm}^{-2}$ (Fig. 4), $-1.0\,\mathrm{fm}^{-2}$ (Fig. 5), $0.0\,\mathrm{fm}^{-2}$ (Fig. 6), $1.0\,\mathrm{fm}^{-2}$ (Fig. 7), and $4.0\,\mathrm{fm}^{-2}$ (Fig. 8), in order.[14] We emphasize that the potentials in Figs. 7 and 8 describe the bound state $|\phi\rangle$ as a BIC at

[13] $\langle k'|V|k\rangle$ of Eq. (21) reduces to $\langle k'|V_0|k\rangle$ of Eq. (13) for $E = -\gamma_0^2 = -5.373\,\mathrm{fm}^{-2}$.

[14] Note that each potential $\langle k'|V|k\rangle$ in Figs. 4–8 generally represents a nonlocal potential in coordinate space, while $\langle k'|V_0|k\rangle$ in Fig. 3 stands for the local Gaussian potential $V_0(r)$ of Eq. (18).

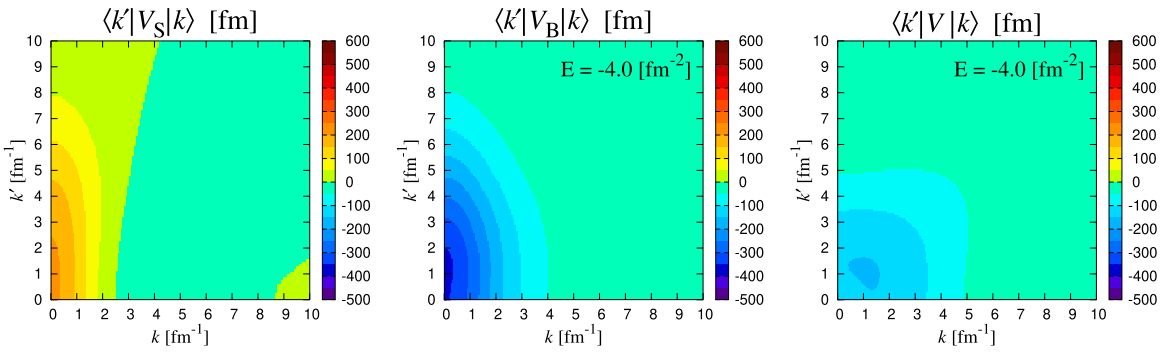

Fig. 4. SB-decomposition $V_{\rm S}+V_{\rm B}=V$ of V that gives the same $\langle k'|T|k\rangle$ and $|\phi\rangle$ as those given by $\langle k'|V_0|k\rangle$ of Fig. 3. $\langle k'|V_{\rm S}|k\rangle$ is the same as $\langle k'|V_{0,{\rm S}}|k\rangle$ in Fig. 3. In the expression (20) for $\langle k'|V_{\rm B}|k\rangle$, the eigenenergy E of $|\phi\rangle$ is chosen to be $E=-4.0\,{\rm fm}^{-2}$ $(=-165.9\,{\rm MeV})$. Other notation is the same as for Fig. 3. See the text.

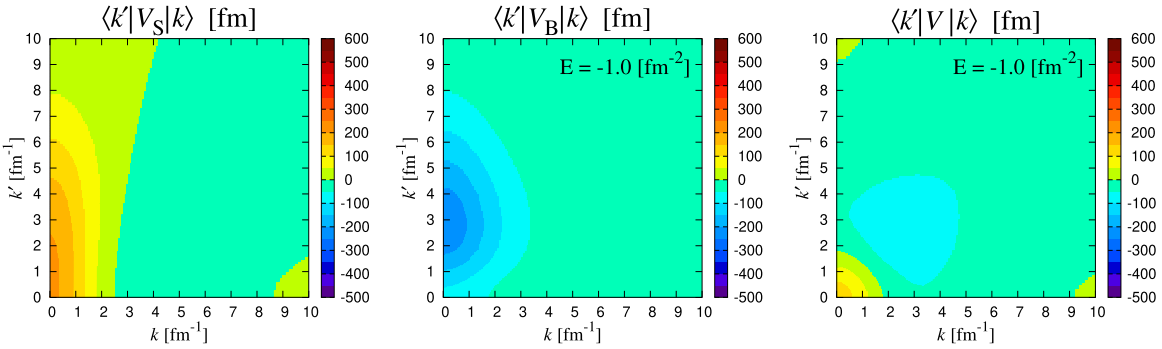

Fig. 5. SB-decomposition $V_{\rm S}+V_{\rm B}=V$. The eigenenergy of $|\phi\rangle$ is $E=-1.0\,{\rm fm}^{-2}$ $(=-41.47\,{\rm MeV})$. Other notation is the same as for Fig. 4. See the text.

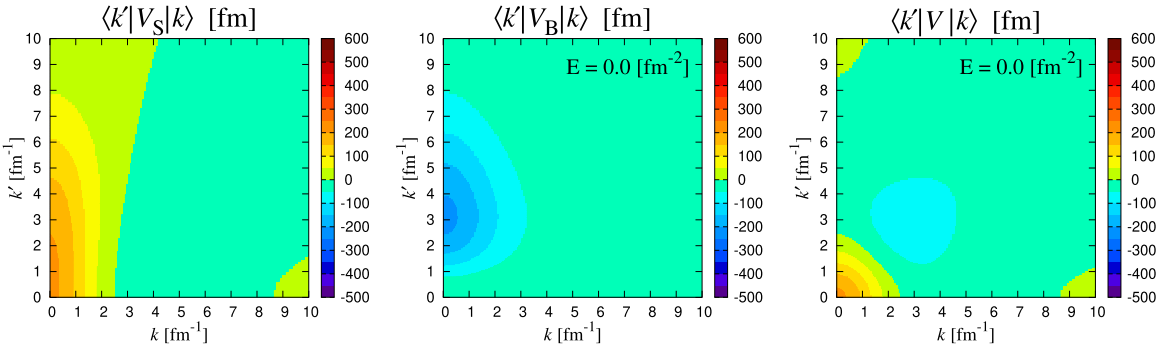

Fig. 6. SB-decomposition $V_{\rm S}+V_{\rm B}=V$. The eigenenergy of $|\phi\rangle$ is $E=0.0\,{\rm fm}^{-2}$ $(=0.0\,{\rm MeV})$, i.e., $|\phi\rangle$ is a zero-energy bound state. Other notation is the same as for Fig. 4. See the text.

positive energies $E=1.0\,{\rm fm}^{-2}$ and $4.0\,{\rm fm}^{-2}$, respectively.[15] In the following, we make several points using Figs. 3–8.

First, in going from Fig. 3 to Fig. 8, we have only varied $\langle k'|V_{\rm B}|k\rangle$ keeping $\langle k'|V_{\rm S}|k\rangle=\langle k'|V_{0,{\rm S}}|k\rangle$ untouched, i.e., we have used the same $\langle k'|T_0|k\rangle$ in Eq. (3) for every $\langle k'|V_{\rm S}|k\rangle$

[15] $\langle k'|V|k\rangle$ in Fig. 6 describes the bound state $|\phi\rangle$ at zero energy. This might sound strange because it is well known that a local potential does not support a zero-energy bound state in the S-wave channel [10,11]. Note, however, that the argument does not apply to nonlocal potentials [12,13], and that $\langle k'|V|k\rangle$ in Fig. 6 is generally nonlocal, as shall soon be shown by Eq. (25).

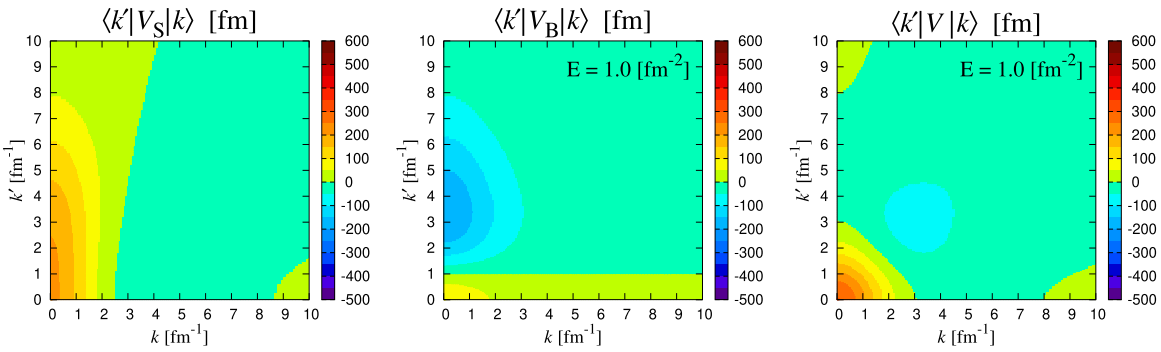

Fig. 7. SB-decomposition $V_S + V_B = V$. The (positive) eigenenergy of $|\phi\rangle$ is $E = K^2 = 1.0\,\mathrm{fm}^{-2}$ (= 41.47 MeV). Other notation is the same as for Fig. 4. See the text.

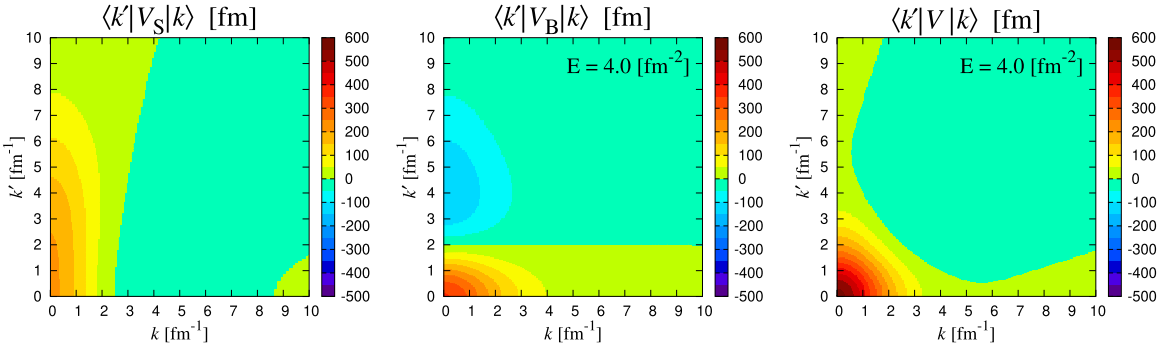

Fig. 8. SB-decomposition $V_S + V_B = V$. The (positive) eigenenergy of $|\phi\rangle$ is $E = K^2 = 4.0\,\mathrm{fm}^{-2}$ (= 165.9 MeV). Other notation is the same as for Fig. 4. See the text.

in Figs. 3–8 as explained in step (ii) in Section 3.2. Conversely, therefore, each $\langle k'|V|k\rangle = \langle k'|V_S|k\rangle + \langle k'|V_B|k\rangle$ in Figs. 3–8 yields the same half-on-shell T-matrix $\langle k'|T|k\rangle = \langle k'|T_0|k\rangle$, and therefore the same phase shift $\delta(k)$. Note also that $\langle k'|V|k\rangle = \langle k'|V_S|k\rangle + \langle k'|V_B|k\rangle$ in each figure is obviously Hermitian as pointed out in Section 2.

Second, as E increases from negative to positive values, it is visible that the strong attraction of $\langle k'|V_B|k\rangle$ around the origin ($k = k' = 0$) in Figs. 3 and 4 gradually turns into the strong repulsion in Fig. 8. We can explain the above observation simply by noting that $\langle k'|V_B|k\rangle$ of Eq. (20) can be approximated by $E\langle 0|\phi\rangle\langle\phi|0\rangle$ around the origin. This in turn means that a positive $\langle 0|V_B|0\rangle$ is a clear indication of a BIC.[16] Because $\langle k'|V_S|k\rangle$ is the same in Figs. 3–8, the above behavior of $\langle k'|V_B|k\rangle$ also explains how $\langle k'|V|k\rangle$ around the origin changes with increasing E.

Third, in order to examine in detail how the k- and k'-dependences of $\langle k'|V_B|k\rangle$ change with increasing E, let us look into Eq. (20). As in the discussion for $\langle k'|V_{0,B}|k\rangle$ in Section 4.2, we can see that the k-dependence of $\langle k'|V_B|k\rangle$ is given by $\phi(k)$ in Fig. 1, which is independent of E and is the same in Figs. 3–8. On the other hand, its k'-dependence is given by $(E - k'^2)\phi(k')$ and varies with E. For a negative energy $E = -\gamma^2$, the factor $E - k'^2 = -\gamma^2 - k'^2$ is negative for any k'. On the other hand, for a positive energy $E = K^2$, the factor $E - k'^2 = K^2 - k'^2$ changes

[16] See also the discussion in Section 9.1.

its sign at $k' = K$, Therefore, because $\phi(k)$ in Fig. 1 is nodeless, $\langle k'|V_{\rm B}|k\rangle$ of Eq. (20) changes its sign only once at $k' = K$ as a function of k'; it is visible that $\langle k'|V_{\rm B}|k\rangle$ changes its sign at $k' = 1.0\,\mathrm{fm}^{-1}$ for each k in Fig. 7, and at $k' = 2.0\,\mathrm{fm}^{-1}$ in Fig. 8. We stress that the above sign change of $\langle k'|V_{\rm B}|k\rangle$ is a clear criterion of a BIC.

Fourth, let us compare the potentials in Figs. 5 and 7, which describe the same state $|\phi\rangle$ as a negative-energy eigenstate ($E = -1.0\,\mathrm{fm}^{-2}$) and a BIC ($E = 1.0\,\mathrm{fm}^{-2}$), respectively. If we look into $\langle k'|V|k\rangle$'s only, we cannot realize that one of them supports a BIC while the other does not. Only by the investigation of $\langle k'|V_{\rm B}|k\rangle$ as in the above do we understand whether a given potential supports a BIC or not.[17]

To summarize, by using the simple method in Section 3.2, we have explicitly constructed potentials that support a BIC, which are shown in Figs. 7 and 8. Further, we have explained major features of these potentials using the SB-decomposition in Section 2, and have shown a criterion to judge whether a given potential has a BIC or not.

4.5. *Numerical results for $V_{\rm S} + V_{\rm B} = V$ in coordinate space*

In order to obtain a better insight into BIC-supporting potentials, we now cast the results in Section 4.4 into coordinate space.

First, let us note that the Fourier transform of $\langle k'|V_0|k\rangle$ of Eq. (19) is given by

$$\begin{aligned}\langle r'|V_0|r\rangle &\equiv \int\frac{k'^2\mathrm{d}k'}{(2\pi)^3}\,4\pi\,j_0(k'r')\int\frac{k^2\mathrm{d}k}{(2\pi)^3}\,4\pi\,j_0(kr)\,\langle k'|V_0|k\rangle\\ &= \frac{\delta(r'-r)}{r'r}V_0(r),\end{aligned}\tag{22}$$

where $V_0(r)$ is the Gaussian potential of Eq. (18). Second, we can easily Fourier transform numerically $\langle k'|V_{\rm S}|k\rangle = \langle k'|V_{0,\rm S}|k\rangle$ in Eq. (10) and $\langle k'|V_{\rm B}|k\rangle$ in Eq. (11) to obtain $\langle r'|V_{\rm S}|r\rangle$ and $\langle r'|V_{\rm B}|r\rangle$, which now read[18]

$$\begin{aligned}\langle r'|V_{\rm S}|r\rangle &= \langle r'|V_{0,\rm S}|r\rangle \;=\; \langle r'|V_0|r\rangle - \langle r'|V_{0,\rm B}|r\rangle\\ &= \frac{\delta(r'-r)}{r'r}V_0(r) \;-\; \int\frac{k'^2\mathrm{d}k'}{(2\pi)^3}\,4\pi\,j_0(k'r')\int\frac{k^2\mathrm{d}k}{(2\pi)^3}\,4\pi\,j_0(kr)\,\left(-\gamma_0^2 - k'^2\right)\phi(k')\phi(k)\\ &= \frac{\delta(r'-r)}{r'r}V_0(r) \;-\; (-\gamma_0^2 + \nabla'^2)\phi(r')\phi(r).\end{aligned}\tag{23}$$

$$\begin{aligned}\langle r'|V_{\rm B}|r\rangle &= \int\frac{k'^2\mathrm{d}k'}{(2\pi)^3}\,4\pi\,j_0(k'r')\int\frac{k^2\mathrm{d}k}{(2\pi)^3}\,4\pi\,j_0(kr)\,(E - k'^2)\phi(k')\phi(k)\\ &= (E + \nabla'^2)\phi(r')\phi(r).\end{aligned}\tag{24}$$

[17] In the search for a BIC, we only have to look for a sign change of $\langle k'|V_{\rm B}|k\rangle$ as a function of k'. We observe the sign change in Fig. 7 at $k' = 1.0\,\mathrm{fm}^{-1}$, while we do not in Fig. 5. We can conclude, therefore, that $\langle k'|V|k\rangle$ in Fig. 7 supports a BIC at $E = k'^2 = 1.0\,\mathrm{fm}^{-2}$, while $\langle k'|V|k\rangle$ in Fig. 5 does not. In the above discussion, we have assumed a system with a single bound state. For a general case with several bound states, see Section 9.2.

[18] Here, we have replaced K^2 with E as in Eq. (20).

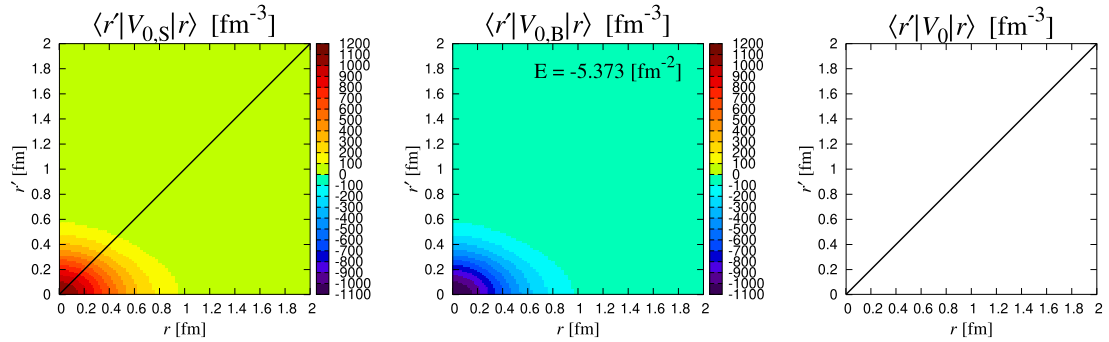

Fig. 9. SB-decomposition $V_{0,\mathrm{S}} + V_{0,\mathrm{B}} = V_0$ in the first line of Eq. (25) with the eigenenergy of $\phi(r)$ being $E = -\gamma_0^2 = -5.373\,\mathrm{fm}^{-2}$ $(= -222.8\,\mathrm{MeV})$, which corresponds to Fig. 3 in momentum space. Left: $\langle r'|V_{\mathrm{S}}|r\rangle = \langle r'|V_{0,\mathrm{S}}|r\rangle$. Middle: $\langle r'|V_{0,\mathrm{B}}|r\rangle$. Right: $\langle r'|V_0|r\rangle = \dfrac{\delta(r'-r)}{r'r}V_0(r)$, which is shown by a thick black line along $r' = r$. Note the same thick black line in the left panel. See the text.

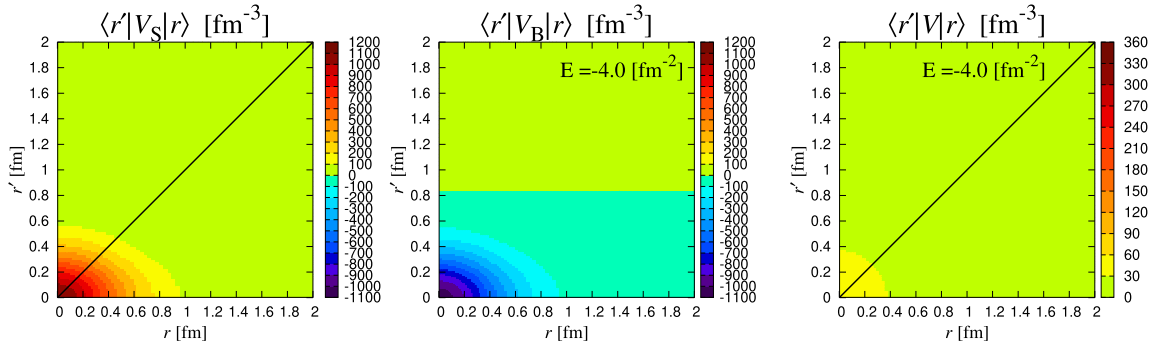

Fig. 10. SB-decomposition $V_{\mathrm{S}} + V_{\mathrm{B}} = V$ in coordinate space that corresponds to Fig. 4. The eigenenergy of $|\phi\rangle$ is $E = -1.0\,\mathrm{fm}^{-2}$ $(= -165.9\,\mathrm{MeV})$. The diagonal solid lines along $r' = r$ in the left and right panels represent the original local potential $\langle r'|V_0|r\rangle = \dfrac{\delta(r'-r)}{r'r}V_0(r)$ in Fig. 9. Other notation is the same as for Fig. 9. See the text.

Here, we have used the notation $\nabla'^2 = \dfrac{\mathrm{d}^2}{\mathrm{d}r'^2} + \dfrac{2}{r'}\dfrac{\mathrm{d}}{\mathrm{d}r'}$. Then, by adding Eqs. (23) and (24), we can express the Fourier transform of Eq. (21) as

$$\begin{aligned}\langle r'|V_{\mathrm{S}}|r\rangle + \langle r'|V_{\mathrm{B}}|r\rangle &= \langle r'|V|r\rangle \\ &= \langle r'|V_0|r\rangle - \langle r'|V_{0,\mathrm{B}}|r\rangle + \langle r'|V_{\mathrm{B}}|r\rangle = \langle r'|V_0|r\rangle + \langle r'|\Delta V_{\mathrm{B}}|r\rangle \\ &= \frac{\delta(r'-r)}{r'r}V_0(r) + \left(E + \gamma_0^2\right)\phi(r')\phi(r). \qquad (25)\end{aligned}$$

The first line of Eq. (25), with $\langle r'|V_{\mathrm{S}}|r\rangle$ and $\langle r'|V_{\mathrm{B}}|r\rangle$ being given by Eqs. (23) and (24) respectively, represents the SB-decomposition of $\langle r'|V|r\rangle$ that supports the bound state $\phi(r)$ at energy E. The third line represents the Fourier transform of Eq. (13) for $\langle k'|V|k\rangle$, and gives the energy-shift operator ΔV_{B} of Eq. (12) in coordinate space:

$$\langle r'|\Delta V_{\mathrm{B}}|r\rangle = \langle r'|V_{\mathrm{B}} - V_{0,\mathrm{B}}|r\rangle = \left(E + \gamma_0^2\right)\phi(r')\phi(r). \qquad (26)$$

It is notable here that $\langle r'|V_{\mathrm{B}}|r\rangle$ of Eq. (24) is purely nonlocal and separable, and so is $\langle r'|\Delta V_{\mathrm{B}}|r\rangle$ of Eq. (26).

In Figs. 9–14, we present the first line of Eq. (25), the SB-decomposition of $\langle r'|V|r\rangle$, corresponding to Figs. 3–8 in momentum space.

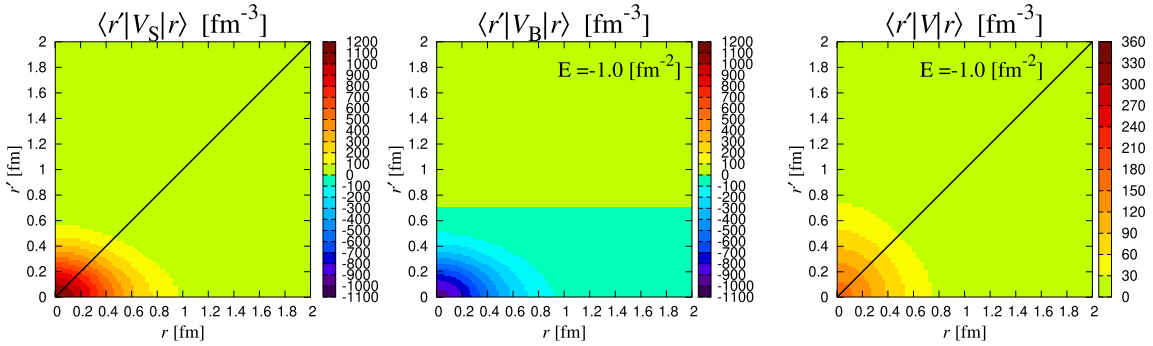

Fig. 11. SB-decomposition $V_S + V_B = V$ in coordinate space that corresponds to Fig. 5. The eigenenergy of $|\phi\rangle$ is $E = -1.0\,\mathrm{fm}^{-2}$ ($= -41.47\,\mathrm{MeV}$). Other notation is the same as for Fig. 9. See the text.

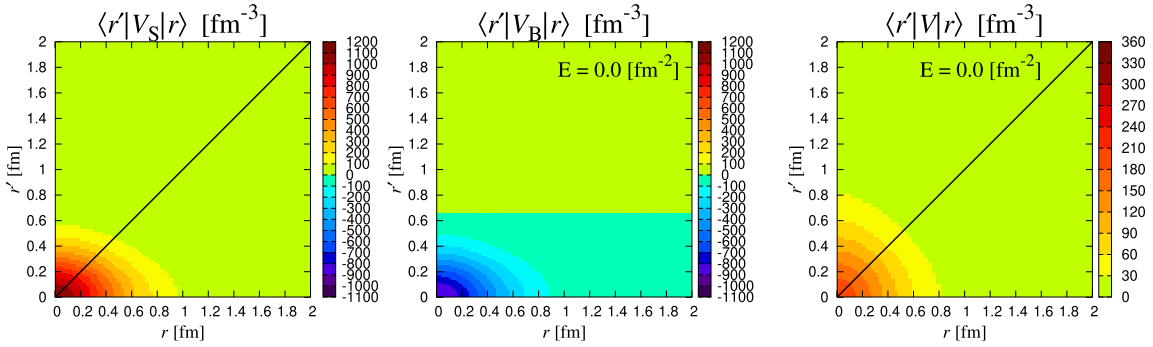

Fig. 12. SB-decomposition $V_S + V_B = V$ in coordinate space that corresponds to Fig. 6. The eigenenergy of $|\phi\rangle$ is $E = 0.0\,\mathrm{fm}^{-2}$ ($= 0.0\,\mathrm{MeV}$). Other notation is the same as for Fig. 9. See the text.

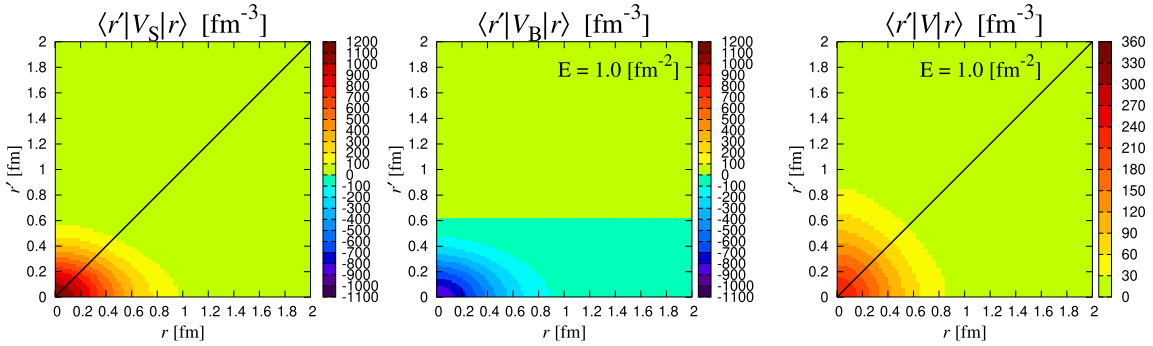

Fig. 13. SB-decomposition $V_S + V_B = V$ in coordinate space that corresponds to Fig. 7. The eigenenergy of $|\phi\rangle$ is $E = K^2 = 1.0\,\mathrm{fm}^{-2}$ ($= 41.47\,\mathrm{MeV}$). Other notation is the same as for Fig. 9. See the text.

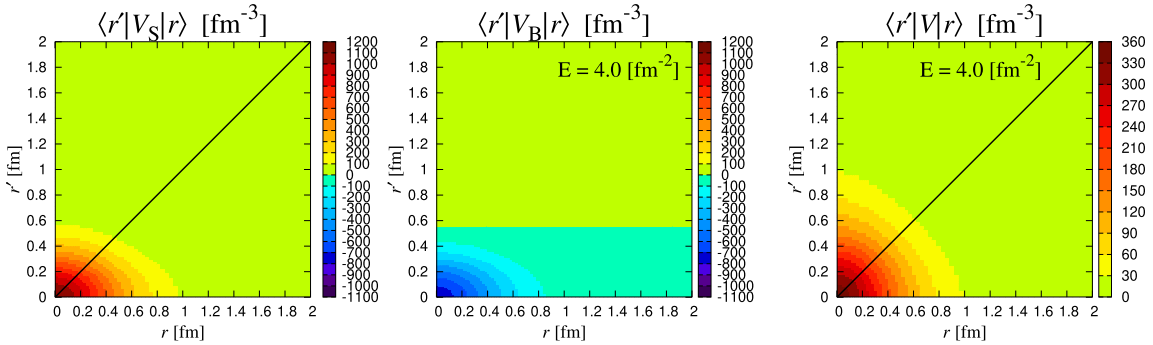

Fig. 14. SB-decomposition $V_S + V_B = V$ in coordinate space that corresponds to Fig. 8. The eigenenergy of $|\phi\rangle$ is $E = K^2 = 4.0\,\mathrm{fm}^{-2}$ ($= -165.9\,\mathrm{MeV}$). Other notation is the same as for Fig. 9. See the text.

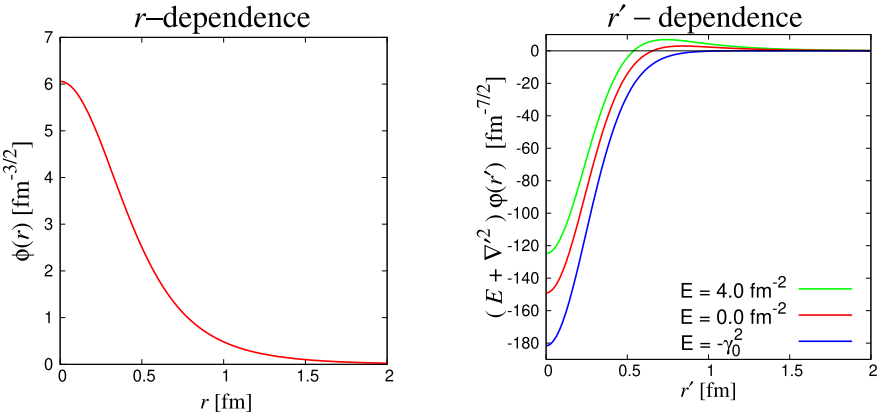

Fig. 15. r- and r'-dependences of $\langle r'|V_S|r\rangle$ and $\langle r'|V_B|r\rangle$. Left: $\phi(r)$ representing the r-dependence of both $\langle r'|V_S|r\rangle$ for $r' \neq r$ and $\langle r'|V_B|r\rangle$. Right: $(E+\nabla'^2)\phi(r')$ representing the r'-dependence of $\langle r'|V_B|r\rangle$ for several values of $E(4.0\,\mathrm{fm}^{-2},\ 0.0\,\mathrm{fm}^{-2},\ -\gamma_0^2 = -5.373\,\mathrm{fm}^{-2})$. The curve $E = -\gamma_0^2$ gives (the minus of) the r'-dependence of $\langle r'|V_S|r\rangle = \langle r'|V_{0,S}|r\rangle$. The curve for $E = 0$ shows $\nabla'^2\phi(r')$. See the text.

First of all, the transformation $V_0 \to V = V_0 + \Delta V_B$ in the last line of Eq. (25) shows that the right panels in Figs. 10–14 represent, apart from the diagonal thick black lines representing $\langle r'|V_0|r\rangle = \dfrac{\delta(r'-r)}{r'r}V_0(r)$, the energy-shift operator $\langle r'|\Delta V_B|r\rangle$ of Eq. (26). For $E = -\gamma_0^2$, $\langle r'|\Delta V_B|r\rangle$ vanishes by definition, and V of Eq. (25) reduces to V_0, to give

$$\langle r'|V_{0,S}|r\rangle + \langle r'|V_{0,B}|r\rangle = \frac{\delta(r'-r)}{r'r}V_0(r), \tag{27}$$

showing that $\langle r'|V_{0,S}|r\rangle$ and $\langle r'|V_{0,B}|r\rangle$ cancel each other exactly for $r' \neq r$, as can be confirmed in Fig. 9.

Now, we examine the r- and r'-dependences of $\langle r'|V_S|r\rangle$ and $\langle r'|V_B|r\rangle$ in these figures. For this purpose, we plot $\phi(r)$ and $(E+\nabla'^2)\phi(r')$ respectively in the left and right panels of Fig. 15.[19] We understand the curves in the right panel in the following manner. First, let us note that $\nabla'^2\phi(r')$, which is given by the curve $E = 0$, has a shorter range than $\phi(r)$ in the left panel, and has a node at $r' = 0.67$ fm.[20] The node appears for $E > -\gamma_0^2$, and moves inward with increasing E.[21]

Let us start with $\langle r'|V_S|r\rangle$, which is independent of E and therefore is common in Figs. 9–14. For $r' \neq r$, $\langle r'|V_S|r\rangle$ reduces to $-\langle r'|V_{0,B}|r\rangle = (\gamma_0^2 - \nabla'^2)\phi(r')\phi(r)$, which is the second term of the right-hand side of Eq. (23). Its r- and r'-dependences are respectively given by $\phi(r)$ in the left panel and by $(-\gamma_0^2 + \nabla'^2)\phi(r')$ in the right panel of Fig. 15. We see that $\langle r'|V_S|r\rangle$ for $r' \neq r$ has no node and that its extension is longer in the r-direction than in the r'-direction, which is explained by the above observations on Fig. 15.

[19] $\phi(r)$ has already been shown in Fig. 1. For convenience, however, we plot it here again.

[20] Observe that $\nabla'^2\phi(r')$ is the Fourier transform of $k'^2\phi(k')$ in Eq. (23) that has a longer rage in momentum space than $\phi(k')$. Therefore, $\nabla'^2\phi(r')$ has a shorter range than $\phi(r')$ in coordinate space. Note also that $\nabla'^2\phi(r')$, mostly the second derivative of $\phi(r')$ in the left panel, naturally has a node.

[21] $\phi(r')$ decays as $\exp(-\gamma_0 r')$ for $r' \to \infty$, and so does $\nabla'^2\phi(r')$. Further, the Schrödinger equation for $\phi(r')$ shows that $(-\gamma_0^2 + \nabla'^2)\phi(r') = V_0(r')\phi(r')$ decays faster than $\exp(-\gamma_0 r')$, showing that leading (exponentially decaying) parts of $-\gamma_0^2\phi(r')$ and $\nabla'^2\phi(r')$ cancel each other exactly for $r' \to \infty$. Therefore, $\phi(r)$ in the left panel and $\nabla'^2\phi(r')$ in the right panel (indicated by $E = 0$) explain that $(E+\nabla'^2)\phi(r')$ shows up with a node for $E > -\gamma_0^2$ while it is nodeless for $E < -\gamma_0^2$.

Now we turn to $\langle r'|V_{\rm B}|r\rangle$ of Eq. (24). Its r-dependence is independent of E, and is given simply by $\phi(r)$ in the same way as $\langle r'|V_{\rm S}|r\rangle$ for $r' \neq r$. Its r'-dependence, on the other hand, is dependent on E, and is given by $(E+\nabla'^2)\phi(r')$ in the right panel of Fig. 15; for $E > -\gamma_0^2$, it has a node that moves inward as E increases, i.e., the node is located at $r' \sim 0.67$ fm for $E = 0.0\,\text{fm}^{-2}$, and $r' \sim 0.54$ fm for $E = 4.0\,\text{fm}^{-2}$. We can thus explain the r'-dependence of $\langle r'|V_{\rm B}|r\rangle$ in Figs. 9–14; $\langle r'|V_{\rm B}|r\rangle$ has a node in the r'-direction for $E > -\gamma_0^2$, which is visible at $r' \sim 0.67$ fm in Fig. 10 for $E = 0.0\,\text{fm}^{-2}$, and goes downward as E increases to $r' \sim 0.54$ fm in Fig. 14 for $E = -4.0\,\text{fm}^{-2}$.

To summarize, we have understood the main behaviors of $\langle r'|V_{\rm S}|r\rangle$ and $\langle r'|V_{\rm B}|r\rangle$, and therefore of $\langle r'|V|r\rangle$ in Figs. 9–14. Note that the present results imply the importance of the nonlocal potential in the physics of BIC; in order to realize a BIC in a given system with a negative-energy bound state, it is crucial to introduce the nonlocal perturbation $\langle r'|\Delta V_{\rm B}|r\rangle$ to the system. We shall soon find a rigorous proof of this observation in Section 8.

5. Numerical example II: General method

In this section, we briefly demonstrate that steps (i)–(iii) of the general method in Section 3.1 are really implementable.

In step (i) of Section 3.1, we arbitrarily adopt the bound state $|\phi\rangle$ in Fig. 1 for simplicity.[22]

In step (ii), with $P_{\rm B} = |\phi\rangle\langle\phi|$, we solve the coupled set of Eqs. (A) and (B) for T. Because we find infinitely many solutions, we arbitrarily choose $\langle k'|T|k\rangle$ in the left panel of Fig. 16, which in turn gives the phase shift $\delta(k)$ in the right panel. Note that they are different from those in Fig. 2. Therefore, the corresponding $\langle k'|V_{\rm S}|k\rangle$ calculated via Eq. (3), which is shown in the left panels of Figs. 17–21, is different from $\langle k'|V_{0,\rm S}|k\rangle$ in Figs. 4–8.

In step (iii), we make $\langle k'|V_{\rm B}|k\rangle$ in the same way as in Section 4; we consider five different eigenenergies of $|\phi\rangle$, $E = -4.0\,\text{fm}^{-2}$, $-1.0\,\text{fm}^{-2}$, $0.0\,\text{fm}^{-2}$, $1.0\,\text{fm}^{-2}$, and $4.0\,\text{fm}^{-2}$, each of which gives $\langle k'|V_{\rm B}|k\rangle$ in Figs. 4–8 in order.

Finally, by adding $\langle k'|V_{\rm S}|k\rangle$ to each $\langle k'|V_{\rm B}|k\rangle$ in Figs. 4–8, we obtain its corresponding $\langle k'|V_{\rm S}|k\rangle + \langle k'|V_{\rm B}|k\rangle = \langle k'|V|k\rangle$ in Figs. 17–21.

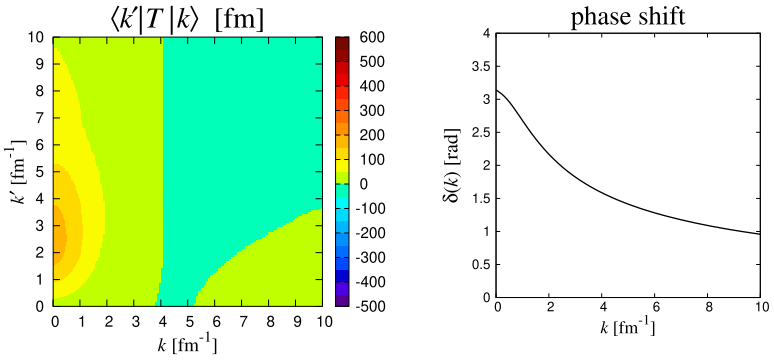

Fig. 16. Arbitrarily chosen solution $\langle k'|T|k\rangle$ to Eqs. (A) and (B) (contour map of the real part, left panel) and its corresponding phase shift $\delta(k)$ (right panel). Notation is the same as for Fig. 2. See the text.

[22] Of course, we could have chosen a different bound state. For the sake of comparison with the results in Section 4, however, we have chosen the same bound state $|\phi\rangle$ as in Section 4. We stress that we do not need knowledge of the Gaussian potential of Eq. (18) in carrying out the general method here, while we used it in implementing the simple method in Section 4.

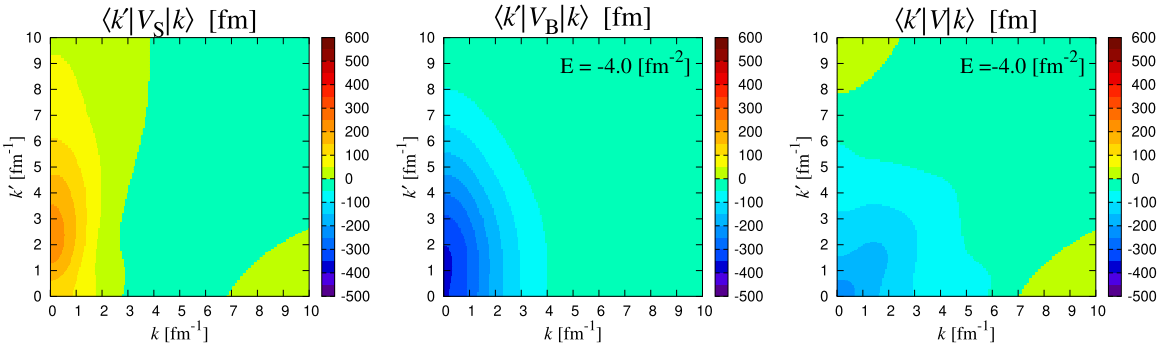

Fig. 17. SB-decomposition $V_{\rm S} + V_{\rm B} = V$ of V. $\langle k'|V_{\rm S}|k\rangle$ is calculated via Eq. (3) using $\langle k'|T|k\rangle$ in Fig. 16. $\langle k'|V_{\rm B}|k\rangle$ is taken from Fig. 4 corresponding to $E = -4.0\,\mathrm{fm}^{-2}$. Other notation is the same as for Fig. 4. See the text.

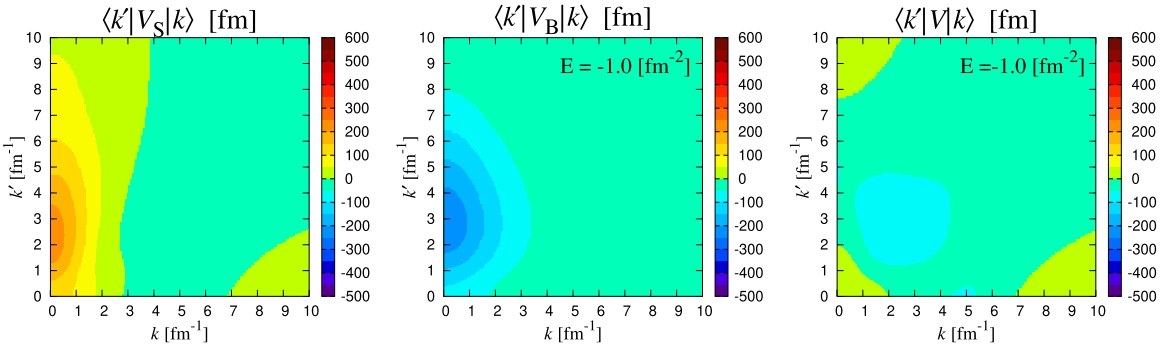

Fig. 18. $\langle k'|V_{\rm B}|k\rangle$ is taken from Fig. 5 corresponding to $E = -1.0\,\mathrm{fm}^{-2}$. Other notation is the same as for Fig. 17. See the text.

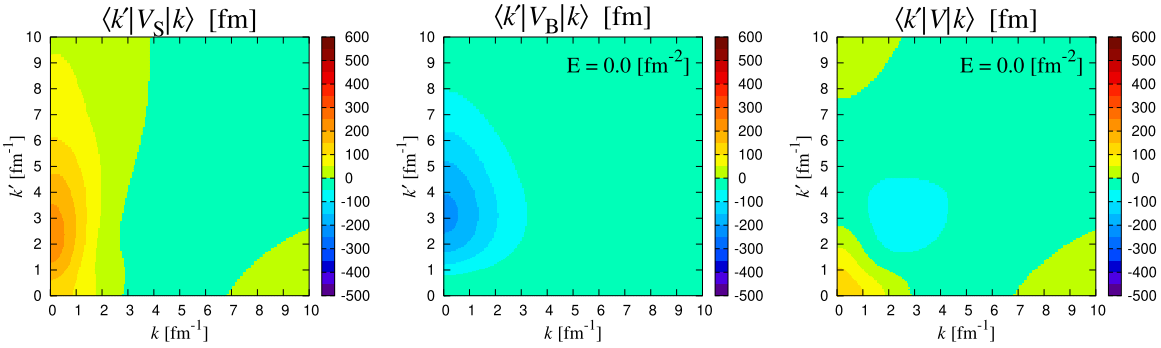

Fig. 19. $\langle k'|V_{\rm B}|k\rangle$ is taken from Fig. 6 corresponding to $E = 0.0\,\mathrm{fm}^{-2}$. Other notation is the same as for Fig. 17. See the text.

Let us note the following: the $\langle k'|V|k\rangle$'s in Figs. 8 and 21, for instance, share the same BIC $|\phi\rangle$ at the same energy ($E = 4.0\,\mathrm{fm}^{-2}$) because they share the same $\langle k'|V_{\rm B}|k\rangle$. On the other hand, they give different $\langle k'|T|k\rangle$'s because their $\langle k'|V_{\rm S}|k\rangle$'s are different. The same argument applies to each pair of Figs. 4 and 17, Figs. 5 and 18, Figs. 6 and 19, and Figs. 7 and 20. In other words, for each $\langle k'|T|k\rangle$ that is compatible with the given bound state $|\phi\rangle$,[23] we can construct a

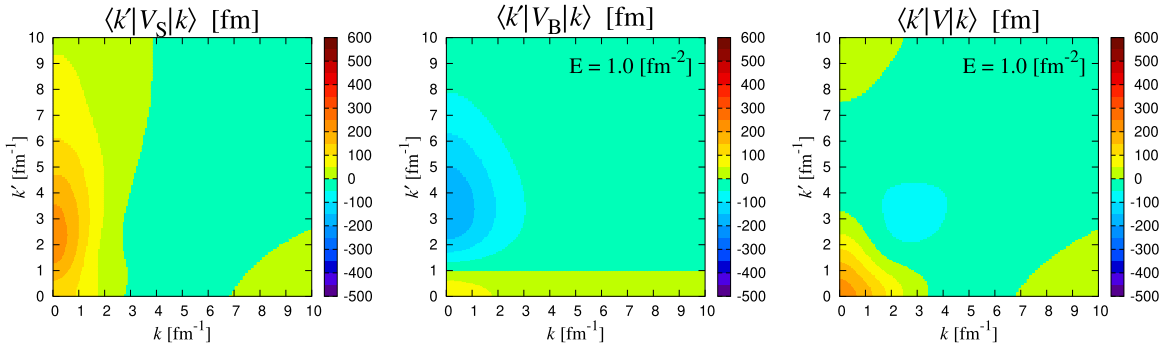

Fig. 20. $\langle k'|V_{\rm B}|k\rangle$ is taken from Fig. 7 corresponding to $E = 1.0\,\mathrm{fm}^{-2}$. Other notation is the same as for Fig. 17. See the text.

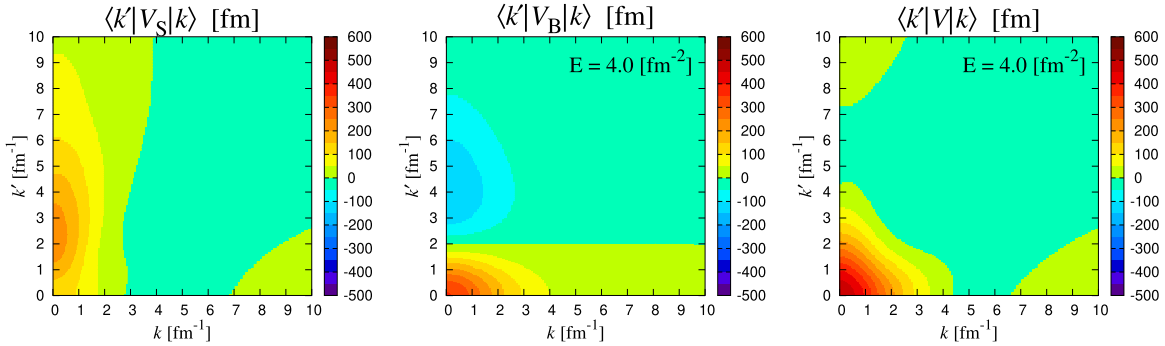

Fig. 21. $\langle k'|V_{\rm B}|k\rangle$ is taken from Fig. 8 corresponding to $E = 4.0\,\mathrm{fm}^{-2}$. Other notation is the same as for Fig. 17. See the text.

Hermitian potential that supports $|\phi\rangle$ at any (positive or negative) eigenenergy as we like. The above observation suggests that BICs are as common as negative-energy bound states if we look into all Hermitian potentials.[24]

6. Discussion on local potentials

In this and following sections, we give some discussions. First, in this section, we examine local potentials in connection with the BIC. It is known that a local potential $V(r)$, which is continuous except at a finite number of finite discontinuities, gives the usual scattering theory if it satisfies [10,11][25]

$$\text{I.}\quad V(r) = O(r^{-3-\epsilon}),\quad \text{as } r \to \infty \ \ (\text{some } \epsilon > 0). \tag{28}$$

$$\text{II.}\quad V(r) = O(r^{-2+\epsilon}),\quad \text{as } r \to 0 \ \ (\text{some } \epsilon > 0). \tag{29}$$

In Section 6.1, we shall see that if a local potential $V(r)$ supports a BIC, it cannot satisfy condition I, and the BIC wave function $\phi(r)$ must have an oscillatory structure. Then in Section 6.2, we show such an example; we explain the only local potential that we know that supports a BIC, i.e., the von Neumann and Wigner potential.

[23] This means that $\langle k'|T|k\rangle$ is a solution to the coupled set of Eqs. (A) and (B) that are fixed by the given bound state $|\phi\rangle$ only, irrespective of its eigenenergy E being negative or positive.

[24] We shall soon give a rigorous proof of this statement in Section 7.1.

[25] The notation $V(r) = O(r^p)$ means that $|V(r)| \leqq c\,|r|^p$ for some constant c.

6.1. *Rarity of BIC for local potentials*

Here we show the following two points.

(i) If a local potential $V(r)$ supports a BIC $\phi(r)$, $V(r)$ does not satisfy condition I of Eq. (28), and $\phi(r)$ is an oscillating function with an infinite number of zeros. Accordingly, $V(r)$ is also an oscillating function.

(ii) If a BIC wave function $\phi(r)$ has at most a finite number of zeros, the potential V of the system cannot be local but must be nonlocal, i.e., $\langle r'|V|r\rangle$ cannot be proportional to $\delta(r'-r)$.

Let us start with point (i). First, we examine a case where a local potential $V(r)$ satisfying conditions I and II supports a BIC wave function $\phi(r)$ at a positive energy $E=k^2$ via

$$(-\nabla^2+V(r))\phi(r)=k^2\phi(r). \tag{30}$$

Now we make an assumption that $\phi(r)$ has only a finite number of zeros. Then, because of condition I for $V(r)$, we can choose R in such a way that $\phi(r)$ has no zeros for $r>R$ and $V(r)\phi(r)$ can be neglected there compared to $k^2\phi(r)$ with any desired accuracy. This means that the solution $\phi(r)$ of Eq. (30) for $r>R$ can be approximated by $j_0(kr+\alpha)$, with α being an appropriate phase. However, because $j_0(kr+\alpha)$ has an infinite number of zeros in the range $r>R$, the above solution $j_0(kr+\alpha)$ contradicts our choice of R. By reduction to absurdity, therefore, we can conclude that our assumption is false, i.e., $\phi(r)$ is an oscillating function with an infinite number of zeros, leading to the second half of point (i).

Second, taking into account the above observation, let us express $\phi(r)$ of Eq. (30) as $\phi(r)=j_0(kr+\alpha)f(r)$ with $f(r)$ being an unknown envelope function that guarantees the normalizability of $\phi(r)$. Then, it is easy to see[26]

$$-\nabla^2\phi(r)=k^2\phi(r)-2\frac{\mathrm{d}j_0(kr+\alpha)}{\mathrm{d}r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}-j_0(kr+\alpha)\frac{\mathrm{d}^2f(r)}{\mathrm{d}r^2}-j_0(kr+\alpha)\frac{2}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}. \tag{31}$$

Now we suppose that $f(r)=O(r^{-n})$ with some $n\geqq 1$ for $r\to\infty$.[27] This means that the leading contribution on the right-hand side as $r\to\infty$ comes from the second term:

$$-2\frac{\mathrm{d}j_0(kr+\alpha)}{\mathrm{d}r}\frac{\mathrm{d}f(r)}{\mathrm{d}r}\;=\;O(r^{-n-1}),\quad \text{as } r\to\infty. \tag{32}$$

Then, by comparing Eqs. (30) and (31), we realize that $V(r)=O(r^{-1})$ as $r\to\infty$. The above observation shows the first half, to complete a proof of point (i).

It is now easy to show point (ii) as a contraposition to point (i). We can thus prove both points (i) and (ii). Note that point (i) suggests that a BIC can rarely occur in a local potential.[28]

6.2. *Existing theory: von Neumann and Wigner potential*

In Ref. [1], von Neumann and Wigner first showed that a BIC is possible by deriving a local potential $V(r)$ that conforms to point (i) in Section 6.1. As briefly explained in Appendix A.1, their potential $V(r)$ in the S-wave channel describes $\phi(r)=j_0(kr)f(r)$, the spherical Bessel function

[26] Note that $\nabla^2=\dfrac{\mathrm{d}^2}{\mathrm{d}r^2}+\dfrac{2}{r}\dfrac{\mathrm{d}}{\mathrm{d}r}$.

[27] It can be easily shown [2] that an exponentially decaying $f(r)$ is not consistent with the Schrödinger equation (30).

[28] This point shall be proven later in a definite manner.

$j_0(kr)$ in an envelope $f(r)$, as a BIC at a positive energy $E = k^2$. Comparing our theory to theirs, we realize the following.

- The wave function of their BIC is tailored from the oscillating plane wave state $j_0(kr)$. Correspondingly, their potential also has an oscillatory structure,

$$V(r) \propto \frac{\sin 2kr}{2kr}, \quad \text{as } r \to \infty, \tag{33}$$

  which does not satisfy condition I of Eq. (28) in accordance with point (i) in Section 6.1, leading to a destructive interference that allows for the existence of the BIC. In our theory, on the other hand, we can choose any normalizable wave function for the BIC. Correspondingly, our BIC-supporting potential is not necessarily of an oscillatory structure, which can be confirmed clearly by Eq. (25).
- The eigenenergy of their BIC, $\phi(r) = j_0(kr)f(r)$, is fixed at $E = k^2$. In our theory, on the other hand, we can assign any eigenenergy E to our bound state.
- Except for the BIC $\phi(r) = j_0(kr)f(r)$ at $E = k^2$, their theory does not give any information on scattering states and other possible bound states.[29] On the other hand, as explained in Section 2, our theory keeps all eigenstates of the system under control by construction.

From the above observation, we realize that the local potential of von Neumann and Wigner is a very special case that cannot be generalized to other BICs. Further, because their BIC is vulnerable to any local perturbation [3,4], we conclude that their BIC is an exceptional phenomenon in the group of all local potentials, which might accord with our intuition. In the following, however, we shall see that the present theory predicts the opposite if we survey the group of all Hermitian potentials.

7. Discussion on general nonlocal potentials

In this section, we discuss various aspects of the present theory that treats general nonlocal potentials.

7.1. *Generality of BIC*

In Section 4.3, we examined the transformation of $\langle k'|V_0|k\rangle = \langle k'|V_{0,\mathrm{S}}|k\rangle + \langle k'|V_{0,\mathrm{B}}|k\rangle$ into $\langle k'|V|k\rangle = \langle k'|V_{0,\mathrm{S}}|k\rangle + \langle k'|V_{\mathrm{B}}|k\rangle$, which is controlled by the single variable E assigned to the normalizable state $\phi(k)$. Depending on E being negative or positive, $\langle k'|V|k\rangle$ describes $\phi(k)$ as a negative- or positive-energy bound state. Therefore, starting from $\langle k'|V_0|k\rangle = \langle k'|V_{0,\mathrm{S}}|k\rangle + \langle k'|V_{0,\mathrm{B}}|k\rangle$ at hand, we can make as many potentials that present $\phi(k)$ as a BIC as those that describe $\phi(k)$ as a negative-energy bound state.

To be precise, let $\mathbb{V}(T_0, \phi)$ be a group of all Hermitian potentials that share the half-on-shell T-matrix $\langle k'|T_0|k\rangle$ and the single bound state $|\phi\rangle$. We distinguish each $V \in \mathbb{V}(T_0, \phi)$ by the eigenenergy E of $|\phi\rangle$.[30] Then, the above argument shows that half elements of $\mathbb{V}(T_0, \phi)$ support $|\phi\rangle$ at positive energies, and the other half at negative energies. Now, let $\mathbb{V}$ represent the group of all Hermitian potentials that have a single bound state. Obviously, $\mathbb{V}$ is a direct sum of all possible subgroups defined similarly to $\mathbb{V}(T_0, \phi)$ in terms of a half-on-shell T-matrix and a

[29] Note that, because their potential does not satisfy condition I of Eq. (28), we cannot even expect the usual description of scattering states.

[30] In the group $\mathbb{V}(T_0, \phi)$, $\langle k'|V_{\mathrm{S}}|k\rangle = \langle k'|V_{0,\mathrm{S}}|k\rangle$ is fixed, and $\langle k'|V_{\mathrm{B}}|k\rangle$ of Eq. (20) alone varies with E. Note that V_0 in Eq. (19) is an element of $\mathbb{V}(T_0, \phi)$ specified by $E = -\gamma_0^2 = -5.373\,\mathrm{fm}^{-2}$.

bound state wave function, i.e.,[31]

$$\begin{aligned}\mathbb{V} = {} & \mathbb{V}(T_0, \phi) \oplus \mathbb{V}(T_1, \phi) \oplus \mathbb{V}(T_2, \phi) \oplus \cdots \\ & \oplus \mathbb{V}\left(T_0', \phi'\right) \oplus \mathbb{V}\left(T_1', \phi'\right) \oplus \mathbb{V}\left(T_2', \phi'\right) \oplus \cdots \\ & \oplus \mathbb{V}\left(T_0'', \phi''\right) \oplus \mathbb{V}\left(T_1'', \phi''\right) \oplus \mathbb{V}\left(T_2'', \phi''\right) \oplus \cdots . \end{aligned} \tag{34}$$

In exactly the same way as $\mathbb{V}(T_0, \phi)$, each subgroup on the right-hand side of Eq. (34) contains as many potentials with a BIC as those without a BIC, so does $\mathbb{V}$ on the left-hand side. The above argument can be easily generalized to systems with two or more bound states, which immediately leads us to the following conclusion: if we examine all Hermitian potentials, most of which are nonlocal, we would find as many BICs as negative-energy bound states. Surprisingly enough, in the group of all possible Hermitian potentials, the BIC is not at all an exceptional phenomenon as predicted by local potentials in Section 6.

7.2. *Coexistence of BIC and scattering state*

Using the method in Section 3.2, let us prepare a potential $\langle k'|V|k\rangle$ that supports a BIC $|\phi\rangle$ at $E = K^2$. Then, the scattering state $|K\rangle_+$ and the BIC $|\phi\rangle$ are degenerate at $E = K^2$ in our system.[32] This means that the Schrödinger equation in coordinate space corresponding to Eq. (14), which is the following integro-differential equation for the S-wave channel,[33]

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\phi(r) + \int_0^\infty r'^2 \mathrm{d}r'\, V(r, r')\phi(r') = K^2\phi(r), \tag{35}$$

has two different solutions that are regular at the origin; one is the BIC $\phi(r)$, and the other is the scattering state $\psi_K^{(+)}(r) = \langle r|K\rangle_+$ with the outgoing boundary condition. This degeneracy implies the existence of a nontrivial solution $\chi(r)$ to Eq. (35) that satisfies $\chi(0) = \chi'(0) = 0$ where $\chi'(r) = \dfrac{\mathrm{d}}{\mathrm{d}r}\chi(r)$ [12–14].[34] Therefore, the conclusion in Section 7.1 means that, for an arbitrary $\phi(r)$ at an arbitrary positive energy $E = K^2$, there is a nonlocal potential $V(r, r')$ that allows a solution $\phi(r) = \chi(r)$ to Eq. (35) satisfying $\chi(0) = \chi'(0) = 0$.

In a system described by a local potential $V(r)$, however, the situation is different. Here, the Schrödinger equation in coordinate space is given by the second-order differential equation:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\phi(r) + V(r)\phi(r) = K^2\phi(r). \tag{36}$$

It is straightforward to show that, using an argument of the uniqueness of solution for an ordinary differential equation, Eq. (36) with the initial condition $\phi(0) = \phi'(0) = 0$ would generally yield the unique solution $\phi(r) \equiv 0$. In other words, a local potential $V(r)$ would not generally

[31] In Eq. (34), the half-on-shell T-matrices $\{T_0, T_1, T_2, \cdots\}$ are "compatible" with the bound state $|\phi\rangle$, i.e., they are solutions to the coupled set of Eqs. (A) and (B) for the given bound state $|\phi\rangle$, and $\{T_0', T_1', T_2', \cdots\}$ are "compatible" with another bound state $|\phi'\rangle$, and so on.

[32] The original potential V_0 describes the scattering state $|K\rangle_+$ at $E = K^2$ and the bound state $|\phi\rangle$ at $E = -\gamma_0^2$, which are of course mutually orthogonal. The transformed potential V describes the scattering state $|K\rangle_+$ in exactly the same way as V_0 by construction. At the same time, it supports $|\phi\rangle$ at $E = K^2$ as explained in Section 3.2. Therefore, $|K\rangle_+$ and $|\phi\rangle$ are eigenstates of $H_0 + V$ with the same eigenenergy, $E = K^2$.

[33] $\langle r|V|r'\rangle = V(r, r')$ is generally nonlocal. See Eq. (25).

[34] In fact, we can prepare such a solution $\chi(r)$ as an appropriate linear combination of $\phi(r)$ of the BIC and $\psi_K^{(+)}(r)$ for our BIC-supporting potential.

allow the degeneracy of the scattering state $\psi_K^{(+)}(r) = \langle r|K\rangle_+$ and the BIC $\phi(r)$ at $E = K^2$, which in turn means that the BIC is a rare phenomenon for local potentials.

Summarizing the above, nonlocal potentials easily allow the coexistence of the scattering state $\psi_K^{(+)}(r)$ and the BIC $\phi(r)$ at the same energy $E = K^2$, while local potentials rarely do. Consequently, BIC is a commonplace event in the group of all Hermitian potentials, most of which are nonlocal, while it is not the case in its subgroup composed of local potentials only.

8. Local versus nonlocal potentials

In Section 7.2, we have seen the difference of the Schrödinger equation between local and nonlocal potentials in connection with the BIC. Here we compare local and nonlocal potentials in terms of the SB-decomposition of Section 2, which will show the importance of nonlocal potentials in the study of BIC from another point of view.

8.1. *Potential group* $\mathbb{V}(t, \phi)$

Let us specify the group of potentials that we are going to examine.

- We choose an on-shell T-matrix

$$t(k) = -\frac{1}{\pi \rho_k} e^{i\delta(k)} \sin \delta(k), \tag{37}$$

  by arbitrarily giving a phase shift $\delta(k)$ that is in accordance with Levinson's theorem.[35]
- We fix a single bound state $|\phi\rangle$ in such a way that it behaves as[36]

$$\phi(r) \propto \exp(-\gamma r), \quad \text{as } r \to \infty. \tag{38}$$

Now we define a group $\mathbb{V}(t, \phi)$; this is a group of all Hermitian potentials that share the on-shell T-matrix $t(k)$ of Eq. (37) and the single bound state $|\phi\rangle$ satisfying Eq. (38). Note that $\mathbb{V}(t, \phi)$ is the group of all solutions to the inverse scattering problem for $t(k)$ with the bound state $|\phi\rangle$. In the following, we examine $\mathbb{V}(t, \phi)$ in terms of the SB-decomposition of Section 2.

First, for any value of the eigenenergy E of $|\phi\rangle$, Eq. (20) gives its corresponding $V_{\rm B}(\phi, E)$.[37] We therefore define

$$\mathbb{V}_{\rm B}(\phi) = \{V_{\rm B}(\phi, E)\} \tag{39}$$

as a group composed of $V_{\rm B}(\phi, E)$'s. Note that, because E is a continuous parameter, $\mathbb{V}_{\rm B}(\phi)$ contains an infinite number of elements.

Second, for the fixed $P_{\rm B} = |\phi\rangle\langle\phi|$, let us take a solution T of Eqs. (A) and (B) with the boundary condition $\langle k|T|k\rangle$ being equal to $t(k)$ of Eq. (37), which in turn gives its corresponding $V_{\rm S}(T, \phi)$ via Eq. (3).[38] Then we define

$$\mathbb{V}_{\rm S}(t, \phi) = \{V_{\rm S}(T, \phi)\} \tag{40}$$

[35] $\rho_k = \dfrac{k}{2}\dfrac{1}{(2\pi)^3}$ is the density of states.

[36] This choice of $|\phi\rangle$ excludes the von Neumann and Wigner potential in Section 6.2.

[37] The present notation $V_{\rm B}(\phi, E)$ explicitly shows that $V_{\rm B}$ is determined by ϕ and E.

[38] The notation $V_{\rm S}(T, \phi)$ shows explicitly that $V_{\rm S}$ is determined by T and ϕ.

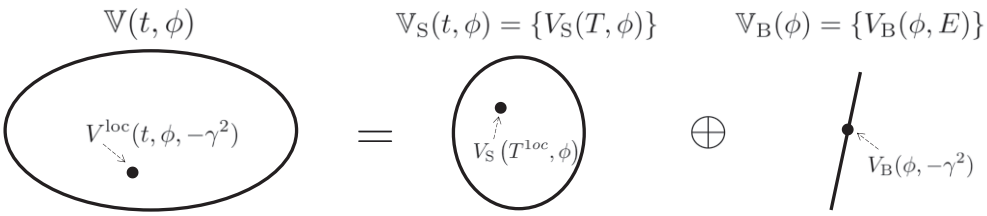

Fig. 22. Schematic expression for Eq. (42). $\mathbb{V}_{\rm B}(\phi)$ is denoted by a single line showing that it is a group with a single parameter E. The only local potential in $\mathbb{V}(t,\phi)$, given by Eq. (43), is indicated in the figure. See the text.

as a group of all possible $V_{\rm S}(T,\phi)$'s thus obtained.[39] Since an infinite number of T's would satisfy Eqs. (A) and (B) with the boundary condition $\langle k|T|k\rangle = t(k)$, $\mathbb{V}_{\rm S}(t,\phi)$ would contain infinitely many elements [7].[40]

Finally, the SB-decomposition expresses each potential in $\mathbb{V}(t,\phi)$ uniquely as

$$V(T,\phi,E) = V_{\rm S}(T,\phi) + V_{\rm B}(\phi,E). \tag{41}$$

Correspondingly, $\mathbb{V}(t,\phi)$ is a direct sum of $\mathbb{V}_{\rm S}(t,\phi)$ and $\mathbb{V}_{\rm B}(\phi)$, i.e.,

$$\mathbb{V}(t,\phi) = \mathbb{V}_{\rm S}(t,\phi) \oplus \mathbb{V}_{\rm B}(\phi), \tag{42}$$

as shown in Fig. 22.[41] In the following, we look for local and nonlocal potentials in $\mathbb{V}(t,\phi)$ of Eq. (42).

8.2. *Local potentials*

Here, we study a possible local potential in $\mathbb{V}(t,\phi)$. In the group of all local potentials, one can rarely find a BIC-supporting potential as explained from the viewpoint of the differential equation in Section 7.2.[42] Here, we explain the above situation from another point of view; we shall see that there is no BIC-supporting local potential in $\mathbb{V}(t,\phi)$.

First, we start with the group $\mathbb{V}_{\rm B}(\phi)$ of Eq. (39). For a local potential satisfying condition I of Eq. (28), Eq. (38) means that $\phi(r)$ has the eigenenergy $E=-\gamma^2$, and therefore that $\phi(r)$ is not a BIC. Then, Eq. (20) uniquely picks up an element $V_{\rm B}(\phi,-\gamma^2) \in \mathbb{V}_{\rm B}(\phi)$.

Second, we turn to the group $\mathbb{V}_{\rm S}(t,\phi)$ of Eq. (40). Let us recall the well-known result of the inverse scattering problem for local potentials [10,15]; there is only a single local solution $V^{\rm loc}(t,\phi,-\gamma^2;r)$ to the inverse scattering problem for a given set of $\{t,\phi,-\gamma^2\}$.[43] Therefore, the only local potential $V^{\rm loc}(t,\phi,-\gamma^2)$ uniquely chooses its corresponding element

[39] Different elements of $\mathbb{V}_{\rm S}(t,\phi)$ are distinguished by different T's that share the on-shell T-matrix $\langle k|T|k\rangle = t(k)$. By writing such T's as $T_1, T_2, \cdots$, we can express the group as $\mathbb{V}_{\rm S}(t,\phi) = \{V_{\rm S}(T_1,\phi), V_{\rm S}(T_2,\phi), \cdots\}$.

[40] In the space orthogonal to $|\phi\rangle$, there are an infinite number of different complete orthonormal sets that share the same asymptotic behaviors (or $t(k)$). Therefore, because knowledge of T is equivalent to that of a complete orthonormal set $\{|k\rangle_+\}$ of scattering states, we can find infinitely many T's satisfying Eqs. (A) and (B).

[41] Note that $\mathbb{V}(t,\phi)$ of Eq. (42) is much larger than $\mathbb{V}(T,\phi)$ introduced in Section 7.1; by choosing a single $V_{\rm S}(T,\phi) \in \mathbb{V}_{\rm S}(t,\phi)$ in Eq. (42), the group $\mathbb{V}(t,\phi)$ reduces to $\mathbb{V}(T,\phi)$.

[42] In fact, we know only a single BIC-supporting local potential, the von Neumann and Wigner potential in Section 6.2, the bound state of which does not satisfy the assumption of an exponentially damping wave function in Eq. (38).

[43] To be precise, the existence of the solution requires the phase shift to be "reasonable" [10].

$V_{\rm S}(T^{\rm loc},\phi)\in\mathbb{V}_{\rm S}(t,\phi)$,[44] to compose $V^{\rm loc}(t,\phi,-\gamma^2)$ in the form of Eq. (41) as

$$V^{\rm loc}(t,\phi,-\gamma^2)=V_{\rm S}(T^{\rm loc},\phi)+V_{\rm B}(\phi,-\gamma^2). \tag{43}$$

For the sake of clarity, we have embedded Eq. (43) in Fig. 22 for Eq. (42). Each element $V(T,\phi,E)\in\mathbb{V}(t,\phi)$ corresponds to a specific choice of $V_{\rm S}(T,\phi)\in\mathbb{V}_{\rm S}(t,\phi)$ and $V_{\rm B}(\phi,E)\in\mathbb{V}_{\rm B}(\phi)$, where both $\mathbb{V}_{\rm S}(t,\phi)$ and $\mathbb{V}_{\rm B}(\phi)$ contain an infinite number of elements. Among all these combinations in Fig. 22, only a single choice, $V_{\rm S}(T^{\rm loc},\phi)\in\mathbb{V}_{\rm S}(t,\phi)$ and $V_{\rm B}(\phi,-\gamma^2)\in\mathbb{V}_{\rm B}(\phi)$, makes the only local potential in $\mathbb{V}(t,\phi)$.

To summarize, with condition I of Eq. (28) for the potential and the assumption of an exponentially damping wave function in Eq. (38), $V^{\rm loc}(t,\phi,-\gamma^2)$ of Eq. (43) is the only local potential in $\mathbb{V}(t,\phi)$, and its bound state $|\phi\rangle$ is not a BIC.

8.3. *Nonlocal potentials*

Here, we turn to nonlocal potentials in $\mathbb{V}(t,\phi)$. Note that all elements of $\mathbb{V}(t,\phi)$ but $V^{\rm loc}(t,\phi,-\gamma^2)$ of Eq. (43) are nonlocal potentials.

Let us arbitrarily choose two elements, $V_{\rm S}(T,\phi)\in\mathbb{V}_{\rm S}(t,\phi)$ and $V_{\rm B}(\phi,E)\in\mathbb{V}_{\rm B}(\phi)$, to compose $V(T,\phi,E)\in\mathbb{V}(t,\phi)$ of Eq. (41). Then, depending on $E<0$ or $E>0$, our potential $V(T,\phi,E)$ describes $|\phi\rangle$ as a negative- or positive-energy bound state, irrespective of the choice of $V_{\rm S}(T,\phi)\in\mathbb{V}_{\rm S}(t,\phi)$. We can conclude, therefore, that half elements of $\mathbb{V}(t,\phi)$ carry positive E and describe $|\phi\rangle$ as a BIC,[45] and they are all nonlocal potentials.

8.4. *Importance of nonlocal potential*

Let $\mathbb{V}$ represent the group of all Hermitian potentials with a single bound state.[46] Then, $\mathbb{V}$ is a direct sum of all subgroups defined similarly to $\mathbb{V}(t,\phi)$ of Eq. (42), i.e.,

$$\begin{aligned}\mathbb{V}=&\,\mathbb{V}(t_0,\phi)\oplus\mathbb{V}(t_1,\phi)\oplus\mathbb{V}(t_2,\phi)\oplus\cdots\\&\oplus\mathbb{V}(t_0,\phi')\oplus\mathbb{V}(t_1,\phi')\oplus\mathbb{V}(t_2,\phi')\oplus\cdots\\&\oplus\mathbb{V}(t_0,\phi'')\oplus\mathbb{V}(t_1,\phi'')\oplus\mathbb{V}(t_2,\phi'')\oplus\cdots.\end{aligned} \tag{44}$$

In Section 8.2, we have seen that there is at most a single local potential in each subgroup on the right-hand side of Eq. (44),[47] and it does not support a BIC. On the other hand, in Section 8.3, we have shown that half elements of each subgroup, which are all nonlocal potentials, support a BIC. In the search for BIC, therefore, we have to examine nonlocal potentials.

9. Observing BIC

Here we discuss how we can observe a BIC or confirm its existence in a system described by a nonlocal potential.

To begin with, let us note the difficulty of observing a BIC directly. On the experimental side, we can observe a BIC only via its interaction with some other systems. On the other hand, however, the interaction necessarily gives a finite width to the BIC, which would inevitably

[44] $T^{\rm loc}$ is the half-on-shell T-matrix corresponding to $V^{\rm loc}(t,\phi,-\gamma^2)$.

[45] This is another way to confirm the results of Section 7.1, the generality of BICs.

[46] We assume condition I of Eq. (28) and an exponentially damping wave function of the bound state as in Eq. (38).

[47] Potentials in $\mathbb{V}(t_0,\phi)$ and $\mathbb{V}(t_0,\phi')$ carry the same on-shell T-matrix t_0. However, they cannot share the same half-on-shell T-matrix; each potential in $\mathbb{V}(t_0,\phi)$ requires a half-on-shell T-matrix satisfying Eqs. (A) and (B) with $P_{\rm B}=|\phi\rangle\langle\phi|$, while $\mathbb{V}(t_0,\phi')$ corresponds to different $P_{\rm B}=|\phi'\rangle\langle\phi'|$.

obscure the BIC to make its empirical identification difficult. On the theoretical side, we note that knowledge of $\langle k'|T|k\rangle$ alone does not give any information on a possible BIC of the system. We can easily prepare two different systems with the same $\langle k'|T|k\rangle$, one of which supports a bound state $|\phi\rangle$ as a BIC at a positive energy $E = K^2$, and the other at a negative energy $E = -\gamma^2$.[48] Naturally, we cannot distinguish between these two systems from their common $\langle k'|T|k\rangle$ only.[49] Therefore, if we look into the scattering data only, we cannot recognize the BIC or its precursor theoretically.

This is not, however, the whole story, as we explain below.

9.1. *Finding N_+*

It is known that Levinson's theorem, $\delta(0) - \delta(\infty) = N\pi$, holds not only for local but also for nonlocal potentials [15,16], where N is the number of normalizable eigenstates of the system. Let us express N as $N = N_- + N_+$ where N_- is the number of negative-energy bound states and N_+ is the number of BICs. Then, as we show below, N and N_- are accessible both theoretically and experimentally, and so is $N_+ = N - N_-$.

First, we explain the experimental observation of N_+ using an example. Suppose that, for a given system, we have observed experimentally the phase shift $\delta(k)$ in Fig. 2 that shows $N = 1$, and also that we do not find a negative-energy bound state, i.e., $N_- = 0$. Then, we can confirm experimentally that $N_+ = N - N_- = 1$, and therefore conclude that the system supports a single BIC.

Second, let us turn to the theoretical observation of N_+. Here, the problem is how to determine N_+ of a system of which the potential is known. To be concrete, we consider a system with $\langle k'|V|k\rangle$ in Fig. 8. Then, we can easily calculate its phase shift $\delta(k)$ in Fig. 2, which in turn gives $N = 1$ via Levinson's theorem. At the same time, we can confirm $N_- = 0$ by diagonalizing the Hamiltonian of Eq. (1) in momentum space. We can thus obtain $N_+ = N - N_- = 1$ theoretically, and conclude that the system under consideration supports a single BIC.[50]

To summarize the above, by making use not only of the scattering data $\delta(k)$, but also of the number N_- of negative-energy bound states that we can observe easily both experimentally and theoretically, we can obtain, by virtue of Levinson's theorem, the number N_+ of BICs in the system.

9.2. *Finding $\phi(k)$ and K^2*

The method in Section 9.1 gives N_+ (and N_-) in a simple manner. However, it does not yield either the wave function or the eigenenergy of the BIC.[51] In the following, therefore, we explain

[48] An example of a pair of such systems can be found in Section 4.4; the potentials in Figs. 4 and 8 yield the same $\langle k'|T|k\rangle$ but give the common bound state $|\phi\rangle$ at different energies $E = -4.0\,\mathrm{fm}^{-2}$ and $4.0\,\mathrm{fm}^{-2}$. In the SB-decomposition, these potentials have the same V_S but differ from each other in V_B. Generally, such a pair of systems can be prepared easily by the method in Section 3.2.

[49] Their common T determines their common V_S only, leaving V_B, and therefore $V = V_S + V_B$ of the system, undetermined.

[50] When $N = 1$, we can easily distinguish the $N_+ = 1$ and $N_- = 1$ cases by looking into the sign of $\langle 0|V_B|0\rangle$ only. For a given $\langle k'|V|k\rangle$, we can easily calculate $\langle k'|T|k\rangle$ and therefore $\langle k'|V_S|k\rangle$, which in turn gives $\langle k'|V_B|k\rangle = \langle k'|V|k\rangle - \langle k'|V_S|k\rangle$. Then, noting that $\langle 0|V_B|0\rangle = E|\langle 0|\phi\rangle|^2$, which is shown by Eq. (20), has the same sign as the eigenenergy E, we realize that $\langle 0|V_B|0\rangle > 0$ means $N_+ = 1$, and that $\langle 0|V_B|0\rangle < 0$ implies $N_- = 1$. The above criterion can be easily confirmed in e.g. Figs. 17, 18, 20, and 21.

[51] For example, suppose that we know $N_+ = 1$ and the phase shift $\delta(k)$ in Fig. 2. Then, $\langle k'|V|k\rangle$ in Fig. 7 and that in Fig. 8 are equally possible as a potential of the system. However the former gives the

a theoretical access to the BIC wave function $\phi(k)$ and its eigenenergy K^2 for a given potential $\langle k'|V|k\rangle$.[52]

Let us start with a system with $N_+ = 1$ and $N_- = 0$. First, using the given potential $\langle k'|V|k\rangle$, we calculate $\langle k'|T|k\rangle$ by the Lippmann–Schwinger equation, which in turn gives $\langle k'|V_S|k\rangle$ by Eq. (3). Then, with knowledge of $\langle k'|V_B|k\rangle = \langle k'|V|k\rangle - \langle k'|V_S|k\rangle$, which we know has the form of Eq. (20), we can easily identify the eigenenergy $E = K^2$ and the wave function $\phi(k)$ of the BIC.

Next, let us turn to general systems with $N_+ \geqq 1$. Obviously, we can easily obtain $\langle k'|V_B|k\rangle$ as in the above. Then, it is convenient to split $\langle k'|V_B|k\rangle$ of Eq. (5) as

$$\begin{aligned}\langle k'|V_B|k\rangle &= \langle k'|V_{B+}|k\rangle + \langle k'|V_{B-}|k\rangle \\ &= \sum_{i=1}^{N_+}\left(K_i^2 - k'^2\right)\langle k'|\phi_i\rangle\langle\phi_i|k\rangle + \sum_{j=N_++1}^{N}\left(-\gamma_j^2 - k'^2\right)\langle k'|\phi_j\rangle\langle\phi_j|k\rangle, \end{aligned} \tag{45}$$

with an obvious notation, where V_{B+} and V_{B-} represent contributions of positive- and negative-energy bound states, respectively. Because we can easily obtain N_- negative-energy bound states $\{|\phi_j\rangle,\ -\gamma_j^2\,;\ j = N_+ + 1, \cdots N\}$, we can calculate their contribution $\langle k'|V_{B-}|k\rangle$ in Eq. (45) to isolate the BIC contribution $\langle k'|V_{B+}|k\rangle$. Then, because $\langle k'|V_{B+}|k\rangle$ has the expression given by the first term in the second line of Eq. (45), it is straightforward numerically to obtain N_+ BICs and their energies, $\{\phi_i(k),\ K_i^2\,;\ i = 1, \cdots N_+\}$. We stress that the above analysis is feasible and clearly shows the usefulness of the SB-decomposition in Section 2.

In the end, let us note that the above problem has been studied in coordinate space [12,13], to give a theoretical route to BICs, $\{\phi_i(r),\ K_i^2\,;\ i = 1, \cdots N_+\}$.[53] However, we believe that our analysis in momentum space via the SB-decomposition is much simpler and more feasible to investigate BICs.

10. Search for BIC

Having established a general theory of BIC-supporting potentials, we now discuss how and where we should search for a BIC in practice. Let us recall the generality of BICs explained in Section 7.1; if we could examine all Hermitian potentials, most of which are nonlocal, we would find as many BICs as negative-energy bound states. Despite great efforts, however, we have not yet discovered a single BIC to date.[54] We believe, however, that the present theory provides us with a good chance to find a BIC in practice.

eigenenergy $K^2 = 1.0\,\mathrm{fm}^{-2}$ of the BIC and the latter gives $K^2 = 4.0\,\mathrm{fm}^{-2}$, showing that we cannot obtain the eigenenergy K^2 from $\delta(k)$ and N_+ only.

[52] One might expect that one could easily obtain $\phi(k)$ and K^2 by numerically diagonalizing the Hamiltonian $H = H_0 + V$ in momentum space. This is not, however, true; the eigenvectors that one would get are linear combinations of $|\phi\rangle$ and $|K\rangle_+$, which are embedded in scattering states.

[53] Here, one transforms the Schrödinger equation (35) in coordinate space into integral equations that incorporate boundary conditions for the physical and regular solutions. Then, by looking for the zeros of the Fredholm determinants of these integral equations, one can derive the eigenenergies $\{E_i = K_i^2\}$ of the BICs. In the same way, by looking for the null space of the integral kernels of these equations, one can obtain the BIC wave functions $\{\phi_i(r)\}$.

[54] In condensed matter physics, quasi-bound states in the continuum (QBICs) can occur with a finite measure and can therefore be observed [17]. This is in contrast to the BICs in Section 6, which are zero-measure phenomena. In Ref. [17], one can find discussions on the relation between the BIC and the QBIC.

10.1. *How to search for BIC*

Here, we outline a possible prescription for how to search for a BIC in practice. Given a potential V, we can SB-decompose it as $V = V_S + V_B$ and explain the behaviors of V_S and V_B both in momentum (Section 4.4) and coordinate (Section 4.5) spaces. Here, V_B has the information on the bound state, and therefore on the BIC, while V_S does not. We examine, therefore, V_B in momentum and coordinate spaces. For simplicity, we consider systems with a single bound state only.

10.1.1. *Search in momentum space.* In momentum space, the k'-dependence of $\langle k'|V_B|k\rangle$ clearly shows whether the system supports a BIC or not, as explained in Sections 4.4 and 9.2.[55] In coordinate space, on the other hand, we cannot realize the existence of a BIC so clearly as in momentum space. We have seen that a node of $\langle r'|V_B|r\rangle$ along the r'-axis moves as the eigenenergy E of the bound state $|\phi\rangle$ varies. However, the above shift of the node does not give a clear criterion of the BIC.

To summarize, we should choose the momentum space to look for theoretical evidence of a BIC in a given potential V, i.e., we had better examine $\langle k'|V_B|k\rangle$ rather than $\langle r'|V_B|r\rangle$.

10.1.2. *Search in coordinate space.* As shown in Sections 6 and 7, we have to examine nonlocal potentials to find a BIC. Because the nonlocality is defined in coordinate space,[56] we can keep it under better control in coordinate space than in momentum space.[57]

First, let us start from a system with a (generally nonlocal) potential $V_{\rm init}(r', r)$ supporting a bound state $\phi(r)$ at a negative energy $-\gamma^2$. By introducing a nonlocal perturbation $(E + \gamma^2)\phi(r')\phi(r)$, we obtain a system with the following potential:

$$V(r', r) = V_{\rm init}(r', r) + (E + \gamma^2)\phi(r')\phi(r). \tag{46}$$

Using the argument leading to Eq. (25), we can show that $V(r', r)$ of Eq. (46) now describes $\phi(r)$ as a BIC at the positive energy E. In other words, only by introducing a nonlocal perturbation proportional to $\phi(r')\phi(r)$, we can transform the negative-energy bound state $\phi(r)$ into a BIC. The SB-decomposition explains that the opposite is also true; any BIC-supporting potential can be obtained in this way from a potential without a BIC.[58]

Second, let us note the following; the generality of BIC in Section 7.1 means that BICs are robust against perturbations.[59] This means that, if a potential $V(r', r)$ supports a BIC, its

[55] We can easily identify the existence of a BIC simply by looking into the sign of $\langle k'|V_B|k\rangle$ at the origin ($k' = k = 0$). If $\langle 0|V_B|0\rangle$ is positive, the system has a BIC. Further, a node $k' = K$ of $\langle k'|V_B|k\rangle$ along the k'-axis gives the (positive) eigenenergy $E = K^2$ of the BIC.

[56] We use "local" and "nonlocal" in coordinate space only.

[57] As an example, let us consider V_0 in Section 4.1; its expression (18) in coordinate space is obviously local, being proportional to $\delta(r' - r)$, while Eq. (19) in momentum space is not proportional to $\delta(k' - k)$. Therefore, it is straightforward to distinguish "local" potentials from "nonlocal" ones in coordinate space, while this is not the case in momentum space. This implies that the coordinate space is more suitable than the momentum space to control the nonlocality of a potential.

[58] Any BIC-supporting potential can be uniquely expressed as $V = V_S + V_B$, where V_B is given by $\langle k'|V_B|k\rangle = (E - k'^2)\langle k'|\phi\rangle\langle\phi|k\rangle$ of Eq. (20) with $E > 0$. By replacing the positive E by a negative one, we can relate the potential with a BIC to a potential without a BIC in a unique fashion.

[59] This is true in the group of all Hermitian nonlocal potentials. If we restrict ourselves to its subgroup of Hermitian local potentials only, the situation is different; it has been shown that the BIC of the von Neumann and Wigner (local) potential is vulnerable to any local perturbation [3,4].

neighboring potential $V(r', r) + \delta V(r', r)$, where $\delta V(r', r)$ is an arbitrary small deviation, would also support a BIC. In other words, for $V(r', r)$ of Eq. (46) to support a BIC, the perturbation on the right-hand side only needs to behave "roughly" as $(E + \gamma^2)\phi(r')\phi(r)$.

We summarize the present section as follows: if we are going to find a BIC in the neighborhood of a system with $V_{\rm init}(r', r)$ supporting a negative-energy bound state, it is absolutely necessary to introduce a "nonlocal" perturbation, which is roughly separable as $\phi(r')\phi(r)$, to the system. For this purpose, we had better work in coordinate space where "nonlocality" is apparent.

10.1.3. *Combined search.* Let us combine the findings in Sections 10.1.1 and 10.1.2 to devise a possible scenario for the search for a BIC.

We first work in coordinate space; we look into every nonlocal potential $V(r', r)$ that is realizable experimentally. Then, if we find a potential $V(r', r)$ that can be put as

$$V(r', r) = V_0(r', r) + \Delta V(r', r), \tag{47}$$

in such a way (i) that $V_0(r', r)$ supports a bound state, and (ii) that $\Delta V(r', r)$ is separable, we regard the potential $V(r', r)$ at hand as a good candidate for a BIC-supporting potential.

Next, we bring the above candidate V into momentum space; we calculate its SB-decomposition $V = V_{\rm S} + V_{\rm B}$ and look into the sign of $\langle k'|V_{\rm B}|k\rangle$ at the origin ($k' = k = 0$). If we find that it is positive, we can conclude that the candidate V at hand is truly a BIC-supporting potential.

10.2. *Where to search for BIC*

Finally, we dare to make a conjecture, on the basis of the result in Section 10.1.3, where we should search for a BIC.

Suppose that we have a system with $V_0(r', r)$ of Eq. (47) that supports a bound state at $E < 0$. Then, the occurrence of a BIC depends crucially on the introduction of a nonlocal separable perturbation $\Delta V(r', r)$. This is, however, admittedly difficult, because fundamental interparticle interactions are described mostly by local potentials. Nonlocal interactions usually come about as an "effective" interaction that incorporates effects of antisymmetrization or some degrees of freedom outside the model space under consideration [12]. We expect, therefore, that the following two scenarios might be possible.

(i) First, we consider the possibility of simulating the above $\Delta V(r', r)$ by a change of the Fock potential of a many-body system. Here, we choose a system that is described as a single particle state $\phi(r)$ on top of a core.[60] We suppose that the core gives a mean (generally nonlocal) potential $V_0(r', r)$ that describes $\phi(r)$ at a negative energy $-\gamma^2$.[61] Now, we introduce a nonlocal perturbation to $\Delta V(r', r)$ by changing the core. One way might be to remove or excite a particle from the core.[62] If the nonlocal perturbation thus introduced has a sufficient overlap with $\phi(r')\phi(r)$, the resultant system might support a BIC.

[60] Take a ^{41}Ca nucleus as an example, which is described mostly as a single neutron in a $0f_{7/2}$ orbit on top of a ^{40}Ca core.

[61] Here, we imagine that the Hartree–Fock mean field $V_{\rm HF}(r', r)$ plays the role of our $V_0(r', r)$ in Eq. (47).

[62] By removing a particle in a state $\varphi(r)$ in the core, for instance, we change the nonlocal Fock potential by $\varphi(r')v(r', r)\varphi(r)$ in the standard notation.

(ii) Second, we may use a nonlocal effective interaction to generate the above $\Delta V(r', r)$. In nuclear physics, for example, we naturally expect that ΛN effective interaction is mostly nonlocal.[63] This implies that we may introduce a highly nonlocal perturbation into a nucleus by replacing a nucleon with a Λ particle. We expect, therefore, that Λ-hypernuclei may be a promising field which to search for a BIC.[64]

We admit that the above scenarios are far from being well designed. However, we hope that they serve as a starting point to set out on a search for a BIC.

11. Summary

In this work, we have presented a general theory to construct a potential that supports a bound state at a positive energy, i.e., a bound state in the continuum (BIC), with numerical examples. Our theory is based on the SB-decomposition of a Hermitian potential, and is able to express any potential supporting a BIC. The present theory has clarified (i) that the BIC is a rare phenomenon if we look into local potentials only, and, astonishingly enough, (ii) that BICs are as common as negative-energy bound states if we look into all Hermitian potentials, most of which are nonlocal. Accordingly, all BIC-supporting potentials in the present theory are nonlocal. In all investigations in search of a BIC, therefore, it is crucial to devise and examine systems with nonlocal potentials. Finally, we have proposed some scenarios to search for a BIC in real systems with possible nonlocal potentials. We believe that the present theory will boost research activities on nonlocal potentials and the BIC in various quantum systems.

A. Existing theories of BIC-supporting potential

In this appendix, we briefly explain two well-known potentials that support a BIC, the von Neumann and Wigner potential and the separable potential.

A.1. von Neumann and Wigner potential

We outline the method of von Neumann and Wigner [1] following Ref. [2]. We consider a local potential $V(r)$ in the S-wave channel, for which the Schrödinger equation in coordinate space reads

$$(-\nabla^2 + V(r))\phi(r) = E\phi(r), \tag{A1}$$

where $\nabla^2 = \dfrac{\mathrm{d}^2}{\mathrm{d}r^2} + \dfrac{2}{r}\dfrac{\mathrm{d}}{\mathrm{d}r}$. Now we look for $V(r)$ that gives a normalizable $\phi(r)$ at a positive energy E via Eq. (A1).

Let us solve Eq. (A1) for $V(r)$:

$$V(r) = E + \frac{\nabla^2\phi(r)}{\phi(r)}, \tag{A2}$$

which explicitly shows that, to obtain a smooth $V(r)$, each zero of $\phi(r)$ must be canceled by vanishing $\nabla^2\phi(r)$. We note that the free state $\phi(r) = j_0(kr)$ with $E = k^2$ naturally gives $V(r) = 0$ by Eq. (A2). Therefore, we look for a BIC solution $\phi(r)$ by modulating the free state $j_0(kr)$ as

$$\phi(r) = j_0(kr)f(r), \tag{A3}$$

[63] The effective interaction between Λ and the nucleon does not originate from a one-pion-exchange process because of the isospin conservation. It necessarily requires at least a two-pion-exchange process that naturally induces a nonlocal potential.

[64] For Λ and Λ-hypernuclei, see Refs. [18,19].

where the envelope $f(r)$ is a smooth function without a zero. Here, we have to require that $f(r)$ be chosen in such a way (i) that $f(r)$ guarantees $\phi(r)$ be normalizable, and (ii) that each zero of $\phi(r)$ be canceled by vanishing $\nabla^2\phi(r)$ in the right-hand side of Eq. (A2). By substituting $\phi(r)$ of Eq. (A3) into Eq. (A2), we obtain

$$V(r) = E - k^2 + 2k\cot(kr)\frac{f'(r)}{f(r)} + \frac{f''(r)}{f(r)}. \tag{A4}$$

Then, we see that the above condition (ii) of $\phi(r)$ has been translated into a condition of $f(r)$, i.e., (ii′) that $\dfrac{f'(r)}{f(r)} = 0$ at $kr = n\pi\,,\ n = 0, 1, \cdots$. We can ensure condition (ii′) by choosing $f(r)$ to be a smooth function of

$$R = 4k\int_0^r \mathrm{d}r' \sin^2(kr') = 2kr - \sin(2kr). \tag{A5}$$

In Ref. [1], taking condition (i) into account, they chose

$$f(r) = \frac{1}{A^2 + R^2}, \tag{A6}$$

where $A(\neq 0)$ is an arbitrary constant. Note also that, for $V(r)$ of Eq. (A4) to tend to zero as $r \to \infty$, we must have $E = k^2$. Summarizing the above, Eq. (A4) gives

$$V(r) = -\frac{128A^2k^2\sin^4(kr)}{(A^2 + (2kr - \sin(2kr))^2)^2} + \frac{96k^2\sin^4(kr) - 16k^2(2kr - \sin(2kr))\sin(2kr)}{A^2 + (2kr - \sin(2kr))^2}, \tag{A7}$$

which behaves as

$$V(r) \propto r^4,\ (r \to 0), \quad V(r) \propto \frac{\sin 2kr}{2kr},\ (r \to \infty). \tag{A8}$$

The above method of von Neumann and Wigner explicitly constructs the local potential $V(r)$ of Eq. (A7) that describes $\phi(r)$ of Eq. (A3) as a BIC at $E = k^2$.

A.2. Separable potential

It is well known that a separable potential can be tuned to support a BIC [15]. Here, we consider the following separable potential in momentum space in the S-wave channel:

$$\langle k'|V|k\rangle = \lambda g(k')g(k), \tag{A9}$$

where λ is a real coupling strength, and $g(k)$ is a real function.

Let us assume a BIC $\phi(k)$ at a positive energy $E = K^2$. Then, the Schrödinger equation (14) takes on the following form:

$$k^2\phi(k) + g(k)\int\frac{p^2\mathrm{d}p}{(2\pi)^3}\,\lambda g(p)\phi(p) = K^2\phi(k), \tag{A10}$$

which directly yields $\phi(p)$ as

$$\phi(p) = \mathcal{N}\frac{g(p)}{K^2 - p^2}, \tag{A11}$$

where $\mathcal{N}$ is a normalization constant. By substituting Eq. (A11) into the Schrödinger equation (A10), we immediately arrive at

$$\int\frac{p^2\mathrm{d}p}{(2\pi)^3}\,\frac{\lambda g(p)^2}{K^2 - p^2} = 1. \tag{A12}$$

Finally, for $\phi(p)$ of Eq. (A11) to be normalizable, we have to require

$$g(K) = 0. \tag{A13}$$

The above observation shows that a BIC is a solution to the set of Eqs. (A12) and (A13), which can be found in the following manner. First, we fix K arbitrarily, and then choose a function

$g(k)$ that vanishes at $k = K$ so that Eq. (A13) is satisfied. Second, by varying the coupling strength λ, we can find $\lambda = \lambda_c$ that satisfies Eq. (A12). The above procedure shows that, for any $g(k)$ with a node at $k = K$, we can always tune the coupling strength λ in such a way that the potential describes $\phi(k)$ of Eq. (A11) as a BIC at $E = K^2$. Conversely, for a fixed $g(k)$ with a node, the BIC only appears at a special value $\lambda = \lambda_c$ of the coupling strength, i.e., the BIC is a rare phenomenon for a system described by a separable potential.

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