File ID	118288
Filename	Thesis

SOURCE (OR PART OF THE FOLLOWING SOURCE):

Type	Dissertation
Title	Partition functions for supersymmetric black holes
Author	J. Manschot
Faculty	Faculty of Science
Year	2008
Pages	154
ISBN	978 90 5629 540 0

FULL BIBLIOGRAPHIC DETAILS:
http://dare.uva.nl/record/286607

AUP-Manschot:AUP/Voorbij 02-09-2008 13:38 Pagina 1

UvA Dissertation

Faculty of Science

This dissertation presents recent discoveries on partition functions for four-dimensional supersymmetric black holes. These partition functions are important tools to explain the entropy of black holes from a microscopic point of view within string theory and M-theory. The results are applied to two central research topics in modern theoretical physics, namely (1) the correspondence between the physics (including gravity) within an Anti-de Sitter space and conformal field theory, and (2) the relation between black holes and topological strings.

Jan Manschot (1980) studied Applied Physics at Delft University of Technology. In 2004, he started his Ph.D. research at the Institute for Theoretical Physics of the University of Amsterdam.

Vossiuspers UvA

9 789056 295400

Partition Functions for Supersymmetric Black Holes

Jan Manschot

V

Partition Functions for Supersymmetric Black Holes

JAN MANSCHOT

University of Amsterdam

Partition Functions for Supersymmetric Black Holes

The research, which is published in this book, has been financially supported by the Foundation of Fundamental Research on Matter (FOM) and has been performed at the Institute for Theoretical Physics of the University of Amsterdam. It has been carried out in the Dutch Research School of Theoretical Physics.

Cover design: René Staelenberg, Amsterdam

ISBN 978 90 5629 540 0
NUR 910

Partition Functions for Supersymmetric Black Holes

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor

aan de Universiteit van Amsterdam

op gezag van de Rector Magnificus

Prof. dr. D. C. van den Boom

ten overstaan van een door het college voor promoties

ingestelde commissie,

in het openbaar te verdedigen in de Agnietenkapel

op donderdag 18 december 2008, te 12:00 uur

door

Jan Manschot

geboren te Alphen aan den Rijn

Promotie commissie:

Promotor:	Prof. dr. E. P. Verlinde
Overige leden:	Prof. dr. J. de Boer
	Prof. dr. R. H. Dijkgraaf
	Prof. dr. G. B. M. van der Geer
	Prof. dr. G. W. Moore
	Dr. K. Skenderis

Faculteit der Natuurwetenschappen, Wiskunde en Informatica

PUBLICATIONS

This thesis is based on the following publications:

J. de Boer, M. C. N. Cheng, R. H. Dijkgraaf, J. Manschot, E. P. Verlinde,
A Farey Tail for Attractor Black Holes,
JHEP **11** (2006) 024, arXiv:hep-th/0608059.

J. Manschot,
AdS_3 Partition Functions Reconstructed,
JHEP **10** (2007) 103, arXiv:0707.1159 [hep-th].

J. Manschot, G. W. Moore,
A Modern Farey Tail,
arXiv:0712.0573 [hep-th].

J. Manschot,
On the Space of Elliptic Genera,
arXiv:0805.4333 [hep-th].

CONTENTS

1. PREFACE

This thesis considers various aspects of black holes, which are objects whose mass is compressed to an extremely small volume. Their distinguishing feature is that nothing, not even light, can escape from within a certain vicinity of this heavy object. Since we cannot look beyond the boundary of this vicinity, it is appropriately called the "horizon". The extraordinary features of black holes raise a variety of puzzles in physics. The puzzles studied in this thesis appear in the analogy of black hole physics with another branch of physics, namely the physics of heat processes or thermodynamics. This analogy strongly suggests that a black hole can be made out of smaller constituents in a large number of ways. This number is called the "degeneracy" of the black hole and is related to the size of the horizon. The understanding of the constituents and the degeneracies have improved tremendously in the last decades. Partition functions enumerate the degeneracies and are therefore useful tools to determine them.

Many branches of physics come together in the proper description of black holes. The attractive force of black holes on other objects ranges typically over macroscopic, and even astronomical, length scales. Therefore, the classical subjects of physics like mechanics, gravity, electromagnetism and thermodynamics are relevant. However, since the mass is confined to an extraordinary small volume, a proper understanding of black holes requires a framework which consistently combines physics of large and small length scales. Such a framework is not fully achieved yet; the most promising candidates are presently string theory and M-theory. Black holes are great (theoretical) laboratories for these theories. This first chapter explains in some more detail the context of the study of black holes and their partition functions.

Black holes and thermodynamics

The first notion of a black hole appeared in a letter by Michell in 1784 to Cavendish, as an object from which nothing, not even light, can escape. In 1795, Laplace suggested independently the existence of black holes. Although the name "black hole" seems very accurate for such an object which does not emit light, it was not introduced until 1967 by Wheeler.

The notion of black hole could be made much more precise when general relativity superseded Newtonian gravity in 1915-16. General relativity was developed by Einstein and provides a revolutionary new vision on gravity. Space and time are no longer a static reference frame, but are deformed by the presence of massive objects. The attractive force between massive objects is a consequence of the deformation of space-time. The precise deformation of space-time is described by the so-called Einstein equations; every space-time must be a solution of them. The first black hole solution was established by Schwarzschild in 1915. In this solution, all mass of the universe is concentrated in a point, which is surrounded by a horizon. General relativity predicts that the horizon area of a black hole with the mass of the earth is 1 cm^2. Another interesting quantity is the surface gravity, which is the strength of the gravitational force at the horizon.

Soon, more complicated black hole geometries were obtained, for example, solutions of a theory which captures both general relativity and electromagnetism. The black holes of this Einstein-Maxwell theory can also carry electromagnetic charges and were first described by Reissner in 1916 and Nordström in 1918. It turns out that the mass of these black holes must be larger then a certain minimum value which is determined by the charge. At the minimum value, the surface gravity vanishes. Such "extremal" black holes are considered in this thesis for reasons to be explained later.

During the 1970's, Bardeen, Bekenstein, Carter, Hawking and others studied the laws which are satisfied by black hole quantities, such as the mass, horizon area and surface gravity. Interestingly, these studies revealed a close resemblance with the laws of thermodynamics. In fact, the three laws of thermodynamics could naturally be extended to incorporate general relativity. The first law of thermodynamics states the conservation of energy in a closed system. In relativity, mass is also a manifestation of energy and must therefore be included in this conservation law. The second law of thermodynamics is a statement about entropy, which is a quantity introduced by Clausius in 1865 to explain the fact that heat always flows from a warm object to a cooler object. The second law states that the entropy of a closed system always increases. Finally, the third law says that the entropy approaches a minimum if the temperature approaches absolute zero. The analogy of black hole physics with thermodynamics suggests that the horizon area should be understood as an entropy and surface gravity as a temperature. With these identifications, the three laws of thermodynamics also hold for black holes. In 1975, Hawking performed a semi-classical calculation which revealed that black holes radiate as objects with a certain temperature. This calculation determined some unspecified constants, and gave further evidence that the analogy with thermodynamics is fundamental.

If thermodynamics had ceased developing after Clausius, this might have been the end of the story. However, Boltzmann claimed in 1877 that the energy spectrum of particles could be discrete, and that entropy should be understood as a measure of the number of microscopic degeneracies of a macroscopic system. His claims were based on the ancient Greek belief that all macroscopic objects are built out of microscopic, indivisible entities (atoms). A macroscopic state can be arranged in many different microscopic ways, which explains the entropy. This elucidates the mysterious second law of thermodynamics: the increase of entropy of a system is now understood as the evolution of the system towards its most probable macroscopic state. Entropy can be accurately calculated for many systems if the constituents are known, for example, mixtures of gasses, or lattices of spins. These microscopic considerations stood at the beginning of one of the biggest revolutions in physics: the invention of quantum mechanics. In this theory, a physical quantity, like energy or momentum, is essentially discrete and is called a quantum number. A quantum mechanical system can realize a given set of quantum numbers only in a finite number of ways (the degeneracy). A specific realization is referred to as a "state".

The discovery of quantum mechanics was soon followed by quantum field theory, which also incorporates special relativity. Quantum field theory is able to unify three of the four fundamental interactions of Nature, namely electromagnetism, the weak interactions and the strong interactions. A fully satisfactory unification of quantum theory with the fourth interaction, which is gravity, has not been discovered yet.

If the analogy of black hole physics and thermodynamics is fundamental, the microscopic understanding of entropy should extend to black holes. However, candidate constituents of a black hole were not known for a long time. The black hole constituents are supposed to be particles of a theory which unifies all four forces of nature. The discovery of such a theory has challenged many theoretical physicists. As mentioned before, string and M-theory are currently the most promising candidates. In fact, black hole entropy can be explained within these theories.

To determine the entropy of a (quantum mechanical) system, partition functions are very useful. Before explaining their use in the context of black holes, two simple examples are given here to give a flavour of what partition functions are. The first example is a function for enumerating the number of ways a throw with two dice can give a certain number. This function is given by $t^2 + 2t^3 + 3t^4 + 4t^5 + 5t^6 + 6t^7 + 5t^8 + 4t^9 + 3t^{10} + 2t^{11} + t^{12}$. The number in front of t^k in the function is the number of ways one can obtain k. The second and almost archetypical example, is the function, which enumerates the number of ways a certain number can written as a sum of smaller numbers. Such a sum is called a partition. For example, the partitions of 3 are : 3, 2+1 and 1+1+1. The corresponding partition function is given by $\prod_{n=1}^{\infty}(1-t^n)^{-1} = 1 + 2t^2 + 3t^3 + 5t^4 + \dots$. In quantum theory, a partition function enumerates the degeneracies of states of a system with given quantum numbers. The exponent of t is then interpreted as a quantum number, for example the energy. The entropy can be directly determined from the coefficients of the partition function. The next section, explains this in some detail for black holes. First a brief introduction is given on how the black holes can be incorporated in a quantum theory, namely in string theory.

A quantum theory of black holes

In string theory, the point like particles of quantum field theory are replaced by little strings, which can be open like $\sim$, or closed like $\bigcirc$. These strings are however so small that a possibly stringy nature of particles cannot be determined with current technology. However, the stringy nature of the particles does spread their interactions in space and time. In this way string theory manages to avoid the singular behavior which arises often in quantum field theory. Originally, string theory was invented in the 70's to solve problems in the theory of strong interactions. Quarks, which interact with each other by the strong force, would be the endpoints of the strings and the string would explain their interaction. The scientific interest of this model decreased when quantum chromodynamics, a specific form of quantum field theory, was able to accurately describe the strong forces. Around 1984, string theory revived when people realized that string theory can possibly capture all the interactions between particles including gravity. An important calculation by Green and Schwarz showed that string theory correctly cancels an "anomaly", which is a certain unwanted deviation of quantum theory from the classical theory. Another great discovery of the 80's is the discovery of the heterotic string, which seemed very promising to describe the four fundamental interactions in the way we experience them. A second "revolution" of string theory happened from 1994-1998. In this era, several different string theories got unified in M-theory. Moreover, D-branes were discovered as the surfaces on which open strings can end. The D-branes and strings appear in M-theory as M-branes. These developments were essential for the first account of black hole entropy in 1996 by Strominger and Vafa.

Another influential development was induced by Maldacena's conjecture in 1997, which relates the fundamental physics of Anti-de Sitter space (including gravity) with a conformal field theory without gravity. An Anti-de Sitter (AdS) space is a space-time with a negative curvature. One of its special properties is its (conformal) boundary. Maldacena and others conjectured in 1997 that string/M-theory on such a space-time is dual to a conformal field theory (CFT) on the boundary. This means that to every state in the AdS-theory corresponds a dual state in the boundary theory. Therefore, the partition functions are expected to be equal. This correspondence is remarkable since the theory in AdS incorporates gravity, and the boundary CFT does not. It has initiated many developments in the study of gravity and improved our understanding considerably.

String and M-theory have several properties which turned out to be essential for the microscopic account of black hole entropy. One of these properties is the necessity of higher dimensions. String theory is a theory in ten dimensions and M-theory in eleven. At first sight, this might seem to contradict directly with our four-dimensional universe. However, some dimensions can be compactified to "invisibly" small length scales. This can be done in many different ways, which result in equally many different four-dimensional theories.

Another important feature, which arises naturally in string theory is supersymmetry. It was originally discovered as a symmetry between two kinds of particles, bosons and fermions, of the quantum theory on the string world sheet. Currently, various supersymmetric field theories are studied in different dimensions, for example supersymmetric gauge theories and supergravities. No experimental evidence of supersymmetry is available yet, but it has certain appealing consequences. For example, it improves the unification of electromagnetism and the weak/strong forces since the coupling constants approach each other at a high energy scale. Another advantage is that some quantities in a supersymmetric theory do not change if the coupling constants are varied. An example is the Witten index, which is a measure of the number of supersymmetric states of the theory. This is actually one of the main ingredients for the microscopic calculation of the entropy. The extremal black holes studied in later chapters are supersymmetric solutions of supergravity. Although the (generalized) Witten index is hard to determine in this regime, the coupling constants can be changed to a regime where the calculation can be done. By supersymmetry, this still provides the correct black hole degeneracy. Four-dimensional ($\mathcal{N} = 2$) supersymmetric theories of gravity can obtained as a compactification of M-theory. The total 11-dimensional geometry decomposes as: $\mathbb{R}^{3,1} \times S^1_{\mathrm{M}} \times X$, where $\mathbb{R}^{3,1}$ represents four-dimensional space-time, S^1_{M} a circle and X a compact six-dimensional manifold.

The last essential ingredient are the branes of string and M-theory. D-branes in string theory are objects on which open strings can end. The D-branes can have various dimensions and source electromagnetic fields in space-time. Naively, one can see the D-branes as some kind of carpets, which float around in space-time. In the low energy limit, the excitations on the branes form a Yang-Mills theory, which is a quantum field theory (without gravity). The elementary objects of M-theory are the three-dimensional M2-brane and six-dimensional M5-brane. They unify the strings and D-branes of string theory.

Now we are ready to explain a black hole in M-theory. It is a bound state of M2- and M5-branes, which reside completely inside the compact dimensions. The M5-branes wrap a four-dimensional manifold in X and S^1_{M}. A crucial part of the entropy calculations is the reduction of the M5-brane degrees of freedom to two-dimensions: time and S^1_{M}, which in the Euclidean signature becomes a torus. Such a reduction of the degrees of freedom to a two-dimensional CFT on a torus, appears frequently in entropy calculations of black objects. This is a consequence of the distinguished properties of conformal field theories, which ultimately enables the determination of the black hole degeneracies. The most important property of a CFT is that its interactions do not depend on the length scale. The theory is even invariant under all transformations which leave the metric invariant up to a scale; these transformations form an infinite dimensional group in two dimensions.

To determine the entropy of the supersymmetric black holes, a special partition function of the CFT is considered. This partition is the so-called elliptic genus, and is a generalized Witten index. The special property of the elliptic genus is that it enumerates supersymmetric states as a function of their electric charges. Because of the conformal symmetry, the partition function does only depend on the shape of the torus (parametrized by τ) and possibly potentials, an observation due to Cardy in 1986. Moreover, the elliptic genus is supposed to be invariant under large coordinate transformations of the torus, which correspond to the modular transformations: $\tau \to \frac{a\tau+b}{c\tau+d}$, $ad - cd = 1$. This invariance brings the elliptic genus in contact with modular invariant functions, which appear often in counting problems and number theory. By modular invariance, the Cardy formula can be derived, which provides the leading behavior of the degeneracies. That this agrees with the entropy calculated in supergravity, is one of the important successes of string theory.

Results of the thesis
The results of this thesis are important for two conjectures related to black hole state counting. The first conjecture is the correspondence between fundamental physics in Anti-de Sitter spaces and the conformal field theory on its boundary. A three-dimensional AdS-space appears naturally in the study of the four-dimensional black holes. The CFT at the boundary is the CFT which is obtained by the reduction of the M5-brane degrees of freedom.

A confirmation of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence would be a proof of the equivalence of the CFT and the gravity partition function, which naturally involves a sum over geometries. Dijkgraaf, Maldacena, Moore and Verlinde suggested in 2000 that CFT partition functions can be related to a Poincaré series, which consecutively can be interpreted as a sum over semi-classical saddle points of AdS_3-gravity. Poincaré series are sums over a coset of the modular group such that a function becomes modular invariant. This thesis improves on the Poincaré series proposed by Dijkgraaf *et al.* for CFT partition functions using the work of Niebur. It is shown how the CFT partition function itself can be rewritten as a Poincaré series, giving evidence that the partition functions on both sides of the correspondence are indeed equal.

The second conjecture, which motivates our study of black hole partition functions, is due to Ooguri, Strominger and Vafa. Based on corrections to the leading black hole entropy, they suggested in 2004 that the black hole partition function is equal to the square of the topological string partition function. Topological string theory is a simplified version of string theory, such that many quantities can actually be calculated, although the necessary mathematical techniques might be quite elaborate. Chapter 5 explains that this conjecture must be viewed as a statement about the corrections to the leading saddle point of the partition function. Moreover, the conjecture is only valid in a certain regime of charges.

While studying the partition functions, also interesting aspects were learned, which were not directly relevant to elucidate the two conjectures. Most interesting are the constraints on the "polar" spectrum of the theory by modular invariance. These constraints are shown to be very powerful for partition functions of topologically twisted Yang-Mills theory, which are closely related to the elliptic genera for black holes. The case of gauge theory on $\mathbb{CP}^2$ is analyzed in some detail. Based on the constraints, a generating function is proposed for the Euler number of moduli spaces of $SU(3)$ instantons on the four-manifold $\mathbb{CP}^2$.

Outline

Chapter 2 continues with the introduction of the subject and the problems in more detail. Three motivations are given for the study of black hole partition functions, namely a microscopic account of the leading black hole entropy, the connection with topological strings and the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. The discussions in this chapter are mainly phenomenological, not much reference is made to higher dimensional string or M-theories. In Chapter 3, the four-dimensional black holes are placed in the more fundamental 11-dimensional M-theory. The degrees of freedom on M5-branes wrapped on Calabi-Yau four-cycles are discussed, and their reduction to two dimensions. In addition, the black hole partition function is defined and the leading black hole entropy is microscopically explained. An important aspect of the partition functions is their connection with vector-valued modular forms. Chapter 4 is devoted to a mathematical analysis of these forms, which is necessary to address the motivations in Chapter 5. It derives a (regularized) Poincaré series for these forms and the space of these forms is discussed. The reader which is mainly interested in the physical implications, might initially postpone reading of this chapter. Chapter 5 interprets the Poincaré series from the point of view of AdS_3-supergravity. Also the connection with topological strings is elucidated. The last section of Chapter 5 applies some additional results obtained in Chapter 4 to gauge theory partition functions on four-manifolds.

2. MOTIVATIONS FOR BLACK HOLE PARTITION FUNCTIONS

This chapter explains three important motivations for the systematic study of black hole partition functions in later chapters of the thesis. The first motivation for the analysis of black hole partition functions is to account microscopically for the macroscopic black hole entropy [1]. Subsecs. 2.1.1 and 2.1.2 introduce black holes and the notion of black hole entropy. The black holes studied in this thesis are solutions of four-dimensional $\mathcal{N} = 2$ supergravity [2]. The entropy of a large class of such black holes was explained in Ref. [3], following the seminal work on the microscopic entropy of D1-D5-brane black holes by Ref. [4]. The second motivation is the conjectured connection between black hole entropy and the topological string free energy [5] and is reviewed in Subsec. 2.1.3.

The third motivation of black hole partition functions is to test the correspondence between fundamental physics in Anti-de Sitter space and a dual conformal field theory [3, 6, 7, 8]. The original conjecture by Maldacena [3] arose historically from the large scientific effort to find a microscopic explanation of black hole entropy in string theory. This third motivation is therefore not completely independent from the first motivation. Sec. 2.2 explains the AdS/CFT correspondence in the context, which is relevant for the later studied black holes. This implies that the AdS-space is three-dimensional and the dual CFT is two-dimensional. AdS_3 is in fact part of the near-horizon geometry of a black string in a five-dimensional M-theory compactification. This black string is massive and sources the four-dimensional black hole. The CFT is the field theory which resides on the black string [3]. The AdS_3/CFT_2 duality motivates a study of AdS_3-gravity in Subsec. 2.2.2. In later chapters, the black hole partition functions are shown to be susceptible of a natural interpretation from the point of view of the AdS_3/CFT_2 correspondence.

The approach in this chapter is phenomenological. The emphasis is on effective field theory in a low number of dimensions, three, four or five. In some cases, for example the explanation of the black hole/topological strings connection and of the AdS_3/CFT_2 correspondence, reference is made to the higher dimensional string- and M-theory. Because of the vast subjects involved, this chapter with motivations cannot possibly be self-contained. Some knowledge of (super)gravity and string theory is assumed. More details about string and M-theory can be found in the textbooks [9, 10] and [11, 12].

2.1 Four-dimensional black holes

This section introduces the four-dimensional black holes whose partition function is analyzed in later chapters. The notion of black hole entropy is introduced in the context of Reissner-Nordström solutions. Subsequently, black hole solutions in $\mathcal{N}=2$ supergravity are discussed. This discussion leads to the connection between topological strings and black holes.

2.1.1 Reissner-Nordström black holes

The Reissner-Nordström black hole solution is a generalization of the Schwarzschild solution to a black hole with electro-magnetic charges. The metric of a charged black hole is not a solution of the vacuum Einstein equations since the electro-magnetic field $F_{\mu\nu}$ gives rise to an additional energy density. The equations of motion of the metric and electro-magnetic field are derived in this case from the Einstein-Maxwell action [13]

$$I_{\rm EM} = \frac{1}{16\pi G_4}\int d^4x\sqrt{|g|}\left(R - F_{\mu\nu}F^{\mu\nu}\right), \tag{2.1}$$

where G_4 is the four-dimensional gravitational constant, $|g|$ is the determinant of the metric $g_{\mu\nu}$ and R is the Riemann curvature. The adopted units are such that the velocity of light c is equal to one. Also G_4 is taken to be one unless otherwise specified. The metric of a black hole with mass M, electric charge q and magnetic charge p is

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M}{r}+\frac{q^2+p^2}{r^2}\right)dt^2 + \left(1-\frac{2M}{r}+\frac{q^2+p^2}{r^2}\right)^{-1}dr^2 \\ &\quad + r^2(d\theta^2+\sin^2\theta d\varphi^2). \end{aligned} \tag{2.2}$$

The electric and magnetic field strength are respectively $F_{tr}=q/r^2$ and $F_{\theta\varphi}=p\sin\theta/r^2$. Since q and p appear at many places as q^2+p^2, we define $Z=q+ip$ such that $q^2+p^2=|Z|^2$. Note that g_{tt} vanishes at two values of r. The black hole has therefore two horizons, with horizon radii r_+ and r_-

$$r_\pm = M \pm \sqrt{M^2-|Z|^2}. \tag{2.3}$$

The conserved charges M, q and p can be determined from the asymptotics of the metric for $r \to \infty$ by the ADM-method [13]. These quantities obey relations known as black hole mechanics [14, 1]. The first law of black hole mechanics for Reissner-Nordström black holes reads

$$\delta M = \frac{\kappa}{2\pi}\frac{1}{4}\delta A + \mu_q \delta q + \mu_p \delta p, \tag{2.4}$$

where A is the horizon area of the outer horizon $A = 4\pi r_+^2$. μ_q and μ_p are respectively $\mu_q = q/r_+$ and $\mu_p = p/r_+$, κ is the surface gravity at the outer horizon

$$\kappa = \frac{\sqrt{M^2 - |Z|^2}}{2M(M + \sqrt{M^2 - |Z|^2}) - |Z|^2}. \tag{2.5}$$

Remarkably, the laws of black hole mechanics show a close resemblance with the classical equations of thermodynamics. Eq. (2.4) should be compared with the first law of thermodynamics

$$\delta E = T\delta S + \mu \delta N + \dots. \tag{2.6}$$

The term with δA in (2.4) is written in the specific form since a semiclassical calculation by Hawking [15] shows that it is natural to view $\kappa/2\pi$ as the temperature T_{BH} of the black hole. Correspondingly, the black hole entropy is given by $S_{\mathrm{BH}} = A/4$. This relation is known as the Bekenstein-Hawking area law. The second and third law of thermodynamics have analogues as well in general relativity [14, 1]. A microscopic explanation of this macroscopic entropy, in a similar spirit as Boltzmann did in 1877 for classical thermodynamics, is one of the major goals in theoretical physics for the last decades. The exponential growth of the number of microstates N_{micro} is given by

$$\log(N_{\mathrm{micro}}) \sim \frac{S_{\mathrm{BH}}}{k_{\mathrm{B}}} = \frac{A}{4G_4\hbar}, \tag{2.7}$$

where the dependence on relevant physical constants is included. One would like to explain this leading behavior of the microstates microscopically.

The geometry of the black hole depends on the value of M with respect to $|Z|^2$. The two horizons of the charged black hole coincide for $M = |Z|$. The metric contains a naked singularity when $M < |Z|$. Such solutions are excluded by the cosmic censorship hypothesis. Therefore, M is bounded from below by

$$M \geq |Z|. \tag{2.8}$$

The black holes with mass $M = |Z|$ are called extremal.

A similar bound as (2.8) has been derived in the connection with dyons in non-abelian gauge theory in the absence of the coupling to gravity by Prasad and Sommerfield [16] and Bogomolny [17]. A bound as (2.8) is therefore known as the Bogomolny-Prasad-Sommerfield (BPS) bound. Ref. [17] establishes the stability of such a state against the emission of vector mesons. Ref. [18] studied the stability of such dyons and proved that the class of dyons studied in [16] saturating the bound $M = |Z|$ are stable against perturbations. The extremal Reissner-Nordström solutions are also expected to be stable. Since their temperature $T_{\mathrm{BH}} = \kappa/2\pi$ vanishes, they do not lose energy by thermal radiation. The next subsection shows that black holes which preserve some supersymmetry satisfy the BPS-bound.

Eq. (2.4) shows that the black hole entropy is a function of the mass M and the charges q and p. In the extremal limit $M \to |Z|$, T_{BH} vanishes but S_{BH} has a well-defined and non-vanishing limit. The explicit dependence of $S_{\mathrm{BH}}(M, q, p)$ on M disappears. The dependence of the entropy on G_4 in the extremal limit is also of interest. We assume that the only significant length scale in the theory is $\sqrt{G_4}$. The dependence of the metric (2.2) on $\sqrt{G_4}$ is made manifest by replacing M by $\sqrt{G_4}M$ and r by $r/\sqrt{G_4}$. Then one finds that the entropy is

$$S_{\mathrm{BH}}(q, p) = \pi |Z|^2, \tag{2.9}$$

which is interestingly independent of G_4! This will turn out to be a convenient property of extremal black holes because one can adjust the strength of gravity without changing the number of states of the black hole. Therefore, it is this "extremal" entropy which we account for microscopically in the following chapters. Since the temperature vanishes in the extremal limit, the entropy could strictly speaking jump between $T_{\mathrm{BH}} = 0$ and $T_{\mathrm{BH}} > 0$. This is assumed not to be case.

The near horizon geometry of the extremal black hole has a universal structure. After a coordinate transformation $r \to r + |Z|$ and the limit $r \to 0$ one finds the metric

$$ds^2 = -\frac{r^2}{|Z|^2} dt^2 + \frac{|Z|^2}{r^2} dr^2 + |Z|^2 (d\theta^2 + \sin^2\theta d\varphi^2), \tag{2.10}$$

This metric describes a product of two spaces. The coordinates t and r are coordinates of two-dimensional Anti-de Sitter space, θ and φ are the coordinates of a two-sphere. More details about AdS-spaces are given in Subsec. 2.2.2.

At first sight, the first law of black hole mechanics (2.4) might appear rather accidentally. Could one derive a first law of black hole mechanics for a more general Lagrangian as (2.1)? The answer to this question is yes. Wald has derived in Ref. [19] the first law for a general diffeomorphism invariant Lagrangian. The equations of motion of the Lagrangian must allow for stationary geometries with a Killing horizon and well-defined conserved charges at infinity. The entropy is derived as a Noether charge and results in the following expression

$$S_{\mathrm{BH}} = -2\pi \int_\Sigma \sqrt{|g|}\varepsilon^{\mu\nu}\varepsilon^{\rho\sigma}\frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}d\Omega, \tag{2.11}$$

if the action does not involve any derivatives of the Riemann tensor. The integration domain, Σ, is an arbitrary spatial cross-section. Evaluation of this integral leads to $S_{\mathrm{BH}} = \frac{A}{4}$ for the Reissner-Nordström solutions. Subsec. 2.1.3 deals with black holes in the presence of R^2-corrections. Then the black hole entropy no longer satisfies the Bekenstein-Hawking area law, but nevertheless a well-defined entropy is associated with a black hole solution.

2.1.2 Black holes in $\mathcal{N} = 2$ supergravity

The Einstein-Maxwell theory discussed in the previous subsection can be extended to a supersymmetric theory of gravity (supergravity). Supergravity contains black hole solutions which are similar to the Reissner-Nordström solutions. The class of solutions which preserve part of the supersymmetry is particularly interesting, these are analogues of extremal black holes in Einstein-Maxwell theory. In the first part of this subsection, the BPS-bound is derived from the supersymmetry algebra. The second part describes black holes in $\mathcal{N} = 2$ supergravity and the evaluation of their entropy.

Relevant concepts of supersymmetry are briefly introduced now, see for more information [20, 21]. Supersymmetry is an anti-commuting symmetry first observed as a symmetry of the worldsheet theory of a fundamental string [22]. The generators of the symmetry are the spinors $\mathcal{Q}^A_\alpha$ and $\bar{\mathcal{Q}}^A_{\dot\alpha}$ in the notation of [20]. The bar denotes Hermitean conjugation. The generators transform bosonic fields into fermionic fields and vice versa. The index A takes values from $1\ldots\mathcal{N}$. The supersymmetry is called "extended" when $\mathcal{N} > 1$. The algebra of the supersymmetry generators is given by

$$\begin{aligned} \{\mathcal{Q}^I_\alpha, \bar{\mathcal{Q}}^J_{\dot\beta}\} &= 2\sigma^\mu_{\alpha\dot\beta}P_\mu\delta^{IJ}, \\ \{\mathcal{Q}^I_\alpha, \mathcal{Q}^J_\beta\} &= 2\varepsilon_{\alpha\beta}Z^{IJ}, \end{aligned} \tag{2.12}$$

where P_μ is the momentum operator, Z^{IJ} is a complex anti-symmetric tensor and $\sigma^\mu_{\alpha\dot\beta}$ are the Pauli matrices. For our case of interest, $\mathcal{N}=2$, Z^{IJ} has only one complex component, the central charge Z. The reason for using the same symbol Z to denote the central charge and $q+ip$ will become clear shortly.

The implications of the supersymmetry algebra (2.12) are most easily derived in the rest frame of the black hole

$$\begin{aligned} \{\mathcal{Q}^I_\alpha, \left(\mathcal{Q}^J_\beta\right)^\dagger\} &= 2M\delta_{\alpha\beta}\delta^{IJ}, \\ \{\mathcal{Q}^I_\alpha, \mathcal{Q}^J_\beta\} &= 2\varepsilon_{\alpha\beta}Z^{IJ}. \end{aligned} \tag{2.13}$$

We can directly deduce from this that for a unitary theory $M \geq 0$. In fact, one can derive a stronger bound for the mass of a state $|M, Z\rangle$ which has mass M and central charge Z. To show this we define the operators a_α, b_α by

$$a_\alpha = \frac{1}{\sqrt{2}}\left(\mathcal{Q}^1_\alpha + \varepsilon_{\alpha\beta}\left(\mathcal{Q}^2_\beta\right)^\dagger\right), \qquad b_\alpha = \frac{1}{\sqrt{2}}\left(\mathcal{Q}^1_\alpha - \varepsilon_{\alpha\beta}\left(\mathcal{Q}^2_\beta\right)^\dagger\right), \tag{2.14}$$

And the 2×2 matrix of anti-commutators

$$\mathbf{M} = \left\langle M, Z \left| \begin{pmatrix} \{a_\alpha, a^\dagger_\alpha\} & \{a_\alpha, b^\dagger_\alpha\} \\ \{b_\alpha, a^\dagger_\alpha\} & \{b_\alpha, b^\dagger_\alpha\} \end{pmatrix} \right| M, Z \right\rangle, \tag{2.15}$$

where the index α is not summed over. Since the matrix of commutators is Hermitian, $\mathbf{M}$ is positive semidefinite. Because the determinant of $\mathbf{M}$ is $4M^2 - 4|Z|^2$, we find that

$$M \geq |Z| \tag{2.16}$$

in a unitary theory. This is reminiscent of the BPS bound (2.8) with $Z = q + ip$. These results are due to [23], where also is shown that the mass of a state which saturates the bound $M = |Z|$ is not affected by quantum corrections. A state $|M, Z\rangle$ which saturates the bound preserves part of the supersymmetry and forms a short representation of the supersymmetry algebra.

Similarly to the discussion in the previous subsection, one would like to determine the dependence of Z on electro-magnetic charges. To explain this dependence some more concepts of supergravity are needed. The field content of $\mathcal{N}=2$ supergravity consists of the metric, gauge fields, scalars and fermions. The supersymmetry algebra groups the different fields in multiplets. The fields in a multiplet transform among each other under the supersymmetries. Several kinds of multiplets exist which each have a different field content; examples are supergravity, vector, chiral and hyper multiplets. Multiplets are nicely understood in the framework of superfields [20]. The number of multiplets and their content have a beautiful interpretation from the point of view of a compactification of higher dimensional string theory [12].

The building stone of a Poincaré supergravity theory is the $\mathcal{N} = 2$ supergravity multiplet, which contains the graviton, a gauge field (the graviphoton), gravitini and auxiliary fields [24]. Poincaré supergravity can be obtained by gauge fixing of superconformal gravity. The Weyl multiplet $\mathbf{W}$ of superconformal gravity is the main ingredient for the Poincaré supergravity multiplet. The lowest component in the superfield expansion of the Weyl multiplet is an auxiliary anti-self-dual anti-symmetric tensor $T^{IJ}_{\mu\nu} = T^-$, which appears in the supersymmetry transformation of the gravitino

$$\delta\psi^I_\mu = 2\mathcal{D}_\mu\epsilon^I - \frac{1}{8}\gamma_\rho\gamma_\sigma T^{\rho\sigma IJ}\gamma_\mu\epsilon_J, \tag{2.17}$$

where $\mathcal{D}_\mu$ is some covariant derivative and γ_μ are the Dirac matrices. The charge related to T^- determines the central charge Z in the algebra since T^- appears in the supersymmetry transformation of the gravitino. The equations of motion determine that $T^{IJ}_{\mu\nu}$ is equal to the anti-self-dual part of the graviphoton field strength. The graviphoton field strength is in turn determined by the equations of motions to be a linear combination of vector multiplet field strengths $F^a_{\mu\nu}$. A vector multiplet $\mathbf{X}^a$ contains besides the gauge field a complex scalar X^a, which is the lowest component of the multiplet. These scalars describe the geometry of the compactification manifold from the point of view of string theory. They are therefore often referred to as “moduli”. The number of vector multiplets is represented by n_{V}. The supergravity multiplet contains an “auxiliary” vector multiplet. The gauge field of this vector multiplet is the graviphoton field strength. The expectation value of the auxiliary vector multiplet scalar is fixed in terms of the dynamical vector multiplet scalars, analogous to the fixing of the graviphoton field strength. The degrees of freedom are conveniently parametrized by $n_{\mathrm{V}} + 1$ scalars X^A and gauge fields $F^A_{\mu\nu}$ with $A = 0 \dots n_{\mathrm{V}}$.

The gauge fields $F^A_{\mu\nu}$ are sourced by objects with charges q_A and p^A. The black holes of our interest have vanishing p^0-charge. The remaining magnetic charges p^a take values in the magnetic lattice Λ. The Dirac-Schwinger-Zwanziger quantization [25, 26, 27, 28] of electromagnetic charges requires that $p^a q_a = p \cdot q \in \mathbb{Z}$. This implies that the electric charges would take values in the dual lattice Λ^* of Λ. Presently, we are interested in black holes which are described in M-theory. Witten has shown that anomaly cancellation on the M2-brane worldvolume requires that $q_a - \frac{d_{ab}p^b}{2} \in \Lambda^*$ [29, 30], with d_{ab} the quadratic form of the magnetic lattice Λ. The anomaly is known as the Freed-Witten anomaly [31].

The different kinds of multiplets can be used as building blocks of a supersymmetric action. These actions are generically intricate expressions, see for example [20, 2, 24]. However the black hole entropy does not depend on all fields in the Lagrangian but only on the gravity multiplet and the vector multiplets. The dependence on other multiplets disappears. The relevant bosonic part of the action contains the metric, $n_{\rm V}+1$ gauge fields and $n_{\rm V}$ scalars

$$\begin{aligned} I = \quad & \frac{1}{16\pi G_4}\int d^4x\sqrt{|g|}\,\Big(R + 2g_{ab}(z,\bar{z})\partial^\mu z^a\partial_\mu \bar{z}^b \\ & + \frac{i}{4}\left(\mathcal{N}_{AB}F^{+A}\wedge F^{+B} - \bar{\mathcal{N}}_{AB}F^{-A}\wedge F^{-B}\right)\Big)\,, \end{aligned} \tag{2.18}$$

The indices a and b run over the number $n_{\rm V}$ of vector multiplets $1\dots n_{\rm V}$. F^{-A} and F^{+B} are respectively the anti-self-dual and self-dual part of the gauge field F^A.[1]

The $n_{\rm V}$ scalars z^a parametrize the moduli space which is Kähler. The $n_{\rm V}$ scalars z^a are conveniently parametrized in terms of the $n_{\rm V}+1$ scalars $X^A(z^a)$ satisfying the constraint

$$N_{AB}X^A\bar{X}^B = -e^{-K(X,\bar{X})} = -\frac{1}{G_4}, \tag{2.19}$$

where $K(X,\bar{X})$ is the Kähler potential.

This is known as conformal gauge or D-gauge [2], it breaks the conformal invariance in superconformal gravity. The kinetic term for the scalars, $g_{ab}(z,\bar{z})\partial^\mu z^a\partial_\mu\bar{z}^b$, is equivalent to $\mathcal{M}_{AB}(X,\bar{X})\partial^\mu X^A\partial_\mu\bar{X}^B$ with the constraint (2.19). The metric $\mathcal{M}_{AB}(X,\bar{X})$ is given by

$$\mathcal{M}_{AB}(X,\bar{X}) = N_{AB} + N_{AC}\bar{X}^C N_{BD}X^D. \tag{2.20}$$

The couplings $\mathcal{N}_{AB}$ and the matrix N_{AB} can be expressed in terms of a projective covariantly holomorphic section $(X^A, F_A(X))$ of an $Sp(2n{+}2)$-vector bundle over the moduli space parametrized by z^a. The inhomogeneous coordinates ("special coordinates") $t^A = X^A/X^0$ are also often

[1] $*^2 = -1$ in a Lorentzian space, therefore we adopt the definition $*F^+ = -iF^+$ and $*F^- = iF^-$ for a self-dual two-form F^+ and anti-self-dual two-form F^- respectively. $F^\pm = \frac{1}{2}(F \pm i * F)$ and such that $F = F^+ + F^-$.

applied in the literature. The geometry of the moduli space of supergravity and the corresponding vector bundle are generically referred to as "special geometry". Special geometry can be understood in a beautiful geometric way in type IIB string theory [32]. An $Sp(2n+2)$-rotation of the symplectic section (X^A, F_A) to a section $(\tilde{X}^A, \tilde{F}_A)$ leaves the equations of motions invariant. The electric and magnetic charges need to be rotated correspondingly. These transformations generalize the electric-magnetic duality transformations of Maxwell theory [33].

The section can often be described in terms of a covariantly holomorphic function $F(X)$. This function carries the name prepotential. We adopt the notation

$$F_A = \partial_A F(X), \qquad F_{AB} = \partial_A \partial_B F(X). \tag{2.21}$$

Then N_{AB} and $\mathcal{N}_{AB}$ are given by [24]

$$N_{AB} = 2\mathrm{Im} F_{AB}, \tag{2.22}$$

$$\mathcal{N}_{AB} = \bar{F}_{AB} + i \frac{N_{AC} X^C N_{BD} X^D}{X^K N_{KL} X^L}. \tag{2.23}$$

Note that $K(X, \bar{X}) = 0$ by the D-gauge (2.19).

The superconformal symmetry, which is present before gauge fixing to Poincaré supergravity, requires that the prepotential $F(X)$ is homogeneous of degree 2. This means that

$$F(\lambda X) = \lambda^2 F(X). \tag{2.24}$$

Differentiating both sides to λ and to X^A, we find the following useful identities

$$X^A F_A(X) = 2F(X), \qquad X^A F_{AB} = F_B, \qquad X^A F_{ABC} = 0. \tag{2.25}$$

The tree-level prepotential is given by

$$F(X) = \frac{D_{abc} X^a X^b X^c}{X^0}. \tag{2.26}$$

D_{abc} is equal to $-\frac{1}{6} d_{abc}$ with d_{abc} the intersection matrix of four-cycles of the compactification manifold [34]. The prepotential receives quantum corrections and instanton corrections. This is discussed in the next section. The action for the vector multiplets can be expressed as a superspace integral of $F(\mathbf{X})$

$$I_{\mathrm{vect}} = \mathrm{Im} \left[\int d^4 x d^4 \theta F(\mathbf{X}) \right]. \tag{2.27}$$

The above brief exposition of $\mathcal{N}=2$ supergravity introduced the elements appearing in the expression for the central charge. The central charge Z associated with a black hole with electric charges q_A and magnetic charges p^A is [35]

$$Z = e^{K(X^A,\bar{X}^A)/2}\left(X^A q_A - F_A p^A\right). \tag{2.28}$$

We observe that Z depends in this situation continuously on the scalars X^A. The scalars can depend on the radius r in the black hole geometry. The black hole mass M is determined at infinity $M = |Z|_{r=\infty}$ as in the case of the Reissner-Nordström solution. For given charges and moduli at infinity, solutions can be constructed which differ considerably for $r < \infty$. The possibility of non-vanishing gradients for the scalar fields complicates the supergravity solutions considerably. In addition, supersymmetric solutions do exist with multiple black hole singularities [36, 37]. The existence of some solutions depends on the values of the moduli at infinity. Regardless of the variety of solutions, the geometry near the horizon of a singularity is universal and equivalent to the near-horizon geometry of the extremal Reissner-Nordström black hole (2.10). Its geometry is a product of $\mathrm{AdS}_2 \times S^2$

$$ds^2 = -\frac{r^2}{|Z|^2}dt^2 + \frac{|Z|^2}{r^2}dr^2 + |Z|^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{2.29}$$

The horizon area of the black hole is $A = 4\pi|Z|^2$ and the black hole entropy is again

$$S_{\mathrm{BH}}(q,p) = \pi|Z|^2. \tag{2.30}$$

The near-horizon metric (2.29) should be complemented with the expressions for the gauge and scalar fields at the horizon. The field strengths have the generic $1/r^2$-behavior. Interestingly, the scalars are fixed at the horizon by the charges

$$q_A = 2\mathrm{Re}\left[i\bar{Z}e^{K/2}F_A\right], \qquad p^A = 2\mathrm{Re}\left[i\bar{Z}e^{K/2}X^A\right]. \tag{2.31}$$

These equations are known as the "attractor equations". They can be derived by the gravitino supersymmetry transformations [38, 39]. When the attractor equations are satisfied, $\mathcal{N}=2$ supersymmetry is restored at the horizon. The attractor equations can also be derived by an extremization of the horizon area [39]. Some references, for example [24, 40], refer to the attractor equations (2.31) as "stabilization equations" since the moduli generically tend to singular values at the horizon. Solutions with constant values of the moduli in the black hole geometry are known as “double extreme” black holes. These solutions contain only a single singularity.

The attractor equations play a crucial role in the context of black hole entropy. The microscopic entropy is in essence a discrete quantity since it represents the number of microstates. The black hole entropy is at first sight however a continuous function of the moduli by (2.28). The attractor mechanism determines the moduli in terms of the charges such that the entropy depends solely on the discrete charges and is therefore a discrete quantity. The entropy can have a discontinuous dependence on the moduli [41, 37], the discontinuous dependence is however exponentially small compared to the leading entropy.

In some cases, the attractor equations can be solved explicitly for the prepotential (2.26). The form of the prepotential suggests that the scalar X^0 and the charges q^0 and p^0 are distinguished from the charges q^a and p^a. The relevant black holes have vanishing p^0-charge. A change of variables to $Y^A = \bar{Z}e^{K/2}X^A$ is particularly convenient [42]. In terms of these variables, $|Z|^2$ reads

$$|Z|^2 = Y^A q_A - F_A(Y)p^A. \tag{2.32}$$

Since $|Z|^2$ is invariant under symplectic transformations, we only need to solve the attractor equations for a black hole with charges $(0, p^a, 0, \hat{q}_0)$. The simultaneous symplectic transformations of the moduli

$$\begin{aligned} Y^0 &\longrightarrow Y^0, \\ Y^a &\longrightarrow Y^a - k^a Y^0, \end{aligned} \tag{2.33}$$

and the charges

$$\begin{aligned} q_0 &\longrightarrow q_0 + k^a q_a - 3D_{abc}k^a k^b p^c, \\ q_a &\longrightarrow q_a - 6D_{abc}k^b p^c, \end{aligned} \tag{2.34}$$

leave $|Z|^2$ invariant. In addition, the transformations of the charges (2.34) leave the combination $\hat{q}_0 = q_0 - \frac{1}{2}q^2$ invariant. The attractor equations (2.31) are easily solved when $q_a = 0$. One finds for the moduli Y^0 and Y^a

$$Y^0 = \frac{1}{2}\sqrt{\frac{D}{\hat{q}_0}}, \qquad Y^a = -\frac{1}{2}ip^a. \tag{2.35}$$

The transformations (2.33) and (2.34) imply that the quantity $|Z|^2$, of a black hole with charges $(0, p^a, q_a, q_0)$, is only a function of p^a and $\hat{q}_0 = q_0-$

$\frac{1}{2}q^2$ with $q^2 = d^{ab}q_a q_b = -(6D_{abc}p^c)^{-1}q_a q_b$. The entropy $S_{\rm BH} = \pi|Z|^2$, expressed solely as a function of the charges, reads [40]

$$S_{\rm BH}(q,p) = 2\pi\sqrt{D\hat{q}_0} = 2\pi\sqrt{D\left(q_0 - \frac{1}{2}q^2\right)}, \tag{2.36}$$

where $D = D_{abc}p^a p^b p^c$. Note that $D < 0$ and that consequently $\hat{q}_0 < 0$ also. The D4-branes form thus bound states with anti-D0-branes [34], which is also natural from a geometric point of view [43, 37]. We define the anti-D0 brane charge $q_{\bar{0}} = -q_0$. Similarly, the quantity $\hat{q}_{\bar{0}}$ is defined as $\hat{q}_{\bar{0}} = -\hat{q}_0 = q_{\bar{0}} + \frac{1}{2}q^2$. Then $S_{\rm BH}$ is given by

$$S_{\rm BH}(q,p) = \pi\sqrt{\frac{2}{3}p^3\hat{q}_{\bar{0}}}. \tag{2.37}$$

2.1.3 A black hole partition function and a Legendre transformation

Classical entropy can generically be obtained by a Legendre transformation of a free energy. The free energy is in many cases determined from a microscopic partition function. In the present context of black hole entropy, Legendre transformations will also proof useful.

A partition function generically treats some quantum numbers in a microcanonical ensemble and some in a (grand) canonical ensemble. For the black holes of interest with $p^0 = 0$, several motivations can be given for the choice of a microcanonical ensemble for the magnetic charges p^A and a canonical ensemble for the electric charges q_A. A first heuristic motivation is the asymmetry between the electric and magnetic charges in the construction of the Dirac monopole. There the magnetic charge leads to a semi-infinite singular string in the electro-magnetic potential, whereas the electric charges do not lead to such non-local effects. The presence of a magnetic charge determines that the $U(1)$-bundle is non-trivial. Therefore, it seems reasonable to keep the magnetic charge fixed and introduce potentials $\phi^A = \phi^{\bar{A}}$ for the electric charges. The partial Legendre transformation has the form

$$S_{\rm BH}(q,p) = \mathcal{F}(\phi,p) - \phi^{\bar{A}}\frac{\partial\mathcal{F}(\phi,p)}{\partial\phi^{\bar{A}}}, \tag{2.38}$$

with

$$q_{\bar{A}} = -q_A = -\frac{1}{\pi}\frac{\partial\mathcal{F}(\phi,p)}{\partial\phi^A}. \tag{2.39}$$

A second motivation is the growth of the entropy (2.36) as a function of $\hat{q}_{\bar{0}}$ and p. Since the growth of $S_{\rm BH}(q,p) = \pi\sqrt{\frac{2}{3}p^3\hat{q}_{\bar{0}}}$ as a function of $\hat{q}_{\bar{0}}$ is $\sim \hat{q}_{\bar{0}}^{1/2}$, the quantity

$$\mathcal{F}(\phi,p) = S_{\rm BH}(q,p) - \pi\phi^{\bar{A}}q_{\bar{A}} \tag{2.40}$$

is $< \infty$ for positive potentials $\phi^{\bar{A}}$ and has a maximum. If we use that $\mathcal{F}(\phi,p)$ is invariant under the symplectic transformations, then $\mathcal{F}(\phi,p) \to -\infty$ for $\hat{q}_{\bar{0}} \to \infty$. A generalization of $\mathcal{F}(\phi,p)$ to a function with potentials for the magnetic charges would not have these properties since $S_{\rm BH}(q,p)$ as a function of p grows as $\sim p^{3/2}$.

A third motivation is that $\mathcal{F}(\phi,p)$ can be expressed as a simple function of the prepotential $F(Y)$ as shown originally by Ooguri, Strominger and Vafa [5]. One finds that

$$\mathcal{F}(\phi,p) = 4\pi \mathrm{Im} F(Y) = \pi \mathrm{Im} F(2Y), \tag{2.41}$$

with the identification

$$Y^A = \frac{1}{2}(\phi^A - ip^A). \tag{2.42}$$

This result is a consequence of the form of the entropy (2.30), the attractor equations (2.31) and the homogeneity properties of $F(Y)$ (2.25).

The partial Legendre transformation leads us to a black hole partition function of the form

$$\mathcal{Z}_{\rm BH}(\phi,p) = \sum_{q_A} c_p(q) e^{\pi\phi^A q_A} = \exp\mathcal{F}(\phi,p), \tag{2.43}$$

$c_p(q)$ is the number of microstates of a black hole with charges q_A and p^A. The microcanonical entropy is given by $S_{\rm BH}(q,p) = \ln c_p(q)$. Note that this partition function is not expected to be convergent, since the sum is over positive and negative charges. The second motivation for the partial Legendre transform implies however the possibility that $\mathcal{Z}_{\rm BH}(\phi,p)$ is related to a convergent generating function. The next chapter discusses a convergent partition function which is related to $\mathcal{Z}_{\rm BH}$ by simple parameter transformations. [2] To this end, $\hat{q}_{\bar{0}}$ must be bounded from below.

[2] In theories with a large amount of supersymmetry, the symmetry between electric and magnetic charges is restored with respect to the black hole entropy. Convergent partition functions with a canonical ensemble for both electric and magnetic charges do exist in that case [44].

The maximum value of $S_{\rm BH}(q,p) - \phi^{\bar{A}} q_{\bar{A}}$ in the sum is well approximated by $\mathcal{F}(\phi,p)$ if the charges are large. Contributions from other states to $\mathcal{F}(\phi,p)$ are exponentially suppressed and at most equal to 1. Therefore the Legendre transformation is a good approximation to the microcanonical entropy $S_{\rm BH}(q,p)$ in the classical limit.

A partition function like (2.43) is also natural from the point of view of microscopic counting of states in string- or M-theory. In that approach the microstates are viewed as states of a D-brane or M-brane system. The magnetic charge p defines in a sense the field theory on the brane, p is analogous to the N of an $SU(N)$ gauge theory. In the two-dimensional SCFT, p determines to a large extent the field content and correspondingly the central charge. The electric charge q on the other hand can vary between the states of the theory. This picture therefore naturally leads to a partition function as (2.43).

The connection between the prepotential $F(X)$ and $\mathcal{F}(\phi,p)$ is a sign of a more fundamental relation. The remaining part of the section explains that the relation (2.41) survives the addition of perturbative and non-perturbative (instanton) corrections. $\mathcal{N}=2$ supergravity is generically not renormalizable. A given action must be considered in the Wilsonian sense, it is only valid up to a given energy scale. Corrections to the classical action involving more derivatives can be calculated in the more fundamental string theory. The prepotential can acquire in this way a dependence on the supergravity multiplet which results in the appearance of R^2-terms in the action. The precise form of such terms is constrained by supersymmetry. A consequence of these terms are corrections to the black hole entropy which invalidate the Bekenstein-Hawking area law. We first discuss the corrections to the macroscopic entropy and then explain the relation with topological strings.

The dependence of the prepotential on the supergravity multiplet is usually denoted by the field $\hat{A}$. Since the superfield $\hat{\mathbf{A}}$ is the square of the Weyl superfield $\mathbf{W}$ from superconformal gravity, $\hat{A}$ is given by the square of the anti-self-dual auxiliary field T^-

$$\hat{A} = (T^-)^2, \tag{2.44}$$

and therefore is itself an auxiliary field. The dependence of $F(X,\hat{A})$ on $\hat{A}$ changes the identities (2.25). The superconformal symmetry determines in the current situation

$$X^A F_A(X,\hat{A}) + 2\hat{A} F_{\hat{A}}(X,\hat{A}) = 2F(X,\hat{A}), \tag{2.45}$$

where $F_{\hat{A}}(X,\hat{A}) = \partial F(X,\hat{A})/\partial \hat{A}$. Using the homogeneity of $F(X,\hat{A})$, the quantity Υ in $F(Y,\Upsilon)$ is given by $\Upsilon = \bar{Z}^2 e^K \hat{A}$.

The vector multiplet action is again calculated as an integral over superspace

$$I_{\text{vect}} = \text{Im}\left[\int d^4x d^4\theta F(\mathbf{X}, \mathbf{W}^2)\right]. \tag{2.46}$$

This leads to terms in the action, which are second order in the Riemann tensor [45]

$$\mathcal{L} = \frac{1}{16\pi} e^{-K(X,\bar{X})} R - \frac{1}{16\pi}\left(iF_{\hat{A}}\hat{C} + \text{h.c.}\right), \tag{2.47}$$

with $\hat{C} = 64 C^{-\mu\nu\rho\sigma} C^{-}_{\mu\nu\rho\sigma}$. The Weyl tensor $C_{\mu\nu\rho\sigma}$ is the traceless part of the Riemann tensor, $C^{-}_{\mu\nu\rho\sigma}$ is anti-self-dual in both pairs $\mu\nu$ and $\rho\sigma$. Eq. (2.47) implies the following terms in the Lagrangian

$$\sum_{g=1}^{\infty} \int d^4x F_g(X)(C^-)^2 (T^-)^{2g-2} + \cdots + \text{c.c.} \tag{2.48}$$

where the $F_{\hat{A}}(X, \hat{A})$ is expanded in $\hat{A}$ as

$$F_{\hat{A}}(X, \hat{A}) = \sum_{g=1}^{\infty} F_g(X) \hat{A}^{g-1}. \tag{2.49}$$

The terms $F_g(X)(C^-)^2(T^-)^{2g-2}$ represent the scattering of two gravitons with $2g-2$ quanta of the auxiliary field.

The R^2-dependence of the Lagrangian changes the area law for the black hole entropy. The entropy is properly calculated by Wald's formalism [19] with Eq. (2.11). One finds for the entropy of the supersymmetric black holes [34, 45, 46]

$$S_{\text{BH}}(q,p) = \pi |Z|^2 - 256\pi \text{Im}\left[F_{\hat{A}}(X, \hat{A})\right], \tag{2.50}$$

$\hat{A}$ is at the horizon equal to $-64\bar{Z}^{-2}e^{-K}$. The entropy can be written as an expansion in $|Z|^{-2}$

$$S_{\text{BH}}(q,p) = \pi \sum_{g=0}^{\infty} a_g |Z|^{2-2g}. \tag{2.51}$$

The first correction to the tree-level prepotential (2.26) is a one-loop effect. The corrected prepotential $F(Y, \Upsilon)$ is

$$F(Y, \Upsilon) = \frac{D_{abc} Y^a Y^b Y^c}{Y^0} + D_a \frac{Y^a}{Y^0} \Upsilon, \tag{2.52}$$

where $D_a = -\frac{1}{24}\frac{1}{64}c_{2a}(X)$ [34] with $c_2(X)$ the second Chern class of the Calabi-Yau X. See for more details Sec. 3.1. At the horizon, the Y-variables are fixed by their attractor values and $\Upsilon = -64$. The entropy for black holes with $p^0 = 0$ can again be expressed in terms of the charges [34, 45, 46]

$$S_{\mathrm{BH}}(q,p) = \pi\sqrt{\frac{2}{3}\left(p^3 + c_2\cdot p\right)\hat{q}_{\bar{0}}}. \qquad (2.53)$$

This expression for the entropy departs already from the Bekenstein-Hawking area law. This was established by Ref. [47] in a first attempt to find agreement between corrections to the leading microscopic entropy derived in [3] and corrections to the horizon area. This entropy will be obtained from an analysis of the partition function in Chapter 3.

The corrected entropy in Eq. (2.50) should have an interpretation as a Legendre transform since it is a classical entropy. Interestingly, the relevant function $\mathcal{F}(\phi,p)$ is again the (now corrected) prepotential. One needs now Eq. (2.49) to show

$$\mathcal{F}(\phi,p) = 4\pi\mathrm{Im}F(Y) = \pi\mathrm{Im}F(2Y). \qquad (2.54)$$

The corrections to the low energy effective supergravity action can be calculated in string theory [48, 49]. Calculation of these corrections is a major topic in string theory and is known as "topological strings". Topological strings are also useful as a substitute toy model for physical string theory since many interesting quantities are explicitly calculable. We suffice with a couple remarks about topological strings since a proper review would lead us too far from our topic. Topological strings were first introduced by Witten in [50]. A major work in the development of the subject is [48]. Recent reviews are [51, 52, 53]. Topological string theory considers maps of a string worldsheet Σ into the Calabi-Yau X. One can show that the amplitudes are independent of the Calabi-Yau metric after a "twisting" of the fields. The relevant twist in our context is the so-called A-twist. The A-twisted theory counts the number of holomorphic maps from the Riemann surface into the Calabi-Yau.

The topological string free energy has a genus expansion

$$F_{\mathrm{top}}(t,\lambda) = \sum_{g=0}^{\infty}\lambda^{2g-2}F_{\mathrm{top},g}(t), \qquad (2.55)$$

with λ the topological string coupling constant and t the complexified Kähler moduli $t = B + iJ$. The topological string partition function is invariant under translations $t \to t+1$. The topological string free energy $F_g(t)$ is a generating function of the Gromov-Witten invariants N_g^q

$$F_{\text{top},g}(t) = \sum_{q \in H_2(X,\mathbb{Z})} N_g^q e^{-2\pi i t\cdot q}. \tag{2.56}$$

The invariants N_g^q generically take values in $\mathbb{Q}$ and count maps of genus g Riemann surfaces into the Calabi-Yau.

The topological string free energy $F_{\text{top}}(\lambda, t)$ and the prepotential are related. The Kähler moduli are given in terms of the projective coordinates by

$$t^A = \frac{X^A}{X^0}. \tag{2.57}$$

The lowest terms in the expansions of $F_{\text{top}}(t,\lambda)$ are [54, 48] [3]

$$F_{\text{top}}(t,\lambda) = \frac{(-2\pi i)^3}{6}\frac{d_{abc}t^a t^b t^c}{\lambda^2} + \frac{-2\pi i}{24} c_{2a} t^a + \cdots \tag{2.58}$$

The first term is the contribution from the constant maps and gives the volume the Calabi-Yau. This has to be compared with the expansion of the prepotential at the attractor point

$$F(Y,\Upsilon)|_{\text{atr.}} = -\frac{1}{6}\frac{d_{abc}Y^aY^bY^c}{Y^0} + \frac{1}{24}c_{2a}\frac{Y^a}{Y^0} + \dots \tag{2.59}$$

We observe from the one-loop contribution that $F_{\text{top}}(t,\lambda) = -2\pi i F(Y,\Upsilon)$ if we make the identification $t^a = \frac{Y^a}{Y^0}$ and $\lambda = \pm\frac{2\pi}{Y^0}$. The variables Y^A are at their attractor values, therefore λ is determined in terms of the charges as $\pm 4\pi\sqrt{6\hat{q}_{\bar{0}}/p^3}$.

Now the relation between the prepotential and the topological string free energy is derived, $\mathcal{F}(\phi,p)$ can be expressed in terms of F_{top}

$$\mathcal{F}(\phi,p) = F_{\text{top}}(t,\lambda) + \overline{F_{\text{top}}(t,\lambda)}. \tag{2.60}$$

This equation is the basis of the conjecture by Ooguri, Strominger and Vafa [5] that the black hole partition function $\mathcal{Z}_{\text{BH}}$ is related to the topological string partition function $\mathcal{Z}_{\text{top}} = \exp(F_{\text{top}})$ by

$$\mathcal{Z}_{\text{BH}} = |\mathcal{Z}_{\text{top}}|^2, \tag{2.61}$$

with the proper identification of variables. The function $\mathcal{Z}_{\text{top}}$ is expected to be divergent which is consistent with $\mathcal{Z}_{\text{BH}}$ being divergent.

[3] The coefficient of F_1 is taken from Ref. [48]; that coefficient differs by a factor of two from the erroneous coefficient in Ref. [54].

Much effort has been put in the recent years to test this conjecture and place it on a firmer footing. Some references are [55, 56, 57, 58, 37], but this list is by no means exhaustive. The conjecture was also one of the motivations for the research presented in [59], which is one of underlying papers for this thesis. In the subsequent chapters, we will perform an analysis of $\mathcal{Z}_{\rm BH}$ and give an interpretation of (2.61). The supergravity approximation of the black hole entropy by a Legendre transform must break down at some point. The instanton corrections of the topological strings might be in some parameter areas of the same order of magnitude as the errors due to the Legendre transform.

The discussion about black holes and their partition functions in this section is led from a (semi)-classical prospective. Since the (super)gravity solution for given charges is essentially unique, (super)gravity does not seem to capture a vast number of degeneracies to explain $S_{\rm BH}(q,p)$ microscopically. To explain the entropy, one would like to relate supergravity to a quantum theory with a Hilbert space such that we can count the degeneracies of states with given charges. However, the quantization of gravity is notoriously difficult. The gravity coupling constant has a positive mass dimension and conventional renormalization methods by counter terms are not applicable. Also the idea of quantizing the Lorentzian metric leads to conceptual issues, for example about causality. Therefore, we do not attempt to find the black hole microstates in four-dimensional supergravity. Instead, we will show in later chapters how the microstates can be identified in 11-dimensional M-theory, a theory proposed in 1995 [60, 61, 62, 63]. Already in the next section, where we review $\mathrm{AdS}_3/\mathrm{CFT}_2$, we will get a flavor how extra dimensions can help us to find the microscopic entropy.

Interestingly, a lot of progress has been made in the recent past to identify the black hole entropy by microstates in the supergravity. These developments started were initiated by the fuzzball proposal which states that every microstate corresponds to a horizon-free non-singular supergravity solution [64]. Finding the microstate solutions is a vast subject by itself which in the rest of the thesis will not be touched upon. See for a recent review [65]. Refs. [66, 67] adapt this approach to four-dimensional $\mathcal{N}=2$ black holes.

2.2 The AdS_3/CFT_2 correspondence

The previous section motivated the analysis of the black hole partition function $\mathcal{Z}(\phi,p)$ by black hole entropy. This section gives a second motivation, namely the AdS/CFT correspondence [68, 6, 7, 8], which suggests that $\mathcal{Z}(\phi, p)$, as a CFT_2 partition function, equals the partition function of M-theory on AdS_3. This section reviews the AdS_3/CFT_2 correspondence in the context of supersymmetric black holes. The second part of the section studies gravity in AdS_3 to learn which features one expects of an AdS_3-gravity path integral.

The AdS/CFT correspondence, as proposed by [68], is not the first connection between AdS_3 and CFT_2 but it is the strongest and has been tested intensively over the last decade. Ref. [69] recognized already more than two decades ago that the algebra of asymptotic symmetries of AdS_3 is a Virasoro algebra, which is familiar from two-dimensional conformal field theories. The value of the central charge of the dual theory can be obtained by study of the asymptotic symmetries and the entropy of black holes in AdS_3 can then be calculated by the Cardy formula. Another relation between AdS_3-gravity and conformal field theory is described in [70, 71]. These references describe how gravity in three dimensions is related to topological gauge theory (Chern-Simons theory) and how Chern-Simons theory is related to conformal field theory.

2.2.1 The correspondence

To realize the connection between the supersymmetric black holes and AdS_3, one needs to describe the black holes in M-theory. Four-dimensional $\mathcal{N} = 2$ supergravity can be obtained as a reduction of M-theory on a six-dimensional Calabi-Yau X times a circle S^1_{M}. A Calabi-Yau is a compact manifold with two, three and four dimensional cycles, more details of Calabi-Yau manifolds are given in Chapter 3. The fundamental objects of M-theory are a two-dimensional M2-brane and a five-dimensional M5-brane. The worldvolume theory of the M2-brane contains 8 scalars which represents the position of the brane. The most important field of the M5-brane worldvolume theory is a self-dual gauge field H.

The massless bosonic fields in 11 dimensions are the metric $g_{\mu\nu}$ and a three-form potential A_3 which is sourced by the M-branes. The various massless fields in four dimensions can be obtained from a reduction of the 11-dimensional fields [72, 73, 74]. For example, the gauge fields $F^a_{\mu\nu}$ are the four-form fields strengths F_4 reduced on a two-cycle A^a in the Calabi-Yau. The moduli X^a can be obtained similarly by reductions of the gauge field and metric. The gauge field $F^0_{\mu\nu}$ and modulus X^0 are related to the compactification circle S^1_{M}.

The electric q_A and magnetic charges p^A have a beautiful interpretation in the M-theory picture [3]. The black hole is sourced by a stack of M5-branes which wrap both a four-cycle in the Calabi-Yau and the circle.

The wrapping number of the four-cycles are p^a, $a = 1 \ldots \dim H_2(X, \mathbb{Z})$. A non-zero flux of H generates the M2-brane charges q_a in the non-compact dimensions. See Chapter 3 for some more explanation. The charge q_0 is related to the momentum around $S^1_{\rm M}$. The black hole entropy is associated with the microstates of the field theory on the M5-brane. The number of microstates of the M5-brane theory is most easily analyzed after a reduction of the M5-brane degrees of freedom to the two dimensions given by time and the circle $S^1_{\rm M}$.

To rely on the low energy limit of M-theory, the radius R of the circle and the size of the Calabi-Yau V_X must be parametrically larger than the 11-dimensional Planck length $\ell_{\rm P}$. The size of the X is not determined by the attractor mechanism. Instead one can show that $\hat{q}_{\bar{0}}/p \sim V_X^{\frac{2}{3}} R^2/\ell_{\rm P}^6$. Therefore, $\hat{q}_{\bar{0}}^3 \gg p^3$ for the low energy approximation to M-theory to make sense [68].

A reduction to five dimensions instead of four dimensions is valid when the magnitude of the Calabi-Yau is much smaller than the radius R of $S^1_{\rm M}$, thus $R^6 \gg V_X$, and we are interested in length scales smaller than R. For the discussion in this section, we consider $R/\ell_{\rm P} \to \infty$ such that effectively five dimensions are non-compact. Moreover, it is assumed that the four-dimensional black hole is represented by a single infinite black string formed by M5-branes. In general we can write for the action of this system

$$I = I_{\rm bulk} + I_{\rm string} + I_{\rm int}, \tag{2.62}$$

where $I_{\rm bulk}$ describes the physics in the five non-compact dimensions, $I_{\rm string}$ the physics on the black string and $I_{\rm int}$ the interactions between these two systems. The bulk action for the metric and the gauge field is

$$S_{\rm bulk} = \frac{1}{16\pi G_5} \int d^5x \sqrt{|g|} (R - F_{\mu\nu}F^{\mu\nu}). \tag{2.63}$$

The theory on the black string can be shown to be a two-dimensional conformal field theory

We will take the low energy limit in two different regimes of the parameters, this limit is known as the decoupling limit. A comparison between the two limits leads us then to the AdS/CFT-correspondence. Invariance of the black string metric under part of the supersymmetry is essential for a valid comparison between the two limits. As in the four-dimensional case, a supersymmetric black object is also extremal. As a consequence, its entropy is determined in terms of its charges and does not depend on G_5. Deformations of the parameters therefore do not change the (leading) number of states.

The low energy limit is the limit where the Planck length ℓ_P approaches 0. To determine the effect of this limit on the action and the metric, one needs to know the dependence of the five-dimensional Newton's constant G_5 and electro-magnetic coupling μ_5 on the Planck length ℓ_P and size of the Calabi-Yau V_{CY}. This can be derived from the 11-dimensional parameters G_{11} and μ_{11}. Since $G_{11} \sim \ell_P^9$, G_5 is given by $G_5 \sim \ell_P^9/V_X$. Since the two-form potential $F_{\mu\nu}$ in Eq. (2.63) descends from the four-form potential F_4 by integrating over a two-cycle in the Calabi-Yau, $\mu_{11}\int d^3xA_3 = \mu_5 \int dxA$. This quantity is dimensionless, such that μ_5 scales as $V_X^{\frac{1}{3}} l_P^{-3}$. The relevant low energy limit $\ell_P \to 0$ is supplemented with the requirement that $V_X/\ell_P^6 = v$ is fixed.

The first regime, $G_5 \gg 0$, is where the infinite string is so massive that it is a black string. The metric for the corresponding geometry is given by

$$ds^2 = f(r)^{-1}(-dt^2 + dz^2) + f(r)^2(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2), \quad (2.64)$$

with $f(r)$ qualitatively

$$f(r) = 1 + \frac{l_P^3 V_X^{-\frac{1}{3}} p}{r}, \quad (2.65)$$

where p is a measure for the number of black strings. We have only expressed the r-dependence important in the different limits. See for the correct expression of the metric for example [75].

An observer at infinity can detect low energy modes from two different sources, namely low energy modes in the bulk but also higher energetic modes originating from the throat of the geometry. These modes are redshifted and will be detected with a lower energy by the observer. The detected energy E is given in terms of the original energy E_p by $E = f(r)^{-\frac{1}{2}} E_p$. The redshift factor for modes originating from deep in the throat is $(UV_X^{\frac{1}{6}}/p)E_p$ with $U^2 = \frac{r}{\ell_P^3}$. Therefore, to analyze modes at a fixed energy E in terms of the compactification scale $V^{\frac{1}{6}}$, one needs to keep U fixed. This limit transforms the metric to $AdS_3 \times S^2$

$$\frac{ds^2}{\ell_P^2} = \left[\frac{U^2 v^{\frac{1}{3}}}{p} \left(-dt^2 + dz^2\right) + \frac{4p}{v^{\frac{1}{3}}} \frac{dU^2}{U^2} + \frac{p^2}{v^{\frac{2}{3}}} (d\theta^2 + \sin^2\theta d\varphi^2) \right]. \quad (2.66)$$

The radius ℓ of AdS_3 is equal to $2\ell_P p/v^{1/3}$, which is equal to twice the radius of the sphere.

The supergravity approximation is only valid if the curvature radii of AdS_3 and S_2 are parametrically larger than the Planck length. This translates into the condition

$$\frac{\ell_\mathrm{P}^2 p}{V_X^{\frac{1}{3}}} = \frac{p}{v^{\frac{1}{3}}} \gg 1. \tag{2.67}$$

The next subsection introduces the concept of a central charge c_L for AdS_3, this quantity is given by $(3\ell/2G_3)$ which becomes for the present situation $\sim \ell^3/G_5 \sim p^3$. This will be confirmed in Chapter 3 from the CFT point of view. In that chapter is also shown that the classical entropy is an accurate estimate of the microcanonical entropy in this limit.

The important feature of this specific low energy limit is the decoupling of the bulk low energy modes and the modes which originate in the throat. The geometry corresponding to the modes of the throat is $\mathrm{AdS}_3 \times S^2$. We will see a similar decoupling when we take the low energy limit in the second regime.

Next, we take the low energy limit is the opposite regime of Eq. (2.67), $p/v^{\frac{1}{3}} \ll 1$. In this regime, the back reaction of the string on the geometry disappears. The space is flat. The interaction Lagrangian I_int has an expansion in positive powers of $G_5 \sim \ell_\mathrm{P}^9/V_X$. The low energy limit will therefore result in the vanishing of I_int. Consequently, the limit again decouples two systems, in this case the bulk theory and the brane theory. Since the bulk theories are equal in both regimes $p/v^{\frac{1}{3}} \gg 1$ and $p/v^{\frac{1}{3}} \ll 1$, Maldacena conjectured in Ref. [68] that the theory on the black string must be dual to the fundamental theory of physics on $\mathrm{AdS}_3 \times S^2 \times \mathrm{CY}$. This duality is known as the AdS/CFT duality. "Dual" implies that every state in the CFT corresponds to a state in (super)gravity. In many cases correlation functions on one side of the correspondence can be obtained by calculations on the other side of the correspondence [6].

Many subtleties are omitted in the above discussion. An example is the role played by the moduli in five dimensions. These scalars are fixed at the horizon, similar to the moduli in four dimensions. However, the number of states can depend discontinuously on the asymptotic moduli. This dependence on the asymptotic moduli is attributed to the Coulomb branch of the theory [5]. The precise mechanism is however still subject of studies, see for example [37, 66].

Soon after the original proposal, the AdS/CFT conjecture was more formalized. Especially the presence of a boundary of AdS-spaces played an important role to make the correspondence more precise. The dual brane theory is a conformal field theory which resides on the boundary of

AdS_p. The boundary conditions of the gravity fields are sources for the fields in the conformal field theory [7]. The AdS/CFT correspondence is supposed to be an identity at the level of partition functions. The partition function of string theory or M-theory on AdS_p times a compact manifold with boundary conditions $[\phi_0]$ is dual to a conformal field theory on ∂AdS with sources ϕ_0

$$\mathcal{Z}_{\mathrm{M-theory}}([\phi_0]) = \mathcal{Z}_{\mathrm{CFT}}(\phi_0) \tag{2.68}$$

or

$$\int_{\phi_0} \mathcal{D}\phi \, e^{-S(\phi)} = \left\langle e^{\int_{\partial \mathrm{AdS}} \mathcal{O}\phi_0} \right\rangle_{\mathrm{CFT}}. \tag{2.69}$$

In Chap. 3, the partition function is calculated on the CFT side of the correspondence. This CFT partition function is shown to admit an interpretation as a gravity partition function. In the next subsection, gravity in AdS_3 is studied. This will give us an idea what to expect of a partition function of gravity in AdS_3.

2.2.2 Gravity in asymptotic AdS_3

In this section we introduce gravity in three dimensions. More specifically we will study three-dimensional metrics which are asymptotically AdS-spaces. We consider various metrics of AdS_3, which have different physical interpretations. The evaluation of the action of a metric is discussed, which is relevant for a semi-classical analysis later. Black hole thermodynamics in three dimensions is reviewed and we comment on a chiral action of general relativity.

We consider metrics with Lorentzian signature. Lorentzian AdS_3 is the Lorentzian analogue of three dimensional hyperbolic space. The curvature of hyperbolic space is constant and negative and so is the curvature of AdS. AdS_3 can be represented as a Lorentzian hyperboloid, which are the points in $\mathbb{R}^{2,2}$ satisfying

$$X_0^2 + X_1^2 - X_2^2 - X_3^2 = \ell^2. \tag{2.70}$$

ℓ is a measure of the "size" of AdS_3. The isometry group of the hyperboloid is $SO(2,2)$. A solution to Eq. (2.70) is given by

$$\begin{aligned} X_0 &= \ell \cosh\rho \cos t/\ell, \quad & X_1 &= \ell\cosh\rho \sin t/\ell, \\ X_2 &= \ell \sinh\rho\cos\varphi, \quad & X_3 &= \ell \sinh\rho \sin\varphi. \end{aligned} \tag{2.71}$$

The domains of ρ, t and φ are respectively $[0,\infty)$, $[0, 2\pi\ell)$ and $[0, 2\pi)$. This coordinate system is known as the global coordinate system. We can "unwrap" the domain of t to $(-\infty, \infty)$, the resulting space-time is causal. The metric of $\mathbb{R}^{2,2}$

$$ds^2 = -dX_0^2 - dX_1^2 + dX_2^2 + dX_3^2, \tag{2.72}$$

induces a metric on AdS_3 given by

$$ds^2 = -\cosh^2\!\rho\, dt^2 + \ell^2 d\rho^2 + \ell^2 \sinh^2\!\rho\, d\varphi^2. \tag{2.73}$$

Another solution of the hyperboloid equation (2.70) is

$$\begin{aligned} X_0 &= (r/\ell)t, \quad X_1 = \left(\ell^2 + (r/\ell)^2(x^2 - t^2 + \ell^2)\right)/2r, \\ X_2 &= (r/\ell)x, \quad X_3 = \left(\ell^2 + (r/\ell)^2(x^2 - t^2 - \ell^2)\right)/2r. \end{aligned} \tag{2.74}$$

This solution leads to the metric

$$ds^2 = \frac{\ell^2}{r^2}dr^2 + \frac{r^2}{\ell^2}\left(-dt^2 + dx^2\right), \tag{2.75}$$

where r is the radial coordinate with domain $(0, \infty)$, t is again the time coordinate and x is a spatial coordinate. These coordinates are called Poincaré coordinates, they cover only half of the hyperboloid. This form of the AdS-metric also appeared in the near-horizon geometry of black holes (2.10) and black strings (2.66).

An important feature of AdS-spaces is the presence of a boundary. The boundary of AdS_3 is not apparent in the global coordinate system. To make the boundary of AdS_3 manifest we change coordinates by $\tan\theta = \sinh\rho$, $0 \leq \theta < \frac{\pi}{2}$. The metric becomes

$$ds^2 = \frac{\ell^2}{\cos^2\!\theta}\left(-dt^2/\ell^2 + d\theta^2 + + \sin^2\!\theta\, d\varphi^2\right). \tag{2.76}$$

This form of the metric suggests a "conformal compactification", an overall rescaling of the metric by $(\ell^2/\cos^2\theta)^{-1}$. Such a rescaling does not change the causal structure of the space. The coordinates of θ, t and φ describe a solid cylinder, which has a natural boundary, represented in this case by the coordinates t

and φ. We usually write that the space with metric (2.73) has a boundary at $\rho = \infty$, with boundary coordinates t and φ. In terms of the Poincaré coordinates (2.75), the boundary is situated at $r = \infty$.

Now we have seen a view features of AdS_3, we will discuss some relevant physical aspects. The metrics (2.73) and (2.75) are solutions of the vacuum Einstein equations with a cosmological constant $\Lambda = -\frac{1}{\ell^2}$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \tag{2.77}$$

Since AdS_3 is a maximally symmetric space, the Riemann curvature tensor is given by

$$R_{\mu\nu\rho\sigma} = -\frac{1}{\ell^2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right). \tag{2.78}$$

The Ricci scalar R of AdS_3 is constant and equal to $-\frac{6}{\ell^2}$.

The equations of motion (2.77) can be derived from an action. An action principle is particularly useful for the transition from classical physics to quantum physics by the path integral approach. A satisfactory quantum theory of gravity is not yet established but we will see that the action principle is useful for a semi-classical analysis of gravity. The action for gravity in AdS_3 is the Einstein-Hilbert action with a cosmological constant

$$I_{\mathrm{bulk}} = \frac{1}{16\pi G_3}\int_{\mathrm{AdS}} d^3x\sqrt{|g|}\left(R - 2\Lambda\right), \tag{2.79}$$

where G_3 is the gravitational constant in three dimensions, g is the determinant of the metric $g_{\mu\nu}$. The action has the subscript "bulk" since the presence of a boundary needs the addition of a term I_{boundary}. This is necessary since variation of R with respect to $g_{\mu\nu}$ results in a total derivative. The boundary term ensures proper transformation properties. The boundary term is an integral of the extrinsic curvature K over the boundary

$$I_{\mathrm{boundary}} = \frac{1}{16\pi G_3}\int_{\partial\mathrm{AdS}} d^2x\sqrt{|h|}K, \tag{2.80}$$

where $h_{\mu\nu}$ is the induced metric from $g_{\mu\nu}$ on the boundary: $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$, with n_μ an orthonormal vector to the boundary. K is given in terms of $h_{\mu\nu}$ and n_μ by $h^\mu_\nu\nabla_\mu n^\nu$. The sum $I(g_{\mu\nu}) = I_{\mathrm{bulk}} + I_{\mathrm{boundary}}$ has proper transformation properties and Eq. (2.77) can be derived.

An action principle for gravity provides a way to determine the relative probability of different metrics. Gravity in three dimensions is distinguished from gravity in higher dimensions since topologically equivalent solutions are related by coordinate transformations. In other words gravity in three dimensions does not have local degrees of freedom. This can simply be seen by a count of the parameters in the Hamiltonian approach. Space time is decomposed in this context in space-like hyperplanes together with a time parameter t describing time-evolution. In a d-dimensional space time, the metric of the space-like hyperplane provides us $d(d-1)/2$ fields, which together with their $d(d-1)/2$ momenta add up to $d(d-1)$ degrees of freedom. These $d(d-1)$ d.o.f. are reduced by $d(d-1)/2$ initial value conditions and d coordinate transformations. So we end up with a total of $d(d-3)/2$ degrees of freedom, which indeed vanishes for $d = 3$. As a consequence, gravity waves do not exist in three dimensions and all AdS_3-spaces are locally equivalent. The geometries are not necessarily globally equivalent, a fact we will explore shortly.

The metric known as "thermal AdS_3" is obtained by introducing a radial coordinate $r = \ell \sinh \rho$ in Eq. (2.73). The metric is given by

$$ds^2 = -\frac{r^2 + \ell^2}{\ell^2} dt^2 + \frac{\ell^2}{r^2 + \ell^2} dr^2 + r^2 d\varphi^2. \tag{2.81}$$

We note that in the limit $r \to \infty$, this metric reduces to AdS_3 metric in Poincaré coordinates (2.75).

We are interested in a path integral approach to quantum gravity, that is in a sum over geometries weighted by the exponent of the action [76]. As in quantum field theory, the path integral is better behaved after a Wick rotation to Euclidean time $t_\mathrm{E} = -it$. The Euclidean form of the metric in Eq. (2.81) is

$$ds_\mathrm{E}^2 = \frac{r^2 + \ell^2}{\ell^2} dt_\mathrm{E}^2 + \frac{\ell^2}{r^2 + \ell^2} dr^2 + r^2 d\varphi^2. \tag{2.82}$$

A finite temperature in Lorentzian space translates to a periodic time after rotation to Euclidean space. We give t_E a periodicity by identifying t_E with $t_\mathrm{E} + 2\pi\tau_2\ell$. The temperature is related to the "length" of time $T^{-1} = 2\pi\tau_2\ell$. We introduce a more general periodicity by combining the translation in t_E by a translation of φ

$$(t_\mathrm{E}, \varphi) \sim (t_\mathrm{E} + 2\pi\tau_2\ell, \varphi + 2\pi\tau_1). \tag{2.83}$$

This identification and the original periodicity in φ can be summarized by the complex coordinate $z = (\varphi + it_\mathrm{E}/\ell)/2\pi$

$$z \sim z + m\tau + n, \qquad (m, n) \in \mathbb{Z}. \tag{2.84}$$

Compactification of the time coordinate transforms the infinite cylinder into a torus, which is conveniently parametrized as a lattice in the complex plane. The complex structure parameter is given by τ with $\mathrm{Re}(\tau) = \tau_1$ and $\mathrm{Im}(\tau) = \tau_2$. Taking $r \to 0$ shows that the one-cycle of the torus parametrized by φ is contractible in the solid torus, whereas the one-cycle parametrized by t is not contractible. We will refer to the metric Eq. (2.82) as the Euclidean metric of thermal AdS_3.

The identification (2.83) is complex in Lorentzian signature. This suggests that we should also take a complex identification for φ. We define

$$(t, \varphi) \sim (t + 2\pi i\tilde{\tau}_2\ell, \varphi + 2\pi i\tilde{\tau}_1), \tag{2.85}$$

with $\tau_1 = i\tilde{\tau}_1$ and $\tau_2 = \tilde{\tau}_2$.

To make a step towards a quantum theory of gravity, we would like to evaluate the Euclidean action $I_{\mathrm{E}}(g_{\mu\nu})$ explicitly, such that we can compare $\exp(-I_{\mathrm{E}}(g_{\mu\nu}))$ for different metrics $g_{\mu\nu}$. Naive evaluation of the integrals leads however to divergences, since asymptotically AdS$_3$-spaces are non-compact. A well-known regularization procedure exists [77, 78, 79] to subtract the infinities. Application of this procedure to thermal AdS$_3$ is straightforward. We make a coordinate transformation (note that this ρ is different from the one appearing in Eq. (2.73))

$$\rho = \frac{\ell}{r + \sqrt{r^2 + \ell^2}}, \tag{2.86}$$

such that the boundary of the solid torus is at $\rho = 0$ and the center of the interior at $\rho = 1$. This transformation brings the metric in Fefferman-Graham [80] form

$$ds^2 = \frac{\ell^2}{4}\frac{d\rho^2}{\rho^2} + \rho^{-1} g_{ij} dx^i dx^j, \tag{2.87}$$

where the indices (i, j) are indices of the boundary coordinates, in this case t and φ. The metric g_{ij} can be expanded as a function of ρ. We obtain in this case

$$g_{tt} = \frac{1}{4}(\rho + 1)^2, \qquad g_{\varphi\varphi} = \frac{\ell^2}{4}(\rho - 1)^2. \tag{2.88}$$

The regularization is now performed by replacing the boundary at $\rho = 0$ by $\rho = \epsilon$. The infinities are in this way easily identified and can be subtracted in a covariant way. For the action of thermal AdS$_3$, we obtain

$$I_{\mathrm{E,thermal}} = -\frac{\pi \ell \tau_2}{4G_3} = \frac{2\pi i}{24}(c_L \tau - c_R \bar{\tau}), \tag{2.89}$$

with $c_L = c_R = \frac{3\ell}{2G_3}$. The expression of $I_{\mathrm{E,thermal}}$ in terms of c_L, c_R and τ is suggested by the close connection between AdS$_3$ gravity and conformal field theory. The obtained value for c_L and c_R confirms the value found earlier by the analysis of asymptotic charges [69].

Another important metric in AdS$_3$ is the rotating black hole metric found by Bañados, Teitelboom and Zanelli [81] and is given by

$$\begin{aligned} ds^2 &= -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} dt^2 + \frac{\ell^2 r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 \\ &\quad + r^2 \left(d\varphi + \frac{r_+ r_-}{r^2} \frac{dt}{\ell} \right)^2, \end{aligned} \tag{2.90}$$

where r_+ and r_- are respectively the radii of the outer horizon and the inner horizon. Note that rotation to Euclidean signature $t \to -it$ makes the metric complex. This is however not harmful, see for example [82]. The limit $r \to r_+$ shows that in the black hole space-time the thermal circle is contractible and the circle parametrized by φ is non-contractible.

The three-dimensional black hole (2.90) satisfies a first law of black hole mechanics similar to (2.4) for four-dimensional charged black holes. The BTZ black hole carries angular momentum J instead of the electromagnetic charges q and p of the Reissner-Nordström solutions. The mass M and the angular momentum J are determined at infinity by the ADM method [13]. M and J are given by $M = \frac{r_+^2+r_-^2}{8G_3\ell^2}$ and the angular momentum $J = -\frac{r_- r_+}{4G_3\ell^2}$. The horizon length, surface gravity and angular velocity are determined at the horizon. The horizon length given by $L = 2\pi r_+$, the surface gravity at the horizon by $\kappa = \frac{r_+^2-r_-^2}{G_3 r_+ \ell^2}$ and the angular velocity at the horizon by $\Omega_\mathrm{H} = -\frac{r_-}{r_+}$. The first law reads in this case

$$\delta M = \frac{1}{2\pi}\kappa\frac{1}{4}\delta A + \Omega_\mathrm{H}\delta J, \tag{2.91}$$

The second and third law of black hole mechanics [14, 1] are also applicable in three dimensions. The temperature is again related to the surface gravity by $T_\mathrm{BH} = \kappa/2\pi$. The entropy of the three-dimensional black hole is correspondingly $L/4G_3 = \pi r_+/2G_3$. This value of the entropy agrees with the Cardy formula and the central charge obtained by studying the asymptotic symmetries. The Cardy formula expresses the leading entropy in terms of the central charge, see for more details Sec. 3.3.2.

Since there are no degrees of freedom in three dimensions, the black hole metric (2.90) can be obtained by a coordinate transformation from the thermal metric (2.81). The following change of variables transforms the thermal AdS metric to the black hole metric

$$t \to i\frac{r_-}{\ell}t + ir_+\varphi, \quad \varphi \to i\frac{r_+}{\ell}\frac{t}{\ell} + i\frac{r_-}{\ell}\varphi, \quad r^2 \to \ell^2\frac{r^2 - r_+^2}{r_+^2 - r_-^2}. \tag{2.92}$$

Similarly as in the case of thermal AdS_3, a finite temperature is equal to a complex periodicity. The periodicities of the rotating black hole are [83]

$$t \sim t + \frac{2\pi i r_+\ell^2}{r_+^2 - r_-^2}, \qquad \varphi \sim \varphi - \frac{2\pi i r_-\ell}{r_+^2 - r_-^2}. \tag{2.93}$$

The black hole has the same periodicities (2.83) as thermal AdS_3 if we express r_+ and r_- in terms of $\tilde\tau_1$ and $\tilde\tau_2$ by

$$r_+ = \frac{\tilde\tau_2 \ell}{\tilde\tau_2^2 - \tilde\tau_1^2}, \qquad r_- = \frac{-\tilde\tau_1 \ell}{\tilde\tau_2^2 - \tilde\tau_1^2}. \tag{2.94}$$

In terms of the Euclidean coordinates $t_{\rm E}$ and τ, the coordinate transformation (2.92) is

$$z \to -\frac{z}{\tau}. \tag{2.95}$$

This shows that the black hole and thermal geometry are related by a modular transformation $\tau \to -1/\tau$. As in the case of thermal AdS_3, we can evaluate the Einstein-Hilbert action. After renormalization we find

$$S_{\rm E,BTZ} = -\frac{\pi r_+}{4G_3}, \tag{2.96}$$

With the given values for τ, this can be written as

$$S_{\rm E,BTZ} = \frac{2\pi i}{24}\left(-c_L\frac{1}{\tau} + c_R\frac{1}{\bar\tau}\right). \tag{2.97}$$

This is of course not unexpected, since (2.95) shows that the geometries are simply related by a modular transformation $\tau \to -1/\tau$.

One of the distinguishing features of three dimensional gravity is that diffeomorphism invariance and Lorentz invariance can be seen as gauge invariance of a proper gauge theory [84, 70]. The gauge fields are respectively the vielbein e^a_μ and the spin connection $\omega^a_{\ b\mu}$. We will view these as one forms e^a and $\omega^a_{\ b}$. We define moreover $\omega^a = \frac{1}{2}\epsilon_{ab}{}^c\omega^b{}_c$. The gauge group of the gauge theory is naturally $SL(2,\mathbb{R})^+ \otimes SL(2,\mathbb{R})^-$. The gauge fields read in terms of e^a and ω^a

$$A^{a\pm} = \omega^a \pm \frac{1}{\ell}e^a. \tag{2.98}$$

The Einstein-Hilbert action (2.79) can be written as the difference of the $SL(2,\mathbb{R})^+$ and $SL(2,\mathbb{R})^-$ Chern-Simons action (up to a boundary term)

$$\begin{aligned} I_{\rm bulk} &= \frac{4}{\ell}\int A^{a+}dA^{a+} + \frac{2}{3}\epsilon_{abc}A^{a+}A^{b+}A^{c+} \\ &\quad -\frac{4}{\ell}\int A^{a-}dA^{a-} + \frac{2}{3}\epsilon_{abc}A^{a-}A^{b-}A^{c-}, \end{aligned} \tag{2.99}$$

where the product is the wedge product.

Our interest in later sections will be in theories which contain besides the Einstein-Hilbert term more fields and interactions in the Lagrangian.

Especially we will be concerned with supersymmetric theories and their supersymmetric states. The symmetry group of AdS_3 is $SO(2,2)$ which is locally equal to $SL(2,\mathbb{R})^+ \otimes SL(2,\mathbb{R})^-$. The symmetry groups are extended to supergroups in a supersymmetric theory. States can be decomposed into representations of the supergroups. To preserve supersymmetries we require that the theory resides in its ground state with respect one chirality. This would imply that the partition function is independent of $\bar{\tau}$ and thus holomorphic.

A second possibility for a holomorphic action of τ is a chiral version of pure gravity. The connection between three-dimensional gravity and Chern-Simons theory suggests a natural candidate for such a theory. The Chern-Simons action $I_{\mathrm{CS}}(A^+)$ of $SL(2,\mathbb{R})^+$ is in terms of the familiar gravity quantities [70]

$$\begin{aligned} I_{\mathrm{CS}}(A^+) &= \frac{2}{\ell} I_{\mathrm{bulk}} + I_{\mathrm{CS}}(\omega) \\ &\quad + \int_{\mathrm{AdS}} \frac{1}{\ell} e^a de^a + \frac{2}{\ell^2} \epsilon_{abc} \omega^a e^b e^c + \frac{1}{\ell} \int_{\partial \mathrm{AdS}} e^a \omega^a, \end{aligned} \tag{2.100}$$

$I_{\mathrm{CS}}(\omega)$ is known as a gravitational Chern-Simons term. Addition of such a term to the action was first studied in [85] and is known as topologically massive gravity.

The vielbeins of the BTZ black hole are given by (with $f^2(r) = (r^2 - r_+^2)(r^2 - r_-^2)/r^2\ell^2$)

$$e^1 = f(r)dt, \qquad e^2 = f^{-1}(r)dr, \qquad e^3 = r(d\varphi - \frac{ir_+ r_-}{r^2} dt). \tag{2.101}$$

They determine the spin connections to be

$$\begin{aligned} \omega^1{}_2 &= \left(f(r)f'(r) + \frac{r_+^2 r_-^2}{r^3 \ell^2} \right) dt + \frac{ir_+ r_-}{r\ell} d\varphi = \frac{r}{\ell^2} dt + \frac{ir_+ r_-}{r\ell} d\varphi, \\ \omega^1{}_3 &= \frac{ir_+ r_-}{r^2 \ell} f^{-1}(r) dr, \\ \omega^2{}_3 &= -f(r) d\varphi. \end{aligned} \tag{2.102}$$

It is suggestive that an evaluation of Eq. (2.100) with the given expressions of the vielbein (2.101) and spin connection (2.102) will lead to a holomorphic action in τ. This hypothesis suggests that the action of the black hole is given by

$$I_{\mathrm{E,BTZ}} = \frac{2\pi i}{24} c_L \left(\frac{-1}{\tau} \right) = -\frac{\pi}{4G_3} \left(r_+ + ir_- \right). \tag{2.103}$$

Refs. [86, 87] study the contribution of the Chern-Simons term to the two-dimensional energy-momentum tensor by a gravity computation. This does agree with the expectations.

A variant of the Chern-Simons action as a function of ω, is the Chern-Simons action as a function of $\Gamma^{\sigma}_{\mu\nu}$. The difference between the two actions is that Lorentz invariance is broken in case of the spin connection and diffeomorphism invariance is broken in case of the Christoffel symbols. The theory gained a degree of freedom which is massive [85], see also [88].

At the classical level, gravity and Chern-Simons gauge theory seem equivalent. However if one quantizes Chern-Simons theory as an attempt to quantize gravity, one encounters the problem that the phase spaces can not be equivalent. Namely, a vanishing expectation value of the gauge field is a valid field configuration in the gauge theory. This would however correspond to a singular configuration from the point of gravity. Asserting that such states have to be allowed is problematic.

Around Eq. (2.95), we have seen that thermal AdS_3 and the BTZ black hole are related by the modular transformation $\tau \to -1/\tau$. We show now that by such modular transformations a much larger class of AdS_3-geometries is related to each other. All these geometries are asymptotically equivalent. They would therefore be relevant for the gravity path integral of AdS_3 in the spirit of Eq. (2.68).

As a start, consider a Lorentzian AdS_3 geometry with periodicities of (t, φ)

$$(t,\varphi) \sim (t,\varphi+2\pi) \sim (t+2\pi i\tilde{\tau}_2, \varphi+2\pi i\tilde{\tau}_1), \tag{2.104}$$

where the complex identification is the "thermal identification". This identification is naturally related to an Euclidean geometry with identifications

$$(t_{\mathrm{E}},\varphi) \sim (t_{\mathrm{E}},\varphi+2\pi) \sim (t_{\mathrm{E}}+2\pi\tau_2, \varphi+2\pi\tau_1), \tag{2.105}$$

The holomorphic one form of the Euclidean geometry is $2\pi dz = d\varphi + idt_{\mathrm{E}}/\ell$. The identifications (2.105) suggest a basis of one cycles given by (α, β) such that

$$\int_\alpha dz = \tau, \qquad \int_\beta dz = 1. \tag{2.106}$$

The two cycles have unit intersection $\alpha \cap \beta = 1$ and $\alpha \cap \alpha = \beta \cap \beta = 0$.

AdS_3 can be viewed topologically as a solid torus. A solid torus contains a contractible cycle. Generically, neither α nor β is contractible. Therefore, it is more convenient to use a basis (A, B) (with $A \cap B = 1$ and $A \cap A = 0 = B \cap B = 0$), where A is chosen to be the primitive contractible cycle. A can be written in terms of the basis (α, β) as $A = c\alpha + d\beta$. The integers (c, d) are relatively prime. Then the intersection $A \cap B = 1$ determines that $B = a\alpha + b\beta$ such that $ad - bc = 1$. Since $A \cap A = 0$, choices of B which differ by a multiple of A are equivalent.

The condition $ad - bc = 1$ shows that the two bases are related by an element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\Gamma := SL(2, \mathbb{Z})$. And the equivalence of choices of B determines that the geometries are in one-to-one correspondence with the left coset $\Gamma_\infty \backslash \Gamma$. The group Γ_∞ is the parabolic subgroup of translations and is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Elements of the coset are determined by the pair of relative prime integers $(c, d) = 1$. The correspondence with the coset is due to [89], earlier was a relation with Γ suggested [90].

The cycle A determines the physics of the geometry. For example, the thermal AdS$_3$-geometry has a spatial contractible cycle $A : \varphi + it_{\mathrm{E}} \sim \varphi + it_{\mathrm{E}} + 2\pi$. In terms of the basis (α, β), A is given by $(0, 1)$. In the BTZ black hole geometry, the time-circle is contractible $A : \varphi + it_{\mathrm{E}} \sim \varphi + it_{\mathrm{E}} + 2\pi\tau_1 + 2\pi i\tau_2$, A is thus $(1, 0)$. Therefore, all geometries correspond to Euclidean black hole geometries except for the geometry with $(c, d) = (0, 1)$. Fig. 2.1 demonstrates the contractability of the different geometries.

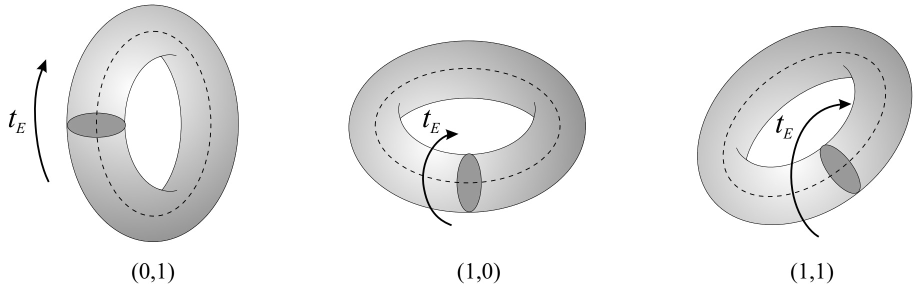

Fig. 2.1: From the left to the right: thermal AdS$_3$ with the β-cycle being contractible, the BTZ black hole geometry with the α-cycle being contractible, and the geometry with with the $\alpha+\beta$ cycle being contractible.

A geometry with contractible cycle $A = c\alpha + d\beta$ can be transformed to the thermal AdS$_3$-geometry. This involves the transformation $z \to \frac{z}{c\tau+d}$ and correspondingly $\tau \to \frac{a\tau+b}{c\tau+d}$. The Einstein-Hilbert action of these geometries is therefore

$$I_{\mathrm{E}}(\gamma) = \frac{2\pi i}{24}\left(c_L \frac{a\tau + b}{c\tau + d} - c_R \frac{a\bar{\tau} + b}{c\bar{\tau} + d}\right), \tag{2.107}$$

with $c_L = c_R = 3\ell/2G_3$.

As remarked earlier, our interest in later sections will be in AdS_3-supergravity . We argued that the evaluation of such an action for a black hole solution would be a holomorphic function of τ. The gravitational part of such an action would therefore be

$$I_{\mathrm{E,hol}}(\gamma) = \frac{2\pi i c_L}{24}\frac{a\tau + b}{c\tau + d}. \tag{2.108}$$

The geometries given by $(c, d) = 1$ should naturally be included in the gravity path integral. The naive Ansatz for such a path integral is

$$\mathcal{Z}_{\mathrm{AdS}}(\tau) = \sum_{\Gamma_\infty\backslash\Gamma} e^{-\frac{2\pi i c_L}{24}\frac{a\tau+b}{c\tau+d}}. \tag{2.109}$$

Whether these are all geometries which should be included in the sum is a priori not sure. Other geometries exist with equivalent asymptotic behavior, for example the cusp geometries. An argument not to include these geometries in the sum is that their volume is finite and does not need to be renormalized as we did for the other geometries. A later analysis will show that the CFT partition function can be rewritten in a form which closely resembles Eq. (2.109).

The path integral (2.109) is unfortunately divergent and a regularization is required. One can determine the divergence and subtract that from the path integral. To this end, rewrite the exponent for $c \neq 0$ as

$$e^{-2\pi i\left(\frac{c_L}{24}\frac{a}{c} - \frac{\frac{c_L}{24}}{c(c\tau+d)}\right)} = e^{\left(-2\pi i\frac{c_L}{24}\frac{a}{c}\right)}\left(\sum_{l=0}^{\infty}\frac{\left(2\pi i\frac{\frac{c_L}{24}}{c(c\tau+d)}\right)^l}{l!}\right). \tag{2.110}$$

Convergence of the sum over (c, d) can be shown for all but the term with $l = 0$. Thus, this term has to be subtracted from the sum. We arrive at

$$\mathcal{Z}_{\mathrm{AdS}}(\tau) = \sum_{\Gamma_\infty\backslash\Gamma} e^{-\frac{2\pi i c_L}{24}\left(\frac{a\tau+b}{c\tau+d}\right)} - r(a, c), \tag{2.111}$$

with

$$r(a,c) = \begin{cases} e^{-\frac{2\pi i c_L}{24}\frac{a}{c}}, & c \neq 0, \\ 0, & c = 0. \end{cases}$$

This is the regularization suggested in Ref. [91] for the partition function of pure gravity in AdS_3. In case of negative integer weight more terms need to be subtracted. This was proposed earlier in Ref. [37]. We propose that this is the proper way to regularize the gravity path integral in AdS_3, since the degeneracies are not changed with respect to the CFT partition function $\mathcal{Z}_{\mathrm{CFT}}(\tau)$ and it holds for general weights depending on the matter content of the theory. These assertions will be further explained in Chap. 4. Another (erroneous) regularization was earlier proposed in [89].

Note that the Laurent series of the series (2.111) must contain a principal part, since the element $(c,d)=(0,1)$ leads to a pole for $\tau\to i\infty$. When $c_L/24\in\mathbb{N}$, the regularized series is related to the modular invariant J-function. If $c_L/24=1$, the series is equal to $2J(\tau)+24$, with $J(\tau)=q^{-1}+196884q+\dots$. The term q^{-1} is referred to as a polar term. Polynomials in $J(\tau)$ are suggested in [92] as the holomorphic square root of partition functions of pure gravity. The corresponding CFT for $c_L/24=1$ is constructed by Frenkel, Lepowski and Meurman in [93] has partition function $|J(\tau)|^2$. Therefore, this CFT is conjectured to be the dual of pure gravity with $c_L/24=1$.

A series as (2.109) is known as a Poincaré series. In mathematics, the Poincaré series generically have no principal part and a positive weight ≥ 2, since the exponent is multiplied by $(c\tau+d)^{-k}$. The series with the regularization proposed above are referred to as a regularized Poincaré series. The weight 0 regularization is in Chapter 4 generalized to non-positive weight modular forms. This is essential to rewrite more general CFT partition functions as a sum over geometries. Such Poincaré series referred to in the physics literature as a Farey tail expansion [89], due to the connection with Farey fractions, see Chap. 4 for more on this connection.

One attractive feature of the Poincaré series (2.109) is that it is well-suited to deduce phase transitions between different AdS_3 geometries [89]. Such phase transitions were first described in four dimensions by Hawking and Page [94] and interpreted in the AdS/CFT context by Witten [95]. We can understand the phase transformations by determining which term in the sum (2.109) contributes most to the partition function. We have

$$|\mathcal{Z}_{\mathrm{AdS}}(\tau)|\leq\sum_{\Gamma_\infty\backslash\Gamma}e^{\frac{2\pi c_L}{24}\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}}. \tag{2.112}$$

So the combination of (c,d) which maximizes $\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}$ determines the term which contributes most to the path integral. This (c,d) describes the dominant classical geometry. Phase transitions occur between geometries by variation of τ. The regularization in (2.111) does not change this interpretation since its magnitude is always equal to unity.

3. BLACK HOLES IN M-THEORY

In the previous chapter, two main motivations were given for the study of black hole partition functions. The aim of this chapter is to explain the solution to the first motivating problem, the microscopic explanation of the leading black hole entropy in $\mathcal{N}=2$ supergravity. This is possible in the setting of 11-dimensional M-theory. The discussion of the AdS/CFT-correspondence showed briefly how the black holes are described in M-theory. Seven dimensions are compactified to a six-dimensional Calabi-Yau manifold X times the "M-theory circle" S^1_{M}. In Euclidean signature, the time-direction of four-dimensional space-time is the circle S^1_{t}. The total 11-dimensional geometry is therefore $\mathbb{R}^3 \otimes T^2 \otimes X$, where the torus T^2 is a product of S^1_{t} and S^1_{M}. The extra dimensions are crucial for the explanation of black hole entropy, since they can accommodate branes whose degrees of freedom account for the black hole entropy.

The black holes are sourced by M5-branes, which wrap T^2 and a four-cycle P inside the Calabi-Yau X. It is assumed that the M5-branes wrap a single Calabi-Yau, such that an appropriate scaling of the parameters results in a single AdS_3-throat in the geometry. Sec. 3.1 studies the low energy worldvolume theory of the M5-branes and the way electric charges arise by fluxes on the M5-brane. The worldvolume theory is a scale invariant theory with $\mathcal{N}=(2,0)$ supersymmetry. In the limit where the typical length scale of X is much smaller than that of T^2, the low energy degrees of freedom of the M5-brane can be reduced to the T^2. There, they form a two-dimensional $\mathcal{N}=(4,0)$ SCFT, also known as Maldacena-Strominger-Witten (MSW) CFT. [1] The degrees of freedom of this CFT account for the entropy of the single centered black holes. Sec. 3.3 defines a partition function for this two-dimensional superconformal field theory and studies some of its properties. Sec. 3.1 reviews the M5-brane worldvolume theory. Subsequently, it determines the number of degrees of freedom on T^2 after the reduction. Sec. 3.2 studies BPS states in the SCFT. The partition function and the microscopic account of S_{BH}

[1] Ref. [3] uses the convention that the supersymmetric side is anti-holomorphic such that the CFT has $(0,4)$ supersymmetry. Here the other convention is used since this is more natural from the point of view of the signature of the charge lattice.

are described in Sec. 3.3.

3.1 M5-branes on Calabi-Yau manifolds

The low energy limit of M-theory is 11-dimensional supergravity with 32 supersymmetries. Eleven is the maximum number of dimensions where a supermultiplet does not contain fields with spin > 2. The presence of an M5-brane breaks half of the supersymmetries. Therefore, only 16 supersymmetries are realized on the M5-brane. One can show that these 16 supersymmetries all have the same chirality such that the theory has $\mathcal{N} = (2,0)$ supersymmetry [96, 97, 12]. The fundamental worldvolume theory is a non-critical string theory in six dimensions. This is natural from the M2-brane point of view. The M5-branes are associated with the boundaries of M2-branes, analogous to the way D-branes are associated with the boundaries (endpoints) of strings [98, 61, 99]. The zero modes of this theory are five scalars, a self-dual three-form field strength H and four chiral fermions. The five scalars are the coordinates X^i in the orthogonal directions to the M5-brane. The self-dual field strength and the fermions form with the scalars a $(2,0)$ tensor multiplet. The full six-dimensional field theory is scale invariant.

Four-dimensional supergravity is obtained by a compactification on a Calabi-Yau manifold X times a circle S^1_{M}. In the Euclidean signature, the time dimension is compactified as a circle S^1_{t}. The two circles combine to the torus T^2. Since the holonomy group of X is $SU(3)$ (see next subsection), $\frac{1}{4}$ of the original supersymmetry is preserved. Therefore, eight supersymmetries remain in four dimensions which is equivalent to $\mathcal{N} = 2$. The electric and magnetic charges in supergravity are naturally associated to M2-branes wrapping two-dimensional surfaces in X and M5-branes wrapping four-dimensional surfaces in X and S^1_{M}. The surfaces are non-contractible (cycles) in X, which guarantees the stability of the charges.

The black holes are bound states of magnetic and electric charges. The M2-branes are embedded in the M5-brane as a non-zero flux of the self-dual gauge field H. The electric charges q_a are generated by this flux. These notions are explained in some more detail in subsection 3.1.3. The surface wrapped by the M5-brane has (complex) codimension one, which is why they are often referred to as "divisor" of X. The next subsection explains the relation between the magnetic charges p^a and the divisor P.

To count the number of states of the M5-brane theory, it is convenient to reduce the low energy degrees of freedom on $T^2 \otimes P$ to T^2. Some requirements need to be satisfied for this reduction to be valid. The low energy approximation to the M5-brane d.o.f. requires that the M5-branes are scarce in X. This translates to the requirement that $p^3 \ll v$. Note that this is the second regime of the discussion on the AdS/CFT correspondence and is opposite to the limit (2.67) where supergravity is valid. Of course, also the typical length scale of X needs to be much smaller than the length scale of T^2. We assume that the order of magnitude of the BPS degeneracies does not vary between these regimes. The leading entropy will be determined by an index, which is invariant under continuous variations of variables which preserve the supersymmetry.

The reduction of the M5-brane worldvolume theory leads to a variety of fields on T^2. The precise number of fields depends on the topological structure of P. Scalars arise on T^2 by two effects: deformations of the divisor in X, and the reduction of the three-form field strength H on two-cycles of P. The number of fermions is similarly related to the geometry of P. All these degrees of freedom together account for the black hole entropy in four dimensions! After a short introduction to Calabi-Yau manifolds, we will perform the reduction and determine the number of modes on T^2. This analysis is due to Ref. [3].

3.1.1 Calabi-Yau manifolds

We introduce complex three-dimensional Calabi-Yau manifolds and several of its properties, see for more details for example [10, 12, 100] and references therein. The complex three-dimensional Calabi-Yau X is a compact Kähler manifold with the additional constraint that its Ricci curvature vanishes. This statement is equivalent to saying that the first Chern class of the tangent bundle vanishes: $c_1(T_X) = 0$. As a consequence, the canonical bundle K_X of X (the line bundle, given by the highest exterior power of the holomorphic cotangent bundle) is also trivial, and therefore a global nowhere vanishing holomorphic $(3,0)$-form Ω does exist. This form is unique up to multiplication by a scalar. The Calabi-Yau condition can be formulated equivalently as the statement that a Kähler metric for X exists, which has holonomy group $SU(3)$. The holonomy group is the group of vector displacements under parallel transport of the vector around closed loops. This shows that X contains a covariantly constant spinor, since the generic holonomy group for six-manifolds is $SO(6) \cong SU(4)$. The existence of this covariantly constant spinor leads to the presence of $\mathcal{N} = 2$ supersymmetry in the resulting supergravity.

If the holonomy group of X is a subgroup of $SU(3)$, then the supergravity contains more supersymmetry. Therefore, we make the restriction that Calabi-Yau threefolds have proper $SU(3)$ holonomy. This is equivalent to the statement that X does not admit holomorphic $(n, 0)$-forms for $0 < n < \dim X = 3$. Or yet another formulation is that the Čech cohomology groups $H^n(X, \mathcal{O}_X) =$ are trivial for $0 < n < 3$; here $\mathcal{O}_X$ is the sheaf of holomorphic functions on X. This definition excludes Abelian varieties, and products of a K3 manifold and an elliptic curve as Calabi-Yau threefolds. This more restrictive definition is also more common in the mathematical literature.

The cohomology groups $H^{p,q}(X)$ form the Dolbeault cohomology ($\bar{\partial}$-cohomology) of (p, q)-forms on X. The spaces $H^{p,q}(X)$ are complex. Dolbeault cohomology on Kähler manifolds is a refinement of de Rham cohomology

$$H^r(X, \mathbb{C}) = \oplus_{p+q=r} H^{p,q}(X). \tag{3.1}$$

One of the basic characteristics of a compact Kähler manifold is the Hodge diamond, which presents the Hodge numbers $h^{p,q}$. These are the complex dimensions of Dolbault cohomology groups $h^{p,q} = \dim H^{p,q}(X)$. The Betti numbers b_r are the real dimensions of the de Rham cohomology groups $H^r(M, \mathbb{R})$, they equal the sum of the Hodge numbers $b_r = \sum_{p+q=r} h^{p,q}$. The *-operator on $H^{p,q}(X)$ and complex conjugation impose relations among the Hodge numbers [101]. In addition, the Calabi-Yau condition states that $h^{3,0} = 1$ and $h^{2,0} = h^{1,0} = 0$. The Hodge diamond is then given by

$$\begin{matrix} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{matrix} \quad = \quad \begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix} \; .$$

Poincaré duality states that the de Rham cohomology group $H^n(X, \mathbb{Z})$ is isomorphic with the homology group $H_{6-n}(X, \mathbb{Z})$. Therefore, the Hodge diamond provides us also the number of independent n-cycles of X, for example $\dim H_4(X) = h^{1,1}$. The homology class of the divisor P wrapped by the M5-brane can be expanded in a set of basis elements $\{A_a\}$ of $H_4(X, \mathbb{Z})$ as $p^a A_a$ with p^a the wrapping numbers. The Poincaré dual two-form of A_a is denoted by α_a. The Poincaré dual of the divisor is then $p^a \alpha_a \in H^2(X, \mathbb{Z})$. We usually refer to the two-form $p^a \alpha_a$ by P, the same symbol as for the divisor. A four-cycle $P = p^a A_a$ might self-intersect in points, where locally three components of P meet. The intersection number is given by $\int_X P^3 = d_{abc}p^a p^b p^c = p^3$ where

$$d_{abc} = \int_X \alpha_a \wedge \alpha_b \wedge \alpha_c. \tag{3.2}$$

As we can see from the macrocoscopic entropy formula (2.36), a non-zero self-intersection of the divisor P is crucial for describing a blackhole with a non-zero entropy.

3.1.2 Electromagnetic charges and their lattices

The electromagnetic charges q and p take values in the lattices $\Lambda^* + p/2$ and Λ, as is argued in section 2.1 from the supergravity perspective. For the following discussion, the shift of Λ^* by $p/2$ is not crucial and therefore we will ignore it. From the M-theory point of view, q and p naively correspond to elements of the homology groups $H_2(X,\mathbb{Z})$ and $H_4(X,\mathbb{Z})$, respectively. A closer inspection shows that the charges are more accurately described by K-theory [102]. This thesis will nevertheless treat the charges in the naive picture. Although, a pairing exists between $H_2(X,\mathbb{Z})$ and $H_4(X,\mathbb{Z})$ (the intersection number), appropriate quadratic forms for $H_2(X,\mathbb{Z})$ are not directly available. We will explain in the following paragraphs how a divisor P leads to natural quadratic forms for charges $k^a \in H_4(X,\mathbb{Z})$ and $q_a \in H_2(X,\mathbb{Z})$.

First, we consider the magnetic charges. The homology group $H_4(X,\mathbb{Z})$ is isomorphic to the cohomology group $H^2(X,\mathbb{Z})$ by Poincaré duality. The inclusion map $i : P \hookrightarrow X$ induces a pull back map i^*

$$i^* : H^r(X,\mathbb{Q}) \to H^r(P,\mathbb{Q}), \tag{3.3}$$

from r-forms on X to r-forms on P. The Lefschetz hyperplane theorem [101] states that for positive divisors of a complex three-dimensional manifold X, the map (3.3) is an isomorphism for $r \leq 1$ and injective for $r = 2$. This implies that $\dim H^1(P)$ is zero-dimensional, and consequently that P does not contain any one-cycles. On the other hand, the map $H^2(X,\mathbb{Q}) \to H^2(P,\mathbb{Q})$ is injective: $H^2(X,\mathbb{Q}) \subseteq H^2(P,\mathbb{Q})$. In the next section we will see that for "black hole divisors" $\dim H^2(P) \gg \dim H^2(X)$.

Since the degree of the forms in $H^2(P)$ is half the real dimension of P, the following inner product on $H^2(P)$ exists

$$Q : H^2(P) \otimes H^2(P) \to \mathbb{C}, \quad Q(\alpha_P, \beta_P) = \int_P \alpha_P \wedge \beta_P, \tag{3.4}$$

where we gave the forms the subscript P to denote that they are two-forms on P. This inner product is non-degenerate and symmetric. When one chooses a basis of $H^2(P)$ in $H^2(P,\mathbb{Z})$, Q can be written as an integral unimodular matrix, $\det Q = \pm 1$. It is integral since it can be related to the intersection matrix of two-cycles of P by Poincaré duality. Since, $H_2(P,\mathbb{Z})$ is also the dual space of $H^2(P,\mathbb{Z})$, Q must be unimodular [103].

The quadratic form for $H^2(P,\mathbb{Z})$ naturally provides us an integral quadratic form D for $H_4(X,\mathbb{Z})$ by the pull back (3.3) on the Poincaré dual forms in $H^2(X,\mathbb{Z})$. The map $D : H^2(X) \otimes H^2(X) \to \mathbb{C}$ is given by

$$D(\alpha,\beta) = \int_P i^*\alpha \wedge i^*\beta = \int_X \alpha \wedge \beta \wedge P. \tag{3.5}$$

The quadratic form, written as a matrix, is $d_{ab} = d_{abc}p^c$ with d_{abc} the earlier used triple intersection number of four-cycles (3.2). We have thus established a lattice Λ for the magnetic charges $k \in H_4(X,\mathbb{Z})$. The lattice Λ is

$$\Lambda = i^* H^2(X,\mathbb{Z}) \subset H^2(P,\mathbb{Z}). \tag{3.6}$$

Having determined the lattice Λ as the lattice of magnetic charges k, the electric charges q naturally take values in the dual lattice Λ^* since $q \cdot p \in \mathbb{Z}$. The quadratic form on Λ^* is $d^{ab} = (d_{ab})^{-1}$ and takes values in $\mathbb{Q}$. On the other hand, M2-brane charges are generated by flux of the self-dual field H on the M5-brane through two-cycles of P. This is a consequence of the coupling of H to the space-time three-form potential C in its low energy effective action

$$\int_{\Sigma\times t} C + \int_{P\times S^1_{\mathrm{M}}\times t} H \wedge C, \tag{3.7}$$

where Σ is a two-cycle. The part of H on P is valued in $H^2(P,\mathbb{Z})$ which would naively lead to charges in $H_2(P,\mathbb{Z})$. We take $\{\alpha_{P,j}\}$ as a basis for $H^2(P,\mathbb{Z})$. The field strength H can be expanded in this basis as

$$H = dz \wedge h_+ \cdot \alpha_P + d\bar{z} \wedge h_- \cdot \alpha_P. \tag{3.8}$$

The momentum of the scalar on the torus is $h_+ + h_-$ and the winding numbers are $h_+ - h_-$. The naive charge generated by this flux would be

$$q_P = \int_{\Sigma_P \times S^1_{\mathrm{M}}} h_+ \cdot \alpha_P + h_- \cdot \alpha_P, \tag{3.9}$$

where Σ_P is some two-cycle in P.

That H takes values in $H^2(P,\mathbb{Z})$ seems at odds with the earlier statement that the electric charges are valued in Λ^*. For example, the dimension of $H^2(P)$ might be much larger than $\dim \Lambda^*$. The solution to the mismatch between $H_2(X)$ and $H^2(P)$ was solved in [3] by realizing that charges in $\mathbb{R}^{3,1}$ corresponding to fluxes which do not lie in $i^*H^2(X,\mathbb{Z})$, are unstable. Fluxes of H which are exact on X are not conserved since they might decay by instanton effects [3, 104].

The space spanned by the lattice $H_2(P,\mathbb{Z})$ can be decomposed in the spaces spanned by Λ and its orthogonal complement $\Lambda_\perp$. The sum of these lattices forms a sublattice in $H_2(P,\mathbb{Z})$

$$\Lambda \oplus \Lambda_\perp \subset H_2(P,\mathbb{Z}). \tag{3.10}$$

An element of $k_P \in H_2(P,\mathbb{Z})$ can be decomposed in a parallel and orthogonal part to Λ

$$k_P = q + q_\perp. \tag{3.11}$$

The vectors q and $q_\perp$ lie respectively in the spaces spanned by Λ and $\Lambda_\perp$. However generically, they are not elements of these lattices but take their values in the dual lattices Λ^* and $\Lambda^*_\perp$. The group Λ^*/Λ is called the glue group of Λ in this context and its elements are gluing vectors [105]. The glue vectors are chosen such that they have minimal length in the coset. Thus, we have the following decomposition of an element $k_P \in H_2(P,\mathbb{Z})$:

$$k_P = k + k_\perp + \mu_P, \tag{3.12}$$

where $k \in \Lambda$, $K_\perp \in \Lambda_\perp$, $\mu_P \in \Lambda^* \oplus \Lambda^*_\perp$. Projecting the fluxes in $H^2(P,\mathbb{Z})$ to $H^2(X,\mathbb{Q})$ leads to charges $q \in \Lambda^*$. The embedding of $i^*H^2(X,\mathbb{Z})$ in $H^2(P,\mathbb{Z})$ is a primitive embedding [106]. For such an embedding, the discriminant group $H^2(P,\mathbb{Z})/(\Lambda \oplus \Lambda_\perp) \subset \Lambda^*/\Lambda \oplus \Lambda^*_\perp/\Lambda_\perp$ and $\Lambda^*/\Lambda \cong \Lambda^*_\perp/\Lambda_\perp$. Ref. [37] claims that $H^2(P,\mathbb{Z})/(\Lambda \oplus \Lambda_\perp) \cong \Lambda^*/\Lambda$ follows from the Nikulin embedding theorem [106].

At this point we would like to comment on the relation between the two-cycles in P and X. Taking the dual of (3.10) gives

$$\Lambda^* \oplus \Lambda^*_\perp \supset H^2(P,\mathbb{Z})^* = H_2(P,\mathbb{Z}). \tag{3.13}$$

Therefore the map from $H_2(P,\mathbb{Z})$ to $H_2(X,\mathbb{Z})$ is not surjective. We denote a two-cycle in P which also lies in X by K (its Poincaré dual two-form in P is also denoted by K). The two-cycles in P are given by a vector $k_a = d_{abc}k^b p^c$, with respect to the chosen basis of $H^2(X,\mathbb{Z})$.

From the Euler characteristic $\chi(K)$ of K, we will now deduce that $k^2 + k \cdot p \in 2\mathbb{Z}$, which makes p a so-called characteristic vector of Λ. To calculate $\chi(K)$, one has to integrate the first Chern class of the tangent bundle $c_1(TK)$ over K. After Eq. (3.28) is shown how $c_1(TP)$ and $c_2(TP)$ can be calculated using the adjunction formula [101]. Similarly one can show that $k^*c(TP) = 1 + k^*c_1(TP) = c(TK)(1+K)$, with k^* the pull back of the inclusion map $k : K \hookrightarrow P$. As a consequence, $c_1(TK) = -(K + k^*c_1(TP))$. Integration of $c_1(TK)$ over K gives

$$\chi(K) = \int_K c_1(TK) = -\int_X PK(K+P) = -k^2 - k \cdot p, \tag{3.14}$$

where Poincaré duality is applied. We can assume that K is a Riemann surface without boundaries if X is a smooth Calabi-Yau threefold. Therefore, we have $\chi(K) = 2 - 2g \in 2\mathbb{Z}$, with g the genus of K and it follows that $k^2 + k \cdot p \in 2\mathbb{Z}$ as claimed.

Another important issue is the signature of the matrix d^{ab}. This is the number of its positive eigenvalues minus the number of negative eigenvalues. The decomposition of the lattice in its positive and negative subspaces will play a significant role in the analysis of the black hole partition function. The signature $\sigma(S)$ of the intersection matrix of a surface S is called the index or Hirzebruch signature. It can be evaluated by the index theorem [101], which can be derived using the Lefschetz decomposition of the cohomology. One finds that

$$\sigma(S) = \sum_{p+q=0(2)} (-1)^p h^{p,q}. \tag{3.15}$$

Since the pairing between $H^{2,0}(S)$ and $H^{0,2}(S)$ is positive, it follows that the intersection matrix restricted to $H^{1,1}(S)$ has only one positive eigenvalue and therefore its signature is $(1, h^{1,1} - 1)$. We are interested in the signature of the inner product of two-forms on P which are also two-forms on X. Since $h^{2,0}(X) = h^{0,2}(X) = 0$, these forms are in $H^{1,1}(P)$ and the signature of d^{ab} is $(1, b_2 - 1)$. This result is known as the (Hodge) index theorem.

The projection of a non-zero vector q to the positive respectively negative definite sublattice is given by q_+ and q_- with $q_+^2 > 0$ and $q_-^2 < 0$. The positive direction in the magnetic lattice is given by the magnetic charge vector p^a since $p^3 = d_{abc}p^ap^bp^c > 0$. The projection of q_a to Λ_+^* is $(p \cdot q/p^3) d_{abc}p^bp^c$.

3.1.3 Reduction of M5-branes

This subsection reduces the low energy degrees of freedom of the M5-brane worldvolume theory to the torus T^2 formed by S^1_{t} and S^1_{M}. The compactification on X preserves $\frac{1}{4}$ of the supersymmetry since X has $SU(3)$-holonomy. The original $\mathcal{N} = (2,0)$ M5-brane theory in six dimensions reduces to an $\mathcal{N} = (4,0)$ SCFT. By an analysis of the divisor, we can determine the number of degrees of freedom on T^2. This provides us the magnitude of the central charges c_L and c_R, which are crucial in the account of the black hole entropy for large charges by the Cardy formula in Sec. 3.3.2.

The degrees of freedom on T^2 arise from basically four different phenomena:

- space-time momenta,
- moduli of the divisor,
- reduction of H on two cycles,
- fermions.

In the following, we will address the reduction of these different modes.

Space-time momentum

The coordinates of the black hole in $\mathbb{R}^3$ contribute three continuous bosonic degrees of freedom $\vec{p}$ to the holomorphic and anti-holomorphic sector of the SCFT.

Moduli of the divisor

The embedding of P in X can vary from point to point on T^2. Therefore, the variables determining P (moduli) arise as scalar degrees of freedom on T^2. To determine the number of moduli, we need to describe the divisor in a more formal way. A divisor is a sum of codimension 1 subvarieties V_i of X

$$P = \sum_i n_i V_i, \tag{3.16}$$

where n_i is the multiplicity of V_i in P. The set of non-zero n_i is finite. The divisor is called effective if $n_i \geq 0$ for all i. In that case, we write $P \geq 0$. Effective divisors are the relevant divisors for us, since $n_i < 0$ would indicate the presence of anti-M5-branes which would break all the supersymmetry. Note that the hypersurfaces V_i, situated at different places in X, might nevertheless be in the same homology class. Therefore, the expansion (3.16) is in terms of the numbers n_i, instead of p^a, which denote the homology class of the divisor in a certain basis of $H_4(X,\mathbb{Z})$.

The degrees of freedom associated to P become more manifest if P is locally considered as the zeros (or poles) of some (meromorphic) function

$h_{P,\alpha} : U_\alpha \to \mathbb{C}$, where the U_α form an open cover of X. The multiplicity of a zero of $h_{P,\alpha}$ is given by $n_i > 0$. The function diverges for $n_i < 0$, then $|n_i|$ is the order of the pole. An effective divisor can be described locally by holomorphic functions. The functions $h_{P,\alpha}$ are determined up to multiplication by non-zero holomorphic functions $g \in \mathcal{O}^*(U_\alpha)$. Heuristically, the divisor moduli are related to variations of the divisor $P \to P'$, such that the homology class of P' is equal to the one of P and P' is also effective. This gives a lot of freedom to choose the functions $h_{P,\alpha}$.

The homology class of the divisor is not changed if the defining functions $h_{P,\alpha}$ are multiplied by a global meromorphic function $f \in \mathcal{M}(X)$. In that case the divisor changes to $P' = P + (f)$. The divisor (f) of a meromorphic function is called a principal divisor. Divisors which differ from each other by a principal divisor are said to be linearly equivalent. All linearly equivalent divisors form naturally an equivalence class. We are interested in the linearly equivalent divisors, which are also effective

$$P' = P + (f) \geq 0. \tag{3.17}$$

The space of all effective divisors, which are linearly equivalent to P, is called a (complete) linear system $|P|$.

We define the space $\mathcal{L}(P)$ as

$$\mathcal{L}(P) = \{f \in \mathcal{M}(X) : f = 0, \text{ or } P + (f) \geq 0\}, \tag{3.18}$$

where $\mathcal{M}(X)$ is the space of meromorphic functions on X. This is a vector space. Since multiplication by a constant does not change the zero locus of a function, the moduli space of the divisor is the projective space $\mathbb{P}(\mathcal{L}(P))$. The dimension of $|P|$ is thus given by

$$\dim |P| = \dim \mathcal{L}(P) - 1 \tag{3.19}$$

To calculate $\dim \mathcal{L}(P)$, we use the connection between divisors and line bundles. The local functions $h_{P,\alpha}$ can be extended to a global meromorphic section s_P of a holomorphic line bundle $\mathcal{P} \to X$. The transition functions $t_{P,\alpha\beta}$ on $U_\alpha \cap U_\beta$ are given by $h_{P,\alpha}/h_{P,\beta}$. The sheaf of holomorphic sections of $\mathcal{P}$ is denoted by $\mathcal{O}(\mathcal{P})$. An effective divisor is the zero locus of a global holomorphic section $s_P \in H^0(X, \mathcal{O}(\mathcal{P}))$, where H^0 is the zeroth Čech cohomology group. The divisor defines in this way the line bundle $\mathcal{P}$ up to isomorphism. In fact, a bijection exists between the linear equivalence classes of divisors and the isomorphism classes of line bundles. Line bundles, which belong to the same isomorphism class, have equal first Chern class $c_1(\mathcal{P})$. Moreover, $c_1(\mathcal{P})$ is the Poincaré dual two-form of the divisor

$$c_1(\mathcal{P}) = P = p^a \alpha_a. \tag{3.20}$$

Therefore, linearly equivalent divisors lie in the same homology class.

The connection between line bundles and divisors shows that there is a correspondence between $\mathcal{L}(P)$ and $H^0(X, \mathcal{O}(\mathcal{P}))$. Therefore, we can also express the linear system as $|P| = \mathbb{P}(H^0(X, \mathcal{O}(\mathcal{P})))$. A calculation of $\dim H^0(X, \mathcal{O}(\mathcal{P}))$ is generically not possible. However, the Hirzebruch-Riemann-Roch theorem provides us the alternating sum

$$\sum_{i=0}^{3} (-1)^i \dim H^i(X, \mathcal{O}(\mathcal{P})) = \int_X \mathrm{ch}(\mathcal{P})\,\mathrm{Td}(TX). \tag{3.21}$$

The conditions that the long wave length approximation to M-theory is valid, ensure that the divisor P is very ample [3], which means that the linear system defines an embedding into projective space. For a very ample divisor $H^i(X, \mathcal{O}(\mathcal{P})) = 0$ for $i > 0$ (page 228 in [107]), and therefore the index gives us $\dim H^0(X, \mathcal{O}(\mathcal{P}))$.

The Todd class $\mathrm{Td}(TX)$ of a Calabi-Yau manifold is given by

$$\mathrm{Td}(TX) = 1 + \frac{1}{12} c_2(X). \tag{3.22}$$

The Calabi-Yau condition $c_1(X) = 0$ implies that $\mathrm{Td}_3(TX) = 0$. The Chern character of the line bundle $\mathcal{P}$ is $e^{c_1(\mathcal{P})}$. As a result we find

$$\dim H^0(X, \mathcal{O}(\mathcal{P})) = \frac{1}{6} p^3 + \frac{1}{12} c_2 \cdot p. \tag{3.23}$$

The real dimension of the linear system $|P|$ is therefore

$$\dim_{\mathbb{R}} |P| = \frac{1}{3} p^3 + \frac{1}{6} c_2 \cdot p - 2, \tag{3.24}$$

which is the number of non-chiral scalars on the torus.

Gauge field scalars

The self-dual three-form H reduced on a two-cycle in P becomes a scalar on the torus. The self-duality of H under the Hodge $*$-operation combined with the (anti-)self-duality of harmonic two-forms on P, shows that these scalars are either holomorphic or anti-holomorphic. A proper understanding of the space of two-forms on P is therefore indispensable to determine the number and chiralities of the scalars in the SCFT.

The number of independent (anti-)self-dual harmonic two-forms is given by $b_2^+(P)$ ($b_2^-(P)$). By Hodge's theorem, $\mathrm{Harm}(P) \cong H^2(P)$, we know that $b_2^+(P) + b_2^-(P) = b_2(P)$. The index or Hirzebruch signature $\sigma(P)$ [101, 100] is

$$\sigma(P) = b_2^+(P) - b_2^-(P). \tag{3.25}$$

The difference $\sigma(P)$ arises also as the signature of the non-degenerate inner product Q (3.4), which exists on the space of two-forms $H^2(P)$. The index $\sigma(P)$ is equal to the number of positive eigenvalues $b_2^+(P)$ of Q minus the number of negative eigenvalues $b_2^-(P)$.

To determine the number of (anti)-chiral scalars on the torus, we decompose H as

$$H = d\phi^j(z,\bar{z}) \wedge \alpha_{P,j}, \tag{3.26}$$

where $\{\alpha_{P,j}\}$ is a basis of harmonic two-forms for $H^2(P)$.

We define the differentials $dz_{\mathrm{L}} = dx + dt$ and $d\bar{z}_{\mathrm{L}} = dx - dt$ on S^1_{M} times Lorentzian time. The Hodge $*$-operator in Lorentzian $\mathbb{R}_{\mathrm{t}} \otimes S^1_{\mathrm{M}}$ acts on these differentials by

$$*dz_{\mathrm{L}} = dz_{\mathrm{L}}, \qquad *d\bar{z}_{\mathrm{L}} = -d\bar{z}_{\mathrm{L}}. \tag{3.27}$$

The self-duality $H = *H$ combined with the (anti)-self-duality of the two-forms $\alpha_{P,j}$ determines that the CFT on the torus contains $b_2^+(P)$ left-moving scalars and $b_2^-(P)$ right-moving scalars. After a Wick rotation to Euclidean time, z_{L} becomes the familiar complex coordinate on the torus. Consequently, we obtain $b_2^+(P)$ holomorphic scalars and $b_2^-(P)$ anti-holomorphic scalars.

By the Lefschetz hyperplane theorem, P does not contain any one- and three-cycles. Therefore, the numbers $b_2^+(P)$ and $b_2^-(P)$ can be obtained by a calculation of the Euler characteristic $\chi(P) = \sum_{i=0}^4 (-1)^i b_i(P) = 2 + b_2(P)$ and the index $\sigma(P)$ of P. In the following calculation of these numbers is assumed that the divisor is smooth. The Euler characteristic is the integral over the second Chern class $c_2(TP)$

$$\chi(P) = \int_P e(TP) = \int_P c_2(TP). \tag{3.28}$$

To determine $c_2(TP)$, we consider the tangent space TP_p of P at a given point p. This space TP_p is a subspace of the tangent space of X at the point p, TX_p. The orthogonal complement of TP_p in TX_p is denoted by TN_p. The bundle TN is the normal bundle of P. The tangent bundle of X restricted to P, $TX|_P$, is simply the sum of TP and TN:

$$TX|_P = TP \oplus TN. \tag{3.29}$$

The total Chern class of a sum of bundles is given by the product of the Chern classes, thus

$$i^*c(TX) = c(TN)\, c(TP), \tag{3.30}$$

where i^* is the pull back of the inclusion map $i : P \hookrightarrow X$. Since X is a Calabi-Yau manifold $c_1(TX) = 0$. The adjunction formula states that the conormal bundle TN^* is equal to $\mathcal{P}^*|_P$, which is the restriction of the dual line bundle of $\mathcal{P}$ to the divisor P. The transition functions of the dual line bundle $\mathcal{P}^*$ are just the inverse of those of $\mathcal{P}$ ($\mathcal{P}^* = [-P]$ in the notation of [101]). From the adjunction formula, we find for the Chern class of the normal bundle $c(TN) = i^*c(\mathcal{P})$. Since $\mathcal{P}$ is a line bundle, $c_i(\mathcal{P}) = 0$ for $i > 1$. Therefore Eq. (3.30) becomes

$$\begin{aligned} 1 + i^*c_2(TX) + \dots =& (1 + i^*c_1(\mathcal{P}))(1 + c_1(TP) + c_2(TP) + \dots) \\ =& 1 + i^*c_1(\mathcal{P}) + c_1(TP) \\ &+ i^*c_1(\mathcal{P})\, c_1(TP) + c_2(TP) + \dots \end{aligned} \tag{3.31}$$

By comparing the first Chern classes on the left and right hand side we find $i^*c_1(\mathcal{P}) = -c_1(TP)$ and similarly $i^*c_2(TX) = i^*c_1(\mathcal{P})\, c_1(TP) + c_2(TP)$. This gives us finally for $c_2(TP)$:

$$c_2(TP) = i^*c_1^2(\mathcal{P}) + i^*c_2(TX). \tag{3.32}$$

After substitution of this formula in the integral (3.28), we find

$$\chi(P) = \int_P i^*P^2 + i^*c_2(TX) = \int_X P^3 + c_2(TX)P = p^3 + c_2 \cdot p. \tag{3.33}$$

The signature $\sigma(P)$ is given by the index theorem as the integral over the L-genus, which in the current case becomes an integral over the first Pontryagin class:

$$\sigma(P) = b_2^+(P) - b_2^-(P) = \int_P L = \int_P \frac{1}{3} p_1(TP). \tag{3.34}$$

Since $p_1(TP) = c_1^2(TP) - 2c_2(TP)$, we obtain

$$\begin{aligned} \sigma(P) &= -\frac{2}{3}\chi(P) + \frac{1}{3}\int_P c_1(TP)^2 \\ &= -\frac{1}{3}p^3 - \frac{2}{3}c_2 \cdot p. \end{aligned} \tag{3.35}$$

Combining (3.33) and (3.35), $b_2^+(P)$ and $b_2^-(P)$ can be evaluated to be

$$b_2^+(P) = \frac{1}{3}p^3 + \frac{1}{6}c_2 \cdot p - 1, \tag{3.36}$$

$$b_2^-(P) = \frac{2}{3}p^3 + \frac{5}{6}c_2 \cdot p - 1. \tag{3.37}$$

This provides us the number of holomorphic and (anti-)holomorphic scalars on T^2 due to the reduction of H on P.

Fermions

After the discussion of the bosonic degrees of freedom, we now analyze the fermions. Supersymmetry in the left-moving sector suggests an equal number of bosons and fermions, which will be confirmed by the calculation of the number of fermions based on the cohomology of P. We briefly review the connection between fermions and the cohomology of the compactification manifold. The algebra of Dirac matrices on a Kähler manifold is given by

$$\{\Gamma^a, \Gamma^{\bar{b}}\} = 2g^{a\bar{b}}, \quad \{\Gamma^a, \Gamma^b\} = 0, \quad \{\Gamma^{\bar{a}}, \Gamma^{\bar{b}}\} = 0, \tag{3.38}$$

where $a, \bar{a}$ are indices of the complex coordinates z^a and $\bar{z}^{\bar{a}}$. The algebra (3.38) resembles a fermionic creation-annihilation algebra [10]. If $|\,\Omega\,\rangle$ is a covariantly constant spinor on P, then one can find new spinors by acting with the creation operator $\Gamma^{\bar{a}}$,

$$|\,\Omega^{\bar{a}}\,\rangle = \Gamma^{\bar{a}}\,|\,\Omega\,\rangle\,, \qquad |\,\Omega^{\bar{a}\bar{b}}\,\rangle = \Gamma^{\bar{a}}\Gamma^{\bar{b}}\,|\,\Omega\,\rangle\,. \tag{3.39}$$

This suggests an identification of the fermions on P with the cohomology groups $H^{0,i}(P)$. The chirality operator anti-commutes with $\Gamma^{\bar{a}}$. Therefore, the cohomology group with i even leads to holomorphic fermions and analogously i odd to anti-holomorphic fermions. Since $b_1(P) = b_3(P) = 0$, the reduction gives only rise to $h^{0,0}(P) + h^{0,2}(P)$ holomorphic fermions, which can be evaluated in terms of p to be

$$h^{0,0}(P) + h^{0,2}(P) = \frac{1}{6}p^3 + \frac{1}{12}c_2 \cdot p. \tag{3.40}$$

Since only holomorphic fermions arise by the reduction, only the holomorphic sector will be supersymmetric. The $U(2)$-holonomy on P leaves only two components from the original eight components invariant. The original M5-brane symmetry is $\mathcal{N} = (2,0)$, therefore the number of fermionic d.o.f. on T^2 is $4(h^{0,0}(P)+h^{2,0}(P))$. This is indeed equal to sum of scalars due to space-time momenta, divisor moduli and gauge field reduction.

3.2 The $\mathcal{N} = (4,0)$ superconformal algebra

Section 2.1 explained that the charges of supersymmetric black holes were subject to a symplectic symmetry. In addition the black hole is a half-BPS state of $\mathcal{N} = 2$ supergravity and preserves four space-time supersymmetries. We expect to find analogues of these phenomena in the $\mathcal{N} = (4,0)$ SCFT. This section explains how symplectic invariance is manifested as spectral flow of the SCFT and how the black hole state can preserve four fermionic symmetries by the presence of fermionic zero-modes. We assume in the following that the reader is familiar with (super)conformal algebras. More details can be found in [9, 10, 11, 12, 108].

We are interested in $\mathcal{N} = (4,0)$ superconformal field theory. The Hamiltonian is given by

$$H = L_0 + \bar{L}_0 - \frac{c_L + c_R}{24}, \tag{3.41}$$

where L_0 and $\bar{L}_0$ are Virasoro operators. The numbers c_L and c_R are the central charges of the holomorphic and anti-holomorphic sector of the theory. Since we know the number of degrees of freedom after the analysis in section 3.1, c_L and c_R can be calculated. A boson contributes 1 to the central charge whereas a fermion contributes $\frac{1}{2}$. After adding up all contributions, we find

$$c_L = p^3 + \frac{1}{2} c_2 \cdot p, \qquad c_R = p^3 + c_2 \cdot p. \tag{3.42}$$

These values of the central charges confirm the value of $3\ell/2G_3$ obtained from three-dimensional supergravity below Eq. (2.67) since for c_L, $c_R \gg 1$ the linear contribution in p can be neglected.

The difference of the Virasoro generators is the momentum, q_0, around the M-theory circle

$$q_0 = L_0 - \bar{L}_0 - \frac{c_L - c_R}{24}. \tag{3.43}$$

This is the q_0-charge from supergravity. The other electric charges are eigenvalues of the $U(1)$-generators $J_{0,a}$.

The structure of the CFT will teach us many properties of the partition function in the next section. An important symmetry of the present $\mathcal{N} = (4,0)$ conformal algebra is "spectral flow". [2] This symmetry relates states with different eigenvalues of L_0, $\bar{L}_0$ and $J_{0,a}$. We have seen that the lattice of the charges of $J_{0,a}$ is not definite, one direction is positive and the others negative. The commutation relations of the Virasoro and

[2] A famous example of spectral flow is the symmetry relating the Ramond en Neveu-Schwarz sector in $\mathcal{N} = 1$ SCFT [12].

$U(1)$-generators are

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c_L}{12}(m^3-m)\delta_{m,-n}, \\
[J_{m,a}, J_{n,b}] &= m d_{ab}\delta_{m,-n}, \\
[L_m, (J_{n,a})_+] &= -m(J_{m+n,a})_+, \\
\left[\bar{L}_m, (J_{n,a})_-\right] &= -m(J_{m+n,a})_-,
\end{aligned} \tag{3.44}$$

The spectral flow symmetry leaves the above algebra invariant. It is given by

$$\begin{aligned}
L_m - \bar{L}_m &\to L_m - \bar{L}_m + k^a J_{m,a} + \frac{1}{2} d_{ab} k^a k^b \delta_{m,0}, \\
L_m + \bar{L}_m &\to L_m + \bar{L}_m + g^{ab} k_a J_{m,b} + \frac{1}{2} g_{ab} k^a k^b \delta_{m,0}, \\
J_{m,a} &\to J_{m,a} + d_{ab} k^b \delta_{m,0}.
\end{aligned} \tag{3.45}$$

The metric g_{ab} is positive definite and is related to d_{ab} by changing the negative signs of d_{ab}, thus $g_{ab}k^a k^b = k_+^2 - k_-^2$. It can be written in terms of the two-forms α_a as

$$g_{ab} = \int_P i^*\alpha_a \wedge * i^*\alpha_b. \tag{3.46}$$

Spectral flow maps a state with charge $q \in p/2 + \Lambda^*$ to another state with charge in $p/2 + \Lambda^*$ if the flow variable k^a lies in the magnetic lattice Λ. Therefore, spectral flow with $k \in \Lambda$ is a symmetry of the spectrum. The quantity

$$L_0 - \bar{L}_0 - \frac{1}{2} d^{ab} J_{0,a} J_{0,b} \tag{3.47}$$

is left invariant by spectral flow. With q_0 given by (3.43), this translates to the statement that $\hat{q}_0 = q_0 - \frac{1}{2}q^2$ is invariant. The charge $\hat{q}_0$ appeared below Eq. (2.34) in the discussion about symplectic invariance in supergravity. We therefore find here that spectral flow is the CFT analogue of symplectic invariance in supergravity.

The black hole states in the $\mathcal{N} = (4,0)$ superconformal field theory are expected to preserve four supersymmetries. Naively, this seems to require black hole states to be supersymmetric ground states of the SCFT, since the ground states preserve four supersymmetries. However, the SCFT actually contains large $\mathcal{N} = 4$ supersymmetry [109, 110], which allows also excited states to be invariant under four fermionic symmetries. The minimal $\mathcal{N} = 4$ superconformal algebra can be derived from a theory with four bosonic currents and four fermionic currents. The bosonic currents are the energy momentum tensor and an $SU(2)$ current algebra which provides three other bosonic currents. We give this $SU(2)$ the superscript $+$. The $SU(2)^+$ current algebra is

$$\left[T_m^{+i}, T_n^{+i}\right] = \epsilon^{ijk} T_{m+n}^{+k} - \frac{1}{2} k^+ m \delta_{m+n} \delta_{ij}, \tag{3.48}$$

with $i, j = 1 \dots 3$, k^+ is the level of the $SU(2)^+$ algebra and is proportional to the central charge $c_L = 6k^+$, m and n take values in $\mathbb{Z}$. The fermionic currents $G^a(z)$, $a = 1 \dots 4$ satisfy an anti-commuting algebra with r, s either integral (Ramond sector) or half-integral (Neveu-Schwarz sector). We are mainly interested in the Ramond sector. The modes with $r = s = 0$ are the supersymmetry generators. Some of these leave a BPS-state invariant.

The minimal $\mathcal{N} = 4$ algebra can be enlarged with an additional $SU(2)^- \oplus U(1)$ current algebra. This gives four more bosonic currents T^{-i}, $i = 1..3$ and U which are completed by four fermionic currents Q^a to form an $\mathcal{N} = 4$ algebra. It is this extended $\mathcal{N} = 4$ algebra which arises in the $\mathcal{N} = (4,0)$ SCFT. The total algebra has one free parameter γ, determining the $SU(2)$-levels k^+ and k^- in terms of the central charge $k^+ = c_L/6\gamma$ and $k^- = c_L/6(1-\gamma)$. The anti-commutation relations of the fermionic generators are [109]

$$\begin{aligned}
\{G_0^a, G_0^b\} &= 2\delta^{ab}\left(L_0 - \tfrac{c_L}{24}\right), \\
\{Q_0^a, Q_0^b\} &= -\tfrac{c_L}{12\gamma(1-\gamma)}\delta^{ab}, \\
\{Q_0^a, G_0^b\} &= 2\left(\alpha_{ab}^{+i} T_0^{+i} - \alpha_{ab}^{-i} T_0^{-i}\right) + \delta^{ab} U_0,
\end{aligned} \tag{3.49}$$

where $\alpha_{ab}^{\pm i} = \pm \delta_{[a}^{i} \delta_{b]}^{4} + \frac{1}{2}\epsilon_{iab}$. Note that in a unitary theory, the operators G_0^a are Hermitian. The Q_0^a are anti-Hermitian operators, $Q_0^{a\dagger} = -Q_0^a$ by the minus sign in the commutator.

We are interested in the number of supersymmetries of an excited state $|u, l^+, l^-\rangle$, where u, l^+ and l^- are respectively the eigenvalues of U, T_0^{+3} and T_0^{-3}. The only non-vanishing eigenvalues of the $SU(2)^\pm$-generators are the eigenvalues corresponding to $T_0^{\pm 3}$. As in Sec. 2.1, we have to determine the zero eigenvalues of the Hermitian (8×8)-matrix $\mathbf{M}$

$$\mathbf{M} = \left\langle u, l^+, l^- \right| \begin{pmatrix} \{G_0^a, G_0^b\} & \{G_0^{a\dagger}, Q_0^b\} \\ \{Q_0^{a\dagger}, G_0^b\} & \{Q_0^a, Q_0^b\} \end{pmatrix} \left| u, l^+, l^- \right\rangle, \tag{3.50}$$

We can find the off-diagonal elements of $\mathbf{M}$ using

$$\{Q_0^a, G_0^b\} = \begin{pmatrix} U & -T_0^{+3} + T_0^{-3} & 0 & 0 \\ T_0^{+3} - T_0^{-3} & U & 0 & 0 \\ 0 & 0 & U & -T_0^{+3} - T_0^{-3} \\ 0 & 0 & T_0^{+3} + T_0^{-3} & U \end{pmatrix}.$$

The eigenvalues of $\mathbf{M}$ can now easily be calculated. The characteristic polynomial reads

$$\left[\left(2L_0 - \frac{c_L}{12} - \lambda\right)\left(-\frac{c_L}{12\gamma(1-\gamma)} - \lambda\right) + \left(u^2 + \left(l^+ - l^-\right)^2\right)\right]^2 \times$$
$$\left[\left(2L_0 - \frac{c_L}{12} - \lambda\right)\left(-\frac{c_L}{12\gamma(1-\gamma)} - \lambda\right) + \left(u^2 + \left(l^+ + l^-\right)^2\right)\right]^2 .$$

Zero eigenvalues ($\lambda = 0$) of $\mathbf{M}$ exist if

$$L_0 - \frac{c_L}{24} = \frac{1}{2}\frac{u^2 + \left(l^+ \pm l^-\right)^2}{\frac{c}{12\gamma(1-\gamma)}}. \tag{3.51}$$

The two possibilities, $+$ and $-$, have both a degeneracy of two. If $l^+ = 0$ or $l^- = 0$, $\lambda = 0$ has degeneracy four and thus four supersymmetries are preserved.

This algebra can be interpreted in terms of the $\mathcal{N} = (4,0)$ M5-brane SCFT. The three non-compact dimensions of space-time and the single right-moving $U(1)$-charge q_+ form the $SU(2)^- \oplus U(1)$ current algebra. This is a "decompactified" version of the algebra because the level k^- is taken to ∞, which has the consequence that $SU(2)^- \oplus U(1) \to U(1)^4$ [110]. The limit $k^- \to \infty$ is equivalent to $\gamma \to 1$. The currents T^{-i}, Q^a and U are rescaled by $\sqrt{\frac{12(1-\gamma)}{c_L}}$. The limit $\gamma \to 1$ is then taken with the requirement that the rescaled operators remain finite. This removes the $SU(2)^+$-charges. The characteristic polynomial becomes

$$\left[\left(2L_0 - \frac{c_L}{12} - \lambda\right)(-1-\lambda) + \left(u^2 + (l^-)^2\right)\right]^4 . \tag{3.52}$$

For a five-brane state $|M, Z\rangle$, the eigenvalue u is the positive direction of the electric charges $u = q_+$ and $(l^-)^2 = \vec{p}^2$. So we find that there exists a zero eigenvalue with degeneracy four if $L_0 - \frac{c_L}{24}$ satisfies [110]

$$L_0 - \frac{c_L}{24} = \frac{1}{2}q_+^2 + \frac{1}{2}\vec{p}^2. \tag{3.53}$$

3.3 An elliptic genus for black holes

This section introduces a partition function for the $\mathcal{N} = (4,0)$ SCFT, which therefore also counts the degeneracies of the black holes. The elliptic genus for $(4,0)$ SCFT's appeared in a series of papers in 2006 [111, 112, 59, 55]. Ref. [37] performs a similar analysis which results in the same partition function from the point of view of IIA string theory. The external parameters, where the CFT depends on, are geometric data of the torus and the coupling to the space-time three form C. The dependence on C will be represented by the parameters t^a. The torus is conveniently represented as the quotient of the complex plane by a lattice L, spanned by generators $\vec{\alpha}$ and $\vec{\beta}$. The complex structure τ can be expressed in terms of these vectors as $\tau = \left(\vec{\alpha}\cdot\vec{\beta} + i|\vec{\alpha}\times\vec{\beta}|\right)/|\vec{\alpha}|^2$. Note that τ takes values in the upper half-plane $\mathcal{H} : \mathrm{Im}(\tau) > 0$. Since the CFT does not depend on the size or on any absolute direction of the lattice vectors, the only torus data the CFT depends on is τ [113, 108].

The previous section explained that the black hole states are half-BPS states of the holomorphic sector. To count these states, we would like to perform a trace over the Hilbert space restricted to the half-BPS states, for example

$$\mathrm{Tr}_{\frac{1}{2}\mathrm{BPS}}\, q^{L_0-\frac{c_L}{24}}\,\bar{q}^{\bar{L}_0-\frac{c_R}{24}}\,y^{J_0}, \tag{3.54}$$

where $q = e(\tau)$, $\bar{q} = e(-\bar{\tau})$ and $y^{J_0} = e(z\cdot J_0)$. The vector z^a are potentials for the electric charges. A canonical way exists to project a sum over a Hilbert space to supersymmetric states. The projection is obtained by the insertion of $(-1)^F$ in the trace, with F the fermion number. For example, for a $\mathcal{N} = 1$ supersymmetric system one can evaluate the trace

$$\mathrm{Tr}\,(-1)^F e^{-\beta H}, \tag{3.55}$$

where β is the inverse temperature and H is the Hamiltonian. It calculates the ground states of the theory with a $\pm$-sign depending on its fermion number. Massive states always come in pairs in a supersymmetric theory. Therefore, the contributions of these states cancel out and the trace is independent of β. This trace is the celebrated Witten index [114]. It is protected against continuous changes of the parameters, which preserve the supersymmetry. Such a deformation of the theory could lift some of the ground states to massive states. However, since the contribution from massive states vanishes, this would not change the value of the index.

The index does not count the absolute number of states, but the index and the number of states are expected to have the same order of magnitude. A simple example of this phenomena is the same order of magnitude of $b^2_+(P) - b^2_-(P)$ and $b^2(P)$. The fact that such a partition function is protected against continuous changes of the parameters is convenient for us, since the supergravity and M5-brane approximation require different regimes of the parameters. Because the index is protected, we can trust the extrapolation of the M5-brane calculation to the black hole regime.

As explained in the previous section, we are not interested in the holomorphic ground states but in half-BPS excited states. To count the half-BPS states we have to change our trace slightly. In fact, a trace with $(-1)^F$ vanishes in the presence of $\mathcal{N} = (4, 0)$ supersymmetry, by the presence of fermionic zero-modes. The half-BPS states lie in short multiplets, and are counted by a trace with insertion of $\frac{1}{2}F^2(-1)^F$ [110, 115]. The fermion number is given by $F = T_0^{+3}$ in terms of the above $(4, 0)$ algebra.

The insertion of $(-1)^F$ is necessary when the trace is over a Hilbert space with periodic fermions (Ramond sector). The insertion ensures the anti-commutating property of the fermions in the trace. The Ramond-Ramond boundary conditions are preserved under large reparametrizations of the torus. See also the next subsection.

A related issue is the fact that bosonic states might not be invariant under monodromies on the torus. This is related to the phenomenon that the shifted electric charges $q - p/2$ take values in Λ^* due to the Freed-Witten anomaly. A state with charge q can be represented in the CFT by a vertex operator $V_q(x)$ (with x the coordinate on T^2)

$$V_q(x) = e^{iq\cdot\phi(x)}. \tag{3.56}$$

One can consider the operator product expansion (OPE) of a vertex operator $V_k(x)$ and $V_q(0)$ with $k \in \Lambda$ and $q \in \Lambda^* + p/2$. One finds

$$V_k(x)V_q(0) \sim x^{k\cdot q}V_{k+q}(0). \tag{3.57}$$

Under a monodromy, $x \to e^{2\pi i}x$, the OPE will pick up a phase $e\,(k \cdot q) = (-1)^{k\cdot p}$. Locality of the OPE requires projection onto states with even $k \cdot p$. Therefore, the elliptic genus will contain a factor $e(p \cdot k/2)$ for it to be modular invariant. For convenience, a term $(-1)^{p\cdot q}$ will instead be inserted. This can be interpret as a half-integral contribution to the fermion number.

After insertion of the factors $\frac{1}{2}F^2(-1)^F$ and $(-1)^{p\cdot q}$, we find that the partition function (3.54) takes the form

$$\chi(\tau, \bar\tau, z) = \mathrm{Tr}_{\mathrm{R}}\, \frac{1}{2}F^2(-1)^{F+p\cdot J_0}q^{L_0-\frac{c_L}{24}}\bar q^{\bar L_0-\frac{c_R}{24}}y^{J_0}. \tag{3.58}$$

The subscript R indicates that the trace is taken in the Ramond sector. Eq. (3.58) can be obtained by differentiating

$$\mathrm{Tr}_{\mathrm{R}}\, q^{L_0-\frac{c_L}{24}}\,\bar{q}^{\bar{L}_0-\frac{c_R}{24}}\,e^{2\pi i v(F+p\cdot J_0)}y^{J_0}, \tag{3.59}$$

two times to v and then set $v=\frac{1}{2}$.

Eq. (3.58) is known as the $\mathcal{N}=(4,0)$ elliptic genus and is the analogue of the elliptic genus for $\mathcal{N}=(2,2)$ SCFT's. An elliptic genus is a refinement of a field theory index, it is a "character valued" index. When the SCFT is a sigma model, the elliptic genus can be specialized to topological quantities as the Euler number, Hirzebruch genus and $\hat{A}$-genus of the target manifold [116, 117]. The elliptic genus has had various applications in the counting of BPS states including the black hole state counting by Strominger and Vafa [4].

The form of $\chi(\tau,\bar\tau,z)$ suggests an integer expansion in terms of q, $\bar q$ and y from which we can deduce the black hole degeneracies. Our aim is to relate $\chi(\tau,\bar\tau,z)$ to $\mathcal{Z}_{\mathrm{BH}}$ (Eq. (2.43)). However, the integer expansion is obstructed in two ways. The first obstruction is the presence of the three space-time momenta, which lead to a term $\vec{p}^2$ in the Hamiltonian. They can simply be integrated over and we find an overall factor $(2\tau_2)^{-\frac{3}{2}}$.

Second, the regularization of the path integral leads to an overall factor

$$\exp\left(\frac{\pi z^2}{2\tau_2}\right), \tag{3.60}$$

and is known as the Quillen anomaly [118, 119, 120]. This can be viewed as a shift in the zero point energies. We find that $\chi(\tau,\bar\tau,z)$ can be written as

$$\chi(\tau,\bar\tau,z)=\frac{1}{(2\tau_2)^{\frac{3}{2}}}\exp\left(\frac{\pi z^2}{2\tau_2}\right)\mathcal{Z}_{\mathrm{CFT}}(\tau,\bar\tau,z), \tag{3.61}$$

where $\mathcal{Z}_{\mathrm{CFT}}(\tau,\bar\tau,z)$ has an expansion in terms of integers $c(q_{\bar 0},q_a)$

$$\mathcal{Z}_{\mathrm{CFT}}(\tau,\bar\tau,z)=\sum_{q_{\bar 0},q_a} c(q_{\bar 0},q_a)\bar{q}^{q_{\bar 0}}y^{q}(q\bar q)^{\frac{1}{2}q_+^2}. \tag{3.62}$$

The eigenvalues of $L_0-\frac{c_L}{24}$ and $\bar{L}_0-\frac{c_R}{24}$ are expressed in terms of $q_{\bar 0}$ and q_+. The degeneracies $c(q_{\bar 0},q_a)$ are denoted in (2.43) by $c_p(q)$. Since the magnetic charge p does not vary within a given CFT, the dependence of the degeneracies on p is omitted here. The value of $c(0,0)$ is $(-1)^{\chi(|P|)}\chi(|P|)$, where $\chi(|P|)=\frac{1}{6}p^3+\frac{1}{12}c_2\cdot p$ is the Euler number of the linear system $|P|$ [37]. We hope that the different uses of the symbol q will not confuse the reader.

The generating function $\mathcal{Z}_{\rm CFT}$ is related to $\mathcal{Z}_{\rm BH}$ (for the black holes which correspond to a single AdS$_3$-throat in five dimensions). However, $\mathcal{Z}_{\rm CFT}$ is a more refined generating function than $\mathcal{Z}_{\rm BH}$, since the Hamiltonian H appears also as a quantum number in the expansion, in contrast to $\mathcal{Z}_{\rm BH}$. The dependence on H easily disappears by setting $\tau_2 = 0$. This has the consequence that $\mathcal{Z}_{\rm CFT}(\tau, \bar\tau, z)$ is no longer convergent. The other identifications of variables are

$$\tau_1 = -\frac{1}{2} i\phi^0, \qquad \tau_2 = 0, \qquad z^a = -\frac{1}{2} i\phi^a, \tag{3.63}$$

The connection between the topological string partition function and $\mathcal{Z}_{\rm CFT}(\tau, \bar\tau, z)$ is further discussed in Chap. 5. When the context is clear, we will omit the subscript CFT in the following from $\mathcal{Z}_{\rm CFT}$.

3.3.1 Modular properties of the elliptic genus

In this section, we analyze symmetries of the partition function. This is very instructive and lets us evaluate ultimately the leading behavior of the entropy for large charges. The most important symmetry of the SCFT partition function is modular invariance. Modular invariance of the partition function is a consequence of the reparametrization invariance of the SCFT. A transformation of

$$\tau \to \gamma(\tau) = \frac{a\tau + b}{c\tau + d}, \qquad \gamma \in \Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}, \tag{3.64}$$

leaves the lattice invariant up to a rescaling. Such a transformation acts on v and z^a by

$$\gamma(v) = \frac{v}{c\tau + d}, \qquad \gamma(z^a) = \gamma(z^a_+) + \gamma(z^a_-) = \frac{z^a_+}{c\tau + d} + \frac{z^a_-}{c\bar\tau + d}. \tag{3.65}$$

The original trace (3.59), being a trace in the R-R sector, should be invariant under these transformations. This means among others that its weight is $(0, 0)$ [3]. Invariance of $\mathcal{Z}(\tau, \bar\tau)$ does not need to be checked for every element in Γ, since Γ is generated by the elements S and T:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{3.66}$$

[3] A function $f(\tau, \bar\tau)$ has weight $(k, \bar k)$ when it transforms as $f(\gamma(\tau), \gamma(\bar\tau)) = (c\tau + d)^k (c\bar\tau + d)^{\bar k} f(\tau, \bar\tau)$ under Γ. The "slash operator" $|_{(k,\bar k)}\gamma$ acts on a function by $f|_{(k,\bar k)}\gamma = (c\tau + d)^{-k}(c\bar\tau + d)^{-\bar k} f(\gamma(\tau), \gamma(\bar\tau))$.

Modular invariance might actually be broken by the presence of a global gravitational anomaly [12]. In that case, $\mathcal{Z}(\tau,\bar\tau,t)$ is invariant under $\gamma\in\Gamma$ up to a phase $\varepsilon(\gamma)$. Thus, in case of the T-transformation

$$T: \quad \mathcal{Z}(\tau+1,\bar\tau+1,z)=\varepsilon(T)\mathcal{Z}(\tau,\bar\tau,z). \tag{3.67}$$

The phase $\varepsilon(T)$ depends on the central charges and the zero-point energies. These work out such that

$$L_0-\bar L_0-\frac{c_L-c_R}{24}=\frac{p^3}{8}+\frac{c_R}{24} \mod \mathbb{Z}. \tag{3.68}$$

Using that the expression in (3.23) must be an integer, we deduce that $\varepsilon(T)=e\left(-\frac{c_2\cdot p}{24}\right)$. The anomaly is thus really a quantum effect, since c_2 arose as the first quantum correction to the prepotential in Chap. 2. Consistency of modular transformations requires that the S-transformation is also accompanied by a unitary prefactor $\varepsilon(S)$. Since both $(ST)^3=-\mathbf{1}$ and $S^2=-\mathbf{1}$ leave τ invariant, $\varepsilon(S)=\varepsilon(T)^{-3}$. The phases for general γ are denoted by $\varepsilon(\gamma)$.

The transformation properties of the generating function $\mathcal{Z}(\tau,\bar\tau,z)$ can be deduced from the invariance of (3.59). To this end, we first realize that the weight of $\chi(\tau,\bar\tau,z)$ is $(2,0)$ since it is obtained from (3.59) by a second derivative to v. After taking into account the transformation properties of the multiplying factor in (3.61) we find

$$\begin{aligned}\mathcal{Z}|_{(\frac12,-\frac32)}S= &\ \ \varepsilon(S)\, e\left(\frac{z_+^2}{2\tau}+\frac{z_-^2}{2\bar\tau}\right)\mathcal{Z}, &(3.69)\\ \mathcal{Z}|_{(\frac12,-\frac32)}T= &\ \ \varepsilon(T)\,\mathcal{Z}. &(3.70)\end{aligned}$$

In addition to these modular transformations, $\chi(\tau,\bar\tau,z)$ transforms nicely under shifts of z. The behavior under a shift $z\to z+l$ with $l\in\Lambda$ can be determined straightforwardly. The transformation rule for the shift $z\to z+\tau k_+ +\bar\tau k_-$ with $k\in\Lambda$ is a consequence of the the spectral flow symmetry (3.45). One finds that

$$\chi(\tau,\bar\tau,z+\tau k_+ +\bar\tau k_- +l)=(-1)^{p\cdot(k+l)}q^{-\frac12 k_+^2}\bar q^{-\frac12 k_-^2}y^{-k}\chi(\tau,\bar\tau,z). \tag{3.71}$$

The transformations (3.69) and (3.71) are very similar to the transformations of Jacobi forms described in detail in [121]. We will generalize some of the techniques of [121] to the present case of interest.

Spectral flow, $q\to q+k$ with $k\in\Lambda$, determines an equivalence class for the charges $q\in p/2+\Lambda^*$. The sum of all coset representatives forms the discriminant group Λ^*/Λ, which is finite and abelian. Its order $|\Lambda^*/\Lambda|$ divides $d = \det d_{ab}$. The representatives μ are chosen such that they have minimal length. They are the projection of μ_P in Eq. (3.12) to Λ^*. Spectral flow as a symmetry of the spectrum, determines that the coefficients $c(q_{\bar{0}}, q_a)$ in (3.62) depend only on the equivalence class of q_a in $\Lambda^*/\Lambda + p/2$ and $\hat{q}_{\bar{0}} = q_{\bar{0}} + \frac{1}{2}q^2 = n - \Delta_\mu$, with Δ_μ defined as the minimum value of $\hat{q}_{\bar{0}}$ for given $q = \mu \mod \Lambda$. We thus have

$$c(q_{\bar{0}}, q_a) = c_\mu(n). \tag{3.72}$$

Since the Fourier coefficients are given by $\tilde{c}_\mu(n)$ and $L_0 - \frac{c_L}{24} = \frac{1}{2}q_+^2$, we can decompose $\chi(\tau, \bar{\tau}, z)$ as

$$\mathcal{Z}(\tau, \bar{\tau}, z) = \sum_{\mu \in \Lambda^*/\Lambda} \overline{h_\mu(\tau)}\Theta_\mu(\tau, \bar{\tau}, z), \tag{3.73}$$

where $h_\mu(\tau)$ and $\Theta_\mu(\tau, \bar{\tau}, z)$ have the expansions

$$h_\mu(\tau) = \sum_{\hat{q}_{\bar{0}} \geq -\frac{c_R}{24}} c_\mu(q_{\bar{0}}, q_a) q^{\hat{q}_{\bar{0}}} = \sum_{n=0}^{\infty} c_\mu(n) q^{n - \Delta_\mu}, \tag{3.74}$$

$$\begin{aligned} \Theta_\mu(\tau, \bar{\tau}, z) \quad = \quad & \sum_{k \in \Lambda + p/2} (-1)^{p \cdot (k+\mu)} \\ & \times e\left(\tau(k+\mu)_+^2/2 + \bar{\tau}(k+\mu)_-^2/2 + (k+\mu)\cdot z\right). \end{aligned} \tag{3.75}$$

Note that all the τ and z-dependence of $\mathcal{Z}(\tau, \bar{\tau}, z)$ is captured by $\Theta_\mu(\tau, \bar{\tau}, z)$. Moreover, all the information of the index is captured by the functions $h_\mu(\tau)$. From the CFT point of view, the decomposition (3.73) can be understood by the Sugawara construction. This construction identifies the part due to the currents of the energy-momentum tensor. This explains also the lower bound on $\hat{q}_{\bar{0}}$: the contribution to the weight of the remaining part of $\bar{L}_0$ can not be negative in a unitary CFT. Therefore, $\hat{q}_{\bar{0}} \geq -c_R/24$, and Δ_μ is given by

$$\Delta_\mu = \frac{c_R}{24} - \left(\frac{\mu^2 + p\cdot\mu}{2} - \left\lfloor \frac{\mu^2 + p\cdot\mu}{2} \right\rfloor \right). \tag{3.76}$$

A more natural decomposition from the point of view of the Sugawara construction might be $\Theta_\mu(\tau, \bar{\tau}, z)/\eta(\bar{\tau})^{b_2 - 1}$ as in [122] for $\mathcal{N} = (2,2)$ elliptic genera. Since the above decomposition provides us a generating function with the black hole degeneracies, this will be used in the rest of the thesis.

This decomposition has an interpretation in AdS_3-gravity also. Eq. (3.73) is then a manifestation of the singleton spectrum. The singleton modes are pure gauge in the bulk of AdS_3, however the presence of the boundary makes them dynamical degrees of freedom. The $U(1)$ degrees of freedom cannot be dynamical in the bulk, since the boundary theory suggests that they would be free. However, the $U(1)$-fields would necessarily couple to gravity in the bulk, and thus not be free. Ref. [123] studied the singleton sector from a topological perspective. That the singleton sector is manifested by the decomposition of the partition function was conjectured in [124] and [125]. Ref. [59] gives an interpretation of these large gauge transformation as the nucleation of M5 and anti-M5-branes.

The Laurent expansion of the functions $h_\mu(\tau)$ contains a principal part (the part where $\hat{q}_{\bar{0}} < 0$). The principal part leads to a pole in $h_\mu(\tau)$ in the limit $\tau \to i\infty$. The functions $h_\mu(\tau)$ are thus meromorphic as function of τ. The limit $\tau \to i\infty$ is under Γ-transformations equivalent to the limit $\tau \to \mathbb{Q}$. The partition function can not have other poles than these, which can be rephrased by saying that $h_\mu(\tau)$ only admits poles at the cusps but not in the upper half-plane $\mathcal{H}$. We define the "polar spectrum" as the spectrum which corresponds to the principal part of $h_\mu(\tau)$. Chapter 4 explains that the degeneracies of the polar spectrum determine the partition function completely. Importantly, modularity imposes in general additional constraints on the polar degeneracies.

Fig. 3.1 presents the polar and non-polar spectrum for the SCFT, which corresponds to one M5-brane on the quintic, *i.e.* the degree five hypersurface in $\mathbb{CP}^4$. The elliptic genus of this SCFT is analyzed in [111]. The lattice Λ is one-dimensional in this case, since $b_2 = 1$. The triple intersection number of the hyperplane of the quintic is 5 and $c_2(X) = 10$. For one M5-brane, $p = 1$, the central charge c_R is 55 and $|\Lambda^*/\Lambda| = 5$. With (5.2), one can determine that $\Delta_0 = \frac{55}{24}$, $\Delta_1 = \Delta_{-1} = \frac{55}{24} - \frac{3}{5}$ and $\Delta_2 = \Delta_{-2} = \frac{55}{24} - \frac{2}{5}$. Using these data, it is straightforward to construct Fig. 3.1.

The theta functions transform among each other under modular transformations. They transform under S and T by

$$\begin{aligned} S \quad &: \quad \Theta_\mu\left(\frac{-1}{\tau}, \frac{-1}{\bar{\tau}}, \frac{z_+}{\tau} + \frac{z_-}{\bar{\tau}}\right) = \frac{1}{\sqrt{|\Lambda^*/\Lambda|}} (-i\tau)^{b_2^+/2} (i\bar{\tau})^{b_2^-/2} \\ &\qquad \times e\left(\frac{z_+^2}{2\tau} + \frac{z_-^2}{2\bar{\tau}}\right) e\left(-\frac{p^2}{4}\right) \sum_{\delta \in \Lambda^*/\Lambda} e(-\delta \cdot \mu) \Theta_\delta(\tau, \bar{\tau}, z), \\ T \quad &: \quad \Theta_\mu(\tau + 1, \bar{\tau} + 1, z) = e\left(\frac{(\mu + p/2)^2}{2}\right) \Theta_\mu(\tau, \bar{\tau}, z). \end{aligned} \tag{3.77}$$

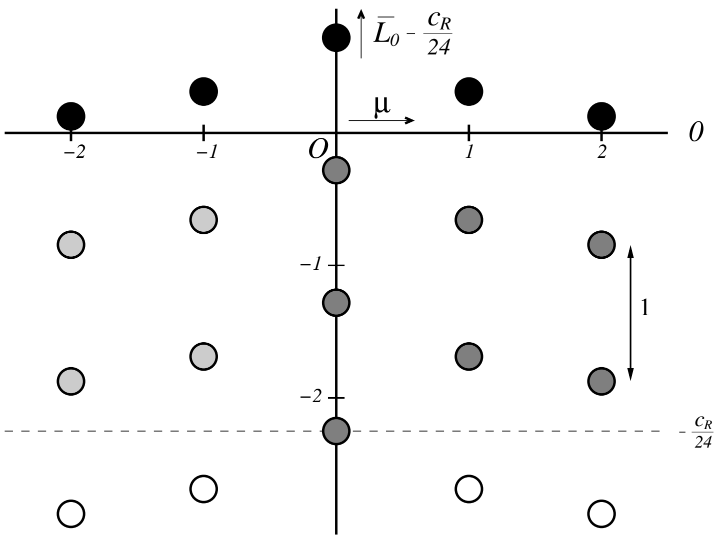

Fig. 3.1: The lowest part of the spectrum for one M5-brane on the quintic. The black-filled circles represent states in the non-polar spectrum. The grey-filled circles correspond to states in the polar spectrum. By modularity, the degeneracies of the states, corresponding to the circles with a light grey filling, are determined by their mirror images with respect to the vertical axis (whose grey filling is darker). No states correspond to the non-filled circles.

Note that p^2 is actually p^3 in (3.77), since the inner product is given by $d_{abc}p^c$. However, since we are working here with quadratic forms, we use the notation p^2. The weight of the theta functions is $(\frac{1}{2}, \frac{b_2-1}{2})$, which is generically half-integral. We must therefore choose a branch of the logarithm for consistent modular transformations [126], which we take to be $\log z := \log|z| + i\arg z$ with $-\pi < \arg z \leq \pi$. The transformation properties of the theta functions provides a representation of the Abelian group Λ^*/Λ, known as the Weil representation [127].

This works out nicely with the meromorphicity of the functions $h_\mu(\tau)$. They have weight $(-\frac{b_2}{2}-1, 0)$ and transform under S and T as

$$\begin{aligned} S \quad &: h_\mu\left(\frac{-1}{\tau}\right) = -\frac{1}{\sqrt{|\Lambda^*/\Lambda|}}(-i\tau)^{-b_2/2-1}\varepsilon(S)^* e\left(-\frac{p^2}{4}\right) \\ &\qquad \times \sum_{\delta\in\Lambda^*/\Lambda} e(-\delta\cdot\mu)h_\delta(\tau), \\ T \quad &: h_\mu(\tau+1) = \varepsilon(T)^* e\left(\frac{(\mu+p/2)^2}{2}\right) h_\mu(\tau). \end{aligned} \tag{3.78}$$

The additional $-$-sign, appearing in the S-transformation, is a consequence of the unitary factor in (3.77). Such a set of modular forms, which transform among each other under Γ, is known as a vector-valued modular form. In a given $h_\mu(\tau)$, $\hat{q}_{\bar{0}}$ takes values in $\frac{c_2\cdot p}{24} + \frac{1}{2}(\mu + p/2)^2$ mod $\mathbb{Z}$. The vector $h_\mu(\tau)$ has length $|\Lambda^*/\Lambda|$. Some elements of the vector are however related. This is a consequence of the transformation of $\mathcal{Z}(\tau,\bar{\tau},z)$ under $S^2 = -\mathbf{1}$

$$\mathcal{Z}(\tau,\bar{\tau},-z) = \varepsilon(S)^2\mathcal{Z}(\tau,\bar{\tau},z). \tag{3.79}$$

Moreover, $-\mathbf{1}$ acts on $\Theta_\mu(\tau,\bar{\tau},z)$ by

$$\Theta_\mu(\tau,\bar{\tau},-z) = (-1)^{p^2}\Theta_\nu(\tau,\bar{\tau},z), \qquad \nu = -\mu \mod \Lambda. \tag{3.80}$$

Since $-\mathbf{1}$ acts trivially on $h_\mu(\tau)$, these equations determine that

$$h_\mu(\tau) = h_\nu(\tau), \qquad \nu = -\mu \mod \Lambda. \tag{3.81}$$

Thus the length d of the vector $h_\mu(\tau)$ can be reduced to the order of the group $\Lambda^*/\Lambda \otimes R$, where R is the reflection group of Λ^*. For a one-dimensional lattice with inner product ℓ, $d = \frac{1}{2}\ell + 1$ for ℓ even, and $d = \frac{1}{2}(\ell+1)$ if ℓ is odd. This is depicted in Fig. 3.1. The degeneracies of the states with negative μ (light grey filling), are equal to the corresponding degeneracies of states with positive μ (dark grey filling). Therefore, there are seven independent polar degeneracies.

The T-transformation of $h_\mu(\tau)$ shows that an integer m exists, such that $h_\mu(\tau+4m) = h_\mu(\tau)$ for every $\mu \in \Lambda^*/\Lambda$. Using the S-transformation and (3.81), one can show that for this m,

$$h_\mu\left(\frac{\tau}{4m\tau+1}\right) = (4m\tau+1)^{-b_2/2-1} h_\mu(\tau). \tag{3.82}$$

Therefore, the forms $h_\mu(\tau)$ are modular forms of the principal congruence subgroup $\Gamma(4m)$ of level $4m$ when b_2 is even. If b_2 is odd, the square root requires the introduction of the group $\Gamma(4m)^*$, see the next chapter for more details. The group $\Gamma(N)$ is given by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}. \tag{3.83}$$

The observation that all elements of $h_\mu(\tau)$ are modular forms of some $\Gamma(4m)$ will be important to determine the dimension of the space of such vector-valued forms in Sec. 4.3.

3.3.2 The microscopic account of black hole entropy

The previous subsection derived a generating function for the degeneracies of supergravity black holes. It is a meromorphic vector-valued modular form with a negative weight. Modular covariance puts constraints on the behavior of the Fourier coefficients and therefore on the spectra of conformal field theories [113]. We now derive the leading order behavior of the Fourier coefficients for large $\hat{q}_{\bar{0}}$ and confirm the supergravity entropy.

Black hole entropies, or equivalently the coefficients $c(q_{\bar{0}}, q_a)$, are determined by the integral

$$e^{S_{\mathrm{BH}}(q,p)} = c(q_{\bar{0}}, q_a) = \int_{\mathcal{C}} d\bar{q}\, dy\, \mathcal{Z}(\tau,\bar{\tau},z)\, \bar{q}^{-q_{\bar{0}}-1} y^{-q-1}. \tag{3.84}$$

The integration over y projects to $h_\mu(\tau)$ with $q = \mu \mod \Lambda$. Since our current aim is an estimate of only the leading behavior of the entropy, we will ignore the index μ. An upper bound for the Fourier coefficients is found by estimating an upper bound of the integrand

$$\begin{aligned} c(q_{\bar{0}}, q_a) &\leq \int_0^1 d\tau_1 \left| q^{-\hat{q}_{\bar{0}}} h(\tau) \right| \\ &= \int_0^1 d\tau_1 \left| q^{-\hat{q}_{\bar{0}}} h\left(\frac{a\tau+b}{c\tau+d}\right) \right|. \end{aligned} \tag{3.85}$$

The non-principal part of $h(\tau)$ is suppressed with $\frac{a\tau+b}{c\tau+d} \in \mathcal{H}$, $\left|h(\frac{a\tau+b}{c\tau+d})\right|$ is therefore well approximated by $e\left(-\frac{c_R}{24}\frac{a\tau+b}{c\tau+d}\right)$. The maximum of the integrand is found by the saddle point method at $c\tau+d = i\sqrt{\frac{c_R}{24\,\hat{q}_{\bar{0}}}}$. Note that the saddle point approximation is more accurate for $\hat{q}_{\bar{0}} \gg c_R$. This is a stronger condition than the condition $\hat{q}_{\bar{0}} \gg p$ in section 2.2 for a reliable low energy approximation of M-theory. Substituting the saddle point value for $(c\tau+d)$ in (3.85), we obtain

$$c(q_{\bar{0}}, q_a) \leq \exp\left(\frac{4\pi}{c}\sqrt{\frac{c_R\,\hat{q}_{\bar{0}}}{24}}\right) \leq \exp\left(4\pi\sqrt{\frac{c_R\,\hat{q}_{\bar{0}}}{24}}\right). \tag{3.86}$$

This result is known as the Cardy formula. After expressing c_R in terms of p, we find

$$S_{\mathrm{BH}}(q,p) = \pi\sqrt{\frac{2}{3}(p^3 + c_2\cdot p)\hat{q}_{\bar{0}}}, \tag{3.87}$$

which equals (2.53). Thus we have shown that the M5-brane degrees of freedom indeed account for the black hole entropy! In the next chapter, a more accurate calculation will be performed which determines the Fourier coefficients with an arbitrary accuracy.

4. VECTOR-VALUED MODULAR FORMS

The previous chapter showed the relevance of vector-valued modular forms for black hole partition functions. This chapter is devoted to a closer study of such modular forms. A couple important results are derived. Sec. 4.1 deduces an expression for the Fourier coefficients, with which they can be determined with an arbitrary accuracy. This is of course useful for more precise studies of black hole entropy. Using this expression, we are able to rewrite the vector-valued form as a (regularized) Poincaré series in Sec. 4.2. This makes the erroneous "Farey Tail transform", introduced by [89], superfluous. In the last part of the chapter, we discuss the space of relevant meromorphic vector-valued modular forms. This chapter takes a more mathematical approach, although at some places we refer to the relevance of the material for previous or future chapters. The reader who is more interested in the physical implications than in the derivations, might postpone reading of this chapter, and proceed directly to Chap. 5, which interprets and applies the results of this chapter in the physical context.

As a start, we state the properties of the vector-valued modular forms $h_\mu(\tau)$ more systematically. In this chapter, the index $\mu = 1, \ldots, d$ is only used as a label, in contrast to the previous chapter where they represented elements in Λ^*/Λ. Since the weight of the modular forms is half-integer in general, non-trivial unitary prefactors arise under Γ-transformations. A proper description of transformation properties of such modular forms requires the introduction of the metaplectic group $\tilde{\Gamma}$, which is a double cover of Γ. An element $\tilde{\gamma} \in \tilde{\Gamma}$ is represented by

$$\tilde{\gamma} = \left(\gamma, \epsilon\sqrt{j(\gamma,\tau)}\right), \qquad \gamma \in \Gamma, \qquad \epsilon = \pm 1, \tag{4.1}$$

where we have defined $j(\gamma,\tau) = c\tau + d$. The branch of the square root is chosen such that $z \in \mathbb{C}$ always has $-\pi < \arg z < \pi$. Generically, elements of $\tilde{\Gamma}$ are denoted with a tilde; in case the tilde is omitted ϵ is taken to be one. The factor $j(\gamma,\tau)$ satisfies the identity

$$j(\gamma\gamma',\tau) = j(\gamma,\gamma'(\tau))j(\gamma',\tau). \tag{4.2}$$

The product of two elements $\tilde\gamma, \tilde\gamma' \in \tilde\Gamma$ is defined by

$$\left(\gamma, \epsilon\sqrt{j(\gamma,\tau)}\right)\cdot\left(\gamma', \epsilon'\sqrt{j(\gamma',\tau)}\right) = \left(\gamma\gamma', \epsilon\epsilon'\sqrt{j(\gamma,\gamma'(\tau))}\sqrt{j(\gamma',\tau)}\right).$$

Generators of $\tilde\Gamma$ are $\tilde S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right)$ and $\tilde T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right)$.

We define the slash operator $|_w\tilde\gamma$ with $\tilde\gamma \in \tilde\Gamma$ acting on a modular form $h(\tau)$ of (possibly half-integer) weight w, by

$$h|_w\tilde\gamma = \epsilon^{-2w} j(\gamma,\tau)^{-w} h\left(\gamma(\tau)\right). \tag{4.3}$$

In the previous chapter, we observed that $h_\mu(\tau)$ is a modular form of $\Gamma(4m)$ if $w \in \mathbb{Z}$. To treat the cases with $w \in \mathbb{Z} + \frac{1}{2}$ half-integer, we define the group $\Gamma(4m)^* \in \tilde\Gamma$ as

$$\Gamma(4m)^* = \left\{ \tilde\gamma = \left(\gamma, \left(\frac{c}{d}\right)\varepsilon_d^{-1} j(\gamma,\tau)^{\frac{1}{2}}\right) \middle|\, \gamma \in \Gamma(4m)\right\}, \tag{4.4}$$

where $\left(\frac{c}{d}\right)$ is the extended Legendre symbol [126] and $\varepsilon_d = \sqrt{\left(\frac{-1}{d}\right)}$,

$$\varepsilon_d = \begin{cases} 1, & d = 1 \mod 4, \\ i, & d = 3 \mod 4. \end{cases} \tag{4.5}$$

Eq. (4.4) gives an explicit expression for ϵ in Eq. (4.1). This expression is derived from the transformation properties of the weight $\frac{1}{2}$ theta function $\Theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$ under $\Gamma_0(4)$, and is therefore consistent with the transformations of half-integer weight forms [126]. Since this expression for ϵ takes values in $(\pm1, \pm i)$, $\Theta(\tau)$, the metaplectic group is actually a four-sheeted cover of Γ. If $w \in \mathbb{Z} + \frac{1}{2}$, the $h_\mu(\tau)$ transform diagonally under $\Gamma(4m)^*$

$$h_\mu|_w\tilde\gamma = h_\mu, \qquad \tilde\gamma \in \Gamma(4m)^*. \tag{4.6}$$

Note that in case $w \in \mathbb{Z}$, an element $\tilde\gamma \in \Gamma(4m)^*$ acts on $h_\mu(\tau)$ as an element $\gamma \in \Gamma(4m)$.

Using the slash operator, we can state the transformation of a vector-valued modular form $h_\mu(\tau)$ in matrix notation as

$$\begin{pmatrix} h_1|_w\tilde\gamma \\ h_2|_w\tilde\gamma \\ \dots \\ \dots \\ h_d|_w\tilde\gamma \end{pmatrix} = \mathbf{M}(\tilde\gamma) \begin{pmatrix} h_1 \\ h_2 \\ \dots \\ \dots \\ h_d \end{pmatrix} \qquad \text{for} \quad \tilde\gamma \in \tilde\Gamma. \tag{4.7}$$

Boldface notation is used to denote matrices. The matrices $\mathbf{M}(\tilde{\gamma})$ with $\tilde{\gamma} \in \Gamma(4m)^*$ are the identity matrix. Therefore, they form a representation of the finite group $\tilde{\Gamma}/\Gamma(4m)^*$. This representation is denoted by $\mathbf{M}$. Consistency of modular transformations requires that

$$\begin{aligned} \epsilon_3^{2w}\mathbf{M}_\mu^\nu(\tilde{\gamma}_3)(c_3\tau + d_3)^w &= \epsilon_1^{2w}\epsilon_2^{2w}\mathbf{M}_\mu^\sigma(\tilde{\gamma}_1)\mathbf{M}_\sigma^\nu(\tilde{\gamma}_2) \\ &\quad \times (c_1\gamma_2(\tau) + d_1)^w(c_2\tau + d_2)^w, \end{aligned} \tag{4.8}$$

for $\gamma_3 = \gamma_1\gamma_2$. The sum over σ is implicit.

The functions $h_\mu(\tau)$ can be written as a Laurent expansion

$$h_\mu(\tau) = \sum_{n=0}^{\infty} c_\mu(n) q^{n-\Delta_\mu}, \tag{4.9}$$

where n takes values in $\mathbb{Z}$ but Δ_μ is generically valued in $\mathbb{Q}$. The expansion contains a principal part, namely the part with $n - \Delta_\mu < 0$. The next section shows that the coefficients $c_\mu(n)$ with $n - \Delta_\mu \geq 0$ can be expressed in terms of those with $n - \Delta_\mu < 0$. The coefficients $c_\mu(n)$, $n - \Delta_\mu < 0$ are called the polar coefficients. Note that for transformations $\gamma_n(\tau) = \tau + n$, $\mathbf{M}(\gamma)_\mu^\nu$ is given by $\delta_\mu^\nu e(-\Delta_\mu n)$.

We define the space $A_{w,\mathbf{M}}(\Gamma(4m)^*, \Delta)$ as the space of meromorphic vector-valued modular forms from $\mathcal{H} \to \mathbb{C}$:

- which transform under $\tilde{\Gamma}$ according to the representation $\mathbf{M}$ and have weight w,
- whose components are meromorphic modular forms of the subgroup $\Gamma(4m)^* \in \tilde{\Gamma}$,
- and whose only possible poles are for $\tau \to i\infty \cup \mathbb{Q}$; the maximum order of the pole is Δ.

The maximum order Δ is equal to $\max \Delta_\mu$, which is given by $c_R/24$ in the context of the $\mathcal{N} = (4,0)$ CFT. The number of polar coefficients is therefore finite. The relevant weight w in (3.78) is $\leq -\frac{3}{2}$. Most of our discussion is valid for weight $w \leq 0$.

4.1 Fourier coefficients by the Rademacher circle method

The Fourier coefficients $c_\mu(m)$ in (4.9) are determined by the integral

$$c_\mu(m) = \int_0^1 d\tau_1\, h_\mu(\tau) q^{-m+\Delta_\mu}. \tag{4.10}$$

In Sec. 3.3.2, a saddle point method was used to obtain the leading behavior of the coefficients. However, by a clever choice of integration contour, one can determine the Fourier coefficients exactly. This method is known as the Rademacher circle method [128] and is beautifully applied to $1/\eta(\tau)$ in Ref. [129]. It is generalized to vector-valued modular forms in Ref. [89]. The right contour is given by Ford circles which are based on Farey fractions.

Farey fractions and Ford circles

A Farey sequence F_N is a set of irreducible rational numbers (Farey fractions) k/c such that $0 \leq k \leq c \leq N$ and their greatest common divisor is equal to one: $(c,k)=1$. They are arranged in increasing order. The first three are given by

$$\begin{aligned} F_1 &= \left\{\frac{0}{1}, \frac{1}{1}\right\}, \\ F_2 &= \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}, \\ F_3 &= \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}. \end{aligned} \tag{4.11}$$

A Farey fraction $\frac{k}{c}$ defines a Ford circle $\mathcal{C}(k,c)$ in $\mathbb{C}$. Its center is given by $\frac{k}{c} + i\frac{1}{2c^2}$ and its radius is $\frac{1}{2c^2}$. Two Ford circles $\mathcal{C}(a,b)$ and $\mathcal{C}(c,d)$ are tangent whenever $ad - bc = \pm 1$. This is the case for Ford circles related to consecutive Farey fractions in a sequence F_N. Fig. 4.1 shows the Ford circles for F_4. The points of tangency of three circles related to three consecutive Farey fractions $\frac{k_1}{c_1} < \frac{k}{c} < \frac{k_2}{c_2}$ are given by

$$\alpha_1(k,c) = \frac{k}{c} - \frac{c_1}{c(c^2+c_1^2)} + \frac{i}{c^2+c_1^2}, \quad \alpha_2(k,c) = \frac{k}{c} + \frac{c_2}{c(c^2+c_2^2)} + \frac{i}{c^2+c_2^2}.$$

After this short digression, we can explain how a Farey sequence F_N provides us a contour of integration for (4.10). The contour starts at $\tau = i$ and follows the circle belonging to $\frac{0}{1}$ until the point where this circle is tangent to the circle corresponding to the next fraction in the Farey sequence. Proceeding in this way, the integration is along a part of every Ford circle corresponding to F_N. Fig. 4.2 shows the path for F_3. The integral over such a contour reads

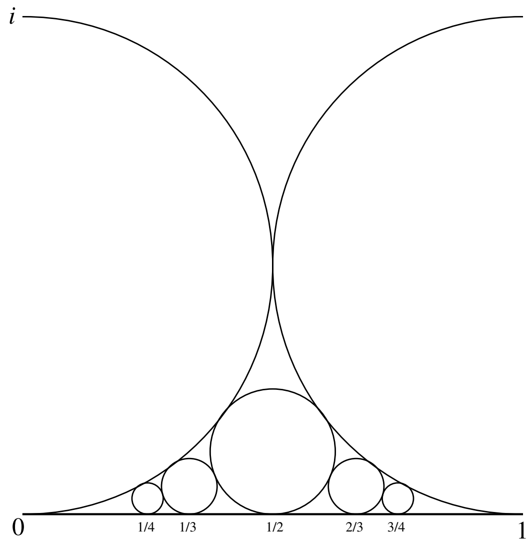

Fig. 4.1: Ford circles for F_4.

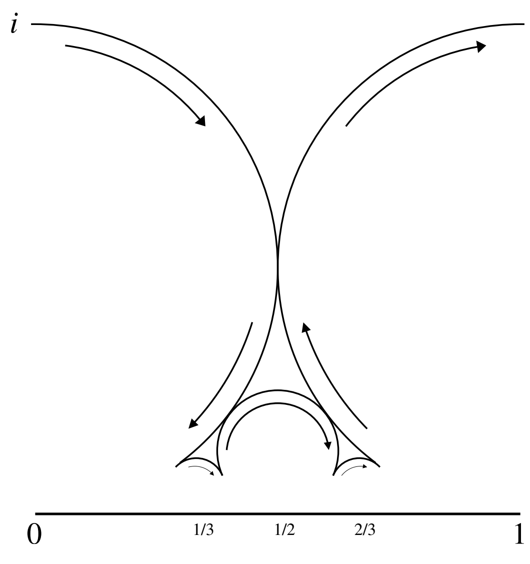

Fig. 4.2: Rademacher contour for F_3.

$$\begin{aligned} c_\mu(m) &= \int_i^{i+1} d\tau\, q^{-m+\Delta_\mu} h_\mu(\tau) \\ &= \frac{1}{2\pi i}\sum_{c=1}^{N} \sum_{\substack{0\le k\le c\\(c,k)=1}} \int_{\beta(k,c)} dq\, h_\mu(\tau) q^{-m+\Delta_\mu-1}, \end{aligned} \tag{4.12}$$

where $\beta(k,c)$ is the arc on $\mathcal{C}(k,c)$ from $\alpha_1(k,c)$ to $\alpha_2(k,c)$. The integral over $\beta(k,c)$ can be mapped by a coordinate transformation to a line element of a "standard" circle $\mathcal{C}_\mathrm{s}$. This is the circle centered at $(\frac{1}{2},0)$ with radius $\frac{1}{2}$. It is parametrized by $\frac{1}{2}+\frac{1}{2}\cos\theta+\frac{i}{2}\sin\theta$ with $0\le\theta<2\pi$. The transformation from $\mathcal{C}(k,c)$ to this circle is given by $z=-ic^2(\tau-\frac{k}{c})$ and its inverse is $\tau=\frac{k}{c}+\frac{iz}{c^2}$. The points of tangency are mapped to

$$z_1(k,c)=\frac{c^2}{c^2+c_1^2}+\frac{icc_1}{c^2+c_1^2}, \qquad z_2(k,c)=\frac{c^2}{c^2+c_2^2}-\frac{icc_2}{c^2+c_2^2}. \tag{4.13}$$

The starting point i of the integration contour on the $\frac{0}{1}$-circle and the end point $1+i$ on the $\frac{1}{1}$-circle are both mapped to one. The two arcs of the $\frac{0}{1}$- and $\frac{1}{1}$-circle can be combined into an integral along $\mathcal{C}_\mathrm{s}$ from $z_1(1,1)=\frac{N^2}{N^2+1}+\frac{iN}{N^2+1}$ to $z_2(1,1)=\frac{N^2}{N^2+1}-\frac{iN}{N^2+1}$. Fig. 4.3 shows for F_3, the image of $\beta(1,3)$ on $\mathcal{C}_\mathrm{s}$ with $z_1(1,3)$ and $z_2(1,3)$). The integral (4.12) is transformed to

$$\begin{aligned} c_\mu(m) &= \sum_{c=1}^{N}\frac{i}{c^2}\sum_{\substack{0<k\le c\\(k,c)=1}}\int_{z_1(k,c)}^{z_2(k,c)} dz\, h_\mu\left(\frac{k}{c}+\frac{iz}{c^2}\right) \\ &\quad\times e\left((-m+\Delta_\mu)\left(\frac{k}{c}+\frac{iz}{c^2}\right)\right). \end{aligned} \tag{4.14}$$

Note that $\gcd(c,k)=(c,k)$ is defined for $k=0$ by $(c,0)=|c|$, thus 0 is included in the sum only when $c=1$. Like in the case of the Cardy formula, we will now rewrite the partition function using a modular transformation $\gamma\in\Gamma$ as $h_\mu(\tau)=j(\gamma,\tau)^{-w}\mathbf{M}^{-1}(\gamma)_\mu^\nu h_\nu(\gamma(\tau))$. In this section, the factor ϵ in (4.3) is taken to be 1, therefore we omit the tildes from the equations. The modular transformation depends on the Farey fraction; γ is given by

$$\gamma=\begin{pmatrix} a & -\frac{ka+1}{c} \\ c & -k\end{pmatrix}=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma, \tag{4.15}$$

with $ka=-1 \mod c$ and we have defined the integers b and d. This transforms $\tau=\frac{k}{c}+\frac{iz}{c^2}$ to $\gamma(\tau)=\frac{a}{c}+\frac{i}{z}$, and $h_\mu\left(\frac{k}{c}+\frac{iz}{c^2}\right)$ can be rewritten to

$$h_\mu\left(-\frac{d}{c}+\frac{iz}{c^2}\right)=\left(\frac{iz}{c}\right)^{-w}\mathbf{M}^{-1}(\gamma)_\mu^\nu h_\nu\left(\frac{a}{c}+\frac{i}{z}\right). \tag{4.16}$$

The contour integral becomes

$$\begin{aligned} c_\mu(m) &= \sum_{c=1}^{N} \sum_{\substack{-c\le d<0\\(c,d)=1}} \frac{i^{1-w}}{c^{2-w}} \mathbf{M}^{-1}(\gamma)_\mu^\nu \\ &\times \int_{z_1(-d,c)}^{z_2(-d,c)} dz\, z^{-w} h_\nu\left(\frac{a}{c}+\frac{i}{z}\right) e\left((-m+\Delta_\mu)\left(-\frac{d}{c}+\frac{iz}{c^2}\right)\right). \end{aligned} \tag{4.17}$$

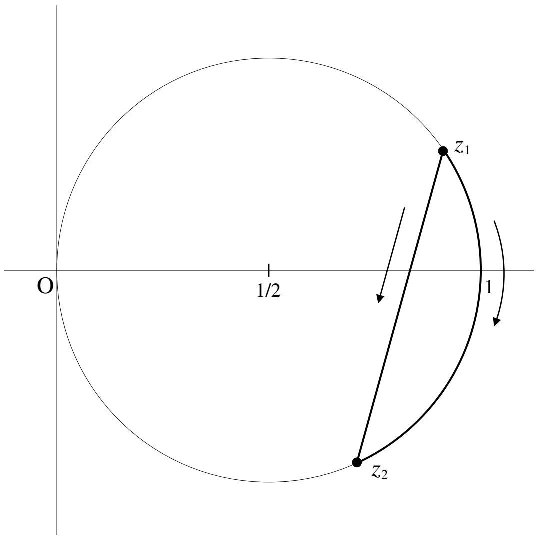

Fig. 4.3: The "standard" circle $\mathcal{C}_\mathrm{s}$ with the image of $\beta(1,3)$ derived from F_3.

What rests is to estimate different parts of the integral and to take eventually the limit $N \to \infty$. We first split $h_\mu(\tau)$ in its principal and non-principal part

$$h_\mu^-(\tau) = \sum_{n-\Delta_\mu<0} c_\mu(n) q^{n-\Delta_\mu}, \qquad h_\mu^+(\tau) = \sum_{n-\Delta_\mu\ge 0} c_\mu(n) q^{n-\Delta_\mu}. \tag{4.18}$$

The two integrals to estimate are $I^+(-d,c)$ and $I^-(-d,c)$

$$I^\pm(-d,c) = \int_{z_1(-d,c)}^{z_2(-d,c)} dz\, z^{-w} h_\nu^\pm\left(\frac{a}{c}+\frac{i}{z}\right) e\left((-m+\Delta_\mu)\left(-\frac{d}{c}+\frac{iz}{c^2}\right)\right).$$

We first determine an upperbound for I^+. Fig. 4.3 shows the standard circle together with the images of the tangent points of the circle corresponding to $\frac{1}{3}$ in F_3. The structure of the figure is however more generally valid. We are free to change the integration along the circle to the straight line connecting z_1 and z_2. At any point on this straight line, z satisfies $|z| \leq \frac{\sqrt{2}c}{N}$. Moreover the length of the line is always $\leq \frac{2\sqrt{2}c}{N}$. For the integrand we find the upperbound

$$\begin{aligned}
&\left| z^{-w} h_\nu^+\left(\frac{a}{c}+\frac{i}{z}\right) e\left((-m+\Delta_\mu)\left(-\frac{d}{c}+\frac{iz}{c^2}\right)\right)\right| \\
&= |z|^{-w} \exp\left(2\pi(m-\Delta_\mu)\left(\frac{\mathrm{Re}(z)}{c^2}\right)\right) \\
&\quad \times \left| \sum_{n-\Delta_\nu\geq 0} c_\nu(n) e\left((n-\Delta_\nu)\left(\frac{a}{c}+\frac{i}{z}\right)\right)\right| \\
&\leq C_1 \left(\frac{c}{N}\right)^{-w},
\end{aligned}$$

where C_1 is some constant. Multiplying this by the maximal length of the line, $\frac{2\sqrt{2}c}{N}$, gives us as upperbound for the integral

$$\left|I^+(-d,c)\right| \leq C_2 \left(\frac{c}{N}\right)^{1-w}, \tag{4.19}$$

where $C_2 = 2\sqrt{2}C_1$. An upperbound for the sums over c and d of $I^+(-d,c)$ is

$$\begin{aligned}
\left| \sum_{c=1}^{N} \sum_{\substack{-c\leq d<0 \\ (c,d)=1}} \frac{i^{1-w}}{c^{2-w}} I^+(-d,c)\right| &\leq \sum_{c=1}^{N} \sum_{\substack{-c\leq d<0 \\ (c,d)=1}} C_2 \frac{1}{c}\left(\frac{1}{N}\right)^{1-w} \\
&\leq C_2 \sum_{c=1}^{N} \left(\frac{1}{N}\right)^{1-w} = C_2 N^w.
\end{aligned}$$

Thus in the limit $N \to \infty$ the contribution of the non-principal part of $h_\mu(\tau)$ vanishes for $w < 0$. A more accurate estimate shows that the contribution from the non-principal part $h_\mu^+(\tau)$ vanishes also when $w = 0$ [130].

Since the contribution from $h_\mu^+(\tau)$ vanishes, the integral over the polar coefficients must provide us the Fourier coefficients. Indeed, $h_\mu^-(\tau)$ is not suppressed by $\mathrm{Im}(\tau)$ and the integral does not vanish. We will show now that if we replace the integration from $z_1(-d,c)$ to $z_2(-d,c)$ by the integral along the whole circle $\mathcal{C}_\mathrm{s}$, the errors will again vanish in the limit $N \to \infty$. The integral along the arc $\int_{z_1}^{z_2}$ equals

$$\int_{z_1}^{z_2} = \int_{\mathcal{C}} - \int_0^{z_1} - \int_{z_2}^{0} = I_{\mathcal{C}_\mathrm{s}}^- - I_1^- - I_2^-. \tag{4.20}$$

$\mathrm{Re}(1/z)$ on the circle is equal to 1. Consequently, an upperbound for the integrand of I_1^- is

$$\begin{aligned}&\left|z^{-w}h_\nu^-\left(\frac{a}{c}+\frac{i}{z}\right)e\left((-m+\Delta_\mu)\left(-\frac{d}{c}+\frac{iz}{c^2}\right)\right)\right|\\&\leq |z|^{-w}\exp\left(2\pi(m-\Delta_\mu)\left(\frac{\mathrm{Re}(z)}{c^2}\right)\right)\sum_{n-\Delta_\nu\leq 0}c_\nu(n)\exp\left(-2\pi(n-\Delta_\nu)\right)\\&\leq C\left(\frac{c}{N}\right)^{-w},\end{aligned} \tag{4.21}$$

for some constant C. The length of the path of integration on the circle from 0 to z_1 is always smaller than $\pi|z_1(a,b)| < \pi\sqrt{2}\frac{b}{N}$. Multiplying the upperbounds on the integrand and the path of integration gives us $I_1^- < \pi\sqrt{2}CN^w$. Therefore, I_1^- vanishes for $N\to\infty$ and $w<0$. The vanishing of I_2^- is shown analogously.

Finally, we come to the integral which does not vanish $I_{\mathcal{C}_\mathrm{s}}^-$. We obtain after taking the limit $N\to\infty$

$$\begin{aligned}c_\mu(m) &= \sum_{c=1}^{\infty}\sum_{\substack{-c\leq d<0\\(c,d)=1}}\frac{i^{1-w}}{c^{2-w}}\mathbf{M}^{-1}(\gamma)_\mu^\nu\sum_{n-\Delta_\nu<0}c_\nu(n)\\&\times\int_{\mathcal{C}_\mathrm{s}}dz\,z^{-w}e\left((n-\Delta_\nu)\left(\frac{a}{c}+\frac{i}{z}\right)+(-m+\Delta_\mu)\left(-\frac{d}{c}+\frac{iz}{c^2}\right)\right)\\&= \sum_{c=1}^{\infty}\frac{i}{c^{2-w}}\sum_{\substack{-c\leq d<0\\(c,d)=1}}\mathbf{K}(\gamma,m-\Delta_\mu,n-\Delta_\nu)\sum_{n-\Delta_\nu<0}c_\nu(n)\\&\times\int_{\mathcal{C}_\mathrm{s}}dz\,z^{-w}e\left((n-\Delta_\nu)\frac{i}{z}+(-m+\Delta_\mu)\frac{iz}{c^2}\right),\end{aligned} \tag{4.22}$$

where we have defined

$$\begin{aligned}&\mathbf{K}(\gamma,m-\Delta_\mu,n-\Delta_\nu)=\\&\qquad i^{-w}\mathbf{M}^{-1}(\gamma)_\mu^\nu e\left((n-\Delta_\nu)\frac{a}{c}+(m-\Delta_\mu)\frac{d}{c}\right).\end{aligned} \tag{4.23}$$

The unitary prefactor $M(\gamma)$ for the function $\eta(\tau)^{-1}$ (which are both scalars) is $M(\gamma)=(-i)^{\frac{1}{2}}e\left(-(\frac{a+d}{24c}+s(-d,c)/2)\right)$, where $s(k,l)$ is a so-called Dedekind sum [129]. Substituting this in (4.23) gives

$$K_{\eta^{-1}}\left(\gamma,m-\frac{1}{24},-\frac{1}{24}\right)=e\left(m\frac{d}{c}+s(-d,c)/2\right), \tag{4.24}$$

which is in agreement with [129].

The sum of $\mathbf{K}(\gamma, m-\Delta_\mu, n-\Delta_\nu)$ over c and d is a generalized Kloosterman sum $\mathbf{K}_c(m-\Delta_\mu, n-\Delta_\nu)$

$$\mathbf{K}_c(m-\Delta_\mu, n-\Delta_\nu) := \qquad (4.25)$$
$$i^{-w} \sum_{\substack{d \mod c \\ (c,d)=1}} \mathbf{M}^{-1}(\gamma)_\mu^\nu e\left((n-\Delta_\nu)\frac{a}{c} + (m-\Delta_\mu)\frac{d}{c}\right),$$

The dependence on a in the exponent and in $\mathbf{M}^{-1}(\gamma)_\mu^\nu$ via γ combine such that the product with the generalized Kloosterman sum is independent of a. The factor of i^{-w} in front of the sum is a consequence of the definition of $\mathbf{M}(\gamma)_\mu^\nu$ in Eq. (4.7).

The (scalar) Kloosterman sum $K_c(m,n)$ [131] is originally defined as

$$K_c(m,n) = \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*} e\left(\frac{(md+n\bar{d})}{c}\right), \qquad (4.26)$$

where $\bar{d}d = 1 \mod c$ [129]. This "scalar" Kloosterman sum appears in the expansion when the unitary prefactor is equal to one and the modular vector has length one. In that case

$$K_c(m,n) = \sum_{\substack{d \mod c \\ (c,d)=1}} K(\gamma, m, n). \qquad (4.27)$$

Because the determinant of $\mathbf{M}(\gamma)$ is a root of unity, the generalized Kloosterman sum is bounded above by the Euler totient function $\phi(c) \leq c$. For later use, we need an estimate for the magnitude of the generalized Kloosterman sum. By the (twisted) multiplicity property of Kloosterman sums, it suffices to show the bound for the case that c is equal to a prime number p. Weil has derived a particularly strong bound for $|K_p(m,n)|$, if $m, n \in \mathbb{Z}$ and with a trivial multiplier system [132]. He derived that $|K_p(m,n)| \leq 2\sqrt{p}$ using knowledge of elliptic curves over finite fields. This has as a consequence that $|K_c(m,n)|$ is of order $\mathcal{O}(c^{\frac{1}{2}+\epsilon})$.

Using more elementary methods, Kloosterman had shown earlier a weaker upperbound [133]. His estimate goes as follows. First, one notices that $K_c(m,n) = K_c(am, \bar{a}n)$ for $(a,c) = 1$. Therefore, we have $S = \sum_{r=0}^{p-1}\sum_{r=0}^{p-1} |K_p(r,s)|^4 \geq (p-1)\,|K_p(m,n)|^4$. One can rearrange the sums, and perform first the sums over r and s. These are easily evaluated using

$$\sum_{r=0}^{p-1} e\left(\frac{rA}{p}\right) = \begin{cases} 0 & \text{if} \quad p|A, \\ p & \text{if} \quad p \not| A. \end{cases} \qquad (4.28)$$

This leads straightforwardly to the estimate $K_p(m,n) < 3^{\frac{1}{4}}p^{\frac{3}{4}}$. We will need a mild generalization of these estimates to the case of non-trivial multiplier systems. A bound as weak as $\mathcal{O}(c^{1-\epsilon})$ with $\epsilon > 0$ on $|K_c(m,n)|$ will suffice. We will not attempt to establish such bound here.

After a coordinate transformation $t = 2\pi(\Delta_\nu - n)\frac{1}{z}$ in Eq. (4.22), we recognize the integral representation of the modified Bessel functions of the first kind $I_\nu(z)$. The integral representation of $I_\nu(z)$ is given by

$$I_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} t^{-\nu-1}e^{t+\frac{z^2}{4t}}\,dt \qquad (\text{for } \alpha > 0, \mathrm{Re}(\nu) > 0). \quad (4.29)$$

The coordinate transformation changes the integration contour in Eq. (4.22) to $\alpha - i\infty < t < \alpha + i\infty$ with $\alpha = 2\pi(\Delta_\nu - n)$. The integral in Eq. (4.22) is thus equal to

$$\begin{aligned} &-(2\pi(\Delta_\nu - n))^{1-w}\int_{\alpha-i\infty}^{\alpha+i\infty} t^{w-2}\exp\left(t + \frac{4\pi^2(m-\Delta_\mu)\,|n-\Delta_\nu|}{c^2 t}\right) \\ &= -2\pi i c^{1-w}\left(\frac{|n-\Delta_\nu|}{m-\Delta_\mu}\right)^{(w-1)/2} I_{1-w}\left(\frac{4\pi}{c}\sqrt{(m-\Delta_\mu)\,|n-\Delta_\nu|}\right), \end{aligned}$$

where we substituted the Bessel function for the integral. Inserting this into our series (4.22), we obtain our final exact result for the Fourier coefficients

$$\begin{aligned} c_\mu(m) \;&=\; 2\pi\sum_\nu\sum_{c=1}^{\infty}\frac{1}{c}\mathbf{K}_c(m-\Delta_\mu, n-\Delta_\nu)_\mu^\nu \sum_{n-\Delta_\nu<0} c_\nu(n) \qquad (4.30) \\ &\times\; \left(\frac{|n-\Delta_\nu|}{m-\Delta_\mu}\right)^{(1-w)/2} I_{1-w}\left(\frac{4\pi}{c}\sqrt{(m-\Delta_\mu)\,|n-\Delta_\nu|}\right). \end{aligned}$$

Remarkably, Eq. (4.30) determines all coefficients $c_\mu(m)$, $m \geq 0$ in terms of the polar coefficients $c_\mu(m)$, $m < 0$. This fact will be important in later discussions. If $m - \Delta_\mu = 0$ we should take a limit as $m - \Delta_\mu \to 0$.

Eq. (4.30) is a major improvement over the estimate by the saddle point method (3.86). It provides us a way to calculate the black hole degeneracies with an arbitrary precision. The asymptotic behavior of the sum is governed by the Bessel functions. In the limit $\lim_{z\to\infty}$, $I_\nu(z)$ behaves as $\frac{1}{\sqrt{2\pi z}}e^z$. Corrections to this leading behavior can be calculated as a perturbative expansion in $\frac{1}{z}$. Neglecting a contribution proportional to e^{-z} we find for ν half-integer

$$\begin{aligned}\lim_{z\to\infty} I_\nu(z) &\sim \frac{e^z}{\sqrt{2\pi z}} \sum_{k=0}^{\nu-1/2} \frac{\Gamma(\nu+\frac{1}{2}+k)}{k!\Gamma(\nu+\frac{1}{2}-k)} \frac{(-1)^k}{(2z)^k} \\ &= \frac{e^z}{\sqrt{2\pi z}} \left(1+\sum_{k=1}^{\nu-1/2} \frac{(-1)^k}{k!(8z)^k} \prod_{l=0}^{k} \left((2\nu)^2-(2k+1)^2\right)\right).\end{aligned} \tag{4.31}$$

Notice that this expansion is finite. Refs. [134, 135] are useful for deriving the above expansion.

We observe that the magnitude of the coefficients is dominated by the term $c = 1$ and and $n = 0$ in (4.30). The corrections to the entropy by the Bessel function are polynomially $1/m$. The sums over c, n and μ give exponentially small contributions to the entropy. Using the weak bound on the Kloosterman sum, we see that for large charges, $c_\mu(m)$ is estimated by

$$\begin{aligned}c_\mu(m) &\sim \frac{1}{\sqrt{2}} \sum_{c=1}^{\infty} \sum_{\nu} \sum_{n-\Delta_\nu<0} c_\nu(n)\, |n-\Delta_\nu|^{\frac{1}{4}-w/2}\, (m-\Delta_\mu)^{w/2-\frac{3}{4}} \\ &\times\ c^\epsilon \exp\left(\frac{4\pi}{c}\sqrt{(m-\Delta_\mu)\,|n-\Delta_\nu|}\right)(1+\mathcal{O}((m-\Delta_\mu)^{-1/2})).\end{aligned}$$

We directly observe consistency with our earlier estimate Eq. (3.86). Moreover, two additional sums appear, one over c and another one over $n-\Delta_\nu<0$.

4.2 A regularized Poincaré series for $h_\mu(\tau)$

This section derives an expansion for $h_\mu(\tau)$, which is similar to a Poincaré series. This will be interpreted in Chap. 5 as a manifestation of gravity in the CFT partition function. The derivation given here appeared earlier in the appendix of [136]. It is in some sense a reversed version of the analysis in Ref. [137]. We start with the vector-valued modular form $h_\mu(\tau)$, and derive the series based on its Fourier coefficients (4.30). Whereas Ref. [137] basically starts at the other end, and determines its Fourier coefficients together with its transformation properties. We take the opportunity to generalize the result to vector-valued modular forms.

Our strategy to derive the Poincaré series is fairly straightforward. We substitute the expression for the Fourier coefficients in the Fourier series for the non-polar part of $h_\mu(\tau)$. Then we use the infinite sum representation of the Bessel function (4.32) and the Lipschitz summation formula (4.34) to rewrite $h_\mu(\tau)$ as a Poincaré series.

The Bessel function $I_\nu(z)$, appearing in the expression for $c_\mu(m)$ (4.30), admits a representation as an infinite sum [134]

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)}. \tag{4.32}$$

We substitute this sum in (4.30) and subsequently substitute (4.30) in $h_\mu^+(\tau)$. We obtain

$$\begin{aligned} h_\mu^+(\tau) &= \sum_{m-\Delta_\mu \geq 0} c_\mu(m) q^{m-\Delta_\mu} \qquad (4.33)\\ &= \sum_{n-\Delta_\nu<0} \sum_{c=1}^{\infty} \sum_{\substack{-c\leq d<0\\ (c,d)=1}} \sum_{k=0}^{\infty} i^{-w} \mathbf{M}^{-1}(\gamma)_\mu^\nu c_\nu(n) \left(\frac{2\pi}{c}\right)^{2k+2-w} \\ &\quad \times \frac{|n-\Delta_\nu|^{k+1-w}}{\Gamma(k+2-w)} e\left((n-\Delta_\nu)\frac{a}{c}\right) \\ &\quad \times \sum_{m-\Delta_\mu\geq 0} \frac{(m-\Delta_\mu)^k}{k!} e\left((m-\Delta_\mu)\left(\tau+\frac{d}{c}\right)\right), \end{aligned}$$

where we interchanged the sum over m with the other four sums and grouped the terms dependent on m.

Lipschitz summation formula
At this point, we have to digress and introduce the Lipschitz summation formula, which is a crucial ingredient for the derivation. Let $\tau \in \mathcal{H}$, $N \in \mathbb{N}$, $0 \leq \alpha < 1$, $p \geq 1$, then

$$\sum_{l=-N}^{N} \frac{e(-l\alpha)}{(\tau+l)^p} = \frac{(-2\pi i)^p}{\Gamma(p)} \sum_{m=0}^{\infty} (m+\alpha)^{p-1} q^{m+\alpha} + E(\tau,p,Q), \tag{4.34}$$

where $Q = N + \frac{1}{2}$ and $E(\tau,p,Q)$ is an error term and given by

$$E(\tau,p,Q) = (iQ)^{1-p} \int_{-\infty}^{\infty} \frac{h(x-i)-h(x+i)}{1+\exp(2\pi xQ)} dx, \tag{4.35}$$

where

$$h(x) = \frac{\exp(2\pi xQ\alpha)}{(x+\frac{\tau}{iQ})^p}.$$

The error tends to 0 for $Q \to \infty$, except for the case $p = 1$, $\alpha = 0$; then we obtain $\lim_{Q\to\infty} E(\tau, 1, Q) = \pi i$. The case $p = 1$, $\alpha = 0$ gives the two well known infinite sums for $\cot \pi\tau$

$$\frac{1}{\tau} + \sum_{l=1}^{\infty} \left(\frac{1}{\tau - l} + \frac{1}{\tau + l} \right) = \pi \cot \pi\tau = \pi i - 2\pi i \sum_{m=0}^{\infty} q^m, \tag{4.36}$$

which can also be proved by using $\sin \pi\tau = \pi\tau \prod_{n=1}^{\infty}(1 - \tau^2/n^2)$.

The proof of Eq. (4.34) uses the function $f(z) = e((z+\tau)\alpha)/(iz)^p(e(z+\tau) - 1)$. This function has poles at $z = -\tau - l$, $l \in \mathbb{Z}$ with residues $(2\pi i)^{-1} e(-l\alpha)/(-i\tau - il)^p$. The right hand side is obtained by integrating along the boundary of the rectangle $-\mathrm{Re}(\tau) \pm Q \pm iM$, which is slit along the positive imaginary axis to avoid a branch cut of $(iz)^p$. The main contribution to the integral comes from this part of the contour. It can be calculated using the Hankel contour integral $\frac{1}{\Gamma(p)} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^t t^{-p} dt$ [134], where $\mathcal{C}$ is the contour which begins at $-\infty - i0^+$, circles the origin in the counterclockwise direction and ends at $-\infty + i0^+$. The horizontal sides do not contribute when $M \to \infty$, the error is accordingly calculated by the integral along the vertical segments.

We apply the Lipschitz summation formula (4.34) in (4.33) to the sum over m, the new summation variable will be denoted by l. The error term $E(\tau, k+1, N + \frac{1}{2})$ vanishes in the limit $N \to \infty$, except when $k = 0$ and $\Delta_\mu \in \mathbb{N}$. When the error term does not vanish, we get an additional constant. This constant is equal to $\frac{1}{2} c_\mu(\Delta_\mu)$ and is given by

$$\frac{1}{2} c_\mu(\Delta_\mu) = \tag{4.37}$$
$$\begin{cases} \pi \sum_{n-\Delta_\nu<0} \frac{(2\pi|n-\Delta_\nu|)^{1-w}}{\Gamma(2-w)} c_\nu(n) \displaystyle\sum_{c=1}^{\infty} c^{w-2} \mathbf{K}_c(0_\mu, n - \Delta_\nu), & \Delta_\mu \in \mathbb{N}, \\ 0, & \Delta_\mu \notin \mathbb{N}, \end{cases}$$

where 0_μ is a vector all of whose components are zero. The fact that the right hand side of Eq. (4.37) is equal to $\frac{1}{2} c_\mu(\Delta_\mu)$ can be shown for example by Eq. (4.30) for $c_\mu(\Delta_\mu)$ and the limiting behavior of the Bessel function for $z \to 0$: $\lim_{z\to 0} I_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}$. We get after interchanging the sum over k and l

$$\begin{aligned} h^+_\mu(\tau) &= \frac{1}{2}c_\mu(\Delta_\mu) + \sum_{n-\Delta_\nu<0}\sum_{c=1}^{\infty}\sum_{\substack{-c\le d<0\\(c,d)=1}}\lim_{N\to\infty}\sum_{l=-N}^{N} \\ &\quad \mathbf{M}^{-1}(\gamma)^\nu_\mu c_\nu(n) e\left((n-\Delta_\nu)\frac{a}{c}\right)\frac{1}{(c\tau+d+cl)^w}e(\Delta_\mu l) \\ &\quad \times\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+2-w)}\left(\frac{2\pi i|n-\Delta_\nu|}{c(c\tau+d+cl)}\right)^{k+1-w}. \end{aligned} \tag{4.38}$$

The exchange of the sum over k and l is allowed because the sums are absolutely convergent for $k>0$. In case $k=0$, the sum over l in the limit $N\to\infty$ is as well convergent. This is shown using the weak bound on the Kloosterman sum, to which we referred earlier.

The sums over c and d can be such that they have an equal upperbound. This is clear for $k>0$, but to show it for $k=0$ is slightly subtle. First, we incorporate the sum over l in the sum over d. Since the sum over l and d is convergent for finite c, we can choose for $|d|$ an upperbound N for which we take the limit $N\to\infty$. We thus get a sum of the form

$$\sum_{c=1}^{\infty}\lim_{N\to\infty}\sum_{\substack{|d|\le N\\(c,d)=1}}\mathbf{M}^{-1}(\gamma)^\nu_\mu\frac{e((n-\Delta_\nu)\frac{a}{c})}{c^{1-w}(c\tau+d)}, \tag{4.39}$$

where we used that $e(\Delta_\mu l)\delta^\nu_\mu = \mathbf{M}^{-1}(\gamma_l)^\nu_\mu$ and Eq. (4.8) to include $e(\Delta_\mu l)$ in $\mathbf{M}^{-1}(\gamma)^\nu_\mu$. Ref. [138] shows that

$$\lim_{K\to\infty}\sum_{c=1}^{K}\lim_{N\to\infty}\sum_{\substack{K<|d|\le N\\(c,d)=1}}\mathbf{M}^{-1}(\gamma)^\nu_\mu\frac{e((n-\Delta_\nu)\frac{a}{c})}{c^{1-w}(c\tau+d)} = 0, \tag{4.40}$$

in case $M(\gamma)=1$ and $(n-\Delta_\nu)=-1$. We can show in a similar way that the generalization holds as well. To this end, define the matrix $\mathbf{g}(d)^\nu_\mu$ (with $-\delta_\nu = n-\Delta_\nu$)

$$\mathbf{g}(d)^\nu_\mu = \begin{cases} \mathbf{M}^{-1}(\gamma)^\nu_\mu e(-\delta_\nu\frac{a}{c}), & \text{for } (c,d)=1,\\ 0, & \text{otherwise.}\end{cases} \tag{4.41}$$

Using that $\mathbf{M}(\gamma_l)^\nu_\mu = \delta^\nu_\mu e(-\delta_\nu l)$ (where δ^ν_μ should not be confused with δ_ν), we observe that $e(-\delta_\mu\frac{d}{c})\mathbf{g}(d)^\nu_\mu$ is periodic in d modulo c. Therefore, $e(-\delta_\mu\frac{d}{c})\mathbf{g}(d)^\nu_\mu$ has a Fourier expansion, and we find for $\mathbf{g}(d)^\nu_\mu$

$$\mathbf{g}(d)^\nu_\mu = \sum_{j=1}^{c}(\mathbf{B}_{j,c})^\nu_\mu\, e\left((j+\delta_\mu)\frac{d}{c}\right), \tag{4.42}$$

with

$$(\mathbf{B}_{j,c})^\nu_\mu = \frac{1}{c} \sum_{\substack{d'=1 \\ (c,d')=1}}^{c} \mathbf{M}^{-1}(\gamma)^\nu_\mu e\left(-\delta_\nu \frac{a}{c} - (j+\delta_\mu)\frac{d'}{c}\right). \tag{4.43}$$

$\mathbf{B}_{j,c}$ contains a Kloosterman sum, and with the weak bound on the vector-valued Kloosterman sums (see the discussion below Eq. (4.25)), we obtain $\mathcal{O}(c^{-\epsilon})$ as a bound for $\mathbf{B}_{j,c}$. The left-hand side of Eq. (4.40) can be written as

$$\lim_{K\to\infty} \sum_{c=1}^{K} \frac{1}{c^{1-w}} \sum_{j=1}^{c} (\mathbf{B}_{j,c})^\nu_\mu \sum_{|d|=K+1}^{\infty} \frac{e((j+\delta_\nu)\frac{d}{c})}{(c\tau+d)}. \tag{4.44}$$

Ref. [138] gives estimates for the sum over d which continue to hold for the generalization after minor modifications. We find that in case $(j+\delta_\nu)/c \in \mathbb{Z}$ for some j, the sum over d has an upperbound given by $\mathcal{O}\left(\frac{c\log(K)}{K}\right)$, otherwise the upperbound is $\mathcal{O}\left(K^{-1}\right)$. The estimates for Eq. (4.40) become respectively, $\lim_{K\to\infty} \mathcal{O}\left(K^{w-\epsilon}\log(K)\right)$ and $\lim_{K\to\infty} \mathcal{O}\left(K^{w-\epsilon}\right)$, which are indeed zero for $w \leq 0$. We therefore have shown that Eq. (4.39) is equal to

$$\lim_{K\to\infty} \sum_{c=1}^{K} \sum_{\substack{|d|\leq K \\ (c,d)=1}} \mathbf{M}^{-1}(\gamma)^\nu_\mu \frac{e((n-\Delta_\nu)\frac{a}{c})}{c^{1-w}(c\tau+d)}. \tag{4.45}$$

The sum over k in Eq. (4.38) is equal to an exponent minus the first terms of the Fourier expansion: $\sum_{k=0}^\infty \frac{z^{k+1-w}}{\Gamma(k+2-w)} = e^z - \sum_{k=0}^{|w|} z^k/k!$, when w is a negative integer. We recognize the regularization of Eq. (2.111) for $w=0$. However, we want to obtain a closed form for general non-positive weight. This can be obtained using the equality

$$\begin{aligned} f(z) = \sum_{k=0}^\infty \frac{z^{k+1-w}}{\Gamma(k+2-w)} &= e^z\left(1 - \frac{1}{\Gamma(1-w)}\int_z^\infty e^{-t}t^{-w}dt\right) \\ &= \frac{e^z}{\Gamma(1-w)}\int_0^z e^{-t}t^{-w}dt, \end{aligned} \tag{4.46}$$

which is valid for general $w < 1$. One can establish Eq. (4.46) by developing the second integral expression in series using successive integration by parts, or by considering the differential equation satisfied by $f(z)$.

We define $R(z) = e^{-z} f(z)$. Inserting this and the equal upperbound for c and d in Eq. (4.38), we obtain

$$\begin{aligned} h_\mu^+(\tau) &= \frac{1}{2}c_\mu(\Delta_\mu) + \sum_{n-\Delta_\nu<0} \lim_{K\to\infty} \sum_{c=1}^{K} \sum_{\substack{|d|\le K \\ (c,d)=1}} \frac{\mathbf{M}^{-1}(\gamma)_\mu^\nu c_\nu(n)}{(c\tau+d)^w} \\ &\times \ e\left((n-\Delta_\nu)\gamma(\tau)\right) R(x), \end{aligned} \tag{4.47}$$

where $x = \frac{2\pi i|n-\Delta_\nu|}{c(c\tau+d)}$. The summand is invariant under $\gamma \to -\gamma$ or equivalently $(c,d) \to (-c,-d)$. We can extend therefore the sum over c to $0 < |c| \le K$, and divide by two. The principal part can be included by extending the sum with $c = 0$. Note that $\gcd(0,d) = |d|$, thus $c = 0$ adds $(c,d) = (0,1)$ and $(c,d) = (0,-1)$ to the sum, which works out nicely with the overall factor of $\frac{1}{2}$. We obtain finally

$$\begin{aligned} h_\mu(\tau) &= \frac{1}{2}c_\mu(\Delta_\mu) + \frac{1}{2} \sum_{n-\Delta_\nu<0} \lim_{K\to\infty} \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_K} j(\gamma,\tau)^{-w} \mathbf{M}^{-1}(\gamma)_\mu^\nu c_\nu(n) \\ &\times e\left((n-\Delta_\nu)\gamma(\tau)\right) R(x), \end{aligned} \tag{4.48}$$

where we have defined $\sum_{|c|\le K} \sum_{\substack{|d|\le K \\ (c,d)=1}} = \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_K}$. This is our analogue of the series in Eq. (2.111) for the scalar modular form with weight zero. The regularization is taken care of by the factor $R(x)$, which reduces for weight zero to the regularization in (2.111). One can say heuristically that $R(x)$ determines how much the exponent $e((n-\Delta_\nu)\gamma(\tau))$ contributes to $h_\mu(\tau)$. Fig. 4.4 presents $R(x)$ as a function of x for real x. We observe that the least polar terms contribute less if the absolute weight $|w|$ increases. Eq. (4.48) will be applied in Chapter 5 to write $\mathcal{Z}_{\mathrm{CFT}}$ as a Poincaré series, which has the claimed interpretation in the spirit of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence.

4.3 The space of meromorphic forms

This section studies the space of the vector-valued modular forms. This space is denoted by $A_{w,\mathbf{M}}(\Gamma(4m)^*, \Delta)$, as explained in the introduction to this chapter. We have seen that an element of $A_{w,\mathbf{M}}(\Gamma(4m)^*, \Delta)$ is determined by its polar coefficients for $w \le 0$ and how it can be constructed by a regularized Poincaré series. This is an interesting result since the polar coefficients might be determined by physics, see Chapter 5. The maximal dimension of $A_{w,\mathbf{M}}(\Gamma_c, \Delta)$ is therefore $p(\mathbf{M}, \Delta)$, the number of polar terms with exponents $\ge -\Delta$. Generically, the dimension is actually smaller, as explained in this section.

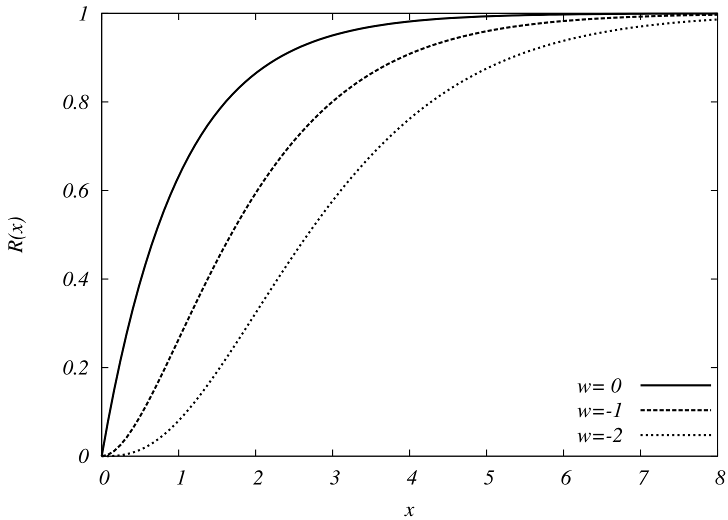

Fig. 4.4: $R(x)$ as function of x for x real.

That the dimension of $A_{w,\mathbf{M}}(\Gamma(4m)^*,\Delta)$ is generically smaller than $p(\mathbf{M},\Delta)$ can be viewed as a manifestation of the Mittag-Leffler problem [101], which is the problem of finding a meromorphic section with prescribed singularities of a line bundle $\mathcal{L}$ over a manifold X. The space of obstructions to find such a section is given by $H^1(X,\mathcal{O}(\mathcal{L}))$, where $\mathcal{O}(\mathcal{L})$ is the sheaf of holomorphic sections of $\mathcal{L}$. Since the modular curve $\mathcal{H}/\Gamma$ is one-dimensional, $H^1(X,\mathcal{O}(\mathcal{L}))$ is by Serre duality related to $H^0(X,\mathcal{O}(K\otimes\mathcal{L}^*))$, with K the canonical bundle. In the present discussion, sections of $\mathcal{L}$ have weight w and therefore sections of $\mathcal{L}^*$ have weight $-w$. Since holomorphic sections of K are cusp forms of weight two, this suggests the appearance of cusp forms of weight $2-w$ as obstructions to the construction of meromorphic forms of weight w. Ref. [139] generalizes these considerations to the vector-valued case and proofs that the obstruction space for vector-valued modular forms $h_\mu(\tau)$ is given by the vector-valued cusp forms $g_\mu(\tau)$ of weight $2-w$ and transforming with the matrices $\bar{\mathbf{M}}$. We denote this space by $S_{2-w,\bar{\mathbf{M}}}(\Gamma(4m)^*)$. We will comment in the end of this section on the magnitude of $p(\mathbf{M},\Delta)$ and $\dim S_{2-w,\bar{\mathbf{M}}}(\Gamma(4m)^*)$.

This section shows that one encounters the presence of constraints on the polar coefficients when one attempts to obtain a negative weight modular form with a prescribed set of polar coefficients by a sum over Γ/Γ_∞ [137, 136]. As in the previous section, such a sum naively completes a function which is not modular covariant to a modular covariant object. However, it is shown that the function generically transforms anomalously. This is a consequence of the regularization of the divergent Poincaré series. In many cases, a proper choice of the polar coefficients avoids the anomalous modular behavior.

To proceed, we define $h_\mu(\tau)$ by the expression given in Eq. (4.48), with a general set of polar coefficients, and consequently deduce its transformation properties in Sec. 4.3.1. Some relevant properties of period functions are reviewed first. Many intermediate steps are given without rigorous proofs, these can be found in Ref. [137]. We discuss the case of scalar modular forms; at the end we simply state the straightforward generalization to vector-valued modular forms. Sec. 4.3.2 explains how the number of constraints can be determined using the Selberg trace formula.

4.3.1 Transformation properties of $h_\mu(\tau)$

For simplicity of exposition we discuss the case of scalar modular forms. We start by explaining what a period function of a cusp form $g(z)$ is. The form $g(z)$ transforms as $g(\gamma(z)) = M^{-1}(\gamma)(cz+d)^{2-w}g(z)$ under $\gamma \in \Gamma$. The period function of $g(z)$, $p(\tau, \bar{y}, \overline{g})$ is defined by

$$p(\tau, \bar{y}, \overline{g}) = \frac{1}{\Gamma(1-w)} \int_{\bar{y}}^{-i\infty} \overline{g(z)} (\bar{z}-\tau)^{-w} d\bar{z}, \qquad y \in \mathcal{H} \cup \mathbb{Q} \cup i\infty. \quad (4.49)$$

Note that in case $-w \in \mathbb{N}$, this expression is a polynomial in τ. Also note that the expression $p(\tau, \bar{y}, \overline{g})$ makes sense for any function $g(z)$ that decays sufficiently rapidly at infinity, e.g. $g(z_1 + iz_2) \sim_{z_2 \to +\infty} C z_2^{\alpha} e^{-Az_2}$ for $A > 0$ will suffice (C and α are undetermined constants). The constituents of the integrand satisfy simple transformation properties: $\gamma(\bar{z}) - \gamma(\tau) = \frac{\bar{z}-\tau}{j(\gamma,\bar{z})j(\gamma,\tau)}$ and $d\gamma(z) = \frac{dz}{j(\gamma,z)^2}$. Using these equations we obtain for $p(\gamma(\tau), \gamma(\bar{y}), \overline{g(z)})$ the transformation rule

$$p(\gamma(\tau), \gamma(\bar{y}), \overline{g}) = j(\gamma, \tau)^w M(\gamma) \left[p(\tau, \bar{y}, \overline{g}) - p(\tau, \gamma^{-1}(\infty), \overline{g}) \right], \quad (4.50)$$

where we have used the fact that $M(\gamma)$ is unitary.

If we choose a constant $\delta > 0$ we can try to construct a cusp form $g^{(\delta)}(z)$ of weight $2-w$ by the Poincaré series

$$g^{(\delta)}(z) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{M(\gamma)(-2\pi i \delta)^{1-w} e(\delta \gamma(z))}{j(\gamma, z)^{2-w}} := \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} g_\gamma^{(\delta)}(z),$$

where we defined $g_\gamma^{(\delta)}(z)$ by the second equality. The prefactor is chosen for later convenience. We will sometimes drop the superscript δ when the context is clear. For $w < 0$ the series is convergent, although it might vanish. The Poincaré series span the space of cusp forms [140].

The period functions are relevant for the transformation properties of regularized Poincaré series, see the next subsection. In that discussion, we make use of the function $t_\gamma(\tau)$ defined by

$$t_\gamma(\tau) := p(\tau, \gamma^{-1}(i\infty), \overline{g_\gamma}).$$

Using the above identities and Eq. (4.2) one can check that $t_\gamma(\tau)$ satisfies the transformation rule with $\tilde{\gamma} \in \Gamma$

$$t_\gamma(\tilde{\gamma}(\tau)) = j(\tilde{\gamma}, \tau)^w M(\tilde{\gamma}) \left[t_{\gamma\tilde{\gamma}}(\tau) - p(\tau, \tilde{\gamma}^{-1}(i\infty), \overline{g_{\gamma\tilde{\gamma}}}) \right], \tag{4.51}$$

Note that $t_\gamma(\tau)$ can be rewritten as

$$t_\gamma(\tau) = \frac{-1}{\Gamma(1-w)} j(\gamma, \tau)^{-w} M^{-1}(\gamma) e(-\delta\gamma(\tau)) \int_x^\infty e^{-z} z^{-w} dz, \tag{4.52}$$

with $x = \frac{2\pi i \delta}{cj(\gamma,\tau)}$ where c is the 21 matrix element of γ. The steps involved are first a transformation of $\bar{z}$ to $\gamma^{-1}(\bar{z})$, then rewriting of the integrand using its modular properties and at last another redefinition of $\bar{z}$.

We study first the transformation properties of a (scalar) modular form with a single polar term $q^{-\delta}$ ($\delta > 0$) for a clear exposition. Eventually we will deduce the transformation law for general $h_\mu(\tau)$. We define the function $s_\gamma(\tau) = j(\gamma, \tau)^{-w} M^{-1}(\gamma) e(-\delta\gamma(\tau))$ and use $t_\gamma(\tau)$ as in Eq. (4.52). Eq. (4.48) is in this case given by

$$h^{(-\delta)}(\tau) = \frac{1}{2} c(\delta) + \frac{1}{2} \lim_{K\to\infty} \sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)_K} s_\gamma(\tau) + t_\gamma(\tau). \tag{4.53}$$

$s_\gamma(\tau)$ satisfies $s_\gamma(\tilde{\gamma}(\tau)) = j(\tilde{\gamma}, \tau)^w M(\tilde{\gamma}) s_{\gamma\tilde{\gamma}}(\tau)$. We obtain with Eq. (4.51)

$$\begin{aligned} h^{(-\delta)}(\tilde{\gamma}(\tau)) \quad &= \quad \frac{1}{2} c(\delta) + \frac{1}{2} M(\tilde{\gamma}) (\tilde{c}\tau + \tilde{d})^w \qquad (4.54) \\ &\quad \times \lim_{K\to\infty} \sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)_K} s_{\gamma\tilde{\gamma}}(\tau) + t_{\gamma\tilde{\gamma}}(\tau) - p(\tau, \tilde{\gamma}^{-1}(-i\infty), \overline{g_{\gamma\tilde{\gamma}}}). \end{aligned}$$

The invariance under $T = \gamma_1$ is obvious from the Fourier expansion and Eq. (4.38). We therefore only need to check the invariance under the other generator of Γ, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The set of elements $(\Gamma_\infty \backslash \Gamma)_K$ is however left invariant under right multiplication of S. Therefore, $\sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)_K} s_{\gamma S}(\tau) + t_{\gamma S}(\tau) = \sum_{\gamma \in (\Gamma_\infty \backslash \Gamma)_K} s_\gamma(\tau) + t_\gamma(\tau)$ holds.

The anomalous terms compared to the usual transformation rule of modular forms are the constant term $\frac{1}{2}c(\delta)$ and the subtraction of period integrals. A careful study of the limit $K \to \infty$ and the period integrals is needed. Lemma 4.4 of Ref. [137] shows that for $y \in \mathcal{H}$

$$\lim_{K\to\infty} \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_K} p(\tau, \bar{y}, \overline{g_\gamma^{(\delta)}}) = p(\tau, \bar{y}, \overline{g^{(\delta)}}) - c(\delta), \tag{4.55}$$

thus the limit $K \to \infty$ and the integral do not commute. This comes about as follows. Calculation of the Fourier coefficients of $g^{(\delta)}$ gives an error term by the Lipschitz summation formula. This error term tends to zero, however the period integral over the error does not vanish and provides us with the offset.

In Eq. (4.54), we however have $y \notin \mathcal{H}$ but $y = \tilde{\gamma}^{-1}(i\infty) \in \mathbb{Q}$. In this case we obtain with Corollary 4.5 of Ref. [137]

$$\begin{aligned}\lim_{K\to\infty} \sum_{\gamma\in(\Gamma_\infty\backslash\Gamma)_K} p(\tau, \tilde{\gamma}^{-1}(i\infty), \overline{g_\gamma^{(\delta)}}) &= p(\tau, \tilde{\gamma}^{-1}(i\infty), \overline{g^{(\delta)}}) \\ &\quad + c(\delta)\left(M^{-1}(\tilde{\gamma})(\tilde{c}\tau + \tilde{d})^{-w} - 1\right).\end{aligned} \tag{4.56}$$

Inserting this result in Eq. (4.54) we find the transformation of $h^{(-\delta)}(\tau)$ under γ

$$h^{(-\delta)}(\gamma(\tau)) = j(\gamma,\tau)^w M(\gamma)\left[h^{(-\delta)}(\tau) - p(\tau, \gamma^{-1}(i\infty), \overline{g^{(\delta)}})\right]. \tag{4.57}$$

Note that in special cases g is zero. This is for example the case for $\delta \in \mathbb{N}$ and $w = 0, -2, -4, -6, -8$ and -12 [141]. A cusp form with weight $12 = 2 - w$ of Γ exists, which explains that in case $w = -10$, we will find a transformation with a non-zero shift.

Extending the above to the case of vector-valued modular forms with multiple polar terms is straightforward. The period function should vanish of course in this case. For a general choice of Δ_μ and polar coefficients $c_\mu(n)$, we obtain the transformation

$$h_\mu(\gamma(\tau)) = (c\tau + d)^w \mathbf{M}(\gamma)_\mu^\nu \left[h_\nu(\tau) - p(\tau, \gamma^{-1}(-i\infty), \overline{g_\nu})\right]. \tag{4.58}$$

with

$$g_\mu(z) = \frac{1}{2} \sum_{n-\Delta_\nu<0} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{\mathbf{M}^{-1}(\gamma)_\mu^\nu} \frac{(2\pi i(n-\Delta_\nu))^{1-w} c_\nu(n) e(|n-\Delta_\nu|\gamma(z))}{(cz+d)^{2-w}}.$$

We observe that transformations of $h_\mu(\tau)$ with generic polar coefficients involve a shift by a period function. Therefore, an $h_\mu(\tau)$ constructed by the regularized Poincaré series transforms generically anomalously compared to Eq. (4.7).

4.3.2 The number of constraints

This subsection determines the number of constraints and the dimension of space of potential partition functions. The analysis appeared earlier in [142].

Since the integration over independent cusp forms $g_\mu(\tau)$ results in independent vector-valued functions, the space of obstructions to the construction of proper $h_\mu(\tau)$ is $S_{2-w,\bar{\mathbf{M}}}(\Gamma_c)$. In other words, we have the following exact sequence

$$0 \to A_{w,\mathbf{M}}(\Gamma(4m)^*, \Delta) \to P(\mathbf{M}, \Delta) \to S_{2-w,\bar{\mathbf{M}}}(\Gamma(4m)^*) \to 0, \qquad (4.59)$$

where $P(\mathbf{M}, \Delta)$ is the space of polar terms, which has dimension $p(\mathbf{M}, \Delta)$. The dimension of $A_{w,\mathbf{M}}(\Gamma(4m)^*, \Delta)$ is therefore calculated as the number of polar terms $p(\mathbf{M}, \Delta)$ minus $\dim S_{2-w,\bar{\mathbf{M}}}(\Gamma(4m)^*)$:

$$\dim A_{w,\mathbf{M}}(\Gamma(4m)^*, \Delta) = p(\mathbf{M}, \Delta) - \dim S_{2-w,\bar{\mathbf{M}}}(\Gamma(4m)^*). \qquad (4.60)$$

First the number of polar terms $p(\mathbf{M}, \Delta)$ is determined. The number of polar terms is given by

$$p(\mathbf{M}, \Delta) = \sum_{\mu=1}^{d} \lceil \Delta_\mu \rceil = -\frac{1}{2} S(\mathbf{M}) + \sum_{\mu=1}^{d} \left\{ \Delta_\mu - ((\Delta_\mu)) + \frac{1}{2} \right\}, \qquad (4.61)$$

where $\lceil \cdot \rceil$ is the ceiling function. In addition, $S(\mathbf{M})$ is the number of Δ_μ which take values in $\mathbb{Z}$ and $((x))$ is an arithmetic function, which is given in terms of the floor and ceiling function by

$$((x)) = x - \frac{\lceil x \rceil + \lfloor x \rfloor}{2} = \begin{cases} \xi - \frac{1}{2}, & \text{if } x = \xi + \mathbb{Z},\ 0 < \xi < 1, \\ 0, & \text{if } x \in \mathbb{Z}. \end{cases} \qquad (4.62)$$

The next step is to determine $\dim S_{2-w,\bar{\mathbf{M}}}(\Gamma(4m)^*)$. For this calculation, we will mainly rely on the methods of Ref. [143]. Dimension formulas for vector-valued modular forms are also mentioned in [139], [144] and [145]. Ref. [136] constructs Jacobi forms from a Poincaré series on the principal part of $h_\mu(\tau)$.

The basic ingredients for the calculation of $\dim S_{2-w,\bar{\mathbf{M}}}(\Gamma(4m)^*)$ are the orthogonality relations for irreducible characters of finite groups and the Selberg trace formula. The relevant finite group appeared in the beginning of the chapter, namely $\tilde{\Gamma}/\Gamma(4m)^*$. The transformation properties of $g_\mu(\tau)$ provide a d-dimensional representation $\bar{\mathbf{M}}$ in terms of the matrices $\bar{\mathbf{M}}(\gamma)$. We define a character of the representation $\mathbf{M}$ in the usual way by

$$\chi_{\mathbf{M}}(\gamma) = \mathrm{Tr}(\mathbf{M}(\gamma)). \qquad (4.63)$$

We label the set of irreducible representations by $\mathbf{R}_i$. The orthogonality relations for characters of finite groups read in this case

$$\frac{1}{|\tilde{\Gamma}/\Gamma(4m)^*|}\sum_{\gamma\in\tilde{\Gamma}/\Gamma(4m)^*}\chi_{\mathbf{R}_i}(\gamma)\overline{\chi_{\mathbf{R}_j}(\gamma)}=\delta_{ij}. \tag{4.64}$$

The multiplicities m_i of the irreducible representations $\mathbf{R}_i$ in $\mathbf{M}$ are given by

$$m_i=\frac{1}{|\tilde{\Gamma}/\Gamma(4m)^*|}\sum_{\gamma\in\tilde{\Gamma}/\Gamma(4m)^*}\chi_{\bar{\mathbf{M}}}(\gamma)\overline{\chi_{\mathbf{R}_j}(\gamma)}. \tag{4.65}$$

The individual vector elements $g_\mu(\tau)$ transform covariantly under $\Gamma(4m)^*$. Therefore they lie in the space of weight $2-w$ cusp forms of $\Gamma(4m)^*$, which is denoted by $S_{2-w}(\Gamma(4m)^*)$. The space $S_{2-w}(\Gamma(4m)^*)$ is closed under transformations of $\gamma\in\tilde{\Gamma}$; such transformations rotate a chosen set of basis elements of $S_{2-w}(\Gamma(4m)^*)$ among each other. As a consequence, $S_{2-w}(\Gamma(4m)^*)$ defines a $\dim S_{2-w}(\Gamma(4m)^*)$-dimensional representation of $\tilde{\Gamma}/\Gamma(4m)^*$, which can similarly be decomposed into the irreducible representations. When the multiplicities of $\mathbf{R}_i$ in $S_{2-w}(\Gamma(4m)^*)$ are s_i, then $\dim S_{2-w,\bar{\mathbf{M}}}(\Gamma(4m)^*)=\sum_i m_i s_i$. The character of the element γ in the $S_{2-w}(\Gamma(4m)^*)$-representation is denoted by

$$\mathrm{Tr}\left[\gamma, S_{2-w}(\Gamma(4m)^*)\right]. \tag{4.66}$$

In terms of the characters , $\dim S_{2-w,\bar{\mathbf{M}}}(\Gamma(4m)^*)$ is now expressed by

$$\begin{aligned}\dim S_{2-w,\bar{\mathbf{M}}}(\Gamma(4m)^*) &= \frac{1}{|\tilde{\Gamma}/\Gamma(4m)^*|} \\ &\times\sum_{\gamma\in\tilde{\Gamma}/\Gamma(4m)^*}\chi_{\bar{\mathbf{M}}}(\gamma)\overline{\mathrm{Tr}\left[\gamma, S_{2-w}(\Gamma(4m)^*)\right]}.\end{aligned} \tag{4.67}$$

The Selberg trace formula provides a way to determine traces as in Eq. (4.66), for example $\mathrm{Tr}\left[\gamma, S_{2-w}(\Gamma(4m)^*\right]-\mathrm{Tr}\left[\gamma^{-1}, M_w(\Gamma(4m)^*\right]$ can be calculated. The space $M_w(\Gamma(4m)^*)$ is the space of holomorphic modular forms of $\Gamma(4m)^*$ with weight w. This is applied by Theorem 5.1 of Ref. [143] to calculate the dimension of the space of a vector-valued cusp forms, with weight $2-w$ and whose transformation matrices $\mathbf{M}(\gamma)$ form a representation of the finite group $\tilde{\Gamma}/\Gamma(4m)^*$. It is a sum, of three (generically fractional) contributions

$$\dim S_{2-w,\bar{\mathbf{M}}}(\Gamma(4m)^*)-\dim M_{w,\mathbf{M}}(\Gamma(4m)^*)=A_{\mathrm{s}}+A_{\mathrm{e}}+A_{\mathrm{p}}, \tag{4.68}$$

where the subscripts "s", "e" and "p" refer respectively to "scalar", "elliptic" and "parabolic". This terminology appears naturally in the derivation of the Selberg trace formula, see for example [146]. The three contributions are given by [143, 144]

$$\begin{aligned} A_{\mathrm{s}} &= \frac{1-w}{12}\chi_{\bar{\mathbf{M}}}(1), \\ A_{\mathrm{e}} &= -\frac{1}{4}\mathrm{Re}\left[e\left(-\frac{w}{4}\right)\chi_{\bar{\mathbf{M}}}\left(S\right)\right] + \frac{2}{3\sqrt{3}}\mathrm{Re}\left[e\left(\frac{5-2w}{12}\right)\chi_{\bar{\mathbf{M}}}\left(ST\right)\right], \\ A_{\mathrm{p}} &= -\frac{1}{2}S(\mathbf{M}) - \sum_{\mu=1}^{d}((\lambda_\mu)). \end{aligned} \tag{4.69}$$

The quantity $S(\mathbf{M})$ is defined by the number of terms equal to 1 in $\chi_{\bar{\mathbf{M}}}(T^n)$ for general n. Note that this definition gives the same number as the definition of $S(\mathbf{M})$ given above (4.62). The numbers λ_μ are the fractional numbers appearing in $\chi_{\bar{\mathbf{M}}}(T^n) = \sum_\mu e(\lambda_\mu n)$. These λ_μ are equal to $\Delta_\mu \mod \mathbb{Z}$. The number of polar terms, (4.61), contains thus a term A_{p}. The trace of the identity matrix $\chi_{\bar{\mathbf{M}}}(1)$ is the dimension d of the representation $\mathbf{M}$. Eqs. (4.68) and (4.69) lead to simpler dimension formulas than for scalar modular of congruence subgroups $\Gamma(N)$. This suggests that vector-valued modular forms might be more fundamental as forms of congruence subgroups.

Eq. (4.68) provides us $\dim S_{2-w,\bar{\mathbf{M}}}(\Gamma(4m)^*)$, since holomorphic modular forms with negative weight do not exist, such that the dimension of $M_{-w,\mathbf{M}}(\Gamma(4m)^*)$ is 0. To calculate $\dim S_{2-w,\bar{\mathbf{M}}}(\Gamma(4m)^*)$, one needs to evaluate $\chi_{\bar{\mathbf{M}}}(\gamma)$ for the relevant γ's and substitute in Eq. (4.69). An equation for $\dim A_{w,\mathbf{M}}(\Gamma(4m)^*, \Delta)$ can now be given. One finds

$$\dim A_{w,\mathbf{M}}(\Gamma(4m)^*, \Delta) = \sum_{\mu=1}^{d} \Delta_\mu + \frac{w+5}{12}d + A_{\mathrm{e}}. \tag{4.70}$$

The first term is generically polynomially in d and $|A_{\mathrm{e}}| < 1$. The calculation of the dimension for the CFT partition functions will be performed in the next chapter.

5. INTERPRETATIONS AND APPLICATIONS

The first chapter motivated the study of black hole partition functions and left us with three main problems. The first problem was to confirm the black hole entropy from a microscopic point of view. This is achieved in the end of Chap. 3, following the original work [3]. The other two main problems are to clarify the connection between black holes and topological strings (Subsec. 2.1.3), and to find manifestations of an AdS_3-gravity partition function in the CFT partition function (Subsec. 2.2.2).

This chapter revisits the motivations and analyzes the two remaining problems using the technical results of the previous chapter. In contrast to the first chapter, first the AdS_3-interpretation is discussed and subsequently the connection with $\mathcal{Z}_{\mathrm{top}}$. The results of Chapter 4 proof also useful in other places where vector-valued modular forms appear. As an application of this, the partition functions of four-dimensional $\mathcal{N} = 4$ supersymmetric twisted Yang-Mills are discussed in Sec. 5.3.

5.1 The AdS_3/CFT_2 correspondence

Subsec. 2.2.2 discusses heuristically pure (chiral) AdS_3 partition functions. The semi-classical path integral of pure gravity is a regularized sum over a set of smooth Euclidean geometries. The sum resembles a Poincaré series, which are sums over Γ/Γ_∞. On the other hand, a CFT partition function is a modular invariant and can be rewritten using Poincaré series. Unfortunately, the discussion in Chapter 2 is not fully satisfactory. Examples of problematic issues are the holomorphic factorization and the appearance of a double sum for pure gravity.

Nevertheless, this section again attempts to write $\mathcal{Z}_{\mathrm{CFT}}$ as a Poincaré series with the interpretation of a semi-classical sum over geometries. The presence of supersymmetry in the M-theory setup makes the prospects for an accurate semi-classical analysis much more promising. Based on the results of Sec. 4.2, $\mathcal{Z}_{\mathrm{CFT}}$ can indeed be written as a Poincaré series. Subsec. 5.1.2 discusses the series from the physical point of view. Some extra attention is paid to the regularization and anomaly cancellation.

5.1.1 $\mathcal{Z}_{\mathrm{CFT}}$ as a Poincaré series

Using the regularized Poincaré series derived in Sec. 4.2, a Poincaré series for $\mathcal{Z}_{\mathrm{CFT}}(\tau,\bar\tau,z)$ is readily written down. Using (4.48), $\mathcal{Z}_{\mathrm{CFT}}(\tau,\bar\tau,z)$ can be expressed as

$$\begin{aligned}\mathcal{Z}_{\mathrm{CFT}}(\tau,\bar\tau,z)= &\ \tfrac{1}{2}\sum_{\mu\in\Lambda^*/\Lambda} c_\mu(\Delta_\mu)\Theta_\mu(\tau,\bar\tau,z) \\ &+\ \tfrac{1}{2}\sum_{\Gamma_\infty/\Gamma}\varepsilon^{-1}(\gamma)j(\gamma,\tau)^{-\frac{1}{2}}j(\gamma,\bar\tau)^{\frac{3}{2}}e\left(-\frac{cz_+^2/2}{c\tau+d}-\frac{cz_-^2/2}{c\bar\tau+d}\right) \\ &\times \sum_{n-\Delta_\mu<0} c_\mu(n)R\left(\frac{2\pi i(n-\Delta_\mu)}{c(c\bar\tau+d)}\right) \\ &\times\ e\left((n-\Delta_\mu)\gamma(\bar\tau)\right)\Theta_\mu(\gamma(\tau),\gamma(\bar\tau),\gamma(z)),\end{aligned} \tag{5.1}$$

where Δ_μ is given by

$$\Delta_\mu=\frac{c_R}{24}-\left(\frac{\mu^2+p\cdot\mu}{2}-\left\lfloor\frac{\mu^2+p\cdot\mu}{2}\right\rfloor\right). \tag{5.2}$$

Furthermore, $n-\Delta_\mu$ is equal to $\hat q_{\bar 0}=q_{\bar 0}+\frac{1}{2}q^2$ with $q=\mu+p/2 \mod \Lambda$. The regularizing factor $R(x)$ is given before in Eq. (4.47).

We see that in the present case of a partition function of BPS states, the semi-classical expansion is remarkably accurate. The function is written as a single sum over Γ/Γ_∞. The series in Eq. (5.1) is however more involved than the gravity path integral described in Sect. 2.2. The elliptic genus contains a theta function which captures the dependence on τ and z. In addition, the principal part can possibly consist of many terms. The remaining part of the section gives an interpretation to the ingredients of (5.1). The interpretation is due to [89]. The improved regularization of [136] gives rises to some new insights.

The dependence on z^a in Eq. (5.1) is a consequence of the fact that we are not dealing with pure gravity but with a reduction of M-theory to AdS_3. The parameters z^a are the boundary conditions for gauge fields on the boundary T^2 [112]. The flat gauge fields lead to a singular line γ in the interior of the solid torus. They correspond to Wilson lines [89]. The theta function arises from the singleton modes [125] and adds the contributions of $U(1)$ degrees of freedom to the spectrum. These are pure gauge in the bulk but dynamical on the boundary. As mentioned earlier, they can be seen as the nucleation of M5-branes [59]. .

Another important difference between (2.111) and (5.1) is the sum over $q_{\bar 0}+\frac{1}{2}q^2<0$. Eq. (5.1) therefore contains many polar terms whereas

(2.111) contains only one. These terms are quantum corrections to the classical action. Although every single term is a non-perturbative correction, the whole sum captures both perturbative and non-perturbative corrections to the supergravity action. The contribution of these polar states to the full elliptic genus, is given by

$$\mathcal{Z}_{\mathrm{CFT}}(\tau,\bar{\tau},z)^- = \sum_{\substack{\mu\in\Lambda^*/\Lambda \\ q_{\bar{0}}+\frac{1}{2}q^2<0}} c_\mu\left(q_{\bar{0}}+\frac{1}{2}q^2\right)\bar{q}^{q_{\bar{0}}+\frac{1}{2}q^2}\Theta_\mu(\tau,\bar{\tau},z). \tag{5.3}$$

These states are interpreted as "light" excitations of thermal AdS$_3$. They are typically Kaluza-Klein modes or (charged) point particles. The charged point particles can be branes wrapping cycles in an orthogonal compact manifold, see the next section for more details on this higher dimensional interpretation of the AdS$_3$ degrees of freedom. The Wilson lines are the worldlines of the (virtual) particles. The contribution of the gas of virtual particles can be determined by a Schwinger calculation. Such a calculation is carried out in the next section. The negative energy of the AdS$_3$-geometry allows a certain amount of particles to be present without a gravitational collapse. However when the energy surpasses the cosmic censorship bound a black hole will form through a Hawking-Page phase transition. In the case of AdS$_3$/CFT$_2$ correspondence, such an interpretation was first proposed in Ref. [147]. The states without gravitational collapse satisfy in this case $M-\frac{1}{2}g^{ab}J_{0,a}J_{0,b}<0$ with $M=L_0+\bar{L}_0-\frac{c_L+c_R}{24}$ [148]. Since $M=q_{\bar{0}}+q_+^2$, the bound corresponds to $q_{\bar{0}}+\frac{1}{2}q^2<0$. The polar states (counted by (5.3)) are thus exactly the states which do not collapse into a black hole in thermal AdS$_3$.

The polar regime is the regime where counting of the degeneracies in supergravity could be reliable. For cases with higher supersymmetry, $\mathcal{N}=(4,4)$ and "large" $\mathcal{N}=(4,4)$, such a comparison between supergravity and CFT side is carried out by Refs. [149, 110]. After a spectral flow transformation to the Neveu-Schwarz sector, they have shown that the supergravity degeneracies indeed match with the CFT degeneracies for negative eigenvalues of $(L_0-\frac{c_L}{24})_{\mathrm{NS}}$. The computations on either side of the correspondence do not match for states with a higher energy. This suggests that gravitational degrees of freedom start contributing at this level, which is in agreement with the Hawking-Page phase transition. Since $n_{\mathrm{NS}}=0$ is the smallest value of n_{NS} which satisfies the cosmic censorship bound this is not surprising [89]. Ref. [112] determines the supergravity modes for the current case of interest: AdS$_3$ arising by an M-theory compactification on $X\times S^2$.

The Poincaré series for the elliptic genus (5.1) has the following physical interpretation. It is is a sum of the polar spectrum over all the black hole geometries. Higher excitations would collapse into the black hole and are therefore excluded, since those states are counted by another classical black hole geometry in the sum.

5.1.2 Regularization and anomaly cancellation

An indispensable ingredient in (5.1) is the convergence factor $R\left(\frac{2\pi i\hat{q}_{\bar{0}}}{c(c\bar{\tau}+d)}\right)$, where $R(z) = \frac{1}{\Gamma(1-w)}\int_0^z e^{-t}t^{-w}dt$ with $w = -b_2/2 - 1$. The presence of this factor is the main novel ingredient compared to the original discussion in [89]. The consequences of this factor are described here in some detail.

First, we check that the factor does not change one of the attractive properties of the series in (2.111) and [89], namely the manifestation of Hawking-Page phase transitions. To see that this is the case, we estimate $\left|R\left(\frac{2\pi i(n-\Delta_\nu)}{c(c\bar{\tau}+d)}\right) - 1\right|$:

$$\left|R\left(\frac{2\pi i(n-\Delta_\nu)}{c(c\bar{\tau}+d)}\right) - 1\right| \leq \frac{e^{-2\pi\frac{c_R}{24}\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}}}{\Gamma(1-w)}\left(\frac{2\pi\frac{c_R}{24}}{|c(c\bar{\tau}+d)|}\right)^{w-1}, \qquad (5.4)$$

where we assumed that $\frac{2\pi|n-\Delta_\nu|}{|c(c\bar{\tau}+d)|} \gg 1$. We observe that the correction is typically exponentially smaller than the exponent of the classical action, and therefore (5.1) predicts phase transitions parametrized by $\Gamma_\infty\backslash\Gamma$. Fig. 4.4 shows that $R(z)$ is indeed exponentially close to one for $z \in \mathbb{R}$ and $z \gg 1$ and how its behavior near $z = 0$ depends on the weight w.

The absolute value $|R(z)|$ takes its values between 0 and 1. Therefore, the factor $R\left(\frac{2\pi i\hat{q}_{\bar{0}}}{c(c\bar{\tau}+d)}\right)$ can be understood heuristically as the "fraction" of light excitations, with a given value of $\hat{q}_{\bar{0}}$ in thermal AdS_3, which contributes to the states of a black hole geometry given by (c, d). This fraction represents stable states in the black hole geometry, whereas the other states are unstable and will collapse into the black hole. From this viewpoint, $R(z)$ can be seen as a smooth cut-off on the contributions of the light excitations in thermal AdS_3 to the geometries with $c \neq 0$, since $R\left(\frac{2\pi i\hat{q}_{\bar{0}}}{c(c\bar{\tau}+d)}\right)$ is exponentially close to 1 for $\hat{q}_{\bar{0}} \gg 1$, and is zero for $\hat{q}_{\bar{0}} = 0$. The geometries with complicated topologies (c and/or $d \gg 1$) are similarly cut-off. Note that $R(z)$ is in general complex, so such an interpretation is heuristic, at best.

In a similar way, the anomalous appearance of the states with $\hat{q}_{\bar{0}} = 0$ in the elliptic genus can be explained. Half of these states are counted by the term

$$\sum_{\mu\in\Lambda^*/\Lambda} \frac{1}{2} c_\mu(0)\Theta_\mu(\tau,\bar\tau,z),$$

which appears separately in Eq. (5.1). Comparison with the Fourier series of the elliptic genus, Eqs. (3.62) and (3.73), shows that the sum over $\Gamma_\infty\backslash\Gamma$ contains an equal term. This suggests that half of the states at $\hat q_{\bar 0} = 0$ correspond to black holes, whereas the other half are stable states in thermal AdS$_3$. Since these stable states in thermal AdS$_3$ do not contribute to the black hole states, their interpretation is more subtle than the states with $\hat q_{\bar 0} < 0$. The way the states at the threshold appear in the partition function leads us to suggest that these excitations are so close to a collapse in thermal AdS$_3$, that they would collapse into the black hole when added to a black hole geometry. A more quantitative description of these phenomena is highly desirable.

Sec. 4.3 explained the transformation properties of $h_\mu(\tau)$ for a general set of polar coefficients. The regularization by $R(z)$ led to an interesting anomaly. The transformation (4.58) determines the following transformation for $\mathcal{Z}$

$$\begin{aligned}\mathcal{Z}|_{(\frac12,-\frac32)}\gamma - \mathcal{Z} = & \qquad (5.5)\\ -\frac{\varepsilon(\gamma)}{\Gamma(2+b_2/2)} \sum_{\mu\in\Lambda^*/\Lambda} & \Theta_\mu(\tau,\bar\tau,z)\int_{\gamma^{-1}(\infty)}^{i\infty} g_\mu(t)(t-\bar\tau)^{\frac{b_2}{2}+1}dt,\end{aligned}$$

where $g_\mu(\tau)$ is a vector-valued cusp form determined by the polar coefficients and the integral is a period function. The shift in the transformation of $\mathcal{Z}$ compared to Eq. (3.69) represents an anomaly under modular transformations. This is a familiar situation in quantum field theory: a divergent quantity is formally invariant, the regularized quantity breaks the invariance, but in a controlled way. Thus the problem of constructing $\mathcal{Z}$ is a kind of anomaly cancellation problem: one must choose the polar coefficients, such that $g_\mu(\tau) = 0$.

In fact, the analogy goes deeper, since the anomaly is related to cohomology theories. As explained in Sect. 4.3, the obstructions are given by $H^1(X\mathcal{O}(\mathcal{L}))$, where $\mathcal{L}$ is the bundle determined by the singularities of $h_\mu(\tau)$. In addition to this, there is the relation with the cohomology theory known as Eichler cohomology. This theory defines a cohomology on the space of functions with at most polynomial growth. The transformation properties of these functions are derived from the period functions. We refer to references [141, 150, 151] for more details.

An interesting question is if and how the anomaly can be canceled. Two different possibilities exist to cancel the anomaly:

1. If the number of polar coefficients is larger than the number of obstructions, the polar coefficients can be chosen such that the anomaly cancels. This possibility is most common in physical situations.

2. The anomaly can be canceled by the addition of a non-holomorphic term to $h_\mu(\tau)$, which grows at most polynomially with τ_2.

The first possibility has the advantage that meromorphicity is preserved. To determine how many polar coefficients can be freely specified, one needs to compare the number of polar terms $p(\mathbf{M}, \frac{c_R}{24})$ and the constraints $\dim S_{3+b_2/2,\bar{\mathbf{M}}}(\Gamma(4m)^*)$, where $\mathbf{M}$ is the representation of $\tilde{\Gamma}/\Gamma(4m)^*$ determined by the transformation properties of $h_\mu(\tau)$. The dimension d of $\mathbf{M}$ is the order of the group $\Lambda^*/\Lambda \otimes R$, with R the reflection group of Λ^*. This quantity is roughly given by $\frac{1}{|R|}|\Lambda^*/\Lambda|$. The number of polar terms $p(\mathbf{M}, \Delta)$ can be calculated by (4.61), with Δ_μ given by (5.2). On the other hand, the number of constraints is calculated by Eqs. (4.68) and (4.69). In the context of AdS_3-gravity, c_R is typically large. In this situation, $p(\mathbf{M}, \frac{c_R}{24})$ grows approximately as $\frac{c_R}{24}d$, and $\dim S_{3+b_2/2,\bar{\mathbf{M}}}(\Gamma(4m)^*)$ as $\frac{4+b_2}{24}d$. We observe that the ratio between the numbers of polar terms and constraints grows linearly with the central charge. This property is also valid for $\mathcal{N} = (2, 2)$ elliptic genera [142]. A general and precise analysis can in principle be carried out. To this end, the traces $\chi_{\bar{\mathbf{M}}}(S)$ and $\chi_{\bar{\mathbf{M}}}(ST)$ need to be calculated. Currently, this is only established for some one-dimensional lattices [142]. Secs. 5.2 and 5.3 calculate the number of polar coefficients and constraints for a view specific cases.

Note that the form of Δ_μ in (5.2) makes it suggestive to write $h_\mu(\tau)$ as $f_\mu(\tau)/\eta(\tau)^{c_R}$. The dimension of the space of meromorphic $h_\mu(\tau)$ can then be calculated as the space of holomorphic $f_\mu(\tau)$ by (4.68), if $c_R \geq b_2 + 2$. These $f_\mu(\tau)$ might capture interesting data as in the case of $SU(2)$ $\mathcal{N} = 4$ Yang-Mills theory on $\mathbb{CP}^2$, see Sec. 5.3 for more details.

The second way to cancel the anomaly is required if the space of constraints is larger or equal to the number of polar terms. Also, physics might prescribe a set of polar degeneracies, which can not be consistently extended to a holomorphic modular form with the required transformation properties. In such cases, the modular anomaly must be treated for a holomorphic anomaly, since diffeomorphism invariance is more fundamental than meromorphy. Eq. (4.50) shows that a non-holomorphic term can be added to $h_\mu(\tau)$, based on the period function (4.49), such that

$$\mathcal{Z}_{\rm c}(\tau,\bar\tau,z)=\mathcal{Z}(\tau,\bar\tau,z)-\sum_{\mu\in\Lambda^*/\Lambda}\Theta_\mu(\tau,\bar\tau,z)\overline{p(\tau,\bar\tau,\overline{g_\mu})} \tag{5.6}$$

transforms covariantly. To study its properties more precisely, we rewrite $\overline{p(\tau,\bar\tau,\overline{g_\mu})}$ as

$$\frac{1}{\Gamma(2+b_2/2)}\int_\tau^{i\infty} g_\mu(z)(z-\bar\tau)^{1+b_2/2}dz = \frac{(2i\tau_2)^{2+b_2/2}}{\Gamma(2+b_2/2)}\int_1^\infty g_\mu(\bar\tau+2ui\tau_2)u^{1+b_2/2}du. \tag{5.7}$$

From the first expression it is clear that $h_{\mu,{\rm c}}(\tau,\bar\tau)=h_\mu(\tau)-p(\tau,\bar\tau,\overline{g_\mu})$ satisfies the holomorphic anomaly equation

$$\frac{\partial}{\partial\bar\tau}h_{\mu,{\rm c}}(\tau,\bar\tau)=\frac{(-2i\tau_2)^{1+b_2/2}}{\Gamma(2+b_2/2)}\overline{g_\mu(\tau)}. \tag{5.8}$$

Of course, such a non-holomorphic correction is far from being unique! The above choice is distinguished by the fact that $h_{\mu,{\rm c}}(\tau,\bar\tau)$ is annihilated by a Laplacian given by $\Delta=\frac{\partial}{\partial\tau}\tau_2^{-1-b_2/2}\frac{\partial}{\partial\bar\tau}$. Note that (5.7) also reduces to a polynomial in τ and $\bar\tau$ for b_2 even.

The holomorphic anomaly described here is similar to the one appearing for weight $\frac{3}{2}$ modular forms discussed in [152, 153]. In physics, such holomorphic anomalies arise in the partition function of $\mathcal{N}=4$ topologically twisted Yang-Mills theory on $\mathbb{CP}^2$ with gauge group $SO(3)$ [154], and also in the context of Donaldson invariants [155]. This suggests that holomorphic anomalies might appear as well in the $(4,0)$ elliptic genus.[1] Sec. 5.3 discusses the holomorphic anomaly for $\mathcal{N}=4$ Yang-Mills in some detail.

Finally, we comment on an ambiguity related to the Poincaré series. We have argued that the states counted by the theta function are pure gauge in the bulk and only dynamical on the boundary. Therefore, these states should not be summed over all different bulk geometries. This interpretation implies that all non-polar states are black hole states. The validity of this statement might be questioned for two reasons. First, the singleton degrees of freedom are not just $\Theta_\mu(\tau,\bar\tau,z)$ but $\Theta_\mu(\tau,\bar\tau,z)/\eta(\bar\tau)^{b_2+2}$ [124]. Here, the descendants of the primaries given by the lattice and also those due to $\vec p$ are included. These descendants are also excitations

[1] Exactly this suggestion has been made previously by D. Gaiotto in a seminar at Princeton, Oct. 13 2006.

on the boundary, and not to be summed over all geometries. From this point of view, the following composition is natural

$$\mathcal{Z} = \sum_{\mu \in \Lambda^*/\Lambda} \overline{f_\mu(\tau)} \frac{\Theta_\mu(\tau, \bar{\tau}, z)}{\eta(\bar{\tau})^{b_2+2}}. \tag{5.9}$$

The weight 0 vector-valued modular form $f_\mu(\tau)$ will be written as a Poincaré series to write it as a gravity partition function.

The next reason to question the statement that all non-polar states are black hole states is that in general the descendants of primaries should not be considered as black hole states [92]. The black hole states are the primaries. The contributions of the descendants can be calculated in (super)gravity by higher loop corrections [156]. Since the descendants are not black hole states, one should sum these descendants over all geometries. In other words, in the Poincaré series for $f_\mu(\tau)$ one wants to remove the condition $n-\Delta_\mu < 0$ and include also the descendants of the polar primaries. Interestingly, these requests are allowed for the decomposition as in Eq. (5.9). Generically, the non-polar terms would lead to obstruction forms with a polar part. In the current case, the period function vanishes since the non-polar obstruction forms can be written as the derivative of a weight zero form. Constraints on the polar terms of $f_\mu(\tau)$ might still exist since cusp forms of weight two do exist for congruence subgroups in general. The inclusion of these modes in the sum does not change the degeneracies of the non-polar states.

5.2 Four-dimensional black holes

The previous section explained how to interpret the series (5.1) from an AdS_3 point of view. Roughly speaking, it is a sum over all geometries of the light excitations in thermal AdS_3. In this section, we connect the AdS_3 discussion to the four dimensional black holes an M-theory and explain how the Poincaré series quite naturally connect $\mathcal{Z}_{BH}$ and $\mathcal{Z}_{top}$. The light AdS_3 excitations include massless supergravity modes as well as M2-branes and anti-M2-branes [57]. In addition, there are other exotica such as M5-black rings, $\mathbb{Z}_r$ quotients of $AdS_3 \times S^2$ and even more complicated geometries. These are all expected to be dual to the multi-centered D6 anti-D6 configurations that played a crucial role in Ref. [37].

The next subsection considers the relation between $\mathcal{Z}_{BH}$ and $\mathcal{Z}_{top}$ in the regime with strong topological string coupling $\lambda \sim \sqrt{\hat{q}_{\bar{0}}/p^3} \gg 1$. This is also the regime which is required for low energy M-theory to be valid.

In Sec. 5.2.2, the opposite regime with a weak coupling is analyzed. In this regime, we will find deviations from the conjecture (2.61).

5.2.1 Strong topological string coupling

The Poincaré series for the $\mathcal{N} = (4,0)$ elliptic genus can be used to elucidate the OSV conjecture [5, 59, 37]. To relate $\mathcal{Z}_{\rm CFT}$ with $\mathcal{Z}_{\rm BH}$, we made the identification of parameters given in (3.63). This determines among others that $\tau = \bar{\tau}$. In the regime of a strong topological string coupling $\tau \sim i\sqrt{p^3/\hat{q}_{\bar{0}}}$. The leading behavior of the partition function can be determined, by the saddle point method. This will identify the perturbative part of the black hole action. Since τ is small, the most contributing geometry is given by the pair $(c,d) = (1,0)$. In this regime, $\mathrm{Re}(x) \to \infty$ in the argument of $R(x)$. Thus the regularization factor introduces only exponentially small corrections. In this way the artificial restriction to $b_2(X)$ even, imposed in Ref. [37], may be removed. By the saddle point technique, we find the following perturbative action

$$\begin{aligned} 2\pi i \quad & \left(\frac{z^2}{2\tau} + \frac{p^3 + c_2\cdot p}{24\tau} + \frac{q^2}{2\tau} + \frac{q\cdot z}{\tau}\right) = \\ & \frac{2\pi i}{6\tau}\left((\tfrac{1}{2}p + q + z)^3 + \tfrac{1}{4}c_2\cdot(\tfrac{1}{2}p + q + z)\right) \\ & + \frac{2\pi i}{6\tau}\left((\tfrac{1}{2}p - q - z)^3 + \tfrac{1}{4}c_2\cdot(\tfrac{1}{2}p - q - z)\right), \end{aligned} \tag{5.10}$$

where we used on the left hand side d_{ab} to define the quadratic terms, while the cubic terms on the right hand side are defined with the help of d_{abc}. Here we recognize precisely the perturbative genus zero and genus one piece of the topological string partition function, clearly giving evidence for (2.61). The elliptic genus is roughly a sum over k and Γ/Γ_∞ of $|\mathcal{Z}_{\rm top}|^2$. Heuristically, the sum over k in the theta function and the sum over Γ/Γ_∞ make the total expression both spectral flow and modular invariant.

We would like to take into account also non-perturbative corrections to the perturbative part. To this end, we would like to integrate out the lightest degrees of freedom, since they have the largest contribution to the effective action. The ratio of the masses of M5-branes and M2-branes wrapped on the Calabi-Yau is $V_{\rm CY}^{\frac{1}{3}}R/\ell_{\rm P}^3$, which is proportional to $\sqrt{\hat{q}_{\bar{0}}/p}$. Since the parameter range for a valid use of low energy M-theory is $\hat{q}_{\bar{0}}/p \gg 1$ (Sec. 2.2), the M2-branes are the light degrees of freedom. Although the previous section mainly emphasized the excitations in AdS_3, the analysis here is performed in $\mathrm{AdS}_2 \otimes S^2$. The M2-branes become then BPS bound states of D0- and D2-branes. These particles can be integrated out by the calculation of pair creation in a constant magnetic field, which is known as a Schwinger calculation [157]. For a supergravity theory, the pair creation in a background magnetic field should be extended with pair creation in a background metric field. However, the presence of supersymmetry reduces the calculation to the familiar field theory computation. The gauge field is the anti-self-dual field T^-. The calculation is carried out by Gopakumar and Vafa in [158, 159] with background geometry $\mathbb{R}^4$.

First pair creation of charged scalar field quanta is considered in a constant background field T^-. The scalar field is considered as a probe field, the back reaction of the geometry is not taken into account. The charge of the scalar field is $n = n_I X^I$, by supersymmetry $|n|$ is equal to the mass m of the scalar field. The $\mathrm{AdS}_2 \otimes S^2$ metric (2.29) is rotated to Euclidean signature. Then we can introduce the dimensionless complex coordinates $z_1 = (t_{\mathrm{E}} + i|Z|^2/r)/\ell_{\mathrm{P}}$ ($\ell_{\mathrm{P}} = 1$ in the following) for the AdS_2 part, and the Fubini-Study metric in terms of z_2 for the S^2 part. The metric in terms of these coordinates is

$$ds^2 = |Z|^2 \left(\frac{dz_1 d\bar{z}_1}{(\mathrm{Im}\, z_1)^2} + \frac{4 dz_2 d\bar{z}_2}{(1+|z_2|^2)^2} \right). \tag{5.11}$$

The Laplacian $\nabla^2 = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu$ takes a particularly simple form for this metric

$$\nabla^2 = \frac{1}{\sqrt{g_{\mathrm{AdS}_2}}} \left(\partial_1 \bar{\partial}_1 + \bar{\partial}_1 \partial_1 \right) + \frac{1}{\sqrt{g_{S^2}}} \left(\partial_2 \bar{\partial}_2 + \bar{\partial}_2 \partial_2 \right), \tag{5.12}$$

where $\sqrt{g_{\mathrm{AdS}}} = |Z|^2/(\mathrm{Im}\, z_1)^2$ and $\sqrt{g_{S^2}} = 4|Z|^2/(1+|z_2|^2)^2$. The anti-self-dual gauge field in this geometry is

$$T^- = Z \left(\frac{dz_1 d\bar{z}_1}{(\mathrm{Im}\, z_1)^2} - \frac{4 dz_2 d\bar{z}_2}{(1+|z_2|^2)^2} \right). \tag{5.13}$$

The corresponding potential in holomorphic gauge, $\partial A^- = 0$, is

$$A^- = Z \left(-\frac{2i dz_1}{\mathrm{Im}\, z_1} + \frac{4 \bar{z}_2 dz_2}{1+|z_2|^2} \right). \tag{5.14}$$

The minimal coupling of the scalar field to the gauge field amounts to changing ∂ to $\partial - \bar{n} A^-$. This transforms the Laplacian to

$$\begin{aligned} \nabla^2_A = \quad & \tfrac{1}{\sqrt{g_{\mathrm{AdS}_2}}} \left(\left(\partial_1 + \tfrac{2i\bar{n}Z}{\mathrm{Im}\, z_1} \right) \bar{\partial}_1 + \bar{\partial}_1 \left(\partial_1 + \tfrac{2i\bar{n}Z}{\mathrm{Im}\, z_1} \right) \right) + \\ & \tfrac{1}{\sqrt{g_{S^2}}} \left(\left(\partial_2 - \tfrac{4\bar{n}Z\bar{z}_2 dz_2}{1+|z_2|^2} \right) \bar{\partial}_2 + \bar{\partial}_2 \left(\partial_2 - \tfrac{4\bar{n}Z\bar{z}_2 dz_2}{1+|z_2|^2} \right) \right). \end{aligned} \tag{5.15}$$

The Hamiltonian of a massive charged scalar with $|n| = m$ is then

$$H = \nabla_A^2 + |n|^2. \tag{5.16}$$

To perform the one-loop calculation, we introduce two sets of raising and lowering operators. The appearance of a double harmonic oscillator is due to the fact that a self-dual gauge field is present. The operators are given by

$$\begin{aligned} a = &\ \bar{\partial}_1, \qquad a^\dagger = \left(\partial_1 + \frac{2i\bar{n}Z}{\mathrm{Im}(z_1)}\right), \\ b = &\ \bar{\partial}_2, \qquad b^\dagger = \left(\partial_2 - \frac{4\bar{n}Z\bar{z}_2}{1+|z_2|^2}\right). \end{aligned} \tag{5.17}$$

The commutation relations read

$$\begin{aligned} [a, a^\dagger] &= \frac{\bar{n}}{\bar{\bar{Z}}}\sqrt{g_{\mathrm{AdS}_2}}, \\ [b, b^\dagger] &= \frac{\bar{n}}{\bar{\bar{Z}}}\sqrt{g_{\mathrm{S}^2}}. \end{aligned} \tag{5.18}$$

The Hamiltonian can be written in terms of these operators as

$$H = \sum_{n_0, n_a} \frac{1}{\sqrt{g_{\mathrm{AdS}_2}}} a^\dagger a + \frac{1}{\sqrt{g_{\mathrm{S}^2}}} b^\dagger b + \frac{\bar{n}}{\bar{\bar{Z}}} + |n|^2, \tag{5.19}$$

where the sum over n_0, n_a is over both positive and negative charges. The energy level of the two harmonic oscillators are denoted by l_1 and l_2. These are the Landau levels of the charged particles in the magnetic field. Note that the number of ground states are given by the number of holomorphic sections of the line bundle. For a line bundle over S^2, this number is equal to the degree plus one.

The contribution to $\mathcal{F}_\mathrm{S}$ by the one-loop computation is calculated as

$$\mathcal{F}_\mathrm{S}(X) = \sum_{n_0, n_a} \ln\det\left(\nabla_A^2 + |n|^2\right) = \mathrm{Tr}\ \ln\left(\nabla_A^2 + |n|^2\right), \tag{5.20}$$

where is used $\det e^A = e^{\mathrm{Tr} A}$. Up to a constant, dependent on ϵ, $\ln a$ is equal to

$$\ln(a) = \int_\epsilon^\infty \frac{ds}{s} e^{-sa}. \tag{5.21}$$

Thus we write for $\ln H$

$$\mathcal{F}_\mathrm{S}(X) = \mathrm{Tr} \int_\epsilon^\infty \frac{ds}{s} e^{-Hs}. \tag{5.22}$$

Performing the sum over the excited states of the oscillators, one obtains

$$\sum_{l_1,l_2=0}^{\infty}\int_{\epsilon}^{\infty}\frac{ds}{s}e^{-(|n|^2+(l_1+l_2+1)\frac{\bar{n}}{\bar{Z}})s}=\frac{1}{4}\int_{\epsilon}^{\infty}\frac{ds}{s}\frac{e^{-ns}}{\sinh^2\frac{s}{2\bar{Z}}} \tag{5.23}$$

The central charge n of the particles is given by $n_0X^0+n_aX^a$. If we make this substitution in the integral and use that $\sum_{n_0\in\mathbb{Z}}e^{2\pi isn_0}=\sum_{k\in\mathbb{Z}}\delta(s-k)$. Then the integral reduces to

$$-\frac{1}{4}\sum_{k>0}\frac{1}{k}\frac{e^{2\pi ikn_at^a}}{\sin^2(\pi k/X^0\bar{Z})}, \tag{5.24}$$

where is used that the ratio X^a/X^0 is equal to the Kähler modulus t^a of the Calabi-Yau. In the following, $\frac{2\pi}{X^0\bar{Z}}=\frac{2\pi}{Y^0}$ is denoted by λ. Shortly will be explained that this combination is indeed naturally identified as the topological string coupling constant λ as below Eq. (2.59)

The D0-D2-brane bound states can carry higher spins. This does not change the calculation qualitatively. One only needs to change $\nabla^2_{A^-}$ to [157, 159]

$$\nabla^2_{A^-}-\bar{n}J_R\cdot T^-, \tag{5.25}$$

where J_R is the operator for the momentum in the $SU(2)_R$ of the $SO(4)=SU(2)_L\otimes SU(2)_R$. The free energy then reads

$$\mathcal{F}_{\mathrm{S}}(X)=-\frac{1}{4}\sum_{n_a}\int_{\epsilon}^{\infty}\frac{ds}{s}\frac{e^{-ns}\mathrm{Tr}\left[(-)^Fe^{-2\frac{s}{\bar{Z}}J_R}\right]}{\sinh^2\frac{s}{2\bar{Z}}}, \tag{5.26}$$

where also a grading $(-)^F$ is inserted, to ensure the projection to supersymmetric states. The spin representation in the super multiplet is given by $\left[\left(\frac{1}{2}\right)+2\,(0)\right]\otimes\left[\left(\frac{1}{2}\right)+2\,(0)\right]^r$. The first term is always present by supersymmetry. The representation is related to the Lefschetz action on the moduli space [159]. An evaluation of the trace under the integral gives $\mathrm{Tr}\left[(-)^Fe^{-2\frac{s}{\bar{Z}}J_R}\right]=(-4)^r\sinh^{2r}\left(\frac{s}{2\bar{Z}}\right)$. Therefore, the integral gives now the following sum over k

$$-\sum_{k>0}\frac{1}{k}\left[2\sin\left(\frac{k\lambda}{2}\right)\right]^{2r-2}e^{2\pi ikn_at^a}. \tag{5.27}$$

For a given D2-brane charge n_a, the representation r might appear with a certain multiplicity. These multiplicities are denoted by $\alpha_r^{n_a}$ and carry the name Gopakumar-Vafa invariants. These are either positive or negative integers and imply non-trivial integrality properties for the Gromov-Witten invariants. We will not enter into the subject of how to compute these integers, since this is outside the scope of thesis. See for example [159, 160]. The total $\mathcal{F}_{\rm S}(X)$ is conveniently written as

$$\mathcal{F}_{\rm S}(X) = F_{\rm GV}(t,\lambda) + \overline{F_{\rm GV}(t,\lambda)} \tag{5.28}$$

with

$$F_{\rm GV}(t,\lambda) = -\sum_{n_a\geq 0}\sum_{k>0,r\geq 0} \alpha_r^{n_a}\frac{1}{k}\left[2\sin\left(\frac{k\lambda}{2}\right)\right]^{2r-2} e^{2\pi i k n_a t^a}, \tag{5.29}$$

where is used that t^a is completely imaginary for a black hole with D4- and D0-brane charges.

Eq. (5.28) is very similar to (2.60). Indeed, Gopakumar and Vafa argued in [158, 159] that the effective action obtained by integrating out the D0-D2-brane states in $\mathbb{R}^4$, reproduces the topological string partition function. The dependence on λ here and in [158, 159] is identical. Therefore, the identification of λ as $2\pi/Y^0$ is confirmed here from the non-perturbative part of the topological string partition function. However, the degeneracies of the D0-D2-brane bound states in $\mathrm{AdS}_2 \otimes S^2$ might be different compared to the states in $\mathbb{R}^4$. A possible sign of this are the different number of ground states on S^2 and $\mathbb{R}^2$.

The equality between the supersymmetric D0-D2-brane partition function and $F_{\rm top}$ is not unconceivable, since both systems are in essence the holomorphic embedding of a Riemann surface in a Calabi-Yau X. This correspondence can be understood in some more detail by an uplift to M-theory, where fundamental strings and D2-branes are both described as M2-branes. A fundamental string corresponds to an M2-brane which wraps $S^1_{\rm M}$ and a D2-brane is an M2-brane which does not wrap $S^1_{\rm M}$. A topological string is a fundamental instanton string and wraps a holomorphic surface in X. This corresponds in M-theory to an instanton M2-brane, which wraps $S^1_{\rm M}$ in addition to the surface in X . In the Euclidean context, the time circle $S^1_{\rm t}$ and $S^1_{\rm M}$ appear on an equal footing and their role might be interchanged. This would exchange instanton M2-branes and non-instanton M2-branes. If M-theory is compactified on $S^1_{\rm t}$ instead of $S^1_{\rm M}$, the original topological string is related to a D0-D2-brane state. The topological string partition function captures thus information about the BPS-sector of IIA string theory and M-theory. In a similar way, topological strings and six-dimensional $U(1)$ gauge theory (Donaldson-Thomas theory) can be related to each other [161].

The relation between the $\mathcal{Z}_{\rm BH}$ and $|\mathcal{Z}_{\rm top}|^2$ has now become less mysterious. The topological strings calculate the effective action, which is equivalently obtained by integrating out the D0-D2-branes from the action. The leading contribution at a given saddle point is therefore dominated by $|\mathcal{Z}_{\rm top}|^2$. The full black hole partition function takes into account all states in the near-horizon geometry and sums over all the saddle points.

$$\begin{aligned}\mathcal{Z}_{\rm BH} \quad \sim \quad & \sum_{k\in\Lambda}\sum_{\Gamma/\Gamma_\infty}(-1)^{p\cdot k} e\left(-\frac{cz^2/2}{c\tau+d}\right)\times \\ & \mathcal{Z}_{\rm top}\left(\gamma(\tau),\frac{\frac{1}{2}p+k+z}{c\tau+d}\right)\mathcal{Z}_{\rm top}\left(\gamma(\tau),\frac{\frac{1}{2}p-k-z}{c\tau+d}\right)\end{aligned} \tag{5.30}$$

This suggests that $\mathcal{Z}_{\rm BH}$ is well approximated by a double sum over the topological string partition function. The sum over k is the sum over spectral flow images, and the sum Γ/Γ_∞ is the sum over geometries. In the sum, appropriate weight factors, unitary factors and possibly modular forms might appear in a precise version of this formula. If one considers the geometries given by $(c,d)=(1,d)$ [59, 37], then the black hole degeneracy can be written as an inverse Laplace transform, as suggested by the original conjecture [5].

Generically, $\mathcal{Z}_{\rm top}$ is divergent and the parameter identification in (3.63) does not make it a convergent quantity. Therefore, the expansion of $\mathcal{Z}_{\rm top}$ must be cut off appropriately in a precise version of (5.30). The regularization by $R(z)$ cuts off the function smoothly. The cut off can be understood nicely in the $\mathrm{AdS}_3\times S^2$ geometry. The cut off is a consequence of the finite number of Landau levels on S^2. A charged, supersymmetric state at the north pole represents as many supersymmetries as an oppositely charged state located at the south pole. This explains the appearance of $|\mathcal{Z}_{\rm top}|^2$ naturally as the contributions of the membranes and anti-membranes to the partition function [57]. If one start filling the Landau levels centered around the north and south pole, they eventually will start to overlap. The cut off on $\mathcal{Z}_{\rm top}$ is then necessary to avoid double counting.

An interesting property of $F_{\rm GV}(t,\lambda)$ is that it can be rewritten as a product formula. To this end, we define $y=e^{2\pi i t^a}$, $q=e^{i\lambda}$. Eq. (5.27) now reads

$$F_{\rm GV}(t,\lambda)=\sum_{n^a\geq 0, r\geq 0, k>0}\alpha_r^{n^a}\frac{1}{k}(-1)^r(q^{k/2}-q^{-k/2})^{2r-2}y^{kn}. \tag{5.31}$$

We split the sum into $r=0$ and $r>0$. For the sum over r we find

$$\begin{aligned}
&\sum_{r\geq 0}\alpha_r^{n^a}(-1)^r(y^{k/2}-y^{-k/2})^{2r-2}= \\
&\alpha_0^{n^a}(y^{k/2}-y^{-k/2})^{-2}+\sum_{r>0}\alpha_r^{n^a}(-1)^r(y^{k/2}-y^{-k/2})^{2r-2}= \\
&\alpha_0^{n^a}\sum_{l>0}ly^{kl}-\sum_{r\geq 0}\alpha_{r+1}^{n^a}(-1)^r\sum_{l=0}^{2r}\binom{2r}{l}(-1)^l y^{k(l-r)}= \\
&\alpha_0^{n^a}\sum_{l\geq 0}ly^{kl}-\sum_{r\geq 0}\alpha_{r+1}^{n^a}\sum_{l=-r}^{r}\binom{2r}{l+r}(-1)^l y^{kl}
\end{aligned}$$

where $\binom{2r}{l+r}=0$ for $l+r>2r$ and $l+r<0$.

The sum over k,r with $n^a=0$ can be rewritten as the $-\alpha_0^0$ times the logarithm of the (non-convergent) McMahon function $M(q)=\prod_{l>0}(1-q^l)^l$. This is the generating function for three-dimensional partitions. The part of the sum with $r\neq 0$ can similarly be rewritten to a product formula. As a result, we find for $\mathcal{Z}_{\text{top}}$

$$\mathcal{Z}_{\text{top}}=M(q)^{-\alpha_0^0}\prod_{r\geq 0,n^a>0,l}\left(1-q^l y^n\right)^{(-1)^l\binom{2r}{l+r}\alpha_r^{n^a}}. \tag{5.32}$$

The appearance of such a product formula for the modes in the near horizon geometry was shown in [57].

The part of the product with $n^a=0$ represents the D0-brane states in IIA string theory and the constant maps in topological string theory. One can determine that $\alpha_0^0=\chi(X)/2$ by the constant part of the prepotential. Therefore, a term $M(q)^{-\chi(X)}$ appears in the black hole partition function. The D0-branes can be viewed as pointlike instantons of a six-dimensional $U(1)$ gauge theory on X. Göttsche has calculated in [162] the partition function of such instantons in $U(1)$ gauge theory on a four-manifold M to be $\eta(\tau)^{-\chi(M)}$. More recently, the appearance of the McMahon function for six-manifolds is more mathematically derived in [163]. Remarkably, the $\eta(\tau)^{-1}$ and $M(q)^{-1}$ respectively calculate two- and three-dimensional partition functions. Thus, a natural structure seems to exist for the counting of $U(1)$ instantons on manifolds with different dimensions. The theory of partitions plays an important role herein.

5.2.2 Weak topological string coupling

The previous section explained to what extend $|\mathcal{Z}_{\text{top}}|^2$ approximates $\mathcal{Z}_{\text{BH}}$ in the regime of strong topological string coupling constant. In this regime, the effects of $R(x)$ could be neglected. On the other hand, in the opposite regime of *weak* topological string coupling, $p^3 \gg \hat{q}_{\bar{0}}$ the value of x goes to zero for the $c = \pm 1, d = 0$ terms in the Poincaré series and the effects of our regularization become significant, introducing further corrections to the OSV formula in this regime.

An interesting phenomenon described in Ref. [37, 164] is the "entropy enigma." This refers to the fact that for charges corresponding to weak topological string coupling, semi-classical multi-centered states exist which contribute to the "large radius BPS degeneracies" $c_\mu(\hat{q}_{\bar{0}})$ with entropies which grow exponentially in p^3 for $p \to \infty$. In particular, they dominate the single centered entropy, the latter growing like $\sqrt{\hat{q}_{\bar{0}} p^3}$. A growth of $\log |c_\mu(\hat{q}_{\bar{0}})| \sim p^3$ for $p \to \infty$ would be a sharp counterexample to the OSV conjecture, and would have other interesting implications. As discussed at length in Ref. [37, 165], since $c_\mu(\hat{q}_{\bar{0}})$ is an index it is conceivable that the exponentially large contributions might cancel, leaving asymptotics $\log |c_\mu(\hat{q}_{\bar{0}})| \sim \sqrt{\hat{q}_{\bar{0}} p^3}$. Ref. [37] argued that such cancellations are unlikely, but left this central question unanswered.

It is interesting to consider this central question in the light of the present paper. The limit of weak topological string coupling can interfere with the parameter regime for a valid use of low-energy M-theory. Nevertheless, the SCFT degeneracies of the charges in this regime can be analyzed. These degeneracies are the "barely polar degeneracies," that is, the coefficients $c_\mu(\hat{q}_{\bar{0}})$ for $\hat{q}_{\bar{0}}$ of order 1 or smaller (compared to p^3). The entropy enigma suggests that these barely polar degeneracies grow like $\exp(kp^3)$ as $p \to \infty$ for some constant k. We are thus led to ask what constraints are imposed by modular invariance on polar degeneracies, and whether the existence of terms with large poles $\sim q^{-p^3/24}$ implies, through anomaly cancellation, that the coefficients of terms with small or order one poles $\sim q^{-1/p}, \cdots, q^{-1}, \cdots, q^{-2}, \ldots$ are large. The Fourier coefficients $g(n)$ of cusp forms (for Γ, with trivial multiplier system) of weight k grow as $n^{k/2}$. Although modular invariance therefore bounds the growth of the polar degeneracies, a lot of freedom remains for these degeneracies. From these heuristic arguments, it is clear that we must look elsewhere for an explanation of exponentially large barely polar degeneracies.

In the following, we will refine a suggestion made in Ref. [37], p. 117. We make a toy model of the polar terms of the $\mathcal{N} = (4, 0)$ elliptic genus by considering a modular form for Γ with trivial multiplier system (for simplicity) and considering the polar terms of the negative weight form $\Phi(\tau)/\eta(\tau)^{c_R}$ where $c_R = p^3 + c_2(X) \cdot p$ and $\Phi(\tau)$ is a non-singular modular form for Γ of positive weight $w_\Phi = \frac{1}{2}c_R - 1 - \frac{1}{2}b_2$. As we remarked above, the leading coefficient $c_{\mu=0}(0)$ is, up to a sign $\sim p^3/6$ and therefore in our toy model $\Phi(\tau)$ will have a nonzero Petersson inner product with the Eisenstein series.

To begin, let us sharpen the comments made in [37] about the barely polar degeneracies of $\eta(\tau)^{-c_R}$ for large c_R. For simplicity we assume that c_R is a positive integer divisible by 24. Let us define Fourier coefficients by

$$\eta(\tau)^{-c_R} = q^{-c_R/24}\sum_{n=0}^{\infty} p_{c_R}(n)q^n. \tag{5.33}$$

We are considering degeneracies for $n = \frac{c_R}{24} + \ell$ with ℓ fixed as $c_R \to \infty$ (and of either sign) so the usual Hardy-Ramanujan analysis ("Cardy formula") is slightly altered. A naive saddle-point analysis proceeds by writing

$$\begin{aligned} p_{c_R}(n) &= \int_{\tau_0}^{\tau_0+1} e^{-2\pi i(n-c_R/24)\tau}\frac{1}{\eta(\tau)^{c_R}}d\tau \\ &\cong \int_{\tau_0}^{\tau_0+1} e^{-2\pi i(n-c_R/24)\tau+\frac{c_R}{2}\log(-i\tau)+\frac{i\pi c_R}{12\tau}}d\tau. \end{aligned} \tag{5.34}$$

In contrast to the usual estimate, it is now the second and third terms in the exponential which dominate the saddle point. In this way we estimate

$$p_{c_R}\left(\frac{c_R}{24}+\ell\right) \sim_{c_R\to\infty} C\cdot c_R^{-1/2}\exp\left(\frac{c_R}{2}\left(1+\log\frac{\pi}{6}\right)+\frac{\pi^2}{3}\ell\right), \tag{5.35}$$

for some constant C. This agrees very well with a numerical analysis of $\log p_{c_R}(c_R/24)$ in Ref. [37] (p.117). Moreover, we see that although the degeneracies grow exponentially with ℓ, the proportionality between $p_{c_R}\left(\frac{c_R}{24}+\ell\right)$ and $p_{c_R}\left(\frac{c_R}{24}+\ell+1\right)$ is not exponential in c_R. This agrees with the earlier statement that the anomaly cancellation bounds the growth of the polar degeneracies. It is interesting to compare with the Rademacher formula for $p_{c_R}(c_R/24)$:

$$\begin{aligned} p_{c_R}\left(\frac{c_R}{24}\right) &= 2\pi\sum_{0\le n<\frac{c_R}{24}} p_{c_R}(n)\frac{\left(2\pi|n-\frac{c_R}{24}|\right)^{1+c_R/2}}{\Gamma(2+c_R/2)} \\ &\quad\times\sum_{c=1}^{\infty} c^{-2-c_R/2}K_c\left(0,n-\frac{c_R}{24}\right). \end{aligned} \tag{5.36}$$

We can use a beautiful formula of Ramanujan: [2]

$$\sum_{c=1}^{\infty} c^{-s} K_c(0,n) = \frac{\sigma_{1-s}(n)}{\zeta(s)}, \tag{5.37}$$

to simplify our formula to:

$$p_{c_R}\left(\frac{c_R}{24}\right) = 2\pi \sum_{0\le n<\frac{c_R}{24}} p_{c_R}(n) \frac{\left(2\pi|n-\frac{c_R}{24}|\right)^{1+c_R/2}}{\Gamma(2+c_R/2)} \frac{\sigma_{-1-c_R/2}(\frac{c_R}{24}-n)}{\zeta(2+c_R/2)}. \tag{5.38}$$

Now, note for large c_R there is a very large denominator from the Gamma function. The factor $\left(2\pi|n-\frac{c_R}{24}|\right)^{1+c_R/2}$ starts very large for $n=0$ and falls exponentially rapidly. Meanwhile, notice that since the index on the divisor sum is negative the factor $\sigma_{-1-c_R/2}(\frac{c_R}{24}-n)$ is a slowly varying function of n, and strictly smaller than $\frac{c_R}{24}-n$. Thus, the sum is dominated by the terms $n=0$. Using Stirling's formula we find that the contribution of the $n=0$ term is

$$\sigma_{-1-c_R/2}\left(\frac{c_R}{24}\right) \times C \cdot c_R^{-1/2} \exp\left(\frac{c_R}{2}\left(1+\log\frac{\pi}{6}\right)\right). \tag{5.39}$$

in agreement with the naive evaluation. Thus we learn that the contribution of the *extreme polar states* in the Rademacher expansion gives the dominant contribution to the constant term.

Now let us turn to the numerator Φ. A similar discussion applies to the contributions of Φ to the barely polar degeneracies. If Φ is a nonsingular modular form of weight w with $\Phi(\tau) = \sum_{n\ge0}\hat\phi(n)q^n$ then a naive saddle point evaluation of the Fourier coefficients $\hat\phi(n)$ gives

$$\hat\phi(n) \sim \pm\frac{\hat\phi(0)}{\sqrt{2\pi}} w^{-w+\frac12} e^{w(1+\log(2\pi))} n^{w-1}\left(1+\mathcal{O}(e^{-4\pi^2 n/w})\right) \tag{5.40}$$

(Although this is naive, numerical checks indicate it is valid.) To estimate the biggest contribution of the Fourier coefficients of Φ to the constant term in $\eta^{-c_R}\Phi$ we apply this to $w = w_\Phi = \frac12 c_R - \frac12 b_2 - 1$ and $n = \frac{c_R}{24}$ yielding, remarkably,

$$C \cdot c_R^{-1/2} \exp\left[\frac{c_R}{2}\left(1+\log\frac{\pi}{6}\right)\right] \tag{5.41}$$

[2] To show this we first relate the relevant Kloosterman sum to the Möbius function $\mu(n)$: $\sum_{\substack{a=1\\(a,c)=1}}^{c} e(n\frac{a}{c}) = \sum_{m|(c,n)} \mu(\frac{c}{m})m$ (See page 160/161 of Ref. [166]). We substitute this identity in the left hand side of Eq. (5.37). Application of $\zeta(s)\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = 1$ leads then to the claimed identity.

having the same order of exponential growth as the barely polar terms of η^{-c_R}. Thus, in our model for polar degeneracies the barely polar degeneracies are indeed expected to grow exponentially in c_R. It is conceivable that this kind of estimate could be rigorously applied to estimate the coefficients near the cosmic censorship bound in the $(4, 0)$ elliptic genus, and it would be very interesting to do so.

In the above, we studied the leading contributions to the partition function. In a few cases, one can go beyond this and determine the full partition function by specifying the polar coefficients. Often, too many polar coefficients are present to be able to determine all of them. However, [111, 167] are able to determine them for a few cases with $b_2 = 1$ by an analysis of the bound states of D4-D0-branes and also of the chiral ring in the CFT. An example of a manifold with $b_2 = 1$ is the quintic, for several quantities were given in Sec. 3.3. It was found that $p(\mathbf{M}, \frac{55}{24}) = 7$. The polar and non-polar spectrum of this SCFT are shown in Fig. 3.1. To calculate the number of constraints, A_s, A_e and A_p can be evaluated to be $\frac{5}{8}$, $-\frac{1}{4}$ and $-\frac{3}{8}$. Therefore, no constraints are present in this case and one needs to specify seven polar coefficients. Then the full elliptic genus is determined, which can be found in [111]. If p is increased to 2, $d = 6$ and the number of polar terms becomes 36. In this case, the ingredients for the number of constraints are $A_\mathrm{s} = 1\frac{1}{4}$, $A_\mathrm{e} = 0$ and $A_\mathrm{p} = -\frac{1}{4}$. Thus, one finds one constraint and 35 coefficients must be specified to determine the full elliptic genus.

Ref. [167] determines the elliptic genus for some other cases where an M5-brane wraps a hypersurface in a Calabi-Yau with $b_2 = 1$. Interestingly, for an M5-brane wrapping the hyperplane section of the bi-cubic in $\mathbb{CP}^5$ is reported that six basis elements suffice to determine the elliptic genus, whereas the number of polar coefficients is seven. Indeed, one can show that one constraint is present for this example. In agreement with [167], no constraints are found in the other examples worked out there.

The next section studies generating functions of the Euler number of instanton moduli spaces. In this case, the contribution of the triple intersection number vanishes which results in viewer polar coefficients. Therefore, modularity is more powerful in that situation.

5.3 Application to partition functions of $\mathcal{N} = 4$ Yang-Mills

In this final section of the chapter, we study the partition functions of twisted supersymmetric $\mathcal{N} = 4$ Yang-Mills on a four-manifold M with gauge group G. The gauge field F appears in the Lagrangian as

$$\frac{1}{g^2}\mathrm{Tr}\,F\wedge *F+\frac{i\theta}{8\pi^2}\mathrm{Tr}\,F\wedge F, \tag{5.42}$$

where g is the gauge coupling constant. The θ-angle measures the instanton number of a solution. The constants g and θ naturally combine to the complex parameter $\tau=\frac{\theta}{2\pi}+\frac{4\pi i}{g^2}$. This supersymmetric field theory is invariant under a strong-weak coupling duality known as S-duality [168], except that its gauge group G is exchanged with the magnetic dual group $\hat{G}$ [169]. S-duality extends to the action of the full modular group Γ on τ, and manifests itself as modular covariance of the partition function of the twisted theory [154]. Remarkably, the transformation properties of the M5-brane elliptic genera closely resemble the transformation properties of partition functions of twisted $\mathcal{N}=4$ supersymmetric Yang-Mills theory on a four-manifold M. Before we start a closer analysis of the partition functions, we briefly review the field content and twisting of $\mathcal{N}=4$ Yang-Mills.

The field content of $\mathcal{N}=4$ Yang-Mills consists of a gauge field A_μ, six scalars ϕ_i, $i=1\dots6$ and eight fermions $\psi_\alpha^{\pm I}$, $\psi_{\dot\alpha}^{\pm I}$, $I=1,2$. The Lorentz group is $SO(4)=SU(2)_L\otimes SU(2)_R$ and the R-symmetry group is $SU(4)$. The gauge field, scalars and fermions transform under these symmetries as $(2,2,1)$, $(1,1,6)$ and $(2,1,\bar{4})\oplus(1,2,4)$. The R-symmetry group can be decomposed as $SU(4)=SU(2)\otimes SU(2)\otimes U(1)$. The four dimensional representations transform under this decomposition as $4=(2,1)^1\oplus(1,2)^{-1}$ and $\bar{4}=(2,1)^{-1}\oplus(1,2)^1$, where the superscript denotes the $U(1)$-charge. The six-dimensional representation of $SU(4)$ decomposes as $6=4\wedge4=(2,2)^0\oplus(1,1)^2\oplus(1,1)^{-2}$. The supersymmetries transform in the same way as the fermions. Table 5.1 summarizes the field content.

Tab. 5.1: Fields and the way they transform under $SO(4)\times SU(4)$, $i=1\dots6$ and $I=1,2$, the superscript indicates the $U(1)$-charge.

Field	Representation
A_μ	$(2,2,1,1)^0$
ϕ_i	$(1,1,2,2)^0\oplus(1,1,1,1)^2\oplus(1,1,1,1)^{-2}$
$\psi_\alpha^{\pm I}$	$(2,1,2,1)^{-1}\oplus(2,1,1,2)^1$
$\psi_{\dot\alpha}^{\pm I}$	$(1,2,2,1)^1\oplus(1,2,1,2)^{-1}$

Instead of $SO(4)$, a different subgroup of $SO(4)\otimes SU(4)$ can be chosen as Lorentz group. This procedure is called "twisting", and is most interesting if the new Lorentz group is chosen such it leaves some components of the supersymmetry generators invariant. Restricting physical states to lie in the cohomology of these invariant generators makes the theory essentially topological. Three different twistings are possible for $\mathcal{N} = 4$ Yang-Mills [170], of which only the Vafa-Witten twist is relevant here [154]. This twist replaces the $SU(2)_R$ of the Lorentz group with the diagonal of $SU(2)_R$ and the two $SU(2)$'s of $SU(4)$. The new $SU(2)_R$ representation is the product of these three representations. This twist commutes not only with the $U(1)$ of the original $SU(4)$, but in fact with an $SU(2) = F$ subgroup of $SU(4)$, which exchanges the original two $SU(2)$'s. For the $SU(2)_L \otimes SU(2)_R \otimes F$ representations of the gauge fields, scalars and fermions, one finds respectively $(2, 2, 1)$, $(1, 3, 1) \oplus (1, 1, 3)$ and $(2, 2, 2) \oplus (1, 3, 2) \oplus (1, 1, 2)$. Thus, indeed two components of the supersymmetry generators are invariant under the new Lorentz group.

Ref. [154] shows that if certain conditions are satisfied, the partition function of the topologically twisted theory is the generating function for the Euler numbers of instanton moduli spaces. In addition is shown that the partition functions are modular forms. For example, if the gauge group is $SU(N)/\mathbb{Z}_N$, with different 't Hooft fluxes valued in $\mathbb{Z}/\mathbb{Z}_N$, then the partition is a vector-valued modular form with weight $-\chi(M)/2$. The partition function of $SU(N)$ is given by the one of $\mathbb{Z}/\mathbb{Z}_N$ with trivial 't Hooft flux, multiplied by N^{-1+b^1} (with b^1 the first Betti number of M). Interestingly, the transformation properties (3.78) are compatible with those of the partition functions of the field theory. This suggests that BPS-states of the gauge theory on M, can be calculated by considering M5-branes wrapping $T^2 \times M$. For small T^2, we would expect that the partition function of the M5-branes is given by the gauge theory compactifications. On the other hand, in the limit where the Kähler class of the T^2 is much larger than those of M, and M is embedded as a rigid divisor in a Calabi-Yau, the conformal field theory analysis should be applicable. Later in this section, we comment in some more detail on this duality. Whether this is true or not, the resemblance shows that the $SU(N)/\mathbb{Z}_N$ partition functions can be combined into a single partition function by adding $U(1)$ degrees of freedom. This gives the partition function of the theory with gauge group $U(N)$, whose magnetic group is $U(N)$ as well.

The space of constraints is more restrictive in this situation than for the $\mathcal{N} = (4, 0)$ SCFT's arising in the context of M-theory black holes. This is caused by the reduced number of polar degeneracies now the contribution to the central charge of the triple intersection number vanishes.

As an illustration, we calculate the number of polar terms and obstructions for the $U(N)$ theory on $\mathbb{CP}^2$. Since $b^2 = 1$, the lattice Λ is one-dimensional and the theta function is holomorphic in τ. Consequently, $h_\mu(\tau)$ has weight $-\frac{3}{2}$. The second Chern class of $\mathbb{CP}^2$ is $3J^2$, with J the hyperplane class. The central charge c_R from the SCFT reduces to the combination $\chi(\mathbb{CP}^2)N = 3N$. The unitary factor $\varepsilon(T)$ is then given by $\varepsilon(T) = e(\frac{N}{8} + \frac{c_R}{24}) = e(\frac{N}{4})$. Note that the index formula cannot be used in this situation, and that therefore $\varepsilon(T) \neq e(-\frac{c_2 \cdot N}{24})$. The theta functions $\Theta_{N,\mu}$ are given by

$$\Theta_{N,\mu}(\tau, z) = \sum_{k\in\mathbb{Z}} e\left(\frac{\tau}{2N}\left(\frac{N}{2} + \mu + kN\right)^2 + \left(\frac{N}{2} + \mu + kN\right)(z + \tfrac{1}{2})\right),$$

From the transformation properties of $\Theta_{N,\mu}$ follows that $h_\mu(\tau)$ transforms as

$$\begin{aligned} S &: h_\mu\left(\frac{-1}{\tau}\right) = -\frac{1}{\sqrt{N}}(-i\tau)^{-\frac{3}{2}} e\left(\frac{N}{2}\right) \sum_{\nu \mod N} e\left(-\frac{\mu\nu}{N}\right) h_\nu(\tau), \\ T &: h_\mu(\tau+1) = e\left(-\frac{N}{4} + \frac{1}{2N}\left(\mu + \frac{N}{2}\right)^2\right) h_\mu(\tau). \end{aligned} \tag{5.43}$$

The functions satisfy moreover $\Theta_{N,\mu}(\tau, -z) = (-)^N \Theta_{N,-\mu}(\tau, z)$ and $h_\mu(\tau) = h_{-\mu}(\tau)$, such that $h_\mu(\tau)$ can be reduced to a vector of length $\frac{N}{2} + 1$ if N is even and $\frac{N+1}{2}$ if N is odd. The elements $h_\mu(\tau)$ are forms of $\Gamma(2N)$ for N even and otherwise $\Gamma(8N)$. The number of polar terms $p(N)$ is given by

$$p(N) = \sum_\mu \left\lceil \frac{N}{8} - \left(\frac{\mu^2}{2N} + \frac{\mu}{2} - \left\lfloor \frac{\mu^2}{2N} + \frac{\mu}{2} \right\rfloor\right)\right\rceil. \tag{5.44}$$

One can straightforwardly determine the properties of the obstruction forms $g_\mu(\tau)$. They have weight $3\frac{1}{2}$, we denote their representation again by $\mathbf{M}$.

For large N, $p(N)$ grows as $\frac{1}{16}N^2$ and the number of constraints as $\frac{5}{48}N$. Evaluation of Eq. (4.70) gives a closed expression for the dimension of the space of forms which satisfy the required properties for $h_\mu(\tau)$. A tedious part is the calculation of A_e. One needs to determine the $d \times d$ matrices $\mathbf{M}(S)$ and $\mathbf{M}(ST)$ from (5.43), and then their traces. If N is even, $N=2m$, $\chi_{\mathbf{M}}(S)$ can be evaluated to

$$\begin{aligned}
\chi_{\mathbf{M}}(S) &= \frac{1}{\sqrt{-2mi}}\left(1-e\left(\frac{m}{2}\right)+\sum_{\mu=1}^{m} e\left(\frac{\mu^2}{2m}\right)+e\left(-\frac{\mu^2}{2m}\right)\right) \\
&= \begin{cases} e\left(\frac{1}{8}\right) & m=0 \mod 2, \\ 0 & m=1 \mod 2, \end{cases}
\end{aligned} \tag{5.45}$$

where the quadratic sums in the first line are evaluated using the Gauss sums $\sum_{\mu=1}^{m} e\left(\frac{\mu^2}{m}\right)$ [166]. Similarly, one finds

$$\chi_{\mathbf{M}}(ST) = \begin{cases} e\left(\frac{1}{12}\right) & m=0 \mod 3, \\ e\left(\frac{1}{4}\right) & m=1 \mod 3, \\ 0 & m=2 \mod 3, \end{cases} \tag{5.46}$$

Evaluating (5.43) gives then

$$\begin{aligned}
\dim\ & A_{-\frac{3}{2},\mathbf{M}}\left(\Gamma(4m)^*, \frac{m}{4}\right) = \\
& \frac{6m^2+13m+7}{24} - \sum_{\mu=0}^{m}\left(\frac{\mu}{2}+\frac{\mu^2}{4m}-\left\lfloor \frac{\mu}{2}+\frac{\mu^2}{4m}\right\rfloor\right) \\
& -\frac{1}{4}\begin{cases} 1 & m=0 \mod 2, \\ 0 & m=1 \mod 2, \end{cases} -\frac{1}{3}\begin{cases} 0 & m=0 \mod 3, \\ 1 & m=1 \mod 3, \\ 0 & m=2 \mod 3. \end{cases}
\end{aligned} \tag{5.47}$$

A similar formula can be derived for the case when N is odd. Table 5.2 presents the number of polar coefficients $p(N)$ and constraints on the polar spectrum for $N=1\dots 10$. The table confirms earlier results for $N=1$ and $N=2$, which are derived using the Weil conjectures [162, 171]. For $N=1$, the space of potential partition functions is one-dimensional, therefore, $h_0(\tau)$ must be proportional to $\eta(\tau)^{-3}$. This agrees with the computation in Ref. [162].

About $N=2$ is a lot more to say. The dimension of the modular vector is now 2, and the table teaches us that it contains one polar term. The multiplier system $\mathbf{M}(\gamma)$ is generated from

$$\mathbf{M}(T) = \begin{pmatrix} e(-1/4) & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{M}(S) = e(-1/8)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{5.48}$$

Interestingly, an obstruction cusp form does exist, and is given by

$$\eta(\tau)^6 \begin{pmatrix} \theta_3(2\tau) \\ \theta_2(2\tau) \end{pmatrix}. \tag{5.49}$$

Tab. 5.2: For $U(N)$ gauge theory on $\mathbb{CP}^2$, the number of polar coefficients $p(N)$ and constraints on the polar spectrum $\dim S_{3\frac{1}{2},\mathbf{M}}$ are listed for $N = 1 \dots 10$.

N	1	2	3	4	5	6	7	8	9	10
$p(N)$	1	1	1	1	3	4	4	5	7	8
$\dim S_{3\frac{1}{2},\mathbf{M}}$	0	1	0	0	1	1	0	0	1	2

Since this form does exist, a modular meromorphic partition function for $N = 2$ does not exist. This is consistent with the known generating function for the Betti numbers of the compactified moduli space calculated in [171]. This generating function can be specialized to a give the Euler numbers of the moduli space. One finds for this generating function

$$\begin{aligned}\tilde{h}_1(\tau) &= \sum_{n=1}^{\infty} c_\mu(n) q^{n-\frac{1}{2}} \\ &= \frac{\sum_{n,m\geq 0}(4n-2m+1)q^{m(m+2(n+1))+n+\frac{3}{4}}}{\theta_3(2\tau)\eta(\tau)^6},\end{aligned} \tag{5.50}$$

where $c_\mu(n) = \chi(\mathcal{M}(c_1, c_2))$ under the identification $\mu = c_1$ and $n = 4c_2 - c_1^2$ and $\mathcal{M}(c_1, c_2)$ is moduli space of rank two bundle over $\mathbb{CP}^2$ with Chern class c_1 and c_2. Twisting of a vector bundle by a line bundle $E \otimes \mathcal{O}(k)$ gives an isomorphism between $\mathcal{M}(c_1, c_2)$ and $\mathcal{M}(c_1 + 2k, c_2 + kc_1 + k^2)$. This is the gauge theory analogue of the spectral flow in the $\mathcal{N} = (4, 0)$ SCFT. The moduli spaces of bundles with equal discriminant $-n = c_1^2 - 4c_2$ and $c_1 \mod 2$ are equal. Stable bundles have positive n. The numbers in (5.50) are thus calculated for the case with non-trivial 't Hooft flux, or more mathematically said, non-trivial second Stiefel-Whitney class. The Fourier coefficients of (5.50) capture interesting data, since $\frac{1}{3}\tilde{h}_1(\tau)\prod_{\ell=1}^{\infty}(1-q^\ell)^6$ is the generating function of the class numbers

$$\sum_{n=1}^{\infty} H(4n-1)q^n. \tag{5.51}$$

A class number $H(n)$ is the number of equivalence classes of quadratic forms

$$\alpha u^2 + \beta uv + \gamma v^2 = 0, \qquad \beta^2 - 4\alpha\gamma = -n \leq 0, \qquad (\alpha, \beta, \gamma) \in \mathbb{Z}. \tag{5.52}$$

Those (α, β, γ) which are related by the transformation $\binom{u}{v} \to \gamma\binom{u}{v}, \gamma \in \Gamma$ are equivalent. $H(0) = -\frac{1}{12}$ by definition. Quadratic forms which are multiples of u^2+v^2 are counted by $\frac{1}{2}$, multiples of u^2+uv+v^2 by $\frac{1}{3}$. The numbers $H(n)$ are only non-zero for $n=0,1 \mod 4$. Class numbers appeared earlier as the Euler numbers of the uncompactified moduli space of rank two bundles over $\mathbb{CP}^2$ [172].

The function $\tilde{h}_1(\tau)$ is not a modular form, which we already deduced from Table 5.2. However, it is close to a modular form. One can show that it transforms together with $\tilde{h}_0(\tau)$ as

$$\begin{aligned}\tilde{h}_\mu\Big|_{-\frac{3}{2}}\gamma &= \mathbf{M}(\gamma)_\mu^\nu \\ &\times\left[\tilde{h}_\nu(\tau)+\frac{3e(-\frac{1}{8})}{2\sqrt{2\pi}\,\eta(\tau)^6}p\left(\tau,\gamma^{-1}(-i\infty),\overline{\theta_{3-\nu}(2\,\cdot)}\right)\right].\end{aligned} \tag{5.53}$$

This non-modular behavior can be understood by the observation that the class numbers are Fourier coefficients of Eisenstein series with weight $\frac{3}{2}$ [152, 153]. Such Eisenstein series are divergent and need to be regularized similarly to the discussion on divergent Poincaré series. The regularization spoils the modular invariance. As in Sec. 5.1, this can be cured by a non-holomorphic addition to $\tilde{h}(\tau)$ [152, 153]. The holomorphic anomaly reads in this case [154]

$$\frac{\partial}{\partial\bar{\tau}}h_\mu(\tau,\bar{\tau})=\frac{3}{16\pi i\tau_2^{3/2}}\frac{1}{\eta(\tau)^6}\overline{\theta_{3-\mu}(2\tau)}, \tag{5.54}$$

where $\theta_{3-\mu}(\tau)=\sum_{n\in\mathbb{Z}+\frac{\mu}{2}}q^{n^2}$ for $\mu=0,1$; these are standard Jacobi theta functions. If we include the $U(1)$-degrees of freedom (or c_1 dependence), we obtain for the partition function of the $U(2)$-theory

$$\mathcal{Z}(\tau,\bar{\tau},z)=h_0(\tau,\bar{\tau})\overline{\theta_2(2\tau,2z)}-h_1(\tau,\bar{\tau})\overline{\theta_3(2\tau,2z)}. \tag{5.55}$$

Note that μ is 0 when the second Stiefel-Whitney class w_2 of the $SO(3)$ bundle is trivial, and equal to 1 when w_2 is non-trivial.

Let us contrast these formulas with what would be expected from the viewpoint of a sum over AdS$_3$-geometries. To this end, consider two M5-branes on a rigid divisor equal to $\mathbb{CP}^2$ in a suitable Calabi-Yau (*e.g.* the Calabi-Yau elliptic fibration over $\mathbb{CP}^2$). We might expect to be able to construct the partition function – in the AdS$_3$ regime – from a Poincaré series based on its principal part. A priori, this partition function does not need to equal $\mathcal{Z}(\tau,\bar{\tau},\bar{z})$ since we might not be able to rely on modular invariance and/or holomorphy. Therefore, we distinguish the AdS$_3$ partition function and denote it by $\mathcal{Z}^{\mathrm{A}}(\tau,\bar{\tau},\bar{z})$. The comparison reduces now to a comparison of the holomorphic part of $h_\mu(\tau,\bar{\tau})$, $\tilde{h}_\mu(\tau)$, with the vector-valued modular form constructed by the Poincaré series. We label the constructed vector-valued modular form by "A": $\tilde{h}^{\rm A}_\mu(\tau)$. The principal part of $\tilde{h}^{\rm A}_\mu(\tau)$ is equal to the principal part of $\tilde{h}_\mu(\tau)$, if we assume that the polar part is not renormalized as we continue to the AdS_3 regime. $\tilde{h}_0(\tau)$ has a polar term equal to $-\frac{1}{4}q^{-\frac{1}{4}}$ while $\tilde{h}_1(\tau)$ does not contain a polar term. Therefore, we attempt to construct with the Poincaré series a modular form of weight $-3/2$, with multiplier system as in [154] and the given polar term. This construction is bound to fail, since the space of obstruction cusp forms is one-dimensional. The constructed function transform as

$$\begin{aligned}\tilde{h}^{\rm A}(\gamma(\tau))|_{-\frac{3}{2}}\gamma &= \mathbf{M}(\gamma)_\mu^\nu \\ &\times\left[\tilde{h}^{\rm A}_\nu(\tau)+\frac{1}{4}p\left(\tau,\gamma^{-1}(-i\infty),\overline{\eta^6\,\theta_{3-\nu}(2\,\cdot)}\right)\right].\end{aligned} \tag{5.56}$$

The factor $\frac{1}{4}$ in front of the period function is a consequence of the coefficient of the polar term.

A simple check whether the Poincaré series can reproduce the gauge theory partition function is a comparison of the anomalies under modular transformations. Even without a detailed analysis, we can observe qualitative differences between the shifts. An important difference is the behavior for $\mathrm{Im}(\tau)\to\infty$. In this limit the shift in Eq. (5.53) grows exponentially whereas the period function in Eq. (5.56) vanishes. This shows clearly that the holomorphic Poincaré series does not equal the generating function of the Euler numbers of instanton moduli spaces. As a consequence of the different modular anomalies, the associated holomorphic anomalies are different. The holomorphic anomaly given by Eq. (5.54) is not annihilated by the Laplacian Δ. Another difference is that for $\mathrm{Im}(\tau)\to\infty$, the right hand side of Eq. (5.54) grows exponentially (for $\mu=0$). This raises the question to what extent the $\mathcal{N}=(4,0)$ SCFT can calculate BPS quantities of the twisted theory. The results of this section are clearly inconclusive.

For the cases $N=3$ and $N=4$, Table 5.2 shows that the partition function exists and is unique up to an overall factor. One can show that for $N=3$, the $h_\mu(\tau)$ are given by

$$h_\mu(\tau)=\frac{1}{2}\frac{\theta_2^5\,\Theta_{3,\mu}\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}(\tau)+\theta_3^5\,\Theta_{3,\mu}\begin{bmatrix}0\\0\end{bmatrix}(\tau)+\theta_4^5\,\Theta_{3,\mu}\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}(\tau)}{\eta(\tau)^9}, \tag{5.57}$$

where $\Theta_{N,\mu} \begin{bmatrix} a \\ b \end{bmatrix} (\tau)$ is defined by

$$\Theta_{N,\mu} \begin{bmatrix} a \\ b \end{bmatrix} (\tau) = \sum_{k \in \mathbb{Z}} e \left(\frac{\tau}{2N} \left(aN + \mu + kN\right)^2 + \left(aN + \mu + kN\right) b \right). \quad (5.58)$$

The overall factor of $\frac{1}{2}$ in (5.57) is such that the first coefficient of the expansion is one. For $N = 4$, $h_\mu(\tau)$ can be written in the form $f_\mu(\tau)/\eta(\tau)^{12}$, where $f_\mu(\tau)$ is a holomorphic vector-valued modular form. It would be interesting to find out, whether the Fourier coefficients for $N = 3$ and $N = 4$ capture any special information, as is the case for $N = 2$ where they are class numbers. Naturally, an application of the described technique to $\mathcal{N} = 4$ Yang-Mills theory on other manifolds is likely to suggest more partition functions, which are currently unknown.

6. DISCUSSION AND CONCLUSION

We have come to the final chapter of the thesis. The first part of the chapter recapitulates shortly the discussions of previous chapter, after which some suggestions for further research are given. In Chapter 2, the concept of black hole entropy was explained. In particular, the special class of black holes was discussed, which played a central role in this thesis. These are the charged supersymmetric black holes of four-dimensional $\mathcal{N} = 2$ supergravity; with one constraint on the magnetic charge, namely $p^0 = 0$. This condition needs to be imposed to account for its entropy by an M-theory set-up as explained in [3]. An important ingredient for this account is the counting of the degrees of freedom by a two-dimensional superconformal field theory. This thesis concentrated on a special CFT partition function, the elliptic genus, to learn about black hole entropy. Three main motivations were given for this study:

- To explain microscopically the leading black hole entropy.
- To analyze to what extend the elliptic genus gives evidence for the conjecture [5], which relates the black hole partition function $\mathcal{Z}_{\mathrm{BH}}$ with the square of the topological string partition function $|\mathcal{Z}_{\mathrm{top}}|^2$. This conjecture arose by the analysis of quantum corrections to the supergravity entropy. It showed that the attractor mechanism suggests nicely that the black hole partition function considers the electric charges in a canonical ensemble and the magnetic charges in a microcanonical ensemble.
- To find evidence for the $\mathrm{AdS}_3/\mathrm{CFT}_2$ conjecture in the CFT partition function. Such evidence would be the discovery of a gravity path integral in the CFT partition function. In AdS_3, such a path integral reduces to a discrete sum over geometries.

The subsequent chapters addressed these motivations. The answer to the first motivation is due to [3] and reviewed in Chapter 3. The black holes are described as a bound state of multiple M5-branes which wrap a divisor P in a Calabi-Yau X times a circle S^1_{M}. The low energy degrees of freedom of the M5-brane can be reduced to two-dimensions, where they combine to a $\mathcal{N} = (4,0)$ SCFT. A generalized elliptic genus can be defined for this SCFT which enumerates the half-BPS states of the theory. The symmetries of the theory determine that this function is similar to a Jacobi form. Therefore, the elliptic genus can be decomposed into a vector-valued modular form and theta functions. Using the modular properties, the Cardy formula for the entropy can be derived, which agrees with the supergravity entropy.

The third motivation could be addressed after the analysis of vector-valued modular forms in Chapter 4. Inspired by Ref. [89], Chapter 5 explains that the elliptic genus can be written as a (regularized) Poincaré series, which can be interpreted as a sum over geometries. The Poincaré series are a sum over the coset $\Gamma_\infty \backslash \Gamma$; and every term in the sum represents a semi-classical saddle point. This thesis improves the technique of Poincaré series proposed in [89], such that the erroneous "Farey Tail transform" has become obsolete. The main new ingredient is the regularization. Generically, this leads to an anomaly, but that can often be canceled by a proper choice of the polar degeneracies.

The sum over classical saddle points has also proven useful to elucidate the connection between $\mathcal{Z}_{\mathrm{BH}}$ and $|\mathcal{Z}_{\mathrm{top}}|^2$, *i.e.* the second motivation. In a given geometry, $|\mathcal{Z}_{\mathrm{top}}|^2$ arises as the partition function of M2- and anti-M2-branes. It can be determined using a Schwinger calculation as in [158, 159]. In the regime of strong topological string coupling constant λ, only a single saddle point contributes, and the elliptic genus confirms the OSV conjecture. The opposite regime of small λ, is more problematic for the OSV conjecture, since then the entropy seems dominated by multi-center black hole solutions. The microscopic entropy of these solutions is also incorporated in the $\mathcal{N} = (4,0)$ SCFT, but $\mathcal{Z}_{\mathrm{BH}} \sim |\mathcal{Z}_{\mathrm{top}}|^2$ is no longer accurate.

Suggestions for future work

The results of this thesis suggest further research in various directions. The rest of the chapter comments on AdS_3-gravity, partition functions of D4-D2-D0-branes and partition functions of $\mathcal{N} = 4$ gauge theory.

The Poincaré series are a suggestive connection between the (super-symmetric) CFT partition function and gravity. Many issues are however not fully understood. For example, why only these geometries appear in the sum and the relation with Lorentzian space-times [173]. Also, the appearance of a double sum over $\Gamma_\infty \backslash \Gamma$ for bosonic AdS_3-gravities. This might suggest a fundamental role for "chiral" gravity which is under current investigation, see e.g. [88, 174, 175], but certainly still needs a better understanding.

Another aspect which deserves certainly more study is the relation between $\mathcal{Z}_{\rm CFT}$ and $\mathcal{Z}_{\rm BH}$. This thesis required the restriction that the black hole solution contains only a single $\rm AdS_3$ throat. Solutions with multiple $\rm AdS_3$ throats [66] do exist however. The existence of these solutions does depend on the moduli at infinity. This translates into discontinuous changes of the degeneracies as a function of the moduli [37, 36]. It would be interesting to understand the precise relation between $\mathcal{Z}_{\rm CFT}(t_{\rm attractor})$ and the general $\mathcal{Z}_{\rm BH}(t_\infty)$. Does the wall-crossing correspond to jumping of $h_\mu(\tau)$ within the relevant space of vector-valued modular forms? Can the CFT capture all wall-crossing phenomena? A clue to these puzzles might be provided by a derivation of $\mathcal{Z}_{\rm CFT}$ from more fundamental principles, as can be done for BPS states of $\mathcal{N}=4$ and $\mathcal{N}=8$ supergravity [44, 176]. Also, the interpretation of attractor flow as renormalization group flow [177] might provide insights into these problems.

As a last suggestion, I would like to advocate the described techniques to determine the space of vector-valued modular forms. Such modular forms appear at many places in physics, for example rational conformal field theories and twisted $\mathcal{N}=4$ Yang-Mills theories. The techniques proof very useful, if one wants to determine the full partition function on the basis of only a small number of coefficients. An application of this to Yang-Mills theory might lead to interesting results for the Euler numbers of instanton moduli spaces of higher rank gauge groups.

BIBLIOGRAPHY

[1] J. D. Bekenstein, "Black Holes and Entropy," *Phys. Rev. D* **7** (1973) 2333–2346.

[2] B. de Wit, P. G. Lauwers, and A. van Proeyen, "Lagrangians of N=2 Supergravity - Matter Systems," *Nucl. Phys.* **B255** (1985) 569.

[3] J. M. Maldacena, A. Strominger, and E. Witten, "Black hole entropy in M-theory," *JHEP* **12** (1997) 002, `hep-th/9711053`.

[4] A. Strominger and C. Vafa, "Microscopic Origin of the Bekenstein-Hawking Entropy," *Phys. Lett.* **B379** (1996) 99–104, `hep-th/9601029`.

[5] H. Ooguri, A. Strominger, and C. Vafa, "Black hole attractors and the topological string," *Phys. Rev.* **D70** (2004) 106007, `hep-th/0405146`.

[6] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," *Phys. Lett.* **B428** (1998) 105–114, `hep-th/9802109`.

[7] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* **2** (1998) 253–291, `hep-th/9802150`.

[8] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys. Rept.* **323** (2000) 183–386, `hep-th/9905111`.

[9] M. Green, J. Schwarz, and E. Witten, *Superstring theory, Volume I.* Cambridge University Press, 1987.

[10] M. Green, J. Schwarz, and E. Witten, *Superstring theory, Volume II.* Cambridge University Press, 1987.

[11] J. Polchinski, *String Theory, Volume I.* Cambridge University Press, Cambridge, 1998.

[12] J. Polchinski, *String Theory, Volume II.* Cambridge University Press, Cambridge, 1998.

[13] R. M. Wald, *General Relativity.* The University of Chicago Press, Chicago and London, 1984.

[14] J. M. Bardeen, B. Carter, and S. W. Hawking, "The four laws of black hole mechanics," *Commun. Math. Phys.* **31** (1973) 161–170.

[15] S. W. Hawking, "Particle Creation by Black Holes," *Commun. Math. Phys.* **43** (1975) 199–220.

[16] M. K. Prasad and C. M. Sommerfield, "An Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon," *Phys. Rev. Lett.* **35** (1975) 760–762.

[17] E. B. Bogomolny, "Stability of Classical Solutions," *Sov. J. Nucl. Phys.* **24** (1976) 449.

[18] S. R. Coleman, S. J. Parke, A. Neveu, and C. M. Sommerfield, "Can One Dent a Dyon?," *Phys. Rev.* **D15** (1977) 544.

[19] R. M. Wald, "Black hole entropy is the Noether charge," *Phys. Rev.* **D48** (1993) 3427–3431, `gr-qc/9307038`.

[20] J. Wess and J. Bagger, *Supersymmetry and Supergravity.* Princeton University Press, 1992.

[21] S. Weinberg, *The Quantum Theory of Fields*, vol. III, Supersymmetry. Cambridge University Press, 2000.

[22] J.-L. Gervais and B. Sakita, "Field theory interpretation of supergauges in dual models," *Nucl. Phys.* **B34** (1971) 632–639.

[23] E. Witten and D. I. Olive, "Supersymmetry Algebras That Include Topological Charges," *Phys. Lett.* **B78** (1978) 97.

[24] T. Mohaupt, "Black hole entropy, special geometry and strings," *Fortsch. Phys.* **49** (2001) 3–161, `hep-th/0007195`.

[25] P. A. M. Dirac, "Quantised singularities in the electromagnetic field," *Proc. Roy. Soc. Lond.* **A133** (1931) 60–72.

[26] J. S. Schwinger, "Magnetic charge and quantum field theory," *Phys. Rev.* **144** (1966) 1087–1093.

[27] D. Zwanziger, "Quantum field theory of particles with both electric and magnetic charges," *Phys. Rev.* **176** (1968) 1489–1495.

[28] J. S. Schwinger, "Sources and magnetic charge," *Phys. Rev.* **173** (1968) 1536–1544.

[29] E. Witten, "On flux quantization in M-theory and the effective action," *J. Geom. Phys.* **22** (1997) 1–13, `arXiv:hep-th/9609122`.

[30] E. Witten, "Five-brane effective action in M-theory," *J. Geom. Phys.* **22** (1997) 103–133, `arXiv:hep-th/9610234`.

[31] D. S. Freed and E. Witten, "Anomalies in string theory with D-branes," `arXiv:hep-th/9907189`.

[32] P. Candelas and X. de la Ossa, "Moduli space of Calabi-Yau manifolds," *Nucl. Phys.* **B355** (1991) 455–481.

[33] B. de Wit, "N = 2 electric-magnetic duality in a chiral background," *Nucl. Phys. Proc. Suppl.* **49** (1996) 191–200, `hep-th/9602060`.

[34] G. Lopes Cardoso, B. de Wit, and T. Mohaupt, "Corrections to macroscopic supersymmetric black-hole entropy," *Phys. Lett.* **B451** (1999) 309–316, `hep-th/9812082`.

[35] A. Ceresole, R. D'Auria, S. Ferrara, and A. van Proeyen, "Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity," *Nucl. Phys.* **B444** (1995) 92–124, `hep-th/9502072`.

[36] F. Denef, "Supergravity flows and D-brane stability," *JHEP* **08** (2000) , `hep-th/0005049`.

[37] F. Denef and G. W. Moore, "Split states, entropy enigmas, holes and halos," `hep-th/0702146`.

[38] S. Ferrara, R. Kallosh, and A. Strominger, "N=2 extremal black holes," *Phys. Rev.* **D52** (1995) 5412–5416, `hep-th/9508072`.

[39] S. Ferrara and R. Kallosh, "Supersymmetry and Attractors," *Phys. Rev.* **D54** (1996) 1514–1524, `hep-th/9602136`.

[40] M. Shmakova, "Calabi-Yau black holes," *Phys. Rev.* **D56** (1997) 540–544, `hep-th/9612076`.

[41] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," *Nucl. Phys.* **B426** (1994) 19–52, hep-th/9407087.

[42] K. Behrndt *et al.*, "Classical and quantum N = 2 supersymmetric black holes," *Nucl. Phys.* **B488** (1997) 236–260, hep-th/9610105.

[43] I. Brunner, M. R. Douglas, A. E. Lawrence, and C. Romelsberger, "D-branes on the quintic," *JHEP* **08** (2000) 015, hep-th/9906200.

[44] R. Dijkgraaf, E. P. Verlinde, and H. L. Verlinde, "Counting dyons in N = 4 string theory," *Nucl. Phys.* **B484** (1997) 543–561, hep-th/9607026.

[45] G. Lopes Cardoso, B. de Wit, and T. Mohaupt, "Deviations from the area law for supersymmetric black holes," *Fortsch. Phys.* **48** (2000) 49–64, hep-th/9904005.

[46] G. Lopes Cardoso, B. de Wit, and T. Mohaupt, "Area law corrections from state counting and supergravity," *Class. Quant. Grav.* **17** (2000) 1007–1015, hep-th/9910179.

[47] K. Behrndt *et al.*, "Higher-order black-hole solutions in N = 2 supergravity and Calabi-Yau string backgrounds," *Phys. Lett.* **B429** (1998) 289–296, hep-th/9801081.

[48] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," *Commun. Math. Phys.* **165** (1994) 311–428, hep-th/9309140.

[49] I. Antoniadis, E. Gava, K. S. Narain, and T. R. Taylor, "Topological amplitudes in string theory," *Nucl. Phys.* **B413** (1994) 162–184, hep-th/9307158.

[50] E. Witten, "On the structure of the topological phase of two-dimensional gravity," *Nucl. Phys.* **B340** (1990) 281–332.

[51] K. Hori *et al.*, *Mirror Symmetry*. Clay Mathematics Monograph, 2003.

[52] A. Neitzke and C. Vafa, "Topological strings and their physical applications," hep-th/0410178.

[53] M. Vonk, "A mini-course on topological strings," hep-th/0504147.

[54] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Holomorphic anomalies in topological field theories," *Nucl. Phys.* **B405** (1993) 279–304, hep-th/9302103.

[55] A. Dabholkar, F. Denef, G. W. Moore, and B. Pioline, "Exact and asymptotic degeneracies of small black holes," *JHEP* **08** (2005) 021, arXiv:hep-th/0502157.

[56] M. Gunaydin, A. Neitzke, B. Pioline, and A. Waldron, "BPS black holes, quantum attractor flows and automorphic forms," *Phys. Rev.* **D73** (2006) 084019, arXiv:hep-th/0512296.

[57] D. Gaiotto, A. Strominger, and X. Yin, "From AdS(3)/CFT(2) to black holes / topological strings," hep-th/0602046.

[58] C. Beasley *et al.*, "Why $Z_{BH} = |Z_{top}|^2$," arXiv:hep-th/0608021.

[59] J. de Boer, M. C. N. Cheng, R. Dijkgraaf, J. Manschot, and E. P. Verlinde, "A farey tail for attractor black holes," *JHEP* **11** (2006) 024, hep-th/0608059.

[60] P. K. Townsend, "The eleven-dimensional supermembrane revisited," *Phys. Lett.* **B350** (1995) 184–187, arXiv:hep-th/9501068.

[61] P. K. Townsend, "D-branes from M-branes," *Phys. Lett.* **B373** (1996) 68–75, arXiv:hep-th/9512062.

[62] E. Witten, "String theory dynamics in various dimensions," *Nucl. Phys.* **B443** (1995) 85–126, arXiv:hep-th/9503124.

[63] P. Horava and E. Witten, "Heterotic and type I string dynamics from eleven dimensions," *Nucl. Phys.* **B460** (1996) 506–524, arXiv:hep-th/9510209.

[64] S. D. Mathur, "The fuzzball proposal for black holes: An elementary review," *Fortsch. Phys.* **53** (2005) 793–827, arXiv:hep-th/0502050.

[65] K. Skenderis and M. Taylor, "The fuzzball proposal for black holes," arXiv:0804.0552 [hep-th].

[66] J. de Boer, F. Denef, S. El-Showk, I. Messamah, and D. van den Bleeken, "Black hole bound states in $AdS_3 \times S^2$," arXiv:0802.2257 [hep-th].

[67] F. Denef, D. Gaiotto, A. Strominger, D. van den Bleeken, and X. Yin, "Black hole deconstruction," arXiv:hep-th/0703252.

[68] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, hep-th/9711200.

[69] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three Dimensional Gravity," *Comm. Math. Phys.* **104** (1986) 207–226.

[70] E. Witten, "(2+1)-Dimensional Gravity as an Exactly Soluble System," *Nucl. Phys.* **B311** (1988) 46.

[71] E. Witten, "Quantum field theory and the Jones polynomial," *Commun. Math. Phys.* **121** (1989) 351.

[72] A. C. Cadavid, A. Ceresole, R. D'Auria, and S. Ferrara, "Eleven-dimensional supergravity compactified on Calabi-Yau threefolds," *Phys. Lett.* **B357** (1995) 76–80, arXiv:hep-th/9506144.

[73] G. Papadopoulos and P. K. Townsend, "Compactification of D = 11 supergravity on spaces of exceptional holonomy," *Phys. Lett.* **B357** (1995) 300–306, arXiv:hep-th/9506150.

[74] S. Ferrara, R. R. Khuri, and R. Minasian, "M-Theory on a Calabi-Yau Manifold," *Phys. Lett.* **B375** (1996) 81–88, arXiv:hep-th/9602102.

[75] A. H. Chamseddine and W. A. Sabra, "Magnetic strings in five dimensional gauged supergravity theories," *Phys. Lett.* **B477** (2000) 329–334, hep-th/9911195.

[76] G. W. Gibbons and S. W. Hawking, "Action Integrals and Partition Functions in Quantum Gravity," *Phys. Rev.* **D15** (1977) 2752–2756.

[77] M. Henningson and K. Skenderis, "The holographic Weyl anomaly," *JHEP* **07** (1998) 023, hep-th/9806087.

[78] K. Skenderis and S. N. Solodukhin, "Quantum effective action from the AdS/CFT correspondence," *Phys. Lett.* **B472** (2000) 316–322, hep-th/9910023.

[79] K. Skenderis, "Lecture notes on holographic renormalization," *Class. Quant. Grav.* **19** (2002) 5849–5876, hep-th/0209067.

[80] C. Fefferman and C. R. Graham, "Conformal Invariants," in *Elie Cartan et les Mathématiques d'aujourd'hui.* Astérisque, 1985.

[81] M. Banados, C. Teitelboim, and J. Zanelli, "The black hole in three-dimensional space-time," *Phys. Rev. Lett.* **69** (1992) 1849–1851, hep-th/9204099.

[82] J. D. Brown, E. A. Martinez, and J. York, James W., "Complex Kerr-Newman geometry and black hole thermodynamics," *Phys. Rev. Lett.* **66** (1991) 2281–2284.

[83] S. Carlip and C. Teitelboim, "Aspects of black hole quantum mechanics and thermodynamics in (2+1)-dimensions," *Phys. Rev.* **D51** (1995) 622–631, gr-qc/9405070.

[84] A. Achucarro and P. K. Townsend, "A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories," *Phys. Lett.* **B180** (1986) 89.

[85] S. Deser, R. Jackiw, and S. Templeton, "Topologically massive gauge theories," *Ann. Phys.* **140** (1982) 372–411.

[86] P. Kraus and F. Larsen, "Holographic gravitational anomalies," *JHEP* **01** (2006) 022, arXiv:hep-th/0508218.

[87] S. N. Solodukhin, "Holography with gravitational Chern-Simons," *Phys. Rev.* **D74** (2006) 024015, arXiv:hep-th/0509148.

[88] W. Li, W. Song, and A. Strominger, "Chiral Gravity in Three Dimensions," arXiv:0801.4566 [hep-th].

[89] R. Dijkgraaf, J. M. Maldacena, G. W. Moore, and E. P. Verlinde, "A black hole farey tail," hep-th/0005003.

[90] J. M. Maldacena and A. Strominger, "AdS(3) black holes and a stringy exclusion principle," *JHEP* **12** (1998) 005, hep-th/9804085.

[91] J. Manschot, "AdS_3 Partition Functions Reconstructed," *JHEP* **10** (2007) 103, `arXiv:0707.1159 [hep-th]`.

[92] E. Witten, "Three-Dimensional Gravity Revisited," `arXiv:0706.3359 [hep-th]`.

[93] B. Frenkel, I, J. Lepowsky, and A. Meurman, "A Natural Representation of the Fischer-Griess Monster With the Modular Function J As Character," *Proc. Natl. Acad. Sci. USA* **81** (1984) 3256.

[94] S. W. Hawking and D. N. Page, "Thermodynamics of Black Holes in anti-De Sitter Space," *Commun. Math. Phys.* **87** (1983) 577.

[95] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," *Adv. Theor. Math. Phys.* **2** (1998) 505–532, `arXiv:hep-th/9803131`.

[96] J. Callan, Curtis G., J. A. Harvey, and A. Strominger, "Worldbrane actions for string solitons," *Nucl. Phys.* **B367** (1991) 60–82.

[97] D. M. Kaplan and J. Michelson, "Zero Modes for the D=11 Membrane and Five-Brane," *Phys. Rev.* **D53** (1996) 3474–3476, `arXiv:hep-th/9510053`.

[98] A. Strominger, "Open p-branes," *Phys. Lett.* **B383** (1996) 44–47, `arXiv:hep-th/9512059`.

[99] K. Becker and M. Becker, "Boundaries in M-Theory," *Nucl. Phys.* **B472** (1996) 221–230, `arXiv:hep-th/9602071`.

[100] M. Nakahara, *Geometry, topology and physics.* Institute of physics publishing, London, 2003.

[101] P. Griffiths and J. Harris, *Principles of Algebraic Geometry.* John Wiley and Sons, 1978.

[102] R. Minasian and G. W. Moore, "K-theory and Ramond-Ramond charge," *JHEP* **11** (1997) 002, `arXiv:hep-th/9710230`.

[103] D. S. Freed and K. K. Uhlenbeck, *Instantons and Four-Manifolds.* Springer-Verlag, 1984.

[104] R. Minasian, G. W. Moore, and D. Tsimpis, "Calabi-Yau black holes and (0,4) sigma models," *Commun. Math. Phys.* **209** (2000) 325–352, `arXiv:hep-th/9904217`.

[105] J. Conway and N. Sloane, *Sphere packings, lattices and groups.* Springer, 1999.

[106] V. V. Nikulin, "Integral Symmetric Bilinear Forms and Some of Their Applications," *Math. USSR Izvestija* **14** (1980) 103–167.

[107] R. Hartshorne, *Algebraic Geometry.* Springer-Verlag, 1977.

[108] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory.* Springer, 1997.

[109] A. Sevrin, W. Troost, and A. van Proeyen, "Superconformal algebras in two dimensions with N=4," *Phys. Lett.* **B208** (1988) 447.

[110] J. M. Maldacena, G. W. Moore, and A. Strominger, "Counting BPS black holes in toroidal type II string theory," `hep-th/9903163`.

[111] D. Gaiotto, A. Strominger, and X. Yin, "The M5-brane elliptic genus: Modularity and BPS states," `hep-th/0607010`.

[112] P. Kraus and F. Larsen, "Partition functions and elliptic genera from supergravity," *JHEP* **0701** (2007) 002, `hep-th/0607138`.

[113] J. L. Cardy, "Operator Content of Two-Dimensional Conformally Invariant Theories," *Nucl. Phys.* **B270** (1986) 186–204.

[114] E. Witten, "Constraints on Supersymmetry Breaking," *Nucl. Phys.* **B202** (1982) 253.

[115] S. Cecotti, P. Fendley, K. A. Intriligator, and C. Vafa, "A new supersymmetric index," *Nucl. Phys.* **B386** (1992) 405–452, `arXiv:hep-th/9204102`.

[116] E. Witten, "Elliptic genera and quantum field theory," *Commun. Math. Phys.* **109** (1987) 525.

[117] T. Eguchi, H. Ooguri, A. Taormina, and S.-K. Yang, "Superconformal algebras and string compactification on manifolds with SU(n) holonomy," *Nucl. Phys. B* **315** (1989) 193.

[118] D. Quillen, "Determinants of Cauchy-Riemann operators on Riemann surfaces," *Funkts. Anal. Prilozh.* **19** (1985) 37.

[119] L. Alvarez-Gaumé, G. W. Moore, and C. Vafa, "Theta functions, modular invariance, and strings," *Commun. Math. Phys.* **106** (1986) 1–40.

[120] I. Antoniadis, E. Gava, K. S. Narain, and T. R. Taylor, "N=2 type II-heterotic duality and higher derivative F terms," *Nucl. Phys.* **B455** (1995) 109–130, `hep-th/9507115`.

[121] M. Eichler and D. Zagier, *The Theory of Jacobi Forms.* Birkhäuser, 1985.

[122] T. Kawai, Y. Yamada, and S.-K. Yang, "Elliptic genera and N=2 superconformal field theory," *Nucl. Phys.* **B414** (1994) 191–212, `hep-th/9306096`.

[123] E. Witten, "AdS/CFT correspondence and topological field theory," *JHEP* **12** (1998) 012, `arXiv:hep-th/9812012`.

[124] S. Gukov, E. Martinec, G. W. Moore, and A. Strominger, "The search for a holographic dual to $AdS_3 \times S^3 \times S^3 \times S^1$," *Adv. Theor. Math. Phys.* **9** (2005) 435–525, `arXiv:hep-th/0403090`.

[125] S. Gukov, E. Martinec, G. W. Moore, and A. Strominger, "Chern-Simons gauge theory and the AdS_3/CFT_2 correspondence," `arXiv:hep-th/0403225`.

[126] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms.* Springer-Verlag, 1993.

[127] R. E. Borcherds, "Automorphic forms with singularities on Grassmannians," *Inventiones mathematicae* **132** (1998) 491, `alg-geom/9609022`.

[128] H. Rademacher and H. S. Zuckerman, "On the Fourier Coefficients of Certain Modular Forms of Positive Dimension," *Annals of Mathematics* **39** (1938) 433–462.

[129] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory.* Springer-Verlag, 1976.

[130] H. Rademacher, "The Fourier Coefficients of the Modular Invariant $J(\tau)$," *American Journal of Mathematics* **60** (1938) 501–512.

[131] A. D. Kloosterman, "On the representation of numbers in the form $ax^2 + by^2 + cz^2 + dt^2$," *Acta Mathematica* **49** (1927) 407.

[132] A. Weil, "On Some Exponential Sums," *Proceedings of the National Academy of Sciences of the United States of America* **34** (1948) 204.

[133] D. R. Heath-Brown, "Arithmetic applications of Kloosterman sums," *Nieuw Archief voor Wiskunde* **1** (2000) 380.

[134] G. Arfken and H. Weber, *Mathematical methods for physicists.* Academic Press, 2001.

[135] H. Meijer and M. de Bruin, "Speciale Functies." Lecture notes, 2000.

[136] J. Manschot and G. W. Moore, "A Modern Farey Tail," `arXiv:0712.0573 [hep-th]`.

[137] D. Niebur, "Construction of Automorphic Forms and Integrals," *Transactions of the Amer. Math. Soc.* **191** (1974) 373–385.

[138] H. Rademacher, "The Fourier Coefficients and the Functional Equation of the Absolute Modular Invariant $J(\tau)$," *American Journal of Mathematics* **61** (1939) 237–248.

[139] R. E. Borcherds, "The Gross-Kohnen-Zagier theorem in higher dimensions," *Duke Math. J.* **97** (1999) 219–233.

[140] P. Sarnak, *Some applications of modular forms.* Cambridge University Press, 1990.

[141] M. I. Knopp, "Rademacher on $J(\tau)$, Poincaré Series of Nonpositive Weights and the Eichler Cohomology," *Notices of the Amer. Math. Soc.* **37** (1990) 385–393.

[142] J. Manschot, "On the space of elliptic genera," `arXiv:0805.4333 [hep-th]`.

[143] N. P. Skoruppa, *Über den Zusammenhang zwischen Jacobi-Formen und Modulformen halbganzen Gewichts.* PhD thesis, Universität Bonn, 1984.

[144] W. Eholzer and N.-P. Skoruppa, "Modular invariance and uniqueness of conformal characters," *Commun. Math. Phys.* **174** (1995) 117–136, `arXiv:hep-th/9407074`.

[145] P. Bantay and T. Gannon, "Vector-valued modular forms for the modular group and the hypergeometric equation," *Comm. in Number Theory and Physics* **1** (2007) 651–680, arXiv:0705.2467 [math.NT].

[146] D. Zagier, "The Eichler-Selberg trace formula on $SL_2(\mathbb{Z})$," in *Introduction to modular forms*, S. Lang, ed. Springer-Verlag, 1976.

[147] E. J. Martinec, "Conformal field theory, geometry, and entropy," arXiv:hep-th/9809021.

[148] M. Cvetic and F. Larsen, "Near horizon geometry of rotating black holes in five dimensions," *Nucl. Phys.* **B531** (1998) 239–255, hep-th/9805097.

[149] J. de Boer, "Large N Elliptic Genus and AdS/CFT Correspondence," *JHEP* **05** (1999) 017, hep-th/9812240.

[150] M. I. Knopp, "Some new results on the Eichler cohomology of automorphic forms," *Bulletin of the Amer. Math. Soc.* **80** (1974) 607–632.

[151] D. Zagier, "Periods of modular forms and Jacobi theta functions," *Invent. Math.* **104** (1991) 449.

[152] D. Zagier, "Nombres de classes et formes modulaires de poids 3/2," *C.R. Acad. Sc. Paris* **281** (1975) 883.

[153] F. Hirzebruch and D. Zagier, "Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus," *Inv. Math.* **36** (1976) 57.

[154] C. Vafa and E. Witten, "A strong coupling test of S-duality," *Nucl. Phys.* **B431** (1994) 3–77, hep-th/9408074.

[155] G. W. Moore and E. Witten, "Integration over the u-plane in Donaldson theory," *Adv. Theor. Math. Phys.* **1** (1998) 298–387, hep-th/9709193.

[156] S. Giombi, A. Maloney, and X. Yin, "One-loop Partition Functions of 3D Gravity," arXiv:0804.1773 [hep-th].

[157] C. Itzykson and J. Zuber, *Quantum Field Theory*. McGraw-Hill, 1980.

[158] R. Gopakumar and C. Vafa, "M-theory and topological strings. I," `hep-th/9809187`.

[159] R. Gopakumar and C. Vafa, "M-theory and topological strings. II," `hep-th/9812127`.

[160] S. Katz, A. Klemm, and C. Vafa, "M-theory, topological strings and spinning black holes," *Adv. Theor. Math. Phys.* **3** (1999) 1445–1537, `hep-th/9910181`.

[161] R. Dijkgraaf, C. Vafa, and E. Verlinde, "M-theory and a topological string duality," `arXiv:hep-th/0602087`.

[162] L. Göttsche, "The Betti numbers of the Hilbert scheme of points on a smooth projective surface," *Math. Ann.* **286** (1990) 193.

[163] K. Behrend and B. Fantechi, "Symmetric obstruction theories and Hilbert schemes of points on threefolds," (2005) , `arXiv:math/0512556`.

[164] F. Denef and G. W. Moore, "How many black holes fit on the head of a pin?," *Gen. Rel. Grav.* **39** (2007) 1539–1544, `arXiv:0705.2564 [hep-th]`.

[165] M.-x. Huang, A. Klemm, M. Marino, and A. Tavanfar, "Black Holes and Large Order Quantum Geometry," `arXiv:0704.2440 [hep-th]`.

[166] T. M. Apostol, *Introduction to Analytic Number Theory.* Springer-Verlag, 1976.

[167] D. Gaiotto and X. Yin, "Examples of M5-brane elliptic genera," *JHEP* **11** (2007) 004, `arXiv:hep-th/0702012`.

[168] C. Montonen and D. I. Olive, "Magnetic Monopoles as Gauge Particles?," *Phys. Lett.* **B72** (1977) 117.

[169] P. Goddard, J. Nuyts, and D. I. Olive, "Gauge Theories and Magnetic Charge," *Nucl. Phys.* **B125** (1977) 1.

[170] J. P. Yamron, "Topological actions from twisted supersymmetric theories," *Phys. Lett.* **B213** (1988) 325.

[171] K. Yoshioka, "The Betti numbers of the moduli space of stable sheaves of rank 2 on $\mathbb{P}^2$," *J. reine. angew. Math.* **453** (1994) 193–220.

[172] A. A. Klyachko, "Moduli of vector bundles and numbers of classes," *Funct. Anal. and Appl.* **25** (1991) 67–68.

[173] K. Skenderis and B. C. van Rees, "Real-time gauge/gravity duality," arXiv:0805.0150 [hep-th].

[174] S. Carlip, S. Deser, A. Waldron, and D. K. Wise, "Cosmological Topologically Massive Gravitons and Photons," arXiv:0803.3998 [hep-th].

[175] D. A. Lowe and S. Roy, "Chiral geometries of (2+1)-d AdS gravity," arXiv:0806.3070 [hep-th].

[176] D. Shih, A. Strominger, and X. Yin, "Counting dyons in N = 8 string theory," hep-th/0506151.

[177] J. de Boer, J. Manschot, K. Papadodimas, and E. P. Verlinde, "The chiral ring of AdS_3/CFT_2 and the attractor mechanism," arXiv:0809.0507 [hep-th].

SUMMARY

Partition Functions for Supersymmetric Black Holes

This thesis presents a number of results on partition functions for four-dimensional supersymmetric black holes. These partition functions are important tools to explain the entropy of black holes from a microscopic point of view. Such a microscopic explanation was desired after the association of a macroscopic entropy to black holes in the 70's, based on the analogies between black hole physics and thermodynamics. The correct microscopic account of black hole entropy was achieved in string theory and M-theory during the 90's, and a crucial role is played by D-branes and M-branes.

The black holes, which are studied in this thesis, are supersymmetric solutions of four-dimensional $\mathcal{N} = 2$ supergravity, which carry both electric and magnetic charges. An important feature of the global geometry is the the near-horizon geometry, which is $\mathrm{AdS}_2 \times S^2$ and where the Kähler moduli are fixed at their attractor values. The horizon area of this class of black holes is given by $S_{\mathrm{BH}} = \pi|Z|^2 = \pi\sqrt{\frac{2}{3}p^3(q_{\bar{0}} + \frac{1}{2}q^2)}$, where p^a and $(q_{\bar{0}}, q_a)$ are respectively the electric and magnetic charges, and Z is the central charge. The combination $\hat{q}_{\bar{0}} = q_{\bar{0}} + \frac{1}{2}q^2$ is required to be positive for black holes. The first motivation to introduce a black hole partition function $\mathcal{Z}_{\mathrm{BH}}$, is to explain S_{BH} microscopically. An analysis of the attractor equations suggest that $\mathcal{Z}_{\mathrm{BH}}$ is naturally expanded in $q_{\bar{0}}$ and q_a, while the p^a are kept fixed. In other more physical words, the electric charges are in a macrocanonical ensemble and the magnetic charges in a microcanonical ensemble.

If higher derivative contributions are included in the supergravity action, the entropy receives corrections. The form of these corrections suggests that $\mathcal{Z}_{\mathrm{BH}}$ is well approximated by the square of the topological string partition function $|\mathcal{Z}_{\mathrm{top}}|^2$. Topological string theory is a simplified version of string theory, which allows the computation of many quantities using elaborate mathematical techniques, for example $\mathcal{Z}_{\mathrm{top}}$ basically enumerates the holomorphic maps of a Riemann surface into a Calabi-Yau threefold. The conjecture that $\mathcal{Z}_{\mathrm{BH}} = |\mathcal{Z}_{\mathrm{top}}|^2$, is the second motivation

for this thesis.

The third motivation is the correspondence between a theory including gravity in Anti-de Sitter (AdS) space and a conformal field theory (CFT) on the boundary of the AdS-space. Part of the near-horizon geometry of the black holes in the eleven dimensions is AdS_3, whose boundary is a two-dimensional torus. The correspondence suggests that the CFT_2 partition function equals the one of the theory in the bulk of AdS_3. Therefore, the CFT_2 partition function should admit an expansion which is natural for an AdS_3-(super)gravity partition function. Dijkgraaf *et al.* proposed in 2000 that an SCFT partition function can be rewritten as a Poincaré series, which is a sum over the coset $\Gamma_\infty \backslash \Gamma$. Every element in the coset corresponds to a semi-classical saddle point geometry, providing therefore evidence for the AdS/CFT correspondence. Chapter 2 explains these notions rather heuristically in a bosonic setting, subsequent chapters are more precise.

Chapter 3 explains how the black holes arise as a solution of 11-dimensional M-theory, and how this can account for the entropy microscopically. Four-dimensional supergravity appears in this context as the reduction of M-theory on a six-dimensional Calabi-Yau X times a circle S^1_{M}. The heavy objects, which source the black holes, are M5-branes. These correspond to the magnetic charges whereas the electric charges are generated by fluxes on the worldvolume of the M5-brane and momentum around S^1_{M}. The six-dimensional M5-branes wrap a four-dimensional divisor P of X together with S^1_{M} and the Euclidean time circle S^1_{t}. The parameters of the theory can be chosen such that the low energy approximation to M-theory is valid. The M5-brane low energy degrees of freedom can be reduced to the T^2, which is formed by the two circles. There, the degrees of freedom combine to an $\mathcal{N} = (4, 0)$ superconformal field theory. The central charges of the holomorphic and anti-holomorphic sector can be determined using index formulas. The relevant partition function for this SCFT is a (modified) elliptic genus. The symmetries of the theory determine that the elliptic genus transforms covariantly under modular transformations, which makes it possible to derive the Cardy formula for the entropy. This gives the correct leading behavior of the entropy. For a specific identification of the parameters, the CFT partition function is equal to the (divergent) black hole partition function.

An important property of the elliptic genera is the decomposition into theta functions and a vector-valued modular form. The principal part of the Laurent expansion of the vector-valued modular form gives rise to the definition of the "polar spectrum" of the SCFT. Chapter 4 is devoted to an analysis of meromorphic vector-valued modular forms. It is shown how the Fourier coefficients can be expressed as an infinite sum over the coset $\Gamma_\infty\backslash\Gamma$. If the polar degeneracies are known, the non-polar degeneracies can be determined with an arbitrary accuracy, improving on the leading order estimate by the Cardy formula. In addition is shown how the vector-valued modular form can be written as a sum over $\Gamma_\infty\backslash\Gamma$. The sum is a regularized Poincaré series, and an improvement of the proposal by Dijkgraaf *et al.*. The regularization leads in general to an anomaly, which is canceled if the polar degeneracies satisfy a number of constraints. This number can be determined using the Selberg trace formula. The dimension of the space of vector-valued modular forms is simply given by the number of polar terms minus the number of constraints.

With the results of Chapter 4, Chapter 5 revisits the motivations of Chapter 2. The regularized Poincaré series confirms the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, since the sum over $\Gamma_\infty\backslash\Gamma$ is suggestive of a semi-classical sum over AdS_3-geometries. If the complex structure τ is varied, the most contributing geometry to $\mathcal{Z}_{\mathrm{BH}}$ might jump, which is a nice manifestation of Hawking-Page phase transitions. The Poincaré series are essentially a sum over $\Gamma_\infty\backslash\Gamma$ of the polar spectrum, which lies classically below the cosmic censorship bound. Therefore, one can view the series heuristically as a sum over all geometries (including black holes) of the states which do not collapse into a black hole.

This is also how the connection between black holes and topological strings can be understood. The degeneracies of charged BPS-particles (M2-branes) in the near-horizon geometry are enumerated by $|\mathcal{Z}_{\mathrm{top}}|^2$. Therefore, $|\mathcal{Z}_{\mathrm{top}}|^2$ appears for every saddle point geometry in $\mathcal{Z}_{\mathrm{BH}}$. The conjecture is now elucidated for strong topological string coupling constant ($\hat{q}_{\bar{0}} \gg p^3$), since then a single AdS_3-geometry dominates the partition function. Also the case of weak topological string couple is discussed. It is shown that for this part of the spectrum, the elliptic genus confirms the leading behavior of two-centered solutions. The approximation $\mathcal{Z}_{\mathrm{BH}} \sim |\mathcal{Z}_{\mathrm{top}}|^2$ is however no longer valid.

The constraints on the polar degeneracies by modularity are strongest if the number of polar terms is small. This is generically not the case for the black hole and AdS_3 applications, which are discussed before. The vector-valued modular forms appear however at many places in theoretical physics, for example rational conformal field theory and $\mathcal{N}=4$ supersymmetric gauge theory on a four-manifolds M. The partition functions of such gauge theories with gauge group $U(N)$ are closely related to M5-brane elliptic genera. If certain conditions are satisfied, the partition function of this theory is the generating function of the Euler characteristic of instanton moduli spaces. Section 5.3 performs an analysis of the partition functions of the $U(N)$ theories on $\mathbb{CP}^2$. This confirms the older results in the literature for $N = 1$ and 2. A new generating function is proposed for the Euler numbers of $SU(3)$ moduli spaces.

SAMENVATTING

Partitiefuncties voor supersymmetrische zwarte gaten

Dit proefschrift presenteert een aantal resultaten voor partitiefuncties van vierdimensionale zwarte gaten. Deze partitiefuncties zijn belangrijke instrumenten om de entropie van zwarte gaten verklaren op een microscopisch niveau. Zo een microscopische verklaring was gewenst nadat een macroscopische entropie was toegekend aand zwarte gaten in de 70'er jaren, gebaseerd op de analogie tussen de fysica van zwarte gaten en thermodynamica. De correcte microscopische verklaring van de entropie van zwarte gaten is bereikt in snaartheorie en M-theorie in de 90'er jaren, en een cruciale rol wordt gespeeld door D-branen en M-branen.

De zwarte gaten, die bestudeerd worden in dit proefschrift, zijn supersymmetrische oplossingen van vierdimensionale $\mathcal{N} = 2$ supergravitatie, die zowel elektrische als magnetische ladingen bevatten. Een belangrijk kenmerk van zo'n oplossing is dat de meetkunde in de nabijheid van de horizon wordt beschreven door de meetkunde van de productruimte van een tweedimensionale Anti-de Sitter ruimte AdS_2 en een tweedimensionale bol S^2. De Kähler moduli zijn daar bovendien bepaald door hun zgn. aantrekkingswaarden. Het horizonoppervlak van deze klasse zwarte gaten wordt gegeven door $S_{\mathrm{BH}} = \pi|Z|^2 = \pi\sqrt{\frac{2}{3}p^3(q_{\bar{0}} + \frac{1}{2}q^2)}$, waar p^a en $(q_{\bar{0}}, q_a)$ respectievelijk de elektrische en magnetische ladingen zijn, en Z de centrale lading is. De combinatie $\hat{q}_{\bar{0}} = q_{\bar{0}} + \frac{1}{2}q^2$ moet positief zijn voor zwarte gaten. De eerste motivatie om een partitiefunctie voor zwarte gaten $\mathcal{Z}_{\mathrm{BH}}$ te introduceren, is om S_{BH} microscopisch te verklaren. Een analyse van de aantrekkingsvergelijkingen suggereert dat $\mathcal{Z}_{\mathrm{BH}}$ op een natuurlijke wijze wordt geëxpandeerd in $q_{\bar{0}}$ en q_a, terwijl de p^a constant blijven. Of in andere woorden, de elektrische ladingen bevinden zich in een canoniek ensemble en de magnetische ladingen in een microcanoniek ensemble.

Wanneer hogere afgeleide bijdragen worden toegevoegd aan de supergravitatie-actie, ontvangt de entropie correcties. De vorm van deze correcties suggereert dat $\mathcal{Z}_{\mathrm{BH}}$ goed wordt benaderd door het kwadraat van de partitiefunctie van topologische snaartheorie $|\mathcal{Z}_{\mathrm{top}}|^2$. Topologische snaartheorie is een gesimplificeerde versie van snaartheorie, waarin de berekening van veel grootheden mogelijk is door gebruik te maken van verregaande wiskundige technieken, bijvoorbeeld $\mathcal{Z}_{\rm top}$ somt de holomorfe afbeeldingen van een Riemann oppervlak naar een Calabi-Yau drievoud op. Het vermoeden dat $\mathcal{Z}_{\rm BH} = |\mathcal{Z}_{\rm top}|^2$, is de tweede motivatie voor dit proefschrift.

De derde motivatie is de overeenkomst tussen een theorie, welke gravitatie bevat in een Anti-de Sitter (AdS) ruimte en een conforme veldentheorie (CVT) op de rand van de AdS-ruimte. Een gedeelte van de meetkunde in de nabijheid van de horizon van zwarte gaten in elf dimensies is AdS_3, welke een tweedimensionale rand heeft. De overeenkomst suggereert dat de CVT_2 partitiefunctie gelijk is aan die van de theorie in het merendeel van AdS_3. Daarom moet de CVT partitiefunctie een expansie toelaten die natuurlijk is voor een partitiefunctie van AdS_3-(super)gravitatie. Dijkgraaf *et al.* stelde in 2000 voor dat een SCVT partitiefunctie kan worden herschreven als een Poincaré reeks, welke een som over de nevenklasse $\Gamma_\infty\backslash\Gamma$ is. Elk element in de nevenklasse komt overeen met een semi-klassieke zadelpuntsruimte, en verschaft daarom een bevestiging van de AdS/CVT overeenkomst. Hoofdstuk 2 legt deze begrippen enigszins heuristisch uit in een bosonische omgeving, de volgende hoofdstukken zijn preciezer.

Hoofdstuk 3 legt uit hoe de zwarte gaten tevoorschijn komen als een oplossing van elfdimensionale M-theorie, en hoe dit de entropie microscopisch kan verklaren. Vierdimensionale supergravitatie verschijnt in deze context als de reductie van M-theorie op een zesdimensionale Calabi-Yau X keer een cirkel $S^1_{\rm M}$. De zware objecten, die de zwarte gaten voeden, zijn de M5-branen. Deze komen overeen met de magnetische ladingen terwijl de elektrische ladingen gegenereert worden door fluxen op het wereldvolume van de M5-braan en impuls rond $S^1_{\rm M}$. De zesdimensionale M5-branen omwikkelen een vierdimensionale divisor P van X samen met $S^1_{\rm M}$ en de Euclidische tijdscirkel $S^1_{\rm t}$. De parameters van de theorie kunnen zodanig worden gekozen dat de lage energie benadering voor M-theorie toegestaan is. De lage energie vrijheidsgraden van de M5-braan kunnen worden gereduceerd naar de torus T^2, die gevormd wordt door de twee cirkels. Daar combineren de vrijheidsgraden tot een $\mathcal{N} = (4,0)$ superconforme veldentheorie. De centrale ladingen van de holomorfe en antiholomorfe sector kunnen worden bepaald met behulp van indexformules. De relevante partitiefunctie voor deze SCVT is een (gemodificeerde) elliptische genus. De symmetrieën van de theorie bepalen dat de elliptische genus covariant transformeert onder modulaire transformaties, wat het mogelijk maakt om de Cardy formule voor de entropie af te leiden. Dit geeft het juiste leidende gedrag voor de entropie. Voor een specifieke identificatie van parameters is de CVT partitiefunctie gelijk aan de (divergente) zwarte gat partitiefunctie.

Een belangrijke eigenschap van de elliptische genera is de decompositie in thetafuncties en een vectorwaardige modulaire vorm. Het hoofdgedeelte van de Laurentontwikkeling van de vectorwaardige modulaire vorm geeft aanleiding tot de definitie van het "polaire spectrum" van de SCVT. Hoofdstuk 4 is gewijd aan de analyse van meromorfe vectorwaardige modulaire vormen. Er wordt getoond hoe de Fourier coëfficiënten uitgedrukt kunnen worden als een oneindige som over de nevenklasse $\Gamma_\infty\backslash\Gamma$. Wanneer de polaire degeneraties bekend zijn, kunnen de niet-polaire degeneraties bepaald worden met een willekeurige nauwkeurigheid, wat een verbetering is ten opzichte van de afschatting voor de leidende orde door de Cardy formule. Aanvullend is laten zien hoe de vectorwaardige modulaire vorm geschreven kan worden als een som over $\Gamma_\infty\backslash\Gamma$. De som is een gereguleerde Poincaré reeks, en een verbetering van het voorstel door Dijkgraaf *et al.*. De regularisatie leidt in het algemeen tot een anomalie, die teniet gedaan wordt wanneer de polaire degeneraties aan een aantal eisen voldoen. Dit aantal kan worden bepaald met behulp van de Selberg spoorformule. De dimensie van de ruimte van vectorwaardige modulaire vormen wordt simpelweg gegeven door het aantal polaire termen min het aantal eisen.

Met de resultaten van Hoofdstuk 4, gaat Hoofdstuk 5 terug naar de motivaties in Hoofdstuk 2. De geregulariseerde Poincaré reeks bevestigt de $\mathrm{AdS}_3/\mathrm{CVT}_2$ overeenkomst, omdat de som over $\Gamma_\infty\backslash\Gamma$ suggestief is voor een semi-klassieke som over AdS_3-geometrieën. Wanneer de complexe structuur τ gevarieerd wordt, kan de meest bijdragende meetkunde aan $\mathcal{Z}_{\mathrm{BH}}$ verspringen, wat Hawking-Page fasetransformaties op een mooie manier tot uiting brengt. De Poincaré reeks is in essentie een som over $\Gamma_\infty\backslash\Gamma$ van het polaire spectrum, welke onder de kosmische censuur grens ligt. Daarom kan men de reeks heuristisch zien als een som over alle meetkundes (inclusief de zwarte gaten) van de toestanden die niet instorten tot een zwart gat.

Dit is tevens hoe de verbinding tussen zwarte gaten en topologische snaren begrepen kan worden. De degeneraties van geladen BPS-deeltjes (M2-branen) in de nabijheid van de horizon zijn opgesomd in $|\mathcal{Z}_{\mathrm{top}}|^2$. Daarom verschijnt $|\mathcal{Z}_{\mathrm{top}}|^2$ voor elke zadelpuntsruimte in $\mathcal{Z}_{\mathrm{BH}}$. Het vermoeden is nu verhelderd voor sterke topologische snaar koppelingsconstante ($\hat{q}_{\bar{0}} \gg p^3$), omdat dan een enkele AdS_3-meetkunde de partitiefunctie domineert. Ook de situatie van een zwakke topologische snaar koppelingsconstante is besproken. Er wordt laten zien dat voor dit gedeelte van het spectrum, de elliptische genus het leidende gedrag van oplossingen met twee middelpunten bevestigt. De benadering $\mathcal{Z}_{\mathrm{BH}} \sim |\mathcal{Z}_{\mathrm{top}}|^2$ is echter niet langer accuraat.

De eisen aan de polaire degeneraties door modulariteit zijn het sterkst wanneer het aantal polaire termen klein is. Dit is in het algemeen niet het geval voor de toepassingen voor zwarte gaten en $\mathrm{AdS}_3/\mathrm{CVT}_2$, die eerder besproken zijn. De vectorwaardige modulaire vormen verschijnen echter op veel plekken in de theoretische natuurkunde, bijvoorbeeld rationale conforme veldentheorie en $\mathcal{N}=4$ supersymmetrische ijktheorie op een vierdimensionale variëteit M. De partitiefuncties van zulke ijktheorieën met ijkgroep $U(N)$ zijn nauw verwant met elliptische genera van M5-branen. Wanneer aan enige condities voldaan is, is de partitiefunctie van deze theorie de genererende functie van de Euler karakteristiek van instanton moduliruimtes. Paragraaf 5.3 voert een analyse uit van de partitiefuncties van $U(N)$ theorieӥ op $\mathbb{CP}^2$. Dit bevestigt de oudere resultaten in de literatuur voor $N=1$ en $N=2$. Een nieuwe genererende functie is voorgesteld voor de Euler getallen van $SU(3)$ moduliruimtes.

ACKNOWLEDGEMENTS

First and most of all, I would like to thank my supervisor Erik Verlinde. I am very grateful that you gave me the opportunity to continue my studies of physics and mathematics during the last four years. Your guidance of my doctorate work was very important in order to successfully finish and continue as a postdoctoral researcher. Your explanations of aspects of high energy physics and string theory in the first year developed gradually into a research project, which resulted eventually in my first high energy publication. In later years, you encouraged me to publish and to solve the problems related to the "Farey Tail transform". This has turned out to be crucial for the writing of the papers, which led to this thesis. Moreover, I want to thank you for the many trips I have made, especially the trips to Jerusalem in 2005-06 and New Jersey in 2007. Many thanks for all of this.

The articles, on which this thesis is based, are written in collaboration with Jan de Boer, Miranda Cheng, Robbert Dijkgraaf, Gregory Moore and Erik Verlinde. I would like to thank you all for these collaborations, which I appreciated a lot, and for sharing your ideas with me. Moreover, the thesis has benefitted significantly from discussions with many people, among which are Xerxes Arsiwalla, Ruth Britto, Sheer El-Showk, Lotte Hollands, Joost Hoogeveen, Matthias Gaberdiel, Gerard van der Geer, Ilies Messamah, Amir Kashani-Poor, Ingmar Kanitschneider, Juan Maldacena, Paul McFadden, Kyriakos Papadodimas, Balt van Rees, Masaki Shigemori, Peter Sarnak, Kostas Skenderis, Marika Taylor, Edward Witten and Don Zagier. From the above, I would like to especially thank Lotte Hollands for sharing the office with me and Balt van Rees for being one of my paranymfs. More generally, I am thankful to all the members of the ITFA during the years 2004-2008 for all the shared the lunches, tea breaks, journal clubs and other meetings. It was a great time.

Naast al de bovengenoemde mensen, die op een professionele manier hebben bijgedragen aan het proefschrift, zijn er veel mensen wiens bijdragen minder concreet te benoemen zijn. Deze mensen hebben ervoor gezorgd dat de afgelopen vier jaar een gelukkige en plezierige tijd was. Dit heeft natuurlijk een positieve uitwerking gehad voor het werk aan de artikelen en het proefschrift. Mijn huisgenoten in Weesp: Annemieke, Renske, Tom en Wytske, wil ik bedanken voor alle gezellige maaltijden, schaakpartijtjes, fietstochtjes, hardloopwedstrijdjes etc.. Mede dankzij jullie, heb ik met veel plezier in Weesp gewoond. Veel avonden heb ik getraind onder leiding van Peter van Wees bij Phanos. Peter, Koen en de andere atleten van de groep, bedankt voor alle trainingen! Verder wil ik ook Arno & Annemarie, Bram & Silke, Gert & Rianne, Joop & Judith, Maarten & Jantien, Wolbrand, Huigert & Jedidja, Jeroen & Susan, Martin & Marijn en Thijs & Thera bedanken voor alle bruiloften en partijen, vakanties, fietstochten etc. in de afgelopen jaren.

Naast al deze mensen ben ik mijn directe familie, ouders, Hanneke & Henry en Karel zeer dankbaar voor alle hulp bij verhuizen, de wederzijdse bezoeken, maar bovenal alle aandacht en medeleven. Karel wil ik in het bijzonder bedanken voor zijn bereidheid om paranymf te willen zijn en voor alle computerhulp. Als laatste wil ik Ruth Britto bedanken, die de totstandkoming van het proefschrift van het meest nabij heeft meegemaakt. Zij verdient de meeste lof voor al haar interesse, geduld en liefde in de afgelopen tijd.

September 2008