APPLICATIONS OF REGGE POLES

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I. INTRODUCTION

Several years ago we began abstracting from the Feynman diagram expansion of field theory a number of exact properties of the scattering amplitudes on the mass-shell, continued sometimes to unphysical values of the various energy and momentum variables. We found the crossing relation and a set of dispersion relations and generalized unitarity formulae. Research on the unitarity and dispersion equations (or analyticity properties) is still going on, for though they are simple in elementary cases, they become fairly complicated when anomalous thresholds and many-particle channels are present. Still, it became clear quite early [1] that these equations would contain enough information to determine all the scattering amplitudes in the Feynman diagram expansion, provided some high-energy boundary conditions were supplied (the problem of subtractions) and some masses and coupling constants specified.

Mandelstam [2] showed us how to solve the subtraction problem in perturbation theory, but it soon became apparent that, for strongly coupled systems at least, the high-energy boundary conditions suitable for individual Feynman diagrams would not work. In particular, they were inconsistent with the existence of dynamical resonances or composite particles of high spin. Meanwhile, Regge [3] has discovered, by means of the Watson-Sommerfeld transformation, how scattering amplitudes behave in Schrödinger theory at large $\cos\vartheta$ when such states are present. In relativistic theory, large $\cos\vartheta$ is high energy in the crossed reaction, and Mandelstam and others therefore suggested that the high-energy boundary condition suitable for a reaction involving the exchange of a dynamical resonance is just the behaviour $s^{\alpha(t)}$ of a Regge trajectory. If $\alpha(t)$ goes negative at large negative t, then there are no subtraction difficulties and no unwanted singularities. The Regge boundary conditions can thus replace those suggested by the Feynman diagrams.

During 1961, increasingly wide applications of this idea were suggested in the realm of strong interactions. Goldberger and Blankenbecler [4] suggested that the nucleon, and one or more of its isobars, might lie on a Regge trajectory. Chew and Frautschi [5] and Gribov [6] [see also Lovelace [7]] suggested that the exchange of a trajectory with the quantum numbers of the vacuum (the Pomeranchuk trajectory P) could explain the phenomena of diffraction scattering, including Pomeranchuk's rules. [Of course, constant asymptotic cross-sections now come from the condition $\alpha_P(0) = 1$, still unexplained, and are not connected with a fixed particle size, but with increasing size and increasing transparency at high energies.] Finally, Chew and Frautschi [8] declared that all the strongly interacting particles should lie on Regge trajectories. They would then all be " dynamical ", and one hopes that the unitarity and dispersion equations for the strongly interacting particles (when they are all written down) could be solved with the Regge boundary conditions, the specification of a single mass, the requirement of consistency, and very little else (perhaps the conservation laws and the condition $\alpha_P(0) = 1$).

My interest in the Regge pole hypothesis became intense during 1961 and has remained so. I am concerned not so much with the distant prospects of calculations, but with the immediate question of working out the properties of the Regge poles and predicting the results of high-energy experiments, or using the Regge poles to correlate high-energy data with our knowledge of low-energy resonances in the crossed reaction. Let me summarize some of the results obtained so far in this type of research; in what follows, we shall discuss most of them in some detail.

I started on the problem in collaboration with Frautschi and Zachariasen [9]. We worked out the rules for "reggeizing" any two-body scattering amplitude not involving anomalous thresholds. We showed that each trajectory has a signature (± 1) that permits it to give poles corresponding to particles or resonances only at every *other* value of J (0, 2, 4, ... or 1/2, 5/2, 9/2, ... etc.). The Regge term is either odd or even in the cosine of the scattering angle in the relevant reaction, depending on the signature. The data on π-N [7] and N-N [9] scattering have been analyzed and some properties of the P trajectory determined. The "spin" $\alpha_P(t)$ passes through 1 at $t = 0$ with a slope of around 1 $(\text{GeV})^{-2}$. Thus there may be a resonance of spin 2 on the trajectory with a mass of something like 1 GeV and there is the apparent problem of a threatened "ghost" with spin 0 at $t \approx -1\ (\text{GeV})^2$.

The coefficients $\beta(t)$ of the Regge poles are not in general real, but by factoring out the threshold dependence of β one can define [9] a coefficient $b(t)$ that is real in a region of t extending down from the lowest threshold.

Now consider the exchange of a trajectory on which a known particle lies (for instance, the ρ trajectory including the ρ meson). The function $\alpha_\rho(t)$ and the functions $b(t)$ for various reactions involving ρ exchange can all be tied down at $t = m_\rho^2$, where $\alpha_\rho \approx 1$ and the various b become the products of coupling constants of the ρ $[b_{\pi\pi\rho\pi\pi}(m_\rho^2) = 4\gamma_{\rho\pi\pi}^2$, $b_{\pi\pi\rho NN}^{(1)} = 2\gamma_{\rho\pi\pi}\gamma_{\rho NN}$, etc.]. Udgaonkar [10] has used observed total cross-sections (that is, imaginary parts of scattering amplitudes at $t = 0$) to obtain information about α_ρ, α_ω, and various b's at $t = 0$. It is most interesting to compare these and other values of b's and α's obtained for $t \leqq 0$ from high-energy scattering amplitudes with known values at positive t. The method of extrapolating b is then of great importance.

It has been shown [11, 12] that the factoring of b coefficients [e.g., $b_{\pi\pi\rho NN}^{(1)} = \eta_{\pi\pi\rho}\eta_{NN\rho}^{(1)}$] holds not only near an actual particle, but all along the trajectory. In fact, the Regge pole is just a kind of virtual state, with outgoing wave boundary conditions imposed in all open channels and decaying wave conditions in all closed channels. The energy squared is t, the angular momentum α, and the coefficients of the asymptotic outgoing or decaying wave terms in all the channels just the "coupling constants" η.

In the present report, we discuss the following results:

Section 2: a restatement, somewhat more explicit, of the "reggeizing" rules.

Section 3: a comment on the role of the signature in producing symmetries at high energies. The relations of Pomeranchuk can be widely generalized.

Section 4 and 5: the general application of the "reggeizing" rules to the $\pi\pi$, πN, and NN channels. Here some of our results have been obtained by Gribov and Pomeranchuk [13], but with one significant error. The NN channel has been treated by Wagner and Sharp [17].

Section 6: the explanation of why the "ghost" is not present along the Pomeranchuk trajectory, even if $\alpha_\rho(t)$ passes through zero at negative t. The absence of the lowest member, or members, from a Regge series is a familiar phenomenon (for instance, in nuclear physics) and perfectly natural. A number of new general principles involving Regge trajectories in the presence of spin are involved in the explanation. All are rather obvious once we look at the case of the Schrödinger equation.

In the Appendix, we refer briefly to the work of Gell-Mann and Udgaonkar [14] on the case of anomalous thresholds, with particular reference to nuclei.

The present report is to be considered a kind of sequel to Refs. [9] and [11] and we shall make use freely of ideas, notations, and results contained in those publications.

II. "REGGEIZING"

Consider a set of two-body t reactions sharing a given set of values of the conserved quantum numbers other than J. We treat the case of no anomalous thresholds. For simplicity, we take the baryon number in the t reactions to be zero. We label all the participating channels by the indices C, D, E, etc. A given reaction may involve several channels, corresponding to different relative orientations of spin and orbital angular momenta.

Assume the S-matrix for these reactions has a Regge pole at $J = \alpha(t)$:

$$e^{-i\pi(J+\frac{1}{2})}S^J_{CD}(t) \approx \xi_C(t)\xi_D(t)[J-\alpha(t)]^{-1}. \quad (2.1)$$

For each reaction, there is a set of invariant scattering amplitudes free of kinematic singularities, such as are used for "mandelstamizing". Call them T_i. Then we may express T_i in terms of the S-matrix:

$$T_i = \sum_{J,C,D} Z^{CDi}_J(t, x_t)[S^J_{CD}(t) - \delta_{CD}], \quad (2.2)$$

where $x_t = \cos\delta$ for the t reaction, and Z_J is a linear combination of functions like $P_J(x_t)$, $x_t P'_J(x_t)$, $P'_{J+1}(x_t)$, etc., with coefficients that are functions of t and algebraic functions of J. We choose each amplitude T_i so that for integral J the Z_J obey $Z_J(-x_t) = (-1)^J Z_J(x_t)$, or else $Z_J(-x_t) = (-1)^{J+1} Z_J(x_t)$. In the former case, we say $\varepsilon_i = +1$ and in the latter case $\varepsilon_i = -1$.

The contribution of the pole (2.1) to the scattering amplitude (2.2) is then easy to determine. We generalize Z_J to non-integral indices, simply by generalizing the Legendre functions and the algebraic functions. Then we have:

$$\text{Contribution to } P_i = \sum_{C,D} (-i\pi)(2\sin\pi\alpha)^{-1}\xi_C(t)\xi_D(t)[Z^{CDi}_{\alpha(t)}(t, x_t) \pm \varepsilon_i Z^{CDi}_{\alpha(t)}(t, -x_t)]. \quad (2.3)$$

Here the $\pm$ is the signature. From (2.3) we obtain, at high s or large x_t, just the rules of Ref.[9].

Apart from the notion of signature, (2.3) is essentially obvious from the work of Regge [3]. In Regge's work, we deal with one channel only and no spin. The single scattering amplitude f is related to S by the single coefficient

$$Z_J = \frac{(2J+1)}{2ik_t} P_J(x_t)$$

and the contribution to the scattering amplitude from the Regge pole is just

$$-i\pi(\sin\pi\alpha)^{-1}\xi^2(t)(2ik_t)^{-1}(2\alpha+1)P_{\alpha(t)}(x_t).$$

However, one comment is in order. We can write P_ν as the sum of two terms:

$$P_\nu(x) = \mathscr{P}_\nu(x) + \mathscr{P}_{-\nu-1}(x), \quad (2.4)$$

where $\mathscr{P}_\nu(x)$, which is essentially a Q function, is defined by

$$\mathscr{P}_\nu(x) \equiv \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu+1)}(2x)^\nu F\left(-\tfrac{1}{2}\nu, \tfrac{1}{2}-\tfrac{1}{2}\nu; \tfrac{1}{2}-\nu; \frac{1}{x^2}\right) \quad (2.5)$$

and F is the hypergeometric function. Now, strictly speaking, the pole in the S-matrix at $J = \alpha$ contributes only the term $\mathscr{P}_\alpha$ rather than the whole function P_α. [See Mandelstam [15].] Whether the remaining term $\mathscr{P}_{-\alpha-1}$ is actually present, depends on what happens to the S-matrix in the "left half-plane" Re $(J+\frac{1}{2})<0$.

Regge and Predazzi [16] have recently discussed the case of Schrödinger equations with repulsive cores, similar to the original problem of Sommerfeld, for which the S-matrix obeys the symmetry principle: $e^{-i\pi J}S^J$ symmetrical about $J = -\frac{1}{2}$. There is then a twin pole for every Regge pole, giving a total contribution

$$\frac{\xi^2}{J-\alpha} - \frac{\xi^2}{J+\alpha+1}$$

to $e^{-i\pi(J+\frac{1}{2})}S^J$, and the twin poles together give

$$(2\alpha+1)\mathscr{P}_\alpha - [2(-\alpha-1)+1]\mathscr{P}_{-\alpha-1} = (2\alpha+1)P_\alpha.$$

If we do not consider the repulsive core case, but rather the situation with an ordinary attractive Yukawa potential [15], then the poles in the right half-plane Re $(J+\frac{1}{2})>0$ have twins in the above sense only at half-integral and integral J and we should not, in general, add precisely a term $\mathscr{P}_{-\alpha-1}$ to $\mathscr{P}_\alpha$ to make P_α.

Since $\mathscr{P}_\alpha$ has poles at the half integers $\neq -\frac{1}{2}$ [and $\mathscr{P}_{-\alpha-1}$ has equal and opposite poles], something must happen in any theory to cancel these. At $\alpha = \frac{1}{2}$, say, either there is a trajectory that acts as a twin (if only for this particular value of α) or else the coefficient vanishes.

In this report, we shall usually not have to choose between the two situations discussed by Mandelstam and by Regge and Predazzi, because we shall be mostly concerned only with the leading term in each Z at large x_t or large s, and with $\alpha > -\frac{1}{2}$. In the spinless case discussed by Regge, it is obvious that the leading term in $\mathscr{P}_\alpha$ dominates the leading term in $\mathscr{P}_{-\alpha-1}$ as long as $\alpha > -\frac{1}{2}$. Actually we can make a similar statement for the more complicated case including spin. For every channel C there is a corresponding channel $\bar{C}$ with the same particles but with the total spin added to the orbital angular momentum in an opposite sense (for example, for $S = 3$, we might have $L = J-2$ instead of $L = J+2$). For the case with spin, the symmetry rule of Regge and Predazzi becomes: $e^{-i\pi J} S^J_{CD}$ symmetrical under $J \to -J-1$, $C \to \bar{C}$, $D \to \bar{D}$. Similarly the Z's always obey the rule

$$Z_J^{CDi} = -Z_{-J-1}^{\bar{C}\bar{D}i} \tag{2.6}$$

and we may break up each Z into $\mathscr{Z}$'s by breaking up each Legendre function P in it into $\mathscr{P}$'s, as follows:

$$Z_J^{CDi} = \mathscr{Z}_J^{CDi} - \mathscr{Z}_{-J-1}^{\bar{C}\bar{D}i} \, . \tag{2.7}$$

We see, then, that the dominant term coming from the $\mathscr{Z}_{-J-1}$ will not catch up with the dominant term coming from the $\mathscr{Z}_J$ unless $J \leqq -\frac{1}{2}$, just as in the simple case. Thus we can, at very large x_t or s, just replace each P by the asymptotic expansion of $\mathscr{P}$ given in (2.5).

In what follows, we shall simply list P's and Z's as the contributions of Regge poles to scattering amplitudes, without prejudice to the question of twins.

III. SIGNATURE AND HIGH-ENERGY SYMMETRY

The fact that each trajectory has a fixed signature gives rise to an important symmetry at high energies, a generalization of Pomeranchuk's rules relating the forward elastic scattering of $\bar{p}$ and p to that of p and p, etc.

Consider the s reaction $a+b \to c+d$, dominated at high energies by a Regge pole in the crossed reaction $a+\bar{c} \to \bar{b}+d$. The same Regge pole also dominates the s reaction with the " lines " for particles a and c reversed: $\bar{c}+b \to \bar{a}+d$. Moreover, the contribution to the two amplitudes is just about the same. When we perform the line reversal, the contribution of the Regge pole to the amplitude T_i is affected as follows.

Since the line reversal amounts to interchanging x_t and $-x_t$, the amplitude acquires the factor $\pm\varepsilon_i$, where $\pm$ is the signature of the Regge trajectory. Furthermore, if the reversed line has spin, then we may want to reverse the spin matrices in order to re-express the amplitude in terms of spin matrices that go forward from initial to final particle as before. For example, suppose both a and c are Dirac particles; then the amplitudes T_i multiply Dirac invariants. In order to obtain the coefficients of the new invariants with the line reversed, we must charge conjugate the Dirac matrices connecting a and c and new factors of $+1$ or -1 will appear [17].

Apart from these various charges of sign, the amplitudes for $a+b \to c+d$ and $a+\bar{c} \to \bar{b}+d$ at fixed t for sufficiently high s are just the same, since they are determined by the same leading Regge pole of definite signature. The difference between the two amplitudes is determined, at high energies, by the leading Regge pole with the opposite signature.

Pomeranchuk's rules are obtained by considering the special case of elastic scattering, $t = 0$, and no spin flip. The imaginary part of the amplitude is just proportional to the total cross-section and so we obtain $\sigma(p+p \to p+p) = \sigma(\bar{p}+p \to \bar{p}+p)$, etc., as $s \to \infty$ because the dominant P trajectory has signature $+1$. [If the dominant trajectory with these quantum numbers had negative signature, the asymptotic cross-sections would have to be equal and opposite !]

But we can just as easily apply the signature rule to a much more complicated case, say the amplitudes for $\pi^- + p \to K^0 + \Lambda$ and $\overline{K^0} + p \to \pi^+ + \Lambda$ at arbitrary momentum transfer t. The leading trajectory near $t = 0$ includes, at positive t, a K^* resonance; probably the resonance lying on the leading trajectory is a vector state, so that the signature is negative. [If there is no vector K^* at low energies but only a scalar one, then the signature of the leading trajectory is presumably positive.] The criterion in any case is the value of α at $t = 0$. If we define two amplitudes (T_i) as the coefficient A of unity and the coefficient B of

$$\tfrac{1}{2}(-i\gamma\cdot q_i - i\gamma\cdot q_f)$$

where q_i and q_f are initial and final meson momenta, then A has $\varepsilon = 1$ and B has $\varepsilon = -1$. [See below in Section IV.] Since we are reversing a spinless meson line, we do not have to worry about reversing spin matrices.

It would be most interesting to be able to compare experimental results for these two reactions at high energies and confirm the existence of the symmetry.

IV. APPLICATION TO π—π, π—N, AND N—N SCATTERING

Let us specialize at first to the quantum numbers of the P trajectory, $I=0$, $(-1)^J G=1$, $(-1)^J P=1$. Then there are three channels altogether for a given J: the 2π channel with $I=0$, $L=J$ and the two $N\bar{N}$ channels with $I=0$, $S=1$, and $L=J+1$ or $L=J-1$. It is convenient to define " helicity " states

$$|v\rangle \equiv \sqrt{\frac{J}{2J+1}}|L=J-1\rangle+\sqrt{\frac{J+1}{2J+1}}|L=J+1\rangle,$$

$$|w\rangle \equiv \sqrt{\frac{J+1}{2J+1}}|L=J-1\rangle-\sqrt{\frac{J}{2J+1}}|L=J+1\rangle$$

for the $N\bar{N}$ system. The three channels are thus labelled π, v, and w respectively. The relativistic scattering amplitudes are treated in terms of the reactions $\pi+\pi\rightarrow\pi+\pi$, $\pi+N\rightarrow\pi+N$, $N+N\rightarrow N+N$. For π-π scattering we have the relativistic transition amplitude $T_{\pi\pi}$ [the same contribution in all three isotopic spin states]. For πN scattering we write

$$T_{\pi N}=A_{\pi N}+B_{\pi N}\left[\frac{-i\gamma\cdot(q_i+q_f)}{2}\right]=A_{\pi N}+B_{\pi N}\ \gamma\cdot q, \tag{4.1}$$

[again the same contribution in both I states]. Finally for NN scattering we write

$$\begin{aligned}T_{NN}=h_1&+h_2[i\gamma^{(1)}\cdot p^{(2)}+i\gamma^{(2)}\cdot p^{(1)}]\\&+h_3[i\gamma^{(1)}\cdot p^{(2)}i\gamma^{(2)}\cdot p^{(1)}]\\&+h_4[i\gamma^{(1)}\cdot p^{(2)}\gamma_5^{(1)}i\gamma^{(2)}\cdot p^{(1)}\gamma_5^{(2)}]\\&+h_5\gamma_5^{(1)}\gamma_5^{(2)}\ .\end{aligned} \tag{4.2}$$

Here $p^{(1)}=\frac{1}{2}[p_i^{(1)}+p_f^{(1)}]$, $p^{(2)}=\frac{1}{2}[p_i^{(2)}+p_f^{(2)}]$, and once more we have the same contribution in both I states.

Ignoring I-spin, we have contributions to 8 amplitudes to express in terms of three coefficients, ξ_π, ξ_v, and ξ_w. Let us begin with $T_{\pi\pi}$. Connecting $T_{\pi\pi}$ with the S-matrix element in the crossed reaction, we find for this case

$$Z_J=-\frac{8\pi\sqrt{t}}{3}\frac{(2J+1)}{2ik_t}P_J(x_t) \tag{4.3}$$

so that the contribution of the Regge pole to $T_{\pi\pi}$ is

$$-\frac{8\pi\sqrt{t}}{3}\frac{(2\alpha+1)}{2ik_t}\frac{(-i\pi)}{2\sin\pi\alpha}\xi_\pi^2(t)[P_\alpha(x_t)+P_\alpha(-x_t)]\ , \tag{4.4}$$

where we have specialized to the positive signature of the P trajectory. Here $k_t=\sqrt{t/_4-m_\pi^2}$ is the C. of M. momentum in the crossed reaction and

$$x_t=-1-\frac{s}{2k_t^2}\rightarrow-\frac{s}{2k_t^2}\ \text{ as }\ s\rightarrow\infty\ . \tag{4.5}$$

Now to obtain the leading term at large s and fixed t we use the expansion (2.5). We get that the

$$\text{contribution to } T_{\pi\pi}\rightarrow\frac{1+e^{-i\pi\alpha}}{2\sin\pi\alpha}2s_0(s/s_0)^\alpha\eta_\pi^2\ , \tag{4.6}$$

where

$$\begin{aligned}\eta_\pi^2\equiv\frac{2\pi^2}{3}e^{i\pi\alpha}k_t^{-2\alpha-1}t^{\frac{1}{2}}s_0^{\alpha-1}(2\alpha+1)\Gamma(\tfrac{1}{2}+\alpha)\pi^{-\frac{1}{2}}\times\\ \times[\Gamma(1+\alpha)]^{-1}\xi_\pi^2\ .\end{aligned} \tag{4.7}$$

Here s_0 is an arbitrary quantity with the dimension of mass squared. If we put it equal to $2m_\pi^2$ and consider the P trajectory, then $\eta_\pi^2=b_{\pi\pi P\pi\pi}$ as defined in Ref.[9] and Eq. (4.6) is identical with our result given there.

By factoring out the threshold dependence $k_t^{2\alpha+1}$ of ξ_π^2 and by multiplying by $t^{\frac{1}{2}}$, we have made η_π^2 a real quantity from threshold ($t=4m_\pi^2$) down through $t=0$ and for negative t as far as α remains real. We can see that in the following way. Below threshold in t, there is, according to the Mandelstam representation, a cut for positive real s [with $x_t>1$] and one for positive real u [with $x_t<-1$]. Since $P_\alpha(x_t)$ has a cut from $-\infty$ to -1, it is reasonable to expect that the $P_\alpha(x_t)$ term in Eq. (4.4) contributes to the u cut, while the $P_\alpha(-x_t)$ term contributes to the s cut. [Actually, the various Regge terms give cuts from $x_t=1$ to ∞ and $-\infty$ to -1, which are more extensive than the Mandelstam cuts, and some cancellation must take place.] Now consider large positive s and consider only the $P_\alpha(x_t)$ term, which contributes $\frac{s_0}{\sin\pi\alpha}(s/s_0)^\alpha\eta_\pi^2$ to $T_{\pi\pi}$. Since we have omitted the term connected with the s cut, our result is real and so η_π^2 is real. In fact, taking $t=0$, using the optical theorem, and putting $\alpha_P(0)=1$ for the Pomeranchuk trajectory, we find

$$\eta_\pi^2(0)=\sigma_{\pi\pi}\,, \tag{4.8}$$

where $\sigma_{\pi\pi}$ is the asymptotic $\pi\pi$ cross-section.

Next, we treat the π-N problem, much as in Ref.[9]. Expressing A and B in terms of the S-matrix in the crossed reaction, we find [18]

$$Z_J^{v\pi A}=\frac{2\pi i}{\sqrt{6}}p_t^{-3/2}k_t^{-1/2}t^{1/2}(2J+1)P_J(x_t)\,, \tag{4.9}$$

$$Z_J^{w\pi A}=-\frac{4\pi i}{\sqrt{6}}p_t^{-3/2}k_t^{-1/2}\frac{m_N(2J+1)}{\sqrt{J(J+1)}}x_tP_J'(x_t)\,, \tag{4.10}$$

$$Z_J^{v\pi B}=0\,, \tag{4.11}$$

$$Z_J^{w\pi B}=\frac{4\pi i}{\sqrt{6}}p_t^{-1/2}k_t^{-3/2}\frac{(2J+1)}{\sqrt{J(J+1)}}P_J'(x_t)\,, \tag{4.12}$$

where

$$p_t=\sqrt{t/4-m_N^2}\quad\text{and}\quad x_t=-1-\frac{s}{2k_tp_t}\to-\frac{s}{2k_tp_t}\,.$$

Substituting into Eq. (2.3) and taking the limit of large s for fixed t, we obtain

$$\text{Contr. to } A\to\frac{1+e^{-i\pi\alpha}}{2\sin\pi\alpha}\frac{s_0}{m_N}\left(\frac{s}{s_0}\right)^\alpha\eta_\pi(\eta_1-\eta_2)\,, \tag{4.13}$$

$$\text{Contr. to } B\to\frac{1+e^{-i\pi\alpha}}{2\sin\pi\alpha}2\left(\frac{s}{s_0}\right)^{\alpha-1}\eta_\pi\eta_2\,, \tag{4.14}$$

where, if $s_0=2m_\pi m_N$, $\eta_\pi\eta_1$ corresponds to $b^{(1)}_{\pi\pi PNN}$ of Ref.[9] and $\eta_\pi\eta_2$ corresponds to $\alpha_P\, b^{(2)}_{\pi\pi PNN}$.

In terms of ξ's we have

$$\eta_\pi\eta_1=\frac{2\pi^2}{\sqrt{6}}e^{i\pi\alpha}(k_tp_t)^{-\alpha-\frac{1}{2}}m_Np_t^{-1}t^{\frac{1}{2}}s_0^{\alpha-1}(2\alpha+1)\frac{\Gamma(\frac{1}{2}+\alpha)}{\sqrt{\pi}\,\Gamma(1+\alpha)}\xi_\pi\left(\xi_v-\sqrt{\frac{\alpha}{\alpha+1}}\frac{t^{\frac{1}{2}}}{2m_N}\xi_w\right), \tag{4.15}$$

$$\eta_\pi\eta_2=\frac{2\pi^2}{\sqrt{6}}e^{i\pi\alpha}(k_tp_t)^{-\alpha-\frac{1}{2}}m_Np_t^{-1}t^{\frac{1}{2}}s_0^{\alpha-1}(2\alpha+1)\frac{\Gamma(\frac{1}{2}+\alpha)}{\sqrt{\pi}\,\Gamma(1+\alpha)}\xi_\pi\xi_w\left(-2\sqrt{\frac{\alpha}{\alpha+1}}\frac{p_t^2}{t^{\frac{1}{2}}m_N}\right). \tag{4.16}$$

Again the η's are real in a region around $t=0$. In fact we have

$$\eta_\pi(0)\eta_1(0)=\sigma_{\pi N} \tag{4.17}$$

in the asymptotic region.

For the N-N problem, we have no contribution to h_5, so there are twelve Z's corresponding to the four amplitudes h_i and the three S-matrix elements S^J_{vv}, S^J_{vw}, S^J_{ww}. These can be worked out using well-known formulae [19, 20] provided we correct a misprint in Ref.[19]. [See Plenary Section V.]

Going to large s and fixed t, we obtain

$$\text{Contr. to } h_1\to\frac{1+e^{-\pi i\alpha}}{2\sin\pi\alpha}2s_0(s/s_0)^\alpha\left(\frac{\eta_1-\eta_2}{2m_N}\right)^2 \tag{4.18}$$

$$\text{Contr. to } h_2\to\frac{1+e^{-i\pi\alpha}}{2\sin\pi\alpha}2s_0(s/s_0)^\alpha\left(\frac{-\eta_2}{s}\right)\left(\frac{\eta_1-\eta_2}{2m_N}\right), \tag{4.19}$$

$$\text{Contr. to } h_3\to\frac{1+e^{-i\pi\alpha}}{2\sin\pi\alpha}2s_0(s/s_0)^\alpha\left(\frac{\eta_2}{s}\right)^2. \tag{4.20}$$

So far we confirm the results of Gribov and Pomeranchuk [13], but they are in error in saying that the P trajectory contributes nothing to h_4. It gives

$$\text{Contr. to } h_4\to\frac{1+e^{-i\pi\alpha}}{2\sin\pi\alpha}2s_0(s/s_0)^\alpha\left(\frac{-t}{2s\alpha}\right)\left(\frac{\eta_2}{s}\right)^2. \tag{4.21}$$

Although this term is rather small at high energies, it is important in the discussion of the ghost problem in Section 4. Note the factor α^{-1}.

We remark that $\eta_1^2(0)=\sigma_{NN}$ asymptotically, so that the rule [10, 11] $\sigma^2_{\pi N}=\sigma_{\pi\pi}\sigma_{NN}$ is confirmed.

V. A SIMPLER TREATMENT OF THE $N\bar{N}$ CHANNELS

For what follows, we need not make use of invariant amplitudes for N-$\bar{N}$ scattering, free of kinematic singularities and easy to deal with in the s reaction

$N+N \to N+N$. Thus we can dispense with the complicated h's and make use of the four simple amplitudes defined in Ref.[19] for n-p scattering, but used here for N-$\bar{N}$ scattering with $I=0$. In the notation of Ref.[19], we use T_{11}, $T_{1,-1}$, $\frac{T_{10}-T_{01}}{\sqrt{2}}$, and T_{00}.

We now introduce two trajectories, the Pomeranchuk trajectory P and another, called Q, with negative signature and with all the same quantum numbers except that $(-1)^J P=-1$ instead of $+1$. Instead of the helicity states v and w, we go back to the partial wave states $|L=J-1\rangle$ and $|L=J+1\rangle$, called $-$ and $+$ respectively, and we add the state $|L=J\rangle$, called 0, to be connected with the Q trajectory. For the P trajectory we use

$$\xi_- = \sqrt{\frac{\alpha_P}{2\alpha_P+1}}\,\xi_v + \sqrt{\frac{\alpha_P+1}{2\alpha_P+1}}\,\xi_w$$

and

$$\xi_+ = \sqrt{\frac{\alpha_P+1}{2\alpha_P+1}}\,\xi_v - \sqrt{\frac{\alpha_P}{2\alpha_P+1}}\,\xi_w$$

instead of ξ_v and ξ_w. For the Q trajectory, there is just ξ_0.

We now make use of Eq. (B.11) of Ref.[19], correcting the misprint in the equation for $T_{1,-1}$: the denominator of the first term should be $L+1$, not $L+2$. For the contributions of the two Regge trajectories to the T's, we have just the following [without bothering to symmetrize in x_t according to the signature]:

$$\text{Contr. to } T_{11} = -\frac{\pi}{2p_t}\frac{1}{\sin\pi\alpha_P}\Big[\xi_-^2(\alpha_P+1)P_{\alpha_P-1}+\xi_+^2\alpha_P P_{\alpha_P+1}-\xi_+\xi_-\sqrt{\alpha_P(\alpha_P+1)}\,(P_{\alpha_P+1}+P_{\alpha_P+1})\Big] - \frac{\pi}{2p_t}\frac{1}{\sin\pi\alpha_Q}\big[\xi_0^2(2\alpha_Q+1)P_{\alpha_Q}\big], \quad (5.1)$$

$$\text{Contr. to } T_{1,-1} = -\frac{\pi}{2p_t}\frac{1}{\sin\pi\alpha_P}\Big[\frac{\xi_-^2}{\alpha_P}P''_{\alpha_P-1}+\frac{\xi_+^2}{\alpha_P+1}P''_{\alpha_P+1}-\frac{\xi_+\xi_-}{\sqrt{\alpha_P(\alpha_P+1)}}(P''_{\alpha_P+1}+P''_{\alpha_P-1})\Big] - \frac{\pi}{2p_t}\frac{1}{\sin\pi\alpha_Q}\Big[\frac{-\xi_0^2(2\alpha_Q+1)}{\alpha_Q(\alpha_Q+1)}P''_{\alpha_Q}\Big], \quad (5.2)$$

$$\text{Contr. to } \frac{T_{10}-T_{01}}{\sqrt{2}} = -\frac{\pi}{2p_t}\frac{1}{\sin\pi\alpha_P}\Big[\xi_-^2\frac{(2\alpha_P+1)}{\alpha_P}P'_{\alpha_P-1}-\frac{\xi_+^2}{\alpha_P+1}(2\alpha_P+1)P'_{\alpha_P+1}\Big] - \frac{\pi}{2p_t}\frac{1}{\sin\pi\alpha_Q}\Big[-\frac{\xi_0^2(2\alpha_Q+1)}{\alpha_Q(\alpha_Q+1)}P'_{\alpha_Q}\Big], \quad (5.3)$$

$$\text{Contr. to } T_{00} = -\frac{\pi}{2p_t}\frac{2}{\sin\pi\alpha_P}\big[\xi_-^2\alpha_P P_{\alpha_P-1}+\xi_+^2(\alpha_P+1)P_{\alpha_P+1}+\xi_+\xi_-\sqrt{\alpha_P(\alpha_P+1)}(P_{\alpha_P-1}+P_{\alpha_P+1})\big]. \quad (5.4)$$

The Q trajectory is not coupled to the 2π channel; for the P trajectory we retain our definition of ξ_π.

The advantage of using the formulae of this Section is that they are complete, while in Section 4 the formulae for N-N scattering are only asymptotic. Using the matrices of Ref.[20], one can obtain complete formulae for the h's of Section 4, but they are rather complicated.

VI. THE ABSENCE OF THE GHOST

Let us assume that α_P passes through zero at a negative value of t. Why is there not a pole in each of the scattering amplitudes at this value of t and thus a ghost? Let us first note a set of mathematical conditions that will avoid the ghost. Although these conditions may look remarkable, we shall see, by looking at a Schrödinger equation problem with several channels and spin, that they are quite ordinary. A large class of all trajectories obey them, and it is a purely dynamical matter whether the leading trajectory belongs to the class. Similar mathematical reasons [21] underlie the absence, in many nuclear rotational series, of the state or states of lowest J.

First, let us return to Section 4 and look at the leading contributions to $T_{\pi\pi}$, $A_{\pi N}$, $B_{\pi N}$, h_1, h_2, and h_3, ignoring h_4 for the moment. We see that if, as $\alpha_P \to 0$, we were to have $\xi_\pi \to 0$ like $\sqrt{\alpha_P}$ and $\xi_v \to 0$ like $\sqrt{\alpha_P}$, while $\xi_w \to$ const., then all terms in η_π^2, $\eta_\pi \eta_1$, $\eta_\pi \eta_2$, $(\eta_1-\eta_2)^2$, $\eta_2(\eta_1-\eta_2)$, and η_2^2 would $\to 0$ like α_P, so that all the amplitudes under discussion would have a factor α_P to cancel the pole in $\dfrac{1}{\sin \pi\alpha_P}$ and there would be no ghost.

Now, in the notation of Section 5, we note that as $\alpha_P \to 0$, $\xi_- \leftrightarrow \xi_w$ and $\xi_+ \leftrightarrow \xi_v$. Let us consider the exact contribution to T_{00} in Eq. (5.4). If $\xi_- \to$ const. and $\xi_+ \propto \sqrt{\alpha_P}$ as $\alpha_P \to 0$, then there is in each term a factor α_P to cancel the pole of $\dfrac{1}{\sin \pi\alpha_P}$.

Next, in Section 4, we turn to the amplitude h_4. Here we see that the above conditions are insufficient. Because of the additional factor of $\dfrac{1}{\alpha_P}$, we would still have a ghost. Another apparently remarkable condition is needed. We can see what that is by looking at Eqs. (5.1)-(5.3). Suppose that as $\alpha_P \to 0$, the Q trajectory has $\alpha_Q \to -1$. Moreover, we suppose that at the same value of t, the quantity

$$\frac{\xi_0^2}{1+\alpha_Q}+\frac{\xi_-^2}{\alpha_P} \quad \text{or} \quad \frac{\xi_0^2}{1+\alpha_Q}+\frac{\xi_w^2}{\alpha_P}$$

remains finite. In Eq. (5.1), for example, we would have, in the limit,

$$\frac{-\pi}{2p_t}\left[\frac{\xi_-^2}{\pi\alpha_P}+\frac{\xi_0^2}{\pi(1+\alpha_Q)}\right]P.$$

The condition that

$$\frac{\xi_0^2}{1+\alpha_Q}+\frac{\xi_-^2}{\alpha_P} \to \text{const.}$$

is sufficient to remove the pole. The same is true of Eqs. (5.2) and (5.3). Evidently, when we go to the asymptotic limit of Section 4, the obnoxious term in h_4 is taken care of, along with everything else.

Now let us understand how these miracles happen. We consider a system of coupled Schrödinger equations including two-body channels like the 2π and $N\bar{N}$ channels under consideration. Take a Regge trajectory like P, which has a wave function in the $\pi\pi$ channel and in the triplet $N\bar{N}$ channels with $L=J+1$ and with $L=J-1$. The asymptotic wave functions are proportional to ξ_π, ξ_+, and ξ_- respectively. The three channels are coupled together by forces; for example the tensor force connects the $+$ and $-$ channels. At $J=1, 2, 3, \ldots$, both the $+$ and $-$ cases correspond to physical situations [e.g., $^3F_2$ and $^3P_2$, $^3D_1$ and $^3S_1$], but at $J=0$ only the $+$ channel is physical ($^3P_0$). If we introduce the notation "a-wave" for $L=-1$ [corresponding to s wave for $L=0$, etc.], then the $-$ channel at $J=0$ gives the physically meaningless state $^3A_0$.

Now at all values of $J>0$, whether integral or not, the three channels couple together and the wave functions mix. However, as $J \to 0$, that is no longer the case. The tensor force matrix element, for example, goes like $\dfrac{\sqrt{J(J+1)}}{2J+1}$. Similarly the coupling between the $-$ channel and the $\pi\pi$ channel goes to zero like $\sqrt{J}$. Thus at $J=0$ the nonsense channel $^3A_0$ and the sensible channels ($^3P_0$ and $\pi\pi$ s-wave) become decoupled. A given Regge trajectory, therefore, as $\alpha \to 0$, becomes either pure "sense" or pure "nonsense". If there *is* a $J=0^+$ state, then $\xi_- \to 0$ like $\sqrt{\alpha}$ and ξ_+, $\xi_\pi \to$ const. The other type of trajectory, which is just as common, chooses nonsense at $\alpha=0$, so that $\xi_- \to$ const. and ξ_+, $\xi_\pi \propto \sqrt{\alpha}$. The P trajectory, if α_P passes through zero, is evidently of the latter type.

Next, we explain the other miracle, the cancellation between P and Q trajectories. We look at the S-matrix elements in two different channels, one relevant to the P trajectory and the other to Q, namely $S_{--}^{J_1}$ and $S_{00}^{J_2}$. As $J_1 \to 0$ and $J_2 \to -1$, these matrix elements become equal. [Both are, of course, unphysical.] The proof can be accomplished in several ways. First, for the Schrödinger equation, we notice that $L(L+1)$ and $\mathbf{L}\cdot\mathbf{S}$ become the same for the two channels as $J_1 \to 0$ and $J_2 \to -1$. We have seen, too, that the tensor force and the coupling to the $\pi\pi$ state disappear. In fact, all the forces in the two channels become identical. So do the orbital angular momenta, both approaching -1.

In the general case, including the relativistic problem, we have only to generalize Froissart's [22] definition of the analytically continued S-matrix to our problem to see that the definitions of S_{--}^0 and S_{00}^{-1} are the same. Thus $e^{-i\pi(J_1+\frac{1}{2})}S_{--}^{J_1}$ and $e^{-i\pi(J_2+\frac{1}{2})}S_{00}^{J_2}$ approach equal and opposite values as $J_1 \to 0$, $J_2 \to -1$. A pole in the first quantity of the form $\xi_-^2(J_1-\alpha_P)^{-1}$ requires a pole in the second of the form $\xi_0^2(J_2-\alpha_Q)^{-1}$ with

$$\frac{\xi_-^2}{\alpha_P}+\frac{\xi_0^2}{1+\alpha_Q}\to \text{const.}$$

as $\alpha_P\to 0$ and $\alpha_Q\to -1$. The second miracle is explained.

Here we are dealing with $J_2<-\frac{1}{2}$ and the question of twin poles becomes really important for the first time. If the Q trajectory really possesses a twin near the value of t we are discussing, then the twin passes through 0 as α_Q passes through -1. The contribution of the twin to h_4, for example, is one order higher in x_t than that of the leading term due to the Q pole itself. The coefficient of this new contribution is proportional to α_Q, cancelling $\sin \pi\alpha_Q$ in the denominator; thus there is no ghost, but there is a term in h_4 larger by one power of s than that given in Eq. (4.21).

The generalization of our story to more complicated crossed channels with higher spin and with many particles is very interesting and gives rise to a situation closely resembling that in nuclei, where the Regge trajectories are familiar as series of " rotational " levels.

APPENDIX

Regge Poles and Nuclear Scattering

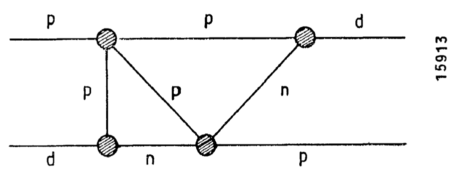
If all the strongly interacting particles lie on Regge trajectories, then there is nothing to distinguish " elementary " particles like nucleons from obviously composite systems like nuclei, except that the latter possess very prominent " anomalous singularities ", corresponding to a spatial extension of their wave functions greatly exceeding the Compton wave lengths of the particles involved. What effect do these singularities have on diffraction scattering and total cross-sections at high energies?

In the scattering of nucleons from nuclei at several GeV, it is apparent that the absorption cross-sections are geometrical and that for scattering of two nuclei with radii R and R' we would obtain a total cross-section of the order of $2\pi(R+R')^2$.

But if we assume that high-energy scattering amplitudes are dominated by the exchange of the P trajectory, then we obtain the factoring property for total cross-sections. There must then be a slow transition from the apparently constant nuclear cross-sections at moderate energies to true asymptotic cross-sections at very high energies, the latter being factorable.

Udgaonkar and Gell-Mann [14] investigated the consequences of a very crude model in which the nucleus is treated as a collection of nucleons and the semi-classical approximation is made. Since each nucleon becomes larger and more transparent at higher energies, eventually the nucleus-nucleus cross-section tends to $AA'\sigma_{NN}$, where A and A' are the atomic numbers of the collision partners. This picture gives an illustration of factorability at very high energies.

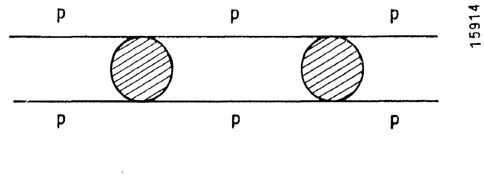
The slow transition is accomplished by having a cut in the angular momentum plane as well as the Pomeranchuk pole, so that besides the asymptotic constant cross-section there are terms in $1/\ln s$, etc. These " eclipse " terms, which eventually vanish at very high energies, come from dispersion diagrams like the following (for p-d scattering):



Here we are dealing with the simple eclipse of neutron by proton and vice versa in the deuteron.

It would be attractive to suppose that the cuts in J due to the eclipse terms are exactly cancelled by cuts from other dispersion diagrams, leaving only Regge poles, even in the case of anomalous singularities. The total cross-sections at moderate energies would then be roughly the asymptotic ones. But how can such a situation be reconciled with the factoring property?

In the absence of anomalous singularities, Amati and Fubini [23] have obtained cuts in J simply from diagrams like



but in that case it is perfectly possible that they cancel against cuts from the other diagrams, leaving just Regge poles.

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DISCUSSION

WEINBERG: Does the vanishing of the denominator $2J+1$ at $J = -\frac{1}{2}$ have anything to do with the difficulty of analytically continuing from $J > -\frac{1}{2}$ to $J < -\frac{1}{2}$?

GELL-MANN: Probably.