A STUDY OF SPHERICALLY SYMMETRIC SPACE-TIMES IN EINSTEIN-CARTAN THEORY OF GRAVITATION

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BY

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DECLARATION BY THE STUDENT

I hereby declare that the dissertation entitled "**A Study of Spherically Symmetric Space-times in Einstein-Cartan theory of Gravitation**" completed and written by me has not formed earlier the basis for the award of any degree or other similar title of this or any other University or examination body.

Research student

Place: Kolhapur

Date:22/01/2019

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DECLARATION BY GUIDE

This is to certify that the thesis entitled "**A Study of Spherically Symmetric Space-times in Einstein-Cartan theory of Gravitation**" which is being submitted herewith for the award of the Degree of **Doctor of Philosophy** in **Mathematics** under the Faculty of Science of Shivaji University, Kolhapur is the result of the original research work completed by **Dadasaheb Rajaram Phadatare** under my supervision and guidance and to the best of my knowledge and belief the work embodied in this thesis has not formed earlier the basis for the award of any Degree or similar title of this or any other University or examining body.

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PREFACE

Einstien's general theory of relativity is one of the most beautiful structures of theoretical physics which describes the mysterious gravitational force in terms of geometry. The success of general theory of relativity is well known as it has passed every unambiguous test both experimentally and observationally. The recent detection of gravitational waves in the space-time as was predicted by Einstein 100 years before cemented the status of general relativity, besides other confirmations of Einstein's predictions of deflection of a ray of light and the perihelion advances of the planet Mercury in the gravitational field of the Sun. However, in spite of all the embracing characters and widely recognized string of success of Einstein's general theory of relativity, it is considered to be still inadequate in the sense that it does not satisfy certain desiderata of the theory of gravitation. For example, it does not incorporate the intrinsic spin of the gravitating matter, it is not free from singularities, it does not incorporate Mach principle etc. Hence, there was a hope that there may be something beyond the Einstein's general theory of relativity yet to be found. To address such issues, several theories of gravitation have been proposed as alternatives to Einstein's general theory of relativity with the hope that the modified theories may satisfy the desiderata of the theory of gravitation. Any new theory of gravitation should be more general and better than the Einstein's general theory of relativity and will have to include the general theory of relativity within itself, satisfying the desiderata for the theory of gravitation, explaining the four fundamental interactions (strong interaction, electromagnetic interaction, weak interaction and gravitational interaction) of nature and their interdependence. None of the theory so far discovered has been completely successful, in the sense that none satisfies the desiderata for the theory of gravitation but all these modified theories of gravitation have gained the attraction of researchers due to many reasons, such as incorporation of intrinsic spin of gravitating matters, adaptability of quantum physics, understanding of Mach principle etc. and good amount of work has been done in these theories in the last more than four decades. Einstein-Cartan theory of gravitation is one such modified theory of gravitation, proposed by Cartan in 1923, by introducing spin- an intrinsic feature of gravitating matter, in the theory. In recent years the Einstein-Cartan theory of gravitation has geared up in receiving the wide attention of researchers to study the role of intrinsic spin of gravitating matter and to study some exact solutions of the field equations of Einstein-Cartan theory of gravitation. The EC theory of gravitation is obtained from Einstein's general relativity by modifying the underlying Reimannian geometry; in which the connections are not symmetric but are asymmetric in character. Hence the underlying geometry of the EC theory of gravitation is non-Riemann due to asymmetric connections arising from the presence of torsion in the space-time. The Riemannian Curvature tensor of a non-Riemannian space, hence forth referred to it as the Riemann-Cartan Curvature tensor. A non-Riemannian space with asymmetric connections is characterized by the metric

$$ds^2 = g_{ij} dx^i dx^j,$$

with Riemann-Cartan Curvature tensor R_{hijk} satisfying the properties

$$\begin{aligned} & R_{hijk} = -R_{ihjk} = -R_{hikj} \ , \\ & R_{hijl} \neq R_{jhli} \ , \\ & R_{hijk} + R_{hjki} + R_{hkij} \neq 0 \ , \\ & R_{hijk;l} + R_{hikl;j} + R_{hilj;k} \neq 0 \ , \end{aligned}$$

where semi colon (;) denotes the covariant derivative on a non-Riemann space with respect to the asymmetric connections, and for a covariant vector A_i, it is defined as

$$A_{i;j} = A_{i,j} - A_k \Gamma^k_{ji} \ ,$$

where $\Gamma^k_{ji} = \{^{k}_{ij}\} - K_{ij}{}^k$, are the components of the asymmetric connections, $\{^{k}_{ij}\}$- are the components of the symmetric Christoffel symbols and $K_{ij}{}^k$ are the components of the contortion tensor satisfying the property

$$K_{i(jk)} = 0 \ .$$

The contortion tensor K_{ijk} can be decomposed in to torsion tensor Q_{ijk} as

$$K_{ijk} = -Q_{ijk} + Q_{jki} - Q_{kij} \ .$$

Conversely

$$Q_{ij}{}^{k} = -\frac{1}{2}(K_{ij}{}^{k} - K_{ji}{}^{k}) \ .$$

The Einstein-Cartan theory of gravitation reduces to Einstein's theory of gravitation in the absence of torsion. There are several investigations who have investigated and studied several aspects of Einstein-Cartan theory of gravitation. Some of them includes Tolman [130], Hehl and his collaborators [50, 51], Trautman [125, 126, 127, 128, 129], Kuchowicz [77, 78, 79, 80, 81, 82], Kerlick [67, 68], Prasanna [100, 101], Kibble [69], Sciama [111], Singh and Yadav [118, 119], Yadav and Prasad [138], Sharif and Iqbal [113], Katkar [58, 59, 61] Katkar and Patil [60], Katkar and Phadatare [63] and many more.

The thesis entitled "**A Study of Spherically Symmetric Space-times in Einstein-Cartan theory of Gravitation** "comprises six chapters and deals with the study of geometry of the non-Riemannian space and the study of some exact solutions of field equations when Weyessenhoff fluid is the source of gravitation in the Einstein-Cartan theory of gravitation. In order to make the thesis self explanatory, we are presenting the review of the concepts, the mathematical tools of differential forms which form the background of our investigations in the research work carried out in the thesis.

Mathematical Techniques of Differential Forms:

Techniques of differential forms is another useful and most powerful analytical tool of modern physics than the old tensor techniques. Instead of forty christoffel symbols in tensor approach, there are only six complex connection 1-forms. The use of the techniques of differential form is well known in the literature as it reduces the complexity of computations. Katkar [61] has extended this technique on a non-Riemannian space to study the geometry of a non-Riemannian space. We use d_* to denote the exterior derivative in the non-Riemannian space-time of EC theory of gravitation. This exterior covariant derivative operator d_* is connection dependent and hence obtained by taking the covariant derivative with respect to the asymmetric connections of a differential form. It satisfies all properties of the exterior derivative operator 'd' of Riemannian space-time except the vanishing of repeated exterior derivative d of a form of any degree.

The operator d_* on a non-Riemannian space converts r-form to $r+1$-form. It is defined as

$$d_* : \wedge^r T_p^* \to \wedge^{r+1} T_p^* \ ,$$

by

$$d_*\tilde{\omega} = \omega_{i_1 i_2 \dots i_{r;k}} d_*x^k \wedge d_*x^{i_1} \wedge d_*x^{i_2} \wedge \dots \wedge d_*x^{i_r} -$$
$$- \omega_{i_1 i_2 \dots i_r} \left[\sum_{p=1}^{r} (-1)^{p-1} d_*x^{i_1} \wedge \dots \wedge d_*x^{i_{p-1}} \wedge d_*^2 x^{i_p} \wedge \dots \wedge d_*x^{i_r} \right] , \tag{1}$$

for any $\tilde{\omega} \in \wedge^r T_p^*$. The operator d_* satisfies the following properties

$$(i) \quad d_*f = f_{,i} d_*x^i \ , \tag{2}$$
$$(ii) \quad d_*(\tilde{\omega} + \tilde{\sigma}) = d_*\tilde{\omega} + d_*\tilde{\sigma} \ , \tag{3}$$
$$(iii) \quad d_*(fg) = d_*f \cdot g + f \cdot d_*g \ , \tag{4}$$
$$(iv) \quad d_*(\tilde{\omega} \wedge \tilde{\sigma}) = d_*\tilde{\omega} \wedge \tilde{\sigma} + (-1)^{\text{deg of } \tilde{\omega}} \tilde{\omega} \wedge d_*\tilde{\sigma} \ , \tag{5}$$
$$(v) \quad d_*(f\tilde{\omega}) = d_*f \wedge \tilde{\omega} + f d_*\tilde{\omega} \ , \tag{6}$$
$$(vi) \quad d_*(d_*\tilde{\omega}) \neq 0 \ , \text{for any form } \tilde{\omega}, \text{ of degree } r \geq 0 \ , \tag{7}$$
$$(vii) \quad d_*(d_*f \wedge d_*g) = d_*^2 f \wedge d_*g - d_*f \wedge d_*^2 g \ . \tag{8}$$

Thus for a 0-form f, we obtain 1-form d_*f as

$$d_*f = f_{;i} d_*x^i \ ,$$

where for a differential function f, we have

$$f_{;i} = f_{/i} = f_{,i} \ .$$

Thus in the case of a scalar function f, we have $d_*f = df$, and the coordinate differentials d_*x^i form a basis of a space of 1-forms, such that $d_*x^i \wedge d_*x^i = 0$.

Using the definition (1) we find

$$d_*^2 f = -f_{;ij} d_* x^i \wedge d_* x^j - f_{;i} d_*^2 x^i \ . \tag{9}$$

Interchanging $i \leftrightarrow j$ in the equation (9) we get

$$d_*^2 f = f_{;ji} d_* x^i \wedge d_* x^j - f_{;i} d_*^2 x^i \ . \tag{10}$$

Adding equations (9) and (10) we get

$$d_*^2 f = -\frac{1}{2}(f_{;ij} - f_{;ji}) d_* x^i \wedge d_* x^j - f_{;k} d_*^2 x^k \ , \tag{11}$$

where we have

$$f_{;ij} - f_{;ji} = 2 f_{;k} Q_{ij}{}^k \ . \tag{12}$$

Hence equation (11) becomes

$$d_*^2 f = -f_{;k} Q_{ij}{}^k d_* x^i \wedge d_* x^j - f_{;k} d_*^2 x^k \ . \tag{13}$$

If f is taken as a coordinate function x^i, then we obtain from equation (13)

$$d_*^2 x^k = -\frac{1}{2} Q_{ij}{}^k d_* x^i \wedge d_* x^j \ . \tag{14}$$

Substituting this in the equation (13) we get

$$d_*^2 f = -\frac{1}{2} f_{;k} Q_{ij}{}^k d_* x^i \wedge d_* x^j \ . \tag{15}$$

We also used a very familiar Newman-Penrose [89] null tetrad formalism and its extension by Jogia and Griffiths [55], especially to find the solutions of the field equations in the Einstein-Cartan theory of gravitation. The approach involves a complex null tetrad consisting of four complex null vector fields

$$e_{(\alpha)}{}^{i} = (l^i, n^i, m^i, \overline{m}^i) \ ,$$

where l^i and n^i are real vector fields and m^i and $\overline{m}^i$ are complex conjugates of each other form a basis at each point of the space-time. The tetrad of the dual basis vector fields is given by

$$e^{(\alpha)}{}_{i} = (n_i, l_i, -\overline{m}_i, -m_i) \ .$$

The basis vectors of the tetrad satisfy the orthonormality conditions

$$l_i n^i = -m_i \overline{m}^i = 1 \ ,$$

and all other inner products are zero.

In our investigations we utilize tetrad components, as they make field equations more transparent, instead of their tensor components. Any vector or a tensor of any rank is expressed as a linear combinations of its tetrad components and conversely. For example

$$\begin{aligned} A_i =& A_\alpha e^{(\alpha)}{}_i \ , \\ A_{ij} =& A_{\alpha\beta} e^{(\alpha)}{}_i e^{(\beta)}{}_j \ , \end{aligned}$$

and conversely, $A_{\alpha\beta} = A_{ij} e_{(\alpha)}{}^i e_{(\beta)}{}^j$ and so on.

Hence the tetrad components of the equation (15) becomes

$$d_*^2 f = -\frac{1}{2} f_{;\sigma} Q_{\alpha\beta}{}^{\sigma} \theta^\alpha \wedge \theta^\beta \ ,$$

where $\theta^\alpha = e^{(\alpha)}{}_i d_* x^i$ is a tetrad basis of the coordinate differentials $d_* x^i$, which form a basis of the space of 1-forms.

In a non-Riemannian space, Katkar [61] has obtain the Cartan's equations of structure, which are used to study the essence of non-Riemannian geometry and are thoroughly used throughout the thesis to facilitate the complex computation of the components of the Riemann-Cartan curvature tensor. These equations are given by

$$\begin{aligned} d_*\theta^\alpha =& -\omega^{0\alpha}{}_\beta \wedge \theta^\beta - \frac{1}{2} Q_{\beta\sigma}{}^\alpha \theta^\beta \wedge \theta^\sigma \ , \\ \Omega^\alpha{}_\beta =& d_*\omega^\alpha{}_\beta + \omega^\alpha{}_\epsilon \wedge \omega^\epsilon{}_\beta + \frac{1}{2} \gamma^\alpha{}_{\beta\sigma} Q_{\epsilon\delta}{}^\sigma \theta^\epsilon \wedge \theta^\delta \ , \end{aligned}$$

where

$$\omega^{0\alpha}{}_\beta = \gamma^{0\alpha}{}_{\beta\sigma} \theta^\sigma \ ,$$

are the tetrad components of the connection 1-form in a Riemannian space-time of Einstein's general theory of relativity; and $\gamma^{0\alpha}{}_{\beta\sigma}$ are the corresponding components of the Ricci's coefficients of rotation; and $\omega^\alpha{}_\beta$, $\Omega^\alpha{}_\beta$ are the tetrad components of connection 1-form and curvature 2-form respectively in the non-Riemannian space-time. They are defined by

$$\omega^\alpha{}_\beta = \gamma^\alpha{}_{\beta\sigma} \theta^\sigma \ ,$$

and

$$\Omega^\alpha{}_\beta = -\frac{1}{2} R_{\delta\epsilon\beta}{}^\alpha \theta^\delta \wedge \theta^\epsilon .$$

Contents of Chapters:

Chapter wise investigations and results obtained in the thesis are presented below.

In the **Chapter 1** review of the literature and some basic concepts, mathematical techniques exploited in the thesis exhibited. In particular, results of the Einstein's general theory of relativity and the techniques of differential form are presented. The chapter is introductory and no original results are claimed in this chapter. Remaining five chapters contain some original results.

In the **Chapter 2**, the inevitability of geometry in the development of theory of gravitation is portrated. Vector identities and their invariance in different theories of gravitation are accomplished. A technique of differential forms, developed by Katkar [61] on a non-Riemannian space, is presented. A formula for the curvature of a non-Riemannian space is derived. A non-Riemannian 2- space is constructed and its curvature is obtained. The results are corroborated by employing the techniques of differential forms on a non-Riemannian space. Maxwell's equations in a more general form are derived.

General relativity has been considered as one of the most difficult subject due to a great deal of complex mathematics. The complexity of the mathematics reflects the complexity of describing space-time curvature and some conceptual issues which are present and even more opaque in the physical 4- dimensions world. Hence in order to gain insight in to these difficult conceptual issues Deser et al. [23, 24, 25] in a series of papers, Giddings et al. [36], and Gott et al. [39, 40] have examined general relativity in lower dimensional spaces and explored some solutions. Studies of general relativity in lower dimensional space-times have proved that solving Einstein's field equations of general relativity in a space-time of reduced dimensionality is rather simple but yields some amusing results that are pedagogical and scientific interests and yet are apparently unfamiliar to most physicists.

In the **Chapter 3**, we study Einstein-Cartan theory of relativity in a 2-dimensional non-Riemannian space. An exposition of a new dyad formalism, consisting of two real null vector fields is developed and employed to construct a 2-dimensional non-Riemannian space. It is claimed that the 2-dimensional non-Riemannian space contains no matter at all; so that there is no gravitational field either.

It is shown that the torsion influences the curvature of the 2-dimensional non-Riemannian space. The results are corroborated by employing the techniques of differential form developed by Katkar [61].

The field equations of EC theory of gravitation are given by Hehl et al. [50, 51]

$$R_{ij} - \frac{1}{2}Rg_{ij} = -kt_{ij} \ , \tag{1}$$

and $$Q_{ij}{}^{k} + \delta_i{}^{k}Q_{jl}{}^{l} - \delta_j{}^{k}Q_{il}{}^{l} = kS_{ij}{}^{k} \ , \tag{2}$$

where ${S_{ij}}^k$ is the spin angular momentum tensor. The spin density of matter is described by an anti-symmetric tensor S_{ij} and is related to the source of torsion according to the equation

$$ {S_{ij}}^k = S_{ij}u^k \ . \tag{3} $$

Frankel condition requires the intrinsic spin of a matter field to be space-like in the rest frame of the fluid. This yields

$$ S_{ij}u^j = 0 \ . \tag{4} $$

The condition (4) implies that the torsion trace vanishes identically and hence the field equations (2) reduces to an algebraic coupling between spin and torsion according to

$$ {Q_{ij}}^k = kS_{ij}u^k \ . \tag{5} $$

In the **Chapter 4**, we consider the non-static spherically symmetric metric in the form

$$ ds^2 = e^{2\nu}dt^2 - e^{2\lambda}dr^2 - B^2(d\theta^2 + sin^2\theta d\phi^2) \ , \tag{6} $$

and the field equations (1) and (2) are solved by using the techniques of differential form on a non-Riemannian space, when Weyssenhoff fluid is the source of gravitation and spin. Two classes A and B of different solutions of the field equations in the EC theory of gravitation are obtained when the Weyssenhoff fluid is the source of gravitation and spin. Many of the previously known solutions for Weyssenhoff fluid in EC theory of gravitation have zero acceleration and vorticity (Kuchowicz [82]). Griffiths and Jogia [42] have claimed some non-zero accelerated solutions. In this chapter we have applied the techniques of differential forms and a class A of non-static solutions with zero acceleration and a class B with non-zero acceleration are obtained. Class A solutions are expanding, shearing and rotating, while the class B solutions are rotating. In class A solutions, the non-zero kinematical parameters, the pressure and the density diverge to infinity, and vanish together at $t = 0$ and at large t respectively. Similar phenomenon is observed in class B solutions at $r = 0$ and at large r respectively. It can be seen that the rotation, the pressure and the density are influenced by the spin of the fluid, while there is no such effect on the expansion, acceleration and the shear. In the absence of the spin our result coincides with the result obtained by Sharif and Iqbal [[113]], and the solution is irrotational.

In the **Chapter 5**, a static spherically symmetric space-time described by the metric

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \ , \qquad (6)$$

is considered and the solution of the field equations when Weyssenhoff fluid is the source of gravitation and spin is obtained. The solution is proved to be rotating with non-zero acceleration, but zero expansion and shear and it is free from singularity. The solution is proved to be of Petrov-type D. Our solution matches with the solution obtained by Prasanna [100] in the absence of the spin.

The **Chapter 6** is devoted to the investigation of solutions of the field equations of EC theory of gravitation when the Weyssenhoff fluid is the source of gravitation. In general, the non-static spherically symmetric solution is expanding, accelerating and rotating but non-shearing. However, the dynamic solution is proved to be expanding and rotating with zero acceleration and shear, where as static solution reduces to the solution obtained by Katkar and Patil [60]. This solution is accelerating and rotating with expansion free and shear free. We see that the spin of the gravitating matter influences the geometry of space-times. The solutions are all Petrov type D.

The following terminologies are used in the presentation of the entire thesis.

1. Equations are numbered by (Chapter number.section.equation number), e.g. (3.2.5) indicates fifth equation in second section of third chapter.

2. References are listed at the end alphabetically and are referred in the text shown in the square bracket.

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Contents

Chapter 1

Theories of Gravitation

1.1 Introduction

"The most valuable theory of my life...

The theory is of incomparable beauty " .

... **Albert Einstein 1915**

It is reported that there are more than forty theories of gravitation. Out of these only the one propounded by Einstein in 1915 is considered as the most popular and successful theory, because of the apparent reason that it has been verified both experimentally and observationally. To know the ingenious work of Einstein, let us digress for a while to see how the concept of our universe has been changing in keeping with the pace of civilization.

Aristotle (390-332 B. C.):

A Greek philosopher and one of the most intellectual leaders of the 4th Century B.C.-Aristotle, only on the basis of experience and commonsense thought that the Earth was stationary at the centre of the universe and the Sun, the Moon, the planets and the stars moved in a circular orbits about the Earth, thus explaining why the Sun and the stars systematically rise in the east and set in the west. Until the 17th century Aristotle's idea came to be regarded as absolute truth. Copernicus (1473-1543) was the first man who challenged the 'geocentric theory'of Aristotle and proposed an exactly a rival frame work called the 'heliocentric theory'. Nearly a century passed before this idea was taken seriously, two astronomers- the German, Johannes Kepler and Italian, Galileo Galilee started publicly to support Copernicus theory. Only on the basis of observations Kepler modified Copernicus's theory, suggesting that the planets moved in elliptical orbit and discovered the exact laws behind the movements of the planets. These laws are stated as follows:

- The orbit of a planet is an ellipse.
- Arial velocity of the planet is constant.
- The square of the period of revolution of the planet is directly proportional to the cube of the semi major axis of the ellipse.

But he did not find the cause responsible for such a well-defined movement. However, he put forward lot many observational data before Newton to find the exact reasons for the well-defined movement of planets.

Newton (1642-1727):

Newton had a great belief that natural phenomena take place according to some definite rules and those rules are well understood. He invented calculus-the branch of mathematics and started using for the description of laws of nature. He defined 3-dimensions space and 1-dimension time independent of each other, and assumed that space, time and mass are absolute, that they are independent of the position of the observers whether at rest or moving with uniform velocity relative to each other and establish a frame of reference with respect to which he studied the laws of nature. The most original contributions of Isaac Newton which essentially laid the foundations of physics were

(*i*) the laws of motion, which describe how matter moves under the action of force and

(*ii*) the law of gravitation, which gives the force of attraction between two masses.

The greatness of Newton was that he formulated these laws mathematically. The Newton's laws of motion are described by the equation

$$\frac{d^2x}{dt^2} = g \ . \tag{1.1.1}$$

This equation is referred as the Newton's equation of motion for a particle falling freely under gravity. As the equation (1.1.1) is independent of mass of a body, which implies that all bodies fall exactly in the same way in the gravitational field, establishing gravitational field is truly a democratic force, it pulls every thing equally irrespective of its mass and composition. Galileo experimentally verified this by dropping two bodies of same size but with different composition from Pisa Tower.

Newton also expressed the law of gravitational attraction between two masses m_1 and m_2, as directly proportional to the product of masses and inversely proportional to the square of the distance between them. Equivalently, he expressed the law mathematically as:

$$\overline{F} = \frac{Gm_1m_2}{r^2}\hat{n} \ , \tag{1.1.2}$$

where G is the universal gravitational constant, $\hat{n}$ is the unit vector in the direction of the force, and r is the distance between the two masses.

The success of Newton's theory of gravitation, based on the Newton's equation of motion and the inverse square law of gravitation, is remarkable. With the help of these laws of motion and the law of gravitation, Newton was not only able to describe the behaviour of the falling apple but also the motion of the projectile on the Earth, the movements of planets around the Sun, the motion of moon around the Earth and so on. The law of gravitation also explains the phenomenon of tides. Newton's theory, even today applies in regions of weak field and scientists use for planning the trajectories of spacecraft visiting planets of the solar system. It works correctly in a domain where the velocities of particles are very small as compare to the velocity of light. However, though the Newton's theory may work beautifully at low speeds, it is considered as unsatisfactory as it breaks at speeds approaching the velocity of light as it is evident from the fact that Newton's equations of motion and the inverse square law of force are covariant under Galilean transformation equations but not the laws of electromagnetism described by Maxwell's equations. This contradiction has doubted the universality of Newton's theory of gravitation, which leads to the foundation of special theory of relativity. However, Newton's laws ruled physics for more than two centuries. It took Einstein to topple from the throne.

Einstein (1879-1955):

Einstein was also a genuine free thinker of 20th century and had a great curiosity about nature. He lived in a deep faith-that there are laws of nature to be discovered. His life long pursuit was to discover them. He had a great belief that "the most incomprehensible thing about nature is that it is comprehensible". Latter it has been established in 1980s by Alain Aspect and his team that the nature really does behave in a non-commonsensical way. Einstein's realism and his optimism are illuminated by his remark "Subtle is a Lord, but malicious He is not". It means that "Nature hides her secret because of her essential loftiness, but not by means of ruse".

Einstein had written two separate theories of relativity, the first one is known as the special theory of relativity and was published in 1905, while the second one is known as the general theory of relativity, was published in 1915.

Special Theory of Relativity

In order to explain the motion of fast moving particles, Einstein developed a new theory in 1905 called the special theory of relativity. The theory of relativity forces us to change fundamentally our ideas of space and time. There is no unique absolute time, but instead each individual has his own personal measure of time that depends on where he is and how he is moving. We must accept that time is not completely separate from and independent of space, but is combined with it to form structure called space-time. The combined space-time structure is called Minkowski space-time. The 4-dimensions Minkowski space-time is characterized by the flat metric given by

$$ds^2 = \eta_{ij} dx^i dx^j \ , \tag{1.1.3}$$

where

$$x^i = (x, y, z, ict) \ ,$$

and

$$\begin{aligned} \eta_{ij} &= 1 \quad \text{when} \quad i = j = 1, 2, 3, \\ &= -1 \quad \text{when} \quad i = j = 4, \\ &= 0 \quad \text{when} \quad i \neq j \ . \end{aligned} \tag{1.1.4}$$

The space-time is flat and hence this theory does not deal with gravitation. The theory is based on two principles:

(*i*) The laws of physics (both for mechanics and electrodynamics), and

(*ii*) the speed of light in free space

must be the same for all observers moving relative to each other with uniform velocity. For this is to be true, space and time can no longer be independent, but rather, they are interdependent of each other in such a way as to keep the speed of light constant for all observations. The special theory of relativity is 'special' in the sense that it is restricted and only describes the behaviour of things moving in straight lines at constant speed. The most remarkable and very strange results of special theory of relativity are

- the length of a moving rod contracts in the direction of motion,
- mass of a moving particle varies with velocity,
- the moving clock slowed down its speed ,
- simultaneity is not an absolute concept.

Another most exciting result of special theory of relativity is mass and energy equivalence relation.

$$E = mc^2 \ , \tag{1.1.5}$$

whose practical devastating power has been demonstrated during the second world war in 1945. It is now understood that the conversion of mass into energy provides the energy source which keeps the Sun and stars shining and is therefore the ultimate source of the energy on which life on Earth depends. However, this theory is still considered as restricted theory because it does not contain gravitational force and deals only with inertial observers. The theory fails to study the relativity of all kinds of motion and restricted to inertial observers only. In view of these constraints Einstein in 1915 generalised the special theory of relativity and put forth a new theory known as the general theory of relativity.

General Theory of Relativity (1915)

A new theory which deals with all types of motion as well as gravitation was needed. Einstein knew that the new theory of gravitation

- should be self-consistent and covariant,
- should resolve the conflict between Newtonian theory of gravitation and the special theory of relativity,
- should reduce to special theory of relativity in the gravity free limit, and
- should have the correct Newtonian limit in the sense that when the velocities involved are very small as compared to the velocity

of light and the gravitational field is weak then the theory should reduce to Newton's theory of gravitation.

He achieved this new theory of gravitation after ten years of his special theory of relativity. The theory is called the general theory of relativity. The name general theory of relativity actually has a double meaning. It is general because it applies to accelerated motion and gravity not just to objects moving in a straight line at constant speed. This is the sense in which Einstein originally used the term. But it is also general in the sense that it applies to every thing- the entire universe and all it contains.

Constancy of velocity of light in free space is one of the radical contributions of Einstein in developing special theory of relativity. Introduction of gravity in the general theory of relativity was ingenious. The key question was how to make gravity interact with light so that its velocity should not change.

In an attempt to achieve gravitational force so as to act on a massless particle without changing its velocity, Einstein observed that the gravitation is an interaction

- which can not be switched on and off at will,
- it is omnipresent,
- ever lasting and

- universal.

Einstein identify this permanent character of gravitation as an intrinsic property of the non-Euclidean nature of space-time region, and announced an astonishing result that

GRAVITATION = SPACE-TIME GEOMETRY.

This law of nature tells that the property of space-time which is responsible for gravity is the curvature of space-time. After an unremitting labour in 1915, Einstein succeeded to formulate this law of nature in the language of mathematics in the form

$$R_{ij} - \frac{1}{2}Rg_{ij} = -kT_{ij} \ , \tag{1.1.6}$$

where R_{ij} is the Ricci tensor, R –the Ricci scalar, g_{ij} are the components of the fundamental metric tensor, T_{ij} is the stress-energy momentum tensor, which is the source of gravitation and k is the coupling constant. In the history of science, general theory of relativity is the only subject without any history entirely created by the efforts of one man Albert Einstein. He described this period as follows: "The years of searching in a dark for a truth that one feels but cannot express ". The following quote from John Gribbin's book [43] specifies that the theory of general relativity is too difficult to comprehend. "If Einstein had not produced the special theory of relativity in 1905, some one else would have done so within a short time, five years or so ". "The

General Theory of Relativity is the startling exception, may be the only one in 20th century. It is agreed by the most eminent theoretical physicists - Dirac has said so without qualification – that if Einstein had not created the General Theory in 1915 no one else would have done so, perhaps not until now, perhaps not for generations".

Einstein's theory goes beyond Newton's theory, but contains Newton's theory within itself. This theory gives a more accurate and comprehensive description of gravitation than the prevailing Newton's theory of gravitation. In its development, Einstein was guided by two principles:

(i) the principle covariance and

(ii) the principle of equivalence.

Principle of covariance helped Einstein to write the physical laws in covariant form so that their forms remain unaltered in all coordinate systems. Equivalently, it means that the physical laws must be expressed in tensorial form. The principle of equivalence – an axiom of indistinguishability between gravity and inertia leads to an intimate relation between metric and gravitation. Einstein's general theory of relativity deals with gravitation, which is one of the four basic interactions in nature which is responsible for most of the phenomena we observe in nature. The success of general theory of relativity is well known as it has passed every unambiguous test both experimentally and observationally. The recent detection of gravitational waves in the space-time as was predicted by Einstein 100 years before cemented the status of general relativity, besides other confirmations of Einstein's predictions of deflection of a ray of light by the gravitational field of the Sun and the perihelion advances of the planet Mercury.

However, in spite of widely recognized success of Einstein's general theory of relativity, it is considered to be inadequate in the sense that it does not satisfy certain 'desirable' features of the theory of gravitation. There was a hope that there may be some thing beyond the Einstein's theory of gravitation yet to be found. For example, understanding of Mach's principle, incorporation of intrinsic spin of gravitating matter, adaptability of quantum mechanics should suggest the link between gravitation and other interactions of physics etc are not substantiated by general theory of relativity. The singularity problem and some other unsatisfactory features exist in general relativity. To address such issues there are several well-known classical theories of gravitation other than Einstein's general theory of relativity obtained by modifying the Einstein's original theory of gravitation. Any new theory of gravity should be better than the Einstein's general theory of relativity and will have to include the general theory of relativity within itself, explaining every thing that the general theory of relativity explains. Few of them are

- Einstein-Cartan theory of gravitation,

- Brans- Dicke Scalar tensor theory of gravitation,
- Bimetric theories gravitation,
- $f(R)$ theory of gravitation,
- $f(R, T)$ theory of gravitation,
- Hoyle – Narlikar theory of gravitation,
- String theory,
- Theory of every thing.

These modified theories of gravitation have been extensively studied by many authors with the hope to unify gravitation and many other effects such as other interactions in nature. Einstein-Cartan theory of gravitation is one such extended theory of gravitation in which spin-an intrinsic feature of gravitating matter, is introduced. In recent years the Einstein-Cartan theory of gravitation has geared up in receiving the wide attention of researchers to study the role of intrinsic spin of gravitating matter and to study some exact solutions of field equations of Einstein-Cartan theory of gravitation.

The thesis entitled "A study of spherically symmetric space-times in Einstein-Cartan theory of gravitation "comprises six chapters and deals with the study of geometry of the non-Riemannian space and the study of some exact solutions of field equations when Weyessenhoff fluid is the source of gravitation in the Einstein-Cartan theory of gravitation. In order to make the thesis self explanatory, we are presenting the review of the concepts, the mathematical tools of differential forms which form the background of our investigations in the research work carried out in the thesis.

Einstein-Cartan Theory of Gravitation

The Einstein-Cartan theory of gravitation is based on 4-dimensions Riemannian space-time with asymmetric connections. This space-time is called non-Rimannian space-time. Non-Riemannian space with asymmetric connection is exhibited by Eisenhart [29]. Cartan [11] was the first author to introduce torsion into gravitational theory, in order to get a possible connection between the intrinsic spin of matter and anti-symmetric part of the affine connection. Cartan considered geometries of space-time with non-symmetric affine connections Γ^k_{ij}, defined by

$$\Gamma^k_{ij} = \{^{\,k}_{ij}\} - K_{ij}{}^k \ , \qquad (1.1.7)$$

where $K_{ij}{}^k$ is known as contorsion tensor, and $\{^{\,k}_{ij}\}$ are symmetric Christoffel symbols. The theory of gravitation with spin and torsion was independently rediscovered by Kibble [69] and Sciama [111]. Its ramifications are due to Trautman [125, 127, 128] and Hehl [48, 49]. The basic difference between the Einstein-Cartan theory of gravitation and the general theory of relativity is that an affine connection compatible with the metric tensor is not necessarily symmetric in general, and the asymmetric part of the connection is coupled with the intrinsic spin of matter. The geometry of the space-time is thus not necessarily Riemannian and both mass and spin are linked up with the geometry. In Einstein's theory of gravitation, mass directly influence the geometry but spin has no such dynamical effect (Trautman [127, 128]). In Einstein's theory of relativity, singularities cannot be prevented (Hawking [45], Hawking and Ellis [46]), however, these can be prevented in the Einstein-Cartan theory by direct influence of spin on the geometry of space-time (Trautman [127], and Hehl, et al. [50]). The Einstein-Cartan theory will reduce to the Einstein's theory of gravitation in the absence of torsion in the space-time geometry.

1.2 Mathematical Pre-requisite

1.2.1 Riemannian Space of Einstein Theory of Gravitation

A space with symmetric connections (usually denoted by Christoffel symbols $\left\{ {k \atop ij} \right\}$) characterized by the pseudo Riemannian metric

$$ds^2 = g_{ij} dx^i dx^j \ , \tag{1.2.1}$$

where g_{ij} are the symmetric components of the fundamental metric tensor play the role of gravitational potentials, is called the Riemannian space. If $\hat{R}^h{}_{ijk}$ is the Riemann curvature tensor of Riemannian space, then we have

$$\hat{R}^h{}_{ijk} = -\frac{\partial}{\partial x^k}\{{}^{\,h}_{ij}\} + \frac{\partial}{\partial x^j}\{{}^{\,h}_{ik}\} - \{{}^{\,h}_{kl}\}\{{}^{\,l}_{ij}\} + \{{}^{\,h}_{jl}\}\{{}^{\,l}_{ik}\} \ , \qquad (1.2.2)$$

and

$$\hat{R}^h{}_{ijk} = g_{hp}\hat{R}^p{}_{ijk} \ . \qquad (1.2.3)$$

The Riemann curvature tensor of Riemannian space satisfies the following properties

$$\hat{R}_{hijk} = -\hat{R}_{ihjk} = -\hat{R}_{hikj} \ ,$$

(skew-symmetry in the first and the second pair of indices)

$\hat{R}_{hijk} = \hat{R}_{jkhi}$, (symmetry in the pair of indices)

and

$$\hat{R}_{hijk} + \hat{R}_{hjki} + \hat{R}_{hkij} = 0 \ .\text{(cyclic property)} \qquad (1.2.4)$$

It also satisfies the Bianchi identities

$$\hat{R}_{hi[jk/l]} = 0 \ , \qquad (1.2.5)$$

where slash (/) denotes the covariant derivative with respect to the symmetric Christoffel symbols.

The contraction of the Bianchi identities yield the dynamical conservation laws

$$T^{ik}_{/k} = 0 \ , \qquad (1.2.6)$$

through the Einstein field equations for gravitation in non-empty space-time

$$\hat{R}_{ij} - \frac{1}{2}\hat{R}g_{ij} = -kT_{ij} \ , \tag{1.2.7}$$

where $\hat{R}_{ij} = g^{hk}\hat{R}_{hijk} = \hat{R}^k{}_{ijk}$ is the symmetric Ricci tensor, $\hat{R} = g^{ij}\hat{R}_{ij}$ is the Ricci curvature scalar, and T_{ij} is the symmetric stress-energy momentum tensor representing the source of gravitation.

1.2.2 Non-Riemann space of Einstein-Cartan Theory of Gravitation

A space with asymmetric connections characterized by the metric (1.2.1) is called a non-Riemann space. The difference between a Riemannian space and a non-Riemannian space is that the connections defined on a Riemannian space are symmetric while those defined on a non-Riemannian space are asymmetric. The non-Riemannian part is defined by the torsion tensor $Q_{jk}{}^l$ defined by

$$Q_{jk}{}^l = \frac{1}{2}(\Gamma^l_{jk} - \Gamma^l_{kj}) \ . \tag{1.2.8}$$

This shows that

$$Q_{jk}{}^l = -Q_{kj}{}^l \ . \tag{1.2.9}$$

Here and in the following, we denote the symmetric Christoffel symbols of first and second kinds by $[ij, k]$ and $\{^{k}_{ij}\}$ respectively, while the asymmetric connections of first and the second kinds are denoted by $\Gamma_{ij,k}$ and Γ^k_{ij}.

The calculus on a non-Riemannian space of Einstein-Cartan theory of gravitation is developed with the help of covariant derivative with respect to asymmetric connections. We denote, in the following, it by a semi-comma(;) and for a covariant vector A_i (a contravariant vector A^i) it is defined as

$$A_{i;j} = A_{i,j} - A_k\Gamma^k_{ij} \ , \tag{1.2.10}$$

and for contravariant vector A^i, we have

$$A^i{}_{;j} = A^i{}_{,j} + A^k\Gamma^i_{kj} \ , \tag{1.2.11}$$

where comma (,) denotes the partial differentiation.

At every point of a Einstein-Cartan space-time, there exists a Lorentz metric g_{ij} which satisfies the metric postulate

$$g_{ij;k} = 0 \ .$$

Generalizing the definition (1.2.10) for the second rank tensor, we obtain

$$g_{ij,k} = g_{hj}\Gamma^h_{ki} + g_{ih}\Gamma^h_{kj} \ . \tag{1.2.12}$$

By cyclic permutation of indices i, j, k in the equations (1.2.12) twice in turn, we obtain two more equations

$$g_{jk,i} = g_{hk}\Gamma^h_{ij} + g_{jh}\Gamma^h_{ik} \ , \tag{1.2.13}$$

and

$$g_{ki,j} = g_{hi}\Gamma^h_{jk} + g_{kh}\Gamma^h_{ji} \ . \tag{1.2.14}$$

Adding equations (1.2.13) and (1.2.14) and subtracting the equation (1.2.12) we obtain on using equation (1.2.8)

$$\{^{\,l}_{ij}\} = \Gamma^l_{ij} - Q_{ij}{}^l + g^{lk} g_{hj} Q_{ik}{}^k + g^{lk} g_{ih} Q_{jk}{}^h \ ,$$

where

$$\{^{\,l}_{ij}\} = g^{lk}[ij, k] \ .$$

This becomes

$$\Gamma^l_{ij} = \{^{\,l}_{ij}\} - (Q_{ij}{}^l + Q_j{}^l{}_i - Q^l{}_{ij}) \ ,$$

or

$$\Gamma^l_{ij} = \{^{\,l}_{ij}\} - K_{ij}{}^l \ , \tag{1.2.15}$$

where

$$K_{ijk} = -Q_{ijk} + Q_{jki} - Q_{kij} \ , \tag{1.2.16}$$

is the contorsion tensor satisfying the property

$$K_{i(jk)} = 0 \ . \tag{1.2.17}$$

Using the equation (1.2.15) in the equations (1.2.8) we get

$$Q_{ij}{}^l = -\frac{1}{2}(K_{ij}{}^l - K_{ji}{}^l) \ . \tag{1.2.18}$$

By virtue of the equations (1.2.15), the definition of the covariant derivative of a vector A_i becomes

$$A_{i;j} = A_{i/j} + A_k K_{ji}{}^k \ . \tag{1.2.19}$$

1.2.3 Ricci Identity in a non-Riemannian Space

Since covariant derivative increases the rank of a tensor by one. Thus, if A_i is a covariant vector then it follows from the equation (1.2.19) that $A_{i;j}$ is a second rank covariant tensor. Hence taking the covariant derivative of $A_{i;j}$ with respect to asymmetric connections we obtain

$$\begin{aligned} A_{i;jk} =& A_{i,jk} - A_{h,k}\Gamma^h_{ji} - A_h \frac{\partial \Gamma^h_{ji}}{\partial x^k} - A_{h,j}\Gamma^h_{ki} - A_{i;h}\Gamma^h_{kj} + \\ &+ A_l \Gamma^l_{jh}\Gamma^h_{ki} + A_l \Gamma^l_{hi}\Gamma^h_{kj} \ . \end{aligned} \tag{1.2.20}$$

Interchanging $j \leftrightarrow k$ in the equation (1.2.20) we get one more equation. Subtracting the result thus obtained from the equation (1.2.20), we get

$$\begin{aligned} A_{i;jk} - A_{i;kj} =& A_h \left[-\frac{\partial \Gamma^h_{ji}}{\partial x^k} + \frac{\partial \Gamma^h_{ki}}{\partial x^j} + \Gamma^h_{jl}\Gamma^l_{ki} - \Gamma^l_{ji}\Gamma^h_{kl} \right] + \\ &+ (\Gamma^h_{jk} - \Gamma^h_{kj})(A_{i,h} - A_l \Gamma^l_{hi}) \ , \\ A_{i;jk} - A_{i;kj} =& A_h R_{kji}{}^h + 2A_{i;h} Q_{jk}{}^h \ , \end{aligned} \tag{1.2.21}$$

where

$$R_{kji}{}^h = \left[-\frac{\partial \Gamma^h_{ji}}{\partial x^k} + \frac{\partial \Gamma^h_{ki}}{\partial x^j} + \Gamma^h_{jl}\Gamma^l_{ki} - \Gamma^l_{ji}\Gamma^h_{kl} \right] \ , \tag{1.2.22}$$

is the Riemann curvature tensor in a non-Riemannian space, we here after called it as Riemann-Cartan curvature tensor. Using the equation (1.2.15) in the equation (1.2.22) and simplifying the equation we obtain

$$\begin{aligned} R_{kji}{}^{h} =& \hat{R}_{kji}{}^{h} + \frac{\partial K_{ji}{}^{h}}{\partial x^{k}} - \frac{\partial K_{ki}{}^{h}}{\partial x^{j}} - \{^{h}_{jl}\} K_{ki}{}^{l} - \{^{l}_{ki}\} K_{jl}{}^{h} + \\ & + \{^{h}_{kl}\} K_{ji}{}^{l} + \{^{l}_{ji}\} K_{kl}{}^{h} + K_{jl}{}^{h} K_{ki}{}^{l} - K_{ji}{}^{h} K_{ji}{}^{l} \ . \end{aligned} \tag{1.2.23}$$

Using the definition of the covariant derivative of contortion tensor we have

$$K_{ji;k}{}^{h} = \frac{\partial K_{ji}{}^{h}}{\partial x^{k}} - K_{li}{}^{h} \Gamma^{l}_{kj} - K_{jl}{}^{h} \Gamma^{l}_{ki} + K_{ji}{}^{l} \Gamma^{h}_{kl} \ . \tag{1.2.24}$$

On using this equation we eliminate the partial derivative term from the equation (1.2.23) and simplifying we get

$$R_{kji}{}^{h} = \hat{R}_{kji}{}^{h} + K_{ji}{}^{h}{}_{;k} - K_{ki}{}^{h}{}_{;j} + 2K_{li}{}^{h} Q^{l}_{kj} + K_{ji}{}^{l} K_{kl}{}^{h} - K_{ki}{}^{l} K_{jl}{}^{h} \ , \tag{1.2.25}$$

where $\hat{R}_{kji}{}^{h}$ is the Riemann curvature tensor in the Riemannian space-time. From the equation (1.2.25) we observe that

$$\begin{aligned} R_{kjih} =& - R_{kjhi} = -R_{jkih} \ , \\ R_{kjih} \neq& R_{ihkj} \ . \end{aligned} \tag{1.2.26}$$

The cyclic property of the Riemann-Cartan curvature tensor in the non-Riemannian space is not true. Its expression is obtain as

$$R_{kji}{}^{h} + R_{jik}{}^{h} + R_{ikj}{}^{h} = 2(Q_{ij}{}^{h}{}_{;k} + Q_{jk}{}^{h}{}_{;j} + Q_{ki}{}^{h}{}_{;i}) -$$
$$- 4(Q_{ij}{}^{l}Q_{kl}{}^{h} + Q_{jk}{}^{l}Q_{il}{}^{h} + Q_{ki}{}^{l}Q_{jl}{}^{h}) \ . \quad (1.2.27)$$

Similarly, the Ricci identity for the second rank covariant tensor A_{ij} is obtain in the form

$$A_{ij;kh} - A_{ij;hk} = A_{pj}R_{hki}{}^{p} + A_{ip}R_{hkj}{}^{p} + 2A_{ij;p}Q_{kh}{}^{p} \ . \quad (1.2.28)$$

1.2.4 Generalized Bianchi identities for torsion and curvature

In Einstein theory of gravitation Bianchi identities are obtained by introducing a locally inertial coordinate system on a Riemannian space. However, in a non-Riemannian space in which connections are asymmetric, there does not exists locally inertial coordinate system. It can be seen by considering a coordinate system x^i in which the asymmetric connections $\Gamma^i_{jk} \neq 0$ at a point $x^i = x^i_0$. Define another coordinate system $\overline{x}^i$ such that

$$\overline{x}^i = (x^i - x^i_0) + \frac{1}{2}(\Gamma^i_{lm})_0(x^l - x^l_0)(x^m - x^m_0) \ , \quad (1.2.29)$$

where the suffix zero indicates that the quantity is evaluated at the pole x^i_0. Differentiating the equation partially with respect to x^k we get

$$\frac{\partial \overline{x}^i}{\partial x^k} = \delta^i_k + \frac{1}{2}\left[(\Gamma^i_{lk})_0 + (\Gamma^i_{kl})_0\right](x^l - x^l_0) \ ,$$

$$\Rightarrow \left(\frac{\partial \overline{x}^i}{\partial x^k}\right)_0 = \delta^i_k \ . \tag{1.2.30}$$

We see that the Jacobin of the transformations $J = \left|\frac{\partial \overline{x}^i}{\partial x^k}\right| \neq 0$ and hence the transformation defined in the equation (1.2.29) is well defined. Differentiating the equation (1.2.29) partially with respect to $\overline{x}^j$ we get

$$\delta^i_j = \frac{\partial x^i}{\partial \overline{x}^j} + \frac{1}{2}(\Gamma^i_{lm})_0 \left[\frac{\partial x^l}{\partial \overline{x}^j}(x^m - x^m_0) + (x^l - x^l_0)\frac{\partial x^m}{\partial \overline{x}^j}\right] \ ,$$

$$\Rightarrow \delta^i_j = \frac{\partial x^i}{\partial \overline{x}^j} + \frac{1}{2}\left[(\Gamma^i_{lm})_0 + (\Gamma^i_{ml})_0\right]\frac{\partial x^l}{\partial \overline{x}^j}(x^m - x^m_0) \ . \tag{1.2.31}$$

It follows from the equation that

$$\left(\frac{\partial x^i}{\partial \overline{x}^j}\right)_0 = \delta^i_j \ .$$

Further differentiating the equation (1.2.31) partially with respect to $\overline{x}^k$, we get

$$\frac{\partial^2 x^i}{\overline{x}^j \overline{x}^k} = -\frac{1}{2}\left[(\Gamma^i_{lm})_0 + (\Gamma^i_{ml})_0\right]\left[(x^m - x^m_0)\frac{\partial^2 x^l}{\overline{x}^j \overline{x}^k} + \frac{\partial x^l}{\partial \overline{x}^j}\frac{\partial x^m}{\partial \overline{x}^k}\right] \ . \tag{1.2.32}$$

Thus the values of the equation at the pole is given by

$$\left(\frac{\partial^2 x^i}{\overline{x}^j \overline{x}^k}\right)_0 = -\frac{1}{2}\left[(\Gamma^i_{lm})_0 + (\Gamma^i_{ml})_0\right]\left(\frac{\partial x^l}{\partial \overline{x}^j}\right)_0\left(\frac{\partial x^m}{\partial \overline{x}^k}\right)_0 \ . \tag{1.2.33}$$

Using the equation (1.2.15) in the equation (1.2.33) we get

$$\left(\frac{\partial^2 \overline{x}^i}{\overline{x}^j \overline{x}^k}\right)_0 = -\left\{{i \atop jk}\right\}_0 + \frac{1}{2}\left[(K_{jk}{}^i)_0 + (K_{kj}{}^i)_0\right] \ . \tag{1.2.34}$$

From the law of transformation of connections at the pole we have

$$\left(\overline{\Gamma}^{l}_{jk}\right)_0 = \left(\frac{\partial \overline{x}^l}{\partial x^i}\right)_0 \left(\frac{\partial x^m}{\partial \overline{x}^j}\right)_0 \left(\frac{\partial x^n}{\partial \overline{x}^k}\right)_0 (\Gamma^i_{mn})_0 + \left(\frac{\partial^2 x^i}{\overline{x}^j \overline{x}^k}\right)_0 \left(\frac{\partial \overline{x}^l}{\partial x^i}\right)_0 \ . \tag{1.2.35}$$

Using the equations (1.2.30) and (1.2.33) in the equation (1.2.35) we get

$$\left(\overline{\Gamma}^{l}_{jk}\right)_0 = (\Gamma^l_{jk})_0 - \left\{{}^{\,l}_{jk}\right\}_0 + \frac{1}{2}\left[(K_{jk}{}^l)_0 + (K_{kj}{}^l)_0\right] \ . \tag{1.2.36}$$

Using the equation (1.2.10) we get

$$\begin{aligned} \left(\overline{\Gamma}^{l}_{jk}\right)_0 &= -\left\{K_{jk}{}^l\right\}_0 + \frac{1}{2}\left[(K_{jk}{}^l)_0 + (K_{kj}{}^l)_0\right] \ , \\ \left(\overline{\Gamma}^{l}_{jk}\right)_0 &= -\frac{1}{2}\left[(K_{jk}{}^l)_0 - (K_{kj}{}^l)_0\right] \ , \\ \Rightarrow \left(\overline{\Gamma}^{l}_{jk}\right)_0 &= \left(Q_{jk}{}^l\right)_0 \neq 0 \ . \end{aligned} \tag{1.2.37}$$

This shows that there does not exist in a non-Riemannian space, a locally inertial coordinate system at a point and in its neighborhood. In order to find the Bianchi identities in the non-Riemannian space, we find the expression for the covariant derivative of Riemann-Cartan curvature tensor as

$$R_{kji}{}^l{}_{;h} = R_{kji}{}^l{}_{/h} + R_{pji}{}^l K_{hk}{}^p + R_{kpi}{}^l K_{hj}{}^p + R_{kjp}{}^l K_{hi}{}^p - R_{kji}{}^p K_{hp}{}^l \ . \tag{1.2.38}$$

By cyclic permutation of the indices k, j, h twice in turn in the equation (1.2.38) we get two more equations. Thus we obtain

$$R_{jhi;k}^{\ \ \ l} = R_{jhi/k}^{\ \ \ l} + R_{phi}{}^{l} K_{kj}{}^{p} + R_{jpi}{}^{l} K_{kh}{}^{p} + R_{jhp}{}^{l} K_{ki}{}^{p} - R_{jhi}{}^{p} K_{kp}{}^{l} \ , \tag{1.2.39}$$

and

$$R_{hki;j}^{\ \ \ l} = R_{hki/j}^{\ \ \ l} + R_{pki}{}^{l} K_{jh}{}^{p} + R_{hpi}{}^{l} K_{jk}{}^{p} + R_{hkp}{}^{l} K_{ji}{}^{p} - R_{hki}{}^{p} K_{jp}{}^{l} \ . \tag{1.2.40}$$

Adding equations (1.2.38), (1.2.39) and (1.2.40) we get

$$R_{kji;h}^{\ \ \ l} + R_{jhi;k}^{\ \ \ l} + R_{hki;j}^{\ \ \ l} = -2 \left(R_{jpi}{}^{l} Q_{kh}{}^{p} + R_{pki}{}^{l} Q_{jh}{}^{p} + R_{hpi}{}^{l} Q_{jk}{}^{p} \right) \ . \tag{1.2.41}$$

From this equation we obtain the relation

$$\left(R^{ik} - \frac{1}{2} R g^{ik} \right)_{;k} = g^{ih} \left(R_{hp}{}^{lk} Q_{lk}{}^{p} - 2 R^{pk} Q_{khp} \right) \ . \tag{1.2.42}$$

This shows that

$$\left(R^{ik} - \frac{1}{2} R g^{ik} \right)_{;k} \neq 0 \ , \tag{1.2.43}$$

where R_{ij} is the Ricci-Cartan tensor obtained by contracting the index h with k in the equation (1.2.25) we obtain

$$R_{ij} = \hat{R}_{ij} + K_{ij}{}^{k}{}_{;k} - K_{kj}{}^{k}{}_{;i} - K_{lj}{}^{k} K_{ki}{}^{l} + K_{ij}{}^{l} K_{kl}{}^{k} \ . \tag{1.2.44}$$

And the Ricci-Cartan curvature scalar is given by

$$R = g^{ij} R_{ij} = \hat{R} + 2 K_{i}{}^{ik}{}_{;k} + K_{l}{}^{ik} k_{ik}{}^{l} + K_{i}{}^{il} K_{kl}{}^{k} \ . \tag{1.2.45}$$

However, in a Riemannian space-time of general relativity, we have

$$\left(R^{ik} - \frac{1}{2}Rg^{ik}\right)_{/k} = 0 \ . \qquad (1.2.46)$$

Consequently, from Einstein's field equations we have

$$T^{ik}{}_{/k} = 0 \ . \qquad (1.2.47)$$

These are called dynamical conservation laws. However, such conservation laws do not hold in the non-Riemannian space of Einstein-Cartan theory of gravitation, as can be seen from the equation (1.2.43).

1.2.5 Field Equations in EC theory of Gravitation

The relevant field equations for curvature and spin are obtained from the action principle by Hehl, et al. [50, 51]. Variation of the action function with respect to the metric tensor g_{ij} yields the equation

$$R_{ij} - \frac{1}{2}Rg_{ij} = -kt_{ij} \ , \qquad (1.2.48)$$

where R_{ij}- is the Ricci-Cartan tensor, t_{ij}- is the energy momentum tensor.

The equation (1.2.48) is not the same as that of the Einstein field equation in Riemann space, because the Ricci-Cartan tensor here is no longer symmetric but instead contains information about the torsion tensor. The right hand side of the equation (1.2.48) cannot be symmetric either, so that t_{ij} must also contain information about the spin tensor. Similarly, the variation of the action with respect to the torsion tensor $Q_{ij}{}^{k}$ yields a new equation

$$Q_{ij}{}^{k} + \delta_i{}^{k} Q_{jl}{}^{l} - \delta_j{}^{k} Q_{il}{}^{l} = k S_{ij}{}^{k} \ , \tag{1.2.49}$$

where $S_{ij}{}^{k}$ is the spin angular momentum tensor. The relation between $S_{ij}{}^{k}$ and t_{ij} is defined by the equation

$$t^{ij} = T^{ij} + (\nabla + 2Q_{kl}{}^{l})(S^{ijk} - S^{jki} + S^{kij}) \ , \tag{1.2.50}$$

where T^{ij} is the stress-energy momentum tensor of matter,and

$$\nabla_k = \hat{\nabla}_k - 2Q_{kl}{}^{l} \ . \tag{1.2.51}$$

The field equation (1.2.48) is an algebraic in character relating to spin angular momentum tensor. Therefore, one can obtain the torsion tensor in terms of spin angular momentum tensor as

$$Q_{ij}{}^{k} = k\left(S_{ij}{}^{k} - \frac{1}{2}\delta_i^k S_{lj}{}^{l} - \frac{1}{2}\delta_j^k S_{il}{}^{l}\right) \ . \tag{1.2.52}$$

The equations (1.2.48) and (1.2.49) together are called the field equations of Einstein-Cartan theory of gravitation.

1.2.6 The Spin Tensor

For the classical description of the spin tensor, Hehl et al.[51] have decomposed the spin angular momentum tensor as

$$S_{ij}{}^{k} = S_{ij} u^{k} \ , \tag{1.2.53}$$

where u^i is the time-like 4-velocity vector; and S_{ij} is the spin tensor antisymmetric in character.

i.e.

$$S_{ij} = -S_{ji} \ . \tag{1.2.54}$$

This spin tensor is orthogonal to the 4-velocity vector.

i.e.,

$$S_{ij}u^j = 0 \ . \tag{1.2.55}$$

This shows that the intrinsic spin of a matter field is space-like in the rest frame of the fluid. The condition (1.2.55) is usually called as Frankel condition. With the help of this condition the field equation (1.2.48) or (1.2.52) gives an algebraic coupling between the spin tensor and torsion tensor as

$$Q_{ij}{}^k = kS_{ij}u^k \ . \tag{1.2.56}$$

Thus the torsion contribution to Einstein-Cartan field equation is entirely described by the spin tensor. Contracting the index j with k in the equation (1.2.56) we see that the torsion trace vanishes.

i.e.

$$Q_i = 0 \ , \tag{1.2.57}$$

where $Q_i = 2Q_{ik}{}^k$. Equivalently, it means that the Frankel condition implies that the torsion trace vanishes identically. If however, if the spin tensor is not u-orthogonal, in this case the trace of the torsion tensor does not vanish, but it is given by

$$Q_{ik}{}^{k} = -\frac{k}{2} S_{ik} u^{k} \ . \tag{1.2.58}$$

Substituting this in the field equation (1.2.49), we get

$$Q_{ij}{}^{k} = \frac{k}{2}[\delta_i{}^{k} S_{jl} u^{l} - \delta_j{}^{k} S_{jl} u^{l} + 2 S_{ij} u^{k}] \ . \tag{1.2.59}$$

The square of the spin scalar is defined as

$$S^2 = \frac{1}{2} S_{ij} S^{ij} \geq 0 \ . \tag{1.2.60}$$

1.3 Newman-Penrose-Jogia-Griffiths Null Formalism

The Newman-Penrose [89] (NP) null tetrad formalism is widely used and proved to be 'amazingly useful' tool in many applications, mainly in finding exact solutions of Einstein field equations. An excellent review on the exact solutions can be found in the book of Kramer Stephani, Herlt and MacCallum [71] and in the study of black holes by S. Chandrashekhar [17]. The approach is extended by Jogia and Griffiths [55] to deal with certain problems in Einstein-Cartan theory of gravitation and also in other theories of gravitation that include torsion. The formalism is widely known as Newman-Penrose-Jogia-Griffiths (NPJG) formalism. Every chapter of the thesis exploits the

NPJG formalism because of its suitability for computational work, its easy adaptability to other formalism and its thorough utilization of the Bianchi identities. The exposition of the NP formalism is available in the following books. Flaherty [30], Carmeli [10], Hawking and Israel [54], Frolov [34], Held [52], Kramer, Stephani, Herlt and MacCallum [71] and S. Chandrashekhar [17]. We describe below in brief the formalism

At each point of a curve $x^i = x^i(s)$ in a 4-dimensional non-Riemann space-time, we introduce a tetrad consisting of four null vector fields. Each vector of a tetrad has four components. A tetrad is denoted by $e_{(\alpha)}$. Thus we have

$$e_{(\alpha)}{}^i = (l^i, n^i, m^i, \overline{m}^i) \ , \alpha = 1, 2, 3, 4. \tag{1.3.1}$$

The vectors l^i and n^i are real null vector fields, while m^i and $\overline{m}^i$ are complex conjugate of each other. These vector fields satisfies the conditions

$$l_i n^i = -m_i \overline{m}^i = 1 \ , \tag{1.3.2}$$

and all other inner products are zero. Greek letters are used to denote tetrad components, while Lattin indices are used to denote tensor indices. The vector fields of the tetrad form a basis at each point of the curve. The tetrad of the dual basis vectors is given by

$$e^{(\alpha)}{}_i = (n_i, l_i, -\overline{m}_i, -m_i) \ . \tag{1.3.3}$$

The basis vectors of the tetrad and its dual satisfy the properties

$$e_{(\alpha)i}e^{(\beta)i} = \delta_{\alpha}^{\beta} \ , \tag{1.3.4}$$

and

$$e_{(\alpha)i}e^{(\alpha)k} = \delta_{i}^{k} \ . \tag{1.3.5}$$

The tensor indices are raised or lowered by using the metric tensor

$$g_{ij} = e_{(\alpha)i}e^{(\alpha)}{}_{j} \ , \tag{1.3.6}$$

while the tetrad indices are raised or lowered by using the tetrad components of the metric tensor $\eta_{\alpha\beta}$ given by

$$\eta_{\alpha\beta} = g_{ij}e_{(\alpha)}{}^{i}e_{(\beta)}{}^{j} \ . \tag{1.3.7}$$

Consequently, the matrix of the tetrad components of the metric tensor is given by

$$\eta_{\alpha\beta} = \eta^{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ . \tag{1.3.8}$$

Using equations (1.3.1) and (1.3.3), we obtain

$$g_{ij} = l_i n_j + n_i l_j - m_i \overline{m}_j - \overline{m}_i m_j \ . \tag{1.3.9}$$

This is called the completeness relation.

1.3.1 NP Spin Coefficients

The Ricci rotation coefficients are defined by

$$\gamma_{\alpha\beta\gamma} = -e_{(\alpha)i;j}e_{(\beta)}{}^{i}e_{(\gamma)}{}^{j} \ , \tag{1.3.10}$$

which is anti-symmetric in first two indices.

i.e.,

$$\gamma_{\alpha\beta\gamma} = -\gamma_{\beta\alpha\gamma} \ . \tag{1.3.11}$$

By expanding covariant derivative in the equation (1.3.10) by using the equation (1.2.19) we have the relation between the components of the Ricci rotation coefficients $\gamma_{\alpha\beta\gamma}$ and the tetrad components of the contortion tensor $K_{\alpha\beta\gamma}$ as follows:

$$\begin{aligned} \gamma_{\alpha\beta\gamma} &= -e_{(\alpha)i/j}e_{(\beta)}{}^{i}e_{(\gamma)}{}^{j} - e_{(\alpha)k}K_{ji}{}^{k}e_{(\beta)}{}^{i}e_{(\gamma)}{}^{j} \ , \\ \Rightarrow \gamma_{\alpha\beta\gamma} &= \gamma^{0}{}_{\alpha\beta\gamma} - K_{\gamma\beta\alpha} \ , \end{aligned} \tag{1.3.12}$$

where

$$K_{\alpha\beta\gamma} = K_{ijk}e_{(\alpha)}{}^{i}e_{(\beta)}{}^{j}e_{(\gamma)}{}^{k} \ , \tag{1.3.13}$$

are the tetrad components of the contortion tensor. The contortion components $K_{\alpha\beta\gamma}$ are the quantities by which the spin coefficients differ from their values in a Riemannian manifold.

The applications of NP-Spin co-efficient formalism in figuring out Einstein's ambitious explanation of gravitation is liberally proclaimed in gravitation radiations (Sachs [108], [109], Goldberg and Sachs [37], Goldberg [38], Newman, Tamburino, Unti [90], Pirani [99], Penrose [97, 98], Zakharov [139], Brooker and Janis [8]), in electromagnetic fields (Debney and Zund [20], [21],[22], Zund [140], [141], [142], Tariq and Tupper [122], [123], Wallace and Zund [133]), in black holes (Price [103], Teukolsky [124], Press and Teukolsky [102], Wald [134], Hawking and Ellis [46], Hajicek [44], Chandrashekhar [15], [16], Carter [13]), in shock waves (Papapetrou [94], [95]), in netrino radiation fields (Collision and Morris [19], Radhakrishna and Rao [104]).

According to Jogia and Griffiths, the 12 complex spin coefficients and 12 complex tetrad components of contortion tensor (with subscript 1) are defined below for ready references.

$$\kappa = \gamma_{311} = l_{i;j} m^i l^j \ , \qquad \pi = -\gamma_{421} = -n_{i;j} \overline{m}^i l^j \ ,$$
$$\rho = \gamma_{314} = l_{i;j} m^i \overline{m}^j \ , \qquad \lambda = -\gamma_{424} = -n_{i;j} \overline{m}^i \overline{m}^j \ ,$$
$$\sigma = \gamma_{313} = l_{i;j} m^i m^j \ , \qquad \mu = -\gamma_{423} = -n_{i;j} \overline{m}^i m^j \ ,$$
$$\tau = \gamma_{312} = l_{i;j} m^i n^j \ , \qquad \nu = -\gamma_{422} = -n_{i;j} \overline{m}^i n^j \ ,$$
$$\epsilon = \frac{1}{2}(\gamma_{211} - \gamma_{431}) = \frac{1}{2}(l_{i;j} n^i l^j - m_{i;j} \overline{m}^i l^j) \ ,$$
$$\alpha = \frac{1}{2}(\gamma_{214} - \gamma_{434}) = \frac{1}{2}(l_{i;j} n^i \overline{m}^j - m_{i;j} \overline{m}^i \overline{m}^j) \ ,$$
$$\beta = \frac{1}{2}(\gamma_{213} - \gamma_{433}) = \frac{1}{2}(l_{i;j} n^i m^j - m_{i;j} \overline{m}^i m^j) \ ,$$
$$\gamma = \frac{1}{2}(\gamma_{212} - \gamma_{432}) = \frac{1}{2}(l_{i;j} n^i n^j - m_{i;j} \overline{m}^i n^j) \ ,$$
$$\kappa_1 = K_{131} = K_{ijk} l^i m^j l^k \ , \qquad \pi_1 = -K_{142} = -K_{ijk} l^i \overline{m}^j n^k \ ,$$

$$\rho_1 = K_{431} = K_{ijk}\overline{m}^i m^j l^k \ , \qquad \lambda_1 = -K_{442} = -K_{ijk}\overline{m}^i\overline{m}^j n^k \ ,$$
$$\sigma_1 = K_{331} = K_{ijk}m^i m^j l^k \ , \qquad \mu_1 = -K_{342} = -K_{ijk}m^i\overline{m}^j n^k \ ,$$
$$\tau_1 = K_{231} = K_{ijk}n^i m^j l^k \ , \qquad \nu_1 = -K_{242} = -K_{ijk}n^i\overline{m}^j n^k \ ,$$
$$\epsilon_1 = \frac{1}{2}(K_{121} - K_{143}) = \frac{1}{2}K_{ijk}l^i(n^j l^k - \overline{m}^j m^k) \ ,$$
$$\alpha_1 = \frac{1}{2}(K_{421} - K_{443}) = \frac{1}{2}K_{ijk}\overline{m}^i(n^j l^k - \overline{m}^j m^k) \ ,$$
$$\beta_1 = \frac{1}{2}(K_{321} - K_{343}) = \frac{1}{2}K_{ijk}m^i(n^j l^k - \overline{m}^j m^k) \ ,$$
$$\gamma_1 = \frac{1}{2}(K_{221} - K_{243}) = \frac{1}{2}K_{ijk}n^i(n^j l^k - \overline{m}^j m^k) \ . \qquad (1.3.14)$$

Then we have

$\kappa = \kappa^0 + \kappa_1, \rho = \rho^0 + \rho_1, \sigma = \sigma^0 + \sigma_1$ etc.

In Einstein-Cartan theory of gravitation the Ricci tensorR_{ij} is not necessarily symmetric and hence it has 16 independent components. These can be expressed in terms of the familiar nine components of a Hermitian 3×3 matrix $\phi_{AB}, (A, B = 0, 1, 2)$ the three complex components ϕ_A and the real parameter Λ. These are defined by (Jogia and Griffiths [55]).

$$\phi_{00} = -\frac{1}{2}R_{ij}l^i l^j \ ,$$
$$\phi_{01} = -\frac{1}{4}R_{ij}(l^i m^j + m^i l^j) \ ,$$
$$\phi_{02} = -\frac{1}{2}R_{ij}m^i m^j \ ,$$
$$\phi_{11} = -\frac{1}{8}R_{ij}(l^i n^j + n^i l^j + m^i\overline{m}^j + \overline{m}^i m^j) \ ,$$
$$\phi_{12} = -\frac{1}{4}R_{ij}(n^i m^j + m^i n^j) \ ,$$

$$
\begin{aligned}
\phi_{22} &= -\frac{1}{2} R_{ij} n^i n^j \ , \\
\phi_0 &= -\frac{1}{4} R_{ij} (l^i m^j - m^i l^j) \ , \\
\phi_1 &= -\frac{1}{4} R_{ij} (l^i n^j - n^i l^j - m^i \overline{m}^j + \overline{m}^i m^j) \ , \\
\phi_2 &= -\frac{1}{4} R_{ij} (\overline{m}^i n^j - n^i \overline{m}^j) \ , \\
\Lambda &= \frac{1}{24} R \ .
\end{aligned}
\tag{1.3.15}
$$

The 20 independent real components of the trace free curvature tensor can be expressed in terms of five complex components, nine components of the Hermitian matrix $\Theta_{AB}(A, B = 0, 1, 2)$ and a real parameter χ. These are defined by

$$
\begin{aligned}
\psi_0 &= -C_{1313} = -C_{1313} l^h m^i l^j m^k \ , \\
\psi_1 &= -\frac{1}{2}(C_{1213} + C_{4313}) = -\frac{1}{2} C_{hijk} (l^h n^i + \overline{m}^h m^i) l^j m^k \ , \\
\psi_2 &= -C_{4213} = -C_{hijk} \overline{m}^h n^i l^j m^k \ , \\
\psi_3 &= -\frac{1}{2}(C_{1242} + C_{4342}) = -\frac{1}{2} C_{hijk} (l^h n^i + \overline{m}^h m^i) \overline{m}^j n^k \ , \\
\psi_4 &= -C_{4242} = -C_{hijk} \overline{m}^h n^i \overline{m}^j n^k \ , \\
\Theta_{00} &= -iC_{1314} = -C_{hijk} l^h m^i l^j \overline{m}^k \ , \\
\Theta_{01} &= -\frac{i}{2}(C_{1312} - C_{1343}) = -\frac{i}{2} C_{hijk} l^h m^i (l^j n^k + \overline{m}^j m^k) \ , \\
\Theta_{02} &= iC_{1323} = iC_{hijk} l^h m^i n^j m^k \ , \\
\Theta_{11} &= \frac{i}{4}(C_{1212} + C_{1243} - C_{4312} - C_{4343}) = \\
&= \frac{i}{4} C_{hijk} (l^h n^i - \overline{m}^h m^i)(l^j n^k + \overline{m}^j m^k) \ ,
\end{aligned}
$$

$$\Theta_{12} = -\frac{i}{2}(C_{2312} + C_{2343}) = -\frac{i}{2}C_{hijk}n^h m^i (l^j n^k + \overline{m}^j m^k) \ ,$$
$$\Theta_{22} = -iC_{2423} = -iC_{hijk}n^h \overline{m}^i n^j m^k \ ,$$
$$\begin{aligned} \chi &= -\frac{i}{2}(C_{1212} + C_{1234} + C_{1423}) = \\ &= -\frac{i}{4}C_{hijk}(l^h n^i + \overline{m}^h m^i)(l^j n^k + \overline{m}^j m^k) - iC_{hijk}n^h \overline{m}^i l^j m^k \ . \end{aligned} \tag{1.3.16}$$

Any arbitrary vector or tensor can be expressed in terms of its tetrad components and conversely, thus we express

$$f_{;i} = f_{;\alpha} e^{(\alpha)}{}_i \ ,$$
$$\text{and} \qquad f_{;\alpha} = f_{;i} e_{(\alpha)}{}^i \ .$$

We write

$$f_{;i} = (f_{;1})_t n_i + (f_{;2})_t l_i - (f_{;3})_t \overline{m}_i - (f_{;4})_t m_i \ ,$$

where the suffix t indicates the tetrad components of the quantity. Thus we have

$$\begin{aligned} (f_{;1})_t &= f_{;i} l^i = Df \ , \\ (f_{;2})_t &= f_{;i} n^i = \Delta f \ , \\ (f_{;3})_t &= f_{;i} m^i = \delta f \ , \\ (f_{;4})_t &= f_{;i} \overline{m}^i = \overline{\delta} f \ . \end{aligned} \tag{1.3.17}$$

Thus we have

$$f_{;i} = Dfn_i + \Delta f l_i - \delta f \overline{m}_i - \overline{\delta} f m_i \ . \tag{1.3.18}$$

1.3.2 Contortion tensor, Torsion tensor and Spin angular momentum tensor

The tetrad formalism provides an algorithm for calculating the curvature tensor of space-time in a form suitable for a variety of applications and providing additional information pertaining to the geometry of space-time. Instead of the tensor components their tetrad components with respect to the tetrad vectors are utilized and this makes the Einstein field equations more transparent.

The contorsion tensor K_{ijk} in terms of its tetrad components is given by

$$K_{ijk} = K_{\alpha\beta\gamma} e_i{}^{(\alpha)} e_j{}^{(\beta)} e_k{}^{(\gamma)} \ , \qquad \alpha \ , \beta \ , \gamma = 1, 2, 3, 4 \ . \tag{1.3.19}$$

By expanding the right hand side of equation (1.3.19) by giving the different values to α , β , γ , δ and using equations (1.3.14) we obtain the expression

$$\begin{aligned} K_{ijk} =& 2[(\epsilon_1 + \overline{\epsilon}_1) n_i l_{[j} n_{k]} + (\gamma_1 + \overline{\gamma}_1) l_i l_{[j} n_{k]} + (\epsilon_1 - \overline{\epsilon}_1) n_i \overline{m}_{[j} m_{k]} + \\ & + (\gamma_1 - \overline{\gamma}_1) l_i \overline{m}_{[j} m_{k]} + \{ \lambda_1 m_i l_{[j} m_{k]} - \kappa_1 n_i \overline{m}_{[j} n_{k]} - \pi_1 n_i l_{[j} m_{k]} - \\ & - \overline{\tau}_1 l_i m_{[j} n_{k]} - \nu_1 l_i l_{[j} m_{k]} + \overline{\sigma}_1 m_i m_{[j} n_{k]} - (\alpha_1 + \overline{\beta}_1) m_i l_{[j} n_{k]} + \\ & + \rho_1 m_i \overline{m}_{[j} n_{k]} + \overline{\mu}_1 m_i l_{[j} \overline{m}_{k]} + (\overline{\alpha}_1 - \beta_1) \overline{m}_i \overline{m}_{[j} m_{k]} \} + c.c] \ , \end{aligned} \tag{1.3.20}$$

where $c.c$ indicates the complex conjugate of the preceding term.

Now the torsion tensor ${Q_{ij}}^k$ in terms of its tetrad components is given by

$$ {Q_{ij}}^k = {Q_{\alpha\beta}}^\gamma {e_i}^{(\alpha)} {e_j}^{(\beta)} {e_{(\gamma)}}^k \ , \quad \alpha,\beta,\gamma = 1,2,3,4. \tag{1.3.21} $$

By giving the different values to α,β,γ and using equation (1.2.18), the equation (1.3.21) gives

$$ \begin{aligned} {Q_{ij}}^k = -\Big[& K_{212} l_{[i}n_{j]}l^k - K_{121} l_{[j}n_{j]}n^k + (K_{314} - K_{431})\overline{m}_{[i}m_{j]}n^k + \\ & + (K_{342} - K_{4321})\overline{m}_{[i}m_{j]}l^k + \Big\{ (K_{124} - K_{214}) l_{[i}n_{j]}m^k + \\ & + (K_{142} - K_{421}) l_{[i}n_{j]}l^k + (K_{413} - K_{143}) m_{[i}n_{j]}\overline{m}^k + \\ & + K_{414} m_{[i}n_{j]}m^k + K_{141} m_{[i}n_{j]}n^k + (K_{421} - K_{241}) l_{[i}m_{j]}n^k - \\ & - K_{242} l_{[i}m_{j]}l^k + (K_{243} - K_{423}) l_{[i}m_{j]}\overline{m}^k - K_{424} l_{[i}m_{j]}m^k + \\ & + K_{434}\overline{m}_{[i}n_{j]}m^k \Big\} + c.c \Big] \ . \end{aligned} \tag{1.3.22} $$

Now using equations (1.3.14) we readily get

$$ \begin{aligned} {Q_{ij}}^k = & (\gamma_1 + \overline{\gamma}_1) l_{[i}n_{j]}l^k + (\epsilon_1 + \overline{\epsilon}_1) l_{[j}n_{j]}n^k + (\rho_1 - \overline{\rho}_1)\overline{m}_{[i}m_{j]}n^k + \\ & + (\mu_1 - \overline{\mu}_1)\overline{m}_{[i}m_{j]}l^k + \Big[(\pi_1 - \alpha_1 - \overline{\beta}_1) m_{[i}n_{j]}l^k - (\pi_1 + \overline{\tau}_1) l_{[i}n_{j]}l^k \\ & + (\rho_1 - \epsilon_1 + \overline{\epsilon}_1) m_{[i}n_{j]}\overline{m}^k + \overline{\sigma}_1 m_{[i}n_{j]}m^k - \overline{\kappa}_1 m_{[i}n_{j]}n^k \\ & + (\overline{\tau}_1 - \alpha_1 - \overline{\beta}_1) l_{[i}m_{j]}n^k - \nu_1 l_{[i}m_{j]}l^k + (\overline{\mu}_1 + \gamma_1 - \overline{\gamma}_1) l_{[i}m_{j]}\overline{m}^k + \\ & + \lambda_1 l_{[i}m_{j]}m^k + (\alpha_1 - \overline{\beta}_1)\overline{m}_{[i}m_{j]}m^k \Big] + c.c \ . \end{aligned} \tag{1.3.23} $$

Similarly, we obtain the expression

$$ {S_{ij}}^k = -\frac{1}{2k}\Big[-(\mu_1 + \overline{\mu}_1) l_{[i}n_{j]}l^k - (\rho_1 + \overline{\rho}_1) l_{[i}n_{j]}n^k + (\mu_1 - \overline{\mu}_1) m_{[i}\overline{m}_{j]}l^k + $$

$$
\begin{aligned}
&+(\rho_1-\overline{\rho}_1)m_{[i}\overline{m}_{j]}n^k+\Big\{\overline{\nu}_1 l_{[i}\overline{m}_{j]}l^k+(2\alpha_1-\pi_1)l_{[i}m_{j]}n^k-\\
&-\kappa_1 n_{[i}\overline{m}_{j]}n^k+(2\overline{\beta}_1-\overline{\tau}_1)m_{[i}n_{j]}l^k+(\mu_1-2\gamma_1)l_{[i}m_{j]}\overline{m}^k+\\
&+(\rho_1-2\epsilon_1)\overline{m}_{[i}n_{j]}m^k+(\pi_1+\overline{\tau}_1)l_{[i}n_{j]}m^k+(\pi_1-\overline{\tau}_1)\overline{m}_{[i}m_{j]}m^k+\\
&+\lambda_1 m_{[i}l_{j]}m^k-\overline{\sigma}_1 m_{[i}n_{j]}m^k\Big\}+c.c\Big]\ . \qquad (1.3.24)
\end{aligned}
$$

The spin tensor is expressed in terms of its three complex tetrad components s_0, s_1 and s_2 as

$$
S_{ij}=-2\left[(s_1+\overline{s}_1)l_{[i}n_{j]}+(s_1-\overline{s}_1)\overline{m}_{[i}m_{j]}-\left(s_0\overline{m}_{[i}n_{j]}+\overline{s}_2 l_{[i}m_{j]}\right)-c.c\right]\ , \qquad (1.3.25)
$$

where the complex tetrad components are defined by

$$
\begin{aligned}
s_0 =&S_{13}=S_{ij}l^i m^j\ ,\\
s_1 =&\frac{1}{2}(S_{12}+S_{43})=\frac{1}{2}S_{ij}\left(l^i n^j+\overline{m}^i m^j\right)\ ,\\
s_2 =&S_{32}=S_{ij}m^i n^j\ .
\end{aligned} \qquad (1.3.26)
$$

We see that the Frenkel condition (1.2.55) is not identically true, but it gives

$$
s_0=s_2\ ,\qquad s_1+\overline{s}_1=0\ . \qquad (1.3.27)
$$

This reduces the number of components of spin tensor from six to three.

We define the time-like vector u^i as $u^i=\frac{1}{\sqrt{2}}(l^i+n^i)$ such that

$u_i u^i = 1$. Now multiplying equation (1.3.25) by u^i we get

$$S_{ij}u^k = -\sqrt{2}\Bigg[(s_1+\overline{s}_1)l_{[i}n_{j]} + (s_1-\overline{s}_1)\overline{m}_{[i}m_{j]} - \left(s_0\overline{m}_{[i}n_{j]} + \overline{s}_2 l_{[i}m_{j]}\right) - - c.c\Bigg](l^k+n^k) \ . \tag{1.3.28}$$

If ${S_{ij}}^k = S_{ij}u^k$, then the corresponding coefficients of the equations (1.3.24) and (1.3.28) must be identical. Hence equating the corresponding coefficients, we obtain the relations

$$\begin{aligned}
&(\rho_1+\overline{\rho}_1) = (\mu_1+\overline{\mu}_1) = -\sqrt{2}k(s_1+\overline{s}_1) \ , \\
&(\rho_1-\overline{\rho}_1) = (\mu_1-\overline{\mu}_1) = -\sqrt{2}k(s_1-\overline{s}_1) \ , \\
&\kappa_1 = 2\beta_1 - \tau_1 = -\sqrt{2}ks_0 \ , \\
&\overline{\nu}_1 = 2\overline{\alpha}_1 - \overline{\pi}_1 = -\sqrt{2}ks_2 \ , \\
&\mu_1 - 2\gamma_1 = 0 \ , \rho_1 - 2\epsilon_1 = 0 \ , \pi_1 + \tau_1 = 0 \ , \\
&\pi_1 - \overline{\tau}_1 = 0 \ , \lambda_1 = 0 \ , \sigma_1 = 0 \ .
\end{aligned} \tag{1.3.29}$$

Now using equations (1.3.27) we obtain

$$\begin{aligned}
\pi_1 =& \tau_1 = \lambda_1 = \sigma_1 = 0 \ , \\
\rho_1 =& \mu_1 = 2\epsilon_1 = 2\gamma_1 = -\sqrt{2}ks_1 \ , \\
\overline{\nu}_1 =& \kappa_1 = 2\overline{\alpha}_1 = 2\beta_1 = -\sqrt{2}ks_0 \ .
\end{aligned} \tag{1.3.30}$$

By virtue of the equations (1.3.27), the expression for S_{ij} becomes

$$S_{ij} = 2\left[2s_1 m_{[i}\overline{m}_{j]} + \overline{s}_0\left(l_{[i}m_{j]} + m_{[i}n_{j]}\right) + c.c\right] \ . \tag{1.3.31}$$

1.4 Techniques of Differential Form in a Riemannian space

The traditional approach of tensors makes heavy use of Christoffel symbols which are forty in number and have no invariant significance under the change of coordinates. Techniques of differential forms is another useful and the most powerful analytical tool of modern mathematics. The use of differential forms can reduce the complexity of computation. There are only six complex connection 1-forms which take care the role of forty Christoffel symbols. In this chapter we have presented this powerful technique on a Riemannian space in which the connections are symmetric Christoffel symbols.

We assume here the readers are familiar with the exterior derivative operator d , which maps r- form to $(r+1)$- form. i.e.,

$$d : \wedge^r T_p^* \to \wedge^{r+1} T_p^* \ ,$$

satisfying the following properties:

$$\begin{aligned}
(i)&\quad df = f_{,i} dx^i \ , \\
(ii)&\quad d(\tilde{\omega} + \tilde{\sigma}) = d\tilde{\omega} + d\tilde{\sigma} \ , \\
(iii)&\quad d(\tilde{\omega} \wedge \tilde{\sigma}) = d\tilde{\omega} \wedge \tilde{\sigma} + (-1)^{\text{deg of } \tilde{\omega}} \tilde{\omega} \wedge d\tilde{\sigma} \ , \\
(iv)&\quad d(f\tilde{\omega}) = df \wedge \tilde{\omega} + f d\tilde{\omega} \ , \\
(v)&\quad d(d\tilde{\omega}) = 0 \ ,
\end{aligned}$$

$$(vi) \quad d(df \wedge dg) = 0 \ . \tag{1.4.1}$$

Here T_p^* is a tangent space of 1 - forms, $\wedge^r T_p^*$ is a set of all r-forms, $\wedge^{r+1} T_p^*$- is set of all $(r+1)$- forms, f and g are a differentiable functions and are also called as 0-forms, $\tilde{\omega}, \tilde{\sigma}$ are forms of any degree, and $\wedge$ is the wedge product and has the following properties.

$$\begin{aligned} (\tilde{\omega} + \tilde{\sigma}) \wedge \tilde{\alpha} &= \tilde{\omega} \wedge \tilde{\alpha} + \tilde{\sigma} \wedge \tilde{\alpha} \ , \\ (\tilde{\omega} \wedge \tilde{\sigma}) \wedge \tilde{\alpha} &= \tilde{\omega} \wedge (\tilde{\sigma} \wedge \tilde{\alpha}) \ , \\ (\tilde{\omega} \wedge \tilde{\sigma}) &= (-1)^{rp} \tilde{\sigma} \wedge \tilde{\omega} \ , \end{aligned} \tag{1.4.2}$$

where r, p are degrees of $\tilde{\omega}$ *and* $\tilde{\sigma}$ respectively. It follows from the property (v) of equation (1.4.1) that

$$d(dx^i) = 0 \ . \tag{1.4.3}$$

The operator d on any form raises the degree of the form by one. Thus the operator d takes a 0-form f to a 1-form df , 1-form $\tilde{\omega}$ to 2-from $d\tilde{\omega}$ and so on and in general any p -form to $(p+1)$ - form for $p \geq 2$. The exterior derivative is independent of the symmetric Christoffel symbols hence it is performed on any p form by taking either the partial derivative or covariant derivative of an associated p^{th} rank tensor. Because of this property of the exterior derivative d , it subsumes ordinary gradient, curl and divergence when operated on a 0-form f and 1-form $\tilde{\omega}$ give the standard vector identities $curl(gradf) = 0$ and $div(curl\omega) = 0$ respectively in Riemannian space. Maxwell's equations also take on a particularly simple and elegant form when expressed in terms of the exterior derivative. Lie derivative is another operator which is independent of the symmetric Christoffel symbols.

1.4.1 Cartan's Equations of Structure in a Riemannian Space

To understand the geometry of a Riemannian space, the Cartan's equations of structure play a vital role. The Cartan's equations of structure facilitate the computation of Riemann curvature tensor. We will elaborate the generalization of these equations in a non-Riemannian space in the Chapter 2. Hence a brief account of these equations in a Riemannian space is presented below.

We denote $\hat{V}_n$ as a Riemannian space with symmetric Christoffel symbols and the metric

$$ds^2 = g_{ij} dx^i dx^j \ , \tag{1.4.4}$$

where g_{ij} is the metric tensor of a Riemannian space. Define a curve in $\hat{V}_n$ and at each point of the curve, one can construct a tetrad $e_{(\alpha)i}$, $\alpha = 1, 2, 3, 4$, consisting of four vector fields which form a basis at each point of the curve. Each vector of the tetrad will have four components denoted by the Lattin index i. Thus for a vector field e_i, one can have an associated basis 1-form θ defined by

$$\theta = e_i dx^i \ .$$

Thus corresponding to four basis vector fields $e^{(\alpha)}{}_i$ of the tetrad, we have four basis 1-forms θ^α, defined by

$$\theta^\alpha = e^{(\alpha)}{}_i dx^i \ , \tag{1.4.5}$$

where $e^{(\alpha)}{}_i$ is called the tetrad of dual basis vectors. The vector fields of the tetrad and the dual tetrad satisfy the orthonormal conditions

$$e^{(\alpha)}{}_i e_{(\beta)}{}^i = \delta^\alpha_\beta \ , \qquad e^{(\alpha)}{}_k e_{(\alpha)}{}^i = \delta^i_k \ .$$

This gives

$$\eta_{\alpha\beta} = e_{(\alpha)i} e_{(\beta)}{}^i = g_{ij} e_{(\alpha)}{}^i e_{(\beta)}{}^j \ , \tag{1.4.6}$$

where $\eta_{\alpha\beta}$ are called the tetrad components of the metric tensor g_{ij}. Conversely, one can also express

$$g_{ij} = \eta_{\alpha\beta} e^{(\alpha)}{}_i e^{(\beta)}{}_j \ . \tag{1.4.7}$$

Similarly, any vector or a tensor of any rank can be expressed as a linear combination of its tetrad components and conversely.

Taking the usual exterior derivative of the equation (1.4.5) we get

$$d\theta^\alpha = e^{(\alpha)}{}_{i/j} dx^j \wedge dx^i \ . \tag{1.4.8}$$

Or

$$d\theta^\alpha = e^{(\alpha)}{}_{i,j} dx^j \wedge dx^i \ ,$$

as the term involving Christoffel symbol vanishes due to the symmetric property of the Christoffel symbol and the skew-symmetric property of the wedge product.

The Ricci's rotation coefficients in a Riemannian space are denoted by $\gamma^{0}{}_{\alpha\beta\gamma}$ and are defined as

$$\gamma^{0}\alpha\beta\gamma = -e_{(\alpha)i/j}e_{(\beta)}{}^{i}e_{(\gamma)}{}^{j} \ . \qquad (1.4.9)$$

Solving the equation (1.4.9) and using the orthonormal conditions (1.4.6), we obtain

$$e^{(\alpha)}{}_{i/j} = -\gamma^{0\alpha}{}_{\beta\gamma}e^{(\beta)}{}_{i}e^{(\gamma)}{}_{j} \ .$$

Substituting this in the equations (1.4.8) we obtain

$$d\theta^{\alpha} = -\ \gamma^{0\alpha}{}_{\beta\gamma}\theta^{\gamma} \wedge \theta^{\beta} \ , \qquad (1.4.10)$$

$$\Rightarrow d\theta^{\alpha} = -\omega^{0\alpha}{}_{\beta} \wedge \theta^{\beta} \ , \qquad (1.4.11)$$

where

$$\omega^{0\alpha}{}_{\beta} = \gamma^{0\alpha}{}_{\beta\gamma}\theta^{\gamma} \ , \qquad (1.4.12)$$

are the tetrad components of connection 1-forms in a Riemannian space. The equation (1.4.11) is known as the Cartan's first equation of structure.

Using the equation (1.4.5), we write the equation(1.4.12) as

$$\omega^{0\alpha}{}_{\beta} = \gamma^{0\alpha}{}_{\beta\sigma}e^{(\sigma)}{}_{i}dx^{i} \ . \qquad (1.4.13)$$

Taking the exterior derivative of (1.4.13) we get

$$d\omega^{0\alpha}{}_{\beta} = \frac{1}{2}\big[-(\gamma^{0\alpha}{}_{\beta\sigma}e^{(\sigma)}{}_{i})_{/j} + (\gamma^{0\alpha}{}_{\beta\sigma}e^{(\sigma)}{}_{j})_{/i}\big]dx^i \wedge dx^j \ .$$

Eliminating the covariant derivative terms we obtain

$$\Omega^{0\alpha}{}_{\beta} = d\omega^{0\alpha}{}_{\beta} + \omega^{0\alpha}{}_{\epsilon} \wedge \omega^{0\epsilon}{}_{\beta} \ , \tag{1.4.14}$$

where

$$\Omega^{0\alpha}{}_{\beta} = -\frac{1}{2}\hat{R}^{\alpha}{}_{\beta\epsilon\delta}\theta^{\epsilon} \wedge \theta^{\delta} \ , \tag{1.4.15}$$

are called tetrad components of curvature 2-forms in a Riemannian space. The equation (1.4.14) is called the Cartan's second equation of structure.

We record below the expressions for the connection 1-forms from equations (1.4.12) in terms of NP spin coefficients for our record for the use in the thesis.

$$\begin{aligned}
\omega^0{}_{12} =& -\left[(\epsilon^0 + \overline{\epsilon}^0)\theta^1 + (\gamma^0 + \overline{\gamma}^0)\theta^2 + (\overline{\alpha}^0 + \beta^0)\theta^3 + (\alpha^0 + \overline{\beta}^0)\theta^4\right] \ , \\
\omega^0{}_{13} =& -\left[\kappa^0\theta^1 + \tau^0\theta^2 + \sigma^0\theta^3 + \rho^0\theta^4\right] \ , \\
\omega^0{}_{23} =& \overline{\pi}^0\theta^1 + \overline{\nu}^0\theta^2 + \overline{\lambda}^0\theta^3 + \overline{\mu}^0\theta^4 \ , \\
\omega^0{}_{34} =& (\epsilon^0 - \overline{\epsilon}^0)\theta^1 + (\gamma^0 - \overline{\gamma}^0)\theta^2 - (\overline{\alpha}^0 - \beta^0)\theta^3 + (\alpha^0 - \overline{\beta}^0)\theta^4 \ .
\end{aligned} \tag{1.4.16}$$

Similarly, from the equation (1.4.10) we find

$$\begin{aligned}
d\theta^1 =& (\gamma^0 + \overline{\gamma}^0)\theta^{12} + (\overline{\alpha}^0 + \beta^0 - \overline{\pi}^0)\theta^{13} + (\alpha^0 + \overline{\beta}^0 - \pi^0)\theta^{14} - \\
& - \overline{\nu}^0\theta^{23} - \nu^0\theta^{24} - (\mu^0 - \overline{\mu}^0)\theta^{34} \ ,
\end{aligned}$$

$$
\begin{aligned}
d\theta^2 =& (\epsilon^0 + \overline{\epsilon}^0)\theta^{12} + \kappa^0\theta^{13} + \overline{\kappa}^0\theta^{14} + (\tau^0 - \overline{\alpha}^0 - \beta^0)\theta^{23} + \\
&+ (\overline{\tau}^0 - \alpha^0 - \overline{\beta}^0)\theta^{24} - (\rho^0 - \overline{\rho}^0)\theta^{34} \ , \\
d\theta^3 =& - (\pi^0 + \overline{\tau}^0)\theta^{12} - (\overline{\rho}^0 + \epsilon^0 - \overline{\epsilon}^0)\theta^{13} - \overline{\sigma}^0\theta^{14} + (\mu^0 - \gamma^0 + \overline{\gamma}^0)\theta^{23} + \\
&+ \lambda^0\theta^{24} + (\alpha^0 - \overline{\beta}^0)\theta^{34} \ , \\
d\theta^4 =& - (\overline{\pi}^0 + \tau^0)\theta^{12} - \sigma^0\theta^{13} - (\rho^0 - \epsilon^0 + \overline{\epsilon}^0)\theta^{14} + \overline{\lambda}^0\theta^{23} + \\
&+ (\overline{\mu}^0 + \gamma^0 - \overline{\gamma}^0)\theta^{24} - (\overline{\alpha}^0 - \beta^0)\theta^{34} \ . \qquad (1.4.17)
\end{aligned}
$$

1.4.2 Weyssenhoff Fluid

The Weyssenhoff fluid is a perfect fluid with spin, where the spin of matter fields is the source of torsion in a Einstein-Cartan theory of gravitation. We assume that the Einstein-Cartan space-time is filled up with Weyssenhoff fluid, which is characterized by the canonical energy momentum tensor, given by

$$
t_{ik} = (p + \rho)u_i u_k - p g_{ik} - u^j \bigtriangledown_h (u^h S_{ij}) u_k \ , \qquad (1.4.18)
$$

where p is an isotropic pressure, ρ is the energy density of matter. We simplify the equation (1.4.18) and write as

$$
t_{ik} = (p + \rho)u_i u_k - p g_{ik} + S_{ij;h} u^j u^h u_k + \theta(S_{ij} u^j) u_k \ , \qquad (1.4.19)
$$

where $\theta = u^i{}_{;i}$ is the expansion scalar. Due to the Frankel's condition (1.2.55), the equation(1.4.19) reduces to

$$
t_{ik} = (p + \rho)u_i u_k - p g_{ik} - S_{ij}\dot{u}^j u_k \ , \qquad (1.4.20)
$$

where $\dot{u}^i = u^i{}_{;k}u^k$ is the acceleration vector. The NP concomitants of the equation (1.4.20) is given by

$$
\begin{aligned}
t_{ik} = & \frac{1}{2}(p+\rho)(l_il_k + l_in_k + n_il_k + n_in_k) - p(l_in_k + n_il_k - m_i\overline{m}_k - \overline{m}_im_k) - \\
& - \frac{1}{2\sqrt{2}}\Big\{\big[s_0(\overline{\tau}^0 + \overline{\tau}_1 + \overline{\kappa}^0 + \overline{\kappa}_1 - \nu^0 - \nu_1 - \pi^0 - \pi_1) + \\
& + \overline{s}_0(\tau^0 + \tau_1 + \kappa^0 + \kappa_1 - \overline{\nu}^0 - \overline{\nu}_1 - \overline{\pi}^0 - \overline{\pi}_1)\big](l_il_k + l_in_k - n_il_k - n_in_k) + \\
& + 2\big[\overline{s}_0(\epsilon^0 + \overline{\epsilon}^0 + \epsilon_1 + \overline{\epsilon}_1 + \gamma^0 + \overline{\gamma}^0 + \gamma_1 + \overline{\gamma}_1) + \\
& + s_1(\overline{\tau}^0 + \overline{\tau}_1 + \overline{\kappa}^0 + \overline{\kappa}_1 - \nu^0 - \nu_1 - \pi^0 - \pi_1)\big](m_il_k + m_in_k) + c.c\Big\} \ .
\end{aligned}
\tag{1.4.21}
$$

By virtue of the equation (1.3.30) the equations (1.4.21) becomes

$$
\begin{aligned}
t_{ik} = & \frac{1}{2}(\rho + p)\,(l_il_k + l_in_k + n_in_k + n_il_k) - p(l_in_k + n_il_k - m_i\overline{m}_k - \overline{m}_im_k) + \\
& + \frac{1}{2\sqrt{2}}\big[\{\overline{s}_0(\overline{\pi}^0 + \overline{\nu}^0 - \kappa^0 - \tau^0) + c.c\}(l_il_k + l_in_k - n_il_k - n_in_k) + \\
& + \{2s_1(\pi^0 + \nu^0 - \overline{\kappa}^0 - \overline{\tau}^0) - 2\overline{s}_0(\epsilon^0 + \overline{\epsilon}^0 + \gamma^0 + \overline{\gamma}^0)\}(m_il_k + m_in_k) + \\
& + c.c\big].
\end{aligned}
\tag{1.4.22}
$$

1.4.3 Kinematical Parameters

In order to study the kinematics of time-like and space-like congruences Greenberg [41] has introduced kinematical parameters for time-like congruences and space-like congruences together with natural transport laws. Radhakrishna et al. [70] has introduced 'complete' optical parameters for null-like congruences. The role of the kinematical parameters is very crucial in the study of universe. The propagation equations of these parameters are studied by Patil [96]. Below we obtain the Newman-Penrose concomitants of the kinematical parameters of the time-like vector field in order to study the solutions of the field equations in Einstein-Cartan theory of gravitation. These are expansion θ, the acceleration $\dot{u}_i$, the shear tensor σ_{ij}, and the rotation tensor W_{ij} and are defined as

$$\theta = u^i{}_{;i}\ , \dot{u}_i = u_{i;j}u^j\ , \sigma_{ij} = u_{(i;j)} - \dot{u}_{(i}u_{j)} - \frac{1}{3}\theta h_{ij}\ ,$$

and

$$W_{ij} = u_{[i;j]} - \dot{u}_{[i}u_{j]}\ , \tag{1.4.23}$$

where $h_{ij} = g_{ij} - u_i u_j$ is the 3-dimensions projection operator and $u_i u^i = 1$.

We define $u_i = \frac{1}{\sqrt{2}}(l_i + n_i)$. Hence the kinematical parameters (1.4.23) become

$$\begin{aligned}
\theta =& \frac{1}{\sqrt{2}}(l^i{}_{;i} + n^i{}_{;i})\ ,\\
\dot{u}_i =& \frac{1}{2}(l_{i;k}l^k + l_{i;k}n^k + n_{i;k}l^k + n_{i;k}n^k)\ ,\\
\sigma_{ij} =& \frac{1}{2\sqrt{2}}\big[(l_{i;j} + l_{j;i} + n_{i;j} + n_{j;i}) - \dot{u}_i(l_i + n_j) - (l_i + n_j)\dot{u}_j\big] -\\
& - \frac{1}{3}\theta h_{ij}\ ,
\end{aligned}$$

and

$$W_{ij} =\frac{1}{2\sqrt{2}}\big[(l_{i;j}-l_{j;i})+(n_{i;j}-n_{j;i})-\dot{u}_i(l_j+n_j)+(l_i+n_i)\dot{u}_j\big] \ , \tag{1.4.24}$$

where

$$\frac{1}{3}\theta h_{ij} =\frac{\theta}{6}\big[(l_in_j+n_il_j)-2(m_i\overline{m}_j+\overline{m}_im_j)-(l_il_j+n_in_j)\big] \ . \tag{1.4.25}$$

The expressions for the covariant derivative of the null vector fields and their intrinsic derivatives along the tetrad vector fields are enumerated in the appendix. Using these equations we readily obtain

$$\begin{aligned}
\theta =&\frac{1}{\sqrt{2}}\big(\epsilon^0+\overline{\epsilon}^0+\epsilon_1+\overline{\epsilon}_1-\gamma^0-\overline{\gamma}^0-\gamma_1-\overline{\gamma}_1-\rho^0-\overline{\rho}^0-\rho_1-\overline{\rho}_1+\\
&+\mu^0+\overline{\mu}^0+\mu_1+\overline{\mu}_1\big) \ ,\\
\dot{u}_i =&\frac{1}{2}\Bigg[(\epsilon^0+\overline{\epsilon}^0+\epsilon_1+\overline{\epsilon}_1+\gamma^0+\overline{\gamma}^0+\gamma_1+\overline{\gamma}_1)(l_i-n_i)-\\
&-(\overline{\tau}^0+\overline{\tau}_1+\overline{\kappa}^0+\overline{\kappa}_1-\nu^0-\nu_1-\pi^0-\pi_1)m_i-c.c\Bigg] \ ,\\
\sigma_{ij} =&\frac{1}{6\sqrt{2}}\Bigg[\big\{2(\gamma^0+\overline{\gamma}^0+\gamma_1+\overline{\gamma}_1-\epsilon^0-\overline{\epsilon}^0-\epsilon_1-\overline{\epsilon}_1)-\\
&-(\rho^0+\overline{\rho}^0+\rho_1+\overline{\rho}_1-\mu^0-\overline{\mu}^0-\mu_1-\overline{\mu}_1)\big\}(l_il_j+n_in_j-\\
&-2l_{(i}n_{j)}-2m_{(i}\overline{m}_{j)})+3\big\{\big(\overline{\kappa}^0+\overline{\kappa}_1-\overline{\tau}^0-\overline{\tau}_1+\nu^0+\nu_1-\\
&-\pi^0-\pi_1-2(\alpha^0+\overline{\beta}^0+\alpha_1+\overline{\beta}_1)\big)\big(l_{(i}m_{j)}-m_{(i}n_{j)}\big)+\\
&+2\big(\overline{\sigma}^0+\overline{\sigma}_1-\lambda^0-\lambda_1\big)m_im_j\big\}+c.c\Bigg] \ ,\\
W_{ij} =&\frac{1}{2\sqrt{2}}\Bigg[2(\rho^0-\overline{\rho}^0+\rho_1-\overline{\rho}_1+\mu^0-\overline{\mu}^0+\mu_1-\overline{\mu}_1)\overline{m}_{[i}m_{j]}-
\end{aligned}$$

$$
- \left\{ 2(\alpha^0 + \overline{\beta}^0 + \alpha_1 + \overline{\beta}_1) + \overline{\kappa}^0 + \overline{\kappa}_1 - \overline{\tau}^0 - \overline{\tau}_1 + \right.
$$

$$
\left. + \nu^0 + \nu_1 - \pi^0 - \pi_1 \right\} \left(l_{[i} m_{j]} + m_{[i} n_{j]} \right) - c.c \Big] \,. \tag{1.4.26}
$$

It should be noticed that, using the conditions (1.3.30), the equations (1.4.26) can be rewritten as

$$
\begin{aligned}
\theta =& \frac{1}{\sqrt{2}} \left(\epsilon^0 + \overline{\epsilon}^0 - \gamma^0 - \overline{\gamma}^0 - \rho^0 - \overline{\rho}^0 + \mu^0 + \overline{\mu}^0 \right) \,, \\
\dot{u}_i =& \frac{1}{2} \left[(\epsilon^0 + \overline{\epsilon}^0 + \gamma^0 + \overline{\gamma}^0)(l_i - n_i) - (\overline{\tau}^0 + \overline{\kappa}^0 - \nu^0 - \pi^0) m_i - c.c \right] \,, \\
\sigma_{ij} =& \frac{1}{6\sqrt{2}} \left[\left\{ 2(\gamma^0 + \overline{\gamma}^0 - \epsilon^0 - \overline{\epsilon}^0) - (\rho^0 + \overline{\rho}^0 - \mu^0 - \overline{\mu}^0) \right\} \cdot \right. \\
& \cdot \left(l_i l_j + n_i n_j - 2 l_{(i} n_{j)} - 2 m_{(i} \overline{m}_{j)} \right) + 3 \left\{ \left(\overline{\kappa}^0 - \overline{\tau}^0 + \nu^0 - \pi^0 - \right. \right. \\
& \left. \left. \left. - 2(\alpha^0 + \overline{\beta}^0) \right) \left(l_{(i} m_{j)} - m_{(i} n_{j)} \right) + 2(\overline{\sigma}^0 - \lambda^0) m_i m_j \right\} + c.c \right] \,, \\
W_{ij} =& \frac{1}{2\sqrt{2}} \left[\left(\overline{\tau}^0 + \pi^0 - \nu^0 - \overline{\kappa}^0 + 4\sqrt{2} k \overline{s}_0 - 2(\alpha^0 + \overline{\beta}^0) \right) \big(l_{[i} m_{j]} + \right. \\
& \left. + m_{[i} n_{j]} \big) + c.c + 2 \big(\rho^0 - \overline{\rho}^0 + \mu^0 - \overline{\mu}^0 - 4\sqrt{2} k s_1 \big) \overline{m}_{[i} m_{j]} \right] \,.
\end{aligned} \tag{1.4.27}
$$

Appendix 1:

$$
\begin{aligned}
l_{i;j} =&(\gamma^0+\overline{\gamma}^0+\gamma_1+\overline{\gamma}_1)l_il_j-(\alpha^0+\overline{\beta}^0+\alpha_1+\overline{\beta}_1)l_im_j-\\
&-(\overline{\alpha}^0+\beta^0+\overline{\alpha}_1+\beta_1)l_i\overline{m}_j+(\epsilon^0+\overline{\epsilon}^0+\epsilon_1+\overline{\epsilon}_1)l_in_j-\\
&-(\overline{\tau}^0+\overline{\tau}_1)m_il_j+(\overline{\sigma}^0+\overline{\sigma}_1)m_im_j+(\overline{\rho}^0+\overline{\rho}_1)m_i\overline{m}_j-\\
&-(\overline{\kappa}^0+\overline{\kappa}_1)m_in_j-(\tau^0+\tau_1)\overline{m}_il_j+(\rho^0+\rho_1)\overline{m}_im_j+\\
&+(\sigma^0+\sigma_1)\overline{m}_i\overline{m}_j-(\kappa^0+\kappa_1)\overline{m}_in_j\ ,\\
n_{i;j} =&-(\epsilon^0+\overline{\epsilon}^0+\epsilon_1+\overline{\epsilon}_1)n_in_j-(\gamma^0+\overline{\gamma}^0+\gamma_1+\overline{\gamma}_1)n_il_j-\\
&-(\overline{\alpha}^0+\beta^0+\overline{\alpha}_1+\beta_1)n_i\overline{m}_j-(\alpha^0+\overline{\beta}^0+\alpha_1+\overline{\beta}_1)n_im_j+\\
&+(\overline{\pi}^0+\overline{\pi}_1)\overline{m}_in_j+(\overline{\nu}^0+\overline{\nu}_1)\overline{m}_il_j-(\overline{\lambda}^0+\overline{\lambda}_1)\overline{m}_i\overline{m}_j-\\
&-(\overline{\mu}^0+\overline{\mu}_1)\overline{m}_im_j+(\pi^0+\pi_1)m_in_j+(\nu^0+\nu_1)m_il_j-\\
&-(\mu^0+\mu_1)m_i\overline{m}_j-(\lambda^0+\lambda_1)m_im_j\ ,\\
m_{i;j} =&-(\kappa^0+\kappa_1)n_in_j-(\tau^0+\tau_1)n_il_j+(\sigma^0+\sigma_1)n_i\overline{m}_j+\\
&+(\rho^0+\rho_1)n_im_j+(\overline{\pi}^0+\overline{\pi}_1)l_in_j+(\overline{\nu}^0+\overline{\nu}_1)l_il_j-\\
&-(\overline{\lambda}^0+\overline{\lambda}_1)l_i\overline{m}_j-(\overline{\mu}^0+\overline{\mu}_1)l_im_j+(\epsilon^0-\overline{\epsilon}^0+\epsilon_1-\overline{\epsilon}_1)m_in_j+\\
&+(\gamma^0-\overline{\gamma}^0+\gamma_1-\overline{\gamma}_1)m_il_j+(\overline{\alpha}^0-\beta^0+\overline{\alpha}_1-\beta_1)m_i\overline{m}_j-\\
&-(\alpha^0-\overline{\beta}^0+\alpha_1-\overline{\beta}_1)m_im_j\ ,\\
\overline{m}_{i;j} =&-(\overline{\kappa}^0+\overline{\kappa}_1)n_in_j-(\overline{\tau}^0+\overline{\tau}_1)n_il_j+(\overline{\rho}^0+\overline{\rho}_1)n_i\overline{m}_j+\\
&+(\overline{\sigma}^0+\overline{\sigma}_1)n_im_j+(\pi^0+\pi_1)l_in_j+(\nu^0+\nu_1)l_il_j-\\
&-(\mu^0+\mu_1)l_i\overline{m}_j-(\lambda^0+\lambda_1)l_im_j-(\epsilon^0-\overline{\epsilon}^0+\epsilon_1-\overline{\epsilon}_1)\overline{m}_in_j+\\
&-(\gamma^0-\overline{\gamma}^0+\gamma_1-\overline{\gamma}_1)\overline{m}_il_j-(\overline{\alpha}^0-\beta^0+\overline{\alpha}_1-\beta_1)\overline{m}_i\overline{m}_j+\\
&+(\alpha^0-\overline{\beta}^0+\alpha_1-\overline{\beta}_1)\overline{m}_im_j\ .
\end{aligned}
$$

Appendix 2: Intrinsic Derivatives of the tetrad vector fields.

$$
\begin{aligned}
l_{i;j}l^j =&(\epsilon^0+\overline{\epsilon}^0+\epsilon_1+\overline{\epsilon}_1)l_i-(\kappa^0+\kappa_1)\overline{m}_i-(\overline{\kappa}^0+\overline{\kappa}_1)m_i\ ,\\
l_{i;j}n^j =&(\gamma^0+\overline{\gamma}^0+\gamma_1+\overline{\gamma}_1)l_i-(\tau^0+\tau_1)\overline{m}_i-(\overline{\tau}^0+\overline{\tau}_1)m_i\ ,\\
l_{i;j}m^j =&(\overline{\alpha}^0+\beta^0+\overline{\alpha}_1+\beta_1)l_i-(\sigma^0+\sigma_1)\overline{m}_i-(\overline{\rho}^0+\overline{\rho}_1)m_i\ ,\\
l_{i;j}\overline{m}^j =&(\alpha^0+\overline{\beta}^0+\alpha_1+\overline{\beta}_1)l_i-(\rho^0+\rho_1)\overline{m}_i-(\overline{\sigma}^0+\overline{\sigma}_1)m_i\ ,
\end{aligned}
$$

$$
\begin{aligned}
n_{i;j}l^j =&-(\epsilon^0+\overline{\epsilon}^0+\epsilon_1+\overline{\epsilon}_1)n_i+(\overline{\pi}^0+\overline{\pi}_1)\overline{m}_i+(\pi^0+\pi_1)m_i\ ,\\
n_{i;j}n^j =&-(\gamma^0+\overline{\gamma}^0+\gamma_1+\overline{\gamma}_1)n_i+(\overline{\nu}^0+\overline{\nu}_1)\overline{m}_i+(\nu^0+\nu_1)m_i\ ,\\
n_{i;j}m^j =&(\overline{\alpha}^0+\beta^0+\overline{\alpha}_1+\beta_1)n_i+(\overline{\lambda}^0+\overline{\lambda}_1)\overline{m}_i+(\mu^0+\mu_1)m_i\ ,\\
n_{i;j}\overline{m}^j =&(\alpha^0+\overline{\beta}^0+\alpha_1+\overline{\beta}_1)n_i+(\overline{\mu}^0+\overline{\mu}_1)\overline{m}_i-(\lambda^0+\lambda_1)m_i\ ,
\end{aligned}
$$

$$
\begin{aligned}
m_{i;j}l^j =&-(\kappa^0+\kappa_1)n_i+(\overline{\pi}^0+\overline{\pi}_1)l_i+(\epsilon^0-\overline{\epsilon}^0+\epsilon_1-\overline{\epsilon}_1)m_i\ ,\\
m_{i;j}n^j =&-(\tau^0+\tau_1)n_i+(\overline{\nu}^0+\overline{\nu}_1)l_i+(\gamma^0-\overline{\gamma}^0+\gamma_1-\overline{\gamma}_1)m_i\ ,\\
m_{i;j}m^j =&-(\sigma^0+\sigma_1)n_i+(\overline{\lambda}^0+\overline{\lambda}_1)l_i-(\overline{\alpha}^0-\beta^0+\overline{\alpha}_1-\beta_1)m_i\ ,\\
m_{i;j}\overline{m}^j =&-(\rho^0+\rho_1)n_i+(\overline{\mu}^0+\overline{\mu}_1)l_i+(\alpha^0-\overline{\beta}^0+\alpha_1-\overline{\beta}_1)m_i\ ,
\end{aligned}
$$

$$
\begin{aligned}
\overline{m}_{i;j}l^j =&-(\overline{\kappa}^0+\overline{\kappa}_1)n_i+(\pi^0+\pi_1)l_i-(\epsilon^0-\overline{\epsilon}^0+\epsilon_1-\overline{\epsilon}_1)\overline{m}_i\ ,\\
\overline{m}_{i;j}n^j =&-(\overline{\tau}^0+\overline{\tau}_1)n_i+(\nu^0+\nu_1)l_i-(\gamma^0-\overline{\gamma}^0+\gamma_1-\overline{\gamma}_1)\overline{m}_i\ ,\\
\overline{m}_{i;j}m^j =&-(\overline{\rho}^0+\overline{\rho}_1)n_i+(\mu^0+\mu_1)l_i+(\overline{\alpha}^0-\beta^0+\overline{\alpha}_1-\beta_1)\overline{m}_i\ ,\\
\overline{m}_{i;j}\overline{m}^j =&-(\overline{\sigma}^0+\overline{\sigma}_1)n_i+(\lambda^0+\lambda_1)l_i-(\alpha^0-\overline{\beta}^0+\alpha_1-\overline{\beta}_1)\overline{m}_i\ .
\end{aligned}
$$

Appendix 3: Metric Equations

$$
\begin{aligned}
\Delta l^i - Dn^i =& (\gamma^0 + \overline{\gamma}^0 + \gamma_1 + \overline{\gamma}_1) l^i - (\overline{\tau}^0 + \pi^0 + \overline{\tau}_1 + \pi_1) m^i - \\
& - (\tau^0 + \overline{\pi}^0 + \tau_1 + \overline{\pi}_1) \overline{m}^i + (\epsilon^0 + \overline{\epsilon}^0 + \epsilon_1 + \overline{\epsilon}_1) n^i \ , \\
\delta l^i - Dm^i =& (\overline{\alpha}^0 + \beta^0 - \overline{\pi}^0 + \overline{\alpha}_1 + \beta_1 - \overline{\pi}_1) l^i - \\
& - (\overline{\rho}^0 + \epsilon^0 - \overline{\epsilon}^0 + \overline{\rho}_1 + \epsilon_1 - \overline{\epsilon}_1) m^i - \\
& - (\sigma^0 + \sigma_1) \overline{m}^i + (\kappa^0 + \kappa_1) n^i \ , \\
\delta n^i - \Delta m^i =& - (\overline{\nu}^0 + \overline{\nu}_1) l^i + (\mu^0 - \gamma^0 + \overline{\gamma}^0 + \mu_1 - \gamma_1 + \overline{\gamma}_1) m^i + \\
& + (\overline{\lambda}^0 + \overline{\lambda}_1) \overline{m}^i + (\tau^0 - \overline{\alpha}^0 - \beta^0 + \tau_1 - \overline{\alpha}_1 - \beta_1) n^i \ , \\
\overline{\delta} m^i - \delta \overline{m}^i =& (\overline{\mu}^0 - \mu^0 + \overline{\mu}_1 - \mu_1) l^i + (\alpha^0 - \overline{\beta}^0 + \alpha_1 - \overline{\beta}_1) m^i - \\
& - (\overline{\alpha}^0 - \beta^0 + \overline{\alpha}_1 - \beta_1) \overline{m}^i + (\overline{\rho}^0 - \rho^0 + \overline{\rho}_1 - \rho_1) n^i \ .
\end{aligned}
$$

Chapter 2

A Geometry of a Non-Riemannian Space

2.1 Introduction

The use of geometry in the development of science in general and physics in particular is well known. The familiar geometry in which parallel lines never meet or diverge, the angles of a triangle add up to 180 degrees, geodesics are straight lines and so on is known as Euclidean geometry and the space on which the geometry rest is called Euclidean space. Newton considered Euclidean space as consisting of 3- dimensions and time as consisting of 1- dimension- the 4th dimension independent of space and developed Newtonian mechanics which is well-known to all. Euclidean space is characterised by the metric

$$ds^2 = \eta_{ij} dx^i dx^j \ , \quad i, j = 1, 2, 3, \tag{2.1.1}$$

where $x^i = (x^1, x^2, x^3) = (x, y, z)$ -space coordinates and

$$\begin{aligned} \eta_{ij} =& 1, \quad \text{when} \quad i = j \ , \\ =& 0, \quad \text{when} \quad i \neq j \ , \end{aligned} \tag{2.1.2}$$

is called Euclidean metric tensor.

By combining 3-dimensions space and 1- dimension time into a single manifold is called the Minkowski space-time and the corresponding geometry is called as pseudo Euclidean geometry. This is the kind of geometry Einstein used in his geometrisation of the special theory of relativity. The 4- dimensions Minkowski space-time is characterised by the flat metric defined by the equation (2.1.1), but in which

$x^i = (x^1, x^2, x^3, x^4) = (x, y, z, ict)$ the space-time coordinates and the Minkowski metric tensor η_{ij} is defined by

$$\begin{aligned} \eta_{ij} =& 1, \quad \text{when} \quad i = j = 1, 2, 3 \ , \\ =& -1, \quad \text{when} \quad i = j = 4 \ , \\ =& 0, \quad \text{when} \quad i \neq j \ . \end{aligned} \tag{2.1.3}$$

In non-relativistic classical mechanics, the use of Euclidean space instead of space-time is appropriate, as time is treated as universal and constant, being independent of the state of motion of an observer. In relativistic contexts, the space-time is our universe. Time cannot be separated from 3- dimensions space.

The first person to go beyond Euclid geometry and appreciate its significance was Carl Friedrich Gauss. Alternate geometries are therefore known as non-Euclidean geometries. Non-Euclidean geometry was independently discovered by the Russian Labochevsky, N. I. in 1829 and by a Hungarian Bolyai, J. The new geometry is known as 'hyperbolic' geometry, in which, the angles of a triangle always add up to less than 180 degrees and many straight lines can be drawn parallel to the given straight line through a point out side the straight line.

Bernhard Riemann realized the possibility of yet another geometry, who comprehensively put across the notion of non- Euclidean geometry in 1851, in which the angles of a triangles always add up to more than 180 degrees and all 'lines of longitudes' cross the equator at right angles and must therefore all be parallel to one another, they all cross each other at poles. Hence no parallel lines exist on such a space.

Einstein uses 4- dimensions pseudo Riemannian space-time, in the sense that the metric of the space-time is positive indefinite, with symmetric connections, called the Christoffel symbols, and developed General Theory of Relativity. Einstein's special theory of relativity describes the way things move about in what is called 'flat space-time'. Einstein's General Theory of Relativity describes how things move in curved space-time, and the curvature in space-time is caused by the presence of matter in the universe.

A Riemannian space-time on which the Riemannian geometry based is characterized by the pseudo Riemannian metric defined by

$$ds^2 = g_{ij} dx^i dx^j \ , \tag{2.1.4}$$

where x^i are the space-time coordinates of an event and g_{ij} are the components of the Riemannian metric tensor, which are functions of coordinates x^i at the point and representing the gravitational potential. We see that the space, time and the gravity are all invisible. The ingenuity of Einstein is that he unified all these invisible quantities into a concise formula given in the equation (2.1.4) and we call it as Riemannian metric. The geometry on such a space-time is called a Riemannian geometry. Riemannian geometry generalises Euclidean geometry to spaces that are not necessarily flat, although they still resemble Euclidean space at each point infinitesimally. As the consequence of this theory, Einstein deduced that the ray of light bends, the perihelion of the planet Mercury advances in the gravitational field of the Sun and made invisible mathematics visible. This is perhaps in consonance with the fictitious description of mathematics as our invisible culture.

In the history of science, Einstein's general theory of relativity is considered to be the most successful theory of gravitation. However, the success of general relativity is decidedly mixed. On one hand it is highly successful which has passed every unambiguous test both experimentally and observationally. On the other hand it is inconsistent with quantum mechanics, not free from singularities and not included the spin of the gravitating matter and so on. To address such issues there are several well known (more than forty theories of gravitation) classical theories of gravitation other than Einstein's general theory of relativity.

Einstein-Cartan theory of gravitation is one such modified theory of gravitation, in which the spin of the gravitating matter is introduced, developed by Cartan with the hopes of avoiding singularities. The underlying geometry for the Einstein-Cartan theory of gravitation is non-Riemannian characterized by the metric (2.1.4) but where in which the connections are asymmetric, through which the torsion is introduced. The theory is also called as the torsion theory of gravitation. Non-Riemannian geometry generalizes Riemannian geometry to spaces in which covariant derivative of a tensor involves torsion term through asymmetric connections.

In this chapter, the technique of differential forms on non-Riemannian space is presented and the essence of non-Riemannian geometry is studied. The material of this chapter is organised as follows. In each of the above mentioned theory of gravitation, we discuss some of the vector identities and their invariance characteristics in the Section 2 and 3. A technique of differential forms, developed by Katkar [61] on a non-Riemannian space, is presented in the Section 4. A formula for the curvature of a non-Riemannian space is derived in the next section. A non-Riemannian 2- space is constructed and its curvature is obtained. The results are corroborated by employing the techniques of differential forms on a non-Riemannian space in the Section 5. Maxwell's equations in a more general form are derived in the last Section.

2.2 Gradient, Divergence and Curl in 3-dimension Euclidean Space

If f is a scalar function of coordinates in R^3, then we are familiar with the standard result

$$gradf = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k \ , \tag{2.2.1}$$

where $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are the components of $gradf$ with respect to the bases indicated. Similarly, if $\overline{F} = F_1 i + F_2 j + F_3 k$ is a vector field in the

3-dimensional Euclidean space, where F_1, F_2, F_3 are the components of the vector field with respect to the basis indicated then we have

$$div\overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \ , \qquad (2.2.2)$$

$$curl\overline{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) k \ , \qquad (2.2.3)$$

where $\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$ are the components of the $curl\overline{F}$ with respect to the basis indicated. In 3-dimension Euclidean space we have the standard vector identities;

$$curl(gradf) = 0 \ \text{ and } \ div(curl\overline{F}) = 0 \ , \qquad (2.2.4)$$

as the two rows of the determinant are identical.

In tensor notations we define the $gradf, div\overline{F}$ and $curl\overline{F}$ in a 3-dimensional Euclidean space as

$$gradf = \frac{\partial f}{\partial x^i}\eta^{ik}\overline{e}_k \ , \quad div\overline{F} = \frac{\partial F_1}{\partial x^k}\eta^{ik} \ , \quad \text{and}$$
$$div\overline{F} = \frac{\partial F^l}{\partial x^k}\eta^{ik}\overline{e}_l \wedge \overline{e}_i \ , \qquad (2.2.5)$$

where $\wedge$ is the wedge product of vectors. Wedge product of two vectors is nothing but their vector product, and η^{ik} is defined in the equation (2.1.2). The tensor notations are useful to extend the definitions (2.2.5) into the higher dimension spaces.

It is evident that definitions (2.2.5) are invariant under the coordinate transformations

$$x^{i'} = \wedge^{i'}{}_{k} x^{k} \ , \tag{2.2.6}$$

where $\wedge^{i'}{}_{k}$ is the matrix of transformation. The matrix $\wedge^{i'}{}_{k}$ and its inverse matrix of transformation $\wedge_{i'}{}^{k}$ satisfy the condition

$$\wedge^{i'}{}_{h} \wedge_{i'}{}^{k} = \delta^{k}_{h} \ . \tag{2.2.7}$$

2.2.1 4-dimensional Euclidean space and time

In the 4-dimensional Euclidean space and time the definition (2.2.5) remains the same, however, the two inertial frames in the Newtonian relativity are connected by the Galilean transformation equations, where the matrix of Galilean transformation is given by

$$\| \wedge^{i'}_{k} \| = \begin{pmatrix} 1 & 0 & 0 & -v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{2.2.8}$$

and v is the uniform velocity.

We see that the matrix of Galilean transformation equations is the particular case of the matrix of the most general transformation. Hence it is obvious that the *gradf* , $div\overline{F}$ and $curl\overline{F}$ are invariant under Galilean transformation equations.

2.2.2 4-dimensions Minkowski Space-Time of Special Relativity

A space-time characterized by the metric (2.1.1) together with (2.1.3) is called the Minkowski flat space-time. The metric can also be represented by

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2 \ . \tag{2.2.9}$$

We choose $x^4 = ct$, where c is the velocity of light and t is time, so that $x^4 = ct$ has the unit of length and $(x^1, x^2, x^3) = (x, y, z)$ are space coordinates. In this notation the Minkowski metric becomes $ds^2 = dx^2+dy^2+dz^2-c^2dt^2$. In the 4-dimension Minkowski space-time of special relativity the $gradf, div\overline{F}$ and $curl\overline{F}$ are defined in the same way as they are defined in the equation (2.2.5). The only difference is in the definition of the metric tensor η_{ij} which is defined in the equation (2.1.3). The definitions (2.2.5) also invariant under Lorentz transformations equations, as the matrix of Lorentz transformation equations is given by

$$\parallel \wedge_k^{i'} \parallel = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma v}{c^2} & 0 & 0 & \gamma \end{pmatrix} , \tag{2.2.10}$$

and $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Lorentz factor.

2.2.3 Exterior Derivative and the Vector Quantities

In the Minkowski space-time (M, η), Amur and Christopher [2] have expressed the $gradf, div\overline{F}$ and $curl\overline{F}$ in terms of exterior derivative as

(i) $df \wedge {}^*d\overline{X} = (gradf)dV$,

(ii) $d\overline{F} \cdot \wedge^* d\overline{X} = div\overline{F}dV$, where $\cdot$ indicates the dot product between two vectors and $\wedge$ denotes the wedge product,

(iii) $-d\overline{F} \wedge \wedge^* d\overline{X} = (curl\overline{F})dV$, where double wedge products are used to indicates the wedge product between differential forms and vector product between vectors and $dV = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$ is the 4-volume and * is the Hodge star operator defined by

$$^*dx^i = (-1)^{i-1} dx^1 \wedge dx^2 \wedge ... \wedge d\hat{x}^i \wedge ... dx^n \ , \tag{2.2.11}$$

where cap over the differential dx^i indicates that the term is to be deleted from the expression and

$$dx^i \wedge {}^*dx^k = \eta^{ik} dV \ .$$

They have also shown that these definitions do not depend on any particular coordinate frame.

2.3 Gradient, divergence and Curl in a Riemannian space of Einstein's General Relativity

The notion of partial derivative of a function from multi-variable calculus is extended in a Riemannian space-time of Einstein's general theory of relativity to the notion of covariant derivative of a tensor with symmetric connections. Furthermore, the covariant derivative of a form of any degree is independent of symmetric connections as the terms vanish due to the product of symmetric connections and the skew-symmetric basis vectors. Consequently, we obtain in Riemannian space, the covariant derivative of a form by taking either the partial derivative or covariant derivative of the associated tensor.

The gradient, divergence and curl of a vector field in a Riemannian space-time of General relativity are defined as

$$\begin{aligned} \hat{grad} f &= f_{/i} = \frac{\partial f}{\partial x^i} = f_{,i} \ , \\ \hat{div} A_i &= \hat{div} A^i = A^i_{/i} = g^{ik} A_{i/k} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} (A^i \sqrt{-g}) \ , \\ \hat{curl} A_i &= A_{i/j} - A_{j/i} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} \ . \end{aligned} \tag{2.3.1}$$

The curl of a vector is the tensor of covariant rank two. These definitions are invariant under the coordinate transformation from x^i to $\overline{x}^i$. We have used overhead cap to denote the terms in a Riemannian space.

2.3.1 The Vector Identities

We have the standard vector identities in vector calculus:

$$\hat{curl}(\hat{grad}f) = 0 \ ,$$

and

$$\hat{div}(\hat{curl}A_i) = 0 \ . \tag{2.3.2}$$

However, in Riemannian space the curl of the gradient of a scalar function f is defined as

$$\hat{curl}(\hat{grad}f) = curl(f_{/i}) = f_{/ij} - f_{/ji} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i} \ ,$$
$$\Rightarrow \hat{curl}(\hat{grad}f) = 0 \ .$$

Similarly

$$\hat{div}(\hat{curl}A_i) = 0 \ . \tag{2.3.3}$$

These identities are well expressed in the techniques of differential forms viz., the exterior derivative. In fact the exterior derivative subsumes the ordinary gradient, curl and the divergence and the two vector identities $\hat{curl}(\hat{grad}f) = 0$ and $\hat{div}(\hat{curl}A_i) = 0$. The same is illustrated below.

2.3.2 Vector Identities in Exterior Derivative

In order to obtain the vector identities in terms of exterior derivative, let f, $\tilde{\omega} = \omega_i dx^i$, $\tilde{\sigma} = f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$ be differential forms of degree 0, 1 and 2 respectively. Then by applying the exterior derivative to each one of these form, we obtain

$$\begin{aligned} df =& f_{,i} dx^i \ , \\ d\tilde{\omega} =& \frac{1}{2}\left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i}\right) dx^j \wedge dx^i \ , \\ d\tilde{\sigma} =& \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dx \wedge dy \wedge dz \ . \end{aligned} \tag{2.3.4}$$

The coefficients of these equations are respectively called $gradf$, $curl\omega$ and $divF$ with respect to a basis indicated. Now taking the exterior derivative of 1-form df, defined above, we obtain the vector identities in the form

$$\begin{aligned} d^2 f =& \frac{1}{2}(f_{/ij} - f_{/ji}) dx^j \wedge dx^i = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i}\right) dx^j \wedge dx^i \ , \\ d^2 f =& 0 \Leftrightarrow c\hat{u}rl(gr\hat{a}df) \equiv 0 \ . \end{aligned} \tag{2.3.5}$$

Similarly, taking the exterior derivative of 2-form $d\tilde{\omega}$ we obtain

$$\begin{aligned} d^2\tilde{\omega} =& -\frac{1}{6}\big[(\omega_{i/jk} - \omega_{i/kj}) - (\omega_{j/ik} - \omega_{j/ki}) + \\ & + (\omega_{k/ij} - \omega_{k/ji})\big] dx^i \wedge dx^j \wedge dx^k \ . \end{aligned} \tag{2.3.6}$$

We see that the coefficients are the components of divergence of curl of a vector. Thus we have

$$d^2\tilde{\omega} = d\hat{i}v(cu\hat{r}l\tilde{\omega}) - \frac{1}{6}\big[(\omega_{i/jk} - \omega_{i/kj}) - (\omega_{j/ik} - \omega_{j/ki}) + \\ + (\omega_{k/ij} - \omega_{k/ji})\big] dx^i \wedge dx^j \wedge dx^k \ . \qquad (2.3.7)$$

By using the Ricci Theorem $\omega_{i/jk} - \omega_{i/kj} = \omega_h \hat{R}^h{}_{ijk}$ we obtain

$$d^2\tilde{\omega} = -\frac{1}{6}\left[\omega_h \hat{R}^h{}_{ijk} - \omega_h \hat{R}^h{}_{jik} + \omega_h \hat{R}^h{}_{kji}\right] dx^i \wedge dx^j \wedge dx^k \ ,$$
$$d^2\tilde{\omega} = -\frac{1}{6}\left[\omega_h \left(\hat{R}^h{}_{ijk} + \hat{R}^h{}_{jki} + \hat{R}^h{}_{kij}\right)\right] dx^i \wedge dx^j \wedge dx^k \ , \qquad (2.3.8)$$
$$d^2\tilde{\omega} = d\hat{i}v(cu\hat{r}l\tilde{\omega}) = 0 \ .$$

Hence the repeated exterior derivative operator in the Riemannian space-time includes the vector identities

$$cu\hat{r}lgr\hat{a}df = 0 \ \text{ and also the } \ d\hat{i}vcu\hat{r}lf = 0 \ ,$$

when applied to 0-form and 1-form respectively.

i. e.

$$d^2 f \equiv cu\hat{r}l(gr\hat{a}df) = 0 \ , d^2\tilde{\omega} = d\hat{i}v(cu\hat{r}l\tilde{\omega}) = 0 \ .$$

We will prove, in the next section that, these identities do not hold good in a non-Riemannian space.

2.4 Gradient, Divergence and Curl in a Non-Riemannian Geometry

In a non-Rimennian space-time the gradient of a function, the divergence of a vector field and the curl of a vector field are defined below:

Let f be a scalar function, then

$$gradf = f_{;i} \ , \tag{2.4.1}$$

where $f_{;i} = f_{/i} = f_{,i}$.

Contracting the indices in the definition of covariant derivative of a contravariant vector, we get

$$A^i{}_{;i} = A^i{}_{/i} - A^k K_{ik}{}^i \ , \tag{2.4.2}$$

$$\Rightarrow divA^i = \hat{div}A^i - A^k K_{ik}{}^i \ . \tag{2.4.3}$$

Using the relation between the contortion tensor and torsion tensor (1.2.18), we obtain $K_{ik}{}^i = -2Q_{ik}{}^i$. Hence the equation (2.4.3) becomes

$$divA^i = \hat{div}A^i + 2A^k Q_{ik}{}^i \ , \tag{2.4.4}$$

However, the Frankel condition suggests that torsion trace vanishes identically. Hence, we have

$$divA^i = \hat{div}A^i \ ,$$

Now the curl of a vector field A_i is defined by

$$curlA_i = A_{i;j} - A_{j;i} = A_{i/j} - A_{j/i} + 2A_k Q_{ij}{}^k \ , \tag{2.4.5}$$

$$\Rightarrow curlA_i = \hat{curl}A_i + 2A_k Q_{ij}{}^k \ ,$$

where

$$\hat{curl}A_i = A_{i/j} - A_{j/i} \ .$$

Replacing the arbitrary vectorA_i by $f_{;i}$ in the above equation we get

$$curl(gradf) = \hat{curl}(\hat{grad}f) + 2f_{;k}Q_{ij}{}^k \ , \tag{2.4.6}$$

where

$$\hat{curl}(\hat{grad}f) = 0 \Rightarrow curl(gradf) = 2f_{;k}Q_{ij}{}^k \ . \tag{2.4.7}$$

2.4.1 Techniques of Differential Forms in a Non-Riemannian Space

Techniques of differential forms in a Riemannian space is wel-known in the literature Israel, W [54], Spivak, Michael [120], Choquet Bruhat et al. [18], Franders, Harley [35], Bernard Schutz [110]. Exterior derivative 'd' defined in such a space is connection independent. Hence, it can be obtained by taking either covariant derivative or the partial derivative (immaterial which) of an associated p^{th} rank tensor of a form of degree $p \geq 0$. However, this is not true in a non-Riemannian space as it involves asymmetric connections. Katkar [61] has introduced a new operator d_* and derived the Cartan's equations of structure. The operator d_* is connection dependent and hence obtained by taking the covariant derivative with respect to asymmetric connections of a differential form of any degree $p \geq 0$. The operator d_* is defined by

$$d_* : \wedge^r T_p^* \to \wedge^{r+1} T_p^* \ ,$$

$$\begin{aligned} d_*\tilde{\omega} = {} & \omega_{i_1 i_2 \dots i_{r;k}} d_* x^k \wedge d_* x^{i_1} \wedge d_* x^{i_2} \wedge \dots \wedge d_* x^{i_r} - \\ & - \omega_{i_1 i_2 \dots i_r} \left[\sum_{p=1}^{r} (-1)^{p-1} d_* x^{i_1} \wedge \dots \wedge d_* x^{i_{p-1}} \wedge d_*^2 x^{i_p} \wedge \dots \wedge d_* x^{i_r} \right] , \end{aligned} \tag{2.4.8}$$

for any $\tilde{\omega} \in \wedge^r T_p^*$. Here the symbols $\wedge^r T_p^*$ and $\wedge^{r+1} T_p^*$ stand for the set of all r-forms and $(r+1)$-forms respectively. The exterior derivative operator d_* satisfies the following properties

$$\begin{aligned} (i) \quad & d_* f = f_{,i} d_* x^i \ , \\ (ii) \quad & d_*(\tilde{\omega} + \tilde{\sigma}) = d_*\tilde{\omega} + d_*\tilde{\sigma} \ , \\ (iii) \quad & d_*(fg) = d_* f \cdot g + f \cdot d_* g \ , \\ (iv) \quad & d_*(\tilde{\omega} \wedge \tilde{\sigma}) = d_*\tilde{\omega} \wedge \tilde{\sigma} + (-1)^{\text{deg of } \tilde{\omega}} \tilde{\omega} \wedge d_*\tilde{\sigma} \ , \\ (v) \quad & d_*(f\tilde{\omega}) = d_* f \wedge \tilde{\omega} + f d_*\tilde{\omega} \ , \\ (vi) \quad & d_*(d_*\tilde{\omega}) \neq 0 \ , \text{for any form } \tilde{\omega}, \text{ of degree } r \geq 0 \ , \\ (vii) \quad & d_*(d_* f \wedge d_* g) = d_*^2 f \wedge d_* g - d_* f \wedge d_*^2 g \ . \end{aligned} \tag{2.4.9}$$

and the coordinate differential $d_* x^i$ form a basis of the space of 1-form, such that $d_* x^i \wedge d_* x^i = 0$.

We prove the last four properties as first three properties are obvious.

Proof: We prove (iv) $d_*(\tilde{\omega} \wedge \tilde{\sigma}) = d_*\tilde{\omega} \wedge \tilde{\sigma} + (-1)^{\text{deg of } \tilde{\omega}}\tilde{\omega} \wedge d_*\tilde{\sigma}$,

(a) we choose $\tilde{\omega} = \omega_i d_* x^j$, $\tilde{\sigma} = \sigma_k d_* x^k$ be two 1-forms. Then by using the definition (2.4.8), we have

$$d_*\tilde{\omega} = \omega_{i;j} d_* x^j \wedge d_* x^i - \omega_i d_*^2 x^k \ , \tag{2.4.10}$$

and

$$d_*\tilde{\sigma} = \sigma_{k;j} d_* x^j \wedge d_* x^k - \sigma_k d_*^2 x^k \ . \tag{2.4.11}$$

Consider the wedge product of $\tilde{\omega}$ and $\tilde{\sigma}$ as

$$\tilde{\omega} \wedge \tilde{\sigma} = \omega_i \sigma_k d_* x^i \wedge d_* x^k \ . \tag{2.4.12}$$

Taking the exterior derivative d_* of (2.4.12) we obtain

$$\begin{aligned} d_*(\tilde{\omega} \wedge \tilde{\sigma}) =& (\omega_{i;j}\sigma_k + \omega_i \sigma_{k;j}) d_* x^j \wedge d_* x^i \wedge d_* x^k - \\ & - \omega_i \sigma_k (d_*^2 x^i \wedge d_* x^k - d_* x^i \wedge d_*^2 x^k) \ . \end{aligned}$$

We rewrite this as

$$\begin{aligned} d_*(\tilde{\omega} \wedge \tilde{\sigma}) =& (\omega_{i;j} d_* x^j \wedge d_* x^i - \omega_i d_*^2 x^i) \wedge \sigma_k d_* x^k - \\ & - \omega_i d_* x^i \wedge (\sigma_{k;j} d_* x^j \wedge d_* x^k - \sigma_k d_*^2 x^k) \ . \end{aligned}$$

Using equations (2.4.10) and (2.4.11), this is nothing but

$$d_*(\tilde{\omega} \wedge \tilde{\sigma}) = d_*\tilde{\omega} \wedge \tilde{\sigma} + (-1)^{\text{deg of } \tilde{\omega}}\tilde{\omega} \wedge d_*\tilde{\sigma} \ .$$

(b) Now let $\tilde{\omega} = \omega_{ij} d_* x^i \wedge d_* x^j$, be 2-form and $\tilde{\sigma}$ a 1-form. Using the definition (2.4.8), we have

$$d_*\tilde{\omega} = \omega_{ij;k} d_* x^k \wedge d_* x^i \wedge d_* x^j - \omega_{ij}(d_*^2 x^i \wedge d_* x^j - d_* x^i \wedge d_*^2 x^j) \ . \tag{2.4.13}$$

Consider

$$\begin{aligned} \tilde{\omega} \wedge \tilde{\sigma} =& (\omega_{ij}\sigma_l) d_* x^i \wedge d_* x^j \wedge d_* x^l \ , \qquad (2.4.14) \\ \Rightarrow d_*(\tilde{\omega} \wedge \tilde{\sigma}) =& (\omega_{ij;k}\sigma_l + \omega_{ij}\sigma_{l;k}) d_* x^k \wedge d_* x^i \wedge d_* x^j \wedge d_* x^l - \\ & - \omega_{ij}\sigma_l (d_*^2 x^i \wedge d_* x^j \wedge d_* x^l - d_* x^i \wedge d_*^2 x^j \wedge d_* x^l + \\ & + d_* x^i \wedge d_* x^j \wedge d_*^2 x^l) \ . \end{aligned}$$

We write this as

$$\begin{aligned} d_*(\tilde{\omega} \wedge \tilde{\sigma}) =& [\omega_{ij;k} d_* x^k \wedge d_* x^i \wedge d_* x^j - \omega_{ij}(d_*^2 x^i \wedge d_* x^j - d_* x^l \wedge d_*^2 x^j)] \wedge \\ & \wedge \sigma_l d_* x^l + \omega_{ij} d_* x^i \wedge d_* x^j \wedge [\sigma_{l;k} d_* x^k \wedge d_* x^l - \sigma_l d_*^2 x^l] \ . \end{aligned}$$

Using the equations (2.4.11) and (2.4.13) we get

$$d_*(\tilde{\omega} \wedge \tilde{\sigma}) = d_*\tilde{\omega} \wedge \tilde{\sigma} + (-1)^{\text{deg of } \tilde{\omega}} \tilde{\omega} \wedge d_*\tilde{\sigma} \ .$$

Thus the result is true for any form $\tilde{\omega}$.

(v) Now we claim that

$$d_*(f\tilde{\omega}) = d_* f \wedge \tilde{\omega} + f d_*\tilde{\omega} \ .$$

Let $\tilde{\omega} = \omega_i d_* x^i$ be a 1-form, and f be a scalar

$\Rightarrow f\tilde{\omega} = f\omega_i d_* x^i$, is a 1-form. Using the definition (2.4.8), we obtain

$$\begin{aligned} d_*(f\tilde{\omega}) =& (f_{;j}\omega_i + f\omega_{i;j}) d_* x^j \wedge d_* x^i - f\omega_i d_*^2 x^i \ , \\ \Rightarrow d_*(f\tilde{\omega}) =& f_{;j} d_* x^j \wedge \omega_i d_* x^i + f(\omega_{i;j} d_* x^j \wedge d_* x^i - \omega_i d_*^2 x^i) \ , \\ \Rightarrow d_*(f\tilde{\omega}) =& d_* f \wedge \tilde{\omega} + f d_* \tilde{\omega} \ . \end{aligned}$$

(vi) Claim: We prove $d_*^2\tilde{\omega} \neq 0$ for any form $\tilde{\omega}$ of degree $r \geq 0$.

(a) Let f be a differentiable function of coordinates. Then by definition,

$$d_* f = f_{;i} d_* x^i \ , \tag{2.4.15}$$

where for a differential function f, we have

$$f_{;i} = f_{/i} = f_{,i} \ . \tag{2.4.16}$$

Thus in the case of a scalar function f, we have $d_* f = df$. Taking the exterior derivative of (2.4.15) and using the definitions (2.4.1), we obtain

$$d_*^2 f = f_{;ij} d_* x^j \wedge d_* x^i - f_{;i} d_*^2 x^i \ . \tag{2.4.17}$$

Interchanging i in to j in the equations (2.4.17) and adding the result thus obtain to the equation (2.4.17), we get

$$d_*^2 f = \frac{1}{2}(f_{;ij} - f_{;ji}) d_* x^j \wedge d_* x^i - f_{;i} d_*^2 x^i \ . \tag{2.4.18}$$

It follows from the definition (1.2.19) that

$$f_{;ij} - f_{;ji} = 2f_{;k}Q_{ij}{}^{k} \ . \tag{2.4.19}$$

Hence the equation (2.4.18) becomes

$$d_*^2 f = f_{;k}Q_{ij}{}^{k} d_*x^j \wedge d_*x^i - f_{;i}d_*^2 x^i \ . \tag{2.4.20}$$

For the co-ordinate functions x^i, we find from the equation (2.4.8)

$$d_*^2 x^k = \frac{1}{2}Q_{ij}{}^{k} d_*x^j \wedge d_*x^i \ . \tag{2.4.21}$$

Consequently, the equation (2.4.20) becomes

$$d_*^2 f = \frac{1}{2}f_{;k}Q_{ij}{}^{k} d_*x^j \wedge d_*x^i \ . \tag{2.4.22}$$

We see from equations (2.4.7) and (2.4.22) that

$$d_*^2 f = \frac{1}{4}curl(gradf) = \frac{1}{2}f_{;k}Q_{ij}{}^{k} d_*x^j \wedge d_*x^i \ . \tag{2.4.23}$$

Consequently, we see that $d_*^2 f \neq 0$, and hence $curl(gradf) \neq 0$ in a non-Riemannian space.

(b) Let $\tilde{\omega} = \omega_i d_*x^i$, be a 1-form. By definition, we have therefore

$$d_*\tilde{\omega} = \omega_{i;j} d_*x^j \wedge d_*x^i - \omega_i d_*^2 x^i \ .$$

Using the equations (2.4.21), we get

$$d_*\tilde{\omega} = -\omega_{i;j} d_*x^i \wedge d_*x^j + \frac{1}{2}\omega_k Q_{ij}{}^{k} d_*x^i \wedge d_*x^j \ . \tag{2.4.24}$$

Interchanging i and j, in the equation (2.4.24) we get

$$d_*\tilde{\omega} = \omega_{j;i} d_*x^i \wedge d_*x^j + \frac{1}{2}\omega_k {Q_{ij}}^k d_*x^i \wedge d_*x^j \ . \tag{2.4.25}$$

Adding equations (2.4.24) and (2.4.25) we obtain

$$d_*\tilde{\omega} = -\frac{1}{2}\big[(\omega_{i;j} - \omega_{j;i}) - \omega_h {Q_{ij}}^h\big] d_*x^i \wedge d_*x^j \ . \tag{2.4.26}$$

This can also be written as

$$d_*\tilde{\omega} = d\tilde{\omega} - \frac{1}{2}\omega_h {Q_{ij}}^h d_*x^i \wedge d_*x^j \ , \tag{2.4.27}$$

where

$$d\tilde{\omega} = -\frac{1}{2}(\omega_{i;j} - \omega_{j;i}) d_*x^i \wedge d_*x^j \ , \tag{2.4.28}$$

representing exterior derivative of 1-form in a Riemann space. Taking the exterior derivative d_* of the equation (2.4.26) and using the definition (2.4.8), we get

$$\begin{aligned} d_*^2\tilde{\omega} = &-\frac{1}{2}\big[(\omega_{i;jk} - \omega_{j;ik}) - \big(\omega_h {Q_{ij}}^h\big)_{;k}\big] d_*x^i \wedge d_*x^j \wedge d_*x^k + \\ &+ \frac{1}{2}\big[(\omega_{i;j} - \omega_{j;i}) - \omega_h {Q_{ij}}^h\big]\big(d_*^2x^i \wedge d_*x^j - d_*x^i \wedge d_*^2x^j\big) \ . \end{aligned}$$

Using the equation (2.4.21) we obtain

$$\begin{aligned} d_*^2\tilde{\omega} = &-\frac{1}{2}\big[(\omega_{i;jk} - \omega_{j;ik}) - \big(\omega_h {Q_{ij}}^h\big)_{;k} - \\ &- \big(\omega_{i;l} - \omega_{l;i} - \omega_h {Q_{il}}^h\big){Q_{jk}}^l\big] d_*x^i \wedge d_*x^j \wedge d_*x^k \ . \end{aligned} \tag{2.4.29}$$

By cyclic permutation of indices i, j, k twice in turn in the equation (2.4.29), we write two more equations

$$d_*^2\tilde{\omega} = -\frac{1}{2}\big[(\omega_{j;ki} - \omega_{k;ji}) - \big(\omega_h Q_{jk}{}^h\big)_{;i} - \\ - \big(\omega_{j;l} - \omega_{l;j} - \omega_h Q_{jl}{}^h\big) Q_{ki}{}^l\big] d_*x^i \wedge d_*x^j \wedge d_*x^k \ , \qquad (2.4.30)$$

and

$$d_*^2\tilde{\omega} = -\frac{1}{2}\big[(\omega_{k;ij} - \omega_{i;kj}) - \big(\omega_h Q_{ki}{}^h\big)_{;j} - \\ - \big(\omega_{k;l} - \omega_{l;k} - \omega_h Q_{kl}{}^h\big) Q_{ij}{}^l\big] d_*x^i \wedge d_*x^j \wedge d_*x^k \ . \qquad (2.4.31)$$

Adding equations (2.4.29), (2.4.30) and (2.4.31) and then using the Ricci identity (1.2.21) and then the cyclic property (1.2.27) we obtain after simplifying

$$d_*^2\tilde{\omega} = -\frac{1}{6}\Big[\omega_h\big\{(Q_{ij}{}^h{}_{;k} + Q_{jk}{}^h{}_{;i} + Q_{ki}{}^h{}_{;j}) - 3(Q_{ij}{}^l Q_{kl}{}^h + Q_{jk}{}^l Q_{il}{}^h + \\ + Q_{ki}{}^l Q_{jl}{}^h)\big\} + Q_{ij}{}^h \omega_{k;h} + Q_{jk}{}^h \omega_{i;h} + Q_{ki}{}^h \omega_{j;h}\Big] d_*x^i \wedge d_*x^j \wedge d_*x^k \ . \qquad (2.4.32)$$

As we know the repeated exterior derivative of 1-form subsumes the $div(curl\tilde{\omega})$, we see from the equation (2.4.25) that $d_*^2\tilde{\omega} \neq 0$, due to the presence of torsion in the space. Consequently, we have

$$div(curl\tilde{\omega}) \neq 0 \ ,$$

in a non-Riemannian space.

(c) Let $\tilde{\omega} = \omega_{ij} d_* x^i \wedge d_* x^j$ be a 2-form, where $\omega_{ij} = -\omega_{ji}$. Using the definition (2.4.8), we find

$$d_* \tilde{\omega} = \omega_{ij;k} d_* x^i \wedge d_* x^j \wedge d_* x^k - \omega_{ij} (d_*^2 x^i \wedge d_* x^j - d_* x^i \wedge d_*^2 x^j) \ .$$

Using the equation (2.4.21) we obtain

$$d_* \tilde{\omega} = (\omega_{ij;k} + \omega_{lk} Q_{ij}{}^l) d_* x^i \wedge d_* x^j \wedge d_* x^k \ . \tag{2.4.33}$$

By cyclic permutation of indices i, j, k twice in turn in the equation (2.4.33), we obtain two more equations as

$$d_* \tilde{\omega} = (\omega_{jk;i} + \omega_{li} Q_{jk}{}^l) d_* x^i \wedge d_* x^j \wedge d_* x^k \ , \tag{2.4.34}$$

and

$$d_* \tilde{\omega} = (\omega_{ki;j} + \omega_{lj} Q_{ki}{}^l) d_* x^i \wedge d_* x^j \wedge d_* x^k \ . \tag{2.4.35}$$

Adding equations (2.4.33), (2.4.34) and (2.4.35) we get

$$\begin{aligned} d_* \tilde{\omega} =& \frac{1}{3} \Big[(\omega_{ij;k} + \omega_{jk;i} + \omega_{ki;j}) + (\omega_{lk} Q_{ij}{}^l + \omega_{li} Q_{jk}{}^l + \\ & + \omega_{lj} Q_{ki}{}^l) \Big] d_* x^i \wedge d_* x^j \wedge d_* x^k \ . \end{aligned} \tag{2.4.36}$$

One can also write the equation (2.4.36) as

$$d_* \tilde{\omega} = d\tilde{\omega} - \frac{1}{3} \left(\omega_{lk} Q_{ij}{}^l + \omega_{li} Q_{jk}{}^l + \omega_{lj} Q_{ki}{}^l \right) d_* x^i \wedge d_* x^j \wedge d_* x^k \ . \tag{2.4.37}$$

where

$$d\tilde{\omega} = \frac{1}{3}(\omega_{ij;k} + \omega_{jk;i} + \omega_{ki;j})d_*x^i \wedge d_*x^j \wedge d_*x^k \ . \tag{2.4.38}$$

Taking the exterior derivative d_* of the equation (2.4.36) and following the same procedure elaborated in (b), we obtain after simplifying

$$\begin{aligned}
{d_*}^2\tilde{\omega} = \frac{1}{12}\Big[&\omega_{ij;kh} - \omega_{ij;hk} + \omega_{jk;ih} - \omega_{jk;hi} + \omega_{ki;jh} - \omega_{ki;hj} + \\
&+ \omega_{kh;ij} - \omega_{kh;ji} + \omega_{hi;kj} - \omega_{hi;jk} + \omega_{jh;ki} - \omega_{jh;ik} - {Q_{ij}}^l\omega_{kh;l} + \\
&+ {Q_{jk}}^l\omega_{hi;l} + {Q_{ki}}^l\omega_{hj;l} - {Q_{kh}}^l\omega_{ij;l} + {Q_{hi}}^l\omega_{jk;l} - {Q_{jh}}^l\omega_{ki;l} + \\
&+ \omega_{lk}({Q_{ij}}^l{}_{;k} - {Q_{hj}}^l{}_{;i} + {Q_{hi}}^l{}_{;j} + {Q_{pj}}^l{Q_{hi}}^p + {Q_{ip}}^l{Q_{hj}}^p + {Q_{hp}}^l{Q_{ji}}^p) + \\
&+ \omega_{li}({Q_{jk}}^l{}_{;h} - {Q_{jh}}^l{}_{;k} + {Q_{kh}}^l{}_{;j} + {Q_{pk}}^l{Q_{ij}}^p + {Q_{jp}}^l{Q_{hk}}^p + {Q_{ph}}^l{Q_{jk}}^p) + \\
&+ \omega_{lj}({Q_{ki}}^l{}_{;h} - {Q_{kh}}^l{}_{;i} - {Q_{hi}}^l{}_{;k} + {Q_{kp}}^l{Q_{hi}}^p + {Q_{pi}}^l{Q_{hk}}^p + {Q_{hp}}^l{Q_{ik}}^p) + \\
&+ \omega_{lh}(-{Q_{jk}}^l{}_{;i} + {Q_{ik}}^l{}_{;j} - {Q_{ij}}^l{}_{;k} + {Q_{ip}}^l{Q_{jk}}^p + {Q_{pk}}^l{Q_{ji}}^p + {Q_{jp}}^l{Q_{ki}}^p)\Big] \cdot \\
&\cdot d_*x^h \wedge d_*x^i \wedge d_*x^j \wedge d_*x^k \ .
\end{aligned} \tag{2.4.39}$$

Using the Ricci identity (1.2.28) and then the cyclic property (1.2.27), we obtain after simplifying

$$\begin{aligned}
{d_*}^2\tilde{\omega} = \frac{1}{6}\Big[&2\omega_{lj}\big\{({Q_{ik}}^l{}_{;h} + {Q_{kh}}^l{}_{;i} + {Q_{hi}}^p{}_{;k}) - \\
&- 3({Q_{ik}}^p{Q_{hp}}^l + {Q_{kh}}^p{Q_{ip}}^l + {Q_{hi}}^p{Q_{kp}}^l)\big\} + \\
&+ {Q_{ih}}^l\omega_{jk;l} + {Q_{ki}}^l\omega_{jh;l} + {Q_{jh}}^l\omega_{ki;l}\Big] d_*x^h \wedge d_*x^i \wedge d_*x^j \wedge d_*x^k \ .
\end{aligned} \tag{2.4.40}$$

This shows that ${d_*}^2\tilde{\omega} \neq 0$, for any $\tilde{\omega}$.

Note: In the absence of torsion term the results reduce to ${d_*}^2\tilde{\omega} = 0$.

(vii) Now we claim $d_*(d_*f \wedge d_*g) \neq 0$.

Let f and g be two 0-forms, then d_*f and d_*g are 1-forms defined respectively by

$$d_*f = f_{;i}d_*x^i \quad \text{and} \quad d_*g = g_{;k}d_*x^k \ .$$

Then

$$d_*f \wedge d_*g = \frac{1}{2}(f_{;i}g_{;k} - f_{;k}g_{;i})d_*x^i \wedge d_*x^k \ , \ \text{is a 2-form}$$

Consider $$d_*(d_*f \wedge d_*g) = d_*^2 f \wedge d_*g - d_*f \wedge d_*^2 g \ .$$

Using the equation (2.4.22), we obtain

$$d_*(d_*f \wedge d_*g) = \frac{1}{2}(f_{;k}g_{;l} - f_{;l}g_{;k}){Q_{ij}}^l d_*x^i \wedge d_*x^j \wedge d_*x^k \ , \qquad (2.4.41)$$

By cyclic permutation of the indices i, j, k twice in turn in the equation (2.4.41) we obtain the following two equations

$$d_*(d_*f \wedge d_*g) = \frac{1}{2}(f_{;i}g_{;l} - f_{;l}g_{;i}){Q_{jk}}^l d_*x^i \wedge d_*x^j \wedge d_*x^k \ , \qquad (2.4.42)$$

and

$$d_*(d_*f \wedge d_*g) = \frac{1}{2}(f_{;j}g_{;l} - f_{;l}g_{;j}){Q_{ki}}^l d_*x^i \wedge d_*x^j \wedge d_*x^k \ , \qquad (2.4.43)$$

Adding equations (2.4.41), (2.4.42) and (2.4.43) we get

$$d_*(d_*f \wedge d_*g) = \frac{1}{6}\big[(f_{;k}g_{;l} - f_{;l}g_{;k})Q_{ij}{}^l + (f_{;i}g_{;l} - f_{;l}g_{;i})Q_{jk}{}^l + \\ + (f_{;j}g_{;l} - f_{;l}g_{;j})Q_{ki}{}^l\big] d_*x^i \wedge d_*x^j \wedge d_*x^k \ . \quad (2.4.44)$$

This is a 3-form. Unlike the identity in the Riemann space of ET of gravitation $d(df \wedge dg) = 0$, it is not zero in the non-Riemannian space of Einstein-Cartan theory of gravitation.

2.4.2 Cartan's Equations of Structure in a Non-Riemann Space

The essence of Riemannian geometry is studied through the Cartan's equations of structure. In this section, we summerise Cartan's equations of structure in a in a non-Riemannian space derived by Katkar [61]. We hope that these equations will be of immense use in the study of the essence of non-Riemannian geometry and also provide a technique of computation of the components of Riemannian curvature tensor which latter can be used to find the solutions of the field equations of the Einstein-Cartan theory of gravitation.

Let V_n be a non-Riemann space with metric defined by

$$ds^2 = g_{ij}dx^i dx^j \ , \quad (2.4.45)$$

where g_{ij} are the components of the metric tensor and the connections involved are asymmetric. Let $x^i = x^i(s)$ be a curve in V_n and s is a parameter of the curve. At each point of the curve, we construct a tetrad $e_{(\alpha)}$, $\alpha = 1, 2, 3, 4$, consisting of four vector fields which form a basis at each point. If θ^α are four basis 1-form corresponding to four basis vector fields $e^{(\alpha)}{}_i$, then we have

$$\theta^\alpha = e^{(\alpha)}{}_i d_* x^i \ , \tag{2.4.46}$$

Applying the derivative operator d_* to the equation (2.4.46), we get

$$d_*\theta^\alpha = e^{(\alpha)}{}_{i;j} d_* x^j \wedge d_* x^i - e^{(\alpha)}{}_i d_*^2 x^i \ . \tag{2.4.47}$$

Using the equation (2.4.21), we get

$$d_*\theta^\alpha = \big(e^{(\alpha)}{}_{i;j} - \frac{1}{2} Q_{ij}{}^k e^{(\alpha)}{}_k\big) d_* x^j \wedge d_* x^i \ . \tag{2.4.48}$$

However, from the equation (2.4.46), we find

$$d_* x^i = \theta^\alpha e_{(\alpha)}{}^i \ . \tag{2.4.49}$$

Using the equation (2.4.49) in the equation (2.4.48), we get

$$d_*\theta^\alpha = \Bigg(- e^{(\alpha)}{}_{i;j} e_{(\beta)}{}^i e_{(\sigma)}{}^j + \frac{1}{2} Q_{ij}{}^k e_{(\beta)}{}^i e_{(\sigma)}{}^j e^{(\alpha)}{}_k \Bigg) \theta^\beta \wedge \theta^\sigma \ .$$

Using the equation (1.3.10), we write this equation as

$$d_*\theta^\alpha = \Bigg(\gamma^\alpha{}_{\beta\sigma} + \frac{1}{2} Q_{\beta\sigma}{}^\alpha \Bigg) \theta^\beta \wedge \theta^\sigma \ , \tag{2.4.50}$$

where

$$Q_{\beta\sigma}{}^\alpha = Q_{ij}{}^k e_{(\beta)}{}^i e_{(\sigma)}{}^j e^{(\alpha)}{}_k \ , \tag{2.4.51}$$

are the tetrad components of the torsion tensor $Q_{ij}{}^{k}$. We write this equation as

$$d_*\theta^\alpha = -\omega^\alpha{}_\beta \wedge \theta^\beta + \frac{1}{2} Q_{\beta\sigma}{}^\alpha \theta^\beta \wedge \theta^\sigma \ , \tag{2.4.52}$$

where

$$\omega^\alpha{}_\beta = \gamma^\alpha{}_{\beta\sigma}\theta^\sigma \ , \tag{2.4.53}$$

are the tetrad components of connection 1-form of a non-Riemann space V_n and $\gamma^\alpha{}_{\beta\sigma}$ are defined in the equation (1.3.12). Using the equation (1.3.12) we write the equation (2.4.53) as

$$\begin{aligned} &\omega^\alpha{}_\beta = \gamma^{0\alpha}{}_{\beta\sigma}\theta^\sigma - K_{\sigma\beta}{}^\alpha\theta^\sigma \ , \\ \Rightarrow \qquad &\omega^\alpha{}_\beta = \omega^{0\alpha}{}_\beta - K_{\sigma\beta}{}^\alpha\theta^\sigma \ , \end{aligned} \tag{2.4.54}$$

where

$$\omega^{0\alpha}{}_\beta = \gamma^{0\alpha}{}_{\beta\sigma}\theta^\sigma \ , \tag{2.4.55}$$

are the tetrad components of connection 1-form of a Riemann space. Using the equations (1.2.16) and (2.4.54) in the equation (2.4.52) we obtain

$$d_*\theta^\alpha = d\theta^\alpha - \frac{1}{2} Q_{\beta\sigma}{}^\alpha \ \theta^\beta \wedge \theta^\sigma \ , \tag{2.4.56}$$

where

$$d\theta^\alpha = -\omega^{0\alpha}{}_\beta \wedge \theta^\beta \ . \tag{2.4.57}$$

The equations (2.4.56) and (2.4.57) are called the Cartan's first equations of structure with respect to a non-Riemann space and a Riemann space respectively.

To arrive at the Cartan's second equation of structure in a non-Riemann space, we first write the equation (2.4.53) by using (2.4.46) as

$$\omega^{\alpha}{}_{\beta} = \gamma^{\alpha}{}_{\beta\sigma} e^{(\sigma)}{}_{i} d_{*}x^{i} \ . \tag{2.4.58}$$

Operating the derivative d_* to the equation (2.4.58) and using the definition (2.4.8) and (2.4.21), we find

$$d_{*}\omega^{\alpha}{}_{\beta} = -\frac{1}{2}\big[(\gamma^{\alpha}{}_{\beta\sigma} e^{(\sigma)}{}_{i})_{;j} - (\gamma^{\alpha}{}_{\beta\sigma} e^{(\sigma)}{}_{j})_{;i} - (\gamma^{\alpha}{}_{\beta\sigma} e^{(\sigma)}{}_{k}) Q_{ij}{}^{k}\big] d_{*}x^{i} \wedge d_{*}x^{j} \ . \tag{2.4.59}$$

From the equation (1.3.10), we find

$$e^{(\alpha)}{}_{k;i} = -\gamma^{\alpha}{}_{\beta\sigma} e^{(\beta)}{}_{k} e^{(\sigma)}{}_{i} \ . \tag{2.4.60}$$

Differentiating the equations (2.4.60) covariantly with respect to x^j, we get

$$\begin{aligned} e^{(\alpha)}{}_{k;ij} &= \big[-(\gamma^{\alpha}{}_{\beta\sigma} e^{(\sigma)}{}_{i})_{;j} + \gamma^{\alpha}{}_{\epsilon\sigma}\gamma^{\epsilon}{}_{\beta\delta} e^{(\sigma)}{}_{i} e^{(\delta)}{}_{j} \big] e^{(\beta)}{}_{k} \ , \\ \Rightarrow (\gamma^{\alpha}{}_{\beta\sigma} e^{(\sigma)}{}_{i})_{;j} &= -\, e^{(\alpha)}{}_{k;ij} e_{(\beta)}{}^{k} + \gamma^{\alpha}{}_{\epsilon\sigma}\gamma^{\epsilon}{}_{\beta\delta} e^{(\sigma)}{}_{i} e^{(\delta)}{}_{j} \ . \end{aligned} \tag{2.4.61}$$

Substituting this equation in the equation (2.4.59), we get

$$d_*\omega^\alpha{}_\beta = \frac{1}{2}\big[(e^{(\alpha)}{}_{k;ij} - e^{(\alpha)}{}_{k;ji})e_{(\beta)}{}^k - \gamma^\alpha{}_{\epsilon\sigma}\gamma^\epsilon{}_{\beta\delta}(e^{(\sigma)}{}_i e^{(\delta)}{}_j - e^{(\delta)}{}_i e^{(\sigma)}{}_j) + $$
$$+ \gamma^\alpha{}_{\beta\sigma}Q_{ij}{}^k e^{(\sigma)}{}_k\big] d_*x^i \wedge d_*x^j \ . \qquad (2.4.62)$$

Using the Ricci identity (1.2.21) and the equations (2.4.49) and (2.4.63) in the equation (2.4.62), we obtain

$$d_*\omega^\alpha{}_\beta = -\frac{1}{2}R_{\delta\epsilon\beta}{}^\alpha\theta^\delta \wedge \theta^\epsilon + \omega^\epsilon{}_\beta \wedge \omega^\alpha{}_\epsilon + \frac{1}{2}\gamma^\alpha{}_{\beta\sigma}Q_{\epsilon\delta}{}^\sigma\theta^\delta \wedge \theta^\epsilon \ , \qquad (2.4.63)$$

where

$$R_{\delta\epsilon\beta}{}^\alpha = R_{jik}{}^h e^{(\alpha)}{}_h e_{(\delta)}{}^j e_{(\epsilon)}{}^i e_{(\beta)}{}^k \ , \qquad (2.4.64)$$

are the tetrad components of the Riemann curvature tensor in a non-Riemannian space. If $\Omega^\alpha{}_\beta$ is the curvature 2-form then it is defined by

$$\Omega^\alpha{}_\beta = -\frac{1}{2}R_{\delta\epsilon\beta}{}^\alpha\theta^\delta \wedge \theta^\epsilon \ . \qquad (2.4.65)$$

Hence the equation (2.4.63) becomes

$$\Omega^\alpha{}_\beta = d_*\omega^\alpha{}_\beta + \omega^\alpha{}_\epsilon \wedge \omega^\epsilon{}_\beta + \frac{1}{2}\gamma^\alpha{}_{\beta\sigma}Q_{\epsilon\delta}{}^\sigma \ \theta^\epsilon \wedge \theta^\delta \ . \qquad (2.4.66)$$

This is known as the Cartan's second equation of structure in a non-Riemannian space.

These Cartan's equations of structure in a more general form can hopefully be used in the following chapters to find the solutions of the field equations of Einstein-Cartan theory of gravitation.

2.4.3 Tetrad Components of Connection 1-form and Curvature 2-form

The Cartan's equations of structure will be used as a technique to compute 36 tetrad components of Riemann curvature tensor in a non-Riemannian space-time of Einstein-Cartan theory of gravitation. For computational purpose, we present below the expressions for the tetrad components of connection 1-forms and curvature 2-forms.

From the definition of connection 1-form (2.4.53), we recall

$$\omega_{\alpha\beta} = \gamma_{\alpha\beta\sigma}\theta^{\sigma} \ . \tag{2.4.67}$$

Using the equation (1.3.12) in the equations (2.4.67) and expanding the summation, we get

$$\begin{aligned}\omega_{\alpha\beta} =&({\gamma^0}_{\alpha\beta 1} + K_{1\alpha\beta})\theta^1 + ({\gamma^0}_{\alpha\beta 2} + K_{2\alpha\beta})\theta^2 + ({\gamma^0}_{\alpha\beta 3} + K_{3\alpha\beta})\theta^3 + \\ &+ ({\gamma^0}_{\alpha\beta 4} + K_{4\alpha\beta})\theta^4 \ .\end{aligned} \tag{2.4.68}$$

By giving different values to $\alpha, \beta = 1, 2, 3, 4$ and using the equation (1.3.14), we readily obtain the expressions for connection 1-form as

$$\begin{aligned}\omega_{12} = {\omega^2}_2 = &-\big[(\epsilon^0 + \bar{\epsilon}^0 + \epsilon_1 + \bar{\epsilon}_1)\theta^1 + (\gamma^0 + \overline{\gamma}^0 + \gamma_1 + \overline{\gamma}_1)\theta^2 + \\ &+ (\overline{\alpha}^0 + \beta^0 + \overline{\alpha}_1 + \beta_1)\theta^3 + (\alpha^0 + \overline{\beta}^0 + \alpha_1 + \overline{\beta}_1)\theta^4\big] \ , \\ \omega_{13} = {\omega^2}_3 = &-\big[\left(\kappa^0 + \kappa_1\right)\theta^1 + \left(\tau^0 + \tau_1\right)\theta^2 + \left(\sigma^0 + \sigma_1\right)\theta^3 + \\ &+ \left(\rho^0 + \rho_1\right)\theta^4\big] \ ,\end{aligned}$$

$$\omega_{23} = \omega^1{}_3 = \left(\overline{\pi}^0 + \overline{\pi}_1\right)\theta^1 + \left(\overline{\nu}^0 + \overline{\nu}_1\right)\theta^2 + \left(\overline{\lambda}^0 + \overline{\lambda}_1\right)\theta^3 +$$
$$+ \left(\overline{\mu}^0 + \overline{\mu}_1\right)\theta^4 \ ,$$
$$\omega_{34} = -\omega^4{}_4 = \Big[\left(\epsilon^0 - \overline{\epsilon}^0 + \epsilon_1 - \overline{\epsilon}_1\right)\theta^1 + \left(\gamma^0 - \overline{\gamma}^0 + \gamma_1 - \overline{\gamma}_1\right)\theta^2 -$$
$$- \left(\overline{\alpha}^0 - \beta^0 + \overline{\alpha}_1 - \beta_1\right)\theta^3 + \left(\alpha^0 - \overline{\beta}^0 + \alpha_1 - \overline{\beta}_1\right)\theta^4\Big] \ . \tag{2.4.69}$$

Similarly Cartan's first equation of structure (2.4.56) on using (1.3.14) yields

$$d_*\theta^1 = (\gamma^0 + \overline{\gamma}^0)\theta^{12} + (\overline{\alpha}^0 + \beta^0 - \overline{\pi}^0)\theta^{13} + (\alpha^0 + \overline{\beta}^0 - \pi^0)\theta^{14} -$$
$$- \overline{\nu}^0\theta^{23} - \nu^0\theta^{24} + (\overline{\mu}^0 - \mu^0)\theta^{34} - \frac{1}{2}Q_{\alpha\beta}{}^1\theta^{\alpha\beta} \ ,$$
$$d_*\theta^2 = (\epsilon^0 + \overline{\epsilon}^0)\theta^{12} + \kappa^0\theta^{13} + \overline{\kappa}^0\theta^{14} - (\overline{\alpha}^0 + \beta^0 - \tau^0)\theta^{23} -$$
$$- (\alpha^0 + \overline{\beta}^0 - \overline{\tau}^0)\theta^{24} - (\rho^0 - \overline{\rho}^0)\theta^{34} - \frac{1}{2}Q_{\alpha\beta}{}^2\theta^{\alpha\beta} \ ,$$
$$d_*\theta^3 = - (\overline{\tau}^0 + \pi^0)\theta^{12} - (\overline{\rho}^0 + \epsilon^0 - \overline{\epsilon}^0)\theta^{13} - \overline{\sigma}^0\theta^{14} -$$
$$- (\gamma^0 - \overline{\gamma}^0 - \mu^0)\theta^{23} + \lambda^0\theta^{24} + (\alpha^0 - \overline{\beta}^0)\theta^{34} - \frac{1}{2}Q_{\alpha\beta}{}^3\theta^{\alpha\beta} \ ,$$
$$d_*\theta^4 = - (\tau^0 + \overline{\pi}^0)\theta^{12} - \sigma^0\theta^{13} + (\epsilon^0 - \overline{\epsilon}^0 - \rho^0)\theta^{14} + \overline{\lambda}^0\theta^{23} +$$
$$+ (\gamma^0 - \overline{\gamma}^0 + \overline{\mu}^0)\theta^{24} - \left(\overline{\alpha}^0 - \beta^0\right)\theta^{34} - \frac{1}{2}Q_{\alpha\beta}{}^4\theta^{\alpha\beta}. \tag{2.4.70}$$

To find the tetrad components of curvature 2-form in a non-Riemannian space, we first record here the tetrad components of the torsion tensor $Q_{ij}{}^k$. We start from the equation (1.2.52) the decomposition of the spin angular momentum tensor ${S_{ij}}^k$ into spin tensor S_{ij} as

$$ {S_{ij}}^k = S_{ij} u^k \ . \tag{2.4.71} $$

We assume in the following that the frenkel condition (1.2.54) does not hold true.

$$ \text{i. e.} \qquad S_{ij} u^j \neq 0 \ . \tag{2.4.72} $$

Contracting j with k in the Einstein-Cartan field equation (1.2.48), and using equations (2.4.71), (2.4.72) we get

$$ {Q_{ik}}^k = -\frac{k}{2} S_{ik} u^k \ . \tag{2.4.73} $$

Substituting this value in the field equation (1.2.48), we get

$$ {Q_{ij}}^k = \frac{k}{2}\left[{\delta^k}_i S_{jl} u^l - {\delta^k}_j S_{il} u^l + 2 S_{ij} u^k \right] \ . \tag{2.4.74} $$

Multiplying the equation (2.4.74) by ${e_{(\alpha)}}^i {e_{(\beta)}}^j {e^{(\gamma)}}_k$ we get

$$ 2{Q_{\alpha\beta}}^\gamma = k\left[{\delta_\alpha}^\gamma S_{\beta\sigma} u^\sigma - {\delta_\beta}^\gamma S_{\alpha\sigma} u^\sigma + 2 S_{\alpha\beta} u^\gamma \right] \ , \tag{2.4.75} $$

where $u^\sigma = u^i {e_i}^{(\sigma)}$, are the tetrad components of the unit time-like vector u^i. For the Newman-Penrose null tetrad defined in the equation (1.3.3) and for the choice of the time-like vector $u^i = \frac{1}{\sqrt{2}}(l^i + n^i)$, such that $u_i u^i = 1$, we have the tetrad components of the time-like vector u^i are given by

$$ u^\sigma = \frac{1}{\sqrt{2}}(1, 1, 0, 0) \ . \tag{2.4.76} $$

From the equation (2.4.75) by giving the values 1, 2, 3, 4 to γ we obtain four equations given by

$$
\begin{aligned}
2Q_{\alpha\beta}{}^{1} =& \frac{k}{\sqrt{2}}[\delta_{\alpha}{}^{1}S_{\beta 1} + \delta_{\alpha}{}^{1}S_{\beta 2} - \delta_{\beta}{}^{1}S_{\alpha 1} - \delta_{\beta}{}^{1}S_{\alpha 2} + 2\sqrt{2}S_{\alpha\beta}] \ , \\
2Q_{\alpha\beta}{}^{2} =& \frac{k}{\sqrt{2}}[\delta_{\alpha}{}^{2}S_{\beta 1} + \delta_{\alpha}{}^{2}S_{\beta 2} - \delta_{\beta}{}^{2}S_{\alpha 1} - \delta_{\beta}{}^{2}S_{\alpha 2} + 2\sqrt{2}S_{\alpha\beta}] \ , \\
2Q_{\alpha\beta}{}^{3} =& \frac{k}{\sqrt{2}}[\delta_{\alpha}{}^{3}S_{\beta 1} + \delta_{\alpha}{}^{3}S_{\beta 2} - \delta_{\beta}{}^{3}S_{\alpha 1} - \delta_{\beta}{}^{3}S_{\alpha 2}] \ , \\
2Q_{\alpha\beta}{}^{4} =& \frac{k}{\sqrt{2}}[\delta_{\alpha}{}^{4}S_{\beta 1} + \delta_{\alpha}{}^{4}S_{\beta 2} - \delta_{\beta}{}^{4}S_{\alpha 1} - \delta_{\beta}{}^{4}S_{\alpha 2}] \ . \qquad (2.4.77)
\end{aligned}
$$

Now giving different values to Greek indices α and $\beta = 1, 2, 3, 4$ in the equations (2.4.77) and using equations (1.3.26), we obtain the tetrad components $Q_{\alpha\beta}{}^{\gamma}$ in terms of 12 complex contortion components which we record as

$$
\begin{aligned}
Q_{12}{}^{1} =& \frac{k}{2\sqrt{2}}S_{12} = -\frac{1}{2}(\gamma_1 + \overline{\gamma}_1) = \frac{k}{2\sqrt{2}}(s_1 + \overline{s}_1) \ , \\
Q_{13}{}^{1} =& \frac{k}{2\sqrt{2}}(S_{13} - S_{23}) = -\frac{1}{2}(-\overline{\pi}_1 + \overline{\alpha}_1 + \beta_1) = \frac{k}{2\sqrt{2}}(s_0 + s_2) \ , \\
Q_{23}{}^{1} =& \frac{k}{\sqrt{2}}S_{23} = \frac{1}{2}\overline{\nu}_1 = -\frac{k}{\sqrt{2}}s_2 \ , \\
Q_{34}{}^{1} =& \frac{k}{\sqrt{2}}S_{34} = \frac{1}{2}(\mu_1 - \overline{\mu_1}) = -\frac{k}{\sqrt{2}}(s_1 - \overline{s}_1) \ , \\
Q_{12}{}^{2} =& \frac{k}{2\sqrt{2}}S_{12} = -\frac{1}{2}(\epsilon_1 + \overline{\epsilon}_1) = \frac{k}{2\sqrt{2}}(s_1 + \overline{s}_1) \ , \\
Q_{13}{}^{2} =& \frac{k}{\sqrt{2}}S_{13} = -\frac{1}{2}\kappa_1 = \frac{k}{\sqrt{2}}s_0 \ , \\
Q_{23}{}^{2} =& -\frac{k}{2\sqrt{2}}(S_{13} - S_{23}) = -\frac{1}{2}(\tau_1 - \overline{\alpha}_1 - \beta_1) = -\frac{k}{2\sqrt{2}}(s_0 + s_2) \ , \\
Q_{34}{}^{2} =& \frac{k}{\sqrt{2}}S_{34} = \frac{1}{2}(\rho_1 - \overline{\rho}_1) = -\frac{k}{\sqrt{2}}(s_1 - \overline{s}_1) \ ,
\end{aligned}
$$

$$
\begin{aligned}
Q_{13}{}^{3} =& -\frac{k}{2\sqrt{2}}S_{12} = \frac{1}{2}(\epsilon_1 - \bar{\epsilon}_1 + \overline{\rho}_1) = -\frac{k}{2\sqrt{2}}(s_1 + \overline{s}_1) \ , \\
Q_{23}{}^{3} =& \frac{k}{2\sqrt{2}}S_{12} = \frac{1}{2}(\gamma_1 - \overline{\gamma}_1 - \mu_1) = \frac{k}{2\sqrt{2}}(s_1 + \overline{s}_1) \ , \\
Q_{34}{}^{3} =& -\frac{k}{2\sqrt{2}}(\overline{S}_{13} + \overline{S}_{23}) = -\frac{1}{2}(\alpha_1 - \overline{\beta}_1) = -\frac{k}{2\sqrt{2}}(\overline{s}_0 - \overline{s}_2) \ , \\
Q_{34}{}^{4} =& \frac{k}{2\sqrt{2}}(S_{13} + S_{23}) = \frac{1}{2}(\overline{\alpha}_1 - \beta_1) = \frac{k}{2\sqrt{2}}(s_0 - s_2) \ , \\
Q_{12}{}^{3} =& 0 \ , Q_{12}{}^{4} = 0 \ , Q_{13}{}^{4} = 0 \ , Q_{23}{}^{4} = 0 \ .
\end{aligned}
\tag{2.4.78}
$$

From equations (2.4.78) we have

$$
\begin{aligned}
&\gamma_1 + \overline{\gamma}_1 = -\frac{k}{\sqrt{2}}(s_1 + \overline{s}_1) \ , && \overline{\pi}_1 - (\overline{\alpha}_1 + \beta_1) = \frac{k}{\sqrt{2}}(s_0 + s_2) \ , \\
&\overline{\nu}_1 = -k\sqrt{2}s_2 \ , && \mu_1 - \overline{\mu}_1 = -k\sqrt{2}(s_1 - \overline{s}_1) \ , \\
&\epsilon_1 + \bar{\epsilon}_1 = -\frac{k}{\sqrt{2}}(s_1 + \overline{s}_1) \ , && \kappa_1 = -k\sqrt{2}s_0 \ , \\
&\tau_1 - (\overline{\alpha}_1 + \beta_1) = \frac{k}{\sqrt{2}}(s_0 + s_2) \ , && \pi_1 + \overline{\tau}_1 = 0 \ , \\
&\epsilon_1 - \bar{\epsilon}_1 + \overline{\rho}_1 = -\frac{k}{\sqrt{2}}(s_1 + \overline{s}_1) \ , && \rho_1 - \overline{\rho}_1 = -\sqrt{2}k(s_1 - \overline{s}_1) \ , \\
&\gamma_1 - \overline{\gamma}_1 - \mu_1 = \frac{k}{\sqrt{2}}(s_1 + \overline{s}_1) \ , && \alpha_1 - \overline{\beta}_1 = \frac{k}{\sqrt{2}}(\overline{s}_0 - \overline{s}_2) \ , \\
&\overline{\pi}_1 + \tau_1 = 0 \ , && \overline{\alpha}_1 - \beta_1 = \frac{k}{\sqrt{2}}(s_0 - s_2) \ , \\
&\sigma_1 = 0 \ , \overline{\lambda}_1 = 0 \ .
\end{aligned}
\tag{2.4.79}
$$

Now solving equations in (2.4.79) by using equations (1.3.27) we obtain

$$\begin{aligned}\rho_1 =&\mu_1 = 2\epsilon_1 = 2\gamma_1 = -\sqrt{2}ks_1 \ , \\ \nu_1 =&2\alpha_1 = -\sqrt{2}k\overline{s}_0 \ , \\ \kappa_1 =&2\beta_1 = -\sqrt{2}ks_0 \ ,\end{aligned}$$

and

$$\pi_1 = \tau_1 = \lambda_1 = \sigma_1 = 0 \ . \tag{2.4.80}$$

Using the equation (2.4.75) in the Cartan's second equation of structure (2.4.66), we get

$$\begin{aligned}\Omega^\alpha{}_\beta =&d_*\omega^\alpha{}_\beta + \omega^\alpha{}_\epsilon \wedge \omega^\epsilon{}_\beta + \frac{k}{4}\big[2\gamma^\alpha{}_{\beta\sigma}S_{\epsilon\delta}u^\sigma + \\ &+ \gamma^\alpha{}_{\beta\epsilon}S_{\delta\sigma}u^\sigma - \gamma^\alpha{}_{\beta\delta}S_{\epsilon\sigma}u^\sigma\big]\theta^\beta \wedge \theta^\sigma \ .\end{aligned} \tag{2.4.81}$$

By giving different values to $\epsilon, \delta = 1, 2, 3, 4$ in the equation (2.4.81) and using the equation (1.3.26) we obtain

$$\begin{aligned}\Omega^\alpha{}_\beta =&d_*\omega^\alpha{}_\beta + \omega^\alpha{}_\epsilon \wedge \omega^\epsilon{}_\beta + \frac{k}{2\sqrt{2}}\big[(s_1+\overline{s}_1)(\gamma^\alpha{}_{\beta 1} + \gamma^\alpha{}_{\beta 2})\theta^{12} + \\ &+ \{\gamma^\alpha{}_{\beta 1}(s_0+s_2) + 2s_0\gamma^\alpha{}_{\beta 2} - (s_1+\overline{s}_1)\gamma^\alpha{}_{\beta 3}\}\theta^{13} + \\ &+ \{\gamma^\alpha{}_{\beta 1}(\overline{s}_0+\overline{s}_2) + 2\overline{s}_0\gamma^\alpha{}_{\beta 2} - (s_1+\overline{s}_1)\gamma^\alpha{}_{\beta 4}\}\theta^{14} + \\ &+ \{-2s_2\gamma^\alpha{}_{\beta 1} - \gamma^\alpha{}_{\beta 2}(s_0+s_2) + (s_1+\overline{s}_1)\gamma^\alpha{}_{\beta 3}\}\theta^{23} + \\ &+ \{-2\overline{s}_2\gamma^\alpha{}_{\beta 1} - \gamma^\alpha{}_{\beta 2}(\overline{s}_0+\overline{s}_2) + (s_1+\overline{s}_1)\gamma^\alpha{}_{\beta 4}\}\theta^{24} + \\ &+ \big\{-2(s_1-\overline{s}_1)(\gamma^\alpha{}_{\beta 1} + \gamma^\alpha{}_{\beta 2}) - \gamma^\alpha{}_{\beta 3}(\overline{s}_0-\overline{s}_2) + \\ &+ \gamma^\alpha{}_{\beta 4}(s_0-s_2)\big\}\theta^{34}\big] \ .\end{aligned} \tag{2.4.82}$$

By giving the values to $\alpha, \beta = 1, 2, 3, 4$ in equation (2.4.82), we obtain

$$
\begin{aligned}
\Omega^1{}_1 =& d_*\omega^1{}_1 + \omega^1{}_\epsilon \wedge \omega^\epsilon{}_1 + \frac{k}{2\sqrt{2}}\big[(s_1 + \overline{s}_1)(\gamma^1{}_{11} + \gamma^1{}_{12})\theta^{12} + \\
&+ \{\gamma^1{}_{11}(s_0 + s_2) + 2s_0\gamma^1{}_{12} - (s_1 + \overline{s}_1)\gamma^1{}_{13}\}\theta^{13} + \\
&+ \{\gamma^1{}_{11}(\overline{s}_0 + \overline{s}_2) + 2\overline{s}_0\gamma^1{}_{12} - (s_1 + \overline{s}_1)\gamma^1{}_{14}\}\theta^{14} + \\
&+ \{-2s_2\gamma^1{}_{11} - \gamma^1{}_{12}(s_0 + s_2) + (s_1 + \overline{s}_1)\gamma^1{}_{13}\}\theta^{23} + \\
&+ \{-2\overline{s}_2\gamma^1{}_{11} - \gamma^1{}_{12}(\overline{s}_0 + \overline{s}_2) + (s_1 + \overline{s}_1)\gamma^1{}_{14}\}\theta^{24} + \\
&+ \Big\{ - 2(s_1 - \overline{s}_1)(\gamma^1{}_{11} + \gamma^1{}_{12}) - \gamma^1{}_{13}(\overline{s}_0 - \overline{s}_2) + \gamma^1{}_{14}(s_0 - s_2)\Big\}\theta^{34}\big] \;, \\
\Omega^1{}_3 =& d_*\omega^1{}_3 + \omega^1{}_\epsilon \wedge \omega^\epsilon{}_3 + \frac{k}{2\sqrt{2}}\big[(s_1 + \overline{s}_1)(\gamma^1{}_{31} + \gamma^1{}_{32})\theta^{12} + \\
&+ \{\gamma^1{}_{31}(s_0 + s_2) + 2s_0\gamma^1{}_{32} - (s_1 + \overline{s}_1)\gamma^1{}_{33}\}\theta^{13} + \\
&+ \{\gamma^1{}_{31}(\overline{s}_0 + \overline{s}_2) + 2\overline{s}_0\gamma^1{}_{32} - (s_1 + \overline{s}_1)\gamma^1{}_{34}\}\theta^{14} + \\
&+ \{-2s_2\gamma^1{}_{31} - \gamma^1{}_{32}(s_0 + s_2) + (s_1 + \overline{s}_1)\gamma^1{}_{33}\}\theta^{23} + \\
&+ \{-2\overline{s}_2\gamma^1{}_{31} - \gamma^1{}_{32}(\overline{s}_0 + \overline{s}_2) + (s_1 + \overline{s}_1)\gamma^1{}_{34}\}\theta^{24} + \\
&+ \Big\{ - 2(s_1 - \overline{s}_1)(\gamma^1{}_{31} + \gamma^1{}_{32}) - \gamma^1{}_{33}(\overline{s}_0 - \overline{s}_2) + \gamma^1{}_{34}(s_0 - s_2)\Big\}\theta^{34}\big] \;, \\
\Omega^2{}_3 =& d_*\omega^2{}_3 + \omega^2{}_\epsilon \wedge \omega^\epsilon{}_3 + \frac{k}{2\sqrt{2}}\big[(s_1 + \overline{s}_1)(\gamma^2{}_{31} + \gamma^2{}_{32})\theta^{12} + \\
&+ \{\gamma^2{}_{31}(s_0 + s_2) + 2s_0\gamma^2{}_{32} - (s_1 + \overline{s}_1)\gamma^2{}_{33}\}\theta^{13} + \\
&+ \{\gamma^2{}_{31}(\overline{s}_0 + \overline{s}_2) + 2\overline{s}_0\gamma^2{}_{32} - (s_1 + \overline{s}_1)\gamma^2{}_{34}\}\theta^{14} + \\
&+ \{-2s_2\gamma^2{}_{31} - \gamma^2{}_{32}(s_0 + s_2) + (s_1 + \overline{s}_1)\gamma^2{}_{33}\}\theta^{23} + \\
&+ \{-2\overline{s}_2\gamma^2{}_{31} - \gamma^2{}_{32}(\overline{s}_0 + \overline{s}_2) + (s_1 + \overline{s}_1)\gamma^2{}_{34}\}\theta^{24} + \\
&+ \Big\{ - 2(s_1 - \overline{s}_1)(\gamma^2{}_{31} + \gamma^2{}_{32}) - \gamma^2{}_{33}(\overline{s}_0 - \overline{s}_2) + \gamma^2{}_{34}(s_0 - s_2)\Big\}\theta^{34}\big] \;,
\end{aligned}
$$

$$
\begin{aligned}
\Omega^3{}_3 =& d_*\omega^3{}_3 + \omega^3{}_\epsilon \wedge \omega^\epsilon{}_3 + \frac{k}{2\sqrt{2}}\big[(s_1+\overline{s}_1)(\gamma^3{}_{31}+\gamma^3{}_{32})\theta^{12} + \\
&+ \{\gamma^3{}_{31}(s_0+s_2) + 2s_0\gamma^3{}_{32} - (s_1+\overline{s}_1)\gamma^3{}_{33}\}\theta^{13} + \\
&+ \{\gamma^3{}_{31}(\overline{s}_0+\overline{s}_2) + 2\overline{s}_0\gamma^3{}_{32} - (s_1+\overline{s}_1)\gamma^3{}_{34}\}\theta^{14} + \\
&+ \{-2s_2\gamma^3{}_{31} - \gamma^3{}_{32}(s_0+s_2) + (s_1+\overline{s}_1)\gamma^3{}_{33}\}\theta^{23} + \\
&+ \{-2\overline{s}_2\gamma^3{}_{31} - \gamma^3{}_{32}(\overline{s}_0+\overline{s}_2) + (s_1+\overline{s}_1)\gamma^3{}_{34}\}\theta^{24} + \\
&+ \big\{ -2(s_1-\overline{s}_1)(\gamma^3{}_{31}+\gamma^3{}_{32}) - \gamma^3{}_{33}(\overline{s}_0-\overline{s}_2) + \\
&+ \gamma^3{}_{34}(s_0-s_2)\big\}\theta^{34}\big] \ . \qquad (2.4.83)
\end{aligned}
$$

From equations (1.3.12) and (1.3.14) we find

$$
\begin{aligned}
\gamma^1{}_{11} =& \gamma_{211} = \epsilon^0 + \overline{\epsilon}^0 + \epsilon_1 + \overline{\epsilon}_1 \ , \quad \gamma^1{}_{12} = \gamma_{212} = \gamma^0 + \overline{\gamma}^0 + \gamma_1 + \overline{\gamma}_1 \ , \\
\gamma^1{}_{13} =& \gamma_{213} = \overline{\alpha}^0 + \beta^0 + \overline{\alpha}_1 + \beta_1 \ , \quad \gamma^1{}_{31} = \gamma_{231} = \overline{\pi}^0 + \overline{\pi}_1 \ , \\
\gamma^1{}_{32} =& \gamma_{232} = \overline{\nu}^0 + \overline{\nu}_1 \ , \quad \gamma^1{}_{33} = \gamma_{233} = \overline{\lambda}^0 + \overline{\lambda}_1 \ , \\
\gamma^1{}_{34} =& \gamma_{234} = \overline{\mu}^0 + \overline{\mu}_1 \ , \quad \gamma^2{}_{21} = \gamma_{121} = -(\epsilon^0 + \overline{\epsilon}^0 + \epsilon_1 + \overline{\epsilon}_1) \ , \\
\gamma^2{}_{22} =& \gamma_{122} = -(\gamma^0 + \overline{\gamma}^0 + \gamma_1 + \overline{\gamma}_1) \ , \\
\gamma^2{}_{23} =& \gamma_{123} = -(\overline{\alpha}^0 + \beta^0 + \overline{\alpha}_1 + \beta_1) \ , \\
\gamma^2{}_{31} =& \gamma_{131} = -(\kappa^0 + \kappa_1) \ , \quad \gamma^2{}_{32} = \gamma_{132} = -(\tau^0 + \tau_1) \ , \\
\gamma^2{}_{33} =& \gamma_{133} = -(\sigma^0 + \sigma_1) \ , \quad \gamma^2{}_{34} = \gamma_{134} = -(\rho^0 + \rho_1) \ , \\
\gamma^3{}_{11} =& -\gamma_{411} = -(\overline{\kappa}^0 + \overline{\kappa}_1) \ , \quad \gamma^3{}_{12} = -\gamma_{412} = -(\overline{\tau}^0 + \overline{\tau}_1) \ , \\
\gamma^3{}_{13} =& -\gamma_{413} = -(\overline{\rho}^0 + \overline{\rho}_1) \ , \quad \gamma^3{}_{14} = -\gamma_{414} = -(\overline{\sigma}^0 + \overline{\sigma}_1) \ , \\
\gamma^3{}_{21} =& -\gamma_{421} = -(\pi^0 + \pi_1) \ , \quad \gamma^3{}_{22} = -\gamma_{422} = -(\nu^0 + \nu_1) \ , \\
\gamma^3{}_{23} =& -\gamma_{423} = (\mu^0 + \mu_1) \ , \quad \gamma^3{}_{24} = -\gamma_{424} = (\lambda^0 + \lambda_1) \ ,
\end{aligned}
$$

$$\begin{aligned}
\gamma^3{}_{31} &= -\,\gamma_{431} = (\epsilon^0 - \overline{\epsilon}^0 + \epsilon_1 - \overline{\epsilon}_1) \ , \\
\gamma^3{}_{32} &= -\,\gamma_{432} = (\gamma^0 - \overline{\gamma}^0 + \gamma_1 - \overline{\gamma}_1) \ , \\
\gamma^3{}_{33} &= -\,\gamma_{433} = -(\overline{\alpha}^0 - \beta^0 + \overline{\alpha}_1 - \beta_1) \ , \\
\gamma^3{}_{34} &= -\,\gamma_{434} = -(\alpha^0 - \overline{\beta}^0 + \alpha_1 - \overline{\beta}_1) \ , \qquad (2.4.84)
\end{aligned}$$

and $\gamma^1{}_{21}, \gamma^1{}_{23}, \gamma^1{}_{22}, \gamma^2{}_{11}, \gamma^2{}_{12}, \gamma^2{}_{13}, \gamma^3{}_{41}, \gamma^3{}_{42}, \gamma^3{}_{43}, \gamma^3{}_{44}$ are all zero. All other Ricci's coefficients of rotation namely $\gamma^1{}_{14}, \gamma^1{}_{41}, \gamma^1{}_{42}, \gamma^1{}_{43}, \gamma^1{}_{44}, \gamma^2{}_{24}$, $\gamma^2{}_{41}, \gamma^2{}_{42}, \gamma^2{}_{43}, \gamma^2{}_{44}, \gamma^4{}_{11}, \gamma^4{}_{12}, \gamma^4{}_{13}, \gamma^4{}_{14}, \gamma^4{}_{22}, \gamma^4{}_{23}, \gamma^4{}_{24}, \gamma^4{}_{41}, \gamma^4{}_{42}, \gamma^4{}_{43}, \gamma^4{}_{44}$, are complex conjugates and are obtain by interchanging 3 and 4 and taking the complex conjugate of the respective terms.

Now using equations (2.4.84) and (1.3.27) in equations (2.4.83), we obtain

$$\begin{aligned}
\Omega^1_{\cdot 1} &= d_*\omega^1_{\cdot 1} + \omega^1_{\cdot 3} \wedge \omega^3_{\cdot 1} + \omega^1_{\cdot 4} \wedge \omega^4_{\cdot 1} + \frac{k}{\sqrt{2}}\big[s_0\big(\epsilon^0 + \overline{\epsilon}^0 + \gamma^0 + \overline{\gamma}^0 + \\
&\quad + \epsilon_1 + \overline{\epsilon}_1 + \gamma_1 + \overline{\gamma}_1\big)(\theta^{13} - \theta^{23}) + \overline{s}_0\big(\epsilon^0 + \overline{\epsilon}^0 + \gamma^0 + \overline{\gamma}^0 + \\
&\quad + \epsilon_1 + \overline{\epsilon}_1 + \gamma_1 + \overline{\gamma}_1\big)(\theta^{14} - \theta^{24}) - 2s_1\big(\epsilon^0 + \overline{\epsilon}^0 + \gamma^0 + \overline{\gamma}^0 + \\
&\quad + \epsilon_1 + \overline{\epsilon}_1 + \gamma_1 + \overline{\gamma}_1\big)\theta^{34}\big] \ , \\
\Omega^1_{\cdot 3} &= d_*\omega^1_{\cdot 3} + \omega^1_{\cdot 1} \wedge \omega^1_{\cdot 3} + \omega^1_{\cdot 3} \wedge \omega^3_{\cdot 3} + \\
&\quad + \frac{k}{\sqrt{2}}\big[s_0\big(\overline{\nu}^0 + \overline{\nu}_1 + \overline{\pi}^0 + \overline{\pi}_1\big)(\theta^{13} - \theta^{23}) + \\
&\quad + \overline{s}_0\big(\overline{\nu}^0 + \overline{\nu}_1 + \overline{\pi}^0 + \overline{\pi}_1\big)(\theta^{14} - \theta^{24}) - \\
&\quad - 2s_1\big(\overline{\nu}^0 + \overline{\nu}_1 + \overline{\pi}^0 + \overline{\pi}_1\big)\theta^{34}\big] \ ,
\end{aligned}$$

$$
\begin{aligned}
\Omega^2_{\cdot 3} &= d_*\omega^2_{\cdot 3} + \omega^2_{\cdot 2}\wedge\omega^2_{\cdot 3} + \omega^2_{\cdot 3}\wedge\omega^3_{\cdot 3} - \\
&- \frac{k}{\sqrt{2}}\big[s_0(\kappa^0+\kappa_1+\tau^0+\tau_1)(\theta^{13}-\theta^{23}) + \\
&+ \overline{s}_0(\kappa^0+\kappa_1+\tau^0+\tau_1)(\theta^{14}-\theta^{24}) - 2s_1(\kappa^0+\kappa_1+\tau^0+\tau_1)\theta^{34}\big]\ , \\
\Omega^3_{\cdot 3} &= d_*\omega^3_{\cdot 3} + \omega^3_{\cdot 1}\wedge\omega^1_{\cdot 3} + \omega^3_{\cdot 2}\wedge\omega^2_{\cdot 3} + \frac{k}{\sqrt{2}}\big[s_0\big(\epsilon^0-\overline{\epsilon}^0+\gamma^0-\overline{\gamma}^0 + \\
&+ \epsilon_1-\overline{\epsilon}_1+\gamma_1-\overline{\gamma}_1\big)(\theta^{13}-\theta^{23}) + \overline{s}_0\big(\epsilon^0-\overline{\epsilon}^0+\gamma^0-\overline{\gamma}^0 + \\
&+ \epsilon_1-\overline{\epsilon}_1+\gamma_1-\overline{\gamma}_1\big)(\theta^{14}-\theta^{24}) - 2s_1\big(\epsilon^0-\overline{\epsilon}^0+\gamma^0-\overline{\gamma}^0 + \\
&+ \epsilon_1-\overline{\epsilon}_1+\gamma_1-\overline{\gamma}_1\big)\theta^{34}\big]\ .
\end{aligned}
\tag{2.4.85}
$$

2.5 Curvature of a Non-Riemannian Space

The formula for the Riemann curvature for a Riemannian space is well known in the literature[28, 135]. Following the same procedure we find in this section, a new formula for the Riemannian Curvature of a non-Riemannian space. We first establish the relation between the asymmetric connections of a non-Riemannian subspace V_n and an enveloping non-Riemannian space V_m. Let the metrics of V_n and V_m be respectively given by

$$ds^2 = g_{ij}dx^i dx^j\ , \tag{2.5.1}$$

and

$$ds^2 = a_{\alpha\beta}dy^\alpha dy^\beta\ , \tag{2.5.2}$$

where g_{ij} and $a_{\alpha\beta}$ are the metric tensors of V_n and V_m respectively such that

$$g_{ij} = a_{\alpha\beta} y^{\alpha}{}_{;i} y^{\beta}{}_{;j} \ . \tag{2.5.3}$$

From this we obtain the relation

$$(\Gamma_{ij,k})_g = -(K_{ijk})_g + \left[(\Gamma_{\alpha\beta,\gamma})_a + (K_{\alpha\beta\gamma})_a\right] y^{\alpha}{}_{,i} y^{\beta}{}_{,j} y^{\gamma}{}_{,k} + a_{\alpha\beta} y^{\alpha}{}_{,ij} y^{\beta}{}_{,k} \ . \tag{2.5.4}$$

We use this relation to find the expression for the Riemann curvature of a non-Riemann space V_n.

2.5.1 Formula for Riemann Curvature of a V_n

Following the method of Weatherburn [135], we obtain in this section, an expression for the Riemannian-Cartan curvature of a non-Riemannian space. We first construct a 2-dimensional geodesic surface S_1, through a point P of V_n, determined by the orientations of the two unit vectors p^i and q^i. The Gussian curvature of this surface at a point is called the Riemannian-Cartan curvature of V_n at that point, for the orientation determined by two vectors p^i and q^i. It is given by

$$\kappa = \kappa_1 + \frac{1}{b}\left[\frac{\partial}{\partial u^1}(K_{212})_b - \frac{\partial}{\partial u^2}(K_{112})_b\right] , \tag{2.5.5}$$

where b is the determinant of the metric tensor of the 2-dimensional surface S_1. It is given by

$$b = (g_{hj} g_{ik} - g_{ij} g_{hk})\, p^h q^i p^j q^k \ ,$$

and

$$\kappa_1 = \frac{\hat{R}_{hijk} p^h q^i p^j q^k}{(g_{hj} g_{ik} - g_{ij} g_{hk}) \, p^h q^i p^j q^k} \ , \tag{2.5.6}$$

is the Riemann-Cartan Curvature, at a point, of a Riemannian space. The formula (2.5.5) determines the Riemann-Cartan curvature of a non-Riemannian space V_n at a point. The quantities $(K_{212})_b$ and $(K_{112})_b$ are the tensor components of contortion tensor of 2-dimensional surface S_1.

In order to construct an example of a 2-dimensional non-Riemannian space, we first develop below the null dyad formalism in a 2-dimensions Riemannian space.

2.5.2 Dyad Formalism in a V_2

In the following we introduce, to work in a 2-dimensional space, two null vector formalism. Hereafter we refer to it as the dyad formalism. Let C be a curve defined in V_2. At each point of the curve we define a dyad of basis vectors as

$$e_{(\alpha)i} = (m_i, \overline{m}_i) \ , \tag{2.5.7}$$

where m_i and $\overline{m}_i$ are complex conjugate null vector fields satisfying the orthonormality conditions

$$\begin{aligned} m_i m^i =& \overline{m}_i \overline{m}^i = 0 \ , \\ m_i \overline{m}^i =& 1 \ . \end{aligned} \tag{2.5.8}$$

Here the Latin indices are used to denote the tensor indices while the Greek indices are used to denote the dyad indices. The dyad of the dual basis vectors is given by

$$e^{(\alpha)}{}_i = (\overline{m}_i, m_i) \ , \tag{2.5.9}$$

where the basis vectors and the dual basis vectors of the dyad satisfy the conditions

$$e^{(\alpha)}{}_i e_{(\alpha)}{}^k = \delta_i^k \ , \text{and} \ e^{(\alpha)}{}_i e_{(\beta)}{}^i = \delta^\alpha{}_\beta \ . \tag{2.5.10}$$

Consequently, we express the dyad components of the metric tensor g_{ij} as

$$\eta_{\alpha\beta} = g_{ij} e_{(\alpha)}{}^i e_{(\beta)}{}^j \ . \tag{2.5.11}$$

This gives

$$\eta_{\alpha\beta} = \eta^{\alpha\beta} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ . \tag{2.5.12}$$

Hence the metric tensor in terms of the basis vectors is defined as

$$g_{ij} = m_i \overline{m}_j + \overline{m}_i m_j \ . \tag{2.5.13}$$

Let $S_{ij}{}^k$ be the spin angular momentum tensor. Hehl et al. [51] have split up this tensor in to spin tensor S_{ij} in the form

$$S_{ij}{}^k = S_{ij} u^k \ . \tag{2.5.14}$$

The spin tensor is anti-symmetric; hence it has just one independent component in the 2-dimension space. We express the spin tensor as a linear combination of the basis vectors of the dyad as

$$\begin{aligned} S_{ij} =& S_{\alpha\beta} e^{(\alpha)}{}_i e^{(\beta)}{}_j \ , \\ S_{ij} =& (S_{12})_d(\overline{m}_i m_j - m_i \overline{m}_j) \ , \\ S_{ij} =& S_d(\overline{m}_i m_j - m_i \overline{m}_j) \ , \end{aligned} \tag{2.5.15}$$

where $S_d = (S_{12})_d$ is the dyad component of spin tensor. In general it is a function of coordinates. For the choice of the time-like vector field

$$u^i = \frac{1}{\sqrt{2}}(m^i + \overline{m}^i) \ ,$$

such that $u^i u_i = 1$, we have from equations (2.5.14) and (2.5.15) that

$$S_{ik}{}^k = S_{ij} u^k = \frac{S_d}{\sqrt{2}}(\overline{m}_i m_j - m_i \overline{m}_j)(m^k + \overline{m}^k) \ . \tag{2.5.16}$$

We see from this equation that

$$S_{ik}{}^k = S_{ik} u^k = \frac{S_d}{\sqrt{2}}(\overline{m}_i - m_i) \neq 0 \ . \tag{2.5.17}$$

We express the torsion tensor $Q_{ij}{}^k$ in terms of its dyad components as

$$Q_{ij}{}^k = Q_{\alpha\beta}{}^\gamma e^{(\alpha)}{}_i e^{(\beta)}{}_j e_{(\gamma)}{}^k \ .$$

This leads to

$$Q_{ij}{}^k = \frac{k S_d}{2\sqrt{2}}(\overline{m}_i m_j - m_i \overline{m}_j)(m^k + \overline{m}^k) \ . \tag{2.5.18}$$

Similarly, we obtain an expression for the contortion tensor ${K_{ij}}^k$

$$ {K_{ij}}^k = \frac{kS_d}{\sqrt{2}}(\overline{m}_i\overline{m}_j m^k - \overline{m}_i m_j \overline{m}^k + m_i\overline{m}_j m^k - m_i m_j \overline{m}^k) \ . \quad (2.5.19) $$

Consider now, the 2-dimensional space V_2 characterized by the metric

$$ ds^2 = r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \ . \quad (2.5.20) $$

We define the basis 1-forms as

$$ \theta^1 = \frac{1}{\sqrt{2}}(rd\theta + irsin\theta d\phi), \quad \theta^2 = \frac{1}{\sqrt{2}}(rd\theta - irsin\theta d\phi) \ . \quad (2.5.21) $$

Hence the metric (2.5.20) becomes

$$ ds^2 = 2\theta^1\theta^2 \ . \quad (2.5.22) $$

The equation $\theta^\alpha = {e^{(\alpha)}}_i dx^i$ yields

$$ \begin{aligned} m_i &= \frac{1}{\sqrt{2}}(r, -irsin\theta) \ , \quad \overline{m}_i = \frac{1}{\sqrt{2}}(r, irsin\theta) \ , \\ m^i &= \frac{1}{\sqrt{2}}(\frac{1}{r}, -\frac{i}{rsin\theta}) \ , \quad \overline{m}^i = \frac{1}{\sqrt{2}}(\frac{1}{r}, \frac{i}{rsin\theta}) \ . \end{aligned} \quad (2.5.23) $$

From equations (2.5.15) and (2.5.23), the tensor components of the spin tensor is obtain as

$$ S_t = -ir^2 sin\theta S_d \ , $$

where $S_t = (S_{12})_t$ and it is a function of θ and ϕ. From this we find

$$ S_d = \frac{i}{r^2} cosec\theta S_t \ . \quad (2.5.24) $$

Thus the tensor components of the asymmetric connection become

$$
\begin{aligned}
&({\Gamma_{11}}^{1})_t = 0 \ , ({\Gamma_{21}}^{1})_t = 0 \ , ({\Gamma_{22}}^{2})_t = 0 \ , \\
&({\Gamma_{12}}^{1})_t = \left(\frac{k}{r}\right) S_t \ , ({\Gamma_{21}}^{2})_t = ({\Gamma_{12}}^{2})_t = cot\theta \ , \\
&({\Gamma_{22}}^{1})_t = - \ sin\theta cos\theta \ , ({\Gamma_{11}}^{2})_t = - \left(\frac{k}{r}\right) cosec^2\theta S_t \ .
\end{aligned}
\tag{2.5.25}
$$

The non-vanishing tensor component of the curvature tensor of a non-Riemannian space is

$$
(R_{1212})_t = r^2 sin^2\theta - (kr) S_{t,2} \ . \tag{2.5.26}
$$

The tensor components of the Ricci tensor become

$$
(R_{11})_t = -1 + \left(\frac{K}{r}\right) cosec^2\theta S_{t,2} \ , (R_{22})_t = -sin^2\theta + \left(\frac{K}{r}\right) S_{t,2} \ , \tag{2.5.27}
$$

and the Ricci scalar takes the form

$$
R = -\frac{2}{r^2} + 2\left(\frac{k}{r^3}\right) cosec^2\theta S_{t,2} \ . \tag{2.5.28}
$$

Finally, the expression for the curvature of a non-Riemannian sphere becomes

$$
\kappa = \frac{1}{r^2} - \left(\frac{k}{r^3}\right) cosec^2\theta S_{t,2} \ , \tag{2.5.29}
$$

$$
\kappa = \kappa_1 - \left(\frac{k}{r^3}\right) cosec^2\theta S_{t,2} \ , \tag{2.5.30}
$$

where $S_t = S_t(\theta, \phi)$ and $\kappa_1 = \frac{1}{r^2}$ is the constant curvature of a Riemannian sphere. We see from the equation (2.5.29) that the curvature of the non-Riemannian sphere is a function of coordinates. It shows that $\frac{\partial\kappa}{\partial x^i} \neq 0$. It follows from the equation(2.5.29) that torsion influences the curvature of the non-Riemannian sphere. We further note that, in the absence of spin tensor or if the component of the spin tensor is only a function θ, then, the Riemann curvature of a non-Riemannian sphere is the same as that of Riemannian sphere. However, if the component of the spin tensor is a function of ϕ, then we see from the equation (2.5.30) that the torsion influences the curvature of the non-Riemannian sphere. For the choice $S_t = \phi$, we have $S_{t,2} = 1$. Consequently, from the equation (2.5.29) we obtain

$$\kappa = \frac{1}{r^2} - \left(\frac{k}{r^3}\right) cosec^2\theta \ . \tag{2.5.31}$$

As $k = \frac{8\pi G}{c^4}$, where $G = 6.66 \times 10^{-11} m^3/kg/sec^2, c = 3 \times 10^8 m/sec$, then the value of the constant $k = 2.0667378 \times 10^{-43}$. At a point $\theta = \frac{\pi}{2}$ of the sphere, the curvature of the non-Riemannian sphere becomes

$$\kappa = \frac{1}{r^2} - \frac{1}{r^3}(2.0667378 \times 10^{-43}) \ . \tag{2.5.32}$$

We see from the equation (2.5.32) that the curvature of the non-Riemannian sphere at a point $\theta = \frac{\pi}{2}$ of the sphere differ from the curvature of a Riemannian sphere by an infinitesimal amount.

The dyad equivalent of the equation (2.4.15) is given by

$$d_*(d_* f) = -\frac{1}{2} f_{;\gamma} Q_{\alpha\beta}{}^{\gamma} \theta^\alpha \wedge \theta^\beta \ , \tag{2.5.33}$$

where

$$f_{;\gamma} = f_{;i} {e_{(\gamma)}}^{i} \ . \tag{2.5.34}$$

Consequently, we obtain

$$d_*^2 f = - \frac{1}{2\sqrt{2}} \Bigg[\left(\frac{1}{r} \frac{\partial f}{\partial \theta} - \frac{i}{r} cosec\theta \frac{\partial f}{\partial \phi} \right) {Q_{\alpha\beta}}^{1} + \\ + \left(\frac{1}{r} \frac{\partial f}{\partial \theta} + \frac{i}{r} cosec\theta \frac{\partial f}{\partial \phi} \right) {Q_{\alpha\beta}}^{2} \Bigg] \theta^{\alpha} \wedge \theta^{\beta} \ . \tag{2.5.35}$$

Using this equation we readily find

$$d_*^2 \theta = - \frac{k}{2r} S_d \theta^1 \wedge \theta^2 \text{ and } \ d_*^2 \phi = 0 \ . \tag{2.5.36}$$

Now operating the new exterior derivative operator d_* to the basis 1-form defined in the equation (2.5.21), we obtain

$$d_* \theta^1 = - \frac{1}{\sqrt{2}} \left(\frac{k S_d}{2} + \frac{1}{r} cot\theta \right) \theta^1 \wedge \theta^2 \ ,$$

and

$$d_* \theta^2 = - \frac{1}{\sqrt{2}} \left(\frac{k S_d}{2} - \frac{1}{r} cot\theta \right) \theta^1 \wedge \theta^2 \ . \tag{2.5.37}$$

We define

$$\overline{m}_{i/j} m^i m^j = \kappa^0 \ , \overline{m}_{i/j} m^i \overline{m}^j = -\overline{\kappa}^0 \ , \tag{2.5.38}$$

where κ^0 is the spin component. We obtain the expression of the covariant derivative of a basis vector of the dyad as

$$m_{i;j} = \left(\overline{\kappa}^0 - \frac{ik cosec\theta}{r^2 \sqrt{2}} S_t \right) m_i m_j - \left(\kappa^0 + \frac{ik cosec\theta}{r^2 \sqrt{2}} S_t \right) m_i \overline{m}_j \ . \tag{2.5.39}$$

The equation (2.4.53) yields the components of connection 1-form as

$$\omega^1{}_1 = -\omega^2{}_2 = -\left(\frac{kS_d}{\sqrt{2}} + \kappa^0\right)\theta^1 - \left(\frac{kS_d}{\sqrt{2}} - \overline{\kappa}^0\right)\theta^2 \ . \tag{2.5.40}$$

Also from the Cartan's first equation of the structure (2.4.52), we have

$$d_*\theta^1 = \left(\overline{\kappa}^0 - \frac{kS_d}{2\sqrt{2}}\right)\theta^1 \wedge \theta^2 \ ,$$

and

$$d_*\theta^2 = -\left(\kappa^0 + \frac{kS_d}{2\sqrt{2}}\right)\theta^1 \wedge \theta^2 \ . \tag{2.5.41}$$

Comparing the corresponding coefficients of the equations (2.5.37) and (2.5.41), we readily get

$$\kappa^0 = \overline{\kappa}^0 = -\frac{cot\theta}{r\sqrt{2}} \ . \tag{2.5.42}$$

Hence the equation (2.5.40) becomes

$$\omega^1{}_1 = -\omega^2{}_2 = -\frac{1}{\sqrt{2}}\left(kS_d - \frac{cot\theta}{r}\right)\theta^1 - \frac{1}{\sqrt{2}}\left(kS_d + \frac{cot\theta}{r}\right)\theta^2 \ . \tag{2.5.43}$$

Now using equation (2.5.43) in the Cartan's second equation of structure (2.4.81), we obtain

$$\Omega^1{}_1 = -\Omega^2{}_2 = \left(\frac{ik}{rsin\theta}S_{d,2} + \frac{1}{r^2}\right)\theta^1 \wedge \theta^2 \ . \tag{2.5.44}$$

Also the components of the curvature 2-form using equation (2.4.65) defined by

$$\Omega^1{}_1 = -(R_{121}{}^1)_d\theta^1 \wedge \theta^2 \ . \tag{2.5.45}$$

Comparing the corresponding coefficients of the equations (2.5.44) and (2.5.45), we obtain the dyad component of the Riemannian curvature tensor as

$$(R_{1212})_d = -\left(\frac{ik}{rsin\theta}S_{d,2} + \frac{1}{r^2}\right) . \tag{2.5.46}$$

We express the Riemannian curvature tensor in terms of its dyad components as

$$R_{hijk} = R_{\alpha\beta\gamma\delta}e^{(\alpha)}{}_h e^{(\beta)}{}_i e^{(\gamma)}{}_j e^{(\delta)}{}_k \ .$$

This becomes

$$R_{hijk} = \left(\frac{k}{r^3}cosec^2\theta S_{t,2} - \frac{1}{r^2}\right)\left(\overline{m}_h m_i \overline{m}_j m_k - \overline{m}_h m_i m_j \overline{m}_k - \right.$$
$$\left. - m_h \overline{m}_i \overline{m}_j m_k + m_h \overline{m}_i m_j \overline{m}_k\right) . \tag{2.5.47}$$

We write this equation as

$$R_{hijk} = \kappa(g_{hj}g_{ik} - g_{ij}g_{hk}) , \tag{2.5.48}$$

where

$$\kappa = \frac{1}{r^2} - \frac{k}{r^3}cosec^2\theta S_{t,2} , \tag{2.5.49}$$

is the Riemann curvature and R_{hijk} is the Riemann curvature tensor of the non-Riemannian sphere V_2. The sphere is non-Riemannian because the curvature tensor of the sphere contains torsion term and satisfies the identities (1.2.27), (1.2.40) and (1.2.41). We see that the equation

(2.5.48) is the same as that of the equation $\hat{R}_{hijk} = \kappa_1(g_{hj}g_{ik} - g_{ij}g_{hk})$. But the corresponding space is not homogeneous, as R_{hijk} involves torsion κ is not constant.

If the Riemann curvature tensor of a non-Riemannian $V_n, n > 2$, satisfies the equation (2.5.48), then it readily follows from the Bianchi identities (1.2.40) that

$$\frac{\kappa_{;i}}{\kappa} = \left(\frac{4}{n-1}\right) Q_{ki}{}^{k} \ ,$$

or

$$\kappa = c exp\left[\left(\frac{4}{n-1}\right) \int Q_{ki}{}^{k} dx^{i}\right] \ , \tag{2.5.50}$$

where c is a constant of integration. If $Q_{ki}{}^{k} = 0$, then $c = \kappa_1$- the constant Riemann curvature of the Riemannian space V_n, for $n > 2$. Thus we have

$$\kappa = \kappa_1 exp\left[\left(\frac{4}{n-1}\right) \int Q_{ki}{}^{k} dx^{i}\right] \ . \tag{2.5.51}$$

In case of 2-dimensional non-Riemannian space, it is remarkable to note that the curvature of a non-Riemannian sphere determined in the equation (2.5.49) is exactly same as the curvature determined from the formula (2.5.5) in the equation (2.5.29).

Using the equation (2.5.23) in the equation (2.5.45) we obtain the tensor component of the Riemann curvature tensor as

$$(R_{1212})_t = r^2 sin^2\theta - (kr) S_{t,2} \ . \tag{2.5.52}$$

This is also exactly the same as the equation (2.5.26). Consequently, the result (2.5.38) for the Riemannian curvature at a point of a non-Riemannian sphere of radius r is corroborated. Contracting the index h with k in the equation (2.5.47) we obtain the expression for the Ricci tensor as

$$R_{ij} = \left(\frac{k}{r^3} cosec^2\theta S_{t,2} - \frac{1}{r^2}\right)(m_i\overline{m}_j + \overline{m}_i m_j) \ . \tag{2.5.53}$$

This gives

$$R = 2\left(\frac{k}{r^3} cosec^2\theta S_{t,2} - \frac{1}{r^2}\right) \ . \tag{2.5.54}$$

From equations (2.5.49) and (2.5.54) we have

$$\kappa = -\frac{R}{2} \ . \tag{2.5.55}$$

We see that the scalar curvature R is not constant, hence the Riemann curvature κ of the non-Riemannian sphere is not constant.

From equations (2.5.13), (2.5.53) and (2.5.54) we obtain

$$R_{ij} = \frac{R}{2} g_{ij} \ . \tag{2.5.56}$$

We see from the equations (2.5.53) and (2.5.54) that R_{ij} and R are the Ricci tensor and the Ricci curvature scalar of the non-Riemannian space that involve spin term.

2.5.3 Geodesics on a non-Riemannian Sphere

Let $t^i = \frac{dx^i}{ds}$ be the unit tangent vector field to a curve C in a non-Riemannian space V_n. Then the intrinsic derivative of the unit tangent vector t^i in the direction of the curve is called the geodesic curvature vector and it is given by $p^i = t^i{}_{;k}\frac{dx^k}{ds}$. A geodesic in V_n is a curve whose geodesic curvature vector at each point of the curve is identically zero. Thus

$$p^i = 0 \Rightarrow t^i{}_{;k}\frac{dx^k}{ds} = 0 \ .$$

This leads to the geodesic equation in a non-Riemannian space V_n

$$\frac{d^2x^i}{ds^2} + \left\{ {i \atop jk} \right\}\frac{dx^j}{ds}\frac{dx^k}{ds} - K_{kj}{}^i\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \ . \qquad (2.5.57)$$

The non vanishing components of the symmetric Christoffel symbols and the tensor components of the contortion tensor are given by

$$\left\{ {1 \atop 22} \right\} = -\ sin\theta cos\theta \ , \left\{ {2 \atop 12} \right\} = cot\theta \ ,$$
$$(K_{12}{}^1)_t = -\left(\frac{k}{r}\right)S_t \ , (K_{11}{}^2)_t = \left(\frac{k}{r}\right)cosec^2\theta S_t \ . \qquad (2.5.58)$$

By virtue of the equations (2.5.58), the geodesics on a non-Riemannian sphere of constant radius r become

$$\frac{d^2\theta}{ds^2} - sin\theta cos\theta\left(\frac{d\phi}{ds}\right)^2 + \frac{k}{r}S_t\frac{d\theta}{ds}\frac{d\phi}{ds} = 0 \ , \qquad (2.5.59)$$
$$\frac{d^2\phi}{ds^2} + 2cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} - \frac{k}{r}S_t cosec^2\theta\left(\frac{d\theta}{ds}\right)^2 = 0 \ . \qquad (2.5.60)$$

From the metric equation (2.5.20) we have

$$\left(\frac{d\theta}{ds}\right)^2 = \frac{1}{r^2} - sin^2\theta\left(\frac{d\phi}{ds}\right)^2 \ , \qquad (2.5.61)$$

where r and K are constants. These non-linear second order ordinary differential equations of geodesics are solved numerically by using software MATHEMATICA 10 and the graphs of the geodesics are drawn. The command "NDSolve" based on explicit Runge-Kutta method is used. If $S_t = \theta$, then we observe that the equations of geodesics on the non-Riemannian sphere are the great circles. Also, if $S_t = \phi$ then the geodesics on the non-Riemannian sphere are indistinguishable from the great circles as it is also evident from the equation (2.5.32) that the curvature of the non-Riemannian sphere differs from the curvature of the Riemannian sphere by an infinitesimal amount.

Some Useful Results in Einstein-Cartan Theory

Let ϕ be a scalar function of co-ordinates defined in a non-Riemannian space of Einstein-Cartan theory of gravitation. Then we have

$$\phi_{;i} = \phi_{/i} = \phi_{,i} \ . \tag{2.5.62}$$

Taking the covariant derivative of the equation (2.5.62) with respect to the asymmetric connections, we obtain

$$\phi_{;ij} = \phi_{/ij} + \phi_{;k} {K_{ji}}^{k} \ .$$

This gives

$$\phi_{;ij} - \phi_{;ji} = \phi_{;k}\left({K_{ji}}^{k} - {K_{ij}}^{k}\right) \ . \tag{2.5.63}$$

Multiplying the equation (2.5.63) by ${e_{(\alpha)}}^i {e_{(\beta)}}^j$, we get

$$\big(\phi_{;ij} - \phi_{;ji}\big) {e_{(\alpha)}}^i {e_{(\beta)}}^j = \phi_{;\sigma}\big({K_{\beta\alpha}}^\sigma - {K_{\alpha\beta}}^\sigma\big) \ , \tag{2.5.64}$$

where

$$\phi_{;\sigma} = \phi_{;i} {e_{(\sigma)}}^i \ .$$

By giving different values to α and β from 1, 2, 3, 4, and using the equations (1.3.14) and (1.3.17), we obtain the equations.

$$\begin{aligned}
\big(\phi_{;ij} - \phi_{;ji}\big) l^i m^j =& - (\overline{\alpha}_1 + \beta_1 - \overline{\pi}_1) D\phi - \kappa_1 \Delta\phi + (\overline{\rho}_1 + \epsilon_1 - \overline{\epsilon}_1)\delta\phi + \\
& + \sigma_1 \overline{\delta}\phi \ , \\
\big(\phi_{;ij} - \phi_{;ji}\big) l^i n^j =& - (\gamma_1 + \overline{\gamma}_1) D\phi - (\epsilon_1 + \overline{\epsilon}_1)\Delta\phi + (\pi_1 + \overline{\tau}_1)\delta\phi + \\
& + (\overline{\pi}_1 + \tau_1)\overline{\delta}\phi \ , \\
\big(\phi_{;ij} - \phi_{;ji}\big) m^i n^j =& - \overline{\nu}_1 D\phi + (\tau_1 - \overline{\alpha}_1 - \beta_1)\Delta\phi + (\mu_1 - \gamma_1 + \overline{\gamma}_1)\delta\phi + \\
& + \overline{\lambda}_1 \overline{\delta}\phi \ , \\
\big(\phi_{;ij} - \phi_{;ji}\big) m^i \overline{m}^j =& (\mu_1 - \overline{\mu}_1) D\phi + (\rho_1 - \overline{\rho}_1)\Delta\phi - (\alpha_1 - \overline{\beta}_1)\delta\phi + \\
& + (\overline{\alpha}_1 - \beta_1)\overline{\delta}\phi \ .
\end{aligned} \tag{2.5.65}$$

We see from the equations (2.5.65) that, in the absence of torsion, these equations are identically zero as was expected on a Riemannian space of Einstein's theory of gravitation.

2.6 Maxwell's Equations in Einstein-Cartan Theory of Gravitation

The Maxwell's equations in Einstein-Cartan theory of gravitation are defined as

$$F^{ij}{}_{;j} = 0\ , \quad \text{and} \tag{2.6.1}$$

$$F_{[ij;k]} = 0\ , \tag{2.6.2}$$

where the covariant derivative is defined with respect to asymmetric connections. Tetrad components of the Maxwell's equations are obtained as follows:

The tetrad components of the electromagnetic field tensor are defined by

$$F^{\alpha\beta} = F^{ij} e^{(\alpha)}{}_i e^{(\beta)}{}_j\ . \tag{2.6.3}$$

From this we obtain

$$\begin{aligned} F^{\alpha\beta}{}_{;\gamma} =& \left(F^{ij} e^{(\alpha)}{}_i e^{(\beta)}{}_j\right)_{;k} e_{(\gamma)}{}^k\ , \\ F^{\alpha\beta}{}_{;\gamma} =& F^{ij}{}_{;k} e^{(\alpha)}{}_i e^{(\beta)}{}_j e_{(\gamma)}{}^k + F^{ij} e^{(\alpha)}{}_{i;k} e_{(\gamma)}{}^k e^{(\beta)}{}_j + F^{ij} e^{(\alpha)}{}_i e^{(\beta)}{}_{j;k} e_{(\gamma)}{}^k\ . \end{aligned}$$

Using equation (2.4.60), we get

$$F^{\alpha\beta}{}_{;\gamma} = F^{ij}{}_{;k} e^{(\alpha)}{}_i e^{(\beta)}{}_j e_{(\gamma)}{}^k - \gamma^{\alpha}{}_{\sigma\gamma} F^{ij} e^{(\sigma)}{}_i e^{(\beta)}{}_j - \gamma^{\beta}{}_{\sigma\gamma} F^{ij} e^{(\alpha)}{}_i e^{(\sigma)}{}_j\ . \tag{2.6.4}$$

Contracting β with γ in the equation (2.6.4) we get

$$F^{\alpha\beta}{}_{;\beta} = F^{ij}{}_{;k} e^{(\alpha)}{}_i \delta^k{}_j - \gamma^\alpha{}_{\sigma\beta} F^{\sigma\beta} - \gamma^\beta{}_{\sigma\beta} F^{\alpha\sigma} \ ,$$

$$\Rightarrow \quad F^{ij}{}_{;j} e^{(\alpha)}{}_i = F^{\alpha\beta}{}_{;\beta} + F^{\beta\sigma}\gamma^\alpha{}_{\beta\sigma} + F^{\alpha\sigma}\gamma^\beta{}_{\sigma\beta} \ , \qquad (2.6.5)$$

$$\text{Thus} \quad F^{ij}{}_{;j} e^{(\alpha)}{}_i = 0 \Rightarrow \quad F^{\alpha\beta}{}_{;\beta} + F^{\beta\sigma}\gamma^\alpha{}_{\beta\sigma} + F^{\alpha\sigma}\gamma^\beta{}_{\sigma\beta} = 0 \ . \qquad (2.6.6)$$

Expanding the summations defined, in the equation (2.6.6), over the repeated indices we get

$$\begin{aligned} &F^{\alpha 1}{}_{;1} + F^{\alpha 2}{}_{;2} + F^{\alpha 3}{}_{;3} + F^{\alpha 4}{}_{;4} + F^{12}(\gamma^\alpha{}_{12} - \gamma^\alpha{}_{21}) + F^{13}(\gamma^\alpha{}_{13} - \gamma^\alpha{}_{31}) + \\ &+ F^{14}(\gamma^\alpha{}_{14} - \gamma^\alpha{}_{41}) + F^{23}(\gamma^\alpha{}_{23} - \gamma^\alpha{}_{32}) + F^{24}(\gamma^\alpha{}_{24} - \gamma^\alpha{}_{42}) + \\ &+ F^{34}(\gamma^\alpha{}_{34} - \gamma^\alpha{}_{43}) + F^{\alpha 1}(\gamma^1{}_{11} + \gamma^2{}_{12} + \gamma^3{}_{13} + \gamma^4{}_{14}) + \\ &+ F^{\alpha 2}(\gamma^1{}_{21} + \gamma^2{}_{22} + \gamma^3{}_{23} + \gamma^4{}_{24}) + F^{\alpha 3}(\gamma^1{}_{31} + \gamma^2{}_{32} + \gamma^3{}_{33} + \gamma^4{}_{34}) + \\ &+ F^{\alpha 4}(\gamma^1{}_{41} + \gamma^2{}_{42} + \gamma^3{}_{43} + \gamma^4{}_{44}) = 0 \ . \qquad (2.6.7) \end{aligned}$$

Now by giving $\alpha = 1$ in the equation (2.6.7) we get

$$\begin{aligned} &F^{12}{}_{;2} + F^{13}{}_{;3} + F^{14}{}_{;4} - F^{12}(\gamma_{423} + \gamma_{324}) + F^{13}(\gamma_{213} + \gamma_{132} - \gamma_{433}) + \\ &+ F^{14}(\gamma_{214} + \gamma_{142} - \gamma_{344}) - F^{23}\gamma_{232} - F^{24}\gamma_{242} + F^{34}(\gamma_{234} - \gamma_{243}) = 0 \ . \end{aligned} \qquad (2.6.8)$$

The equations

$$F^{\epsilon\sigma} = \eta^{\epsilon\alpha}\eta^{\sigma\beta}F_{\alpha\beta} \ ,$$

yields

$$F^{12} = -\, F_{12} \ , F^{13} = -F_{24} \ , F^{14} = -F_{23} \ ,$$

$$F^{23} = - F_{14} \ , F^{24} = -F_{13} \ , F^{34} = -F_{34} \ . \tag{2.6.9}$$

Define the tetrad components of the electromagnetic field tensor

$$\begin{aligned} \phi_0 =& F_{13} = F_{ij} l^i m^j \ , \\ \phi_1 =& \frac{1}{2}(F_{12} - F_{34}) = \frac{1}{2} F_{ij}(l^i n^j - m^i \overline{m}^j) \ , \\ \phi_2 =& - F_{24} = -F_{ij} n^i \overline{m}^j \ , \end{aligned} \tag{2.6.10}$$

where

$$F_{ij} = 2\big[- 2Re\phi_1 l_{[i} n_{j]} + 2iIm\phi_1 m_{[i}\overline{m}_{j]} + (\phi_2 l_{[i} m_{j]} + \overline{\phi}_0 m_{[i} n_{j]}) + c.c.\big] \ . \tag{2.6.11}$$

Using equations (2.6.9) and (2.6.10) in the equation (2.6.8) we obtain

$$\delta\phi_2 - \Delta\phi_1 = -(\nu^0 + \nu_1)\phi_0 + 2(\mu^0 + \mu_1)\phi_1 + (\tau^0 + \tau_1 - 2\beta^0 - 2\beta_1)\phi_2 \ .$$

Similarly, by giving $\alpha = 2, 3, 4$ in the equation (2.6.7) we obtain the equations

$$D\phi_1 - \overline{\delta}\phi_0 = (\pi^0 + \pi_1 - 2\alpha^0 - 2\alpha_1)\phi_0 + 2(\rho^0 + \rho_1)\phi_1 - (\kappa^0 + \kappa_1)\phi_2 \ ,$$
$$D\phi_2 - \overline{\delta}\phi_1 = -(\lambda^0 + \lambda_1)\phi_0 + 2(\pi^0 + \pi_1)\phi_1 + (\rho^0 + \rho_1 - 2\epsilon^0 - 2\epsilon_1)\phi_2 \ ,$$

and

$$\delta\phi_1 - \Delta\phi_0 = (\mu^0 + \mu_1 - 2\gamma^0 - 2\gamma_1)\phi_0 + 2(\tau^0 + \tau_1)\phi_1 - (\sigma^0 + \sigma_1)\phi_2 \ . \tag{2.6.12}$$

The Maxwell's equations for different fields characterized by Debney and Zund [20, 21] reduce to

(I) Type A field ($\phi_1 \neq 0$, $\phi_0 = \phi_2 = 0$)

$$
\begin{aligned}
D\phi_1 =& 2(\rho^0 + \rho_1)\phi_1 \ , \quad \Delta\phi_1 = -2(\mu^0 + \mu_1)\phi_1 \ , \\
\delta\phi_1 =& 2(\tau^0 + \tau_1)\phi_1 \ , \quad \overline{\delta}\phi_1 = -2(\pi^0 + \pi_1)\phi_1 \ .
\end{aligned} \tag{2.6.13}
$$

(II) Type B field ($\phi_2 \neq 0$, $\phi_0 = \phi_1 = 0$)

$$
\begin{aligned}
& D\phi_2 = (\rho^0 + \rho_1 - 2\epsilon^0 - 2\epsilon_1)\phi_2 \ , \\
& \delta\phi_2 = (\tau^0 + \tau_1 - 2\beta^0 - 2\beta_1)\phi_2 \ , \\
& (\sigma^0 + \sigma_1) = 0 \ , (\kappa^0 + \kappa_1) = 0 \ .
\end{aligned} \tag{2.6.14}
$$

(III) Type C field ($\phi_0 \neq 0$, $\phi_1 = \phi_2 = 0$)

$$
\begin{aligned}
& \Delta\phi_0 = -(\mu^0 + \mu_1 - 2\gamma^0 - 2\gamma_1)\phi_0 \ , \\
& \overline{\delta}\phi_0 = -(\pi^0 + \pi_1 - 2\alpha^0 - 2\alpha_1)\phi_0 \ , \\
& (\lambda^0 + \lambda_1) = 0 \ , (\nu^0 + \nu_1) = 0 \ .
\end{aligned} \tag{2.6.15}
$$

On using the equations (1.3.29) we obtain the Maxwell's equations in the form

$$
\begin{aligned}
& D\phi_1 - \overline{\delta}\phi_0 = (\pi^0 - 2\alpha^0 + \sqrt{2}k\overline{s}_2)\phi_0 + 2(\rho^0 - \sqrt{2}ks_1)\phi_1 - (\kappa^0 - \sqrt{2}ks_0)\phi_2 \ , \\
& \Delta\phi_1 - \delta\phi_2 = (\nu^0 - \sqrt{2}ks_2)\phi_0 + 2(\mu^0 - \sqrt{2}ks_1)\phi_1 - (\tau^0 - 2\beta^0 - \sqrt{2}ks_0)\phi_2 \ , \\
& \delta\phi_1 - \Delta\phi_0 = (\mu^0 - 2\gamma^0)\phi_0 + 2\tau^0\phi_1 - \sigma^0\phi_2 \ , \\
& D\phi_2 - \overline{\delta}\phi_1 = -\lambda^0\phi_0 + 2\pi^0\phi_1 + (\rho^0 - 2\epsilon^0)\phi_2 \ .
\end{aligned} \tag{2.6.16}
$$

Similarly, to find the tetrad components of the Maxwell's equation (2.6.2), we start with the equation

$$F_{\alpha\beta} = F_{ij} {e_{(\alpha)}}^i {e_{(\beta)}}^j \ , \tag{2.6.17}$$
$$\Rightarrow F_{\alpha\beta;\gamma} = (F_{ij} {e_{(\alpha)}}^i {e_{(\beta)}}^j)_{;k} {e_{(\gamma)}}^k \ .$$

By virtue of the equation (2.4.60), the above equation reduces to

$$F_{\alpha\beta;\gamma} = F_{ij;k} {e_{(\alpha)}}^i {e_{(\beta)}}^j {e_{(\gamma)}}^k + F_{\sigma\beta} {\gamma^\sigma}_{\alpha\gamma} + F_{\alpha\sigma} {\gamma^\sigma}_{\beta\gamma} \ ,$$
$$\Rightarrow F_{ij;k} {e_{(\alpha)}}^i {e_{(\beta)}}^j {e_{(\gamma)}}^k = F_{\alpha\beta;\gamma} - F_{\alpha\sigma} {\gamma^\sigma}_{\beta\gamma} - F_{\sigma\alpha} {\gamma^\sigma}_{\alpha\gamma} \ . \tag{2.6.18}$$

Similarly, we obtain

$$F_{jk;i} {e_{(\alpha)}}^i {e_{(\beta)}}^j {e_{(\gamma)}}^k = F_{\beta\gamma;\alpha} - F_{\beta\sigma} {\gamma^\sigma}_{\gamma\alpha} - F_{\sigma\gamma} {\gamma^\sigma}_{\beta\alpha} \ , \tag{2.6.19}$$

and

$$F_{ki;j} {e_{(\alpha)}}^i {e_{(\beta)}}^j {e_{(\gamma)}}^k = F_{\gamma\alpha;\beta} - F_{\gamma\sigma} {\gamma^\sigma}_{\alpha\beta} - F_{\sigma\alpha} {\gamma^\sigma}_{\gamma\beta} \ . \tag{2.6.20}$$

Adding equations (2.6.18), (2.6.19) and (2.6.20) we get

$$(F_{ij;k} + F_{jk;i} + F_{ki;j}) {e_{(\alpha)}}^i {e_{(\beta)}}^j {e_{(\gamma)}}^k = F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} -$$
$$- F_{\alpha\sigma}({\gamma^\sigma}_{\beta\gamma} - {\gamma^\sigma}_{\gamma\beta}) - F_{\sigma\beta}({\gamma^\sigma}_{\alpha\gamma} - {\gamma^\sigma}_{\gamma\alpha}) - F_{\sigma\gamma}({\gamma^\sigma}_{\beta\alpha} - {\gamma^\sigma}_{\alpha\beta}) \ .$$

Using equations (2.6.2) and (1.3.12), we get

$$\begin{aligned} F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = & F_{\alpha\sigma}\eta^{\sigma\epsilon}({\gamma^0}_{\epsilon\beta\gamma} - {\gamma^0}_{\epsilon\gamma\beta} + K_{\beta\gamma\epsilon} - K_{\gamma\beta\epsilon}) + \\ & + F_{\sigma\beta}\eta^{\sigma\epsilon}({\gamma^0}_{\epsilon\alpha\gamma} - {\gamma^0}_{\epsilon\gamma\alpha} + K_{\alpha\gamma\epsilon} - K_{\gamma\alpha\epsilon}) + \\ & + F_{\sigma\gamma}\eta^{\sigma\epsilon}({\gamma^0}_{\epsilon\beta\alpha} - {\gamma^0}_{\epsilon\alpha\beta} + K_{\beta\alpha\epsilon} - K_{\alpha\beta\epsilon}) \ . \end{aligned}$$

Expanding the double summations defined on the right hand side of the above equation, we obtain

$$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = F_{\alpha 1}({\gamma^0}_{2\beta\gamma} - {\gamma^0}_{2\gamma\beta} + K_{\beta\gamma 2} - K_{\gamma\beta 2}) +$$
$$+ F_{\alpha 2}({\gamma^0}_{1\beta\gamma} - {\gamma^0}_{1\gamma\beta} + K_{\beta\gamma 1} - K_{\gamma\beta 1}) - F_{\alpha 3}({\gamma^0}_{4\beta\gamma} - {\gamma^0}_{4\gamma\beta} + K_{\beta\gamma 4} -$$
$$- K_{\gamma\beta 4}) - F_{\alpha 4}({\gamma^0}_{3\beta\gamma} - {\gamma^0}_{3\gamma\beta} + K_{\beta\gamma 3} - K_{\gamma\beta 3}) + F_{1\beta}({\gamma^0}_{2\alpha\gamma} - {\gamma^0}_{2\gamma\alpha} +$$
$$+ K_{\alpha\gamma 2} - K_{\gamma\alpha 2}) + F_{2\beta}({\gamma^0}_{1\alpha\gamma} - {\gamma^0}_{1\gamma\alpha} + K_{\alpha\gamma 1} - K_{\gamma\alpha 1}) - F_{3\beta}({\gamma^0}_{4\alpha\gamma} -$$
$$- {\gamma^0}_{4\gamma\alpha} + K_{\alpha\gamma 4} - K_{\gamma\alpha 4}) - F_{4\beta}({\gamma^0}_{3\alpha\gamma} - {\gamma^0}_{3\gamma\alpha} + K_{\alpha\gamma 3} - K_{\gamma\alpha 3}) +$$
$$+ F_{1\gamma}({\gamma^0}_{2\beta\alpha} - {\gamma^0}_{2\alpha\beta} + K_{\beta\alpha 2} - K_{\alpha\beta 2}) + F_{2\gamma}({\gamma^0}_{1\beta\alpha} - {\gamma^0}_{1\alpha\beta} + K_{\beta\alpha 1} -$$
$$- K_{\alpha\beta 1}) - F_{3\gamma}({\gamma^0}_{4\beta\alpha} - {\gamma^0}_{4\alpha\beta} + K_{\beta\alpha 4} - K_{\alpha\beta 4}) - F_{4\gamma}({\gamma^0}_{3\beta\alpha} -$$
$$- {\gamma^0}_{3\alpha\beta} + K_{\beta\alpha 3} - K_{\alpha\beta 3}) \ . \qquad (2.6.21)$$

By giving different values to α , β , γ from 1, 2, 3, 4 and using equations (1.3.29), (2.6.9) and (2.6.10) we readily obtain the same set of Maxwell's equations derived in (2.6.16).

2.7 Conclusion

A technique of differential form on a non-Riemannian space is developed with the help of the new derivative operator d_* introduced by Katkar [61] and Cartan's equations of structure are derived in a more general form. This new technique will definitely be used to study the indispensable qualities of the non-Riemannian geometry and will also be exploited to find the solutions of the field equations of the Einstein-Cartan's theory of gravitation. The new derivative operator d_* and the Cartan's equations of structure in the non-Riemannian space will form basis for development of the non-Riemannian geometry.

Chapter 3

Einstein-Cartan Relativity in 2-Dimensional Non-Riemannian Space

3.1 Introduction

Einstein's theory of general relativity is one of the cornerstones of modern theoretical physics and has been considered as one of the most beautiful structures of theoretical physics not just in its conceptual ingenuity and mathematical elegance but also in its ability to explain real physical phenomena. It is the most successful theory of gravitation in which the gravitation as a universal force can be described by a curvature of space-time consisting of three spatial dimensions and one time that has led Einstein to formulate his famous field equations of general relativity which are non-linear second order partial differential equations. General relativity has been considered as one of the most difficult subject due to a great deal of complex mathematics. The complexity of the mathematics reflects the complexity of describing space-time curvature and some conceptual issues which are present and even more opaque in the physical 4- dimensions world. Hence in order to gain insight in to these difficult conceptual issues Deser et al. [23, 24, 25] in a series of papers, Giddings et al. [36], and Gott et al. [39, 40] have examined general relativity in lower dimensional spaces and explored some solutions. Studies of general relativity in lower dimensional space-times have proved that solving Einstein's field equations of general relativity in a space-time of reduced dimensionality is rather simple but yields some amusing results that are pedagogical and scientific interests and yet are apparently unfamiliar to most physicists.

A.D. Boozer [9] and R. D. Mellinger Jr. [84] have examined the general relativity in (1+1) dimensions. Einstein-Cartan theory of gravitation is one of the extensions of the general theory of relativity developed by Cartan [12] in a non-Riemannian space-time. It is only in the last couple of decades, the Einstein-Cartan theory has caught the imagination of researchers for constructing models with spin for the primary purpose of overcoming singularities. In this chapter we intend to study the Einstein-Cartan theory of relativity in a 2-dimensional non-Riemannian space.

The material of the chapter is organized as follows. In the Section 2, an exposition of a new dyad formalism, consisting of two real null vector fields is given and we have employed this dyad formalism and constructed a 2-dimensional non-Riemannian space and shown that the 2-dimensional non-Riemannian space contains no matter at all, so that there is no gravitational field either but torsion influences the curvature of the 2- dimensional non-Riemannian space.

In the Section 3, the results obtained in the Section 2 are corroborated by employing the techniques of differential form developed by Katkar in [61]. Some conclusions are drawn in the last section.

3.2 Dyad Formalism:

Consider a 2-dimensional space characterized by an indefinite metric

$$ds^2 = f^2(x,t)dx^2 - h^2(x,t)dt^2 \ , \tag{3.2.1}$$

where

$$\begin{aligned} g_{11} = f^2, \ \ g_{22} = -h^2, \ \ g = -f^2h^2 \ , \\ g^{11} = f^{-2}, \ \ g^{22} = -h^{-2} \ . \end{aligned} \tag{3.2.2}$$

We define a basis 1-form as

$$\theta^1 = \frac{1}{\sqrt{2}}[f(x,t)dx + h(x,t)dt] \ , \theta^2 = \frac{1}{\sqrt{2}}[f(x,t)dx - h(x,t)dt] \ . \tag{3.2.3}$$

In terms of the basis 1-forms the metric (3.2.1) becomes

$$ds^2 = 2\theta^1\theta^2 \ . \tag{3.2.4}$$

In order to construct a 2- dimensional non-Riemann space, we introduce, in the following two null vector formalism. This formalism facilitates to introduce torsion in to the space and the space becomes non-Riemannian.

Consider a curve in a space. At each point of the curve, we define a dyad of basis vectors as

$$e_{(\alpha)i} = (l_i, n_i) \ . \tag{3.2.5}$$

Where l_i and n_i are real null vector fields satisfying the ortho-normality conditions

$$\begin{aligned} l_i l^i = n_i n^i = 0 \ , \\ l_i n^i = 1 \ . \end{aligned} \tag{3.2.6}$$

Here the Latin indices are used to denote the tensor indices while the Greek indices are used to denote the dyad indices. Any vector (or tensor) can always be expressed in terms of the dyad components of the vector (tensor) and vice versa. Thus we express

$$\begin{aligned} A_\alpha = A_i e_{(\alpha)}{}^i \ , A_{\alpha\beta} = A_{ij} e_{(\alpha)}{}^i e_{(\beta)}{}^j \ , \\ A_i = A_\alpha e^{(\alpha)}{}_i \ , A_{ij} = A_{\alpha\beta} e^{(\alpha)}{}_i e^{(\beta)}{}_j \ , \end{aligned} \tag{3.2.7}$$

where $e^{(\alpha)}{}_i$ is the dyad of the dual basis vectors satisfying the conditions

$$e^{(\alpha)}{}_i e_{(\alpha)}{}^k = \delta^k{}_i \ , \quad \text{and} \quad e^{(\alpha)}{}_i e_{(\beta)}{}^i = \delta^\alpha{}_\beta \ . \tag{3.2.8}$$

This gives

$$e^{(\alpha)}{}_i = (n_i, l_i) \ . \tag{3.2.9}$$

Consequently, we express the dyad components of the metric tensor g_{ij} as

$$\eta_{\alpha\beta} = g_{ij} e_{(\alpha)}{}^i e_{(\beta)}{}^j \ . \tag{3.2.10}$$

This gives

$$\eta_{\alpha\beta} = \eta^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{3.2.11}$$

Hence the metric tensor in terms of the basis vectors is defined as

$$g_{ij} = l_i n_j + n_i l_j \ . \tag{3.2.12}$$

The tetrad indices can be raised and lowered by the dyad components of the metric tensor $\eta_{\alpha\beta}$, while the tensor indices are raised and lowered by the metric tensor g_{ij}. The equation

$$\theta^{\alpha} = e_i{}^{(\alpha)} dx^i \ ,$$

yields

$$\begin{aligned} l_i &= \frac{1}{\sqrt{2}}(f, -h) \ , n_i = \frac{1}{\sqrt{2}}(f, h) \ , \\ l^i &= \frac{1}{\sqrt{2}}(f^{-1}, h^{-1}) \ , n^i = \frac{1}{\sqrt{2}}(f^{-1}, -h^{-1}) \ . \end{aligned} \tag{3.2.13}$$

The spin tensor is anti-symmetric; hence it has just one independent component in the 2-dimension space. We express the spin tensor as a linear combination of the basis vectors of the dyad as

$$\begin{aligned} S_{ij} &= S_{\alpha\beta} e^{(\alpha)}{}_i e^{(\beta)}{}_j \ . \\ S_{ij} &= (-S_{12})_d (l_i n_j - n_i l_j) \ , \\ S_{ij} &= S_d (l_i n_j - n_i l_j) \ , \end{aligned} \tag{3.2.14}$$

where $S_d = (-S_{12})_d$ is the dyad component of spin tensor. In general it is a function of coordinates. The tensor component of the spin tensor is obtain from equations (3.2.13) and(3.2.14) as

$$S_d = \frac{1}{fh} S_t \ . \tag{3.2.15}$$

Similarly, we express the spin angular momentum tensor in terms of the basis vectors of the dyad as

$$S_{ij}{}^{k} = -\left[(S_{12}{}^{1})_{d}l^{k} + (S_{12}{}^{2})_{d}n^{k}\right](l_{i}n_{j} - n_{i}l_{j}) \ . \qquad (3.2.16)$$

For the choice of the time like vector field $u^i = \frac{1}{\sqrt{2}}(l^i + n^i)$ such that $u^i u_i = 1$, we have from the equations (3.2.14)

$$S_{ij}u^{k} = \left[(\frac{S_d}{\sqrt{2}})l^{k} + (\frac{S_d}{\sqrt{2}})n^{k}\right](l_{i}n_{j} - n_{i}l_{j}) \ . \qquad (3.2.17)$$

It follows from the equations (1.2.52), (3.2.16) and (3.2.17) that

$$(S_{12}{}^{1})_{d} = (S_{12}{}^{2})_{d} = -(\frac{S_d}{\sqrt{2}}) \ . \qquad (3.2.18)$$

Hence we have from equations (3.2.16),(3.2.17) and(3.2.18)

$$S_{ij}{}^{k} = S_{ij}u^{k} = \frac{1}{\sqrt{2}}S_{d}(l_{i}n_{j} - n_{i}l_{j})(l^{k} + n^{k}) \ . \qquad (3.2.19)$$

We express the torsion tensor $Q_{ij}{}^{k}$ in terms of its dyed components as

$$Q_{ij}{}^{k} = Q_{\alpha\beta}{}^{\gamma}e_{i}{}^{(\alpha)}e_{i}{}^{(\beta)}e^{k}{}_{(\gamma)} \ . \qquad (3.2.20)$$

This yields

$$Q_{ij}{}^{k} = \left[-(Q_{12}{}^{1})_{d}\ l^{k} - (Q_{12}{}^{2})_{d}\ n^{k}\right](l_{i}n_{j} - n_{i}l_{j}) \ . \qquad (3.2.21)$$

We approximate the values of the dyad components of torsion tensor to the dyad components of the spin tensor as

$$(Q_{12}{}^{1})_{d} = (Q_{122})_{d} = -\frac{kS_d}{2\sqrt{2}}\ , (Q_{12}{}^{2})_{d} = (Q_{121})_{d} = -\frac{kS_d}{2\sqrt{2}} \ . \qquad (3.2.22)$$

Consequently, the equation (3.2.21) becomes

$$Q_{ij}{}^{k} = \frac{kS_d}{2\sqrt{2}}(l_i n_j - n_i l_j)(l^k + n^k) \ . \tag{3.2.23}$$

Similarly, the tensor components of Contortion tensor are obtain from the equations (3.2.13) and (3.2.21) as

$$(Q_{12}{}^{1})_t = (Q_{122})_t = \frac{kS_t}{2f} \ , (Q_{12}{}^{2})_t = (Q_{121})_t = 0 \ . \tag{3.2.24}$$

We now express the contortion tensor $K_{ij}{}^{k}$ as the linear combinations of the basis vectors of the dyad as

$$K_{ijk} = -(K_{112})_d \ n_i(l_j n_k - n_j l_k) - (K_{212})_d \ l_i(l_j n_k - n_j l_k) \ . \tag{3.2.25}$$

From the relation

$$K_{\alpha\beta\gamma} = -Q_{\alpha\beta\gamma} + Q_{\beta\gamma\alpha} - Q_{\gamma\alpha\beta} \ , \tag{3.2.26}$$

we obtain

$$(K_{212})_d = (K_{112})_d = \frac{kS_d}{\sqrt{2}} \ . \tag{3.2.27}$$

Hence the equation (3.2.25) becomes

$$K_{ijk} = \frac{kS_d}{\sqrt{2}}(l_j n_k - n_j l_k)(l_i + n_i) \ . \tag{3.2.28}$$

The equations (3.2.13) and (3.2.28) yield the tensor components of the Contortion tensor and are given by

$$(K_{11}{}^{2})_t = (K_{12}{}^{1})_t = -k\frac{f}{h^2}S_t \ . \tag{3.2.29}$$

For the given metric, the non vanishing components of the symmetric Christoffel symbols are are given by

$$\begin{aligned} \left\{ {1 \atop 11} \right\} &= \frac{f_{,1}}{f} \ , \left\{ {2 \atop 11} \right\} = \frac{f}{h^2} f_{,2} \ , \left\{ {1 \atop 12} \right\} = \frac{f_{,2}}{f} \ , \\ \left\{ {1 \atop 22} \right\} &= \frac{h}{f^2} h_{,1} \ , \left\{ {2 \atop 12} \right\} = \frac{h_{,1}}{h} \ , \left\{ {2 \atop 22} \right\} = \frac{h_{,2}}{h} \ . \end{aligned} \tag{3.2.30}$$

Thus the tensor components of the asymmetric connections becomes

$$\begin{aligned} &(\Gamma_{11}{}^1)_t = \frac{f_{,1}}{f} \ , (\Gamma_{22}{}^1)_t = \frac{h}{f^2} h_{,1} \ , (\Gamma_{12}{}^2)_t = (\Gamma_{21}{}^2)_t = \frac{h_{,1}}{h} \ , \\ &(\Gamma_{12}{}^1)_t = \frac{f_{,2}}{f} - \frac{k}{f} S_t \ , (\Gamma_{11}{}^2)_t = \frac{f}{h^2} f_{,2} + k \frac{f}{h^2} S_t \ , \\ &(\Gamma_{21}{}^1)_t = \frac{f_{,2}}{f} \ , (\Gamma_{22}{}^2)_t = \frac{h_{,2}}{h} \ . \end{aligned} \tag{3.2.31}$$

Due to equation (3.2.31), the expression for the Riemann curvature tensor becomes

$$(R_{121}{}^2)_t = -\frac{h_{,11}}{h} + \frac{f}{h^2} f_{,22} + (\frac{h_{,1}}{h})(\frac{f_{,1}}{f}) - \frac{f}{h^3} f_{,2} h_{,2} + \frac{kf}{h^3}(hS_{t,2} - S_t h_{,2}) \ .$$

From this equation, we obtain the covariant components of the Riemann curvature tensor of a non-Riemannian space as

$$(R_{1212})_t = hh_{,11} - ff_{,22} - \frac{h}{f} f_{,1} h_{,1} + \frac{f}{h} f_{,2} h_{,2} - \frac{kf}{h}(hS_{t,2} - S_t h_{,2}) \ . \tag{3.2.32}$$

This equation can also be written as

$$(R_{1212})_t = (\hat{R}_{1212})_t - \frac{k}{t^2}(S_t + tS_{t,t}) \ , \tag{3.2.33}$$

where

$$(\hat{R}_{1212})_t = hh_{,11} - ff_{,22} - \frac{h}{f}f_{,1}h_{,1} + \frac{f}{h}f_{,2}h_{,2} \ .$$

The tensor components of the Ricci tensor and the Ricci scalar are given by

$$(R_{11})_t = h^{-2}(R_{1212})_t \ , (R_{12})_t = 0 \ , (R_{22})_t = -f^{-2}(R_{1212})_t \ , \qquad (3.2.34)$$

$$(R_{11})_t = \frac{h_{,11}}{h} - \frac{f}{h^2}f_{,22} - \frac{1}{fh}f_{,1}h_{,1} + \frac{f}{h^3}f_{,2}h_{,2} - \frac{kf}{h^3}(hS_{t,2} - S_t h_{,2}) \ ,$$

$$(R_{22})_t = -\frac{h}{f^2}h_{,11} + \frac{1}{f}f_{,22} + \frac{h}{f^3}f_{,1}h_{,1} - \frac{1}{fh}f_{,2}h_{,2} + \frac{k}{fh}(hS_{t,2} - S_t h_{,2}) \ , \qquad (3.2.35)$$

and

$$R = 2[\frac{1}{hf^2}h_{,11} - \frac{1}{fh^2}f_{,22} - \frac{1}{hf^3}f_{,1}h_{,1} + \frac{1}{fh^3}f_{,2}h_{,2} - \frac{k}{fh^3}(hS_{t,2} - S_t h_{,2})] \ . \qquad (3.2.36)$$

We see from equations (3.2.35), (3.2.36) that

$$R_{ij} = \frac{R}{2}g_{ij} \ . \qquad (3.2.37)$$

This is true for any 2-space. This shows that the Ricci tensor and the Ricci scalar terms cancel in the field equation of the Einstein-Cartan theory of gravitation. In other words, in 2- dimensions space, the Einstein tensor vanishes identically and from Einstein-Cartan field equations, we get $t_{ij} = 0$.

Curvature of a non-Riemannian Space

Katkar [62] has obtained the formula for the Riemann curvature of a non-Riemannian space in the form

$$\kappa = \kappa_1 + \frac{1}{b}\left[\frac{\partial}{\partial u^1}(K_{212})_t - \frac{\partial}{\partial u^2}(K_{112})_t\right] \ , \tag{3.2.38}$$

where $b = (g_{hj}g_{ik} - g_{ij}g_{hk})p^h q^i p^j q^k$ is the determinant of the metric tensor of the 2-dimensional surface determined by the orientations of the two unit vectors p^i and q^i, and

$$\kappa_1 = \frac{\hat{R}_{hijk}p^h q^i p^j q^k}{(g_{hi}g_{ik} - g_{ij}g_{hk})p^h q^i p^j q^k} \ , \tag{3.2.39}$$

is the Riemann Curvature of Riemannian space, at a point, for the orientations determined by the two unit vectors p^i and q^i . The formula (3.2.39) gives the curvature of a Riemannian space as

$$\kappa_1 = -\frac{1}{hf^2}h_{,11} + \frac{1}{fh^2}f_{,22} + \frac{1}{hf^3}f_{,1}h_{,1} - \frac{1}{fh^3}f_{,2}h_{,2} \ . \tag{3.2.40}$$

Consequently, the curvature of a non-Riemannian space becomes

$$\kappa = -\frac{1}{hf^2}h_{,11} + \frac{1}{fh^2}f_{,22} + \frac{1}{hf^3}f_{,1}h_{,1} - \frac{1}{fh^3}f_{,2}h_{,2} + \frac{k}{f^2h^2}(fS_{t,2} + S_t f_{,2}) \ . \tag{3.2.41}$$

We see that curvature of the non-Riemannian 2-space is influenced by the torsion. In the absence of torsion, we see from the equations (3.2.40) and (3.2.41) that $\kappa = \kappa_1$. We also observe from the equations

(3.2.36) and (3.2.41) that

$$\kappa \neq -\frac{R}{2} \, . \tag{3.2.42}$$

If the components of the spin tensor are zero, then the results (3.2.36) and (3.2.42) reduce to the results of Riemann space.

From the tetrad components of the Riemann curvature tensor $R_{\alpha\beta\gamma\delta} = R_{hijk} e^h_{(\alpha)} e^i_{(\beta)} e^j_{(\gamma)} e^k_{(\delta)}$, we obtain

$$(R_{1212})_d = \frac{1}{f^2 h^2}(R_{1212})_t. \tag{3.2.43}$$

Consequently, from the equation $R_{hijk} = R_{\alpha\beta\gamma\delta} e_h{}^{(\alpha)} e_i{}^{(\beta)} e_j{}^{(\gamma)} e_k{}^{(\delta)}$, we obtain the expression for the Riemannian curvature tensor of a non-Riemannian space as

$$\begin{aligned} R_{hijk} = \big[-\frac{1}{hf^2} h_{,11} + \frac{1}{fh^2} f_{,22} + \frac{1}{hf^3} f_{,1} h_{,1} - \frac{1}{fh^3} f_{,2} h_{,2} + \\ + \frac{k}{fh^3}(hS_{t,2} - S_t h_{,2}) \big] (g_{hj} g_{ik} - g_{ij} g_{hk}) \, . \end{aligned} \tag{3.2.44}$$

This equation, due to the equation (3.2.36), becomes

$$R_{hijk} = -\frac{R}{2}(g_{hj} g_{ik} - g_{ij} g_{hk}) \, . \tag{3.2.45}$$

If the Riemann curvature tensor R_{hijk} of any non-Riemann space $V_n, n > 2$, satisfies the Bianchi identities (1.2.27), then we obtain

$$R = c exp\left[\left(\frac{4}{n-1} \right) \int Q_{hi}{}^h \; dx^i \right] \, , \tag{3.2.46}$$

where c is a constant of integration. If ${Q_{hi}}^h = 0 \Rightarrow c = \hat{R}$. Hence

$$R = \hat{R} exp \left[\left(\frac{4}{n-1} \right) \int {Q_{hi}}^h \; dx^i \right] . \tag{3.2.47}$$

Where as in the case of Riemannian space, we have

$$\hat{R}_{hijk} = \kappa_1 (g_{hj} g_{ik} - g_{ij} g_{hk}) . \tag{3.2.48}$$

This gives $\hat{R} = n(n-1)\kappa_1$, where κ_1 is the constant Riemann curvature of a Riemannian space. Hence we have finally,

$$R = n(1-n)\kappa_1 exp \left[\left(\frac{4}{n-1} \right) \int {Q_{hi}}^h \; dx^i \right] . \tag{3.2.49}$$

Contracting the index h with k in the equation (3.2.44) we get

$$R_{ij} = [\frac{1}{hf^2} h_{,11} - \frac{1}{fh^2} f_{,22} - \frac{1}{hf^3} f_{,1} h_{,1} + \frac{1}{fh^3} f_{,2} h_{,2} - \frac{k}{fh^3}(hS_{t,2} - S_t h_{,2})] g_{ij} . \tag{3.2.50}$$

This is nothing but $R_{ij} = \frac{R}{2} g_{ij}$. This shows that the Einstein tensor $G_{ij} = R_{ij} - \frac{R}{2} g_{ij}$ vanishes identically.

3.3 Techniques of Differential Forms

The Katkar [61] has introduced a new operator d_* on a non-Riemannian space and applied to a form of any degree. It converts $p-$ form to $p+1-$ form and is obtained by taking the covariant derivative of an associated with p^{th} ordered skew symmetric tensor with respect to the asymmetric connections. We note here that unlike the exterior derivative operator in a Riemannian space, the repetition of the new derivative operator d_* on any form ϕ of any degree is not zero. i. e.,

$$d_*{}^2\phi \neq 0.$$

However, the operator d_* satisfies all other properties of the exterior derivative. For the scalar function ϕ, the operator d_* gives

$$d_*\phi = \phi_{;i} d_* x^i \ . \tag{3.3.1}$$

Where for the scalar function ϕ, we have

$$\phi_{;i} = \phi_{/i} = \phi_{,i} \ . \tag{3.3.2}$$

Hence we have $d_*\phi = d\phi$ and $d_*x^i = dx^i$, where d is the usual exterior derivative defined in a Riemannian space in which the connections are the symmetric Christoffel symbols. However,the action of the repeated operator d_* on the scalar function ϕ gives

$$d_*(d_*\phi) = -\phi_{;l} Q_{ij}{}^l d_* x^i \wedge d_* x^j - \phi_{;k} d_*{}^2 x^k \ .$$

For the coordinate functions $\phi = x^i$, this equation becomes

$$d_*{}^2 x^k = -\frac{1}{2} Q_{ij}{}^k d_* x^i \wedge d_* x^j \ . \tag{3.3.3}$$

Consequently, the above equation yields

$$d_*(d_*\phi) = -\frac{1}{2}\phi_{;k} Q_{ij}{}^k d_* x^i \wedge d_* x^j \ . \tag{3.3.4}$$

The dyad equivalent of this equation is given by

$$d_*(d_*\phi) = -\frac{1}{2}\phi_{;\gamma}Q_{\alpha\beta}{}^{\gamma}\theta^{\alpha}\wedge\theta^{\beta} \ , \tag{3.3.5}$$

where

$$\phi_{;\gamma} = \phi_{;i}e^{i}{}_{(\gamma)} \ ,$$

i. e. $(\phi_{;1})_d = \frac{1}{\sqrt{2}}(f^{-1}\frac{\partial\phi}{\partial x} + h^{-1}\frac{\partial\phi}{\partial t}) \ , \quad (\phi_{;2})_d = \frac{1}{\sqrt{2}}(f^{-1}\frac{\partial\phi}{\partial x} - h^{-1}\frac{\partial\phi}{\partial t}) \ .$

Consequently, we obtain

$$\begin{aligned} d_*{}^2\phi = &-\frac{1}{2\sqrt{2}}\big[(f^{-1}\frac{\partial\phi}{\partial x} + h^{-1}\frac{\partial\phi}{\partial t})Q_{\alpha\beta}{}^{1} + \\ &+ (f^{-1}\frac{\partial\phi}{\partial x} - h^{-1}\frac{\partial\phi}{\partial t})Q_{\alpha\beta}{}^{2}\big]\theta^{\alpha}\wedge\theta^{\beta} \ . \end{aligned} \tag{3.3.6}$$

From this equation, we readily find

$$d_*{}^2x = \frac{k}{2f^2h}S_t \ \theta^1\wedge\theta^2, \quad \text{and} \quad d_*{}^2t = 0 \ . \tag{3.3.7}$$

Now operating the new exterior derivative operator d_* to the basis 1-from defined in the equation (3.2.3), we obtain

$$\begin{aligned} d_*\theta^1 &= \frac{1}{2\sqrt{2}fh}\big[2(f_{,2} - h_{,1}) + kS_t\big]\theta^1\wedge\theta^2 \ , \\ d_*\theta^2 &= \frac{1}{2\sqrt{2}fh}\big[2(f_{,2} + h_{,1}) + kS_t\big]\theta^1\wedge\theta^2 \ . \end{aligned} \tag{3.3.8}$$

From the Cartan's first equation of structure of the non-Riemannian space, we have

$$d_*\theta^{(\alpha)} = -\omega^{\alpha}{}_{\beta} \ \wedge\theta^{\beta} + \frac{1}{2}Q_{\sigma\beta}{}^{\alpha} \ \theta^{\sigma}\wedge\theta^{\beta} \ , \tag{3.3.9}$$

where

$$\omega^{\alpha}{}_{\beta} = \gamma^{\alpha}{}_{\beta\gamma}\theta^{\gamma} \ , \tag{3.3.10}$$

and

$$\gamma^{\alpha}{}_{\beta\gamma} = \gamma^{0\alpha}{}_{\beta\gamma} - K_{\gamma\beta}{}^{\alpha} \ , \tag{3.3.11}$$

where $\gamma^{\alpha}{}_{\beta\gamma}$ are Ricci's coefficients of rotation and are defined by

$$\gamma^{\alpha}{}_{\beta\gamma} = -e^{(\alpha)}{}_{i;j}e_{(\beta)}{}^{i}e_{(\gamma)}{}^{j} \ ,$$
$$\gamma^{\alpha}{}_{\beta\gamma} = -e^{(\alpha)}{}_{i/j}e_{(\beta)}{}^{i}e_{(\gamma)}{}^{j} - e^{(\alpha)}{}_{k}K_{ji}{}^{k}e_{(\beta)}{}^{i}e_{(\gamma)}{}^{j} \ ,$$

where

$$\gamma^{0\alpha}{}_{\beta\gamma} = -e^{(\alpha)}{}_{i/j}e_{(\beta)}{}^{i}e_{(\gamma)}{}^{j} \ , \tag{3.3.12}$$

are the Ricci's rotation coefficients in the Riemannian space. From the equation (3.3.12) we find

$$\gamma^{01}{}_{11} = -e^{(1)}{}_{i/j}e_{(1)}{}^{i}e_{(1)}{}^{j} \ ,$$
$$\gamma^{01}{}_{11} = l_{i/j}n^{i}\ l^{j} \quad \text{and} \quad \gamma^{01}{}_{12} = l_{i/j}n^{i}n^{j} \ .$$

We define

$$l_{i/j}n^{i}\ l^{j} = \kappa^{0} \ , \ \ l_{i/j}n^{i}n^{j} = \nu^{0} \ , \tag{3.3.13}$$

where κ^0 and ν^0 are the spin components . The components of the Ricci's coefficients of rotation are given by

$$\gamma^{1}{}_{11} = \gamma^{01}{}_{11} - (K_{11}{}^{1})_{d} \ , \ \ \gamma^{1}{}_{12} = \gamma^{01}{}_{12} - (K_{21}{}^{1})_{d} \ ,$$

$$\Rightarrow \gamma^{1}{}_{11} = -\gamma^{2}{}_{21} = \left(\kappa^{0} + \frac{k}{\sqrt{2fh}} S_t\right) \ , \ \gamma^{1}{}_{12} = -\gamma^{2}{}_{22} = \left(\nu^{0} + \frac{k}{\sqrt{2fh}} S_t\right) \ . \tag{3.3.14}$$

Using the equations (3.3.14), we obtain the expression of the covariant derivative of a basis vector of the dyad as

$$l_{i;j} = \left(\nu^{0} + \frac{kS_t}{\sqrt{2fh}}\right) l_i l_j + \left(\kappa^{0} + \frac{kS_t}{\sqrt{2fh}}\right) l_i n_j \ . \tag{3.3.15}$$

The equations (3.3.10) and (3.3.14) yield the components of connection 1-form as

$$\omega^{1}{}_{1} = -\omega^{2}{}_{2} = \left(\kappa^{0} + \frac{k}{\sqrt{2fh}} S_t\right) \theta^1 + \left(\nu^{0} + \frac{kt^2}{\sqrt{2fh}} S_t\right) \theta^2 \ . \tag{3.3.16}$$

Also from the Caratn's first equation of the structure (3.2.9), we obtain

$$d_*\theta^1 = \left(\nu^{0} + \frac{k}{2\sqrt{2fh}} S_t\right) \theta^1 \wedge \theta^2 \ \ , d_*\theta^2 = \left(\kappa^{0} + \frac{k}{2\sqrt{2fh}} S_t\right) \theta^1 \wedge \theta^2 \ . \tag{3.3.17}$$

Comparing the equations (3.3.8) and (3.3.17), we readily get

$$\kappa^0 = \frac{1}{\sqrt{2fh}}(f_{,2} + h_{,1}) \ , \ \nu^0 = \frac{1}{\sqrt{2fh}}(f_{,2} - h_{,1}) \ . \tag{3.3.18}$$

Hence the equation (3.3.16) becomes

$$\omega^{1}{}_{1} = -\omega^{2}{}_{2} = \frac{1}{\sqrt{2fh}} \left[(h_{,1} + f_{,2} + kS_t)\theta^1 + (f_{,2} - h_{,1} + kS_t)\theta^2\right] \ . \tag{3.3.19}$$

The Cartan's second equation of structure in the non-Riemannian space, when the spin tensor is not u-orthogonal is given by Katkar

[61]

$$\Omega^{\alpha}{}_{\beta} = d_* \omega^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\sigma} \wedge \omega^{\sigma}{}_{\beta} + \\ + \frac{K}{4}\left[2\gamma^{\alpha}{}_{\beta\sigma} S_{\delta\gamma} u^{\sigma} + \gamma^{\alpha}{}_{\beta\delta} S_{\gamma\sigma} u^{\sigma} - \gamma^{\alpha}{}_{\beta\gamma} S_{\delta\sigma} u^{\sigma}\right] \theta^{\delta} \wedge \theta^{\gamma} \ . \tag{3.3.20}$$

From this we obtain

$$\Omega^{1}{}_{1} = -\Omega^{2}{}_{2} = d_* \omega^{1}{}_{1} - \frac{k S_t}{2\sqrt{2} f h}\left[(\kappa^0 + \nu^0) + \frac{2k}{\sqrt{2} f h} S_t\right] \theta^1 \wedge \theta^2 \ .$$

On using equation (3.3.18) we get

$$\Omega^{1}{}_{1} = -\Omega^{2}{}_{2} = d_* \omega^{1}{}_{1} - \frac{k}{2 f^2 h^2} S_t \left[f_{,2} + k S_t\right] \theta^1 \wedge \theta^2 \ . \tag{3.3.21}$$

Operating the new exterior derivative operator d_* to the equation (3.3.19) we find

$$d_* \omega^{1}{}_{1} = \left[\frac{1}{h f^3} f_{,1} h_{,1} - \frac{1}{h f^2} h_{,11} + \frac{1}{f h^2} f_{,22} - \frac{1}{f h^3} f_{,2} h_{,2} - \frac{k}{f h^3} S_t h_{,2} + \\ + \frac{k}{f h^2} S_{t,2} + \frac{k}{2 f^2 h^2} S_t (f_{,2} + k S_t)\right] \theta^1 \wedge \theta^2 \ . \tag{3.3.22}$$

Consequently, the equation (3.3.21) becomes

$$\Omega^{1}{}_{1} = -\Omega^{2}{}_{2} = \left[\frac{1}{h f^3} f_{,1} h_{,1} - \frac{1}{h f^2} h_{,11} + \frac{1}{f h^2} f_{,22} - \frac{1}{f h^3} h_{,11} + \\ + \frac{k}{f h^3} (h S_{t,2} - S_t h_{,2})\right] \theta^1 \wedge \theta^2 \ . \tag{3.3.23}$$

The components of the curvature 2- form are defined by

$$\Omega^{1}{}_{1} = -\frac{1}{2} R_{\alpha\beta 1}{}^{1} \ \theta^{\alpha} \wedge \theta^{\beta} \ , \tag{3.3.24}$$

$$\Omega^1{}_1 = -(R_{121}{}^1)_d \, \theta^1 \wedge \theta^2 \ . \qquad (3.3.25)$$

Comparing the corresponding coefficients of the equations (3.3.23) and (3.3.25), we obtain the dyad component of the Riemannian curvature tensor as

$$\begin{aligned}(R_{1212})_d = (R_{121}{}^1)_d = -\Bigg[&\frac{1}{hf^3}f_{,1}h_{,1} - \frac{1}{hf^2}h_{,11} + \frac{1}{fh^2}f_{,22} - \frac{1}{fh^3}f_{,2}h_{,2} + \\ &+ \frac{k}{fh^3}(hS_{t,2} - S_t h_{,2})\Bigg] \ . \end{aligned} \qquad (3.3.26)$$

Hence, the Riemann Curvature tensor of the Non-Riemannian 2-space becomes

$$\begin{aligned}R_{hijk} = \Bigg[&\frac{1}{hf^3}f_{,1}h_{,1} - \frac{1}{hf^2}h_{,11} + \frac{1}{fh^2}f_{,22} - \frac{1}{fh^3}f_{,2}h_{,2} + \\ &+ \frac{k}{fh^3}(hS_{t,2} - S_t h_{,2})\Bigg](g_{hj}g_{ik} - g_{ij}g_{hk}) \ . \end{aligned} \qquad (3.3.27)$$

This on using the equation (3.2.36) we get

$$R_{hijk} = -\frac{R}{2}(g_{hj}g_{ik} - g_{ij}g_{hk}) \ . \qquad (3.3.28)$$

The result is equivalent to (3.2.45). We see from the equation (3.2.35) that

$$\frac{R_{11}}{f^2} = -\frac{R_{22}}{h^2} \ . \qquad (3.3.29)$$

In the 2- dimensional space, Ricci tensor and the Curvature tensor has only one independent component. We express the Riemann curvature tensor R_{hijk} in terms of the Ricci tensor R_{ij} alone as

$$R_{hijk} = -\left[g_{hj}R_{ik} - g_{hk}R_{ij} - g_{ij}R_{hk} + g_{ik}R_{hj}\right] + \\ + \frac{(R_{lm}g^{lm})}{2}(g_{hj}g_{ik} - g_{ij}g_{hk}) \ . \tag{3.3.30}$$

3.4 Conclusion

Introduction of dyad formalism facilitates the complexity of computation. A 2- dimensional non-Riemannian space is constructed with the help of the dyad formalism. It is shown that the Einstein tensor of 2-dimensional non-Riemannian vanishes, hence the corresponding space contains no matter at all, so that there is no gravitational field either but the curvature of the space is influenced by the torsion. The results are corroborated by the method of differential forms.

Chapter 4

Non-Static Spherically Symmetric Space-Times in Einstein-Cartan Theory

4.1 Introduction

Relativistic cosmology is the study of large scale structure of the physical world. Relativists construct mathematical models as the solutions of the Einstein's field equations that represents the universe as a whole concentrating on its large scale features. Einstein's general theory of relativity is one cornerstones of modern theoretical physics and has been enormously successful not just in describing all kinds of motion and in describing gravitation as a manifestation of curvature of the space-time but it has been served as a basis for different mathematical models of the universe. It is still challenging problem to understand the exact physical situation of the physical world at early stages evolution of the universe.

In recent years there has been immense interest in constructing the mathematical models in Einstein's general theory of relativity and also in the several alternative theories of gravitations which are of vital importance for the better understanding of the large scale structure of the universe.

It is only in the last few decades, the Einstein-Cartan theory has caught the imagination of research workers for constructing models with spin for the primary purpose of overcoming singularities. Kopczynski [72] has constructed a two parameters family of spherically symmetric models without singularities in the frame work of Einstein-Cartan theory of gravitation.

Spherically symmetric or cylindrically symmetric perfect fluid models with spin have been obtained by Prasanna [100, 101]. Non-singular Bianchi type-I cosmological models incorporating spin in which a magnetic field is present have been obtained by Raychaudhari [105]. Some spatially homogeneous Bianchi type VI, VII dust distributions with spin have been discovered by Tsoubelis [131]. Many of the previously known solutions reviewed by Kuchowicz [82] for Weyssenhoff fluids in the Einstein-Cartan theory of gravitation have zero acceleration and vorticity. Non-zero accelerating solutions in the framework of Einstein-Cartan theory have been claimed by Griffiths and Jogia [42]. The authors Tolman [130], Florides [31, 32, 33], Eflinger [27], Kyle and Martin [83], Whittaker [136], Shah [112], Vaidya [132], Wilson [137], Trautman [125], Bonnor and Wickramasuriya [7], Bailyn and Eimerl [4], Trautman [126, 127, 128, 129], Omote [91], Isham et al. [53], Tafel [121], Kopczynski [73], Adler [1], Krori and Barua [76], Nduka [87], Sing and Yadav [119], Chakravarti and De [14], Mehra [85] , Pandey et al.[93], Koppar et al. [74], Singh, P. and Griffiths, J. B.[114], T Singh et al. [116], Singh, T. and Prasad, U [117] C.J. G. Junevicus [56], Kuchowicz [78, 79, 80, 81], W Arkuszewski et al. [3], Kuchowicz [82], N, Duka [26], Raychaudhuri and Banerji [107], Singh and Yadav [118], Krori et al. [75], Pandey and Prasad [92], M. L. Bedran and M. M. Som [5], Nurgaliev and Ponomarev [88], Kalyanshetti and Waghmode [66], Yadav and Prasad [138], Kar and S SenGupta [57], Sharif and Iqbal [113], Katkar [59, 61], Katkar and Patil [60], Katkar and Phadatare [64] are some of the research workers who have investigated several aspects of the solutions of the Einstein-Cartan field equations.

Motivated by the above investigations, in this chapter, two different classes of solutions of the field equations are obtained, when Weyssenhoff fluid is the source of gravitation. The material of the chapter is organized as follows. In the Section 2, the non-static spherically symmetric metric is considered and the tetrad components of connection 1-form, curvature 2-form are derived. Consequently, the tetrad components of the Riemann curvature tensor and Ricci tensor are derived. In the Section 3 specific solutions are obtained. Finally some conclusions are drawn in the last section.

4.2 Non-Static Spherically Symmetric Metric

Consider the Non-static spherically symmetric metric in the form

$$ds^2 = e^{2\nu}dt^2 - e^{2\lambda}dr^2 - B^2(d\theta^2 + sin^2\theta d\phi^2) \ , \tag{4.2.1}$$

where λ , ν and B are functions of r and t only. Define the tetrad basis θ^α for the metric (4.2.1) as

$$\theta^1 = \frac{1}{\sqrt{2}}(e^\nu dt + e^\lambda dr) \ ,$$

$$\theta^2 = \frac{1}{\sqrt{2}}(e^{\nu}dt - e^{\lambda}dr) \ ,$$
$$\theta^3 = -\frac{1}{\sqrt{2}}(Bd\theta - iBsin\theta d\phi) \ , \tag{4.2.2}$$

where θ^4 is a complex conjugate of θ^3. Hence the metric (4.2.1) can be written as

$$ds^2 = 2\theta^1\theta^2 - 2\theta^3\theta^4, \tag{4.2.3}$$

using equations (2.4.46) and (4.2.2) we obtain readily the components of the basis vector fields as

$$l_i = \frac{1}{\sqrt{2}}\left(-e^{\lambda}, 0, 0, e^{\nu}\right) \ ,$$
$$n_i = \frac{1}{\sqrt{2}}\left(e^{\lambda}, 0, 0, e^{\nu}\right) \ , \tag{4.2.4}$$
$$m_i = \frac{1}{\sqrt{2}}\left(0, B, iBsin\theta, 0\right) \ ,$$

where $\overline{m}_i$ is a complex conjugate of m_i. The contra variant components of the null basis vectors are obtain by raising the index by the metric tensor as

$$l^i = g^{ik}l_k = \frac{1}{\sqrt{2}}\left(e^{-\lambda}, 0, 0, e^{-\nu}\right) \ ,$$

similarly, we obtain

$$n^i = \frac{1}{\sqrt{2}}\left(-e^{-\lambda}, 0, 0, e^{-\nu}\right) \ , \tag{4.2.5}$$
$$m^i = -\frac{1}{\sqrt{2}}\left(0, B^{-1}, iB^{-1}cosec\theta, 0\right) \ .$$

The tetrad components of the equation (2.4.22) are given by

$$d_*^2 f = -\frac{1}{2} f_{;\gamma} Q_{\alpha\beta}{}^{\gamma} \theta^{\alpha} \wedge \theta^{\beta} \ . \tag{4.2.6}$$

For the metric (4.2.1), this can be conveniently rewritten in the form

$$\begin{aligned} d_*^2 f = & -\frac{1}{2\sqrt{2}} \big[e^{-\lambda} \frac{\partial f}{\partial r} (Q_{\alpha\beta}{}^1 - Q_{\alpha\beta}{}^2) - B^{-1} \frac{\partial f}{\partial \theta} (Q_{\alpha\beta}{}^3 + Q_{\alpha\beta}{}^4) - \\ & - iB^{-1} cosec\theta \frac{\partial f}{\partial \phi} (Q_{\alpha\beta}{}^3 - Q_{\alpha\beta}{}^4) + e^{-\nu} \frac{\partial f}{\partial t} (Q_{\alpha\beta}{}^1 + Q_{\alpha\beta}{}^2) \big] \theta^{\alpha} \wedge \theta^{\beta} \ . \end{aligned} \tag{4.2.7}$$

From this equation, we readily find the expressions for the repeated d_* derivative of the co-ordinate functions as

$$\begin{aligned} d_*{}^2 r = & -\frac{e^{-\lambda}}{2\sqrt{2}} \left(Q_{\alpha\beta}{}^1 - Q_{\alpha\beta}{}^2 \right) \theta^{\alpha} \wedge \theta^{\beta} \ , \\ d_*{}^2 \theta = & \frac{B^{-1}}{2\sqrt{2}} \left(Q_{\alpha\beta}{}^3 + Q_{\alpha\beta}{}^4 \right) \theta^{\alpha} \wedge \theta^{\beta} \ , \\ d_*{}^2 \phi = & \frac{iB^{-1} cosec\theta}{2\sqrt{2}} \left(Q_{\alpha\beta}{}^3 - Q_{\alpha\beta}{}^4 \right) \theta^{\alpha} \wedge \theta^{\beta} \ , \\ d_*{}^2 t = & -\frac{e^{-\nu}}{2\sqrt{2}} (Q_{\alpha\beta}{}^1 + Q_{\alpha\beta}{}^2) \theta^{\alpha} \wedge \theta^{\beta}. \end{aligned} \tag{4.2.8}$$

Now operating d_* to the equations (4.2.2) and using the equations (4.2.8) we readily, get

$$\begin{aligned} d_* \theta^1 = & \frac{1}{\sqrt{2}} e^{-(\lambda+\nu)} \left(\nu' - \dot{\lambda} \right) \theta^{12} - \frac{1}{2} Q_{\alpha\beta}{}^1 \theta^{\alpha\beta} \ , \\ d_* \theta^2 = & \frac{1}{\sqrt{2}} \left(e^{-\lambda} \nu' + e^{-\nu} \dot{\lambda} \right) \theta^{12} - \frac{1}{2} Q_{\alpha\beta}{}^2 \theta^{\alpha\beta} \ , \\ d_* \theta^3 = & \frac{1}{\sqrt{2}} \Big[(e^{-\lambda} \frac{B'}{B} + e^{-\nu} \frac{\dot{B}}{B}) \theta^{13} - (e^{-\lambda} \frac{B'}{B} - e^{-\nu} \frac{\dot{B}}{B}) \theta^{23} + \end{aligned}$$

$$+ B^{-1}cot\theta \;\; \theta^{34}\Big] - \frac{1}{2}Q_{\alpha\beta}{}^{3}\theta^{\alpha\beta} \;,$$

$$d_*\theta^4 = \frac{1}{\sqrt{2}}\Big[\big(e^{-\lambda}\frac{B'}{B} + e^{-\nu}\frac{\dot{B}}{B}\big)\theta^{14} - \big(e^{-\lambda}\frac{B'}{B} - e^{-\nu}\frac{\dot{B}}{B}\big)\theta^{24} -$$

$$- B^{-1}cot\theta \;\; \theta^{34}\Big] - \frac{1}{2}Q_{\alpha\beta}{}^{4}\theta^{\alpha\beta} \;, \tag{4.2.9}$$

where we have used

$$\theta^{\alpha\beta} = \theta^{\alpha} \wedge \theta^{\beta} \;,$$

where the dot denotes partial derivative with respect to time 't' and the prime indicates partial derivative with respect to the coordinate 'r'.

From equations (2.4.70) and (4.2.9) we obtain after simplifying the values of Newman-Penrose spin coefficients in Riemann space-time as

$$\begin{aligned}
\kappa^0 =& \lambda^0 = \sigma^0 = \pi^0 = \tau^0 = \nu^0 = 0 \;, \\
\rho^0 =& -\frac{1}{\sqrt{2}}\left(e^{-\lambda}\frac{B'}{B} + e^{-\nu}\frac{\dot{B}}{B}\right) \;, \mu^0 = -\frac{1}{\sqrt{2}}\left(e^{-\lambda}\frac{B'}{B} - e^{-\nu}\frac{\dot{B}}{B}\right) \\
\epsilon^0 =& \frac{1}{2\sqrt{2}}\left(e^{-\lambda}\nu' + e^{-\nu}\dot{\lambda}\right) \;, \gamma^0 = \frac{1}{2\sqrt{2}}\left(e^{-\lambda}\nu' - e^{-\nu}\dot{\lambda}\right) \;, \\
\alpha^0 =& -\beta^0 = \frac{B^{-1}}{2\sqrt{2}}cot\theta.
\end{aligned} \tag{4.2.10}$$

By virtue of the equations (1.3.30) and (4.2.10) from equations (2.4.69), we obtain

$$\omega_{12} = -\frac{1}{\sqrt{2}}\left[\big(e^{-\lambda}\nu' + e^{-\nu}\dot{\lambda}\big)\theta^1 + \big(e^{-\lambda}\nu' - e^{-\nu}\dot{\lambda}\big)\theta^2 - 2ks_0\theta^3 - 2k\overline{s}_0\theta^4\right] \;,$$

$$
\begin{aligned}
\omega_{13} =& \frac{1}{\sqrt{2}}\big[2ks_0\theta^1 + \big(e^{-\lambda}\frac{R'}{R} + e^{-\nu}\frac{\dot{R}}{R} + 2ks_1\big)\theta^4\big] \ , \\
\omega_{23} =& -\frac{1}{\sqrt{2}}\big[2ks_0\theta^2 + \big(e^{-\lambda}\frac{R'}{R} - e^{-\nu}\frac{\dot{R}}{R} - 2ks_1\big)\theta^4\big] \ , \qquad (4.2.11) \\
\omega_{34} =& -\frac{1}{\sqrt{2}}\left[2ks_1(\theta^1 + \theta^2) + R^{-1}cot\theta(\theta^3 - \theta^4)\right] .
\end{aligned}
$$

Now by using equations (1.3.30), (4.2.10) and (4.2.11) we obtain from the equations (2.4.85) the tetrad components of curvature 2-form. These are listed below:

$$
\begin{aligned}
\Omega^1{}_1 =& -\big[e^{-2\lambda}\left(\nu'' - \lambda'\nu' + \nu'^2\right) - e^{-2\nu}\left(\ddot{\lambda} - \dot{\lambda}\dot{\nu} + \dot{\lambda^2}\right) + 4k^2s_0\overline{s}_0\big]\theta^{12} + \\
& + \big[ke^{-\lambda}(s_{0,r} + 2s_0\nu') + ke^{-\nu}(s_{0,t} + 2s_0\frac{\dot{B}}{B}) - 2k^2s_0s_1\big]\theta^{13} + \\
& + \big[ke^{-\lambda}(\overline{s}_{0,r} + 2\overline{s}_0\nu') + ke^{-\nu}(\overline{s}_{0,t} + 2\overline{s}_0\frac{\dot{B}}{B}) + 2k^2\overline{s}_0s_1\big]\theta^{14} - \\
& - \big[ke^{-\lambda}(s_{0,r} + 2s_0\nu') - ke^{-\nu}(s_{0,t} + 2s_0\frac{\dot{B}}{B}) + 2k^2s_0s_1\big]\theta^{23} - \\
& - \big[ke^{-\lambda}(\overline{s}_{0,r} + 2\overline{s}_0\nu') - ke^{-\nu}(\overline{s}_{0,t} + 2\overline{s}_0\frac{\dot{B}}{B}) - 2k^2\overline{s}_0s_1\big]\theta^{24} + \\
& + \big[kB^{-1}cot\theta(s_0 - \overline{s}_0) + 4ks_1e^{-\lambda}(\frac{B'}{B} - \nu')\big]\theta^{34} \ , \qquad (4.2.12) \\
\Omega^1{}_3 =& -\big[ke^{-\lambda}\left(s_{0,r} + 2s_0\nu'\right) + ke^{-\nu}(s_{0,t} + 2s_0\dot{\lambda}) + 2k^2s_0s_1\big]\theta^{12} + \\
& + \big\{ks_{1,r}e^{-\lambda} + ks_{1,t}e^{-\nu} + 2k^2s_1^2 + \frac{ks_1}{B}\big[e^{-\lambda}\nu'B + e^{-\nu}(2\dot{B} + \dot{\lambda}B)\big] - \\
& - \frac{e^{-2\lambda}}{2B}(B'' - \lambda'B' + \nu'B') + \frac{e^{-2\nu}}{2B}\left(\ddot{B} - \dot{\nu}\dot{B} + \dot{\lambda}\dot{B}\right)\big\}\theta^{14} + \\
& + \big[ks_0B^{-1}cot\theta - 2k^2s_0^2\big]\theta^{23} - \big\{ks_{1,r}e^{-\lambda} - ks_{1,t}e^{-\nu} - 2k^2s_1^2 + \\
& + 2k^2s_0\overline{s}_0 + ks_0B^{-1}cot\theta - \frac{e^{-2\lambda}}{2B}\left(B'' - \lambda'B' - \nu'B'\right) -
\end{aligned}
$$

$$- \frac{e^{-2\nu}}{2B}(\ddot{B} - \dot{\nu}\dot{B} - \dot{\lambda}\dot{B}) - \frac{e^{-(\lambda+\nu)}}{B}(\dot{\lambda}B' + \dot{B}\nu^{'} - \dot{B}^{'})+$$

$$+ \frac{ks_1}{B}\big[e^{-\lambda}(2B' - \nu^{'}B) - e^{-\nu}\left(2\dot{B} - \dot{\lambda}B\right)\big]\big\}\theta^{24}+$$

$$+ \big[\frac{ks_0}{B}(e^{-\lambda}B^{'} - e^{-\nu}\dot{B}) - 2k^2 s_0 s_1\big]\theta^{34} \ , \tag{4.2.13}$$

$$\Omega^2{}_3 = \big[ke^{-\lambda}(s_{0,r} + 2s_0\nu') - ke^{-\nu}(s_{0,t} + 2s_0\dot{\lambda}) - 2k^2 s_0 s_1\big]\theta^{12}-$$

$$- (2k^2 s_0^2 + ks_0 B^{-1} cot\theta)\theta^{13} + \big\{ks_{1,r}e^{-\lambda} + ks_{1,t}e^{-\nu} + 2k^2 s_1^2-$$

$$- 2k^2 s_0\overline{s}_0 + ks_0 B^{-1} cot\theta + \frac{e^{-2\lambda}}{2B}(B^{"} - \lambda' B^{'} - \nu' B^{'})+$$

$$+ \frac{e^{-2\nu}}{2B}(\ddot{B} - \dot{\nu}\dot{B} - \dot{\lambda}\dot{B}) - \frac{e^{-(\lambda+\nu)}}{B}(\dot{\lambda}B' + \dot{B}\nu^{'} - \dot{B}^{'})+$$

$$+ \frac{ks_1}{B}\big[e^{-\lambda}(2B' - \nu^{'}B) + e^{-\nu}(2\dot{B} - \dot{\lambda}B)\big]\big\}\theta^{14}-$$

$$- \big\{ks_{1,r}e^{-\lambda} - ks_{1,t}e^{-\nu} - 2k^2 s_1^2 + \frac{ks_1}{B}\big[e^{-\lambda}\nu' B - e^{-\nu}(2\dot{B} + \dot{\lambda}B)\big]+$$

$$+ \frac{e^{-2\lambda}}{2B}\left(B^{"} - \lambda' B^{'} + \nu' B^{'}\right) - \frac{e^{-2\nu}}{2B}(\ddot{B} - \dot{\nu}\dot{B} + \dot{\lambda}\dot{B})\big\}\theta^{24}+$$

$$+ \big[\frac{ks_0}{B}\big(e^{-\lambda}B^{'} + e^{-\nu}\dot{B}\big) + 2k^2 s_0 s_1\big]\theta^{34} \ , \tag{4.2.14}$$

$$\Omega^3{}_3 = -2ke^{-\lambda}(s_{1,r} + s_1\nu')\theta^{12} + \frac{ks_0}{B}(B^{'}e^{-\lambda} - \dot{B}e^{-\nu} + 2ks_1 B)\theta^{13}-$$

$$- \frac{k\overline{s}_0}{B}(B^{'}e^{-\lambda} - \dot{B}e^{-\nu} - 2ks_1 B)\theta^{14} + \frac{ks_0}{B}\big(B^{'}e^{-\lambda} + \dot{B}e^{-\nu}-$$

$$- 2ks_1 B\big)\theta^{23} - \frac{k\overline{s}_0}{B}(B^{'}e^{-\lambda} + \dot{B}e^{-\nu} + 2ks_1 B)\theta^{24}+$$

$$+ \frac{1}{B^2}[1 - e^{-2\lambda}B'^2 + e^{-2\nu}\dot{B}^2 - 4k^2 {s_1}^2 B^2]\theta^{34} \ . \tag{4.2.15}$$

The expressions for $\Omega^1{}_4, \Omega^2{}_4, \Omega^4{}_4$ are obtained by interchanging the suffixes 3 and 4 and taking the complex conjugate of the right hand sides of the equations in (4.2.13), (4.2.14) and (4.2.15) respectively.

4.2.1 Tetrad Components of Riemann-Cartan Curvature Tensor

From the equation (2.4.65) we obtain

$$\Omega_{\alpha\beta} = R_{12\alpha\beta}\theta^{12} + R_{13\alpha\beta}\theta^{13} + R_{14\alpha\beta}\theta^{14} + R_{23\alpha\beta}\theta^{23} + R_{24\alpha\beta}\theta^{24} + R_{34\alpha\beta}\theta^{34} \ , \tag{4.2.16}$$

where

$$\Omega_{\alpha\beta} = -\Omega_{\beta\alpha} \ , \quad \alpha, \beta = 1, 2, 3, 4.$$

By giving different values to $\alpha, \beta = 1, 2, 3, 4$ in the equation (4.2.16) and then equating the corresponding coefficients of basis 2-forms of equations(4.2.12),(4.2.13),(4.2.14) and(4.2.15) we readily obtain the tetrad components of Riemann-Cartan curvature tensor as

$$R_{1212} = -\left[e^{-2\lambda}\left(\nu'' - \lambda'\nu' + \nu'^2\right) - e^{-2\nu}\left(\ddot{\lambda} - \dot{\lambda}\dot{\nu} + \dot{\lambda^2}\right) + 4k^2 s_0 \overline{s}_0\right] \ ,$$

$$R_{1312} = ke^{-\lambda}(s_{0,r} + 2s_0\nu') + ke^{-\nu}(s_{0,t} + 2s_0\frac{\dot{B}}{B}) - 2k^2 s_0 s_1 \ ,$$

$$R_{2312} = -\left[ke^{-\lambda}(s_{0,r} + 2s_0\nu') - ke^{-\nu}(s_{0,t} + 2s_0\frac{\dot{B}}{B}) + 2k^2 s_0 s_1\right],$$

$$R_{3412} = kB^{-1}cot\theta(s_0 - \overline{s}_0) + 4ks_1 e^{-\lambda}\left(\frac{B'}{B} - \nu^{'}\right) \ ,$$

$$R_{1213} = ke^{-\lambda}\left(s_{0,r} + 2s_0\nu'\right) - ke^{-\nu}(s_{0,t} + 2s_0\dot{\lambda}) - 2k^2 s_0 s_1 \ ,$$

$$R_{1313} = -\left(2k^2 s_0^2 + ks_0 B^{-1}cot\theta\right) \ ,$$

$$R_{1413} = ks_{1,r}e^{-\lambda} + ks_{1,t}e^{-\nu} + 2k^2 s_1^2 - 2k^2 s_0\overline{s}_0 + ks_0 B^{-1}cot\theta +$$
$$+ \frac{e^{-2\lambda}}{2B}\left(B" - \lambda' B^{'} - \nu' B^{'}\right) + \frac{e^{-2\nu}}{2B}\left(\ddot{B} - \dot{\nu}\dot{B} - \dot{\lambda}\dot{B}\right) -$$

$$
\begin{aligned}
&- \frac{e^{-(\lambda+\nu)}}{B}(\dot{\lambda}B' + \dot{B}\nu^{'} - \dot{B}^{'}) + \frac{ks_1}{B}\big[e^{-\lambda}\left(2B' - \nu^{'}B\right) + \\
&+ e^{-\nu}\left(2\dot{B} - \dot{\lambda}B\right)\big] \ , \\
R_{2413} =& - ks_{1,r}e^{-\lambda} + ks_{1,t}e^{-\nu} + 2k^2s_1^2 - \frac{ks_1}{B}\big[e^{-\lambda}\nu'B - e^{-\nu}\left(2\dot{B} + \dot{\lambda}B\right)\big] - \\
&- \frac{e^{-2\lambda}}{2B}(B" - \lambda'B^{'} + \nu'B^{'}) + \frac{e^{-2\nu}}{2B}\left(\ddot{B} - \dot{\nu}\dot{B} + \dot{\lambda}\dot{B}\right) \ , \\
R_{3413} =& \frac{ks_0}{B}\left(e^{-\lambda}B^{'} + e^{-\nu}\dot{B}\right) + 2k^2s_0s_1 \ , \\
R_{1223} =& - \left[ke^{-\lambda}\left(s_{0,r} + 2s_0\nu'\right) + ke^{-\nu}(s_{0,t} + 2s_0\dot{\lambda}) + 2k^2s_0s_1\right] \ , \\
R_{1423} =& ks_{1,r}e^{-\lambda} + ks_{1,t}e^{-\nu} + 2k^2s_1^2 + \frac{ks_1}{B}\big[e^{-\lambda}\nu'B + e^{-\nu}(2\dot{B} + \dot{\lambda}B)\big] - \\
&- \frac{e^{-2\lambda}}{2B}\left(B" - \lambda'B^{'} + \nu'B^{'}\right) + \frac{e^{-2\nu}}{2B}\left(\ddot{B} - \dot{\nu}\dot{B} + \dot{\lambda}\dot{B}\right) \ , \\
R_{2323} =& ks_0B^{-1}cot\theta - 2k^2s_0^2 \ , \\
R_{2423} =& - ks_{1,r}e^{-\lambda} + ks_{1,t}e^{-\nu} + 2k^2s_1^2 - 2k^2s_0\overline{s}_0 - \frac{ks_0}{B}cot\theta + \\
&+ \frac{e^{-2\lambda}}{2B}\left(B" - \lambda'B^{'} - \nu'B^{'}\right) + \frac{e^{-2\nu}}{2B}\left(\ddot{B} - \dot{\nu}\dot{B} - \dot{\lambda}\dot{B}\right) + \\
&+ \frac{e^{-(\lambda+\nu)}}{B}\left(\dot{\lambda}B' + \dot{B}\nu^{'} - \dot{B}^{'}\right) - \frac{ks_1}{B}\big[e^{-\lambda}\left(2B' - \nu^{'}B\right) - \\
&- e^{-\nu}\left(2\dot{B} - \dot{\lambda}B\right)\big] \ , \\
R_{3423} =& \frac{ks_0}{B}\left(e^{-\lambda}B^{'} - e^{-\nu}\dot{B}\right) - 2k^2s_0s_1 \ , \\
R_{1234} =& - 2ke^{-\lambda}\left(s_{1,r} + s_1\nu'\right) \ , \\
R_{1334} =& \frac{ks_0}{B}\left(B^{'}e^{-\lambda} - \dot{B}e^{-\nu} + 2ks_1B\right) \ , \\
R_{2334} =& \frac{ks_0}{B}\left(B^{'}e^{-\lambda} + \dot{B}e^{-\nu} - 2ks_1B\right) \ , \\
R_{3434} =& \frac{1}{B^2}\left[1 - e^{-2\lambda}B'^2 + e^{-2\nu}\dot{B}^2 - 4k^2{s_1}^2B^2\right] \ , \qquad (4.2.17)
\end{aligned}
$$

and

$$R_{2313} = R_{1323} = 0 \ .$$

The complex conjugates of above equations are obtained by interchanging the suffixes 3 and 4 and taking the complex conjugates of the right hand sides of the respective equations.

4.2.2 Tetrad Components of Ricci-Cartan Tensor and Ricci-Cartan Curvature Scalar

The tetrad components of the Ricci-Cartan tensor and Ricci-Cartan curvature scalar are defined by

$$R_{\alpha\beta} = \eta^{\nu\epsilon} R_{\nu\alpha\beta\epsilon} \ \ , R = \eta^{\alpha\beta} R_{\alpha\beta} \ ,$$

$$\Rightarrow \qquad R_{\alpha\beta} = R_{1\alpha\beta 2} + R_{2\alpha\beta 1} - R_{3\alpha\beta 4} - R_{4\alpha\beta 3} \ . \qquad (4.2.18)$$

Using equations (4.2.17) we obtain from equations (4.2.18) expressions for Ricci-Cartan tensors

$$\begin{aligned} R_{11} = {} & kB^{-1}cot\theta(s_0 + \bar{s}_0) + 4k^2 {s_1}^2 - 4k^2 s_0 \bar{s}_0 + \frac{e^{-2\lambda}}{B}\big(B^{''} - \lambda^{'} B^{'} - \\ & - \nu^{'} B^{'}\big) + \frac{e^{-2\nu}}{B}(\ddot{B} - \dot{\nu}\dot{B} - \dot{\lambda}\dot{B}) + \frac{2}{B} e^{-(\lambda+\nu)} \left(\dot{B}^{'} - \nu^{'}\dot{B} - \dot{\lambda}B^{'}\right) \ , \\ R_{12} = R_{21} = {} & -\frac{e^{-2\lambda}}{B}\big[\big(\nu'' - \lambda'\nu' + \nu'^2\big)B + B^{''} - \lambda^{'} B^{'} + \nu^{'} B^{'}\big] + \\ & + 4k^2 {s_1}^2 - 4k^2 s_0 \bar{s}_0 + \frac{e^{-2\nu}}{B}\big[\big(\ddot{\lambda} - \dot{\lambda}\dot{\nu} + \dot{\lambda}^2\big)B + \ddot{B} - \dot{\nu}\dot{B} + \dot{\lambda}\dot{B}\big] \ , \end{aligned}$$

$$R_{13} = ke^{-\lambda}\big(s_{0,r} + 2s_0\nu^{'} + s_0\frac{B^{'}}{B}\big) - ke^{-\nu}\big(s_{0,t} + 2s_0\dot{\lambda} - s_0\frac{\dot{B}}{B}\big) \ ,$$

$$R_{22} = -kB^{-1}cot\theta(s_0 + \overline{s}_0) + 4k^2{s_1}^2 - 4k^2 s_0\overline{s}_0 + \frac{e^{-2\lambda}}{B}\big(B^{''} - \lambda^{'}B^{'} -$$

$$- \nu^{'}B^{'}\big) + \frac{e^{-2\nu}}{B}\big(\ddot{B} - \dot{\nu}\dot{B} - \dot{\lambda}\dot{B}\big) - \frac{2}{B}e^{-(\lambda+\nu)}\big(\dot{B}^{'} - \nu^{'}\dot{B} - \dot{\lambda}B^{'}\big) \ ,$$

$$R_{23} = ke^{-\lambda}\big(s_{0,r} + 2s_0\nu^{'} + s_0\frac{B^{'}}{B}\big) + ke^{-\nu}\big(s_{0,t} + 2s_0\dot{\lambda} + s_0\frac{\dot{B}}{B}\big) \ ,$$

$$R_{31} = ke^{-\lambda}\big(s_{0,r} + 2s_0\nu^{'} + s_0\frac{B^{'}}{B}\big) + ke^{-\nu}\big(s_{0,t} + 3s_0\frac{\dot{B}}{B}\big) \ ,$$

$$R_{32} = ke^{-\lambda}\big(s_{0,r} + 2s_0\nu^{'} + s_0\frac{B^{'}}{B}\big) - ke^{-\nu}\big(s_{0,t} + 3s_0\frac{\dot{B}}{B}\big) \ ,$$

$$R_{34} = 2ks_{1,t}e^{-\nu} + 2ks_1e^{-\nu}\big(2\frac{\dot{B}}{B} + \dot{\lambda}\big) + \frac{e^{-2\lambda}}{B}\big(B^{''} - \lambda^{'}B^{'} + \nu^{'}B^{'} +$$

$$+ \frac{B'^2}{B}\big) - \frac{e^{-2\nu}}{B}\big(\ddot{B} - \dot{\nu}\dot{B} + \dot{\lambda}\dot{B} + \frac{\dot{B}^2}{B}\big) - \frac{1}{B^2} \ ,$$

$$R_{43} = -2ks_{1,t}e^{-\nu} - 2ks_1e^{-\nu}\big(2\frac{\dot{B}}{B} + \dot{\lambda}\big) + \frac{e^{-2\lambda}}{B}\big(B^{''} - \lambda^{'}B^{'} + \nu^{'}B^{'} +$$

$$+ \frac{B'^2}{B}\big) - \frac{e^{-2\nu}}{B}\big(\ddot{B} - \dot{\nu}\dot{B} + \dot{\lambda}\dot{B} + \frac{\dot{B}^2}{B}\big) - \frac{1}{B^2} \ ,$$

$$R_{33} = 0 \ . \qquad (4.2.19)$$

The Ricci-Cartan curvature scalar is given by

$$R = 2\big\{ - \frac{e^{-2\lambda}}{B}\big[\big(\nu'' - \lambda'\nu' + \nu'^2\big)B + 2B^{''} - 2\lambda^{'}B^{'} + 2\nu^{'}B^{'} + \frac{B'^2}{B}\big] +$$

$$+ 4k^2{s_1}^2 - 4k^2 s_0\overline{s}_0 + \frac{e^{-2\nu}}{B}\big[(\ddot{\lambda} - \dot{\lambda}\dot{\nu} + \dot{\lambda}^2)B + 2\ddot{B} - 2\dot{\nu}\dot{B} + 2\dot{\lambda}\dot{B} +$$

$$+ \frac{\dot{B}^2}{B}\big] + B^{-2}\big\} \ . \qquad (4.2.20)$$

4.3 Einstein-Cartan's Field Equations

We start with the tetrad representation of the field equations (1.2.48) as

$$R_{\alpha\beta} - \frac{R}{2}\eta_{\alpha\beta} = -kt_{\alpha\beta} \ , \tag{4.3.1}$$

where $R_{\alpha\beta}$ are the asymmetric components of Ricci-Cartan tensor and $t_{\alpha\beta}$ are likewise asymmetric tetrad components of the energy momentum tensor and are defined by

$$t_{\alpha\beta} = t_{ij}{e_{(\alpha)}}^i {e_{(\beta)}}^j \ . \tag{4.3.2}$$

From this, we find, by using equations (1.4.22) and (4.2.10)

$$\begin{aligned} t_{11} =& t_{22} = \frac{1}{2}(\rho + p), t_{12} = t_{21} = \frac{1}{2}(\rho - p), \\ t_{34} =& t_{43} = p \ \ , t_{31} = t_{32} = \nu' e^{-\lambda} s_0 \ , \\ t_{41} =& t_{42} = \nu' e^{-\lambda} \overline{s}_0 \ , \end{aligned} \tag{4.3.3}$$

and all other tetrad components of the energy-momentum tensor are zero.

Using equations (4.2.19), (4.2.20) and (4.3.3) in the equations (4.3.1), the independent field equations of Einstein-Cartan theory of gravitation are obtained below

$$kB^{-1} cot\theta(s_0 + \overline{s}_0) + 4k^2 {s_1}^2 - 4k^2 s_0 \overline{s}_0 + \frac{e^{-2\lambda}}{B}\big(B'' - \lambda' B' - \nu' B'\big) +$$
$$+ \frac{e^{-2\nu}}{B}\big(\ddot{B} - \dot{\nu}\dot{B} - \dot{\lambda}\dot{B}\big) + \frac{2}{B} e^{-(\lambda+\nu)}(\dot{B}' - \nu'\dot{B} - \dot{\lambda}B') = -\frac{k}{2}(\rho + p) \ ,$$

$$
\begin{aligned}
&\frac{e^{-2\lambda}}{B}\big(B^{''}-\lambda^{'}B^{'}+\nu^{'}B^{'}+\frac{B^{\prime 2}}{B}\big)-\frac{e^{-2\nu}}{B}(\ddot{B}-\dot{\nu}\dot{B}+\dot{\lambda}\dot{B}+\frac{\dot{B}^{2}}{B})-\\
&-B^{-2}=-\frac{k}{2}(\rho-p)\ ,\\
&e^{-\lambda}\big(s_{0,r}+2s_0\nu^{'}+s_0\frac{B^{'}}{B}\big)-e^{-\nu}\big(s_{0,t}+2s_0\dot{\lambda}-s_0\frac{\dot{B}}{B}\big)=0\ ,\\
&-kB^{-1}cot\theta(s_0+\overline{s}_0)+4k^2{s_1}^2-4k^2s_0\overline{s}_0+\frac{e^{-2\lambda}}{B}(B^{''}-\lambda^{'}B^{'}-\nu^{'}B^{'})+\\
&+\frac{e^{-2\nu}}{B}\Big(\ddot{B}-\dot{\nu}\dot{B}-\dot{\lambda}\dot{B}\Big)-\frac{2}{B}e^{-(\lambda+\nu)}(\dot{B}^{'}-\nu^{'}\dot{B}-\dot{\lambda}B^{'})=-\frac{k}{2}(\rho+p)\ ,\\
&e^{-\lambda}\big(s_{0,r}+2s_0\nu^{'}+s_0\frac{B^{'}}{B}\big)+e^{-\nu}\big(s_{0,t}+2s_0\dot{\lambda}+s_0\frac{\dot{B}}{B}\big)=0\ ,\\
&e^{-\lambda}\big(s_{0,r}+3s_0\nu^{'}+s_0\frac{B^{'}}{B}\big)+e^{-\nu}\big(s_{0,t}+3s_0\frac{\dot{B}}{B}\big)=0\ ,\\
&e^{-\lambda}\big(s_{0,r}+3s_0\nu^{'}+s_0\frac{B^{'}}{B}\big)-e^{-\nu}\big(s_{0,t}+3s_0\frac{\dot{B}}{B}\big)=0\ ,\\
&2ks_{1,t}e^{-\nu}+2ks_1e^{-\nu}\big(2\frac{\dot{B}}{B}+\dot{\lambda}\big)-\frac{e^{-2\lambda}}{B}\big[B^{''}-\lambda^{'}B^{'}+\nu^{'}B^{'}+\\
&+B\big(\nu^{''}-\lambda^{'}\nu^{'}+\nu^{'2}\big)\big]+\frac{e^{-2\nu}}{B}\big[(\ddot{\lambda}-\dot{\lambda}\dot{\nu}+\dot{\lambda}^2)B+\ddot{B}-\dot{\nu}\dot{B}+\dot{\lambda}\dot{B}\big]+\\
&+4k^2{s_1}^2-4k^2s_0\overline{s}_0=-kp\ ,\\
&-2ks_{1,t}e^{-\nu}-2ks_1e^{-\nu}(2\frac{\dot{B}}{B}+\dot{\lambda})-\frac{e^{-2\lambda}}{B}\big[B^{''}-\lambda^{'}B^{'}+\nu^{'}B^{'}+\\
&+B(\nu^{''}-\lambda^{'}\nu^{'}+\nu^{'2})\big]+\frac{e^{-2\nu}}{B}\big[(\ddot{\lambda}-\dot{\lambda}\dot{\nu}+\dot{\lambda}^2)B+\ddot{B}-\dot{\nu}\dot{B}+\dot{\lambda}\dot{B}\big]+\\
&+4k^2{s_1}^2-4k^2s_0\overline{s}_0=-kp\ . \qquad (4.3.4)
\end{aligned}
$$

We see from equations (4.3.4) that these equations are consistent provided that $s_0=0$. Hence out of the sixteen field equations, there exists only five independent field equations and are given below:

$$-\frac{1}{B^2}+\frac{2}{B}e^{-2\lambda}\left(\nu^{'}B^{'}+\frac{B^{'2}}{2B}\right)-\frac{2}{B}e^{-2\nu}\left(\ddot{B}-\dot{\nu}\dot{B}+\frac{\dot{B}^2}{2B}\right)-4k^2s_1^2=kp\ , \tag{4.3.5}$$

$$\frac{1}{B^2}-\frac{2}{B}e^{-2\lambda}\left(B^{''}-\lambda^{'}B^{'}+\frac{B^{'2}}{2B}\right)+\frac{2}{B}e^{-2\nu}\left(\dot{\lambda}\dot{B}+\frac{\dot{B}^2}{2B}\right)-4k^2s_1^2=k\rho\ , \tag{4.3.6}$$

$$e^{-2\lambda}\left[B^{''}-\lambda^{'}B^{'}-\nu^{'}B^{'}+B\left(\nu^{''}-\lambda^{'}\nu^{'}+\nu^{'2}\right)-\frac{B^{'2}}{B}\right]+$$
$$+e^{-2\nu}\left[\ddot{B}-\dot{\nu}\dot{B}-\dot{\lambda}\dot{B}-B(\ddot{\lambda}-\dot{\lambda}\dot{\nu}+\dot{\lambda}^2)+\frac{\dot{B}^2}{B}\right]+\frac{1}{B}=0\ , \tag{4.3.7}$$

$$\dot{B}^{'}-\dot{B}\nu^{'}-B^{'}\dot{\lambda}=0\ , \tag{4.3.8}$$

$$s_{1,t}+s_1\left(2\frac{\dot{B}}{B}+\dot{\lambda}\right)=0\ . \tag{4.3.9}$$

Solving the equation (4.3.9), we obtain

$$s_1=h_1\frac{e^{-\lambda}}{B^2}\ , \tag{4.3.10}$$

where h_1 is an arbitrary constant.

4.4 Specific Solutions

Case A:

To solve the non-linear equations, we assume for simplicity sake $B=B(t)=t$ and $\lambda=\lambda(t)$. This class of solutions is identified with the well-known Kantowski-Sachs class [65] of cosmological models. Hence the equation (4.3.7) becomes

$$\ddot{\lambda} - \dot{\lambda}\dot{\nu} + \dot{\lambda}^2 + \frac{1}{t}(\dot{\lambda} + \dot{\nu}) - \frac{1}{t^2} = \frac{e^{2\nu}}{t^2} \ . \tag{4.4.1}$$

To solve the equation (4.4.1), we define

$$y(t) = e^{\lambda} \ , \qquad x(t) = e^{-2\nu} \ . \tag{4.4.2}$$

Then (4.4.1) may be written as

$$\dot{x} + \left[\frac{2\left(t^2\ddot{y} + t\dot{y} - y\right)}{t\left(t\dot{y} - y\right)}\right] x = \frac{2y}{t\left(t\dot{y} - y\right)} \ . \tag{4.4.3}$$

This is linear equation in x, provided y is known. Hence its solution is given by

$$x(t) = exp[-F(t)] \left\{ \int^t exp[F(u)]g(u)du + c \right\} \ ,$$

where

$$F(t) = \int^t f(u)du \ , \qquad f(t) = \frac{2\left(t^2\ddot{y} + t\dot{y} - y\right)}{t\left(t\dot{y} - y\right)} \ ,$$

$$g(t) = \frac{2y}{t\left(t\dot{y} - y\right)} \ , \tag{4.4.4}$$

and c being constant of integration.

The remaining equations (4.3.5) and (4.3.6) give p and ρ as

$$kp = -\frac{1}{t^2} - \frac{x}{t^2}\left(1 + \frac{\dot{x}}{x}t\right) - 4k^2 s_1^2 \ , \tag{4.4.5}$$

$$k\rho = \frac{1}{t^2} + \frac{x}{t^2}\left(1 + 2\frac{\dot{y}}{y}t\right) - 4k^2 s_1^2 \ . \tag{4.4.6}$$

We choose y in such a way that equation (4.4.3) can be immediately integrated. We assume that y satisfies the Cauchy equation

$$t^2\ddot{y} + t\dot{y} + (1-q^2)y = 0 \ . \qquad (4.4.7)$$

Case i : When $-1 < q < 1$

In this case the solution of the equation (4.4.7) is obtained as

$$e^{\lambda} = y = A_1 cos(\eta logt) + A_2 sin(\eta logt) \ , \qquad (4.4.8)$$

where $\eta = \sqrt{1-q^2}$ and A_1, A_2 are arbitrary constants. Using this value of y in the equation (4.4.3), we get

$$\dot{x}+2\frac{(\eta^2+1)[A_1 cos(\eta logt) + A_2 sin(\eta logt)]}{t[(A_1+A_2\eta)cos(\eta logt) + (A_1-A_2\eta)sin(\eta logt)]}x = \\ = -2\frac{[A_1 cos(\eta logt) + A_2 sin(\eta logt)]}{t[(A_1-A_2\eta)cos(\eta logt) + (A_2+A_1\eta)sin(\eta logt)]} \ . \qquad (4.4.9)$$

We obtain the solution of this equation as

$$x = e^{-2\nu} = -\frac{1}{\eta^2+1} + \frac{c}{t^2}\Big[(A_1\eta + A_2)sin(\eta logt) + (A_1 - A_2\eta)cos(\eta logt)\Big]^{-2}. \qquad (4.4.10)$$

The metric (4.2.1) becomes

$$ds^2 = \left[-\frac{1}{\eta^2+1} + \frac{c}{t^2}\Big[(A_1\eta + A_2)sin(\eta logt) + (A_1 - A_2\eta)cos(\eta logt)\Big]^{-2}\right]^{-1} \cdot \\ \cdot dt^2 - \Big[A_1 cos(\eta logt) + A_2 sin(\eta logt)\Big]^2 dr^2 - t^2(d\theta^2 + sin^2\theta d\phi^2). \qquad (4.4.11)$$

Now from equations (4.4.5) and (4.4.6), we obtain pressure and density as

$$\begin{aligned} kp =& -\frac{\eta^2}{t^2(\eta^2+1)} + \frac{c}{t^4}\left[\frac{2(\eta^2+1)+F_1}{{F_1}^3}\right] - 4k^2 s_1^2 \ , \\ k\rho =& \frac{1}{t^2}\left[\frac{G_1}{(\eta^2+1)F_2}\right] + \frac{c}{t^4}\left[\frac{H_1}{{F_1}^2 F_2}\right] - 4k^2 s_1^2 \ , \end{aligned} \tag{4.4.12}$$

where

$$\begin{aligned} F_1 =& (A_1\eta + A_2) sin(\eta logt) + (A_1 - A_2\eta) cos(\eta logt) \ , \\ G_1 =& (A_1\eta^2 - 2A_2\eta) cos(\eta logt) + (A_2\eta^2 + 2A_1\eta) sin(\eta logt) \ , \\ H_1 =& (A_1 + 2A_2\eta) cos(\eta logt) + (A_2 - 2A_1\eta) sin(\eta logt) \ , \\ F_2 =& A_1 cos(\eta logt) + A_2 sin(\eta logt) \ , \\ G_2 =& (2A_1 + A_2\eta) cos(\eta logt) + (2A_2 - A_1\eta) sin(\eta logt) \ , \end{aligned}$$

and

$$s_1^2 = \frac{h_1}{t^4}\left[F_2\right]^{-2} \ . \tag{4.4.13}$$

The kinematical parameters defined in the equation (1.4.27) using equation (4.2.10), for the space-time metric (4.4.11) read as

$$\begin{aligned} \theta =& \frac{1}{t^2}\left[\frac{c}{{F_1}^2} - \frac{t^2}{\eta^2+1}\right]^{1/2}\left[\frac{G_2}{F_2}\right] \ , \\ \sigma_{11} =& \sigma_{22} = -\sigma_{12} = -\sigma_{34} = \frac{1}{3t^2}\left[\frac{c}{{F_1}^2} - \frac{t^2}{\eta^2+1}\right]^{1/2}\left[\frac{F_1}{F_2}\right] \ , \\ \dot{u}_1 =& \dot{u}_2 = \dot{u}_3 = \dot{u}_4 = 0 \ , \\ W_{34} =& -W_{43} = -2ks_1 \ . \end{aligned} \tag{4.4.14}$$

Case ii : When $q = 1$

In this case from equation (4.4.7), we obtain the solution as

$$e^{\lambda} = y = B_1 + B_2 logt \ , \tag{4.4.15}$$

where B_1 and B_2 are arbitrary constants. For this value of y the equation (4.4.3), becomes

$$\dot{x} - \frac{2(B_1 + B_2 logt)}{t(B_2 - B_2 logt - B_1)} x = 2 \frac{B_1 + B_2 logt}{t[B_2 - B_1 - B_2 logt]} \ . \tag{4.4.16}$$

The solution of this equation yields

$$e^{-2\nu} = x = -1 + \frac{c}{t^2} [B_2(1 - logt) - B_1]^{-2} \ . \tag{4.4.17}$$

The metric (4.2.1) becomes

$$ds^2 = \left[-1 + \frac{c}{t^2} [B_2(1 - logt) - B_1]^{-2} \right]^{-1} dt^2 - (B_1 + B_2 logt)^2 dr^2 - \\ - t^2 (d\theta^2 + sin^2\theta d\phi^2) \ . \tag{4.4.18}$$

Now from equations (4.4.5) and (4.4.6), we obtain pressure and density as

$$kp = \frac{c}{t^4} [B_1 + B_2(1 + logt)][B_1 - B_2(1 - logt)]^{-3} - 4k^2 s_1^2 \ ,$$

$$k\rho = -\frac{2B_2}{t^2(B_1 + B_2 logt)} + \frac{c}{t^4} \left[\frac{(B_1 + 2B_2 + B_2 logt)}{(B_1 + B_2 logt)[B_1 - B_2(1 - logt)]^2} \right] - \\ - 4k^2 s_1^2 \ , \tag{4.4.19}$$

and $s_1^2 = \dfrac{h_1}{t^4} [B_1 + B_2 logt]^{-2}$. (4.4.20)

By using equations (4.2.10) in (1.4.27) the tetrad components of the kinematical parameters in the space-time metric (4.4.18) read as

$$
\begin{aligned}
\theta =& \frac{1}{t^2}\left\{c[B_2(1-logt)-B_1]^{-2}-t^2\right\}^{1/2}\left[\frac{2B_1+B_2(2logt+1)}{B_1+B_2logt}\right] \ , \\
\sigma_{11} =& \sigma_{22} = -\sigma_{12} = -\sigma_{34} = \frac{1}{3t^2}\left\{c[B_2(1-logt)-B_1]^{-2}-t^2\right\}^{1/2} \cdot \\
& \cdot \left[\frac{B_1+B_2(logt-1)}{B_1+B_2logt}\right] \ , \\
\dot{u}_1 =& \dot{u}_2 = \dot{u}_3 = \dot{u}_4 = 0 \ , \\
W_{34} =& - W_{43} = -2ks_1 \ .
\end{aligned} \tag{4.4.21}
$$

Case iii : When $q < -1$ and $q > 1$

In this case, solution of the equation (4.4.7) is given by

$$
e^{\lambda} = y = D_1 t^{\beta} + D_2 t^{-\beta} \ , \tag{4.4.22}
$$

where $\beta = \sqrt{q^2 - 1}$ and D_1, D_2 are arbitrary constants. For this value of y the equation (4.4.3), becomes

$$
\dot{x} + 2\frac{(\beta^2-1)(D_1t^{\beta}+D_2t^{-\beta})}{t[(\beta-1)D_1t^{\beta}-(\beta+1)D_2t^{-\beta}]}x = 2\frac{(D_1t^{\beta}+D_2t^{-\beta})}{t[(\beta-1)D_1t^{\beta}-(\beta+1)D_2t^{-\beta}]} \ . \tag{4.4.23}
$$

The solution of this equation yields

$$
e^{-2\nu} = x = \frac{1}{\beta^2-1} + \frac{c}{t^2}\left[D_1(\beta-1)t^{\beta} - D_2(\beta+1)t^{-\beta}\right]^{-2} \ . \tag{4.4.24}
$$

The metric (4.2.1) becomes

$$
ds^2 = \left[\frac{1}{\beta^2-1} + \frac{c}{t^2}\left[D_1(\beta-1)t^{\beta} - D_2(\beta+1)t^{-\beta}\right]^{-2}\right]^{-1} dt^2 -
$$

$$- \left(D_1 t^{\beta} + D_2 t^{-\beta}\right)^2 dr^2 - t^2(d\theta^2 + sin^2\theta d\phi^2) \ . \qquad (4.4.25)$$

Now from equations (4.4.5) and (4.4.6), we obtain pressure and density as

$$\begin{aligned} kp =& -\frac{1}{t^2}\left(\frac{\beta^2}{\beta^2-1}\right) + \frac{c}{t^4}\left[\frac{(2\beta^2-\beta-1)D_1 t^{\beta} + (2\beta^2+\beta-1)D_2 t^{-\beta}}{[D_1(\beta-1)t^{\beta} - D_2(\beta+1)t^{-\beta}]^3}\right] - \\ & - 4k^2 s_1^2 \ , \\ k\rho =& \frac{\beta}{t^2(\beta^2-1)}\left[\frac{(\beta+2)D_1 t^{\beta} + (\beta-2)D_2 t^{-\beta}}{D_1 t^{\beta} + D_2 t^{-\beta}}\right] - 4k^2 s_1^2 + \\ & + \frac{c}{t^4}\left[\frac{(1+2\beta)D_1 t^{\beta} + (1-2\beta)D_2 t^{-\beta}}{(D_1 t^{\beta} + D_2 t^{-\beta})[D_1(\beta-1)t^{\beta} - D_2(\beta+1)t^{-\beta}]^2}\right] \ , \qquad (4.4.26) \end{aligned}$$

where in this case

$$s_1^2 = \frac{h_1}{t^4}[D_1 t^{\beta} + D_2 t^{-\beta}]^{-2} \ . \qquad (4.4.27)$$

For the space-time metric (4.4.25), the kinematical parameters take the form

$$\begin{aligned} \theta =& \frac{1}{t^2}\left\{c[D_1(\beta-1)t^{\beta} - D_2(\beta+1)t^{-\beta}]^{-2} + \frac{t^2}{\beta^2-1}\right\}^{1/2} \cdot \\ & \cdot \left[\frac{D_1(2+\beta)t^{\beta} + D_2(2-\beta)t^{-\beta}}{D_1 t^{\beta} + D_2 t^{-\beta}}\right] \ , \\ \sigma_{11} =& \sigma_{22} = -\sigma_{12} = -\sigma_{34} = \frac{1}{3t^2}\Bigg\{c[D_1(\beta-1)t^{\beta} - D_2(\beta+1)t^{-\beta}]^{-2} + \\ & + \frac{t^2}{\beta^2-1}\Bigg\}^{1/2}\left[\frac{D_1(1-\beta)t^{\beta} + D_2(1+\beta)t^{-\beta}}{D_1 t^{\beta} + D_2 t^{-\beta}}\right] \ , \\ \dot{u}_1 =& \dot{u}_2 = \dot{u}_3 = \dot{u}_4 = 0 \ , \\ W_{34} =& - W_{43} = -2ks_1 \ . \qquad (4.4.28) \end{aligned}$$

If we take the **special case for which** $\lambda = d_3$ (constant) then from equation (4.4.1) we obtain

$$ds^2 = \left(\frac{t^2}{d_1 - t^2}\right) dt^2 - d_2 dr^2 - t^2(d\theta^2 + sin^2\theta d\phi^2) \ . \tag{4.4.29}$$

Where d_1 is an arbitrary constant. The pressure and density are given by

$$kp = k\rho = \frac{d_1}{t^4} - 4k^2 s_1^2 \ . \tag{4.4.30}$$

The kinematical parameters for the metric (4.4.29) reduce to

$$\begin{aligned}
\theta =& \frac{2}{t^2}\left[d_1 - t^2\right]^{1/2} \ , \\
\sigma_{11} =& \sigma_{22} = -\sigma_{12} = -\sigma_{34} = \frac{1}{3t^2}\left[d_1 - t^2\right]^{1/2} \ , \\
\dot{u}_1 =& \dot{u}_2 = \dot{u}_3 = \dot{u}_4 = 0 \ , \\
W_{34} =& - W_{43} = -2ks_1 \ .
\end{aligned} \tag{4.4.31}$$

If the spin component $s_1 = 0$, our results agree with the result obtained by Sharif and Iqbal [113].

Hence in all above cases in (A) our solutions are non-static and have non-zero expansion, shear and rotation but zero acceleration.

Case B:

When $B = B(r) \Rightarrow \lambda = \lambda(r)$ and $\nu = \nu(r)$. In particular $B(r) = r$. In this case the equation (4.3.7) gives

$$\nu^{''} - \lambda^{'}\nu^{'} + \nu^{'2} - \frac{1}{r}(\lambda^{'} + \nu^{'}) - \frac{1}{r^2} = -\frac{e^{2\lambda}}{r^2} \ . \tag{4.4.32}$$

To solve the equation (4.4.32), we define

$$y(r) = e^{\nu} \qquad x(r) = e^{-2\lambda} \; . \tag{4.4.33}$$

Then (4.4.32) may be written as

$$x^{'} + \frac{2(r^2y^{''} - ry^{'} - y)}{r(ry^{'} + y)}x = -\frac{2y}{r^2y^{'} + yr} \; . \tag{4.4.34}$$

This is linear equation in x, provided y is known. Hence its solution is given by

$$x(r) = exp[-F(r)]\left\{\int^{r} exp[F(r)]g(r)dr + c\right\} \; ,$$

where

$$f(r) = \frac{2(r^2y^{''} - ry^{'} - y)}{r(ry^{'} + y)} \; , \qquad F(r) = \int^{r} f(u)du \; ,$$
$$g(r) = -\frac{2y}{r^2y^{'} + yr} \; , \tag{4.4.35}$$

and c is constant of integration.

The remaining equations (4.4.5) and (4.4.6) gives p and ρ as

$$kp = -\frac{1}{r^2} + \frac{2\nu^{'}}{r}e^{-2\lambda} + \frac{1}{r^2}e^{-2\lambda} - 4k^2s_1^2 \; , \tag{4.4.36}$$

and

$$k\rho = \frac{1}{r^2} + \frac{2\lambda^{'}}{r}e^{-2\lambda} - \frac{1}{r^2}e^{-2\lambda} - 4k^2s_1^2 \; . \tag{4.4.37}$$

We choose y in such a manner that equation (4.4.34) can be immediately integrated. We assume that y satisfies the Cauchy equation

$$r^2y^{''} - ry^{'} + (1 - q^2)y = 0 \; . \tag{4.4.38}$$

Case i: When $-1 < q < 1$

In this case the solution of the equation (4.4.38) is obtained as

$$e^{\nu} = y = a_1 r^{1+q} + a_2 r^{1-q} \ , \tag{4.4.39}$$

where a_1 , a_2 are arbitrary constants. For this value of y, the equation (4.4.34) becomes

$$x^{'} + \frac{2(q^2-2)(a_1 r^{2q} + a_2)}{r[(q+2)a_1 r^{2q} + (2-q)a_2]} x = -\frac{2(a_1 r^{2q} + a_2)}{r[(q+2)a_1 r^{2q} + (2-q)a_2]} \ . \tag{4.4.40}$$

Solving this equation by using the Mathematical software "Mathematica 10", we obtain the value of x in the form

$$e^{-2\lambda} = x = \frac{1}{2-q^2} + c r^{2\frac{(2-q^2)}{(2-q)}} \left[(2+q)a_1 r^{2q} + (2-q)a_2\right]^{-2\frac{(2-q^2)}{(4-q^2)}}. \tag{4.4.41}$$

Hence the metric (4.2.1) becomes

$$ds^2 = [a_1 r^{1+q} + a_2 r^{1-q}]^2 dt^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) - $$
$$- \left[\frac{1}{2-q^2} + c r^{2\frac{(2-q^2)}{(2-q)}} \left[(2+q)a_1 r^{2q} + (2-q)a_2\right]^{-2\frac{(2-q^2)}{(4-q^2)}}\right]^{-1} dr^2 \ . \tag{4.4.42}$$

Now from equations (4.4.36) and (4.4.37), we obtain pressure and density as

$$kp = \frac{-1}{r^2} + \frac{[a_1(3+2q)r^{2q} + a_2(3-2q)]}{r^2(a_1 r^{2q} + a_2)} \left[-\frac{1}{q^2-2} + \right.$$

$$+ cr^{2\frac{(2-q^2)}{(2-q)}}\left[(2+q)a_1r^{2q}+(2-q)a_2\right]^{-2\frac{(2-q^2)}{(4-q^2)}}\Bigg] - 4k^2s_1^2 \ ,$$

$$k\rho = \frac{1}{r^2} + \frac{1}{r^2}\left\{\frac{1}{q^2-2} + cr^{2\frac{(2-q^2)}{(2-q)}}\left[(2+q)a_1r^{2q}+(2-q)a_2\right]^{-2\frac{(2-q^2)}{(4-q^2)}}\right\} \cdot$$
$$\cdot\left[1+\frac{2(2-q^2)}{2-q} - \frac{4q(2-q^2)a_1r^{2q}}{(2-q)[(2+q)a_1r^{2q}+(2-q)a_2]}\right] \ , \quad (4.4.43)$$

and

$$s_1^2 = \frac{h_1}{r^4}\left[\frac{1}{2-q^2} + cr^{2\frac{(2-q^2)}{(2-q)}}\left[(2+q)a_1r^{2q}+(2-q)a_2\right]^{-2\frac{(2-q^2)}{(4-q^2)}}\right] \ . \quad (4.4.44)$$

The tetrad components of the kinematical parameters cited in the equation (1.4.27) for the space-time metric (4.4.42) yield

$$\theta = 0 \ ,$$
$$\sigma_{\alpha\beta} = 0 \ , \quad \forall \quad \alpha, \beta \ .$$

$$\dot{u}_1 = -\dot{u}_2 = -\frac{1}{r\sqrt{2}}\left[\frac{(1+q)a_1r^{2q}+(1-q)a_2}{a_1r^{2q}+a_2}\right] \cdot$$
$$\cdot\left\{\frac{1}{2-q^2} + cr^{2\frac{(2-q^2)}{(2-q)}}\left[(2+q)a_1r^{2q}+(2-q)a_2\right]^{-2\frac{(2-q^2)}{(4-q^2)}}\right\},$$
$$W_{34} = -W_{43} = -2ks_1 \ . \quad (4.4.45)$$

Case ii: When $q = 1$

In this case the solution of the equation (4.4.38) is given by

$$e^{\nu} = y = b_1r^2 + b_2 \ , \quad (4.4.46)$$

where b_1, b_2 are arbitrary constants. For this value of y, the equation (4.4.34) becomes

$$x' - \frac{2(b_1 r^2 + b_2)}{r(3b_1 r^2 + b_2)} x = -2\frac{b_1 r^2 + b_2}{r(3b_1 r^2 + b_2)} \ . \tag{4.4.47}$$

Solving this equation we obtain the value of x in the form

$$e^{-2\lambda} = x = 1 + \frac{c_1 r^2}{(3b_1 r^2 + b_2)^{2/3}} \ . \tag{4.4.48}$$

The metric (4.2.1) takes the form

$$ds^2 =[b_1 r^2 + b_2]^2 dt^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) - \left[1 + \frac{c_1 r^2}{(3b_1 r^2 + b_2)^{2/3}}\right]^{-1} dr^2 \ , \tag{4.4.49}$$

with the pressure and density given by

$$\begin{aligned} kp =& \frac{4b_2}{(b_1 + b_2 r^2)} + \frac{c_1}{(b_1 + 3b_2 r^2)^{2/3}} + \frac{4cb_2 r^2}{(b_1 + b_2 r^2)(b_1 + 3b_2 r^2)^{2/3}} - 4k^2 s_1^2 \ , \\ k\rho =& -\frac{c_1(3b_1 + 5b_2 r^2)}{(b_1 + 3b_2 r^2)^{5/3}} - 4k^2 s_1^2 \ , \end{aligned} \tag{4.4.50}$$

and

$$s_1^2 = \frac{h_1}{r^4}\left[1 + \frac{c_1 r^2}{(3b_1 r^2 + b_2)^{2/3}}\right] \ . \tag{4.4.51}$$

We record below the tetrad components of the kinematical parameters for the space-time metric (4.4.49) as

$$\begin{aligned} \theta =& 0 \ , \\ \sigma_{\alpha\beta} =& 0 \ , \quad \forall \quad \alpha, \beta \ . \end{aligned}$$

$$\dot{u}_1 = -\,\dot{u}_2 = -\frac{\sqrt{2}b_1 r}{(b_1 + b_2 r^2)}\left[1 + \frac{c_1 r^2}{(3b_1 r^2 + b_2)^{2/3}}\right] ,$$

$$W_{34} = -\,W_{43} = -2ks_1 \; . \tag{4.4.52}$$

Hence in all above cases in (B) our solutions are non-static and have non-zero acceleration, rotation but zero shear and expansion.

Discussions

Two classes A and B of different solutions of the field equations in the EC theory of gravitation are obtained when the Weyssenhoff fluid is the source of gravitation and spin. Many of the previously known solutions for Weyssenhoff fluid in EC theory of gravitation have zero acceleration and vorticity (Kuchowicz [82]). Griffiths and Jogia [42] have claimed some non-zero accelerated solutions. In this chapter we have applied the techniques of differential forms and a class A of non-static solutions with zero acceleration and a class B with non-zero acceleration are obtained. Class A solutions are expanding, shearing and rotating, while the class B solutions are rotating. In class A solutions, the non-zero kinematical parameters, the pressure and the density diverge to infinity, and vanish together at $t = 0$ and at large t respectively. Similar phenomenon is observed in class B solutions at $r = 0$ and at large r respectively. It can be seen that the rotation, the pressure and the density are influenced by the spin of the fluid, while there is no such effect on the expansion, acceleration and the shear. In the absence of the spin the result (4.4.29) coincides with the result obtained by Sharif and Iqbal [113], and the solution is irrotational.

Chapter 5

A Static Spherically Symmetric Solutions in Einstein-Cartan Theory of Gravitation

5.1 Introduction

The well known Einstein's general relativity theory provides a unified description of gravity as the geometric property of space-time. The recent detection of gravitational waves in the space-time as was predicted by Einstein 100 years before cemented the status of general relativity, besides other confirmations of Einstein's predictions of deflection of a ray of light by the gravitational field of the Sun and the perihelion advances of the planet Mercury. In general relativity theory the underlying Riemannian space-time deals with the case where connections are symmetric admitting a Riemann curvature tensor $\hat{R}_{hijk}$ which satisfies the properties

$$\begin{aligned}
\hat{R}_{(hi)jk} = \hat{R}_{hi(jk)} = 0 \ , \\
\hat{R}_{hijk} = \hat{R}_{jkhi} \ , \\
\hat{R}_{hijk} + \hat{R}_{hjki} + \hat{R}_{hkij} = 0 \ , \\
\hat{R}_{hijk;l} + \hat{R}_{hikl;j} + \hat{R}_{hilj;k} = 0 \ ,
\end{aligned} \tag{5.1.1}$$

where $\hat{R}_{hijk}$ is computed from symmetric Christoffel symbols in the usual way. It is wel-known that vanishing of the divergence of Einstein tensor in Einstein's general relativity theory follows from the Bianchi identities and from it follows the dynamical conservation laws. It has been shown that the Einstein's field equations evolve singularities which is rather unsatisfactory feature.

The successful geometrization of gravitation in Einstein's general relativity stimulated the interest of the great mathematician E. Cartan [11, 12] who suggested a more general geometrical frame work incorporating the notion of torsion as well as curvature.The modification of Einstein's general relativity theory allowing space-time to have torsion in addition to curvature is known as Einstein-Cartan theory of gravitation. When cosmological models with torsion were first studied, it was hoped that the inclusion of torsion would help to avoid singularities. For a long time, Cartan's modified theory of gravity was unfamiliar to physicists and did not attract any attention. But the role of Cartan was soon recognized, when the theory of gravitation with spin and torsion was independently rediscovered by Sciama [111] and Kibble [69]. Since then the Einstein-Cartan theory of gravitation gained the attention of researchers and become a very active field of research. Now the theory has gained a strong theoretical ground both geometrically and physically through the investigation of various authors like Tolman [130], Kuchowicz [78], Trautman [129], Hehl [48, 49], Kerlick [67, 68], Bohmer [6], Prasanna [100], Hehl and Collaborators [51], Singh, T. and Yadav [115], Katkar [61], Katkar and Patil [60], Kalyanshetti and Waghmode [66] in the form of viable rival theory to Einstein's general relativity theory.

In Einstein-Cartan theory of gravitation the underlying geometry is non-Riemannian due to asymmetric connections arising from the presence of torsion in the space-time.

In this chapter a static spherically symmetric solution of Einstein-Cartan field equations is obtained by using the techniques of differential forms. The material of the chapter is organised as below. In the Section 2, the static spherically symmetric metric is considered and following the tetrad algorithm the tetrad components of connection 1-forms, curvature 2-forms, the Riemann curvature tensor and the Ricci tensor are derived. A solutions of the field equations are obtained in the Section 3, and the chapter is concluded along with some discussions in the last Section 4.

5.1.1 The Kinematical Parameters

The kinematical parameters viz; the expansion θ, the acceleration vector $\dot{u}_i$, the shear tensor σ_{ij} and the rotation tensor W_{ij} are respectively defined as

$$\begin{aligned} \theta &= u^i{}_{;i} \ , \\ \dot{u}_i &= u_{i;k} u^k \ , \\ \sigma_{ij} &= u_{(i;j)} - \dot{u}_{(i} u_{j)} + \frac{1}{3}\theta h_{ij} \ , \end{aligned}$$

and

$$W_{ij} = u_{[i;j]} - \dot{u}_{[i} u_{j]} \ , \qquad (5.1.2)$$

where $h_{ij} = g_{ij} - u_i u_j$ is the 3-dimension projection operator. We use the definition of the covariant derivative (1.2.19) for the unit flow vector u_i and by virtue of equations (1.2.16), (1.2.49), (1.2.53) and (1.2.55) find

$$u_{i;j} = u_{i/j} + kS_{ij} \ . \tag{5.1.3}$$

It is obvious exercise to find the Newman-Penrose concomitants of the kinematical parameters by using the equation (5.1.3) as

$$\begin{aligned}
\theta =& \frac{1}{\sqrt{2}}\left(\epsilon^0 + \overline{\epsilon}^0 - \gamma^0 - \overline{\gamma}^0 - \rho^0 - \overline{\rho}^0 + \mu^0 + \overline{\mu}^0\right) \ , \\
\dot{u}_i =& \frac{1}{2}\left[(\epsilon^0 + \overline{\epsilon}^0 + \gamma^0 + \overline{\gamma}^0)(l_i - n_i) - (\overline{\tau}^0 + \overline{\kappa}^0 - \nu^0 - \pi^0)m_i - c.c.\right] \ , \\
\sigma_{ij} =& \frac{1}{6\sqrt{2}}\big\{\big[2(\gamma^0 + \overline{\gamma}^0 - \epsilon^0 - \overline{\epsilon}^0) - (\rho^0 + \overline{\rho}^0 - \mu^0 - \overline{\mu}^0)\big]\big(l_i l_j + n_i n_j - \\
& - 2l_{(i}n_{j)} - 2m_{(i}\overline{m}_{j)}\big) + 3\big[\big(\overline{\kappa}^0 - \overline{\tau}^0 + \nu^0 - \pi^0 - \\
& - 2(\alpha^0 + \overline{\beta}^0)\big)\big(l_{(i}m_{j)} - m_{(i}n_{j)}\big) + 2(\overline{\sigma}^0 - \lambda^0)m_i m_j\big] + c.c.\big\} \ , \\
W_{ij} =& \frac{1}{2\sqrt{2}}\big[\big(\overline{\tau}^0 + \pi^0 - \overline{\kappa}^0 - \nu^0 - 2(\alpha^0 + \overline{\beta}^0)\big)\big(l_{[i}m_{j]} + m_{[i}n_{j]}\big) + c.c. + \\
& + 2\big(\rho^0 - \overline{\rho}^0 + \mu^0 - \overline{\mu}^0\big)\overline{m}_{[i}m_{j]}\big] + kS_{ij} \ .
\end{aligned} \tag{5.1.4}$$

5.2 Static Spherically Symmetric Metric

Consider a static spherically symmetric metric in the form

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \ , \tag{5.2.1}$$

where λ and ν are functions of r only. Define the tetrad basis 1-forms θ^α for the metric (5.2.1) as

$$\begin{aligned}
\theta^1 =& \frac{1}{\sqrt{2}}(e^{\frac{\nu}{2}}dt + e^{\frac{\lambda}{2}}dr) \ , \\
\theta^2 =& \frac{1}{\sqrt{2}}(e^{\frac{\nu}{2}}dt - e^{\frac{\lambda}{2}}dr) \ , \\
\theta^3 =& -\frac{1}{\sqrt{2}}(rd\theta - irsin\theta d\phi) \ ,
\end{aligned} \tag{5.2.2}$$

where θ^4 is a complex conjugate of θ^3. Hence the metric (5.2.1) can be written as

$$ds^2 = 2\theta^1\theta^2 - 2\theta^3\theta^4. \tag{5.2.3}$$

Using equations (2.4.46) and (5.2.2) we obtain readily the components of the basis vector fields as

$$\begin{aligned}
l_i =& \frac{1}{\sqrt{2}}\left(-e^{\frac{\lambda}{2}}, 0, 0, e^{\frac{\nu}{2}}\right) \ , \\
n_i =& \frac{1}{\sqrt{2}}\left(e^{\frac{\lambda}{2}}, 0, 0, e^{\frac{\nu}{2}}\right) \ , \\
m_i =& \frac{1}{\sqrt{2}}\left(0, r, irsin\theta, 0\right) \ ,
\end{aligned} \tag{5.2.4}$$

where $\overline{m}_i$ is a complex conjugate of m_i. The contravariant components of the null basis vectors are obtain by raising the index by the metric tensor as

$$l^i = g^{ik}l_k = \frac{1}{\sqrt{2}}\left(e^{-\frac{\lambda}{2}}, 0, 0, e^{-\frac{\nu}{2}}\right) \ ,$$

similarly, we obtain

$$n^i = \frac{1}{\sqrt{2}}\left(-e^{-\frac{\lambda}{2}}, 0, 0, e^{-\frac{\nu}{2}}\right) \ , \tag{5.2.5}$$
$$m^i = -\frac{1}{\sqrt{2}}\left(0, r^{-1}, ir^{-1}cosec\theta, 0\right) \ ,$$

where $\overline{m}^i$ is obtained from m^i by taking its complex conjugate. The tetrad form of the equation (2.4.22) becomes

$$d_*^2 f = -\frac{1}{2} f_{,\gamma} Q_{\alpha\beta}{}^{\gamma} \theta^\alpha \wedge \theta^\beta \ , \tag{5.2.6}$$

We can also write this equation as

$$d_*^2 f = -\frac{1}{2}\left[Df Q_{\alpha\beta}{}^1 + \Delta f Q_{\alpha\beta}{}^2 + \delta f Q_{\alpha\beta}{}^3 + \overline{\delta} f Q_{\alpha\beta}{}^4\right]\theta^\alpha \wedge \theta^\beta \ , \tag{5.2.7}$$

where

$$\begin{aligned} Df &= f_{;i} l^i = \frac{1}{\sqrt{2}}(e^{-\lambda/2} f_{,r} + e^{-\nu/2} f_{,t}) \ , \\ \Delta f &= f_{;i} n^i = \frac{1}{\sqrt{2}}(-e^{-\lambda/2} f_{,r} + e^{-\nu/2} f_{,t}) \ , \\ \delta f &= f_{;i} m^i = -\frac{1}{\sqrt{2}}(r^{-1} f_{,\theta} + ir^{-1} cosec\theta f_{,\phi}) \ , \\ \overline{\delta} f &= f_{;i} \overline{m}^i = -\frac{1}{\sqrt{2}}(r^{-1} f_{,\theta} - ir^{-1} cosec\theta f_{,\phi}) \ . \end{aligned} \tag{5.2.8}$$

Hence the equation (5.2.7) becomes

$$\begin{aligned} d_*^2 f = &-\frac{1}{2\sqrt{2}}\Big[e^{-\lambda/2} f_{,r}(Q_{\alpha\beta}{}^1 - Q_{\alpha\beta}{}^2) - r^{-1} f_{,\theta}(Q_{\alpha\beta}{}^3 + Q_{\alpha\beta}{}^4) - \\ &- ir^{-1} cosec\theta f_{,\phi}(Q_{\alpha\beta}{}^3 - Q_{\alpha\beta}{}^4) + e^{-\nu/2} f_{,t}(Q_{\alpha\beta}{}^1 + Q_{\alpha\beta}{}^2)\Big]\theta^\alpha \wedge \theta^\beta \ . \end{aligned} \tag{5.2.9}$$

It is evident from equations (5.2.9) that

$$
\begin{aligned}
d_*^2 r =& -\frac{e^{-\frac{\lambda}{2}}}{2\sqrt{2}}\left(Q_{\alpha\beta}{}^1 - Q_{\alpha\beta}{}^2\right)\theta^\alpha \wedge \theta^\beta \ , \\
d_*^2 \theta =& \frac{r^{-1}}{2\sqrt{2}}\left(Q_{\alpha\beta}{}^3 + Q_{\alpha\beta}{}^4\right)\theta^\alpha \wedge \theta^\beta \ , \\
d_*^2 \phi =& \frac{ir^{-1}cosec\theta}{2\sqrt{2}}\left(Q_{\alpha\beta}{}^3 - Q_{\alpha\beta}{}^4\right)\theta^\alpha \wedge \theta^\beta \ , \\
d_*^2 t =& -\frac{e^{-\frac{\nu}{2}}}{2\sqrt{2}}(Q_{\alpha\beta}{}^1 + Q_{\alpha\beta}{}^2)\theta^\alpha \wedge \theta^\beta .
\end{aligned}
\tag{5.2.10}
$$

Now operating d_* to the equations (5.2.2) and using the equations (5.2.10) we readily get

$$
\begin{aligned}
d_*\theta^1 &= \left(\frac{1}{2\sqrt{2}}\nu' e^{-\frac{\lambda}{2}}\right)\theta^{12} - \frac{1}{2}Q_{\alpha\beta}{}^1\theta^{\alpha\beta} \ , \\
d_*\theta^2 &= \left(\frac{1}{2\sqrt{2}}\nu' e^{-\frac{\lambda}{2}}\right)\theta^{12} - \frac{1}{2}Q_{\alpha\beta}{}^2\theta^{\alpha\beta} \ , \\
d_*\theta^3 &= \frac{1}{\sqrt{2}}\left[r^{-1}e^{-\frac{\lambda}{2}}(\theta^{13} - \theta^{23}) + r^{-1}cot\theta \ \ \theta^{34}\right] - \frac{1}{2}Q_{\alpha\beta}{}^3\theta^{\alpha\beta} \ , \\
d_*\theta^4 &= \frac{1}{\sqrt{2}}\left[r^{-1}e^{-\frac{\lambda}{2}}(\theta^{14} - \theta^{24}) - r^{-1}cot\theta \ \ \theta^{34}\right] - \frac{1}{2}Q_{\alpha\beta}{}^4\theta^{\alpha\beta} \ ,
\end{aligned}
\tag{5.2.11}
$$

where we have used $\theta^{\alpha\beta} = \theta^\alpha \wedge \theta^\beta$.

Now from equations (5.2.11) and (2.4.70) we obtain, after equating the corresponding coefficients and simplifying the values of NP spin coefficients of Riemann space-time as

$$
\begin{aligned}
\kappa^0 =& \lambda^0 = \sigma^0 = \pi^0 = \tau^0 = \nu^0 = 0 \ , \\
\rho^0 =& \mu^0 = -\frac{1}{\sqrt{2}}r^{-1}e^{-\frac{\lambda}{2}} \ ,
\end{aligned}
$$

$$\begin{aligned}\epsilon^0 =&\gamma^0 = \frac{1}{4\sqrt{2}}\nu' e^{-\frac{\lambda}{2}} \ ,\\ \alpha^0 =& -\beta^0 = \frac{r^{-1}}{2\sqrt{2}} cot\theta.\end{aligned} \tag{5.2.12}$$

By virtue of the equations (1.3.30) and (5.2.12), we find from the equations (2.4.69)

$$\begin{aligned}\omega_{12} =& -\frac{1}{2\sqrt{2}}\left[\nu' e^{-\frac{\lambda}{2}}(\theta^1-\theta^2) - 4ks_0\theta^3 - 4k\overline{s}_0\theta^4\right] \ ,\\ \omega_{13} =& \frac{1}{\sqrt{2}}\left[2ks_0\theta^1 + (r^{-1}e^{-\frac{\lambda}{2}} + 2ks_1)\theta^4\right] \ ,\\ \omega_{23} =& -\frac{1}{\sqrt{2}}\left[2ks_0\theta^2 + (r^{-1}e^{-\frac{\lambda}{2}} - 2ks_1)\theta^4\right] \ ,\\ \omega_{34} =& -\frac{1}{\sqrt{2}}\left[2ks_1(\theta^1+\theta^2) + r^{-1}cot\theta(\theta^3-\theta^4)\right].\end{aligned} \tag{5.2.13}$$

Now using equations (1.3.30), (5.2.12) and (5.2.13) we obtain from the equation (2.4.85) the tetrad components of curvature 2-form. These are listed below:

$$\begin{aligned}\Omega^1{}_1 =& -\left[\frac{e^{-\lambda}}{4}(2\nu'' - \lambda'\nu' + \nu'^2) + 4k^2 s_0\overline{s}_0\right]\theta^{12} + \left(ks_{0,r}e^{-\frac{\lambda}{2}} + ks_0\nu' e^{-\frac{\lambda}{2}} -\right.\\ & \left. - 2k^2 s_0 s_1\right)\theta^{13} + \left(k\overline{s}_{0,r}e^{-\frac{\lambda}{2}} + k\overline{s}_0\nu' e^{-\frac{\lambda}{2}} + 2k^2\overline{s}_0 s_1\right)\theta^{14} - \left(ks_{0,r}e^{-\frac{\lambda}{2}} + \right.\\ & \left. + ks_0\nu' e^{-\frac{\lambda}{2}} + 2k^2 s_0 s_1\right)\theta^{23} - \left(k\overline{s}_{0,r}e^{-\frac{\lambda}{2}} + k\overline{s}_0\nu' e^{-\frac{\lambda}{2}} - 2k^2\overline{s}_0 s_1\right)\theta^{24} +\\ & + \left[kr^{-1}cot\theta(s_0-\overline{s}_0) + 2ks_1 e^{-\frac{\lambda}{2}}(2r^{-1} - \nu')\right]\theta^{34} \ ,\end{aligned} \tag{5.2.14}$$

$$\begin{aligned}\Omega^1{}_3 =& -\left(ks_{0,r}e^{-\frac{\lambda}{2}} + ks_0\nu' e^{-\frac{\lambda}{2}} + 2k^2 s_0 s_1\right)\theta^{12} + \left[\frac{e^{-\lambda}}{4r}(\lambda'-\nu') + \right.\\ & \left. + \frac{1}{2}ks_1\nu' e^{-\frac{\lambda}{2}} + ks_{1,r}e^{-\frac{\lambda}{2}} + 2k^2{s_1}^2\right]\theta^{14} - \left(2k^2{s_0}^2 - ks_0 r^{-1}cot\theta\right)\theta^{23} -\\ & - \left[\frac{e^{-\lambda}r^{-1}}{4}(\lambda'+\nu') - \frac{1}{2}ks_1\nu' e^{-\frac{\lambda}{2}} + ks_{1,r}e^{-\frac{\lambda}{2}} + 2ks_1 r^{-1}e^{-\frac{\lambda}{2}} - 2k^2{s_1}^2 + \right.\end{aligned}$$

$$+ 2k^2 s_0 \bar{s}_0 + k s_0 r^{-1} cot\theta \Big] \theta^{24} + \left(k s_0 r^{-1} e^{-\frac{\lambda}{2}} - 2k^2 s_0 s_1\right) \theta^{34}, \tag{5.2.15}$$

$$\Omega^2{}_3 = \left(k s_{0,r} e^{-\frac{\lambda}{2}} + k s_0 \nu' e^{-\frac{\lambda}{2}} - 2k^2 s_0 s_1\right) \theta^{12} - \left(2k^2 s_0^2 + k s_0 r^{-1} cot\theta\right) \theta^{13} -$$
$$- \left[\frac{e^{-\lambda} r^{-1}}{4}(\lambda' + \nu') + \frac{1}{2} k s_1 \nu' e^{-\frac{\lambda}{2}} - k s_{1,r} e^{-\frac{\lambda}{2}} - 2k s_1 r^{-1} e^{-\frac{\lambda}{2}} - 2k^2 {s_1}^2 + \right.$$
$$\left. + 2k^2 s_0 \bar{s}_0 - k s_0 r^{-1} cot\theta\right] \theta^{14} + \left[\frac{e^{-\lambda} r^{-1}}{4}(\lambda' - \nu') - \frac{1}{2} k s_1 \nu' e^{-\frac{\lambda}{2}} - \right.$$
$$\left. - k s_{1,r} e^{-\frac{\lambda}{2}} + 2k^2 {s_1}^2\right] \theta^{24} + \left(k s_0 r^{-1} e^{-\frac{\lambda}{2}} + 2k^2 s_0 s_1\right) \theta^{34} \ , \tag{5.2.16}$$

$$\Omega^3{}_3 = -e^{-\frac{\lambda}{2}} \left(2k s_{1,r} + k s_1 \nu'\right) \theta^{12} + k s_0 \left(r^{-1} e^{-\frac{\lambda}{2}} + 2k s_1\right) \theta^{13} -$$
$$- k \bar{s}_0 \left(r^{-1} e^{-\frac{\lambda}{2}} - 2k s_1\right) \theta^{14} + k s_0 \left(r^{-1} e^{-\frac{\lambda}{2}} - 2k s_1\right) \theta^{23} -$$
$$- k \bar{s}_0 \left(r^{-1} e^{-\frac{\lambda}{2}} + 2k s_1\right) \theta^{24} + \left[r^{-2}(1 - e^{-\lambda}) - 4k^2 {s_1}^2\right] \theta^{34} \ . \tag{5.2.17}$$

The expressions for $\Omega^1{}_4, \Omega^2{}_4, \Omega^4{}_4$ are obtained by interchanging the suffixes 3 and 4 and taking the complex conjugate of the right hand sides of the equations in (5.2.15), (5.2.16) and (5.2.17) respectively.

5.2.1 Tetrad Components of Riemann-Cartan Curvature Tensor

From the equation (2.4.65) we obtain

$$\Omega_{\alpha\beta} = R_{12\alpha\beta}\theta^{12} + R_{13\alpha\beta}\theta^{13} + R_{14\alpha\beta}\theta^{14} + R_{23\alpha\beta}\theta^{23} + R_{24\alpha\beta}\theta^{24} + R_{34\alpha\beta}\theta^{34} \ , \tag{5.2.18}$$

where

$$\Omega_{\alpha\beta} = -\Omega_{\beta\alpha} \ , \quad \alpha, \beta = 1, 2, 3, 4.$$

By giving different values to $\alpha, \beta = 1, 2, 3, 4$ in the equation (5.2.18) and then equating the corresponding coefficients of basis 2-forms of equations (5.2.14), (5.2.15), (5.2.16) and (5.2.17) we readily obtain the tetrad components of Riemann-Cartan curvature tensor as

$$\begin{aligned}
R_{1212} =& -\frac{e^{-\lambda}}{4}\left(2\nu'' - \lambda'\nu' + \nu'^2\right) - 4k^2 s_0 \overline{s}_0 \ ,\\
R_{1312} =& k s_{0,r} e^{-\frac{\lambda}{2}} + k s_0 \nu' e^{-\frac{\lambda}{2}} - 2k^2 s_0 s_1 \ ,\\
R_{2312} =& -\left(k s_{0,r} e^{-\frac{\lambda}{2}} + k s_0 \nu' e^{-\frac{\lambda}{2}} + 2k^2 s_0 s_1\right) \ ,\\
R_{3412} =& k r^{-1} cot\theta \left(s_0 - \overline{s}_0\right) + 2k s_1 e^{-\frac{\lambda}{2}}\left(2r^{-1} - \nu^{'}\right) \ ,\\
R_{1213} =& k s_{0,r} e^{-\frac{\lambda}{2}} + k s_0 \nu' e^{-\frac{\lambda}{2}} - 2k^2 s_0 s_1 \ ,\\
R_{1313} =& -\left(2k^2 s_0^2 + k s_0 r^{-1} cot\theta\right) \ ,\\
R_{1413} =& -\frac{e^{-\lambda} r^{-1}}{4}\left(\lambda' + \nu'\right) - \frac{1}{2} k s_1 \nu' e^{-\frac{\lambda}{2}} + k s_{1,r} e^{-\frac{\lambda}{2}} + 2k s_1 r^{-1} e^{-\frac{\lambda}{2}} +\\
& + 2k^2 {s_1}^2 - 2k^2 s_0 \overline{s}_0 + k s_0 r^{-1} cot\theta \ ,\\
R_{2413} =& \frac{e^{-\lambda} r^{-1}}{4}\left(\lambda' - \nu'\right) - \frac{1}{2} k s_1 \nu' e^{-\frac{\lambda}{2}} - k s_{1,r} e^{-\frac{\lambda}{2}} + 2k^2 {s_1}^2 \ ,\\
R_{3413} =& k s_0 r^{-1} e^{-\frac{\lambda}{2}} + 2k^2 s_0 s_1 \ ,\\
R_{1223} =& -\left(k s_{0,r} e^{-\frac{\lambda}{2}} + k s_0 \nu' e^{-\frac{\lambda}{2}} + 2k^2 s_0 s_1\right) \ ,\\
R_{1423} =& \frac{e^{-\lambda} r^{-1}}{4}(\lambda' - \nu') + \frac{1}{2} k s_1 \nu' e^{-\frac{\lambda}{2}} + k s_{1,r} e^{-\frac{\lambda}{2}} + 2k^2 {s_1}^2 \ ,\\
R_{2323} =& -\left(2k^2 {s_0}^2 - k s_0 r^{-1} cot\theta\right) \ ,
\end{aligned}$$

$$
\begin{aligned}
R_{2423} =& -\frac{e^{-\lambda}r^{-1}}{4}\left(\lambda' + \nu'\right) + \frac{1}{2}ks_1\nu' e^{-\frac{\lambda}{2}} - ks_{1,r}e^{-\frac{\lambda}{2}} - 2ks_1 r^{-1}e^{-\frac{\lambda}{2}} + \\
& + 2k^2 {s_1}^2 - 2k^2 s_0 \overline{s}_0 - ks_0 r^{-1} cot\theta \ , \\
R_{3423} =& ks_0 r^{-1} e^{-\frac{\lambda}{2}} - 2k^2 s_0 s_1 \ , \\
R_{1234} =& - e^{-\frac{\lambda}{2}}\left(2ks_{1,r} + ks_1\nu'\right) \ , \\
R_{1334} =& ks_0 \left(r^{-1}e^{-\frac{\lambda}{2}} + 2ks_1\right) \ , \\
R_{2334} =& ks_0 \left(r^{-1}e^{-\frac{\lambda}{2}} - 2ks_1\right) \ , \\
R_{3434} =& r^{-2}\left(1 - e^{-\lambda}\right) - 4k^2 {s_1}^2 \ ,
\end{aligned}
\tag{5.2.19}
$$

and

$$R_{2313} = R_{1323} = 0 \ .$$

The complex conjugates of above equations are obtained by interchanging the suffixes 3 and 4 and taking the complex conjugates of the right hand sides of the respective equations.

5.2.2 Tetrad Components of Ricci-Cartan Tensor and Ricci-Cartan Curvature Scalar

The tetrad components of the Ricci-Cartan tensor and Ricci-Cartan curvature scalar are defined by

$$R_{\alpha\beta} = \eta^{\nu\epsilon} R_{\nu\alpha\beta\epsilon} \quad , R = \eta^{\alpha\beta} R_{\alpha\beta} \ ,$$

$$\Rightarrow \qquad R_{\alpha\beta} = R_{1\alpha\beta 2} + R_{2\alpha\beta 1} - R_{3\alpha\beta 4} - R_{4\alpha\beta 3} \ . \tag{5.2.20}$$

Using equations (5.2.19) we obtain from equations (5.2.20) expressions for Ricci-Cartan tensors

$$
\begin{aligned}
R_{11} =& - e^{-\lambda}r^{-1}(\frac{\lambda' + \nu'}{2}) + kr^{-1}cot\theta(s_0 + \overline{s}_0) + 4k^2 {s_1}^2 - 4k^2 s_0\overline{s}_0 \ ,\\
R_{12} =& - \frac{e^{-\lambda}}{4}\left(2\nu'' - \lambda'\nu' + \nu'^2\right) + \frac{r^{-1}e^{-\lambda}}{2}\left(\lambda' - \nu'\right) + 4k^2 {s_1}^2 - 4k^2 s_0\overline{s}_0 \ ,\\
R_{13} =& ks_{0,r}e^{-\frac{\lambda}{2}} + ks_0 e^{-\frac{\lambda}{2}}(\nu' + r^{-1}) - 4k^2 s_0 s_1 \ ,\\
R_{21} =& - \frac{e^{-\lambda}}{4}\left(2\nu'' - \lambda'\nu' + \nu'^2\right) + \frac{r^{-1}e^{-\lambda}}{2}(\lambda' - \nu') + 4k^2 {s_1}^2 - 4k^2 s_0\overline{s}_0 \ ,\\
R_{22} =& - e^{-\lambda}r^{-1}\left(\frac{\lambda' + \nu'}{2}\right) - kr^{-1}cot\theta(s_0 + \overline{s}_0) + 4k^2 {s_1}^2 - 4k^2 s_0\overline{s}_0 \ ,\\
R_{23} =& ks_{0,r}e^{-\frac{\lambda}{2}} + ks_0 e^{-\frac{\lambda}{2}}(\nu' + r^{-1}) + 4k^2 s_0 s_1 \ ,\\
R_{31} =& ks_{0,r}e^{-\frac{\lambda}{2}} - ks_0 r^{-1}e^{-\frac{\lambda}{2}}(\nu' - r^{-1}) \ ,\\
R_{32} =& ks_{0,r}e^{-\frac{\lambda}{2}} - ks_0 r^{-1}e^{-\frac{\lambda}{2}}\left(\nu' - r^{-1}\right) \ , \qquad\qquad (5.2.21)\\
R_{34} =& - r^{-2} + e^{-\lambda}\left[r^{-2} - \frac{r^{-1}}{2}(\lambda' - \nu')\right] \ ,\\
R_{33} =& 0 \ .
\end{aligned}
$$

The Ricci-Cartan curvature scalar is given by

$$
\begin{aligned}
R =& - e^{-\lambda}\left(\nu'' - \frac{\lambda'\nu'}{2} + \frac{\nu'^2}{2} + 2\frac{(\nu' - \lambda')}{r} + 2r^{-2}\right) + 2r^{-2} + 8k^2 s_1^2 - \\
& - 8k^2 s_0\overline{s}_0 \ . \qquad\qquad (5.2.22)
\end{aligned}
$$

By virtue of the equations (5.2.12) the equations (5.1.4) reduce to

$$
\begin{aligned}
\theta =& 0 \ ,\\
\dot{u}_i =& \frac{1}{2\sqrt{2}}e^{-\frac{\lambda}{2}}\nu'(l_i - n_i) \ ,
\end{aligned}
$$

$$\sigma_{ij} = 0 \ , \tag{5.2.23}$$

$$W_{ij} = 2k[2s_1 m_{[i}\overline{m}_{j]} + \overline{s}_0(l_{[i}m_{j]} + m_{[i}n_{j]}) + c.c.] \ .$$

5.3 Field Equations and Solutions

To find the information of the pertaining space-time geometry, we continue to use tetrad formalism. Thus the tetrad form of the Einstein-Cartan field equations (1.2.48) is given by

$$R_{\alpha\beta} - \frac{R}{2}\eta_{\alpha\beta} = -kt_{\alpha\beta} \ , \tag{5.3.1}$$

where the tetrad components of the energy momentum tensor (1.4.22) are obtained as

$$\begin{aligned} t_{11} &= t_{22} = \frac{1}{2}(\rho + p), \quad t_{12} = t_{21} = \frac{1}{2}(\rho - p), \\ t_{34} &= t_{43} = p \ , \quad t_{31} = t_{32} = \frac{1}{2}\nu' e^{-\frac{\lambda}{2}} s_0 \ , \\ t_{41} &= t_{42} = \frac{1}{2}\nu' e^{-\frac{\lambda}{2}} \overline{s}_0 \ , \end{aligned} \tag{5.3.2}$$

and the remaining tetrad components are zero.

Using above equations (5.2.21), (5.2.22) and (5.3.2) in equation (5.3.1) the independent field equations for gravitation in Einstein-Cartan theory are obtained as

$$-e^{-\lambda}r^{-1}\left(\frac{\lambda' + \nu'}{2}\right) + 4k^2{s_1}^2 - 4k^2 s_0\overline{s}_0 + kr^{-1}cot\theta(s_0 + \overline{s}_0) = -\frac{k}{2}(\rho + p) \ ,$$

$$-r^{-2} + e^{-\lambda}\left[r^{-2} - \frac{r^{-1}}{2}\left(\lambda' - \nu'\right)\right] = -\frac{k}{2}(\rho - p) \ ,$$

$$ks_{0,r}e^{-\frac{\lambda}{2}} + ks_0e^{-\frac{\lambda}{2}}\left(\nu' + r^{-1}\right) - 4k^2s_0s_1 = 0 \ ,$$
$$- e^{-\lambda}r^{-1}\left(\frac{\lambda' + \nu'}{2}\right) + 4k^2{s_1}^2 - 4k^2s_0\bar{s}_0 - kr^{-1}cot\theta(s_0 + \bar{s}_0) = -\frac{k}{2}(\rho + p) \ ,$$
$$ks_{0,r}e^{-\frac{\lambda}{2}} + ks_0e^{-\frac{\lambda}{2}}\left(\nu' + r^{-1}\right) + 4k^2s_0s_1 = 0 \ , \tag{5.3.3}$$
$$ks_{0,r}e^{-\frac{\lambda}{2}} + \frac{1}{2}ks_0e^{-\frac{\lambda}{2}}\left(3\nu' - 2r^{-1}\right) = 0 \ ,$$
$$- \frac{e^{-\lambda}}{4}\left(2\nu'' - \lambda'\nu' + \nu'^2\right) + \frac{r^{-1}e^{-\lambda}}{2}(\lambda' - \nu') + 4k^2{s_1}^2 - 4k^2s_0\bar{s}_0 = -kp \ .$$

We see from equations (5.3.3) that these equations are consistent provided that $s_0 = 0$. Hence out of the seven field equations, there exists only three independent field equations and are given below:

$$e^{-\lambda}r^{-1}\left(\nu' + r^{-1}\right) - r^{-2} - 4k^2{s_1}^2 = kp \ , \tag{5.3.4}$$
$$e^{-\lambda}r^{-1}\left(\lambda' - r^{-1}\right) + r^{-2} - 4k^2{s_1}^2 = k\rho \ , \tag{5.3.5}$$
$$e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r} - \frac{\nu'\lambda'}{4} - \frac{(1 + r\nu')}{r^2}\right) + r^{-2} = 0 \ . \tag{5.3.6}$$

By Birkhoff's theorem outside the field the solution is represented by Schwarzschild metric and is given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right) + \left(1 - \frac{2m}{r}\right)dt^2 \ , \tag{5.3.7}$$

where m is a constant associated with the mass of sphere. Hence we use the boundary conditions

$$(e^{\nu})_{r=a} = \left(e^{-\lambda}\right)_{r=a} = \left(1 - \frac{2m}{a}\right) \ , \tag{5.3.8}$$

and $p = 0$ at $r = a$.

5.4 Specific Solutions

The set of Einstein-Cartan field equations is highly non-linear and is formidable to solve by any analytical method. Hence we restrict ourselves to the following cases.

Case (I): We assume $e^{\nu}{=}{A_1}^2$, where A_1 is an arbitrary constant.

We obtain from the assumed equation $\nu = 2logA_1$, and hence from equation (5.3.6) we have

$$e^{-\lambda}\left(\frac{\lambda^{'}}{2r}+\frac{1}{r^2}\right)-r^{-2}=0\ .$$

Solving this equation we obtain

$$e^{-\lambda}=1+A_2r^2\ .$$

Hence the space-time metric (5.2.1) becomes

$$ds^2=-\frac{1}{(1+A_2r^2)}dr^2-r^2\left(d\theta^2+sin^2\theta d\phi^2\right)+{A_1}^2dt^2\ . \qquad (5.4.1)$$

Where the arbitrary constants A_1 and A_2 can be determined by matching the solutions at the boundary $r = a$ to Schwarzschild exterior solution. They are obtained as

$$A_1=\left(1-\frac{2m}{a}\right)^{1/2}\ ,\qquad A_2=-\frac{2m}{a^3}=-\frac{1}{R^2}.$$

The pressure and density are given by

$$kp=-\left(4k^2s_1^2+\frac{1}{R^2}\right),\qquad k\rho=-\left(4k^2s_1^2-\frac{3}{R^2}\right)\ . \qquad (5.4.2)$$

We also see in this case that the kinematical parameters θ, $\dot{u}_i$, σ_{ij} vanish and $W_{ij} = 4ks_1 m_{[i}\overline{m}_{j]}$. Hence we have non expanding, non-shearing, non-accelerating and rotating solution. Our solutionis analogous to the solution obtained by Prasanna [100].

Case (II): Now we take $e^{-(\lambda+\nu)}{=}{B_1}^2$, where B_1 is a constant.

Here from assumed equation, we find $e^{-\lambda} = {B_1}^2 e^{\nu}$ and $\lambda^{'} = -\nu^{'}$. Hence the equation (5.3.6) becomes

$$ {B_1}^2 e^{\nu} \left(\frac{\nu^{''}}{2} + \frac{\nu^{'2}}{2} - \frac{1}{r^2} \right) + \frac{1}{r^2} = 0 \ . $$

The solution of this equation is given by

$$ e^{-\lambda} = 1 + \frac{A{B_1}^2}{r} + B{B_1}^2 r^2 \ . $$

Hence the geometry of the space-time, in this case, is described by the metric

$$ ds^2 = -\left(1 + \frac{B_2}{r} + B_3 r^2\right)^{-1} dr^2 - r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right) + + \frac{1}{B_1^2}\left(1 + \frac{B_2}{r} + B_3 r^2\right) dt^2 \ , \tag{5.4.3} $$

where the constants $B_2 = A{B_1}^2$ and $B_3 = B{B_1}^2$ are specified by matching the solution to the exterior Schwarzschild solution at the boundary $r = a$. They are obtained as

$$ B_2 = -m \ , \qquad B_3 = -\frac{m}{a^3} \ . $$

Hence the pressure and density are given by

$$kp = -\left(4k^2 s_1^2 + \frac{3m}{a^3}\right), \qquad k\rho = -\left(4k^2 s_1^2 - \frac{3m}{a^3}\right) \ . \tag{5.4.4}$$

We see, in this case that, the expansion and the shear vanish while acceleration and rotation are not. Hence the solution is rotating with non-zero acceleration but expansion free and shear free with negative pressure and density.

Case III: We now assume $e^{-\lambda} = 1 - \frac{r^2}{R^2}$.

This gives

$$\lambda' = \frac{2r}{R^2}\left(1 - \frac{r^2}{R^2}\right)^{-1} \ .$$

Hence the equation (5.3.6) reduces to

$$\left(\frac{\nu''}{2} + \frac{\nu'^2}{4}\right)\left(1 - \frac{r^2}{R^2}\right) - \frac{\nu'}{2r} = 0 \ .$$

Solving this equation, we obtain

$$e^{\nu} = \frac{1}{4}\left[D_2 - D_1 R^2 \left(1 - \frac{r^2}{R^2}\right)^{1/2}\right]^2 \ .$$

Hence the metric (5.2.1) becomes

$$ds^2 = -\frac{1}{\left(1 - \frac{r^2}{R^2}\right)} dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right) + \\ + \left[D_4 - D_3\left(1 - \frac{r^2}{R^2}\right)^{1/2}\right]^2 dt^2 \ . \tag{5.4.5}$$

where the constants $D_3 = \frac{D_1 R^2}{2}$ and $D_4 = \frac{D_2}{2}$ are specified by matching the solution to the exterior Schwarzschild solution at the boundary $r = a$. Thus we have

$$\frac{2m}{r} = \frac{a^2}{R^2} , \quad D_3 = \frac{1}{2} , \quad D_4 = \frac{3}{2}\left(1 - \frac{a^2}{R^2}\right)^{1/2} . \tag{5.4.6}$$

The pressure and density in this case becomes

$$kp = \frac{\frac{1}{R^2}\left[3D_3\left(1 - \frac{r^2}{R^2}\right)^{1/2} - D_4\right]}{D_4 - D_3\left(1 - \frac{r^2}{R^2}\right)^{1/2}} - 4k^2 s_1^2 , \tag{5.4.7}$$

$$k\rho = \left(\frac{3}{R^2} - 4k^2 s_1^2\right) . \tag{5.4.8}$$

From the equations (5.2.23) we have $\theta = 0$, $\sigma_{ij} = 0$ and $\dot{u}_i \neq 0$, $W_{ij} \neq 0$. Thus our solution in this case has the same interpretation as solution determined in the case (II). The solution (5.4.5) is analogous to the solution claimed by Prasanna [100]. We also notice that at the boundary $r = a$, the pressure $kp = -4k^2 s_1^2$. If however, in the absence of spin, the pressure vanishes at $r = a$. This is a classical result of Einstein theory of gravitation.

Case (IV): We assume $e^{\nu}\frac{\nu'}{r} = E_1$, where E_1 is a constant.

Here, we have obtained from assumed equation

$$e^{\nu}\nu' = E_1 r .$$

Integrating we get

$$\nu = log\left(E_1 r^2 + E_2\right) ,$$

where E_2 is another constant.

Consequently from the equation (5.3.6), we obtain

$$e^{-\lambda}\left[\lambda' + \frac{2(2{E_1}^2r^4 + 2E_1E_2r^2 + {E_2}^2)}{r(E_1r^2+E_2)(2E_1r^2+E_2)}\right] = \frac{2}{r}\left(\frac{E_1r^2+E_2}{2E_1r^2+E_2}\right) .$$

Solving this equation, we obtain the solution as

$$e^{-\lambda} = \frac{\left(E_3r^2+1\right)\left(E_1r^2+E_2\right)}{(2E_1r^2+E_2)} .$$

By substituting the values of e^{λ} and e^{ν} in equation (5.2.1) we get

$$ds^2 = -\frac{\left(1+\frac{2E_1}{E_2}r^2\right)}{\left(1+\frac{E_1}{E_2}r^2\right)(1+E_3r^2)}dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right) + + E_2\left(1+\frac{E_1}{E_2}r^2\right)dt^2 . \tag{5.4.9}$$

This metric is analogous to the metric obtained by Prasanna [100]. The arbitrary constants E_3, E_1 and E_2 are determined by the boundary conditions at $r = a$. They are given by

$$E_1 = \frac{m}{a^3} , \quad E_2 = \left(1-\frac{3m}{a}\right) , \quad E_3 = -\frac{m}{a^3} ,$$

with the pressure and density given by

$$kp = \frac{3m^2}{a^4}\left(\frac{1-\frac{r^2}{a^2}}{1-\frac{3m}{a}+\frac{2m}{a^3}r^2}\right) - 4k^2s_1^2 , \tag{5.4.10}$$

$$k\rho = \frac{m}{a^3}\left(\frac{6-\frac{9m}{a}-\frac{3m}{a^3}r^2}{1-\frac{3m}{a}+\frac{2m}{a^3}r^2}\right) - \frac{12m^2r^2}{a^7}\left[\frac{1-\frac{r^2}{a^2}}{\left(1-\frac{3m}{a}+\frac{2m}{a^3}r^2\right)^2}\right] - 4k^2s_1^2 . \tag{5.4.11}$$

The equation of state is obtain by eliminating r between the equations (5.4.10) and (5.4.11) as

$$3m^2a^3k\left[a(\rho-5p)-m(\rho-9p)+4p\right]-4k^3s_1^2a^6\left[2am\rho-3p(4m-3a)\right]+$$
$$+4k^2p^2a^6(3m-a)=\left[18m^3(a-2m)+12k^2s_1^2m^2a^3(m-4)+\right.$$
$$\left.+32k^4s_1^4a^7(m+2)\right]. \qquad (5.4.12)$$

It follows from the equations(5.2.23) that the solution in this case is accelerating as well as rotating with zero expansion and shear.

Case (V): We assume $e^{\nu}=Ar^{2n}$, where A is a constant.

In this case assumed equation gives

$$\nu=logA+2nlogr\ .$$

The equation (5.3.6) gives

$$e^{-\lambda}\left[\lambda^{'}+\frac{2}{r}\left(\frac{1+2n-n^2}{n+1}\right)\right]=\frac{2}{r(n+1)}\ .$$

Solving this equation we obtain

$$e^{-\lambda}=\left(\frac{1+B\left(1+2n-n^2\right)r^{2N}}{1+2n-n^2}\right)\ .$$

Now substituting the values of e^{λ} and e^{ν} in equation (5.2.1) we get

$$ds^2=-\left(\frac{1+2n-n^2}{1+(1+2n-n^2)Br^{2N}}\right)dr^2-r^2\left(d\theta^2+sin^2\theta d\phi^2\right)+Ar^{2n}dt^2\ , \qquad (5.4.13)$$

with

$$N = \left(\frac{1 + 2n - n^2}{1 + n} \right) \ ,$$

where the arbitrary constants A and B are specified by matching the solution to the exterior Schwarzschild solution at the boundary $r = a$. They are given by

$$A = a^{-2n} \left(1 - \frac{2m}{a} \right) , \qquad B = \left(1 - \frac{2m}{a} - \frac{1}{1 + 2n - n^2} \right) a^{-2N} \ . \tag{5.4.14}$$

Hence from equations (5.3.4) and (5.3.5) the pressure and density are given by

$$\begin{aligned} kp =& - 4k^2 s_1^2 + \frac{1}{r^2} \left(\frac{n^2}{1 + 2n - n^2} \right) + B \left(2n + 1 \right) r^{2n \left(\frac{1-n}{1+n} \right)} \ , \\ k\rho =& - 4k^2 s_1^2 + \frac{1}{r^2} \left(\frac{2n - n^2}{1 + 2n - n^2} \right) - B \left(\frac{3 + 5n - 2n^2}{1 + n} \right) r^{2n \left(\frac{1-n}{1+n} \right)} \ . \end{aligned} \tag{5.4.15}$$

At $r = a$, $p = 0$ gives us n in terms of m , a and s_1 as

$$n = \left(\frac{m}{a} \right) \left(1 - \frac{2m}{a} \right)^{-1} + 2k^2 s_1^2 a^2 \ . \tag{5.4.16}$$

We notice from equation (5.4.15) that when $n = -1$, the density ρ becomes infinite and $\frac{m}{a}$ is given by

$$\frac{m}{a} = \left(\frac{1 + 2k^2 s_1^2 a^2}{1 + 4k^2 s_1^2 a^2} \right) \ . \tag{5.4.17}$$

Following the treatment of Tolman [130] and Prasanna [100], we consider the special case $n = \frac{1}{2}$, for which $\frac{m}{a}$ becomes

$$\frac{m}{a} = \frac{1}{4}\left(\frac{1 - 4k^2 s_1^2 a^2}{1 - 2k^2 s_1^2 a^2}\right) \ , \tag{5.4.18}$$

showing that mass and spin are linked up with the geometry, Raychaudhuri [106]. Consequently,

$$A = \frac{1}{a}\left[1 - \frac{1}{2}\left(\frac{1 - 4k^2 s_1^2 a^2}{1 - 2k^2 s_1^2 a^2}\right)\right] \ ,$$

and

$$B = \frac{3}{7}\left[1 - \frac{7}{6}\left(\frac{1 - 4k^2 s_1^2 a^2}{1 - 2k^2 s_1^2 a^2}\right)\right] a^{-7/3} \ . \tag{5.4.19}$$

Hence, the metric (5.4.13) becomes

$$ds^2 = -\frac{7}{4 + (\frac{r}{a})^{7/3}\left[3 - \frac{7}{2}\left(\frac{1-4k^2 s_1^2 a^2}{1-2k^2 s_1^2 a^2}\right)\right]} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + \\ + \frac{r}{a}\left[1 - \frac{1}{2}\left(\frac{1 - 4k^2 s_1^2 a^2}{1 - 2k^2 s_1^2 a^2}\right)\right] dt^2 \ . \tag{5.4.20}$$

The pressure and the density are given by

$$kp = \frac{1}{7r^2} + 2Br^{1/3} - 4k^2 s_1^2 \ ,$$

and

$$k\rho = \frac{3}{7r^2} - \frac{10}{3} Br^{1/3} - 4k^2 s_1^2 \ . \tag{5.4.21}$$

Eliminating B between the equations (5.4.21) we readily get

$$r^2\left(3k\rho + 5kp + 32k^2 s_1^2\right) = 2 \ . \tag{5.4.22}$$

The equation of state is obtain by eliminating r between the equations (5.4.21) and (5.4.22) as

$$\left(3kp - k\rho + 8k^2 s_1^2\right)\left(5kp + 3k\rho + 32k^2 s_1^2\right)^{1/6} = \\ = 4(2)^{1/6} a^{-7/3}\left[1 - \frac{7}{6}\left(\frac{1 - 4k^2 s_1^2 a^2}{1 - 2k^2 s_1^2 a^2}\right)\right]. \quad (5.4.23)$$

It is evident from equations (5.2.23) that the solution in this case is accelerating as well as rotating with zero expansion and shear. The solution (5.4.20) matches with the solution of Prasanna [100] in the absence of spin.

Conclusion: A class of five different static spherically symmetric solutions propounded in the above five cases are all expansion free and shear free. One of these solutions is rotating with zero acceleration, while all other are rotating with non-zero acceleration. Our solution match with solutions obtained by Prasanna [100]. We have also seen that the pressure and the density have been influenced by the spin of the matter. However, in the absence of spin the pressure vanishes on the boundary and the result coincides with the standard result of Einstein theory of gravitation. Using Goldberg Sachs theorem we claim that the solutions are Petrov-type D.

Chapter 6

Non-Static Conformally Flat Spherically Symmetric Space-Times in Einstein-Cartan Theory of Gravitation

6.1 Introduction

Amongst the reported forty theories of gravitation, Einstein's general theory of relativity is considered as one of the most successful theory. This theory describes the mysterious gravitational force in terms of geometry of space-time. In spite of all its wel-known embrassing characters and the string of success, it is still considered to be inadequate as it does not satisfy certain desirable features. Hence there was hope that there may be something beyond the Einstein's general theory of relativity yet to be found. In search of a new theory with the hope that the new theory may satisfy the desirable features of the original theory, several theories of gravitation have been proposed as alternatives to Einstein's theory of gravitation. All these modified theories of gravitation have gained the attraction of researchers and good amount of work has been done in these theories in the last more than four decades. Amongst all these ramifications, Einstein-Cartan theory of gravitation is the one proposed by Cartan [11, 12].

The historical development of the Einstein-Cartan theory of gravitation and the comprehensive account of the work done by various researchers is exhibited in the previous chapters.

Cosmology is another branch of the theory of relativity in which researchers work for the physical world as the solution of the field equations. The aim of the study of cosmology is to understand the past, present and future of the universe and to know four mysterious forces in nature, their interdependence and their consequences on the universe.

In the purview of Einstein-Cartan theory of gravitation, several authors have investigated different aspects of the solutions of the field equations Einstein-Cartan theory of gravitation. Explicit solutions of the Einstein's field equations for static fluid spheres have been obtained by Tolman [130] and noticed that some of these solutions can be used in the investigations of Stellar Structure. By using Hehl's approach and Tolman's technique three solutions have been presented by Prasanna [100] with special reference to a perfect fluid distribution and shown that a space-time metric similar to the Schwarzschild interior solution will no longer represent a homogeneous fluid sphere in the presence of spin density, and at the boundary of the fluid sphere the hydrostatic pressure is discontinuous. Kuchowicz [77] has reviewed most of the previously known solutions for Weyssenhoff fluids in Einstein-Cartan theory of gravitation and addressed on the question as to whether or not such models have singularities. All these solutions have zero acceleration and vorticity. Some non-zero accelerated solutions have been obtained by Griffiths and Jogia [42].

Singh and Yadav [119] have studied the Einstein-Cartan field equations for the interior of a fluid sphere in an analytic form by the method of quadrature. Some other solutions have also been obtained under certain assumptions. Several of these solutions may be applicable to the investigations of Stellar interiors where high central density and pressure are significant. Kalyanshetti and Waghmode [66] considered the static, conformally flat spherically symmetric perfect-fluid distribution in Einstein-Cartan theory and obtained the field equations. These field equations are solved by adopting Hehl's approach with the assumption that the spins of the particles composing the fluid are all aligned in the radial direction only and the reality conditions are discussed. Yadav and Prasad [138] have obtained general solution representing conformally flat non-static spherically symmetric perfect fluid distribution in Einstein-Cartan theory. The explicit expressions for pressure, density, expansion, rotation, shear and non-vanishing components of flow vector have also been found. Sharif and Iqbal [113] have investigated solutions of the Einstein's field equations for the case of a non-static spherically symmetric perfect fluid using different equations of state. The properties of some exact spherically symmetric perfect fluid solutions which contain shear are obtained. Katkar [59] by adopting the Newman-Penrose-Jogia-Griffith formalism, the field equations in Einstein–Cartan theory for matter with spin creating torsion in space–time are solved in a spherically symmetric space-time by assuming only one non-vanishing component of spin. The exact solution might be the prototype for more realistic models. Katkar and Patil [60] have obtained exact solution of Einstein-Cartan field equations for static, conformally flat spherically symmetric space-time and it is proved to be Petrov-type D.

In this chapter, non-static conformally flat, Petrov-type D, spherically symmetric solutions of the Einstein-Cartan field equations; when Weyssenhoff fluid is the source of spin are obtained. In general the solution is expanding, accelerating, rotating and non-shearing. However, the dynamic solution is expanding and rotating with zero acceleration and shear; whereas the static solution is accelerating and rotating with zero expansion and shear. The work done in the chapter is organized as follows: In the Section 2, the non-static conformally flat spherically symmetric metric is considered and the tetrad components of connection 1-form, curvature 2-form are derived. Consequently, the tetrad components of the Riemann curvature tensor and Ricci tensor are derived and the components of expansion, acceleration, rotation and shear tensor are obtained. In the Section 3, the Einstein-Cartan field equations are formulated and solutions are obtained, the solution is shown to be Petrov-type D and finally some conclusions are drawn.

6.2 Non-Static Conformally Flat Spherically Symmetric Metric

Consider the non-static conformally flat spherically symmetric space-time in the form

$$ds^2 = e^{2\mu}\left[dt^2 - dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2)\right] \ , \tag{6.2.1}$$

where μ is a function of r and t only. Define the tetrad basis 1-forms θ^α for the metric (6.2.1) as

$$\begin{aligned} \theta^1 =& \frac{e^\mu}{\sqrt{2}}(dt + dr) \ , \\ \theta^2 =& \frac{e^\mu}{\sqrt{2}}(dt - dr) \ , \\ \theta^3 =& -\frac{e^\mu r}{\sqrt{2}}(d\theta - isin\theta d\phi) \ , \end{aligned} \tag{6.2.2}$$

where θ^4 is a complex conjugate of θ^3. Hence the metric (6.2.1) can be written as

$$ds^2 = 2\theta^1\theta^2 - 2\theta^3\theta^4 \ . \tag{6.2.3}$$

Using the equation (6.2.2) we obtain readily the components of the basis vector fields as

$$\begin{aligned} l_i =& \frac{e^\mu}{\sqrt{2}}(-1, 0, 0, 1) \ , \\ n_i =& \frac{e^\mu}{\sqrt{2}}(1, 0, 0, 1) \ , \end{aligned} \tag{6.2.4}$$

$$m_i = \frac{e^{\mu} r}{\sqrt{2}} (0, 1, isin\theta, 0) \ ,$$

where $\overline{m}_i$ is a complex conjugate of m_i. The contravariant components of the null basis vectors are obtain by raising the index by the metric tensor as

$$l^i = g^{ik} l_k = \frac{e^{-\mu}}{\sqrt{2}} (1, 0, 0, 1) \ ,$$

similarly, we obtain

$$n^i = \frac{e^{-\mu}}{\sqrt{2}} (-1, 0, 0, 1) \ , \tag{6.2.5}$$
$$m^i = -\frac{e^{-\mu} r^{-1}}{\sqrt{2}} (0, 1, icosec\theta, 0) \ .$$

The tetrad form of the equation (2.4.22) becomes

$$d_*{}^2 f = -\frac{1}{2} f_{,\gamma} Q_{\alpha\beta}{}^{\gamma} \theta^{\alpha} \wedge \theta^{\beta} \ , \tag{6.2.6}$$

where $Q_{\alpha\beta}{}^{\gamma}$ are the tetrad components of the torsion tensor $Q_{ij}{}^{k}$ and are given by

$$Q_{\alpha\beta}{}^{\gamma} = k S_{\alpha\beta} u^{\gamma} \ .$$

For coordinate functions x^i, the equation (6.2.6) becomes

$$d_*{}^2 x^i = -\frac{1}{2} e^i_{(\gamma)} Q_{\alpha\beta}{}^{\gamma} \theta^{\alpha} \wedge \theta^{\beta} \ . \tag{6.2.7}$$

It is evident from equations (6.2.5) and (6.2.7) that

$$\begin{aligned} d_*{}^2 r =& -\frac{e^{-\mu}}{2\sqrt{2}}\left(Q_{\alpha\beta}{}^1 - Q_{\alpha\beta}{}^2\right)\theta^\alpha\wedge\theta^\beta\ , \\ d_*{}^2\theta =& \frac{e^{-\mu}r^{-1}}{2\sqrt{2}}\left(Q_{\alpha\beta}{}^3 + Q_{\alpha\beta}{}^4\right)\theta^\alpha\wedge\theta^\beta\ , \\ d_*{}^2\phi =& \frac{ir^{-1}e^{-\mu}cosec\theta}{2\sqrt{2}}\left(Q_{\alpha\beta}{}^3 - Q_{\alpha\beta}{}^4\right)\theta^\alpha\wedge\theta^\beta\ , \\ d_*{}^2 t =& -\frac{e^{-\mu}}{2\sqrt{2}}(Q_{\alpha\beta}{}^1 + Q_{\alpha\beta}{}^2)\theta^\alpha\wedge\theta^\beta. \end{aligned} \tag{6.2.8}$$

Now operating d_* to the equations (6.2.2) and using the equations (6.2.8) we readily, get

$$\begin{aligned} d_*\theta^1 =& \frac{1}{\sqrt{2}}e^{-\mu}\left(\mu' - \dot{\mu}\right)\theta^{12} - \frac{1}{2}Q_{\alpha\beta}{}^1\theta^{\alpha\beta}\ , \\ d_*\theta^2 =& \frac{1}{\sqrt{2}}e^{-\mu}\left(\mu' + \dot{\mu}\right)\theta^{12} - \frac{1}{2}Q_{\alpha\beta}{}^2\theta^{\alpha\beta}\ , \\ d_*\theta^3 =& \frac{e^{-\mu}}{\sqrt{2}}\Big[\left(\dot{\mu} + \mu' + r^{-1}\right)\theta^{13} + \left(\dot{\mu} - \mu' - r^{-1}\right)\theta^{23} + \\ & + r^{-1}cot\theta\ \ \theta^{34}\Big] - \frac{1}{2}Q_{\alpha\beta}{}^3\theta^{\alpha\beta}\ , \\ d_*\theta^4 =& \frac{e^{-\mu}}{\sqrt{2}}\Big[\left(\dot{\mu} + \mu' + r^{-1}\right)\theta^{14} + \left(\dot{\mu} - \mu' - r^{-1}\right)\theta^{24} - \\ & - r^{-1}cot\theta\ \ \theta^{34}\Big] - \frac{1}{2}Q_{\alpha\beta}{}^4\theta^{\alpha\beta}\ , \end{aligned} \tag{6.2.9}$$

where we have used

$$\theta^{\alpha\beta} = \theta^\alpha\wedge\theta^\beta\ ,$$

and the dot denotes partial derivative with respect to time 't' and the prime indicates partial derivative with respect to the coordinate 'r'.

Now from equations (6.2.9) and (2.4.70) we obtain after simplifying the values of NP spin coefficients in Riemann space-time as

$$\begin{aligned}
\kappa^0 =&\lambda^0 = \sigma^0 = \pi^0 = \tau^0 = \nu^0 = 0 \ , \\
\rho^0 =& -\frac{e^{-\mu}}{\sqrt{2}}\left(\dot{\mu} + \mu' + r^{-1}\right) \ , \mu^0 = \frac{e^{-\mu}}{\sqrt{2}}\left(\dot{\mu} - \mu' - r^{-1}\right) \ , \\
\epsilon^0 =& \frac{e^{-\mu}}{2\sqrt{2}}\left(\dot{\mu} + \mu'\right) \ , \gamma^0 = -\frac{e^{-\mu}}{2\sqrt{2}}\left(\dot{\mu} - \mu'\right) \ , \\
\alpha^0 =& -\beta^0 = \frac{e^{-\mu}}{2\sqrt{2}} r^{-1} cot\theta.
\end{aligned} \tag{6.2.10}$$

By virtue of the equations (6.2.10), (1.3.30), the equations (2.4.69) reduces to

$$\begin{aligned}
\omega_{12} =& -\frac{1}{\sqrt{2}}\left[e^{-\mu}\left(\dot{\mu} + \mu'\right)\theta^1 - e^{-\mu}\left(\dot{\mu} - \mu'\right)\theta^2 - 2ks_0\theta^3 - 2k\overline{s}_0\theta^4\right] \ , \\
\omega_{13} =& \frac{1}{\sqrt{2}}\left\{2ks_0\theta^1 + \left[e^{-\mu}\left(\dot{\mu} + \mu' + r^{-1}\right) + 2ks_1\right]\theta^4\right\} \ , \\
\omega_{23} =& \frac{1}{\sqrt{2}}\left\{-2ks_0\theta^2 + \left[e^{-\mu}\left(\dot{\mu} - \mu' - r^{-1}\right) + 2ks_1\right]\theta^4\right\} \ , \\
\omega_{34} =& -\frac{1}{\sqrt{2}}\left[2ks_1(\theta^1 + \theta^2) + r^{-1}e^{-\mu}cot\theta(\theta^3 - \theta^4)\right].
\end{aligned} \tag{6.2.11}$$

Now using equations (1.3.30), (6.2.10) and (6.2.11), from equations (2.4.85) we obtain the tetrad components of curvature 2-form. These are listed below:

$$\begin{aligned}
\Omega^1{}_1 =& \left[e^{-2\mu}\left(\ddot{\mu} - \mu''\right) - 4k^2 s_0\overline{s}_0\right]\theta^{12} + \\
&+ \left\{ke^{-\mu}\left[s_{0,r} + s_{0,t} + 2s_0\left(\mu' + \dot{\mu}\right)\right] - 2k^2 s_0 s_1\right\}\theta^{13} + \\
&+ \left\{ke^{-\mu}\left[\overline{s}_{0,r} + \overline{s}_{0,t} + 2\overline{s}_0\left(\mu' + \dot{\mu}\right)\right] + 2k^2\overline{s}_0 s_1\right\}\theta^{14} -
\end{aligned}$$

$$- \Big\{2k^2 s_0 s_1 + ke^{-\mu}\big[s_{0,r} - s_{0,t} + 2s_0\,(\mu' - \dot\mu)\,\big]\Big\}\theta^{23} + \Big\{2k^2 \overline{s}_0 s_1 -$$

$$- ke^{-\mu}\big[\overline{s}_{0,r} - \overline{s}_{0,t} + 2\overline{s}_0\,(\mu' - \dot\mu)\,\big]\Big\}\theta^{24} + \Big\{4ks_1 e^{-\mu} r^{-1} +$$

$$+ ke^{-\mu} r^{-1} cot\theta\,(s_0 - \overline{s}_0)\,\Big\}\theta^{34}\ , \qquad (6.2.12)$$

$$\Omega^1{}_3 = -\Big\{ke^{-\mu}\big[s_{0,r} + s_{0,t} + 2s_0\,(\dot\mu + \mu')\,\big] + 2k^2 s_0 s_1\Big\}\theta^{12} + \Big\{ks_{1,r}e^{-\mu} +$$

$$+ ks_{1,t}e^{-\mu} + 2k^2 s_1^2 + ks_1 e^{-\mu}\,(3\dot\mu + \mu') + \frac{e^{-2\mu}}{2}\Big(\ddot\mu - \mu'' - 2\mu' r^{-1} +$$

$$+ \dot\mu^2 - \mu'^2\Big)\Big\}\theta^{14} + \big[-2k^2 s_0^2 + ks_0 r^{-1} e^{-\mu} cot\theta\big]\theta^{23} - \Big\{ks_{1,r}e^{-\mu} -$$

$$- ks_{1,t}e^{-\mu} + ks_0 r^{-1} e^{-\mu} cot\theta + 2k^2 s_0 \overline{s}_0 - \frac{e^{-2\mu}}{2}\Big(\ddot\mu - 2\dot\mu' + \mu'' - \dot\mu^2 -$$

$$- \mu'^2 + 2\dot\mu\mu'\Big) - ks_1 e^{-\mu}\Big(\dot\mu - \mu' - 2r^{-1}\Big) - 2k^2 s_1^2\Big\}\theta^{24} -$$

$$- \big[2k^2 s_0 s_1 + ks_0 e^{-\mu}(\dot\mu - \mu' - r^{-1})\big]\theta^{34}\ , \qquad (6.2.13)$$

$$\Omega^2{}_3 = \Big\{ke^{-\mu}\,[s_{0,r} - s_{0,t} - 2s_0\,(\dot\mu - \mu')] - 2k^2 s_0 s_1\Big\}\,\theta^{12} - \big[2k^2 s_0^2 +$$

$$+ ks_0 r^{-1} e^{-\mu} cot\theta\big]\theta^{13} + \Big\{ks_{1,r}e^{-\mu} + ks_{1,t}e^{-\mu} + ks_1 e^{-\mu}\Big(\dot\mu + \mu' +$$

$$+ 2r^{-1}\Big) - 2k^2 s_0 \overline{s}_0 + 2k^2 s_1^2 + ks_0 r^{-1} e^{-\mu} cot\theta + \frac{e^{-2\mu}}{2}\Big(\ddot\mu + 2\dot\mu' + \mu'' - \dot\mu^2 -$$

$$- \mu'^2 - 2\dot\mu\mu'\Big)\Big\}\theta^{14} - \Big\{ks_{1,r}e^{-\mu} - ks_{1,t}e^{-\mu} - 2k^2 s_1^2 - ks_1 e^{-\mu}\,(3\dot\mu - \mu') -$$

$$- \frac{e^{-2\mu}}{2}\Big(\ddot\mu - \mu'' - 2\mu' r^{-1} + \dot\mu^2 - \mu'^2\Big)\Big\}\theta^{24} + \big[2k^2 s_0 s_1 + ks_0 e^{-\mu}\Big(\dot\mu +$$

$$+ \mu' + r^{-1}\Big)\big]\theta^{34}\ , \qquad (6.2.14)$$

$$\Omega^3{}_3 = -2ke^{-\mu}\,(s_{1,r} + s_1\mu')\,\theta^{12} + \big[2k^2 s_0 s_1 - ks_0 e^{-\mu}\,\Big(\dot\mu - \mu' - r^{-1}\Big)\big]\,\theta^{13} +$$

$$+ \big[2k^2 \overline{s}_0 s_1 + k\overline{s}_0 e^{-\mu}\,\Big(\dot\mu - \mu' - r^{-1}\Big)\big]\,\theta^{14} - \big[2k^2 s_0 s_1 - ks_0 e^{-\mu}\Big(\dot\mu + \mu' +$$

$$+ r^{-1}\Big)\big]\theta^{23} - \big[2k^2 \overline{s}_0 s_1 + k\overline{s}_0 e^{-\mu}\,\Big(\dot\mu + \mu' + r^{-1}\Big)\big]\,\theta^{24} + \big[e^{-2\mu}\Big(\dot\mu^2 - \mu'^2 -$$

$$- 2\mu' r^{-1}\Big) - 4k^2 s_1^2\big]\theta^{34}\ . \qquad (6.2.15)$$

The expressions for $\Omega^1{}_4, \Omega^2{}_4, \Omega^4{}_4$ are obtained by interchanging the suffixes 3 and 4 and taking the complex conjugate of the right hand sides of the equations in (6.2.13), (6.2.14) and (6.2.15) respectively.

6.2.1 Tetrad Components of Riemann-Cartan Curvature Tensor

From the equation (2.4.65) we obtain

$$\Omega_{\alpha\beta} = R_{12\alpha\beta}\theta^{12} + R_{13\alpha\beta}\theta^{13} + R_{14\alpha\beta}\theta^{14} + R_{23\alpha\beta}\theta^{23} + R_{24\alpha\beta}\theta^{24} + R_{34\alpha\beta}\theta^{34} \ , \tag{6.2.16}$$

where

$$\Omega_{\alpha\beta} = -\Omega_{\beta\alpha} \ , \quad \alpha, \beta = 1, 2, 3, 4.$$

By giving different values to $\alpha, \beta = 1, 2, 3, 4$ in the equation (6.2.16) and then equating the corresponding coefficients of basis 2-forms of equations(6.2.12), (6.2.13), (6.2.14) and(6.2.15) we readily obtain the tetrad components of Riemann-Cartan curvature tensor as

$$R_{1212} = e^{-2\mu}\left(\ddot{\mu} - \mu''\right) - 4k^2 s_0 \overline{s}_0 \ ,$$

$$R_{1312} = ke^{-\mu}\left[s_{0,r} + s_{0,t} + 2s_0\left(\mu' + \dot{\mu}\right)\right] - 2k^2 s_0 s_1 \ ,$$

$$R_{2312} = -\left\{ke^{-\mu}\left[s_{0,r} - s_{0,t} + 2s_0\left(\mu' - \dot{\mu}\right)\right] + 2k^2 s_0 s_1\right\} \ ,$$

$$R_{3412} = 4ks_1 e^{-\mu} r^{-1} + ke^{-\mu} r^{-1} cot\theta\left(s_0 - \overline{s}_0\right) \ ,$$

$$R_{1213} = ke^{-\mu}\left[s_{0,r} - s_{0,t} - 2s_0\left(\dot{\mu} - \mu'\right)\right] - 2k^2 s_0 s_1 \ ,$$

$$R_{1313} = -\left[2k^2 s_0^2 + ks_0 r^{-1} e^{-\mu} cot\theta\right] \ ,$$

$$
\begin{aligned}
R_{1413} =& ks_{1,r}e^{-\mu} + ks_{1,t}e^{-\mu} + ks_1e^{-\mu}\left(\dot{\mu} + \mu' + 2r^{-1}\right) - 2k^2s_0\overline{s}_0 + 2k^2s_1^2 + \\
&+ ks_0r^{-1}e^{-\mu}cot\theta + \frac{e^{-2\mu}}{2}\left(\ddot{\mu} + 2\dot{\mu}^{'} + \mu'' - \dot{\mu}^2 - \mu'^2 - 2\dot{\mu}\mu'\right) \ , \\
R_{2413} =& ks_{1,t}e^{-\mu} - ks_{1,r}e^{-\mu} + 2k^2s_1^2 + ks_1e^{-\mu}\left(3\dot{\mu} - \mu'\right) + \\
&+ \frac{e^{-2\mu}}{2}\left(\ddot{\mu} - \mu'' - 2\mu'r^{-1} + \dot{\mu}^2 - \mu'^2\right) \ , \\
R_{3413} =& 2k^2s_0s_1 + ks_0e^{-\mu}\left(\dot{\mu} + \mu' + r^{-1}\right) \ , \\
R_{1223} =& -\left\{ke^{-\mu}\left[s_{0,r} + s_{0,t} + 2s_0\left(\dot{\mu} + \mu'\right)\right] + 2k^2s_0s_1\right\} \ , \\
R_{1423} =& ks_{1,t}e^{-\mu} + ks_{1,r}e^{-\mu} + 2k^2s_1^2 + ks_1e^{-\mu}\left(3\dot{\mu} + \mu'\right) + \\
&+ \frac{e^{-2\mu}}{2}\left(\ddot{\mu} - \mu'' - 2\mu'r^{-1} + \dot{\mu}^2 - \mu'^2\right) \ , \\
R_{2323} =& -\left[2k^2s_0^2 - ks_0r^{-1}e^{-\mu}cot\theta\right] \ , \\
R_{2423} =& ks_{1,t}e^{-\mu} - ks_{1,r}e^{-\mu} + ks_1e^{-\mu}\left(\dot{\mu} - \mu' - 2r^{-1}\right) - 2k^2s_0\overline{s}_0 + 2k^2s_1^2 - \\
&- ks_0r^{-1}e^{-\mu}cot\theta + \frac{e^{-2\mu}}{2}\left(\ddot{\mu} - 2\dot{\mu}^{'} + \mu'' - \dot{\mu}^2 - \mu'^2 + 2\dot{\mu}\mu'\right) \ , \\
R_{3423} =& -\left[2k^2s_0s_1 + ks_0e^{-\mu}(\dot{\mu} - \mu' - r^{-1})\right] \ , \\
R_{1234} =& -2ke^{-\mu}\left(s_{1,r} + s_1\mu'\right) \ , \\
R_{1334} =& 2k^2s_0s_1 - ks_0e^{-\mu}\left(\dot{\mu} - \mu' - r^{-1}\right) \ , \\
R_{2334} =& -\left[2k^2s_0s_1 - ks_0e^{-\mu}\left(\dot{\mu} + \mu' + r^{-1}\right)\right] \ , \\
R_{3434} =& e^{-2\mu}\left(\dot{\mu}^2 - \mu'^2 - 2\mu'r^{-1}\right) - 4k^2s_1^2 \ , \qquad (6.2.17)
\end{aligned}
$$

and

$$
R_{2313} = R_{1323} = 0 \ .
$$

The complex conjugates of above equations are obtained by interchanging the suffixes 3 and 4 and taking the complex conjugates of the right hand sides of the respective equations.

6.2.2 Tetrad Components of Ricci-Cartan Tensor and Ricci-Cartan Curvature Scalar

The tetrad components of the Ricci-Cartan tensor and Ricci-Cartan curvature scalar are defined by

$$R_{\alpha\beta} = \eta^{\nu\epsilon} R_{\nu\alpha\beta\epsilon} \quad , R = \eta^{\alpha\beta} R_{\alpha\beta} \ ,$$

$$\Rightarrow \qquad R_{\alpha\beta} = R_{1\alpha\beta 2} + R_{2\alpha\beta 1} - R_{3\alpha\beta 4} - R_{4\alpha\beta 3} \ . \qquad (6.2.18)$$

Using equations (6.2.17) we obtain from equation (6.2.18) expressions for Ricci-Cartan tensors

$$\begin{aligned}
R_{11} =& e^{-2\mu}\left(\ddot{\mu} + 2\dot{\mu}' + \mu'' - \dot{\mu}^2 - 2\dot{\mu}\mu' - \mu'^2\right) + kr^{-1}e^{-\mu}cot\theta\left(s_0 + \overline{s}_0\right) + \\
& + 4k^2{s_1}^2 - 4k^2 s_0\overline{s}_0 \ , \\
R_{12} =& R_{21} = e^{-2\mu}\left(2\ddot{\mu} - 2\mu'' - 2\mu' r^{-1} + \dot{\mu}^2 - \mu'^2\right) + 4k^2 s_1^2 - 4k^2 s_0\overline{s}_0 \ , \\
R_{13} =& = R_{32} = ke^{-\mu}\left[s_{0,r} - s_{0,t} - s_0\left(3\dot{\mu} - 3\mu' - r^{-1}\right)\right] \ , \\
R_{22} =& e^{-2\mu}\left(\ddot{\mu} - 2\dot{\mu}' + \mu'' - \dot{\mu}^2 + 2\dot{\mu}\mu' - \mu'^2\right) - kr^{-1}e^{-\mu}cot\theta\left(s_0 + \overline{s}_0\right) + \\
& + 4k^2{s_1}^2 - 4k^2 s_0\overline{s}_0 \ , \\
R_{23} =& = R_{31} = ke^{-\mu}\left[s_{0,r} + s_{0,t} + s_0\left(3\dot{\mu} + 3\mu' + r^{-1}\right)\right] \ , \\
R_{34} =& 2ks_{1,t}e^{-\mu} + 6ks_1 e^{-\mu}\dot{\mu} - e^{-2\mu}(\ddot{\mu} - \mu'' - 4\mu' r^{-1} + 2\dot{\mu}^2 - 2\mu'^2) \ ,
\end{aligned}$$

$$R_{43} = -2ks_{1,t}e^{-\mu} - 6ks_1e^{-\mu}\dot{\mu} - e^{-2\mu}(\ddot{\mu} - \mu'' - 4\mu' r^{-1} + 2\dot{\mu}^2 - 2\mu'^2) \ , \tag{6.2.19}$$

$$R_{33} = 0 \ .$$

The Ricci-Cartan curvature scalar is given by

$$R = 6e^{-2\mu}\left(\ddot{\mu} - \mu'' - 2\mu' r^{-1} + \dot{\mu}^2 - \mu'^2\right) + 8k^2{s_1}^2 - 8k^2 s_0\bar{s}_0 \ . \tag{6.2.20}$$

6.2.3 The Kinematical Parameters

By virtue of the equation (6.2.10), the kinematical parameters viz; the expansion θ, the acceleration vector $\dot{u}_i$, the shear tensor σ_{ij} and the rotation tensor W_{ij} defined in equation (1.4.27), take the form

$$\begin{aligned} \theta &= 3e^{-\mu}\dot{\mu} \ , \\ \dot{u}_i &= \frac{1}{\sqrt{2}}e^{-\mu}\mu'(l_i - n_i) \ , \\ \sigma_{ij} &= 0 \ , \\ W_{ij} &= 4ks_1 m_{[i}\overline{m}_{j]} + 2k\bar{s}_0\left(l_{[i}m_{j]} + m_{[i}n_{j]}\right) + c.c. \end{aligned} \tag{6.2.21}$$

This will be of great use in the interpretations of the solutions obtained below.

6.3 Einstein-Cartan's Field Equations

We start with the tetrad representation of the field equations (1.2.48) as

$$R_{\alpha\beta} - \frac{R}{2}\eta_{\alpha\beta} = -kt_{\alpha\beta} \ , \tag{6.3.1}$$

where the tetrad components of the energy momentum tensor

$$t_{\alpha\beta} = t_{ij}e_{(\alpha)}{}^{i}e_{(\beta)}{}^{j} \ , \tag{6.3.2}$$

are obtained by using equation (1.4.22) and (6.2.10) as

$$\begin{aligned} t_{11} =& t_{22} = \frac{1}{2}(\rho + p) \ , \ t_{12} = t_{21} = \frac{1}{2}(\rho - p), \\ t_{34} =& t_{43} = p \ , \ t_{31} = t_{32} = \mu' e^{-\mu} s_0 \ , \\ t_{41} =& t_{42} = \mu' e^{-\mu} \overline{s}_0 \ , \end{aligned} \tag{6.3.3}$$

and all other tetrad components of the energy-momentum tensor are zero.

Using above equations (6.2.19), (6.2.20) and (6.3.3) in equation (6.3.1), the independent field equations for gravitation in Einstein-Cartan theory are given by

$$\begin{aligned} & e^{-2\mu}\left(\ddot{\mu} + 2\dot{\mu}' + \mu'' - \dot{\mu}^2 - 2\dot{\mu}\mu' - {\mu'}^2\right) + kr^{-1}e^{-\mu}cot\theta\left(s_0 + \overline{s}_0\right) + \\ & + 4k^2{s_1}^2 - 4k^2 s_0\overline{s}_0 = -\frac{k}{2}(\rho + p) \ , \\ & e^{-2\mu}(\mu'' - \ddot{\mu} - 2\dot{\mu}^2 + 2{\mu'}^2 + 4\mu' r^{-1}) = -\frac{k}{2}(\rho - p) \ , \\ & s_{0,r} - s_{0,t} - s_0\left(3\dot{\mu} - 3\mu' - r^{-1}\right) = 0 \ , \end{aligned}$$

$$e^{-2\mu}\left(\ddot{\mu}-2\dot{\mu}'+\mu''-\dot{\mu}^2+2\dot{\mu}\mu'-\mu'^2\right)-kr^{-1}e^{-\mu}cot\theta\,(s_0+\overline{s}_0)+$$
$$+\,4k^2{s_1}^2-4k^2s_0\overline{s}_0=-\frac{k}{2}(\rho+p)\ ,$$
$$s_{0,r}+s_{0,t}+s_0\left(3\dot{\mu}+3\mu'+r^{-1}\right)=0\ ,$$
$$s_{0,r}+s_{0,t}+s_0\left(3\dot{\mu}+4\mu'+r^{-1}\right)=0\ ,$$
$$s_{0,r}-s_{0,t}+s_0\left(4\mu'-3\dot{\mu}+r^{-1}\right)=0\ ,$$
$$e^{-2\mu}(2\ddot{\mu}-2\mu''-2\mu'r^{-1}+\dot{\mu}^2-\mu'^2)+2ks_{1,t}e^{-\mu}+6ks_1e^{-\mu}\dot{\mu}+$$
$$+\,4k^2{s_1}^2-4k^2s_0\overline{s}_0=-kp\ ,$$
$$e^{-2\mu}(2\ddot{\mu}-2\mu''-2\mu'r^{-1}+\dot{\mu}^2-\mu'^2)-2ks_{1,t}e^{-\mu}-6ks_1e^{-\mu}\dot{\mu}+$$
$$+\,4k^2{s_1}^2-4k^2s_0\overline{s}_0=-kp\ . \tag{6.3.4}$$

We see from equations (6.3.4) that these equations are consistent provided that $s_0=0$. Hence out of the nine field equations (6.3.4), there exists only five independent field equations these are given by

$$\dot{\mu}'-\dot{\mu}\mu'=0\ , \tag{6.3.5}$$
$$\mu''-\mu'^2-\mu'r^{-1}=0\ , \tag{6.3.6}$$
$$s_{1,t}+3s_1\dot{\mu}=0\ , \tag{6.3.7}$$
$$-\,kp=e^{-2\mu}\left(2\ddot{\mu}-3\mu'^2+\dot{\mu}^2-4\mu'r^{-1}\right)+4k^2s_1^2\ , \tag{6.3.8}$$
$$-\,k\rho=3e^{-2\mu}\left(\mu'^2-\dot{\mu}^2+2\mu'r^{-1}\right)+4k^2s_1^2\ . \tag{6.3.9}$$

The solution of equation (6.3.7) is given by

$$s_1=de^{-3\mu}\ . \tag{6.3.10}$$

where d is a constant of integration.

(I) General Case: When $\mu = \mu(r,t)$. The general integral of the equation (6.3.5) is given by

$$e^{-\mu} = \phi(r) + \psi(t) + c \ , \tag{6.3.11}$$

where ϕ and ψ are function of r and t respectively and c is any arbitrary constant independent of r and t. The solution (6.3.11) must satisfies the equation (6.3.6). Hence we have

$$\phi''(r) - \frac{1}{r}\phi'(r) = 0 \ . \tag{6.3.12}$$

Solving the equation (6.3.12) we obtain

$$\phi(r) = a_1 r^2 + a_2 \ , \tag{6.3.13}$$

where a_1 and a_2 are constants of integration.

Also by integrating the equation (6.3.5) with respect to r we obtain

$$\mu(r,t) = ln\,|\dot{\mu}(r,t)A(t)| \ \ , \tag{6.3.14}$$

where $A(t)$ is a constant of integration with respect to r and it may involve the time t explicitly., so that $\dot{\mu} \neq 0$. Similarly, integrating the equation (6.3.5) with respect to t, we obtain

$$\mu(r,t) = ln\,|\mu'(r,t)B(r)| \ \ , \tag{6.3.15}$$

where $B(r)$ is a constant of integration with respect to t it may be a function of r.

From equations (6.3.14) and (6.3.15), we have

$$\mu(r,t) = ln\,|\dot{\mu}(r,t)A(t)| = ln\,|\mu'(r,t)B(r)|\ . \qquad (6.3.16)$$

From equation (6.3.11), we obtain

$$\begin{aligned}
\mu^{'} =& - e^{\mu}\phi'(r)\ , \quad \dot{\mu} = -e^{\mu}\dot{\psi}(t)\ , \\
\mu^{''} =& - e^{\mu}[\phi^{''}(r) - e^{\mu}\phi'^{2}(r)]\ , \quad \ddot{\mu} = -e^{\mu}[\ddot{\psi}(t) - e^{\mu}\dot{\psi}^{2}(t)]\ , \\
\dot{\mu}' =& e^{2\mu}\phi'(r)\dot{\psi}(t)\ . \qquad (6.3.17)
\end{aligned}$$

From equations (6.3.11) and (6.3.16) we write

$$[\phi(r) + \psi(t) + c]^{-1} = \dot{\mu}(r,t)A(t) = \mu'(r,t)B(r)\ . \qquad (6.3.18)$$

This gives

$$A(t)\dot{\psi}(t) = B(r)\phi'(r) = -1\ . \qquad (6.3.19)$$

This yields

$$\phi'(r) = -\ B^{-1}(r)\ , \qquad (6.3.20)$$

$$\text{and} \quad \dot{\psi}(t) = -\ A^{-1}(t)\ . \qquad (6.3.21)$$

Integrating the equation (6.3.20) with respect to r we get

$$\phi(r) = a_1 r^2 + a_2 = -\int B^{-1}(r)dr + a_3\ . \qquad (6.3.22)$$

Differentiating the equation (6.3.22) we get

$$2a_1 r = -\ B^{-1}(r)\ ,$$

$$\Rightarrow B(r) = -\frac{1}{2a_1 r} \ . \tag{6.3.23}$$

Similarly, from the equation (6.3.21), we obtain

$$\psi(t) = -\int A^{-1}(t)dt + b_3 \ . \tag{6.3.24}$$

We choose

$$A(t) = \frac{1}{2b_1 t} \ . \tag{6.3.25}$$

For this choice of $A(t)$, the equation (6.3.24) gives

$$\psi(t) = -b_1 t^2 + b_2 \ , \tag{6.3.26}$$

where b_1, b_2 are constants of integration with respect to t. Thus the general integral (6.3.11) becomes

$$\begin{aligned} e^{-\mu} =& a_1 r^2 - b_1 t^2 + a_2 + b_2 + c \ , \\ \Rightarrow e^{-\mu} =& a_1 r^2 - b_1 t^2 + c_1 \ , \end{aligned} \tag{6.3.27}$$

where $c_1 = a_2 + b_2 + c$. Hence the metric (6.2.1) becomes

$$ds^2 = [a_1 r^2 - b_1 t^2 + c_1]^{-2}[dt^2 - dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2)] \ . \tag{6.3.28}$$

The pressure and density become

$$kp = 4a_1(a_1 - b_1)r^2 + 8b_1(a_1 - b_1)t^2 - 4c_1(2a_1 + b_1) - 4k^2 s_1^2 \ , \tag{6.3.29}$$

$$k\rho = -12b_1(a_1 - b_1)t^2 + 12a_1 c_1 - 4k^2 s_1^2 \ . \tag{6.3.30}$$

In the absence of time t, the solution (6.3.28) reduces to the solution obtained by Katkar and Patil [60].

From these equations (6.3.29) and (6.3.30), we obtain

$$k(p+\rho) = 4a_1(a_1-b_1)r^2 - 4b_1(a_1-b_1)t^2 + 4c_1(a_1-b_1) - 8k^2s_1^2 \ . \tag{6.3.31}$$

Multiplying equation (6.3.29) by 3 and the equation (6.3.30) by 2 and adding we get

$$k(3p+2\rho) = 12a_1(a_1-b_1)r^2 - 12b_1c_1 - 20k^2s_1^2 \ . \tag{6.3.32}$$

Without loss of generality, we set $c_1 = a_2 + b_2 + c = 0$. Consequently, from equations (6.3.30) and (6.3.32), we find

$$12a_1(a_1-b_1) = \frac{k(3p+2\rho)+20k^2s_1^2}{r^2} \ , \tag{6.3.33}$$

$$\text{and} \quad -12a_1(a_1-b_1) = \frac{k\rho+4k^2s_1^2}{t^2} \ . \tag{6.3.34}$$

Solving the equations (6.3.33) and (6.3.34) for a_1 and b_1 we find

$$a_1 = \frac{[k(3p+2\rho)+20k^2s_1^2]t}{2r\sqrt{3}\{[k(3p+2\rho)+20k^2s_1^2]t^2 + r^2[k\rho+4k^2s_1^2]\}^{1/2}} \ , \tag{6.3.35}$$

$$b_1 = -\frac{(k\rho+4k^2s_1^2)r}{2t\sqrt{3}\{[k(3p+2\rho)+20k^2s_1^2]t^2 + r^2[k\rho+4k^2s_1^2]\}^{1/2}} \ . \tag{6.3.36}$$

Substituting these values of a_1 and b_1 in the equation (6.3.28) we get

$$ds^2 = \frac{4[(3p+2\rho+20ks_1^2)t^2+(\rho+4ks_1^2)r^2]}{3k\left(p+\rho+8ks_1^2\right)^2 r^2t^2}\Bigg[dt^2-dr^2- \\ -r^2(d\theta^2+sin^2\theta d\phi^2)\Bigg] \ . \qquad (6.3.37)$$

If now μ is a function of t alone, in this case the space-time (6.2.1) becomes

$$ds^2 = [-b_1t^2+c_1]^{-2}[dt^2-dr^2-r^2(d\theta^2+sin^2\theta d\phi^2)] \ . \qquad (6.3.38)$$

where $c_1 = b_2 + c$, with pressure and density given by

$$-kp = e^{-2\mu}(2\ddot{\mu}+\dot{\mu}^2)+4k^2s_1^2 \ , \qquad (6.3.39)$$

$$-k\rho = -3e^{-2\mu}\dot{\mu}^2+4k^2s_1^2 \ . \qquad (6.3.40)$$

The constants b_1 and c_1 are obtained by solving equations (6.3.39) and (6.3.40) with the help of (6.3.27) as

$$\begin{aligned} b_1 &= \frac{1}{2\sqrt{3}t}(k\rho+4k^2s_1^2)^{1/2} \ , \\ c_1 &= -\frac{t}{2\sqrt{3}(k\rho+4k^2s_1^2)^{1/2}}(3kp+2k\rho+20k^2s_1^2) \ . \end{aligned} \qquad (6.3.41)$$

Hence, the solution (6.3.38) reduces to

$$ds^2 = \left\{\frac{4[k\rho+4k^2s_1^2]}{3t^2(kp+k\rho+8k^2s_1^2)^2}\right\}\Big[dt^2-dr^2-r^2(d\theta^2+sin^2\theta d\phi^2)\Big] \ . \qquad (6.3.42)$$

6.3.1 Petrov classification of the solution

The free gravitational field is characterized by the completely trace free Weyl curvature tensor C_{hijk}. It has 20 independent components in the Einstein-Cartan theory of gravitation. These can be expressed in terms of the five complex components of the Weyl tensor $\psi_A, (A = 0, 1, 2, 3, 4)$, nine components of a Hermitian 3×3 matrix $\Theta_{AB}, (A, B = 0, 1, 2)$ and a real parameter χ.[Jogia and Griffiths [55]].

We follow the notations of Jogia and Griffiths and found that

$$\begin{aligned}
\psi_0 =&\psi_1 = \psi_3 = \psi_4 = 0 \ , \quad \Theta_{01} = \Theta_{02} = \Theta_{21} = 0 \ , \\
\psi_2 =&-\frac{1}{3}[3ke^{-\mu}(s_{1,r} + s_1\mu^{'}) + 2k^2s_1^2] \ , \\
\Theta_{00} =&ike^{-\mu}[s_{1,r} + s_1(\dot{\mu} + \mu^{'} + 2r^{-1})] \ , \\
\Theta_{11} =&\frac{ik}{2}e^{-\mu}(s_{1,r} + s_1\mu^{'} + 2s_1r^{-1}) \ , \\
\Theta_{22} =&ike^{-\mu}[s_{1,r} - s_1(\dot{\mu} - \mu^{'} - 2r^{-1})] \ , \\
\chi =&-\frac{i}{6}[3ke^{-\mu}(s_{1,r} + s_1\mu^{'} - 2s_1r^{-1}) - 2k^2s_1^2] \ .
\end{aligned} \qquad (6.3.43)$$

This is the Petrov-type D solution. For vanishing of spin s_1, we see that all the components of the Weyl tensor vanish and the space-time metric reduces to the conformally flat space-time and the solution will be Petrov-type 0.

Discussion

It is evident from the equations (6.2.21) that the non-static spherically symmetric solutions are expanding, accelerating and rotating but non-shearing. The solutions are proved to be Petrov-type D. However, the dynamic solution (6.3.42) is expanding and rotating with zero acceleration and shear; whereas the static solution is accelerating and rotating with expansion free and shear free. We see that the spin of the gravitating matter influences the geometry of space-times.

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