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On $(\bar{m}, m)$-Conformal Mappings

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Abstract

Conformal mappings between Riemannian spaces $\bar{\mathbb{R}}_N$ and $\mathbb{R}_N$ are defined by the explicit transformation of the metric tensor of the space $\bar{\mathbb{R}}_N$ to the metric tensor of the space $\mathbb{R}_N$. Geodesic mapping between these two Riemannian spaces is a transformation that transforms any geodesic line of the space $\bar{\mathbb{R}}_N$ to a geodesic line of the space $\mathbb{R}_N$. In this research, we defined an m-conformal line of a Riemannian space, which is geodesic if $m = 0$. Based on this definition, we involved the concept of $(\bar{m}, m)$-conformal mapping as a transformation $\bar{\mathbb{R}}_N \to \mathbb{R}_N$ in which any $\bar{m}$-conformal line of the space $\bar{\mathbb{R}}_N$ transforms to an m-conformal line of the space $\mathbb{R}_N$. The result of this research is the establishment of three invariants for these mappings. At the end of this research, we gave an example of a scalar geometrical object which may be used in physics.

Keywords: mapping; Riemannian space; invariant; variation

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1. Introduction

The theory of Riemannian spaces was given by L. P. Eisenhart [1]. Based on his research, this theory was developed to such a degree that it can be studied as a theoretical concept or a tool for applications in physics.

Many authors have developed the theory of mappings between Riemannian spaces. Some of them include J. Mikeš, with his research group [2–6], N. S. Sinyukov [7–9], U. C. De [10,11], among many others.

Motivated by Eisenhart's definition of a generalized Riemannian space, the theory of mappings of generalized Riemannian spaces has been developed. The most significant researchers in of subject are M. S. Stanković [12–16], M. Lj. Zlatanović [17–19], Lj. S. Velimirović [16,20], among many others.

An article [21] by N. Vesić developed the well known methodology for obtaining invariants of geometric mappings. In many later papers, like in this research paper, the article [15] and the methodology presented in [21] are applied, and the result is at least one new invariant for the analyzed mapping. In this research, we will review the results of Vesić's article and apply them to obtain invariants for a special mapping of a Riemannian space, which will be defined.

Conformal transformations are important for different applications in physics [22]. The Weyl tensor is composed by itself for generating the integrand for Einstein–Hilbert action. But this integrand leads the study of R^2-cosmology. We are interested in finding invariants for mappings which stay within the valid research topic of R^1-cosmology.

The integrand in Einstein–Hilbert action in [23] is very similar to invariants which Vesić obtained in [21]. In this research, we will study the Einstein–Hilbert action generated by Vesić's invariants.

In this manuscript, we aim to generalize the concept of conformal mappings by transforming the basic equation of conformal mapping between two Riemannian spaces. These results will provide different opportunities for theoretical research into this new class of mappings, and for its applications in cosmology as well.

1.1. Tensors as Indexed Geometrical Objects

An indexed magnitude $X^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$, $\alpha_i, \beta_j = 0, \ldots, N-1$, presented in the Cartesian coordinate system $(O, x^0, \ldots, x^{N-1})$, is a tensor of the type (p, q) if under a change of coordinate system to $(0', x'^0, \ldots, x'^{N-1})$, where the value $X^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}$ transforms as [2,9]

$$X^{\alpha'_1\ldots\alpha'_p}_{\beta'_1\ldots\beta'_q} = x^{\alpha'_1}_{\alpha_1} \ldots x^{\alpha'_p}_{\alpha_p} x^{\beta_1}_{\beta'_1} \ldots x^{\beta_q}_{\beta'_q} X^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q},$$

where $x^{\alpha'_i}_{\alpha_i} = \dfrac{\partial x'^{\alpha_i}}{\partial x^{\alpha_i}}$, $x^{\beta_j}_{\beta'_j} = \dfrac{\partial x^{\beta_j}}{\partial x'^{\beta_j}}$, and Einstein's Summation Convention is used for repeated indices. Scalar functions are tensors of the type $(0,0)$.

From the last definition, we conclude that a partial derivative of a tensor of the type $(0,0)$ forms a tensor of the type $(0,1)$. Partial derivatives of a tensor of any another type do not form tensors.

1.2. Riemannian Spaces

An N-dimensional manifold $\mathcal{M} = \mathcal{M}(u^0, \ldots, u^{N-1})$ equipped with a symmetric metric tensor $g_{\mu\nu}$, $g_{\mu\nu} = g_{\nu\mu}$ is the Riemannian space $\mathbb{R}_N$ (see [2,9]). The Greek indices μ, ν, ... take values of 0,..., $N-1$.

Remark 1. *For studies in physics, especially in cosmology, it is important to make a distinction between indices which denote time and indices which denote space. It is standardized in physics that the Greek indices present both time and space, but Latin indices correspond to space. In our study, we will not make such a distinction, but for possibly easier applications, we will use the Greek indices.*

We assume that the matrix $[g_{\mu\nu}]$ is non-singular, i.e., det $[g_{\mu\nu}] \neq 0$. Hence, the contravariant metric tensor is defined as $[g^{\mu\nu}] = [g_{\mu\nu}]^{-1}$. This means that $g^{\mu\alpha} g_{\nu\alpha} = \delta^\mu_\nu$.

Christoffel symbols of the second kind,

$$\Gamma^\pi_{\mu\nu} = \frac{1}{2} g^{\pi\alpha} (g_{\mu\alpha,\nu} - g_{\mu\nu,\alpha} + g_{\nu\alpha,\mu}),$$

where a comma denotes partial differentiation, are the affine connection coefficients of the space $\mathbb{R}_N$.

The Christoffel symbols are not tensors because they change as

$$\Gamma^{\pi'}_{\mu'\nu'} = x^{\pi'}_\pi x^\mu_{\mu'} x^\nu_{\nu'} \Gamma^\pi_{\mu\nu} + x^{\pi'}_\pi x^\pi_{\mu'\nu'},$$

where $x^\pi_{\mu'\nu'} = \dfrac{\partial^2 x^\pi}{\partial x'^\mu \partial x'^\nu}$.

The Christoffel symbols $\Gamma^{\pi}_{\mu\nu}$ and their traces $\Gamma_{\mu} = \Gamma^{\alpha}_{\mu\alpha}$ are not tensors.

1.3. Geodesic Lines of Riemannian Spaces

A curve $\ell = (\ell^{\mu}) = \ell(t)$ in Riemannian space $\mathbb{R}_N$ is geodesic if its tangential vector $\lambda^{\mu} = \dfrac{d\ell^{\mu}}{dt}$ satisfies the forthcoming system of differential equations

$$\frac{d\lambda^{\mu}}{dt} + \Gamma^{\mu}_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta} = \rho\lambda^{\mu}, \tag{1}$$

where ρ is a scalar function. In one paper (Mikeš et al. [2], pp. 88–89), the definition of geodesics is discussed.

In spaces with a positive definite metric, the geodesic line is the shortest line between two points on manifolds by the path in it. The FLRW metric is not a positive definite one; light (the fastest unit in cosmology) moves along geodesic lines.

We are interested in obtaining some new invariants for a special geometric mapping and to present the corresponding action in four-dimensional cosmology.

1.4. Motivation

In cosmology, a geodesic line is defined in the case of homogeneous Equation (1), i.e., in the case of $\rho = 0$ (see [24], page 30, Equation (2.18)). For geodesics defined in such a way, we will prove that a geometrical object which determines a transformation of Christoffel symbols under geodesic mapping vanishes, (for the concrete methodology, see the sections on geodesic mappings in [2,9]).

After connecting the invariance of the Ricci tensor under geodesic mappings, we find that the standard Einstein–Hilbert action [24] $S = \dfrac{1}{2\kappa}\int d^4x\sqrt{-g}R$ is nothing more than action for the Lagrangian which is equal to the invariant $R_{\alpha\beta}$ composed by $g^{\alpha\beta}$, and for topological invariance multiplied by $\sqrt{-g}$.

Perturbations in cosmology are nothing more than summing the Friedmann–Lemaitre–Robertson–Walker metric $\bar{g}_{\mu\nu}$ with a tensor $\delta g_{\mu\nu}$ (named perturbation). In this way, any geodesic line of the initial space equipped with the FLRW metric transforms to a curve of perturbed space (equipped with metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$).

Geodesics are defined by the non-homogeneous Equation (1) [1,2,9]. All special curves (F-planar, almost geodesic, ... [2,9]) are generalizations of geodesics.

Until 2020, all invariants for mappings (Weyl projective tensor, Weyl conformal tensor, Weyl F-plannar tensor,...) were created in such a way that their traces by contravariant and covariant indices vanish. In (Vesić, 2020 [21]), the methodology for obtaining another invariant (whose traces do not vanish) was presented.

This other invariant, which does not lose the Ricci tensor after contraction, motivated us to pursue this research. The other motivation for our research is the significance of the conformal mappings of Riemannian spaces presented in [22]. The Lagrangian of Einstein–Hilbert action studied in [23] is very similar to the invariants for mappings obtained in [21], which is another motivation for our research, presented below.

This manuscript is organized as follows:

1. We will review the preferred methodology for obtaining invariants of geometric mappings from [15,21]. In this review, we will present the corresponding invariants for geodesic and conformal mappings of an N-dimensional Riemannian space.
2. In the next section, we will generalize the concept of conformal mapping by defining a special curve of a Riemannian space. A mapping from which any geodesic line from an initial space transforms to a curve of this special class of the deformed space will be the subject of our research.

3. We will obtain the basic equations for mappings mentioned in previous studies. After that, we will obtain the invariants for this mapping.
4. At the end of this paper, we will present the cosmology which corresponds to one of the invariants obtained herein and present the corresponding Einstein equations.

2. Review of Invariants for Geometric Mappings

In this section, we will review the process for obtaining invariants of mappings defined on N-dimensional Riemannian spaces [15,21]

Let $f:\bar{\mathbb{R}}_N\to\mathbb{R}_N$ be a mapping whose basic equation is

$$\Gamma^\pi_{\mu\nu}=\bar{\Gamma}^\pi_{\mu\nu}+\psi_\nu\delta^\pi_\mu+\psi_\mu\delta^\pi_\nu+\omega^\pi_{\mu\nu}-\bar{\omega}^\pi_{\mu\nu}. \tag{2}$$

After contracting this equation by π and ν, we obtain $\Gamma_\mu=\bar{\Gamma}_\mu+(N+1)\psi_\mu+\omega_\mu-\bar{\omega}_\mu$, i.e.,

$$\psi_\mu=\frac{1}{N+1}\big(\Gamma_\mu-\omega_\mu\big)-\frac{1}{N+1}\big(\bar{\Gamma}_\mu-\bar{\omega}_\mu\big). \tag{3}$$

With respect to (3), the basic Equation (2) is transformed to

$$\begin{aligned}\Gamma^\pi_{\mu\nu}=\bar{\Gamma}^\pi_{\mu\nu}+\omega^\pi_{\mu\nu}+\frac{1}{N+1}\big(\delta^\pi_\mu\Gamma_\nu+\delta^\pi_\nu\Gamma_\mu\big)-\frac{1}{N+1}\big(\delta^\pi_\mu\omega_\nu+\delta^\pi_\nu\omega_\mu\big)\\ -\bar{\omega}^\pi_{\mu\nu}-\frac{1}{N+1}\big(\delta^\pi_\mu\bar{\Gamma}_\nu+\delta^\pi_\nu\bar{\Gamma}_\mu\big)+\frac{1}{N+1}\big(\delta^\pi_\mu\bar{\omega}_\nu+\delta^\pi_\nu\bar{\omega}_\mu\big).\end{aligned} \tag{4}$$

The basic Equation (4) is expressed in the form $\Gamma^\pi_{\mu\nu}=\bar{\Gamma}^\pi_{\mu\nu}+d^\pi_{\mu\nu}-\bar{d}^\pi_{\mu\nu}$, where

$$\begin{aligned}d^\pi_{\mu\nu}&=\omega^\pi_{\mu\nu}+\frac{1}{N+1}\big(\delta^\pi_\mu\Gamma_\nu+\delta^\pi_\nu\Gamma_\mu\big)-\frac{1}{N+1}\big(\delta^\pi_\mu\omega_\nu+\delta^\pi_\nu\omega_\mu\big),\\ \bar{d}^\pi_{\mu\nu}&=\bar{\omega}^\pi_{\mu\nu}+\frac{1}{N+1}\big(\delta^\pi_\mu\bar{\Gamma}_\nu+\delta^\pi_\nu\bar{\Gamma}_\mu\big)-\frac{1}{N+1}\big(\delta^\pi_\mu\bar{\omega}_\nu+\delta^\pi_\nu\bar{\omega}_\mu\big).\end{aligned}$$

The basic invariants for the mapping f of the Thomas and Weyl type are $\mathcal{T}^\pi_{\mu\nu}=\Gamma^\pi_{\mu\nu}-d^\pi_{\mu\nu}$ and $\mathcal{W}^\pi_{\mu\nu\sigma}=R^\pi_{\mu\nu\sigma}-d^\pi_{\mu\nu|\sigma}+d^\pi_{\mu\sigma|\nu}+d^\alpha_{\mu\nu}d^\pi_{\alpha\sigma}-d^\alpha_{\mu\sigma}d^\pi_{\alpha\nu}$, and the corresponding $\bar{\mathcal{T}}^\pi_{\mu\nu}$ and $\overline{\mathcal{W}}^\pi_{\mu\nu\sigma}$. The invariants $\mathcal{T}^\pi_{\mu\nu}$ and $\mathcal{W}^\pi_{\mu\nu\sigma}$ are directly expressed as

$$\mathcal{T}^\pi_{\mu\nu}=\Gamma^\pi_{\mu\nu}-\omega^\pi_{\mu\nu}-\frac{1}{N+1}\big(\delta^\pi_\mu\Gamma_\nu+\delta^\pi_\nu\Gamma_\mu\big)+\frac{1}{N+1}\big(\delta^\pi_\mu\omega_\nu+\delta^\pi_\nu\omega_\mu\big), \tag{5}$$

$$\begin{aligned}\mathcal{W}^\pi_{\mu\nu\sigma}=&R^\pi_{\mu\nu\sigma}-\omega^\pi_{\mu\nu|\sigma}+\omega^\pi_{\mu\sigma|\nu}+\omega^\alpha_{\mu\nu}\omega^\pi_{\alpha\sigma}-\omega^\alpha_{\mu\sigma}\omega^\pi_{\alpha\nu}\\ &-\frac{1}{(N+1)^2}\delta^\pi_\nu\Big((N+1)\big(\Gamma_{\mu|\sigma}-\omega_{\mu|\sigma}+\omega^\alpha_{\mu\sigma}(\Gamma_\alpha-\omega_\alpha)\big)+\big(\Gamma_\mu-\omega_\mu\big)\big(\Gamma_\sigma-\omega_\sigma\big)\Big)\\ &+\frac{1}{(N+1)^2}\delta^\pi_\sigma\Big((N+1)\big(\Gamma_{\mu|\nu}-\omega_{\mu|\nu}+\omega^\alpha_{\mu\nu}(\Gamma_\alpha-\omega_\alpha)\big)+\big(\Gamma_\mu-\omega_\mu\big)\big(\Gamma_\nu-\omega_\nu\big)\Big),\end{aligned} \tag{6}$$

and the corresponding $\bar{\mathcal{T}}^\pi_{\mu\nu}$ and $\overline{\mathcal{W}}^\pi_{\mu\nu\sigma}$.

The equality $0=\mathcal{W}^\alpha_{\alpha\nu\sigma}-\overline{\mathcal{W}}^\alpha_{\alpha\nu\sigma}$ is equivalent to the equality

$$\omega_{\nu|\sigma}-\omega_{\sigma|\nu}=\bar{\omega}_{\nu\|\sigma}-\bar{\omega}_{\sigma\|\nu},$$

which proves that the anti-symmetric part of $\omega_{\mu|\nu}$ by μ and ν is an invariant for the mapping f.

The equivalent forms of invariants, $\mathcal{W}^\pi_{\mu\nu\sigma}$ and $\overline{\mathcal{W}}^\pi_{\mu\nu\sigma}$ are

$$\begin{cases}\mathcal{W}^\pi_{\mu\nu\sigma}=R^\pi_{\mu\nu\sigma}-\omega^\pi_{\mu\nu|\sigma}+\omega^\pi_{\mu\sigma|\nu}+\omega^\alpha_{\mu\nu}\omega^\pi_{\alpha\sigma}-\omega^\alpha_{\mu\sigma}\omega^\pi_{\alpha\nu}+\delta^\pi_\nu X_{\mu\sigma}-\delta^\pi_\sigma X_{\mu\nu},\\ \overline{\mathcal{W}}^\pi_{\mu\nu\sigma}=\bar{R}^\pi_{\mu\nu\sigma}-\bar{\omega}^\pi_{\mu\nu\|\sigma}+\bar{\omega}^\pi_{\mu\sigma\|\nu}+\bar{\omega}^\alpha_{\mu\nu}\bar{\omega}^\pi_{\alpha\sigma}-\bar{\omega}^\alpha_{\mu\sigma}\bar{\omega}^\pi_{\alpha\nu}+\delta^\pi_\nu\bar{X}_{\mu\sigma}-\delta^\pi_\sigma\bar{X}_{\mu\nu},\end{cases}$$

where

$$\begin{cases} X_{\mu\nu} = -\frac{1}{N+1}\Big(\Gamma_{\mu|\nu} - \omega_{\mu|\nu} + \omega^{\alpha}_{\mu\nu}(\Gamma_{\alpha} - \omega_{\alpha})\Big) - \frac{1}{(N+1)^2}(\Gamma_{\mu} - \omega_{\mu})(\Gamma_{\nu} - \omega_{\nu}), \\ \bar{X}_{\mu\nu} = -\frac{1}{N+1}\Big(\bar{\Gamma}_{\mu\|\nu} - \bar{\omega}_{\mu\|\nu} + \bar{\omega}^{\alpha}_{\mu\nu}(\bar{\Gamma}_{\alpha} - \bar{\omega}_{\alpha})\Big) - \frac{1}{(N+1)^2}(\bar{\Gamma}_{\mu} - \bar{\omega}_{\mu})(\bar{\Gamma}_{\nu} - \bar{\omega}_{\nu}). \end{cases}$$

The invariance $\mathcal{W}^{\pi}_{\mu\nu\sigma} = \overline{\mathcal{W}}^{\pi}_{\mu\nu\sigma}$ is equivalent to the equality $0 = \mathcal{W}^{\pi}_{\mu\nu\sigma} - \overline{\mathcal{W}}^{\pi}_{\mu\nu\sigma}$, i.e.,

$$\begin{aligned} 0 = {} & (R^{\pi}_{\mu\nu\sigma} - \bar{R}^{\pi}_{\mu\nu\sigma}) - (\omega^{\pi}_{\mu\nu|\sigma} - \bar{\omega}^{\pi}_{\mu\nu\|\sigma}) + (\omega^{\pi}_{\mu\sigma|\nu} - \bar{\omega}^{\pi}_{\mu\sigma\|\nu}) + (\omega^{\alpha}_{\mu\nu}\omega^{\pi}_{\alpha\sigma} - \bar{\omega}^{\alpha}_{\mu\nu}\bar{\omega}^{\pi}_{\alpha\sigma}) \\ & - (\omega^{\alpha}_{\mu\sigma}\omega_{\alpha} - \bar{\omega}^{\alpha}_{\mu\sigma}\bar{\omega}_{\alpha}) + \delta^{\pi}_{\nu}(X_{\mu\sigma} - \bar{X}_{\mu\sigma}) - \delta^{\pi}_{\sigma}(X_{\mu\nu} - \bar{X}_{\mu\nu}). \end{aligned} \tag{7}$$

After contracting (7) by π and ν, one obtains

$$\begin{aligned} X_{\mu\sigma} - \bar{X}_{\mu\sigma} = {} & -\frac{1}{N-1}R_{\mu\sigma} + \frac{1}{N-1}(\omega_{\mu|\sigma} - \omega^{\alpha}_{\mu\sigma|\alpha} - \omega^{\alpha}_{\mu\beta}\omega^{\beta}_{\sigma\alpha} + \omega^{\alpha}_{\mu\sigma}\omega_{\alpha}) \\ & + \frac{1}{N-1}\bar{R}_{\mu\sigma} - \frac{1}{N-1}(\bar{\omega}_{\mu\|\sigma} - \bar{\omega}^{\alpha}_{\mu\sigma\|\alpha} - \bar{\omega}^{\alpha}_{\mu\beta}\bar{\omega}^{\beta}_{\sigma\alpha} + \bar{\omega}^{\alpha}_{\mu\sigma}\bar{\omega}_{\alpha}). \end{aligned} \tag{8}$$

If we substitute (8) into (7), we will obtain the relation $0 = W^{\pi}_{\mu\nu\sigma} - \bar{W}^{\pi}_{\mu\nu\sigma}$, for

$$\begin{aligned} W^{\pi}_{\mu\nu\sigma} = {} & R^{\pi}_{\mu\nu\sigma} - \frac{1}{N-1}(\delta^{\pi}_{\nu}R_{\mu\sigma} - \delta^{\pi}_{\sigma}R_{\mu\nu}) - \omega^{\pi}_{\mu\nu|\sigma} + \omega^{\pi}_{\mu\sigma|\nu} + \omega^{\alpha}_{\mu\nu}\omega^{\pi}_{\alpha\sigma} - \omega^{\alpha}_{\mu\sigma}\omega^{\pi}_{\alpha\nu} \\ & + \frac{1}{N-1}\delta^{\pi}_{\nu}(\omega_{\mu|\sigma} - \omega^{\alpha}_{\mu\sigma|\alpha} - \omega^{\alpha}_{\mu\beta}\omega^{\beta}_{\sigma\alpha} + \omega^{\alpha}_{\mu\sigma}\omega_{\alpha}) \\ & + \frac{1}{N-1}\delta^{\pi}_{\sigma}(\omega_{\mu|\nu} - \omega^{\alpha}_{\mu\nu|\alpha} - \omega^{\alpha}_{\mu\beta}\omega^{\beta}_{\nu\alpha} + \omega^{\alpha}_{\mu\nu}\omega_{\alpha}), \\ \bar{W}^{\pi}_{\mu\nu\sigma} = {} & \bar{R}^{\pi}_{\mu\nu\sigma} - \frac{1}{N-1}(\delta^{\pi}_{\nu}\bar{R}_{\mu\sigma} - \delta^{\pi}_{\sigma}\bar{R}_{\mu\nu}) - \bar{\omega}^{\pi}_{\mu\nu\|\sigma} + \bar{\omega}^{\pi}_{\mu\sigma\|\nu} + \bar{\omega}^{\alpha}_{\mu\nu}\bar{\omega}^{\pi}_{\alpha\sigma} - \bar{\omega}^{\alpha}_{\mu\sigma}\bar{\omega}^{\pi}_{\alpha\nu} \\ & + \frac{1}{N-1}\delta^{\pi}_{\nu}(\bar{\omega}_{\mu\|\sigma} - \bar{\omega}^{\alpha}_{\mu\sigma\|\alpha} - \bar{\omega}^{\alpha}_{\mu\beta}\bar{\omega}^{\beta}_{\sigma\alpha} + \bar{\omega}^{\alpha}_{\mu\sigma}\bar{\omega}_{\alpha}) \\ & + \frac{1}{N-1}\delta^{\pi}_{\sigma}(\bar{\omega}_{\mu\|\nu} - \bar{\omega}^{\alpha}_{\mu\nu\|\alpha} - \bar{\omega}^{\alpha}_{\mu\beta}\bar{\omega}^{\beta}_{\nu\alpha} + \bar{\omega}^{\alpha}_{\mu\nu}\bar{\omega}_{\alpha}). \end{aligned} \tag{9}$$

The traces of invariant $W^{\pi}_{\mu\nu\sigma}$ are

$$W^{\alpha}_{\alpha\nu\sigma} = -\frac{N-2}{N-1}(\omega_{\nu|\sigma} - \omega_{\sigma|\nu}), \quad W^{\alpha}_{\mu\alpha\sigma} = -W^{\alpha}_{\mu\sigma\alpha} = 0.$$

The next theorem was proven above.

Theorem 1. *Let $f : \bar{\mathbb{R}}_N \to \mathbb{R}_N$ be a mapping between Riemannian spaces $\bar{\mathbb{R}}_N$ and $\mathbb{R}_N$. The geometrical objects $\mathcal{T}^{\pi}_{\mu\nu}$ and $\mathcal{W}^{\pi}_{\mu\nu\sigma}$ given by* (5) *and* (6) *are the basic invariants of the Thomas and Weyl type for the mapping f. The geometrical object $W^{\pi}_{\mu\nu\sigma}$ given by* (9) *is the derived invariant of the Weyl type for the mapping f. The geometrical object $\omega_{\mu|\nu} - \omega_{\nu|\mu}$ is an invariant for the mapping f.*

2.1. Review of Geodesic and Conformal Mappings

In this section, we will review the necessary results presented in [2,9] about geodesic and conformal mappings and their preferment with respect to the methodology for obtaining invariants of geometric mappings presented in [21], and its form, presented in [15], necessary for this research.

A mapping $f : \bar{\mathbb{R}}_N \to \mathbb{R}_N$ in which any geodesic line of the space $\bar{\mathbb{R}}_N$ transforms to a geodesic line of the space $\mathbb{R}_N$ is geodesic. The basic equation of mapping f is

$$\Gamma^{\pi}_{\mu\nu} = \bar{\Gamma}^{\pi}_{\mu\nu} + \psi_{\nu}\delta^{\pi}_{\mu} + \psi_{\mu}\delta^{\pi}_{\nu},$$

In comparison with (4), we conclude that $\omega^{\pi}_{\mu\nu} = 0$ and $\bar{\omega}^{\pi}_{\mu\nu} = 0$. That means that for geodesic mappings, the equalities $\omega_{\mu} = 0$ and $\bar{\omega}_{\mu} = 0$ are satisfied. Hence, the invariants for geodesic mapping f are

$$T^{\pi}_{\mu\nu} = \Gamma^{\pi}_{\mu\nu} - \frac{1}{N+1}\big(\delta^{\pi}_{\mu}\Gamma_{\nu} + \delta^{\pi}_{\nu}\Gamma_{\mu}\big), \tag{10}$$

$$\mathcal{W}^{\pi}_{\mu\nu\sigma} = R^{\pi}_{\mu\nu\sigma} - \frac{1}{(N+1)^2}\delta^{\pi}_{\nu}\big(\Gamma_{\mu|\sigma} + \Gamma_{\mu}\Gamma_{\sigma}\big) + \frac{1}{(N+1)^2}\delta^{\pi}_{\sigma}\big(\Gamma_{\mu|\nu} + \Gamma_{\mu}\Gamma_{\nu}\big), \tag{11}$$

$$W^{\pi}_{\mu\nu\sigma} = R^{\pi}_{\mu\nu\sigma} - \frac{1}{N-1}\big(\delta^{\pi}_{\nu}R_{\mu\sigma} - \delta^{\pi}_{\sigma}R_{\mu\nu}\big). \tag{12}$$

For the Weyl projective tensor $W^{\pi}_{\mu\nu\sigma}$, the next identities are satisfied $W^{\alpha}_{\alpha\nu\sigma} \equiv 0$, $W^{\alpha}_{\mu\alpha\sigma} \equiv 0$, $W^{\alpha}_{\mu\nu\alpha} \equiv 0$, but $\mathcal{W}^{\alpha}_{\alpha\nu\sigma} \equiv 0$, $\mathcal{W}^{\alpha}_{\mu\alpha\sigma} = -\mathcal{W}^{\alpha}_{\mu\sigma\alpha} = R_{\mu\sigma} - \dfrac{N-1}{(N+1)^2}\big(\Gamma_{\mu|\sigma} + \Gamma_{\mu}\Gamma_{\sigma}\big) \not\equiv 0$.

The basic equation of conformal mapping $f : \bar{\mathbb{R}}_N \to \mathbb{R}_N$ is

$$\Gamma^{\pi}_{\mu\nu} = \bar{\Gamma}^{\pi}_{\mu\nu} + \psi_{\nu}\delta^{\pi}_{\mu} + \psi_{\mu}\delta^{\pi}_{\nu} - g_{\mu\nu}g^{\pi\alpha}\psi_{\alpha}. \tag{13}$$

After contracting (13) by π and ν, we obtain

$$\psi_{\mu} = \frac{1}{N}\Gamma_{\mu} - \frac{1}{N}\bar{\Gamma}_{\mu}. \tag{14}$$

Substituting (14) into the basic Equation (13), one can transform it to the form

$$\Gamma^{\pi}_{\mu\nu} = \bar{\Gamma}^{\pi}_{\mu\nu} + \frac{1}{N}\big(\delta^{\pi}_{\mu}\Gamma_{\nu} + \delta^{\pi}_{\nu}\Gamma_{\mu} - g_{\mu\nu}g^{\pi\alpha}\Gamma_{\alpha}\big) - \frac{1}{N}\big(\delta^{\pi}_{\mu}\bar{\Gamma}_{\nu} + \delta^{\pi}_{\nu}\bar{\Gamma}_{\mu} - \bar{g}_{\mu\nu}\bar{g}^{\pi\alpha}\bar{\Gamma}_{\alpha}\big).$$

This relation is equivalent to

$$\Gamma^{\pi}_{\mu\nu} = \bar{\Gamma}^{\pi}_{\mu\nu} + \delta^{\pi}_{\mu}\psi_{\nu} + \delta^{\pi}_{\nu}\psi_{\mu} - \frac{1}{N}g_{\mu\nu}g^{\pi\alpha}\Gamma_{\alpha} + \frac{1}{N}\bar{g}_{\mu\nu}\bar{g}^{\pi\alpha}\bar{\Gamma}_{\alpha}. \tag{15}$$

It is well known that the mapping $f : \bar{\mathbb{R}}_N \to \mathbb{R}_N$ is conformal if and only if the Christoffel symbols $\bar{\Gamma}^{\pi}_{\mu\nu}$ and $\Gamma^{\pi}_{\mu\nu}$ satisfy (13). It is not hard to prove that the relation (13) is identically satisfied for $\psi_{\mu} = \dfrac{1}{N}\Gamma_{\mu} - \dfrac{1}{N}\bar{\Gamma}_{\mu}$. Moreover, (15) is equivalent to (2) for $\omega^{\pi}_{\mu\nu} = -\dfrac{1}{N}g_{\mu\nu}g^{\pi\alpha}\Gamma_{\alpha}$ and $\bar{\omega}^{\pi}_{\mu\nu} = -\dfrac{1}{N}\bar{g}_{\mu\nu}\bar{g}^{\pi\alpha}\bar{\Gamma}_{\alpha}$.

Hence, the corresponding basic invariants of the Thomas and Weyl type, and the derived invariant of the Weyl type for conformal mapping, f are

$$\begin{aligned}
\mathcal{T}^{\pi}_{\mu\nu} &= \Gamma^{\pi}_{\mu\nu} - \frac{1}{N}\delta^{\pi}_{\mu}\Gamma_{\nu} + \frac{1}{N}\delta^{\pi}_{\nu}\Gamma_{\mu} + \frac{1}{N}g_{\mu\nu}g^{\pi\alpha}\Gamma_{\alpha},\\
\mathcal{W}^{\pi}_{\mu\nu\sigma} &= R^{\pi}_{\mu\nu\sigma} + \frac{1}{N^2}g_{\mu\nu}g^{\pi\alpha}\big(N\Gamma_{\alpha|\sigma} + \Gamma_{\alpha}\Gamma_{\sigma}\big) - \frac{1}{N^2}g_{\mu\sigma}g^{\pi\alpha}\big(N\Gamma_{\alpha|\nu} + \Gamma_{\alpha}\Gamma_{\nu}\big)\\
&\quad - \frac{1}{N^2}\delta^{\pi}_{\nu}\big(N\Gamma_{\mu|\sigma} + \Gamma_{\mu}\Gamma_{\sigma} - g_{\mu\sigma}g^{\alpha\gamma}\Gamma_{\alpha}\Gamma_{\gamma}\big)\\
&\quad + \frac{1}{N^2}\delta^{\pi}_{\sigma}\big(N\Gamma_{\mu|\nu} + \Gamma_{\mu}\Gamma_{\nu} - g_{\mu\nu}g^{\alpha\gamma}\Gamma_{\alpha}\Gamma_{\gamma}\big),
\end{aligned}$$

$$\begin{aligned}
W^{\pi}_{\mu\nu\sigma} &= R^{\pi}_{\mu\nu\sigma} - \frac{1}{N-1}\big(\delta^{\pi}_{\nu}R_{\mu\sigma} - \delta^{\pi}_{\sigma}R_{\mu\nu}\big)\\
&+ \frac{1}{N^2}g_{\mu\nu}g^{\pi\alpha}\big(N\Gamma_{\alpha|\sigma} + \Gamma_{\alpha}\Gamma_{\sigma}\big) - \frac{1}{N^2}g_{\mu\sigma}g^{\pi\alpha}\big(N\Gamma_{\alpha|\nu} + \Gamma_{\alpha}\Gamma_{\nu}\big)\\
&+ \frac{1}{N^2(N-1)}\delta^{\pi}_{\nu}\Big(N\Gamma_{\mu|\sigma} - \Gamma_{\mu}\Gamma_{\sigma} - g_{\mu\sigma}g^{\alpha\gamma}\big(N\Gamma_{\alpha|\gamma} - \Gamma_{\alpha}\Gamma_{\gamma}\big)\Big)\\
&- \frac{1}{N^2(N-1)}\delta^{\pi}_{\sigma}\Big(N\Gamma_{\mu|\nu} - \Gamma_{\mu}\Gamma_{\nu} - g_{\mu\nu}g^{\alpha\gamma}\big(N\Gamma_{\alpha|\gamma} - \Gamma_{\alpha}\Gamma_{\gamma}\big)\Big).
\end{aligned}$$

2.2. What Is Given byVesić's Method [21]

If $f : \bar{\mathbb{R}}_N \to \mathbb{R}_N$ is a mapping between Riemannian spaces $\bar{\mathbb{R}}_N$ and $\mathbb{R}_N$, whose deformation tensor is $P^{\pi}_{\mu\nu} = \Gamma^{\pi}_{\mu\nu} - \bar{\Gamma}^{\pi}_{\mu\nu}$, the transformation rule of the curvature tensor $\bar{R}^{\pi}_{\mu\nu\sigma}$ to $\mathbb{R}^{\pi}_{\mu\nu\sigma}$ of the spaces $\bar{\mathbb{R}}_N$ and $\mathbb{R}_N$ is

$$R^{\pi}_{\mu\nu\sigma} = \bar{R}^{\pi}_{\mu\nu\sigma} + P^{\pi}_{\mu\nu\|\sigma} - P^{\pi}_{\mu\sigma\|\nu} + P^{\alpha}_{\mu\nu}P^{\pi}_{\alpha\sigma} - P^{\alpha}_{\mu\sigma}P^{\pi}_{\alpha\nu}. \tag{16}$$

If the mapping f is geodesic, then it is $P^{\pi}_{\mu\nu} = \psi_{\nu}\delta^{\pi}_{\mu} + \psi_{\mu}\delta^{\pi}_{\nu}$. After substituting this relation into (16), we will obtain [2,9]

$$R^{\pi}_{\mu\nu\sigma} = \bar{R}^{\pi}_{\mu\nu\sigma} + \delta^{\pi}_{\mu}\big(\psi_{\nu\|\sigma} - \psi_{\sigma\|\nu}\big) + \delta^{\pi}_{\nu}\big(\psi_{\mu\|\sigma} - \psi_{\mu}\psi_{\sigma}\big) - \delta^{\pi}_{\sigma}\big(\psi_{\mu\|\nu} - \psi_{\mu}\psi_{\nu}\big). \tag{17}$$

When one contracts (17) by π and μ, and by π and ν, the next relations will be obtained

$$\begin{aligned}
\psi_{\nu\|\sigma} - \psi_{\sigma\|\nu} &= -\frac{1}{N+1}\big(R^{\alpha}_{\alpha\nu\sigma} - \bar{R}^{\alpha}_{\alpha\nu\sigma}\big) = 0,\\
\psi_{\mu\|\sigma} - \psi_{\mu}\psi_{\sigma} &= \frac{1}{N-1}R_{\mu\sigma} - \frac{1}{N-1}\bar{R}_{\mu\sigma}.
\end{aligned}$$

The last two equalities, together with (17), give

$$R^{\pi}_{\mu\nu\sigma} = \bar{R}^{\pi}_{\mu\nu\sigma} + \frac{1}{N-1}\big(\delta^{\pi}_{\nu}R_{\mu\sigma} - \delta^{\pi}_{\sigma}R_{\mu\nu}\big) - \frac{1}{N-1}\big(\delta^{\pi}_{\nu}\bar{R}_{\mu\sigma} - \delta^{\pi}_{\sigma}R_{\mu\nu}\big).$$

From the last equality, the invariance of Weyl projective tensor $W^{\pi}_{\mu\nu\sigma}$ given by (12) is confirmed. Because $W^{\alpha}_{\alpha\nu\sigma} = 0$, $W^{\alpha}_{\mu\alpha\sigma} = 0$, and $W^{\alpha}_{\mu\nu\alpha} = 0$, we are not able to obtain an another invariant for the mapping f which contains $R^{\pi}_{\mu\nu\sigma}$ as a variable of a monic polynomial.

Vesić's approach [21] gives the significance. Namely, if $\Gamma^{\pi}_{\mu\nu} = \bar{\Gamma}^{\pi}_{\mu\nu} + d^{\pi}_{\mu\nu} - \bar{d}^{\pi}_{\mu\nu}$, for tensors $d^{\pi}_{\mu\nu}$ and $\bar{d}^{\pi}_{\mu\nu}$ of the type $(1,2)$, and symmetric by μ and ν, it was directly concluded that $\mathcal{T}^{\pi}_{\mu\nu} = \bar{\mathcal{T}}^{\pi}_{\mu\nu}$ for $\mathcal{T}^{\pi}_{\mu\nu} = \Gamma^{\pi}_{\mu\nu} - d^{\pi}_{\mu\nu}$ and the corresponding $\bar{\mathcal{T}}^{\pi}_{\mu\nu}$.

In the next step, the following relation was analyzed:

$$\bar{\mathcal{W}}^{\pi}_{\mu\nu\sigma} := \bar{\mathcal{T}}^{\pi}_{\mu\nu,\sigma} - \bar{\mathcal{T}}^{\pi}_{\mu\sigma,\nu} + \bar{\mathcal{T}}^{\alpha}_{\mu\nu}\bar{\mathcal{T}}^{\pi}_{\alpha\sigma} - \bar{\mathcal{T}}^{\alpha}_{\mu\sigma}\bar{\mathcal{T}}^{\pi}_{\alpha\nu} = \mathcal{T}^{\pi}_{\mu\nu,\sigma} - \mathcal{T}^{\pi}_{\mu\sigma,\nu} + \mathcal{T}^{\alpha}_{\mu\nu}\mathcal{T}^{\pi}_{\alpha\sigma} - \mathcal{T}^{\alpha}_{\mu\sigma}\mathcal{T}^{\pi}_{\alpha\nu} =: \mathcal{W}^{\pi}_{\mu\nu\sigma}.$$

The last equality is equivalent to $\bar{\mathcal{W}}^{\pi}_{\mu\nu\sigma} = \mathcal{W}^{\pi}_{\mu\nu\sigma}$, where $\mathcal{W}^{\pi}_{\mu\nu\sigma} = R^{\pi}_{\mu\nu\sigma} - d^{\pi}_{\mu\nu|\sigma} + d^{\pi}_{\mu\sigma|\nu} + d^{\alpha}_{\mu\nu}d^{\pi}_{\alpha\sigma} - d^{\alpha}_{\mu\sigma}d^{\pi}_{\alpha\nu}$, and the corresponding $\bar{\mathcal{W}}^{\pi}_{\mu\nu\sigma}$.

In the case of geodesic mapping f, the tensor $d^{\pi}_{\mu\nu}$ is $d^{\pi}_{\mu\nu} = \dfrac{1}{N+1}\big(\delta^{\pi}_{\mu}\Gamma_{\nu} + \delta^{\pi}_{\nu}\Gamma_{\mu}\big)$. In the case of a conformal mapping, the tensor $d^{\pi}_{\mu\nu}$ is $d^{\pi}_{\mu\nu} = \dfrac{1}{N}\big(\delta^{\pi}_{\mu}\Gamma_{\nu} + \delta^{\pi}_{\nu}\Gamma_{\mu}\big)$.

The corresponding basic invariants $\mathcal{W}^{\pi}_{\mu\nu\sigma}$ are not trace-free, nor is the Ricci-tensor vanished in their trace. The same holds for any of the above obtained basic invariants. That will help us to create an R^1-cosmological model as an example in this manuscript.

Furthermore, the invariance $\bar{\mathcal{W}}^{\pi}_{\mu\nu\sigma} = \mathcal{W}^{\pi}_{\mu\nu\sigma}$ gives us the chance to obtain an another invariant for analyzed mapping as a monic polynomial of curvature tensor $R^{\pi}_{\mu\nu\sigma}$. The Weyl projective and Weyl conformal tensors are examples of these other invariants.

Hence, by the methodology presented by H. Weyl, we are able to obtain only trace-free invariants for mappings. By Vesić's methodology, we obtained a novel invariant which makes it possible for linear cosmological models to be created from these invariants. Both of these invariants will present some significant magnitudes in physics, but we will not talk about these details in this paper.

3. Generalized Concept of Conformal Mappings

In this section, we are ready to define a curve of new kind in space $\mathbb{R}_N$. An m-conformal line of space $\mathbb{R}_N$ is a curve $\ell = \ell(t)$ whose tangential vector (λ^μ) satisfies the system of differential equations

$$\frac{d\lambda^\mu}{dt} + \Gamma^\mu_{\alpha\beta}\lambda^\alpha\lambda^\beta = \rho\lambda^\mu + m g_{\alpha\beta} g^{\mu\gamma}\Gamma_\gamma\lambda^\alpha\lambda^\beta,$$

where ρ and m are scalar functions. The function m is the *conformality coefficient*.

A 0-conformal line of space $\mathbb{R}_N$ is a geodesic line of this space.

A $(-\frac{1}{N}, -\frac{1}{N})$-conformal mapping $f : \bar{\mathbb{R}}_N \to \mathbb{R}_N$ is the conformal mapping of space $\bar{\mathbb{R}}_N$.

A $(0,0)$-conformal mapping $f : \bar{\mathbb{R}}_N \to \mathbb{R}_N$ is geodesic mapping.

3.1. Invariants for $(\bar{m}, m)$-Conformal Mappings

A mapping $f : \bar{\mathbb{R}}_N \to \mathbb{R}_N$ in which any $\bar{m}$-conformal line of space $\bar{\mathbb{R}}_N$ transforms to an m-conformal line of space $\mathbb{R}_N$ is the $(\bar{m}, m)$-conformal mapping.

The $(0, m)$-conformal mapping $f : \bar{\mathbb{R}}_N \to \mathbb{R}_N$ transmits any geodesic line of space $\bar{\mathbb{R}}_N$ to an m-conformal line of space $\mathbb{R}_N$. The $(\bar{m}, 0)$-conformal mapping $f : \bar{\mathbb{R}}_N \to \mathbb{R}_N$ transforms any $\bar{m}$-conformal line of space $\bar{\mathbb{R}}_N$ to a geodesic line of the space $\mathbb{R}_N$.

Let us consider an $\bar{m}$-conformal line of the space $\bar{\mathbb{R}}_N$ and an m-conformal mapping of the space $\mathbb{R}_N$,

$$\begin{cases} \dfrac{d\lambda^\mu}{dt} + \bar{\Gamma}^\mu_{\alpha\beta}\lambda^\alpha\lambda^\beta = \bar{\rho}\lambda^\mu + \bar{m}\bar{g}_{\alpha\beta}\bar{g}^{\mu\gamma}\bar{\Gamma}_\gamma\lambda^\alpha\lambda^\beta, \\ \dfrac{d\lambda^\mu}{dt} + \Gamma^\mu_{\alpha\beta}\lambda^\alpha\lambda^\beta = \rho\lambda^\mu + m g_{\alpha\beta} g^{\mu\gamma}\Gamma_\gamma\lambda^\alpha\lambda^\beta. \end{cases} \tag{18}$$

Based on the expressions $\rho = \rho_\alpha\lambda^\alpha$, $\bar{\rho} = \bar{\rho}_\alpha\lambda^\alpha$, $\lambda^\mu = \delta^\mu_\alpha\lambda^\alpha$, from system (18), one obtains

$$\Big((\Gamma^\mu_{\alpha\beta} - \bar{\Gamma}^\mu_{\alpha\beta}) - \frac{1}{2}(\rho_\alpha\delta^\mu_\beta + \rho_\beta\delta^\mu_\alpha) + \frac{1}{2}(\bar{\rho}_\alpha\delta^\mu_\beta + \bar{\rho}_\beta\delta^\mu_\alpha) - \big(m g_{\alpha\beta} g^{\mu\gamma}\Gamma_\gamma - \bar{m}\bar{g}_{\alpha\beta}\bar{g}^{\mu\gamma}\bar{\Gamma}_\gamma\big)\Big)\lambda^\alpha\lambda^\beta = 0.$$

The next lemma was proven above.

Lemma 1. *A mapping $f : \bar{\mathbb{R}}_N \to \mathbb{R}_N$ is an $(\bar{m}, m)$-conformal one if and only if*

$$\Gamma^\pi_{\mu\nu} = \bar{\Gamma}^\pi_{\mu\nu} + \delta^\pi_\mu\psi_\nu + \delta^\pi_\nu\psi_\mu + m g_{\mu\nu} g^{\pi\alpha}\Gamma_\alpha - \bar{m}\bar{g}_{\mu\nu}\bar{g}^{\pi\alpha}\bar{\Gamma}_\alpha, \tag{19}$$

where m is a scalar function and ψ_μ is a 1-form.

3.2. Invariants for $(\bar{m}, m)$-Conformal Mappings

Let $f : \bar{\mathbb{R}}_N \to \mathbb{R}_N$ be an $(\bar{m}, m)$-conformal mapping. After contracting its basic Equation (19) by π and ν, we obtain

$$\psi_\mu = -\frac{m-1}{N+1}\Gamma_\mu - \frac{\bar{m}-1}{N+1}\bar{\Gamma}_\mu. \tag{20}$$

After substituting (20) into the basic Equation (19), we obtain

$$\begin{aligned}\Gamma^{\pi}_{\mu\nu} = \bar{\Gamma}^{\pi}_{\mu\nu} - \frac{m-1}{N+1}\big(\delta^{\pi}_{\mu}\Gamma_{\nu} + \delta^{\pi}_{\nu}\Gamma_{\mu}\big) + m g_{\mu\nu} g^{\pi\alpha}\Gamma_{\alpha} \\ + \frac{\bar{m}-1}{N+1}\big(\delta^{\pi}_{\mu}\bar{\Gamma}_{\nu} + \delta^{\pi}_{\nu}\bar{\Gamma}_{\mu}\big) - \bar{m}\bar{g}_{\mu\nu}\bar{g}^{\pi\alpha}\bar{\Gamma}_{\alpha}.\end{aligned} \tag{21}$$

(21) is equivalent to the equality $\mathcal{T}^{\pi}_{\mu\nu} = \bar{\mathcal{T}}^{\pi}_{\mu\nu}$, for

$$\mathcal{T}^{\pi}_{\mu\nu} = \Gamma^{\pi}_{\mu\nu} - m g_{\mu\nu} g^{\pi\alpha}\Gamma_{\alpha} + \frac{m-1}{N+1}\big(\delta^{\pi}_{\mu}\Gamma_{\nu} + \delta^{\pi}_{\nu}\Gamma_{\mu}\big), \tag{22}$$

and the corresponding $\bar{\mathcal{T}}^{\pi}_{\mu\nu}$.

From the basic Equation (21) of $(\bar{m}, m)$-mapping $f : \bar{\mathbb{R}}_N \to \mathbb{R}_N$, one obtains

$$\begin{cases} d^{\pi}_{\mu\nu} = -\dfrac{m-1}{N+1}\big(\delta^{\pi}_{\mu}\Gamma_{\nu} + \delta^{\pi}_{\nu}\Gamma_{\mu}\big) + m g_{\mu\nu} g^{\pi\alpha}\Gamma_{\alpha}, \\ \bar{d}^{\pi}_{\mu\nu} = -\dfrac{\bar{m}-1}{N+1}\big(\delta^{\pi}_{\mu}\bar{\Gamma}_{\nu} + \delta^{\pi}_{\nu}\bar{\Gamma}_{\mu}\big) + \bar{m}\bar{g}_{\mu\nu}\bar{g}^{\pi\alpha}\bar{\Gamma}_{\alpha}. \end{cases} \tag{23}$$

Remark 2. *From (23) we have $\omega^{\pi}_{\mu\nu} = g_{\mu\nu} g^{\pi\alpha}\Gamma_{\alpha}$ and $\bar{\omega}^{\pi}_{\mu\nu} = \bar{m}\bar{g}_{\mu\nu}\bar{g}^{\pi\alpha}\bar{\Gamma}_{\alpha}$, but it will be more simple to use the equality $\mathcal{W}^{\pi}_{\mu\nu\sigma} = R^{\pi}_{\mu\nu\sigma} - d^{\pi}_{\mu\nu|\sigma} + d^{\pi}_{\mu\sigma|\nu} + d^{\alpha}_{\mu\nu}d^{\pi}_{\alpha\sigma} - d^{\alpha}_{\mu\sigma}d^{\pi}_{\alpha\nu}$.*

For $d^{\pi}_{\mu\nu}$ given by (23), and with respect to $\Gamma_{\mu|\nu} = \Gamma_{\nu|\mu}$, one obtains

$$\begin{aligned} -d^{\pi}_{\mu\nu|\sigma} + d^{\pi}_{\mu\sigma|\nu} &= \frac{1}{N+1}\delta^{\pi}_{\mu}\big(\Gamma_{\nu}m_{\sigma} - \Gamma_{\sigma}m_{\nu}\big) + \frac{1}{N+1}\delta^{\pi}_{\nu}\big(\Gamma_{\mu}m_{\sigma} + (m-1)\Gamma_{\mu|\sigma}\big) \\ &\quad - \frac{1}{N+1}\delta^{\pi}_{\sigma}\big(\Gamma_{\mu}m_{\nu} + (m-1)\Gamma_{\mu|\nu}\big) - g^{\pi\alpha}\Gamma_{\alpha}\big(g_{\mu\nu}m_{\sigma} - g_{\mu\sigma}m_{\nu}\big) \\ &\quad - m g^{\pi\alpha}\big(g_{\mu\nu}\Gamma_{\alpha|\sigma} - g_{\mu\sigma}\Gamma_{\alpha|\nu}\big), \\ d^{\alpha}_{\mu\nu}d^{\pi}_{\alpha\sigma} - d^{\alpha}_{\mu\sigma}d^{\pi}_{\alpha\nu} &= m^2 g^{\pi\alpha}\Gamma_{\alpha}\big(g_{\mu\nu}\Gamma_{\sigma} - g_{\mu\sigma}\Gamma_{\nu}\big) \\ &\quad + \frac{1}{(N+1)^2}\delta^{\pi}_{\nu}\Big((N+1)m(m-1)g_{\mu\sigma}g^{\alpha\gamma}\Gamma_{\alpha}\Gamma_{\gamma} - (m-1)^2\Gamma_{\mu}\Gamma_{\sigma}\Big) \\ &\quad - \frac{1}{(N+1)^2}\delta^{\pi}_{\sigma}\Big((N+1)m(m-1)g_{\mu\nu}g^{\alpha\gamma}\Gamma_{\alpha}\Gamma_{\gamma} - (m-1)^2\Gamma_{\mu}\Gamma_{\nu}\Big), \end{aligned}$$

and the corresponding differences $-\bar{d}^{\pi}_{\mu\nu\|\sigma} + \bar{d}^{\pi}_{\mu\sigma\|\nu}$ and $\bar{d}^{\alpha}_{\mu\nu}\bar{d}^{\pi}_{\alpha\sigma} - \bar{d}^{\alpha}_{\mu\sigma}\bar{d}^{\pi}_{\alpha\nu}$.

Hence, we obtain the following geometrical objects

$$\begin{aligned} \mathcal{W}^{\pi}_{\mu\nu\sigma} &= R^{\pi}_{\mu\nu\sigma} - g^{\pi\alpha}\Gamma_{\alpha}\Big(g_{\mu\nu}m_{\sigma} - g_{\mu\sigma}m_{\nu} - m^2\big(g_{\mu\nu}\Gamma_{\sigma} - g_{\mu\sigma}\Gamma_{\nu}\big)\Big) \\ &\quad - m g^{\pi\alpha}\big(g_{\mu\nu}\Gamma_{\alpha|\sigma} - g_{\mu\sigma}\Gamma_{\alpha|\nu}\big) + \frac{1}{N+1}\delta^{\pi}_{\mu}\big(\Gamma_{\nu}m_{\sigma} - \Gamma_{\sigma}m_{\nu}\big) \\ &\quad + \frac{1}{(N+1)^2}\delta^{\pi}_{\nu}\Big((N+1)(m-1)\big(\Gamma_{\mu|\sigma} + m g_{\mu\sigma}g^{\alpha\gamma}\Gamma_{\alpha}\Gamma_{\gamma}\big) + \Gamma_{\mu}m_{\sigma} - (m-1)^2\Gamma_{\mu}\Gamma_{\sigma}\Big) \\ &\quad - \frac{1}{(N+1)^2}\delta^{\pi}_{\sigma}\Big((N+1)(m-1)\big(\Gamma_{\mu|\nu} + m g_{\mu\nu}g^{\alpha\gamma}\Gamma_{\alpha}\Gamma_{\gamma}\big) + \Gamma_{\mu}m_{\nu} - (m-1)^2\Gamma_{\mu}\Gamma_{\nu}\Big), \end{aligned} \tag{24}$$

$$\begin{aligned} \bar{\mathcal{W}}^{\pi}_{\mu\nu\sigma} &= \bar{R}^{\pi}_{\mu\nu\sigma} - \bar{g}^{\pi\alpha}\bar{\Gamma}_{\alpha}\Big(\bar{g}_{\mu\nu}\bar{m}_{\sigma} - \bar{g}_{\mu\sigma}\bar{m}_{\nu} - \bar{m}^2\big(\bar{g}_{\mu\nu}\bar{\Gamma}_{\sigma} - \bar{g}_{\mu\sigma}\bar{\Gamma}_{\nu}\big)\Big) \\ &\quad - \bar{m}\bar{g}^{\pi\alpha}\big(\bar{g}_{\mu\nu}\bar{\Gamma}_{\alpha\|\sigma} - \bar{g}_{\mu\sigma}\bar{\Gamma}_{\alpha\|\nu}\big) + \frac{1}{N+1}\delta^{\pi}_{\mu}\big(\bar{\Gamma}_{\nu}\bar{m}_{\sigma} - \bar{\Gamma}_{\sigma}\bar{m}_{\nu}\big) \\ &\quad + \frac{1}{(N+1)^2}\delta^{\pi}_{\nu}\Big((N+1)(\bar{m}-1)\big(\bar{\Gamma}_{\mu\|\sigma} + \bar{m}\bar{g}_{\mu\sigma}\bar{g}^{\alpha\gamma}\bar{\Gamma}_{\alpha}\bar{\Gamma}_{\gamma}\big) + \bar{\Gamma}_{\mu}\bar{m}_{\sigma} - (\bar{m}-1)^2\bar{\Gamma}_{\mu}\bar{\Gamma}_{\sigma}\Big) \\ &\quad - \frac{1}{(N+1)^2}\delta^{\pi}_{\sigma}\Big((N+1)(\bar{m}-1)\big(\bar{\Gamma}_{\mu\|\nu} + \bar{m}\bar{g}_{\mu\nu}\bar{g}^{\alpha\gamma}\bar{\Gamma}_{\alpha}\bar{\Gamma}_{\gamma}\big) + \bar{\Gamma}_{\mu}\bar{m}_{\nu} - (\bar{m}-1)^2\bar{\Gamma}_{\mu}\bar{\Gamma}_{\nu}\Big). \end{aligned} \tag{25}$$

The traces of the geometrical objects $\mathcal{W}^{\pi}_{\mu\nu\sigma}$ and $\bar{\mathcal{W}}^{\pi}_{\mu\nu\sigma}$ from Equations (24) and (25) by π and μ are $\mathcal{W}^{\alpha}_{\alpha\nu\sigma} = -\dfrac{N}{(N+1)^2}\big(\Gamma_\nu m_\sigma - \Gamma_\sigma m_\nu\big)$ and $\bar{\mathcal{W}}^{\alpha}_{\alpha\nu\sigma} = -\dfrac{N}{(N+1)^2}\big(\bar{\Gamma}_\nu \bar{m}_\sigma - \bar{\Gamma}_\sigma \bar{m}_\nu\big)$. The traces of these geometrical objects by π and ν are

$$\begin{aligned}
\mathcal{W}^{\alpha}_{\mu\alpha\sigma} = {} & R_{\mu\sigma} - \frac{2m+N-1}{N+1}\Gamma_{\mu|\sigma} + m g_{\mu\sigma} g^{\alpha\gamma}\Gamma_{\alpha|\gamma} - \left(\frac{2}{N+1}m + \frac{N-1}{N+1}\right) g_{\mu\sigma} g^{\alpha\gamma}\Gamma_\alpha\Gamma_\gamma \\
& + \left(\frac{N^2+N+2}{(N+1)^2}m^2 + \frac{2N-2}{(N+1)^2}m - \frac{N-1}{(N+1)^2}\right)\Gamma_\mu\Gamma_\sigma + g_{\mu\sigma} g^{\alpha\gamma}\Gamma_\alpha m_\gamma \\
& - \frac{1}{N+1}\Gamma_\sigma m_\gamma - \frac{N^2+1}{(N+1)^2}\Gamma_\mu m_\sigma, \\
\bar{\mathcal{W}}^{\alpha}_{\mu\alpha\sigma} = {} & \bar{R}_{\mu\sigma} - \frac{2\bar{m}+N-1}{N+1}\bar{\Gamma}_{\mu\|\sigma} + \bar{m}\bar{g}_{\mu\sigma} \bar{g}^{\alpha\gamma}\bar{\Gamma}_{\alpha\|\gamma} - \left(\frac{2}{N+1}\bar{m} + \frac{N-1}{N+1}\right) \bar{g}_{\mu\sigma} \bar{g}^{\alpha\gamma}\bar{\Gamma}_\alpha\bar{\Gamma}_\gamma \\
& + \left(\frac{N^2+N+2}{(N+1)^2}\bar{m}^2 + \frac{2N-2}{(N+1)^2}\bar{m} - \frac{N-1}{(N+1)^2}\right)\bar{\Gamma}_\mu\bar{\Gamma}_\sigma + \bar{g}_{\mu\sigma} \bar{g}^{\alpha\gamma}\bar{\Gamma}_\alpha \bar{m}_\gamma \\
& - \frac{1}{N+1}\bar{\Gamma}_\sigma \bar{m}_\gamma - \frac{N^2+1}{(N+1)^2}\bar{\Gamma}_\mu \bar{m}_\sigma.
\end{aligned}$$

Because $\mathcal{W}^{\pi}_{\mu\nu\sigma} = -\mathcal{W}^{\pi}_{\mu\sigma\nu}$ and $\bar{\mathcal{W}}^{\pi}_{\mu\nu\sigma} = -\bar{\mathcal{W}}^{\pi}_{\mu\sigma\nu}$, the traces of $\mathcal{W}^{\pi}_{\mu\nu\sigma}$ and $\bar{\mathcal{W}}^{\pi}_{\mu\nu\sigma}$ by π and σ are $\mathcal{W}^{\alpha}_{\mu\nu\alpha} = -\mathcal{W}^{\alpha}_{\mu\alpha\nu}$ and $\bar{\mathcal{W}}^{\alpha}_{\mu\nu\alpha} = -\bar{\mathcal{W}}^{\alpha}_{\mu\alpha\nu}$.

After substituting the previously obtained geometrical objects necessary for the derived invariant $W^{\pi}_{\mu\nu\sigma}$, we get

$$\begin{aligned}
W^{\pi}_{\mu\nu\sigma} = {} & R^{\pi}_{\mu\nu\sigma} - \frac{1}{N-1}\big(\delta^{\pi}_{\nu} R_{\mu\sigma} - \delta^{\pi}_{\sigma} R_{\mu\nu}\big) - g^{\pi\alpha}\Gamma_\alpha\big(g_{\mu\nu} m_\sigma - g_{\mu\sigma} m_\nu\big) \\
& - m g^{\pi\alpha}\big(g_{\mu\nu}\Gamma_{\alpha|\sigma} - g_{\mu\sigma}\Gamma_{\alpha|\nu}\big) + m^2 g^{\pi\alpha}\Gamma_\alpha\big(g_{\mu\nu}\Gamma_\sigma - g_{\mu\sigma}\Gamma_\nu\big) \\
& + \frac{1}{N-1}\delta^{\pi}_{\nu}\Big(m\Gamma_{\mu|\sigma} + \Gamma_\mu m_\sigma - m^2\Gamma_\mu\Gamma_\sigma - g_{\mu\sigma} g^{\alpha\gamma}\big(m\Gamma_{\alpha|\gamma} + \Gamma_\alpha m_\gamma - m^2\Gamma_\alpha\Gamma_\gamma\big)\Big) \\
& - \frac{1}{N-1}\delta^{\pi}_{\sigma}\Big(m\Gamma_{\mu|\nu} + \Gamma_\mu m_\nu - m^2\Gamma_\mu\Gamma_\nu - g_{\mu\nu} g^{\alpha\gamma}\big(m\Gamma_{\alpha|\gamma} + \Gamma_\alpha m_\gamma - m^2\Gamma_\alpha\Gamma_\gamma\big)\Big),
\end{aligned} \tag{26}$$

and the corresponding $\bar{W}^{\pi}_{\mu\nu\sigma}$.

The traces $W_{\nu\sigma} = W^{\alpha}_{\alpha\nu\sigma}$ and $W_{\mu\sigma} = W^{\alpha}_{\mu\alpha\sigma}$ of the invariant $W^{\pi}_{\mu\nu\sigma}$ given by (26) are $W_{\nu\sigma} = -\dfrac{N-2}{N-1}\big(\Gamma_\nu m_\sigma - \Gamma_\sigma m_\nu\big)$ and $W_{\mu\sigma} = 0$.

The next theorem was proven above.

Theorem 2. *Let $f : \bar{\mathbb{R}}_N \to \mathbb{R}_N$ be an $(\bar{m}, m)$-mapping. The geometrical object $\mathcal{T}^{\pi}_{\mu\nu}$ given by (22) is the basic invariant for mapping f of the Thomas type. The geometrical object $\mathcal{W}^{\pi}_{\mu\nu\sigma}$ given by (24) is the basic invariant for mapping f of the Weyl type. The geometrical object $W^{\pi}_{\mu\nu\sigma}$ given by (26) is the derived invariant for mapping f of the Weyl type.*

3.3. Lagrangian Caused by $(\bar{m}, m)$-Conformal Mappings

In [23], the Lagrangian is taken to be

$$\mathcal{L}_N = \sqrt{-g}\, g^{\mu\nu}\left(G^{\alpha}_{\mu\nu,\alpha} + \frac{1}{N-1}G^{\alpha}_{\alpha\mu}G^{\beta}_{\beta\nu} - G^{\alpha}_{\beta\mu}G^{\beta}_{\alpha\nu}\right),$$

for $G^{\pi}_{\mu\nu} = \Gamma^{\pi}_{\mu\nu} - \dfrac{1}{2}\big(\delta^{\pi}_{\mu}\Gamma_\nu + \delta^{\pi}_{\nu}\Gamma_\mu\big)$.

The value $G^{\pi}_{\mu\nu}$ is similar to the Thomas projective parameter (10). As we mentioned above, the basic invariant $\mathcal{W}_{\mu\sigma} = \mathcal{W}^{\alpha}_{\mu\alpha\sigma}$ for the geodesic mapping given by (11) reduces to the Ricci tensor $R_{\mu\sigma}$.

We obtained the basic invariant $\mathcal{W}^{\pi}_{\mu\nu\sigma}$ for an $(\bar{m}, m)$-conformal mapping. In this example, we will study the transformation of $\bar{\Gamma}^{\pi}_{\mu\nu}$ to $\Gamma^{\pi}_{\mu\nu}$ such that any $\bar{m}$-conformal line of an N-dimensional Riemannian space $\bar{\mathbb{R}}_N$ equipped with an FLRW-metric whose square form is $ds^2 = -dt^2 + a^2\left(dx^{1^2} + \ldots + dx^{N-1^2}\right)$, where $a = a(t)$ is the scale factor, transforms to an m-conformal line of the perturbed space.

To simplify our computing, we will start from the basic equation of the form $\Gamma^{\pi}_{\mu\nu} = \bar{\Gamma}^{\pi}_{\mu\nu} + d^{\pi}_{\mu\nu} - \bar{d}^{\pi}_{\mu\nu}$, where

$$\begin{cases} d^{\pi}_{\mu\nu} = -\dfrac{m-1}{N+1}\left(\delta^{\pi}_{\mu}\Gamma_{\nu} + \delta^{\pi}_{\nu}\Gamma_{\mu}\right) + m g_{\mu\nu} g^{\pi\alpha}\Gamma_{\alpha}, \\ \bar{d}^{\pi}_{\mu\nu} = -\dfrac{\bar{m}-1}{N+1}\left(\delta^{\pi}_{\mu}\bar{\Gamma}_{\nu} + \delta^{\pi}_{\nu}\bar{\Gamma}_{\mu}\right) + \bar{m}\bar{g}_{\mu\nu}\bar{g}^{\pi\alpha}\bar{\Gamma}_{\alpha}. \end{cases}$$

is given by (23).

The corresponding basic invariant of the Weyl type in the perturbed space is

$$\mathcal{W}_{\mu\sigma} = \mathcal{W}^{\alpha}_{\mu\alpha\sigma} = R_{\mu\sigma} - d_{\mu|\sigma} + d^{\alpha}_{\mu\sigma|\alpha} + d^{\alpha}_{\mu\beta}d^{\beta}_{\sigma\alpha} - d^{\alpha}_{\mu\sigma}d_{\alpha}.$$

The corresponding Einstein–Hilbert action is

$$2\kappa S^{+}_{0} = \int d^N x\sqrt{-g}g^{\alpha\beta}\mathcal{W}_{\alpha\beta},$$

i.e.,

$$\begin{aligned} 2\kappa S^{+}_{0} = &\int d^N x\sqrt{-g}R - \int d^N x\sqrt{-g}g^{\alpha\beta}d_{\alpha|\beta} + \int d^N x\sqrt{-g}g^{\alpha\beta}d^{\gamma}_{\alpha\beta|\gamma} \\ &+ \int d^N x\sqrt{-g}g^{\alpha\beta}d^{\gamma}_{\alpha\delta}d^{\delta}_{\beta\gamma} - \int d^N x\sqrt{-g}g^{\alpha\beta}d^{\gamma}_{\alpha\beta}d_{\gamma}. \end{aligned} \tag{27}$$

Because $\Gamma_{\mu} = \left(\ln\sqrt{-g}\right)_{,\mu} = \dfrac{\sqrt{-g}_{,\mu}}{\sqrt{-g}}$, we get $\sqrt{-g}_{,\mu} = \sqrt{-g}\Gamma_{\mu}$.

For this reason, if τ^{α} is a tensor of the type $(1,0)$, we obtain

$$\sqrt{-g}\tau^{\alpha}_{|\alpha} = \sqrt{-g}\tau^{\alpha}_{,\alpha} + \sqrt{-g}\Gamma^{\alpha}_{\beta\alpha}\tau^{\beta} = \sqrt{-g}\tau^{\alpha}_{,\alpha} + \sqrt{-g}_{,\beta}\tau^{\beta} = \left(\sqrt{-g}\tau^{\alpha}\right)_{,\alpha}.$$

Because $\delta g^{\mu\nu} = 0$ at the border of integration, and based on the Stokes Theorem, we conclude that the variation $\delta\left(\int d^N x\sqrt{-g}d^{\alpha}_{|\alpha}\right)$ vanishes. For this reason, and because $g^{\alpha\beta}d_{\alpha|\beta} = \left(g^{\alpha\beta}d_{\alpha}\right)_{|\beta}$ and $g^{\alpha\beta}d^{\gamma}_{\alpha\beta|\gamma} = \left(g^{\alpha\beta}d^{\gamma}_{\alpha\beta}\right)_{|\gamma}$, the variations of the second and third integral of (27) are equal to zero.

The variation of the scalar curvature R is

$$\begin{aligned} \delta R &= \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\left(\left(\delta\Gamma^{\alpha}_{\mu\alpha}\right)_{,\nu} - \left(\delta\Gamma^{\alpha}_{\mu\nu}\right)_{,\alpha} + \delta\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta} + \Gamma^{\beta}_{\mu\alpha}\delta\Gamma^{\alpha}_{\nu\beta} - \delta\Gamma^{\alpha}_{\mu\sigma}\Gamma_{\alpha} - \Gamma^{\alpha}_{\mu\sigma}\delta\Gamma_{\alpha}\right) \\ &= \left(\delta\Gamma_{\mu}|\right)_{|\sigma} - \left(\delta\Gamma^{\alpha}_{\mu\sigma}\right)_{|\alpha}. \end{aligned}$$

For this reason, because $\delta\sqrt{-g} = -\dfrac{1}{2}\sqrt{-g}\delta g^{\mu\nu}g_{\mu\nu}$, and with respect to the Stokes Theorem, we obtain

$$\delta\left(\int d^N x\sqrt{-g}R\right) = \int d^N x\sqrt{-g}\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right).$$

Because $\bar{m}$ and m are scalars, the variation $\delta m = m - \bar{m}$ is a scalar too. With respect to the tensor $\delta g^{\mu\nu}$ of the type $(0,2)$, and the quotient rule, there exists a tensor $M_{\mu\nu}$ of the type $(0,2)$ such that $\delta m = M_{\mu\nu}\delta g^{\mu\nu}$.

Because $0 = \delta(\delta^{\alpha}_{\beta}) = \delta(g^{\alpha\gamma}g_{\beta\gamma}) = \delta g^{\alpha\gamma}g_{\beta\gamma} + g^{\alpha\gamma}\delta g_{\beta\gamma}$, we conclude that

$$\delta g_{\beta\delta} = -\delta g^{\alpha\gamma}g_{\alpha\beta}g_{\gamma\delta}.$$

The variation of the vector $\Gamma_\alpha = \Gamma^{\beta}_{\alpha\beta}$ is

$$\begin{aligned}
\delta\Gamma_\alpha &= \frac{1}{2}\delta\left(g^{\beta\delta}(g_{\alpha\beta,\delta} - g_{\alpha\delta,\beta} + g_{\beta\delta,\alpha})\right) = \frac{1}{2}\delta g^{\beta\delta}g_{\beta\delta,\alpha} + \frac{1}{2}g^{\beta\delta}\left(\delta g_{\beta\delta}\right)_{,\alpha} \\
&= \frac{1}{2}\delta g^{\beta\delta}g_{\beta\delta,\alpha} - \frac{1}{2}g^{\epsilon\zeta}\left(\delta g^{\beta\delta}g_{\beta\epsilon}g_{\delta\zeta}\right)_{,\alpha} \\
&= \frac{1}{2}\delta g^{\beta\delta}g_{\beta\delta,\alpha} - \frac{1}{2}\left(\delta g^{\beta\delta}\right)_{,\alpha}g_{\beta\delta} - \frac{1}{2}\delta g^{\beta\delta}g_{\beta\delta,\alpha} - \frac{1}{2}\delta g^{\beta\delta}g_{\beta\delta,\alpha} = -\frac{1}{2}\left(\delta g^{\beta\delta}g_{\beta\delta}\right)_{,\alpha}.
\end{aligned}$$

For $d^{\pi}_{\mu\nu}$ given by (23), the following equalities hold:

$$\begin{aligned}
g^{\alpha\beta}d^{\gamma}_{\alpha\delta}d^{\delta}_{\beta\gamma} &= -\left(\frac{N^2+N-2}{(N+1)^2}m^2 + \frac{2(N^2+N-2)}{(N+1)^2}m + \frac{N+3}{(N+1)^2}\right)g^{\alpha\gamma}\Gamma_\alpha\Gamma_\gamma, \\
g^{\alpha\beta}d^{\gamma}_{\alpha\beta}d_\gamma &= \left(\frac{N^2+N-2}{(N+1)^2}m + \frac{2}{N+1}\right)g^{\alpha\gamma}\Gamma_\alpha\Gamma_\gamma.
\end{aligned}$$

Hence, the difference $I = g^{\alpha\beta}d^{\gamma}_{\alpha\delta}d^{\delta}_{\beta\gamma} - g^{\alpha\beta}d^{\gamma}_{\alpha\beta}d_\gamma$ between the last two integrands in (27) is

$$I = -\left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)g^{\alpha\gamma}\Gamma_\alpha\Gamma_\gamma.$$

The variation of term $\sqrt{-g}I$ is

$$\begin{aligned}
\delta(\sqrt{-g}I) &= \frac{1}{2}\sqrt{-g}\delta g^{\mu\nu}g_{\mu\nu}\left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)g^{\alpha\gamma}\Gamma_\alpha\Gamma_\gamma \\
&- \sqrt{-g}\delta g^{\mu\nu}\left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)\Gamma_\mu\Gamma_\nu \\
&- \frac{N^2+N-2}{(N+1)^2}\sqrt{-g}\delta g^{\mu\nu}M_{\mu\nu}(2m-3)g^{\alpha\gamma}\Gamma_\alpha\Gamma_\gamma \\
&+ \sqrt{-g}\left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)g^{\alpha\gamma}\Gamma_\alpha\left(\delta g^{\mu\nu}g_{\mu\nu}\right)_{,\gamma}
\end{aligned}$$

The following equality is satisfied

$$\begin{aligned}
&\sqrt{-g}\left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)g^{\alpha\gamma}\Gamma_\alpha\left(\delta g^{\mu\nu}g_{\mu\nu}\right)_{,\gamma} \\
&= \left(\sqrt{-g}\left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)g^{\alpha\gamma}\Gamma_\alpha\delta g^{\mu\nu}g_{\mu\nu}\right)_{,\gamma} \\
&- \sqrt{-g}\left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)\delta g^{\mu\nu}g_{\mu\nu}g^{\alpha\gamma}\Gamma_\alpha\Gamma_\gamma \\
&- \sqrt{-g}\frac{N^2+N-2}{(N+1)^2}\delta g^{\mu\nu}g_{\mu\nu}(2m-3)g^{\alpha\gamma}\Gamma_\alpha m_\gamma \\
&- \sqrt{-g}\left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)\delta g^{\mu\nu}g_{\mu\nu}g^{\alpha\gamma}\left(\Gamma_{\alpha|\gamma} - \Gamma_\alpha\Gamma_\gamma\right).
\end{aligned}$$

Finally, the variation of Einstein–Hilbert action (27) vanishes if and only if

$$\begin{aligned}
0 = R_{\mu\nu} - \frac{1}{2}Rg_{\nu\nu} &- \left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)\Gamma_\mu\Gamma_\nu \\
&- \frac{1}{2}\left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)g_{\mu\nu}g^{\alpha\gamma}\Gamma_\alpha\Gamma_\gamma \\
&- \frac{N^2+N-2}{(N+1)^2}(2m-3)g^{\alpha\gamma}\Gamma_\alpha\left(M_{\mu\nu}\Gamma_\gamma + g_{\mu\nu}m_\gamma\right) \\
&- \left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)g_{\mu\nu}g^{\alpha\gamma}\left(\Gamma_{\alpha|\gamma} - \Gamma_\alpha\Gamma_\gamma\right).
\end{aligned}$$

The corresponding energy–momentum tensor is

$$\begin{aligned}
T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\nu\nu} &- \left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)\Gamma_\mu\Gamma_\nu \\
&- \frac{1}{2}\left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)g_{\mu\nu}g^{\alpha\gamma}\Gamma_\alpha\Gamma_\gamma \\
&- \frac{N^2+N-2}{(N+1)^2}(2m-3)g^{\alpha\gamma}\Gamma_\alpha\left(M_{\mu\nu}\Gamma_\gamma + g_{\mu\nu}m_\gamma\right) \\
&- \left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)g_{\mu\nu}g^{\alpha\gamma}\left(\Gamma_{\alpha|\gamma} - \Gamma_\alpha\Gamma_\gamma\right).
\end{aligned} \tag{28}$$

If the scalar functions m and $\bar{m}$ are equal, which means that is $\delta m = 0$, the energy–momentum tensor $T_{\mu\nu}$ given by (28) reduces to

$$\begin{aligned}
T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\nu\nu} &- \left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)\Gamma_\mu\Gamma_\nu \\
&- \frac{1}{2}\left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)g_{\mu\nu}g^{\alpha\gamma}\Gamma_\alpha\Gamma_\gamma \\
&- \frac{N^2+N-2}{(N+1)^2}(2m-3)g^{\alpha\gamma}\Gamma_\alpha g_{\mu\nu}m_\gamma \\
&- \left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)g_{\mu\nu}g^{\alpha\gamma}\left(\Gamma_{\alpha|\gamma} - \Gamma_\alpha\Gamma_\gamma\right).
\end{aligned}$$

If m and $\bar{m}$ are equal numbers, which means $M_{\mu\nu} = 0$ and $m_\mu = \bar{m}_\mu = 0$, the energy–momentum tensor $T_{\mu\nu}$ given by (28) reduces to

$$\begin{aligned}
T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\nu\nu} &- \left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)\Gamma_\mu\Gamma_\nu \\
&- \frac{1}{2}\left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)g_{\mu\nu}g^{\alpha\gamma}\Gamma_\alpha\Gamma_\gamma \\
&- \left(\frac{N^2+N-2}{(N+1)^2}m^2 - \frac{3(N^2+N-2)}{(N+1)^2}m + \frac{N+5}{(N+1)^2}\right)g_{\mu\nu}g^{\alpha\gamma}\left(\Gamma_{\alpha|\gamma} - \Gamma_\alpha\Gamma_\gamma\right).
\end{aligned}$$

The free parameters $M_{\mu\nu}$, m_μ and m make possible some cosmological measuring to be presented exactly by the model. In this way, the problems with Einstein's model of cosmology that makes relatively high numerical errors in some experiments may be attempted to be solved.

4. Conclusions

In this paper we reviewed the methodology for obtaining invariants of geometric mappings, already published by Vesić [21] and developed by Vesić, Stanković, and Mihajlović [15]. We also defined the m-conformal line of a Riemannian space. The conformal mappings of Riemannian spaces are generalized to $(\bar{m}, m)$-conformal mappings. This represents a completely new approach in the field of conformal mappings that is very important, especially from an application perspective.

At the end of this research, we introduced one example from cosmology that illustrates how the results can be applied in further physical research.

In future research, we will determine the general equations of motion for various cosmological models defined in Riemannian spaces of different dimensions. In the simplest case, these general equations of motion will reduce to those obtained in classical mechanics. Furthermore, since the geometric mappings transform all curves of a given class in the initial Riemannian space into curves of a corresponding class in the deformed space, and since the transformation laws for the Christoffel symbols and the associated invariants are determined based on this, we will use these transformed curves to identify the corresponding mechanical properties of a particle moving along such a curve. A special case will be the Navier–Stokes equations, when the moving particle is considered to be a fluid particle.

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