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PAPER

# Identify spin property of relativistic electrons in fully relativistic laser fields

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## Abstract

A semiclassical method is developed to study the spin evolution of a relativistic electron in an fully relativistic laser pulse. Different from the previous classical method which is based on the direct generalization of nonrelativistic spin precession equation, we perform first-principle calculations on the mean values of various spin operators with respect to a relativistic electron wave packet. It is demonstrated, via theoretical derivation and numerical simulation, that although the Foldy–Wouthuysen operator merits the single-particle interpretation, its mean value obviously deviates from the result of the classical method, which sheds light on not only the understanding of relativistic spin itself but also broad related applications. To achieve a direct observation of such effect, a feasible experimental setup utilizing the asymmetric field of a single-cycle laser is proposed. In such geometry, the deviation is evidenced in the total change of spin which can be easily measured after the interaction.

## 1. Introduction

With the undergoing constructions and commissionings of multi-Petawatt (PW) laser facilities such as ELI [1], Apollon [2], and SULF [3], the peak laser intensity is projected to reach higher-than-ever $10^{23}$ W cm$^{-2}$. Under these circumstances, Quantum ElectroDynamics (QED) effects, including spin effects, high energy photon emission and pair production, begin to play a dominant role in laser–plasma interaction (see references [4, 5] for detailed reviews on strong-field QED effects). These powerful lasers serve as unparalleled experimental platforms for testing various QED effects. The spin angular momentum, as one of the fundamental concepts in quantum mechanics (QM), has recently spurred huge interest in the strong-field QED community [6–13]. To include the spin effects in fully relativistic laser–plasma interaction, the first and foremost question that needs to be addressed is how to correctly describe the spin evolution of electrons in the laser fields. Actually, there are two ways to consider the spin dynamics of relativistic electrons in an external electromagnetic field. One is described by a classical vector whose motion in the external electromagnetic field satisfies the Thomas–Bargmann–Michel–Telegdi (T-BMT) equation [14], which has been employed to model spin precession in fully relativistic laser background [7–9], although the derivation of the T-BMT equation does require that the electromagnetic field should be spatially homogeneous [15]. Based on this approach, there have already been several works proposed to obtain highly polarized lepton beams [7–9]. The other is the quantum operator method, which has been proposed to study spin dynamics with the relativistic spin operator in various forms. This method is based on either numerically solving time-dependent Dirac equation [16, 17] or determining time-dependent

coefficients of the Volkov wave function [18, 19] which is the exact solution of the Dirac equation in a plane wave background field. However, the quantum treatment of spin dynamics requires an additional high-resolution spatiotemporal grid of quantum wave functions, which is difficult to be implemented in particle-in-cell codes that are widely used for studying ultrarelativistic laser–plasma interaction.

In the nonrelativistic Schrödinger QM, the spin of an electron (spin-1/2) is defined by the $2 \times 2$ Pauli matrices $\sigma_i (i = 1, 2, 3)$ which are the generators of a two-dimensional irreducible representation of SU(2) group and are Hermitian, unitary, and traceless. The spin of a nonrelativistic electron can be described equivalently by a three-dimensional vector which we tentatively call here 'the classical spin vector' following reference [18], the components of which are exactly the expectation of spin operators. The spin dynamics defined by the Schrödinger–Pauli equation also fully corresponds to the rotation of classical spin vector in external magnetic fields. However, the clear correspondence between the quantum-mechanical spin operator and the classical spin vector is only a special feature of nonrelativistic QM. Actually, the case in the relativistic regime is much more complicated [20–25]. The direct relativistic generalization of the Pauli spin operator is not commutative with the Dirac Hamiltonian, which leads to the non-conservation of the corresponding spin of a free electron.

The central question in the quantum method is to determine the *relativistic* spin operator, or equivalently the relativistic position operator. This often neglected problem dates back to the early days of QM. In recent several years, the same issue has been encountered by researchers in quantum information [26], electron vortices [27–30], etc. Actually, there are still ongoing debates on which one is the correct operator [23, 27, 31]. Especially, reference [27] holds the opposite view with reference [23]. The former claims that the Czachor operator [32] is superior to the Foldy–Wouthuysen (FW) operator, while the latter regards that the FW operator is the proper relativistic spin operator, except in a few cases. In references [16, 31], seven propositions for the relativistic spin operator are summarized and their properties are analyzed mathematically. Therefore the physical nature of relativistic electron spin is still obscure, whilst it is inevitable in many related fields where the relativity and quantum effects are both important. These operators are basically proposed from the theoretical point of view related to the group-theoretical structure of relativistic QM and have not been strictly tested by experiments. Thus, they have been used to discuss different problems related to the spin of relativistic electrons. As for which kind of operator to use under certain circumstances, there is no strict general conclusion at present, which can only be analyzed individually.

In this paper, in order to explore which operator is suitable to describe the electron spin in fully relativistic laser fields, the spin dynamics of a relativistic electron wave packet interacting with a linearly polarized laser pulse are studied in the framework of relativistic QM. We develop a semiclassical method by evolving a Volkov wave packet, which is described by an ensemble of particles following a Gaussian distribution. Each particle in the ensemble carries a spinor whose coordinates are obtained from the Lorentz equation. In this way, the mean values of the spin operator in various forms can be computed with respect to the Volkov wave packet. Results are compared with the classical model described by the T-BMT equation, which has been widely used in many recent works. We find that in free space, the Pauli and Czachor operators give the electron spin in the laboratory frame while the others describe the electron spin in the stationary frame. In the weakly relativistic regime, these operators have little difference on the description of relativistic electron spin. When the laser field enters the fully relativistic regime, the electron spin described by these operators will be obviously different from each other. In particular, the prediction of the FW operator deviates from that of the classical T-BMT equation, which is apparently different from results in weakly relativistic regime. These seemingly contradicting predictions can only be examined by the experimental observation. Therefore, based on our theoretical analysis, we propose a feasible experimental setup to identify this effect by using the asymmetric field of a single-cycle laser pulse, where the measurement is enabled by detecting whether the electron spin changes or not after the interaction.

The paper is organized as follows. In the next section, we will briefly review the classical spin vector of electron and the T-BMT equation. In section 3, the expectations of various proposed relativistic spin operators for free Dirac particles are discussed. In section 4, we propose a semiclassical method to calculate the evolution of electron spin in a plane-wave background field with the Volkov wave function. In section 5, we simulate the evolution of electron spin in laser fields of different parameters based on classical T-BMT equation and the semiclassical method. The numerical results are compared and discussed. A feasible experimental setup is proposed to discern the Pauli operator and the FW operator. Finally, the results and conclusions of this paper are summarized in section 6.

## 2. Classical spin vector and T-BMT equation

As well known, the spin is an intrinsic property of a quantum particle, which characterizes the mutual transformation between components of the spinor wave function of a particle under spatial rotation transformation. In general, the spin of a particle can be an integer or a half-integer (in the unit of $\hbar$), corresponding to bosons or fermions, respectively. In this paper, we only consider the electron whose spin is $\hbar/2$. Hereafter, we will use the natural units $\hbar = c = 1$ unless given explicitly. In the nonrelativistic QM or in the stationary reference frame of electron in relativistic QM, the electron spin can be polarized in an arbitrary direction $\mathbf{n} = (\sin\vartheta\cos\varphi,\ \sin\vartheta\sin\varphi, \cos\vartheta)$, which means that the spinor wave functions are the eigenvectors of operator $\boldsymbol{\sigma}\cdot\mathbf{n}$ with eigenvalues $\pm1$, which are

$$\chi_+ = \begin{pmatrix} \cos\dfrac{\vartheta}{2}\mathrm{e}^{-\mathrm{i}\varphi} \\ \sin\dfrac{\vartheta}{2} \end{pmatrix}, \qquad \chi_- = \begin{pmatrix} \sin\dfrac{\vartheta}{2}\mathrm{e}^{-\mathrm{i}\varphi} \\ -\cos\dfrac{\vartheta}{2} \end{pmatrix}. \tag{1}$$

The classical spin vector $\boldsymbol{\zeta}$ can be obtained by calculating the expectation value of spin operator $\boldsymbol{\sigma}$ (a factor $\hbar/2$ is omitted for convenience) with respect to $\chi_+$, which is found to be $\boldsymbol{\zeta} = \mathbf{n}$. The classical spin vector could be understood as the intrinsic magnetic moment of the electron up to a constant factor, i.e. $\boldsymbol{\mu} = g\mu_\mathrm{B}\boldsymbol{\zeta}$, where $g$ is the gyromagnetic factor and $\mu_\mathrm{B} = e\hbar/2mc$ is the Bohr magneton, $e$ is the charge and $m$ is the mass of the electron. In a constant magnetic field $\mathbf{H}$, the intrinsic magnetic moment or classical spin vector of the electron will precess as follows,

$$\frac{\mathrm{d}\boldsymbol{\zeta}}{\mathrm{d}t} = \frac{2\mu_\mathrm{B}}{\hbar}\boldsymbol{\zeta}\times\mathbf{H}. \tag{2}$$

This nonrelativistic precession equation can be extended to the case of relativity by taking the classical spin vector as spatial components of a four-dimensional tensor like the orbital angular momentum or as spatial parts of a four-dimensional vector [33, 34]. The extensions by these two methods have been proved to be equivalent in the description of spin precession for a relativistic electron, both of which lead to the famous T-BMT equation. Here, the vector $\boldsymbol{\zeta}$ is considered as the spatial parts of a four-dimensional spin vector $a^\mu$ in the rest frame of the electron, which takes the form [34]

$$a^\mu = (a^0, \boldsymbol{a}) = \left(\frac{\boldsymbol{\zeta}\cdot\mathbf{p}}{m}, \boldsymbol{\zeta} + \frac{\mathbf{p}(\boldsymbol{\zeta}\cdot\mathbf{p})}{m(E_\mathrm{p}+m)}\right), \tag{3}$$

in a general reference frame where the electron moves with velocity $\boldsymbol{v} = \mathbf{p}/E_\mathrm{p}$ ($\mathbf{p}$ and $E_\mathrm{p}$ are the momentum and energy of electron, respectively). Then, we can derive the T-BMT equation in two ways. One is to construct the precession equation of classical spin vector from the perspective of Lorentz-covariance [14]. The other is to transform the electromagnetic field of the laboratory frame into the rest frame of the electron by Lorentz transformation and then put it into the precession equation (2). After that, we transform the equation back to the laboratory system with the contribution from the Thomas precession reduced [15]. However, in the latter method, it is necessary to assume that the two reference frames are connected by only one Lorentz transformation. The covariant form of the T-BMT equation appears as [34]

$$\frac{\mathrm{d}a^\mu}{\mathrm{d}\tau} = 2\mu F^{\mu\nu}a_\nu - 2\mu' u^\mu F^{\nu\lambda}u_\nu a_\lambda, \tag{4}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = (\mathbf{E}, \mathbf{H})$ is the electromagnetic field tensor, $u^\mu = p^\mu/m$ is the four-velocity of electron, $\mu$ is the magnetic moment of electron, $\mu' = \mu - \mu_\mathrm{B}$ is the abnormal magnetic moment, $\tau = t/\gamma$ is the proper time of electron with $\gamma$ the Lorentz factor here. Since the ratio of the abnormal magnetic moment of electron with the normal magnetic moment is about $a_\mathrm{e} = (g-2)/2 \simeq 0.001\,16$ (see table 2 in Mane *et al* 2005 [15]) which is tiny, here the abnormal magnetic moment $\mu'$ is neglected.

After the abnormal magnetic moment is neglected, we can see from (4) that $a_\mu\,\mathrm{d}a^\mu/\mathrm{d}\tau = 0$, which means the magnitude of the spin polarization vector keeps invariant with the electron moving and only the direction of the spin vector changes over time. The spatial part of this equation which determines the variation of direction of electron spin polarization can then be obtained as

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \frac{2\mu m}{E_\mathrm{p}}\boldsymbol{a}\times\mathbf{H} + \frac{2\mu m}{E_\mathrm{p}}(\boldsymbol{a}\cdot\boldsymbol{v})\mathbf{E}. \tag{5}$$

The precession equation of the classical spin vector in the stationary frame of electron $\boldsymbol{\zeta}$ can be derived as

$$\frac{\mathrm{d}\boldsymbol{\zeta}}{\mathrm{d}t} = \boldsymbol{\zeta} \times \left( \frac{2\mu m}{E_\mathrm{p}} \mathbf{H} + \frac{2\mu m}{E_\mathrm{p} + m} \mathbf{E} \times \boldsymbol{v} \right). \tag{6}$$

Here, it is worth noting that these precession equations of the classical spin vector are derived implicitly on the base of the Ehrenfest theorem and the semiclassical approximation of the nonrelativistic QM [14, 15]. When generalized to relativistic QM, the spin operator is neither uniquely defined nor representation independent, which should be treated carefully.

## 3. Basic equation and spin operators for a free Dirac particle

In relativistic QM, the state of an electron is described with the Dirac bispinor wave function $\psi$, the dynamics of which is governed by the Dirac equation

$$\mathrm{i}\frac{\partial}{\partial t}\psi = H_\mathrm{D}\psi, \tag{7}$$

with the Dirac Hamiltonian

$$H_\mathrm{D} = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m, \tag{8}$$

where $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$ and $\beta = \gamma^0$ are the Dirac matrices. One type of the positive-energy wave function in the Dirac representation appears as (with a factor $\exp(\mathrm{i}p \cdot x)$ implied)

$$u_\mathrm{p}^s = \sqrt{\frac{E_\mathrm{p} + m}{2m}} \begin{pmatrix} \chi_s \\ \dfrac{\sigma \cdot \mathbf{p}}{E_\mathrm{p} + m}\chi_s \end{pmatrix}, \tag{9}$$

where $s = \pm 1$ denote different spin states of the electron, $\chi_s$ are the nonrelativistic two-spinors (1) and the Dirac bispinor is normalized by $\bar{u}_\mathrm{p}^r u_\mathrm{p}^s = \delta^{rs}$, or equivalently ${u_\mathrm{p}^r}^\dagger u_\mathrm{p}^s = (E_\mathrm{p}/m)\delta^{rs}$. Another positive-energy wave function can be constructed as

$$w_\mathrm{p}^s = \sqrt{\frac{E_\mathrm{p} + m}{2E_\mathrm{p}}} \begin{pmatrix} \chi_s \\ \dfrac{\sigma \cdot \mathbf{p}}{E_\mathrm{p} + m}\chi_s \end{pmatrix} \tag{10}$$

which follows the normalization ${w_\mathrm{p}^r}^\dagger w_\mathrm{p}^s = \delta^{rs}$ (not to be confused with the negative-energy wave function). Both of them have been used in the literature. The former is employed in references [6, 35] while the latter is used in references [16, 18, 31]. In fact, these two normalization conditions essentially reflect the coordinate system where we are considering the problem. If it is considered in the laboratory system, the former is used; if it is considered in the stationary frame of the electron, the latter is used.

As in Schrödinger's nonrelativistic theory, the scalar product in the relativistic QM is defined by

$$\langle\psi|\phi\rangle = \int \mathrm{d}^3x\, \psi^\dagger(x)\phi(x), \tag{11}$$

and the expectation of an operator $\hat{O}$ is defined as

$$\langle\psi|\hat{O}|\psi\rangle = \int \mathrm{d}^3x\, \psi^\dagger(x)\hat{O}\psi(x), \tag{12}$$

both of which hold in Dirac representation as well as in FW representation, although the spin and position operators in the two representations have different forms.

As we have mentioned above, various relativistic spin operators have been discussed in the literature, and there is still controversy about which operator can correctly describe the relativistic electron spin. Now let's analyze some conclusions in the literature in detail [23]. We know that for a free Dirac particle, its space–time symmetry can be described by Poincaré group. The Poincaré group has ten generators, including the angular momentum vector operator $\mathbf{J}$, which describes the spatial rotation transformation. The corresponding physical quantity is the total angular momentum of Dirac particles. As well known, the

**Table 1.** Brief summary of spin operators' definitions, normalization of wave functions and their expectations of spin operators with respect to the free Dirac bispinor.

| Name | Definition | Normalization | Expectation |
|---|---|---|---|
| Pauli | $\hat{\mathbf{S}}_{\mathrm{P}}=\frac{1}{2}\boldsymbol{\Sigma}$ | $\bar{u}u=1$ | $u^{\dagger}\hat{S}u=\frac{1}{2}\boldsymbol{a}$ |
| Czachor [32] | $\hat{\mathbf{S}}_{\mathrm{Cz}}=\frac{m^2}{2p_0^2}\boldsymbol{\Sigma}+\frac{im\beta}{2p_0^2}\mathbf{p}\times\boldsymbol{\alpha}+\frac{\mathbf{p}\cdot\boldsymbol{\Sigma}}{2p_0^2}\mathbf{p}$ | $\bar{u}u=1$ | $u^{\dagger}\hat{S}u=\frac{1}{2}\boldsymbol{a}$ |
| Foldy–Wouthysen [20] | $\hat{\mathbf{S}}_{\mathrm{FW}}=\frac{1}{2}\boldsymbol{\Sigma}+\frac{i\beta}{2p_0}\mathbf{p}\times\boldsymbol{\alpha}-\frac{\mathbf{p}\times(\boldsymbol{\Sigma}\times\mathbf{p})}{2p_0(p_0+m)}$ | $w^{\dagger}w=1$ | $w^{\dagger}\hat{S}w=\frac{1}{2}\boldsymbol{\zeta}$ |
| Pryce [36] | $\hat{\mathbf{S}}_{\mathrm{Pr}}=\frac{1}{2}\beta\boldsymbol{\Sigma}+\frac{1}{2}\boldsymbol{\Sigma}\cdot\mathbf{p}(1-\beta)\frac{\mathbf{p}}{\mathbf{p}^2}$ | $w^{\dagger}w=1$ | $w^{\dagger}\hat{S}w=\frac{1}{2}\boldsymbol{\zeta}$ |
| Chakrabarti [37] | $\hat{\mathbf{S}}_{\mathrm{Ch}}=\frac{1}{2}\boldsymbol{\Sigma}+\frac{\mathrm{i}}{2m}\boldsymbol{\alpha}\times\mathbf{p}+\frac{\mathbf{p}\times(\boldsymbol{\Sigma}\times\mathbf{p})}{2m(p_0+m)}$ | $w^{\dagger}w=1$ | $w^{\dagger}\hat{S}w=\frac{1}{2}\boldsymbol{\zeta}$ |

total angular momentum is the vector sum of spin and orbital angular momentum, $\mathbf{J}=\mathbf{l}+\mathbf{s}=\boldsymbol{x}\times\mathbf{p}+\mathbf{s}$. We can see that $\mathbf{l}$ and $\mathbf{s}$ operators, or equivalently $\boldsymbol{x}$ and $\mathbf{s}$ operators, can not be completely determined just following the consideration of symmetry. They can be arbitrarily selected, as long as $\mathbf{J}$ remains unchanged. In other words, the use of the ten-parameter Poincaré group leaves room for different definitions of the position and spin operators. In reference [23], the different choices of relativistic position, spin, and orbital angular momentum operators are discussed in detail and three self-consistent sets of fundamental FW operators which have different meanings are given. Among the three sets of operators, the projected operator can be used to describe Berry phase effects, while the canonical FW operators which are just named FW operators in other references should be used in most cases.

The main reason is that the FW operator has many advantages. For example, the position, spin, orbital angular momentum, and other operators in FW representation can well correspond to the classical physical quantities, and only the wave functions in FW representation can be interpreted as probability amplitudes, and it will not lead to the spurious *zitterbewegung* at all.

However, the FW operator is criticized for some characteristics such that it leads to separate conservation of spin and orbital angular momentum of a relativistic free electron and no spin–orbit interaction (SOI) of free particles. For this reason, Bliokh *et al* [27] argue that the projected operator (corresponding to the Czachor operator in reference [31]) is more reasonable than the FW operator because it can give the SOI naturally. Seven proposals for the relativistic spin operators are summarized in reference [31] and their properties are analyzed mathematically. Although the FW spin operator corresponding to the classical spin vector is considered to be the correct operator to describe the relativistic electron spin in many cases [18, 23], the electron in the fully relativistic laser field is usually extremely relativistic and other operators cannot be completely excluded. For this reason, we consider five spin operators which are more likely to describe the real physical spin of a relativistic electron, i.e. Pauli, Czachor, FW, Pryce, and Chakrabarti operators, named following reference [31].

By substituting these operators into equation (12), the classical spin vectors corresponding to these operators are obtained by calculating the expectation values of relativistic free electrons. The results are shown in table 1. We can see clearly from table 1 that for a free relativistic electron, the expectation of the Pauli and Czachor operators correspond to the classical spin vector in the laboratory frame, while the expectation of FW, Chakrabarti, and Pryce operators correspond to the classical spin vector in the stationary frame of the electron. This also makes us understand why we need to use bispinor (9) when calculating the expectation values of the first two operators, and use bispinor (10) when calculating the expectation values of the last three operators. This is consistent with the results from the analysis of positive energy subspace of Dirac bispinors in reference [31]: the Pauli and Czachor operators yield the same expectation value in the positive-energy subspace, and FW, Pryce, and Chakrabarti operators are also equivalent in the positive-energy subspace of free-particle states. At the same time, we confirm the discussion in reference [18] that the FW operator is equivalent to the Lorentz transformation of the Pauli operator. We also calculate two other operators in table 1 of reference [31], the Frenkel [38] and Fradkin–Good [22] operators, with their expectation values, $\langle\mathbf{S}_{\mathrm{F}}\rangle=\frac{p_0^2}{2m^2}\boldsymbol{a}-\frac{\mathbf{p}\mathbf{p}\cdot\boldsymbol{a}}{2m^2}$ where bispinor (9) is employed and $\langle\mathbf{S}_{\mathrm{FG}}\rangle=\frac{1}{2}\boldsymbol{\zeta}$ where bispinor (10) is used, respectively. The expectation of the Frenkel operator is neither the classical spin vectors in the laboratory system nor the electron rest system, which means that it is unlikely to be the correct operator to describe the relativistic electron spin. In the semiclassical approximation, the Fradkin–Good operator always coincides with the Pryce operator. Thus, neither of them will be considered in the later discussion.

When calculating the expectation values of these spin operators, some basic expressions are needed, which are actually 16 independent Lorentz structures of Dirac bispinor space in quantum field theory. When we use bispinor (9), they are

$$\bar{u}_{\rm p}^{s}\gamma^{\mu}u_{\rm p}^{s}=\frac{p^{\mu}}{m}=\gamma v^{\mu}, \tag{13}$$

$$\bar{u}_{\rm p}^{s}\gamma^{\mu}\gamma^{5}u_{\rm p}^{s}=a^{\mu}, \tag{14}$$

$$\bar{u}_{\rm p}^{s}\gamma^{\mu}\gamma^{\nu}u_{\rm p}^{s}=g^{\mu\nu}+\frac{\mathrm{i}}{m}\epsilon^{\mu\nu\rho\sigma}p_{\rho}a_{\sigma}, \tag{15}$$

where $\bar{u}_{\rm p}^{s}=u_{\rm p}^{s\dagger}\gamma^{0}$ is the Dirac conjugation. When bispinor (10) is used, the corresponding results will have an additional factor of $1/\gamma$ ($\gamma=E_{\rm p}/m$ is the Lorentz factor, not to be confused with the Dirac $\gamma$ matrices), they become

$$\bar{w}_{\rm p}^{s}\gamma^{\mu}w_{\rm p}^{s}=\frac{p^{\mu}}{E_{\rm p}}=v^{\mu}, \tag{16}$$

$$\bar{w}_{\rm p}^{s}\gamma^{\mu}\gamma^{5}w_{\rm p}^{s}=\frac{1}{\gamma}a^{\mu}, \tag{17}$$

$$\bar{w}_{\rm p}^{s}\gamma^{\mu}\gamma^{\nu}w_{\rm p}^{s}=\frac{1}{\gamma}g^{\mu\nu}+\mathrm{i}\epsilon^{\mu\nu\rho\sigma}v_{\rho}a_{\sigma}. \tag{18}$$

The vector current corresponds to the four-momentum or four-velocity, while the axial-vector current corresponds to the four-dimensional polarization vector, respectively. Any other operators acting on the bispinor space can be decomposed into a linear combination of these structures.

Summing up the expectation values of these operators, all of them can be used to describe the spin of free relativistic electrons. However, the Pauli and Czachor operators describe the spin of electrons in the laboratory system, while the FW, Chakrabarti, and Pryce operators correspond to the spin of an electron in its static reference system. They can be related by a Lorentz transformation. In other words, they are equivalent for describing the spin of relativistic free electrons. Of course, this does not guarantee that the equivalence still exists when there is an external field. The expectations of different spin operators may evolve differently in an external electromagnetic field.

## 4. Relativistic spin evolution in external plane-wave fields

In this section, we present a semiclassical approach to describe the electron-spin dynamics based on relativistic QM in the strong electromagnetic field of a plane wave laser. As we know, when there is a background of strong electromagnetic fields, the Dirac equation can be solved accurately only in a few cases, such as plane wave, constant crossed field, etc. Among these exact solutions, the Volkov solution obtained in the plane wave background field is the most widely used one, with its form as follows,

$$\psi_{\rm p}^{s}(x)=\left(1+\frac{e\not{k}\not{A}(\eta)}{2p\cdot k}\right)u_{\rm p}^{s}\,\mathrm{e}^{-\mathrm{i}S_{\rm p}(x)}, \tag{19}$$

with $k_{\mu}$ the external electromagnetic wave vector, $A^{\mu}$ the four-vector potential, $\eta=k\cdot x=k^{\mu}x_{\mu}=\omega t-\boldsymbol{k}\cdot\boldsymbol{x}$, and the Feynman slash notation $\not{A}=\gamma_{\mu}A^{\mu}$ is used. The Volkov wavefunction depends on time via the field $A^{\mu}(x)$ and the exponential factor $S_{\rm p}(x)$, where $x^{\mu}=(t,\boldsymbol{x})$. $u_{\rm p}^{s}$ is the free Dirac bispinor (9) which satisfies

$$\not{p}u_{\rm p}^{s}=mu_{\rm p}^{s}. \tag{20}$$

If the other bispinor (10) with different normalization is employed, we can just replace $u_{\rm p}^{s}$ in (19) by (10). The exponential factor $S_{\rm p}(x)$ is

$$S_{\rm p}(x)=p\cdot x+g_{\rm p}(\eta), \tag{21}$$

$$g_{\rm p}(\eta)=\int_{0}^{\eta}\left(\frac{ep\cdot A(\varphi)}{p\cdot k}-\frac{e^{2}A^{2}(\varphi)}{2p\cdot k}\right)\mathrm{d}\varphi. \tag{22}$$

The factor

$$\mathfrak{R} = 1 + \frac{e\not{k}\not{A}(\eta)}{2p\cdot k} \tag{23}$$

is usually referred to as the Ritus matrix. The Volkov wave function is normalized by [39]

$$\langle \psi_{\rm p}^{r}(x)|\psi_{\rm p}^{s}(x)\rangle = \int \bar{\psi}_{\rm p}^{r}(x)\psi_{\rm p}^{s}(x){\rm d}^3\boldsymbol{x} = V\delta^{rs}. \tag{24}$$

The Volkov wave function completely describes the electron dynamics in a plane-wave background field. We know from QM that any physical observable can be obtained by calculating the expectation of the corresponding operator with respect to it. Therefore, the expectation of spin operator $\hat{\mathbf{S}}$ can be calculated by the following formula [40]

$$\langle \hat{\mathbf{S}}\rangle = \int \psi_{\rm p}^{s\dagger}(x)\hat{\mathbf{S}}\psi_{\rm p}^{s}(x){\rm d}^3\boldsymbol{x}. \tag{25}$$

The expectation value (25) is generally not easy to calculate. In order to simplify the calculation, we follow the method used in the derivation of the T-BMT equation in reference [15] where the semiclassical approximation is assumed. Under this assumption, we shall approximately replace the operators $\boldsymbol{x}$ and $\mathbf{p}$ with the c-number expectation values $\langle\boldsymbol{x}\rangle$ and $\langle\mathbf{p}\rangle$ which denote the classical coordinate and momentum values of the electron along its classical trajectory, that is, $\langle\boldsymbol{\Sigma}(\boldsymbol{x},\mathbf{p})\rangle \simeq \boldsymbol{\Sigma}(\langle\boldsymbol{x}\rangle,\langle\mathbf{p}\rangle)$ under the semiclassical approximation. Meanwhile, the integral contributes a factor of $V$, the total space volume, which would be canceled out by the same factor in the normalization condition (24), see appendix A.

The key to calculate the expectation (25) is to find the integrand function $\psi_{\rm p}^{s\dagger}\hat{\mathbf{S}}\psi_{\rm p}^{s}$. The spin operator $\hat{\mathbf{S}}$ can be any spin operator shown in table 1. These operators are in the form of Dirac representation, so the wave function in the integral is (19). Obviously, different operators may lead to different expectations. Among these operators, the Pauli operator is the simplest one, which does not depend on the momentum $\mathbf{p}$ of the electron. Generally, the following covariant generalization can greatly simplify the calculation, i.e.

$$\boldsymbol{\Sigma} = \gamma^0\boldsymbol{\gamma}\gamma^5 \Rightarrow S^\mu = \gamma^0\gamma^\mu\gamma^5. \tag{26}$$

The expectation with respect to the Volkov wave function (19) can be calculated as [6]

$$\langle S^\mu\rangle = a^\mu - eA^\mu\frac{k\cdot a}{p\cdot k} + \frac{k^\mu}{p\cdot k}\left(eA\cdot a - \frac{e^2A^2k\cdot a}{2p\cdot k}\right), \tag{27}$$

which is exactly the solution of the covariant T-BMT equation (4). It is worth noting that the momentum $p^\mu$ in (27) which comes from the Volkov wave function (19) denotes the free momentum of the electron when the external electromagnetic field is absent, which has been clearly explained in the famous review of Ritus [41] and has been widely accepted by the strong-field QED community. The space–time coordinate $x^\mu$ implied in $A(\eta)$ is the position of the electron along its classical trajectory.

In principle, the calculation of other spin operators is similar, but the strict derivation is very difficult because all of them depend on the momentum operator of the electron. Fortunately, with the help of the semiclassical approximation mentioned above, we can replace the coordinate $\boldsymbol{x}$ and momentum operators $\mathbf{p}$ contained in the expected formula (25) with classical kinematic quantities, so that the integral of coordinates will be removed, and the rest of the calculation becomes the matrix operation in the spinor space. The matrix calculation of spinor space can be completed by use of expressions (13)–(18). In order to make the expectation of the spin obtained from the Volkov solution be the same as that of the free electron when the external field tends to zero, we should choose the same bispinor for each operator as that in the free electron case.

The classical coordinate and momentum of the electron in a plane-wave electromagnetic field can also be obtained theoretically by means of the Volkov wave function. However, the more convenient way is to obtain these quantities directly through the classical Lorentz equation, which is the method we will employ in the next section. Therefore, to sum up, our semiclassical method is to use the Lorentz equation to calculate the coordinates and momentum of the electron at a certain moment in the plane wave field and then use the Volkov wave function to calculate the expectation of the electron spin operator at this moment

under the semiclassical approximation, so as to obtain the evolution of the spin at any time. A lengthy calculation leads to the result, in the explicit form,

$$\begin{aligned}\langle\hat{\mathbf{S}}_{\mathrm{FW}}\rangle = &\frac{m}{2\pi_0}\frac{m}{p_0}\left(\boldsymbol{a}-e\mathbf{A}\frac{k\cdot a}{p\cdot k}+\mathbf{k}\frac{eA\cdot a}{p\cdot k}-\mathbf{k}\frac{e^2A^2k\cdot a}{2(p\cdot k)^2}\right)\\&+\frac{\boldsymbol{\pi}\boldsymbol{\pi}}{2\pi_0(\pi_0+m)}\cdot\frac{m}{p_0}\left(\boldsymbol{a}-e\mathbf{A}\frac{k\cdot a}{p\cdot k}+\mathbf{k}\frac{eA\cdot a}{p\cdot k}-\mathbf{k}\frac{e^2A^2k\cdot a}{2(p\cdot k)^2}\right)\\&-\frac{1}{2m\pi_0}\boldsymbol{\pi}\times\frac{m}{p_0}\left\{\epsilon^{0j\rho\sigma}p_\rho a_\sigma+\frac{e}{p\cdot k}\left(k^j\epsilon^{0\tau\rho\sigma}A_\tau p_\rho a_\sigma-k^0\epsilon^{j\tau\rho\sigma}A_\tau p_\rho a_\sigma\right.\right.\\&\left.-A^j\epsilon^{0\tau\rho\sigma}k_\tau p_\rho a_\sigma+A^0\epsilon^{j\tau\rho\sigma}k_\tau p_\rho a_\sigma\right)-\frac{e^2}{2(p\cdot k)^2}\left[2(k^0A^j-k^jA^0)\right.\\&\left.\left.\times\epsilon^{\alpha\beta\rho\sigma}A_\alpha k_\beta p_\rho a_\sigma+A^2\left(k^j\epsilon^{0\tau\rho\sigma}k_\tau p_\rho a_\sigma-k^0\epsilon^{j\tau\rho\sigma}k_\tau p_\rho a_\sigma\right)\right]\right\}.\end{aligned}\tag{28}$$

The notations in this equation needs some explanation. $\pi_0$ is the zero-component of the kinematic four-momentum of the electron, $\pi_\mu=(\langle p_0\rangle,-\langle\mathbf{p}\rangle)=(\pi_0,-\boldsymbol{\pi})$, which turns out to be [6, 41]

$$\pi_\mu=p_\mu-eA_\mu+k_\mu\left(\frac{eA\cdot p}{k\cdot p}-\frac{e^2A\cdot A}{2k\cdot p}\right).\tag{29}$$

$p_\mu$ is the initial momentum of electron in the Volkov wave function (19), and $p_0=E_\mathrm{p}$ is the zero-component of $p_\mu$. In order to simplify the notation we use simultaneously the bold characters and the Latin index $j$ ($j=1,2,3$) to represent the spatial three-vectors, while the Greek alphabets denotes four-dimensional indices.

Taking the derivative of equation (28) with respect to time, we find that it is not the solution of the T-BMT equation (6) (see appendix B), which means that the relativistic electron spin described by the FW operator is different from the classical spin vector described by the T-BMT equation. Nevertheless, the discrepancy will disappear under the nonrelativistic limit. In the next section, we will illustrate this difference by simulating some cases under specific parameters of external plane-wave laser fields. In order to make the results more reliable, we will use wave packet instead of Volkov wave function to simulate the electron in the laser fields.

## 5. Results and discussions

As shown in the schematic figure 1(a), consider the case where a Volkov wave packet head-on collides with an intense laser pulse. The laser propagation direction is the $+x$ axis. Namely, the wave vector of the laser pulse $k^\mu=(\omega,\omega,0,0)$, where $\omega=1.55$ eV is the angular frequency of a typical Ti:sapphire high-intensity femtosecond laser device. The four-vector potential of the laser electromagnetic field is expressed as

$$A^\mu=A\varepsilon^\mu f(\eta)\sin\eta\tag{30}$$

$$f(\eta)=\begin{cases}\cos^2\left(\dfrac{\eta-\eta_0}{2N_\mathrm{c}}\right), & |\eta-\eta_0|<N_\mathrm{c}\pi\\ 0, & \text{else}\end{cases},\tag{31}$$

where $\eta=\omega(x-t)+\phi_0$ is the phase, $x$ here is the coordinate along $x$-axis, $A$ is the amplitude, and $\varepsilon^\mu=(0,0,1,0)$ is the polarization direction. The laser temporal profile is controlled by $\eta_0$ and $N_\mathrm{c}$, which indicates the laser duration $\tau=N_\mathrm{c}T_0$, where $T_0=2\pi/\omega$ is the period. $\eta_0$ represents the initial location of the laser peak. To avoid the initial overlap of the laser field and the wave packet, $\eta_0=-10\pi$ is used. Henceforth, the laser amplitude is varied to illustrate the transition from the nonrelativistic regime to the relativistic regime. Finally, a feasible experimental setup to discern different spin operators, in which a single-cycle laser is employed, is proposed.

To study the spin evolution of an electron in this laser field, the semiclassical method described in the previous section is employed to bridge the Lorentz equation and the Dirac equation, which is based on the Ehrenfest theorem and the semiclassical approximation. For plane-wave Volkov wave function, the outline of our method is as follows: (a) initialize $\boldsymbol{x}$ and $\mathbf{p}$ coordinate for the a classical particle. (b) Integrate the Lorentz equation to obtain the whole trajectory. (c) For each time $t$, substitute $\boldsymbol{x}(t)$ and $\mathbf{p}(t)$ into equation (19) to obtain the Volkov wave function. (d) Substitute $\boldsymbol{x}(t)$ and $\mathbf{p}(t)$ into the spin operators

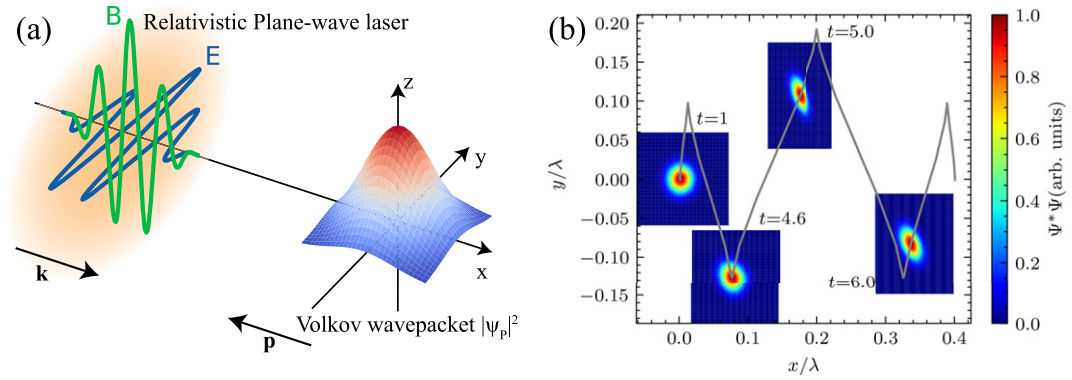


**Figure 1.** (a) Schematic view of the process. A Volkov wave packet head-on collides with a plane-wave laser pulse. (b) Evolution of the electron wave packet distribution in the linearly polarized plane wave at different times $t = 1.0, 4.6, 5.0, 6.0T_0$. The pseudocolor plots denote the wave packet's probability density at time $t$. The solid grey line indicates the trajectory of the center of the wave packet. The electron wave packet is initially at rest and the laser propagates from left to right. The normalized amplitude of the laser $a_0 = 1$ and the duration $\tau = 4T_0$.

table 1. (e) Calculate the expectation value following equation (25). The procedure of our method is described in detail in the following.

Initially, a Volkov wave packet is formulated as

$$\Psi = N \int \exp\left(-\frac{(\mathbf{p}-\mathbf{p}_0)^2}{\delta^2}\right) \psi_{\mathbf{p}} \mathrm{d}^3 p, \tag{32}$$

where $N$ is the normalization constant, $\delta = \hbar/0.02\lambda$ is the width of the Gaussian distribution in momentum space and $\mathbf{p}_0$ is the initial momentum. The wave packet (equation (32)) is constructed by superimposing the positive-energy solutions (equation (19)) at all momentums $\mathbf{p}$ weighted by a Gaussian distribution. To simulate the Gaussian wave packet in the semiclassical framework, a Monte-Carlo trajectory method is employed. The wave packet is represented by an ensemble of one million classical electrons initially in the $z = 0$ plane, whose coordinates and momentums obey the three-dimensional Gaussian distributions as shown in equation (32). It has been proven by Fu *et al* [17] that this method could well reproduce the results from quantum split-operator method. The polarization of the electron wave packet is arbitrarily chosen as $\vartheta = \pi/4$ and $\varphi = -\pi/4$ and we choose $\chi_+$ as the eigenvector to construct the Volkov solutions (equation (1)). The spatial motion of each electron in the ensemble is described by the Lorentz equation

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = e\mathbf{E} + e\mathbf{v} \times \mathbf{B}. \tag{33}$$

In our calculation, the Lorentz equation is solved via the famous LSODA algorithm (see section 3.6 in [42]), which is now shipped with many standard scientific libraries in programming languages like Python (SciPy), Matlab, and Mathematica etc. We have also performed benchmark test for solving the Lorentz equation with this method and confirmed its validity in our calculations. Furthermore, for the problem considered here, we have decreased the stepping of the algorithm by ten and even one hundred times respectively to ensure the convergence of our method.

The Volkov wave function at a certain time $t$ is calculated by updating the vector potential $A^\mu(x)$ in the Ritus matrix (equation (23)), where $x$ is the location of electron at time $t$ from the Lorentz equation. Here the spin-induced Stern–Gerlach force is neglected because it has already been shown that the Lorentz force dominates over the Stern–Gerlach force with the parameters used here [15, 43]. The radiation reaction (RR) effect is also not considered because the field strength and the Lorentz factor we choose here are not strong enough to manifest RR.

Before we dive into the problem of spin evolution, it is of benefit to check if our method could reproduce the known results for wave packet dynamics in the linearly polarized plane wave. Figure 1(b) gives the wave packet distribution at different timesteps, along with the classical trajectory obtained from the Lorentz equation. A localized wave packet in a linearly polarized plane-wave field undergoes an oscillation in the polarization direction and a drift in the laser propagation direction [17, 44], which indicates that our method is a valid way to study the wave packet evolution in a laser field.

Subsequently, the evolution of spin can be computed by taking the expectation value of different spin operators using equation (25). Different operators are considered separately. It is analytically and numerically found that the Pauli and the Czachor operators yield the same dynamics because the

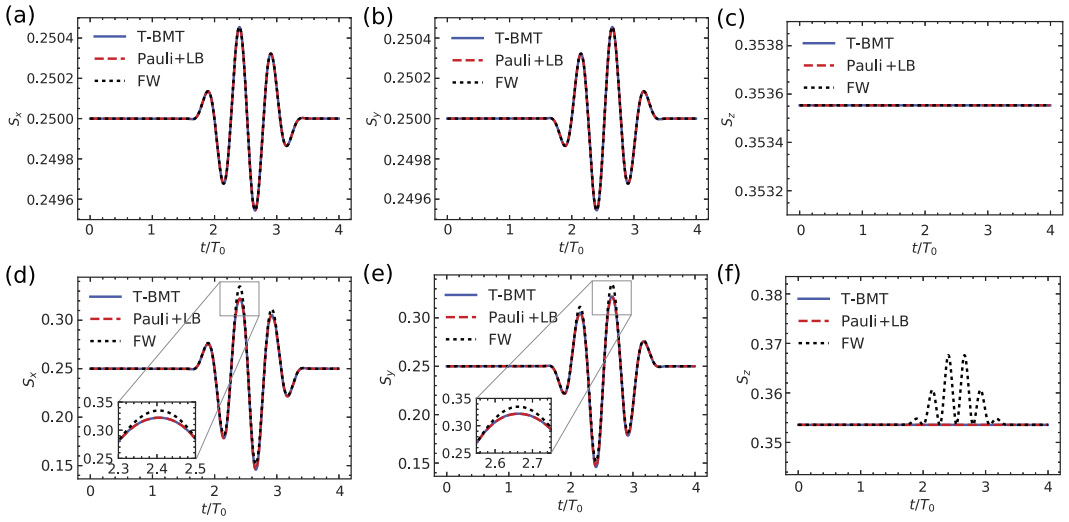


**Figure 2.** Evolution of the spin components $s_x$ (a) and (d), $s_y$ (b) and (e), and $S_z$ (c) and (f) of an Volkov wave packet head-on collides a linearly polarized laser field in different models. The T-BMT (blue solid) results are calculated with equation (5). The red lines are obtained by taking the expectation value of the Pauli operator and performing a Lorentz boost afterward, which is denoted as 'Pauli + LB'. The black dotted lines represent the evolution of the expectations of the FW operator. We sampled 1000 points along the trajectory for both 'Pauli + LB' and FW operator. The initial kinetic momentum of the wave packet $\mathbf{p}_0 = 5m\hat{\mathbf{x}}$. The initial polarization direction of the wave packet is $\vartheta = \pi/4$ and $\varphi = \pi/4$, which indicates the initial classical spin vector $\boldsymbol{\zeta} = (0.25, 0.25, 0.3535)$. In (a)–(c), the laser amplitude $a_0 = 0.01$, denoting the mildly relativistic regime (the electron is relativistic but the laser amplitude is not). All three models produce the same results. In (d)–(f), the laser amplitude $a_0 = 2.0$, which indicates the onset of the fully relativistic effect. The T-BMT results can be reproduced with the Pauli operator combined with a Lorentz boost, while the FW operator slightly deviates from the other two models.

parameters of interest are within the positive energy subspace. For similar reasons, the FW, the Pryce, and the Chakrabarti operators predict the same dynamics. For simplicity, the results from two typical operators, which are the Pauli operator and the FW operator, are demonstrated here. As it is demonstrated in the last section, the Pauli operator describes the spin in the laboratory frame. Therefore, a Lorentz boost is applied to the results of the Pauli operator, which we denote as 'Pauli + LB'. Besides, in the classical T-BMT model, the evolution of spin is described by equation (6), which is solved simultaneously with the Lorentz equation.

The simulation results are shown in figure 2. To begin with, the case, where the laser amplitude is not relativistic, is shown in figures 2(a)–(c). The dimensionless laser amplitude $a_0 = eA/m$ (not to be confused with the polarization vector in section 2) is set to 0.01 and the Volkov wave packet has an initial kinetic momentum $\mathbf{p}_0 = 5m\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ denotes the unit vector in the $x$-axis. It is obvious that at the nonrelativistic limit, the classical T-BMT equation, the Pauli + LB model, and the FW operator give the same result, which verifies the consistency of different models. However, the discrepancies appear when the relativistic effect becomes important. In figures 2(d)–(f), the laser amplitude $a_0$ is changed to 2.0, which enters the fully relativistic regime. With such parameters, the FW operator starts to predict different results from the other two, which is proved analytically in the previous section.

To distinguish the correct operator in the experiment, the biggest problem is to measure the electron spin dynamics inside the laser field, which is still an open question [31, 45]. What is meaningful is the total change of spin, which can be obtained by measuring the spin before and after the interaction with lasers and comparing the measured data. The spin of a relativistic particle can be measured using the Mott scattering, the Møller scattering, or the Compton scattering, The problem is now turned into how to make the total change of spin nonzero. Referring to the Lawson–Woodward theorem, which states that an electron cannot gain any energy from interaction with an ideal plane-wave laser pulse, a similar conjecture can be made that an ideal plane wave laser pulse can not leave a total change of spin. Hence, a single-cycle laser [46], which breaks the symmetry, could in principle be used to discern different spin operators. The theoretical and numerical analysis is given in the following.

For simplicity, the initial polarization direction $\boldsymbol{\zeta}$ is set as $-z$, which is parallel to the magnetic field of the laser pulse. In this way, using the relation $\boldsymbol{\zeta} \cdot \mathbf{p_0} = 0$ and $\boldsymbol{\zeta} \cdot \mathbf{A} = 0$, where $\mathbf{A}$ is the vector potential, the $z$-component of equation (28) can be simplified as

$$\langle \hat{\mathbf{S}}_{\mathrm{FW}} \rangle_z = \frac{E_{\mathrm{kin}}}{E_0} \zeta_z, \tag{34}$$

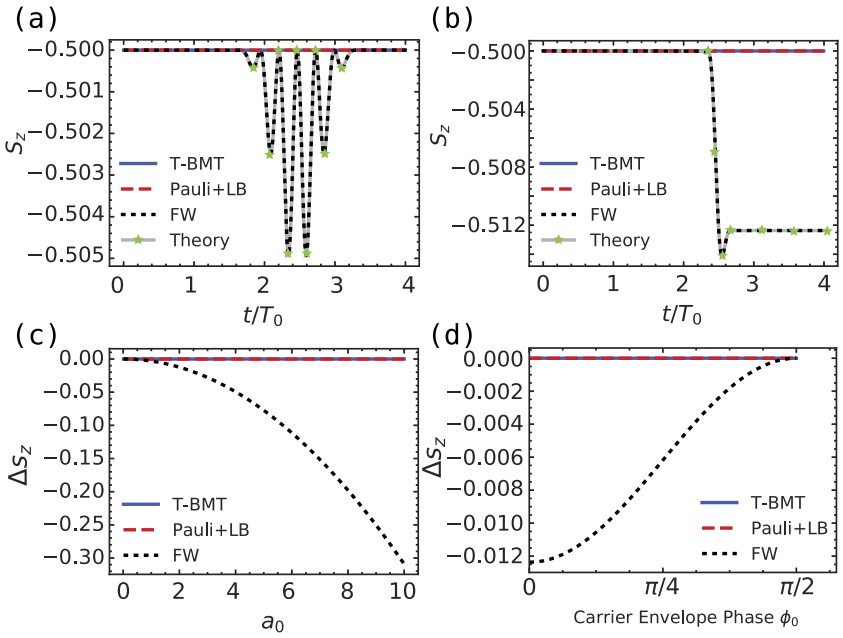


**Figure 3.** Evolution of the spin component $S_z$ in a multi-cycle ($N_c = 4$) laser field (a) and a single-cycle laser field (b). A total change of $S_z$ is found in the single-cycle laser case. The initial momentum of the wave packet is $\mathbf{p}_0 = 10m\hat{\mathbf{x}}$. The laser amplitude $a_0 = 2.0$. The initial polarization direction of the wave packet is $-z$. Other parameters are the same with figure 2. Solid grey lines with star markers are obtained using equation (34). (c) The total change of spin as a function of the laser amplitude. (d) The total change of spin as a function of the laser carrier envelope phase $\phi_0$.

where $E_{\text{kin}}$ is the kinetic energy and $E_0$ is the initial energy of the electron. This expression indicates an oscillation of spin in the magnetic field direction (see the dotted lines in figures 3(a) and (b)). In the T-BMT model, however, the spin does not change along the magnetic field direction as shown by the blue solid lines in figure 3. Shown by the red dashed lines are the results from the Pauli + LB model, which apparently also remain unchanged. When the cycles of the laser pulse is long enough, $E_{\text{kin}}$ is always equal to $E_0$ after the electron exits the laser pulse, as can be seen from figure 3(a). Thus, a single-cycle laser is preferable to make the electron obtain a total change of spin. Figure 3(b) shows that the electron receives a total change of 0.012, which can be measured by the techniques mentioned earlier. The dependence of the total change of $S_z$ on the laser parameters are also investigated in figures 3(c) and (d). As shown in equation (34), the total spin change increases as the laser amplitude $a_0$ increases. The total change of $S_z$ is also dependent on the carrier envelope phase $\phi_0$ of the laser, since the asymmetry is maximized for $\phi_0 = 0$. When $\phi_0 = \pi/2$, the temporal profile of the single-cycle laser becomes symmetric and hence the total change of spin is zero.

After we have finished our work, we find that Aleksandrov *et al* [18] have also studied relativistic spin dynamics in a strong unipolar laser field recently. They found that in a unipolar laser field with similar parameters ($a \approx 0.07$) as in figures 2(a)–(c), the evolution of relativistic electron spin described by the FW operator is consistent with the T-BMT equation while the Pauli operator gives a different result, which seems contradictory to the results here. This contradictory might comes from the semiclassical approximation applied here. However, we should mention that the consistency between the expectation value of Pauli spin operator with respect to the Volkov wave function and the T-BMT equation has been proved analytically in section II B of reference [6], which can be also seen from equations (B.5) and (B.6) of appendix B.

## 6. Conclusion

In nonrelativistic QM, the spin operator of the electron is completely determined, which is the famous Pauli matrices. However, when it is extended to relativistic QM, the definition of the electron spin operator is not unique and there is no universally accepted spin operator in the Dirac theory so far. From the physical point of view, the stationary electron and the electron with definite non-zero momentum have different symmetries under the spatial rotation transformation. The former corresponds to the SO(3) group, while the latter only has the SO(2) rotational symmetry around momentum direction. It can be seen that although spin is the 'internal' degree of freedom of a Dirac particle, it is related to the direction of

momentum in the case of relativity. This results in more than one spin operators can be defined when we describe the relativistic electron spin, as long as the projection of these operators in the direction of momentum gives the correct physical description, i.e. helicity. For this reason, although a lot of discussions about relativistic electron spin operators have been carried out in literature [18, 23, 27, 47–49], and the FW operator is thought to correspond with classical physical quantities, we still can not determine which operator is generally applicable in fully relativistic lasers.

To investigate which operator can correctly describe the relativistic electron spin dynamics in fully relativistic laser fields, we propose a semiclassical method under the framework of relativistic QM and results are compared with the classical T-BMT equation. We find that all these operators can well describe the spin of free relativistic electrons. Among them, the Pauli and Czachor operators describe the spin of electrons in the laboratory system, while the FW, Pryce, and Chakrabarti operators describe the spin of electrons in its rest frame. When there is an external field, the equivalence of these operators no longer holds. When the laser field enters the fully relativistic regime, the electron spin described by these operators turn out to be obviously different. In particular, the prediction of the FW operator is different from that of the classical T-BMT equation, while the classical result can be well produced by the Pauli operator with a Lorentz boost, which in contradiction with results of reference [18]. This discrepancy might come from the semiclassical approximation used in the calculations here. Based on the theoretical analysis, we also propose a feasible experimental setup to identify this effect by exploiting the asymmetric field of single-cycle laser pulses. In future work, we hope to extend our model to study the influence of collective effects on the spin dynamics in laser–plasma interactions [50, 51]. This study is of fundamental importance for the spin effects in future fully relativistic laser–plasma experiments and related fields, such as electron vortices, quantum information and quantum spintronics, etc.

## Acknowledgments

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## Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

## Appendix A. Semiclassical calculation for expectation of spin operators

As we can see from equation (25), the Volkov wave function and its Hermitian conjugation always appear at the same time in the calculation of the expectation values. Under the assumption of semiclassical approximation, the phase factor $S_{\rm p}(x)$ of the Volkov wave function is canceled out exactly by the opposite phase of the conjugated wave function since that phase factor $S_{\rm p}(x)$ is a real function. For instance, The expectation value of spin operator $\hat{\mathbf{S}}(\mathbf{p})$ with respect to the Volkov wave function $\psi^s_{\rm p}(x)$ is

$$\begin{aligned}\langle\hat{\mathbf{S}}(\mathbf{p})\rangle &= \int \mathrm{d}^3\boldsymbol{x}\,\psi^{s\dagger}_{\rm p}(x)\hat{\mathbf{S}}(\mathbf{p})\psi^s_{\rm p}(x)\\ &= \int \mathrm{d}^3\boldsymbol{x}\,u^{s\dagger}_{\rm p}\left(1-\frac{e\not{k}\not{A}}{2p\cdot k}\right)\mathrm{e}^{+\mathrm{i}S_{\rm p}(x)}\hat{\mathbf{S}}(\mathbf{p})\left(1+\frac{e\not{k}\not{A}}{2p\cdot k}\right)u^s_{\rm p}\,\mathrm{e}^{-\mathrm{i}S_{\rm p}(x)}\\ &\approx u^{s\dagger}_{\rm p}\left(1-\frac{e\not{k}\not{A}(\langle\boldsymbol{x}\rangle)}{2p\cdot k}\right)\hat{\mathbf{S}}(\langle\mathbf{p}\rangle)\left(1+\frac{e\not{k}\not{A}(\langle\boldsymbol{x}\rangle)}{2p\cdot k}\right)u^s_{\rm p},\end{aligned}\tag{A.1}$$

where the semiclassical approximation is assumed in the last step where $\langle\boldsymbol{x}\rangle$ and $\langle\mathbf{p}\rangle$ correspond to the classical coordinate and momentum of the electron, respectively. In addition, a factor of the total space volume $V$ is canceled out by considering the normalization of the Volkov wave function (see equation (24)), as has been pointed out in the main text. In this way, the phase factors are canceled out and the integrals in the phase factors need not to be calculated explicitly which greatly simplifies the calculations.

## Appendix B. Time derivative of equation (28)

The time derivative of equation (28) can be calculated analytically with the help of the software *Mathematica*, which is simply described as below.

We begin by noting that $\tau = \frac{1}{\gamma}t$, $\mathrm{d}\tau = \frac{1}{\gamma}\mathrm{d}t$, then we have

$$\frac{\mathrm{d}A(\phi)}{\mathrm{d}\tau} = \frac{\mathrm{d}A}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = A'\frac{\mathrm{d}(k_\mu x^\mu)}{\frac{1}{\gamma}\mathrm{d}t} = \gamma A' k\cdot v, \tag{B.1}$$

where $x^\mu$ is the positron of electron and $v^\mu$ is its velocity. The differential of kinematic momentum with respect to $\tau$ is

$$\begin{aligned}\frac{\mathrm{d}\pi^\mu}{\mathrm{d}\tau} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(p^\mu - eA^\mu + k^\mu\frac{eA\cdot p}{p\cdot k} - k^\mu\frac{e^2A^2}{2p\cdot k}\right)\\ &= \gamma k\cdot v\left[-eA'^\mu + k^\mu\frac{eA'\cdot p}{p\cdot k} - k^\mu\frac{e^2A'\cdot A}{p\cdot k}\right],\end{aligned} \tag{B.2}$$

here $p^\mu$ is a constant which is the 'initial' momentum of electron when external field is turned off. Notice that $k\cdot A = k\cdot A^{'} = k\cdot k = 0$, we have

$$\begin{aligned}F^{\mu\nu}\pi_\nu &= (\partial^\mu A^\nu - \partial^\nu A^\mu)\left(p_\nu - eA_\nu + k_\nu\frac{eA\cdot p}{p\cdot k} - k_\nu\frac{e^2A^2}{2p\cdot k}\right)\\ &= \frac{\gamma mk\cdot v}{e}\left(-eA'^\mu + k^\mu\frac{eA'\cdot p}{p\cdot k} - k^\mu\frac{e^2A'\cdot A}{p\cdot k}\right).\end{aligned} \tag{B.3}$$

Hence, the kinematic momentum follows the Lorentz equation

$$\frac{\mathrm{d}\pi^\mu}{\mathrm{d}\tau} = \frac{e}{m}F^{\mu\nu}\pi_\nu. \tag{B.4}$$

Similarly, the expectation value of Pauli spin operator $\Sigma^\mu$ with respect to the Volkov wave function, which appears as

$$S^\mu_{\mathrm{Pauli}} = a^\mu - eA^\mu\frac{k\cdot a}{p\cdot k} + k^\mu\frac{eA\cdot a}{p\cdot k} - k^\mu\frac{e^2A^2k\cdot a}{2(p\cdot k)^2}, \tag{B.5}$$

can be proved to satisfy the T-BMT equation with the abnormal magnetic moment ignored, i.e.

$$\frac{\mathrm{d}S^\mu_{\mathrm{Pauli}}}{\mathrm{d}\tau} = 2\ \mu F^{\mu\nu}S_{\nu,\mathrm{Pauli}} = \frac{e}{m}F^{\mu\nu}S_{\nu,\mathrm{Pauli}}. \tag{B.6}$$

Then we have

$$\frac{\mathrm{d}\pi^0}{\mathrm{d}\tau} = \frac{e}{m}F^{0\nu}\pi_\nu, \qquad \frac{\mathrm{d}\boldsymbol{\pi}}{\mathrm{d}\tau} = \frac{e}{m}F^{j\nu}\pi_\nu, \tag{B.7}$$

and

$$\frac{\mathrm{d}\boldsymbol{S}_{\mathrm{Pauli}}}{\mathrm{d}\tau} = \frac{e}{m}F^{j\nu}S_{\nu,\mathrm{Pauli}}. \tag{B.8}$$

Under the semiclassical approximation, The expectation of FW spin operator with respect to the Volkov wave function is

$$\begin{aligned}\mathbf{S}_{\mathrm{FW}} &= \frac{m}{2\pi_0}\frac{m}{p_0}\boldsymbol{S}_{\mathrm{Pauli}} + \frac{\boldsymbol{\pi}\boldsymbol{\pi}}{2\pi_0(\pi_0+m)}\cdot\frac{m}{p_0}\boldsymbol{S}_{\mathrm{Pauli}} + \frac{\mathrm{i}}{2\pi_0}\boldsymbol{\pi}\times\langle\beta\boldsymbol{\alpha}\rangle\\ &= \frac{m}{2\pi_0}\frac{m}{p_0}\left(\boldsymbol{a} - e\mathbf{A}\frac{k\cdot a}{p\cdot k} + \mathbf{k}\frac{eA\cdot a}{p\cdot k} - \mathbf{k}\frac{e^2A^2k\cdot a}{2(p\cdot k)^2}\right)\\ &\quad + \frac{\boldsymbol{\pi}\boldsymbol{\pi}}{2\pi_0(\pi_0+m)}\cdot\frac{m}{p_0}\left(\boldsymbol{a} - e\mathbf{A}\frac{k\cdot a}{p\cdot k} + \mathbf{k}\frac{eA\cdot a}{p\cdot k} - \mathbf{k}\frac{e^2A^2k\cdot a}{2(p\cdot k)^2}\right)\\ &\quad - \frac{1}{2m\pi_0}\boldsymbol{\pi}\times\frac{m}{p_0}\left\{\epsilon^{0j\rho\sigma}p_\rho a_\sigma + \frac{e}{p\cdot k}\left(k^j\epsilon^{0\tau\rho\sigma}A_\tau p_\rho a_\sigma\right.\right.\\ &\quad \left. - k^0\epsilon^{j\tau\rho\sigma}A_\tau p_\rho a_\sigma - A^j\epsilon^{0\tau\rho\sigma}k_\tau p_\rho a_\sigma + A^0\epsilon^{j\tau\rho\sigma}k_\tau p_\rho a_\sigma\right)\\ &\quad \left. - \frac{e^2}{2(p\cdot k)^2}\left[2(k^0A^j - k^jA^0)\times\epsilon^{\alpha\beta\rho\sigma}A_\alpha k_\beta p_\rho a_\sigma + A^2\left(k^j\epsilon^{0\tau\rho\sigma}k_\tau p_\rho a_\sigma - k^0\epsilon^{j\tau\rho\sigma}k_\tau p_\rho a_\sigma\right)\right]\right\}.\end{aligned} \tag{B.9}$$

As we can see from table 1 in the main text, the FW operator describes the spin of electron in its stationary system where $S^0_{\rm FW}=0$. Then we take the time derivative of equation (B.9) and obtain

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{S}_{\rm FW}}{\mathrm{d}\tau}=&\frac{m}{2\pi_0}\frac{m}{p_0}\frac{\mathrm{d}\boldsymbol{S}_{\rm Pauli}}{\mathrm{d}\tau}-\frac{m}{2\pi_0^2}\frac{m}{p_0}\boldsymbol{S}_{\rm Pauli}\frac{\mathrm{d}\pi_0}{\mathrm{d}\tau}+\frac{\boldsymbol{\pi}\boldsymbol{\pi}}{2\pi_0(\pi_0+m)}\cdot\frac{m}{p_0}\frac{\mathrm{d}\boldsymbol{S}_{\rm Pauli}}{\mathrm{d}\tau}\\&+\frac{\boldsymbol{\pi}}{2\pi_0(\pi_0+m)}\cdot\frac{m}{p_0}\boldsymbol{S}_{\rm Pauli}\frac{\mathrm{d}\boldsymbol{\pi}}{\mathrm{d}\tau}+\frac{\boldsymbol{\pi}}{2\pi_0(\pi_0+m)}\frac{\mathrm{d}\boldsymbol{\pi}}{\mathrm{d}\tau}\cdot\frac{m}{p_0}\boldsymbol{S}_{\rm Pauli}\\&-\frac{\boldsymbol{\pi}\boldsymbol{\pi}(2\pi_0+m)}{2[\pi_0(\pi_0+m)]^2}\cdot\frac{m}{p_0}\boldsymbol{S}_{\rm Pauli}\frac{\mathrm{d}\pi_0}{\mathrm{d}\tau}+\frac{\mathrm{d}\pi_0}{\mathrm{d}\tau}\frac{1}{2m\pi_0^2}\boldsymbol{\pi}\times\frac{m}{p_0}\left\{\epsilon^{0j\rho\sigma}p_\rho a_\sigma+\frac{e}{p\cdot k}\left(k^j\epsilon^{0\tau\rho\sigma}A_\tau p_\rho a_\sigma\right.\right.\\&\left.-k^0\epsilon^{j\tau\rho\sigma}A_\tau p_\rho a_\sigma-A^j\epsilon^{0\tau\rho\sigma}k_\tau p_\rho a_\sigma+A^0\epsilon^{j\tau\rho\sigma}k_\tau p_\rho a_\sigma\right)\\&\left.-\frac{e^2}{2(p\cdot k)^2}\left[2(k^0A^j-k^jA^0)\times\epsilon^{\alpha\beta\rho\sigma}A_\alpha k_\beta p_\rho a_\sigma+A^2\left(k^j\epsilon^{0\tau\rho\sigma}k_\tau p_\rho a_\sigma-k^0\epsilon^{j\tau\rho\sigma}k_\tau p_\rho a_\sigma\right)\right]\right\}\\&-\frac{1}{2m\pi_0}\frac{\mathrm{d}\boldsymbol{\pi}}{\mathrm{d}\tau}\times\frac{m}{p_0}\left\{\epsilon^{0j\rho\sigma}p_\rho a_\sigma+\frac{e}{p\cdot k}\left(k^j\epsilon^{0\tau\rho\sigma}A_\tau p_\rho a_\sigma\right.\right.\\&\left.-k^0\epsilon^{j\tau\rho\sigma}A_\tau p_\rho a_\sigma-A^j\epsilon^{0\tau\rho\sigma}k_\tau p_\rho a_\sigma+A^0\epsilon^{j\tau\rho\sigma}k_\tau p_\rho a_\sigma\right)-\frac{e^2}{2(p\cdot k)^2}\left[2(k^0A^j-k^jA^0)\epsilon^{\alpha\beta\rho\sigma}A_\alpha k_\beta p_\rho a_\sigma\right.\\&\left.\left.+A^2\left(k^j\epsilon^{0\tau\rho\sigma}k_\tau p_\rho a_\sigma-k^0\epsilon^{j\tau\rho\sigma}k_\tau p_\rho a_\sigma\right)\right]\right\}-\frac{1}{2m\pi_0}\boldsymbol{\pi}\times\frac{m}{p_0}\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\epsilon^{0j\rho\sigma}p_\rho a_\sigma+\frac{e}{p\cdot k}\left(k^j\epsilon^{0\tau\rho\sigma}A_\tau p_\rho a_\sigma\right.\right.\\&\left.-k^0\epsilon^{j\tau\rho\sigma}A_\tau p_\rho a_\sigma-A^j\epsilon^{0\tau\rho\sigma}k_\tau p_\rho a_\sigma+A^0\epsilon^{j\tau\rho\sigma}k_\tau p_\rho a_\sigma\right)\\&\left.-\frac{e^2}{2(p\cdot k)^2}\left[2(k^0A^j-k^jA^0)\times\epsilon^{\alpha\beta\rho\sigma}A_\alpha k_\beta p_\rho a_\sigma+A^2\left(k^j\epsilon^{0\tau\rho\sigma}k_\tau p_\rho a_\sigma-k^0\epsilon^{j\tau\rho\sigma}k_\tau p_\rho a_\sigma\right)\right]\right\}\end{aligned}\tag{B.10}$$

Put equations (B.7) and (B.8) into equation (B.10), we obtain a complex expression for $\frac{\mathrm{d}\boldsymbol{S}_{\rm FW}}{\mathrm{d}\tau}$. If $\mathbf{S}_{\rm FW}$ satisfies the T-BMT equation, then we should have

$$\frac{\mathrm{d}\boldsymbol{S}_{\rm FW}}{\mathrm{d}\tau}-\frac{e}{m}F^{j\nu}S_{\nu,\rm FW}=\frac{\mathrm{d}\boldsymbol{S}_{\rm FW}}{\mathrm{d}\tau}-\frac{e}{m}(\mathbf{k}\times\mathbf{A})\cdot\boldsymbol{S}_{\rm FW}=0.\tag{B.11}$$

When $\frac{\mathrm{d}\boldsymbol{S}_{\rm FW}}{\mathrm{d}\tau}$ and $\boldsymbol{S}_{\rm FW}$ are replaced with equations (B.9) and (B.10), we find that the $\frac{\mathrm{d}\boldsymbol{S}_{\rm FW}}{\mathrm{d}\tau}-\frac{e}{m}(\mathbf{k}\times\mathbf{A})\cdot\boldsymbol{S}_{\rm FW}$ is not equal to 0 at least for some specific values of $p^\mu$, $a^\mu$, $k^\mu$ and $A^\mu$. Therefore, $\boldsymbol{S}_{\rm FW}$ can not be the solution of T-BMT equation.

As we can also see from the numerical results in figure 3, solid grey lines with star markers are obtained from equation (34) which is a special case of equation (28), and shows the time evolution of spin expectation obtained from theory is different from that predicted by the T-BMT equation. Hence, equation (28) can not be the solution of the T-BMT equation.

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