

# Overview and perspectives on metric-affine gravity

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**Abstract.** The main purpose of this work is to give an overview of a generalization of the theory of general relativity, namely metric-affine gravity. We rederive an expression for the Lie derivative of the metric in the case of metric-affine theory and discuss some consequences of such an expression. As a gauge theory of gravitation it may be considered as an upshot of a gauging procedure of the general affine group, or its double covering. A historical approach of such a theory is also contained including the key results. One concludes with some perspectives on the calculation of topological observables in that theory viewed as topological gravity theory.

## 1. Introduction

The general theory of relativity (GR for short) based strikingly on the principle of equivalence as well as the covariance principle provides a well comprehension of gravitational phenomena, starting with the anomalous perihelion advance of Mercury observed in 1859 by Le Verrier [1](see also [2]) to the binary pulsar PSR J1915+1606 experiments which have been realised for some 45 years [3] (for a short review see [4]) and therefore the 1993 Nobel Prize in Physics was awarded to the physicists who discovered it [5]. We should notice that the confrontation between this experimental observation and Einstein's general relativity predictions is of one part in  $10^{14}$  [6] which was extremely exquisite precision in physics experiments. Moreover, the general relativity is an unrivalled theory but from the history of science we have learned that there is no theory which can withstand the fact that oneday it will be superseded by another one ; even Einstein himself predicted this consequence when he talked about the evolution of gravitational theories :

*'...That which you call agnostic in your position is present also in mine, specifically, in the following form : No matter how we draw a complex from nature for simplicity's sake, its theoretical treatment will ultimately never prove to be (adequately) right. Newton's theory, for ex., seems to describe the gravitational field completely with the potential  $\varphi$ . This description proves to be insufficient ; the  $g_{\mu\nu}$  functions take its place. But I do not doubt that the day will come when this approach will also have to give way to a principally different one for reasons that we do not anticipate today. I believe that this process of securing the theory has no limit'*

**From a letter of A. Einstein to F. Klein [7] April 4, 1917 (Berlin).**



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In order to answer the questions about the possibility of quantizing gravity, the physical community has been probably divided into four lines of viewing at the gravitational microworld, that is, String Theory, Loop Quantum Gravity, breaking of Lorentz invariance or modifying GR to obtain a gauge theory of gravity (i.e. gauging the gravitation [8]). The success of describing the physical forces in nature other than gravitational one, namely electromagnetic, weak and strong nuclear forces as classical (thereafter quantized) gauge field theories has stimulated some visions to extend the gauge procedure to gravitation in order achieve the beautiful picture of unification. So it is legally ambitious to think about reformulating GR by means of a gauge principle and if this aim is achieved, it will be a deep paradigm shift in our understanding of the unified picture of physical phenomena. As an alternative to the general theory of relativity describing a Riemannian spacetime with vanishing torsion and vanishing nonmetricity, one can imagine an extension of such a theory in a general non-Riemannian spacetime with nonvanishing torsion and nonmetricity, which may be considered as tentative to unify the macroscopic and microscopic scales of physics. Therefore, a plethora of physicists have had indications that the theory of metric-affine gravity (for an exhaustive review see [9]) is a good candidate to be the theory which will supersede general relativity but without any conclusive evidence.

In the next section we will exhibit an overview of the theory of metric-affine gravity with focusing on the gauge approach. Furthermore, we rederive an important expression of the Lie derivative of the metric tensor in the presence of torsion and nonmetricity (previously derived in a more simple manner in [10]) starting from its geometrical definition with a discussion of some subcases. We will also present our perspective on the calculation of topological invariants related to the metric-affine gravity (MAG for short). We will then briefly conclude.

## 2. Towards a gauge theory of gravity

The first attempt to build a gauge field theory of gravity was introduced by H. Weyl in order to unify Electromagnetism with GR in 1918 (for a review, see [11]). The gauge group in that theory was the group of scaling invariance  $\mathbb{R}$ , which is a noncompact feature of spacetime. In 1929, the same author reformulated his theory with a compact gauge group, i.e.,  $U(1)$  and by this procedure, he withdrew the scale invariance. This later group played a vital role in QED. We present in the below the key results in the development of the gauge idea :

- The formulation of Quantum Mechanics by E. Schrödinger and the quantum interpretation of phase invariance instead of the scaling one.
- V. Fock, on the invariant form of the wave and motion equations for a charged point-mass.
- The reappearance of Weyl's idea with respect to the invariance of the wave function instead of the invariance of vectors' lengths [12].
- The seminal work of Yang and Mills [13].
- Utiyama's approach [14] : gauging the Lorentz group  $SO(1,3)$  with the assumption about the inexplicable symmetry of the connection and the inconsistency of the angular momentum current conservation appeared in the gauging procedure of the Lorentz group with the energy-momentum conservation in GR (see [15]).
- Kibble-Sciama approach [16, 18]: gauging the Poincaré group  $ISO(1,3) = T(4) \otimes SO(1,3)$  which leads to the Einstein-Cartan theory  $U_4$  ; this point of view is well-explained by quoting Kibble himself [17]:

'...In general relativity, curvature is sourced by energy and momentum. In the Poincaré gauge theory, in its basic version, additionally torsion is sourced by spin.'

- The appearance of a substantial Poincaré gauge theory by *Hehl et al.* [19].
- Formulation of a Yang-Mills gauge theory of gravity of the de Sitter group  $SO(3,2)$ , by introducing a nonpropagating field [20] in the space of dimension  $D=5$  and returning to the usual space of dimension  $D=4$  by a suitable spontaneous symmetry breaking [21].
- Formulation of a gauge theory of gravity of the group  $O(5)$  and the emergence of the connection of the manifold as well as the tetrad as dynamical variables after symmetry breaking [22].

### 2.1. Overview of metric-affine gravity

A general action of metric-affine gravity is given by :

$$S_{MAG}[g_{\mu\nu}, \Gamma^\lambda_{\alpha\beta}, \Psi] = S_G[g_{\mu\nu}, \Gamma^\lambda_{\alpha\beta}] + S_M[g_{\mu\nu}, \Gamma^\lambda_{\alpha\beta}, \Psi] \quad (1)$$

where  $S_G$ ,  $S_M$  are the gravity, matter parts of the action respectively, and by  $\Psi$  we denote the field which gathers all matter fields. If one drops out the dependence of the matter action  $S_M$  in the connection  $\Gamma$ , one obtains the action of the Palatini formulation of gravity. Moreover, if we go further and get rid of the connection from the gravity action  $S_G$ , we will have the metric formulation of gravitation (Einstein's general relativity is an example).

In fact, besides the energy-momentum tensor of matter expressed by

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \quad (2)$$

the dependence in the connection of the matter action creates the notion of the *Hypermomentum* tensor [23], which is related to the spin of particles and defined by :

$$\Delta_\lambda^{\alpha\beta} := -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta \Gamma^\lambda_{\alpha\beta}} \quad (3)$$

with  $g \equiv \det(g_{\mu\nu})$ . Turn now into the gauge formulation of metric-affine gravity. To control the geometry of the spacetime manifold, one has to know the three ingredients of a gauge theory of MAG [15] : the coframe  $\vartheta^\alpha$ , the linear connection  $\Gamma^\beta_\alpha$  and the metric  $g_{\mu\nu}$ . The first one is a translational gauge potential and the second (i.e. the linear connection) is nothing but the  $GL(4, R)$  gauge potential. As far as the metric tensor is concerned, the reason of its existence as ingredient is of physical importance that is, measurements of distances and angles.

General relativity may be considered as a gauge theory of the translation group  $T(4)$  if one looks at the energy-momentum conservation law as an invariance of the physical system under translations in space and time. This is nothing but the teleparallel approach of gravity which is similar to GR with spacetime curvature induced by torsion [15]. If we relax the constraint of metricity, i.e.,  $Q = 0$  but take into account just the trace of the nonmetricity tensor, namely  $TrQ \neq 0$ , the result is the Weyl-Cartan  $Y_4$ -theory . For a nonvanishing curvature and nonvanishing torsion, one obtains the Einstein-Cartan  $U_4$ -theory. Finally, the relaxation of all the constraints leads to the metric-affine theory of gravity as mentioned in the table 1 below.

**Table 1.** Classification of spacetime theories by means of the field strengths [8], with  $R, Q$  and  $S$  denote the curvature, nonmetricity and torsion, respectively.

Spacetime theory	Ingredients
GR	$R \neq 0, \quad Q = 0, \quad S = 0$
Teleparallel gravity $GR_{  }$	$R = 0, \quad Q = 0, \quad S \neq 0$
Weyl's theory	$R \neq 0, \quad TrQ \neq 0, \quad S = 0$
Einstein-Cartan theory (EC)	$R \neq 0, \quad Q = 0, \quad S \neq 0$
MAG	$R \neq 0, \quad Q \neq 0, \quad S \neq 0$

### 3. A geometrical survey of MAG

#### 3.1. The Lie derivative of the metric tensor in the general case of MAG

Now, we consider the Lie derivative of the metric tensor along a vector  $\xi$  which is geometrically given by :

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu} \partial_\mu \xi^\lambda + g_{\mu\lambda} \partial_\nu \xi^\lambda \quad (4)$$

One can easily express the covariant derivative of the metric with respect to the affine connection  $\Gamma$  as :

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\alpha_{\mu\lambda} g_{\alpha\nu} - \Gamma^\alpha_{\nu\lambda} g_{\mu\alpha} \quad (5)$$

We also know that :

$$\nabla_\mu \xi^\alpha = \partial_\mu \xi^\alpha + \Gamma^\alpha_{\lambda\mu} \xi^\lambda \quad (6)$$

If we insert the partial derivative of the metric and the contravariant vector  $\xi$  in the expression of the Lie derivative of the metric (4) one obtains :

$$\mathcal{L}_\xi g_{\mu\nu} = -\xi^\lambda Q_{\lambda\mu\nu} + g_{\alpha\nu} \xi^\lambda \Gamma^\alpha_{\mu\lambda} + g_{\mu\alpha} \xi^\lambda \Gamma^\alpha_{\nu\lambda} - g_{\alpha\nu} \Gamma^\alpha_{\lambda\mu} \xi^\lambda - g_{\mu\alpha} \Gamma^\alpha_{\lambda\nu} \xi^\lambda \quad (7)$$

which can be easily rewritten as :

$$\mathcal{L}_\xi g_{\mu\nu} = -\xi^\lambda Q_{\lambda\mu\nu} + 2g_{\alpha\nu} \xi^\lambda \Gamma^\alpha_{[\mu\lambda]} + 2g_{\mu\alpha} \xi^\lambda \Gamma^\alpha_{[\nu\lambda]} + \tilde{\nabla}_\mu \xi_\nu + g_{\lambda\nu} N^\lambda_{\alpha\mu} \xi^\alpha + \tilde{\nabla}_\nu \xi_\mu + g_{\mu\lambda} N^\lambda_{\alpha\nu} \xi^\alpha \quad (8)$$

with  $Q_{\lambda\mu\nu}$  is the nonmetricity tensor given by :  $Q_{\lambda\mu\nu} = -\nabla_\lambda g_{\mu\nu}$  and  $\tilde{\nabla}$  is the covariant derivative operator with respect to the metric connection (Christoffel symbols)  $\tilde{\Gamma}$ . Note also that  $N^\lambda_{\mu\nu} := \Gamma^\lambda_{\mu\nu} - \tilde{\Gamma}^\lambda_{\mu\nu}$  is a tensor and is called the *distortion* tensor. If we define the *torsion* tensor  $S_{\mu\nu}^\lambda$  as the antisymmetric part of the affine connection in the two lower indices i.e.,  $S_{\mu\nu}^\lambda := \Gamma^\lambda_{[\mu\nu]}$  we arrive at :

$$\mathcal{L}_\xi g_{\mu\nu} = -\xi^\lambda Q_{\lambda\mu\nu} + 2g_{\alpha\nu}\xi^\lambda S_{\mu\lambda}{}^\alpha + 2g_{\mu\alpha}\xi^\lambda S_{\nu\lambda}{}^\alpha + \tilde{\nabla}_\mu \xi_\nu + g_{\lambda\nu} N^\lambda{}_{\alpha\mu} \xi^\alpha + \tilde{\nabla}_\nu \xi_\mu + g_{\mu\lambda} N^\lambda{}_{\alpha\nu} \xi^\alpha \quad (9)$$

One can remark from the expression above that the Lie derivative of the metric in the case of metric-affine gravity is nothing but the Lie derivative of the metric in metric gravity (e.g. Einstein gravity), that is,  $\mathcal{L}_\xi g_{\mu\nu}^{(GR)} = \tilde{\nabla}_\mu \xi_\nu + \tilde{\nabla}_\nu \xi_\mu$  modulo an extra term (E.T.) which can be expressed by :

$$E.T. = -\xi^\lambda Q_{\lambda\mu\nu} + 2g_{\alpha\nu}\xi^\lambda S_{\mu\lambda}{}^\alpha + 2g_{\mu\alpha}\xi^\lambda S_{\nu\lambda}{}^\alpha + g_{\lambda\nu} N^\lambda{}_{\alpha\mu} \xi^\alpha + g_{\mu\lambda} N^\lambda{}_{\alpha\nu} \xi^\alpha \quad (10)$$

After some algebra, one arrives at :

$$E.T. = 0 \quad (11)$$

Therefore, the equation (9) becomes :

$$\mathcal{L}_\xi g_{\mu\nu} = \tilde{\nabla}_\mu \xi_\nu + \tilde{\nabla}_\nu \xi_\mu \equiv \mathcal{L}_\xi g_{\mu\nu}^{(GR)} \quad (12)$$

Now, we want to destroy the covariant derivative operator  $\tilde{\nabla}$  with respect to the metric connection  $\tilde{\Gamma}$  and replace it with covariant derivative operator  $\nabla$  with respect to the affine connection  $\Gamma$ . To achieve this purpose, one utilizes the expression :

$$\tilde{\nabla}_\mu \xi_\nu = \nabla_\mu \xi_\nu + N^\lambda{}_{\nu\mu} \xi_\lambda \quad (13)$$

Then, the equation (12) becomes

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + 2N^\lambda{}_{\nu\mu} \xi_\lambda \quad (14)$$

After some algebra and using the fact that

$$N^\lambda{}_{(\mu\nu)} = \frac{1}{2} g^{\lambda\alpha} (2Q_{(\mu\nu)\lambda} - Q_{\lambda\mu\nu}) - 2g^{\lambda\alpha} S_{\lambda(\mu\nu)} \quad (15)$$

Finally, we arrive at the expression :

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + (2Q_{(\mu\nu)\lambda} - Q_{\lambda\mu\nu}) \xi^\lambda - 4S_{\lambda(\mu\nu)} \xi^\lambda \quad (16)$$

From this equation we can distinguish two cases :

- (i) For a completely antisymmetric torsion tensor, i.e.,  $S_{\lambda\mu\nu} = S_{[\lambda\mu\nu]}$ , one has  $S_{[\lambda\mu\nu]} = N_{[\lambda\mu\nu]}$  and then the equation (14) becomes

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \quad (17)$$

This expression is similar to the one derived in the case of Einstein's gravity (eqn (12)) but with respect to the affine connection  $\Gamma = \tilde{\Gamma} + N$ . Thus it is obvious that a completely antisymmetric torsion tensor has no effect on the form of the Lie derivative of the metric.

(ii) If the torsion and the nonmetricity are related via the relation

$$2Q_{(\mu\nu)\lambda} - Q_{\lambda\mu\nu} = 4g^{\lambda\alpha} S_{\lambda(\mu\nu)} \quad (18)$$

One obtains the same equation (17) as in the case (i). In fact the equation (18) is a direct result from the completely antisymmetry of the torsion tensor mentioned in (i) because of the relation  $S_{[\lambda\mu\nu]} = N_{[\lambda\mu\nu]}$ . In the same spirit of the consequences of such an expression on the autoparallels discussed in [10], we see that in the presence of both torsion and nonmetricity, as long as the motion of a particle is concerned, the situation that a test particle recognizes torsion but does not recognize nonmetricity will be the same as that a test particle recognizes nonmetricity and does not recognize torsion.

### 3.2. Geometrical interpretation of torsion and nonmetricity

- In the presence of torsion, the infinitesimal parallelograms fail to be closed. The result of the parallel transport in a 2-dimensional space with torsion is a pentagon and the vector describing the deformation of parallelograms is proportional to torsion [24]:

$$A^\lambda = 2S_{\mu\nu}{}^\lambda \tilde{u}^\mu u^\nu \quad (19)$$

with  $\tilde{u}^\mu$ ,  $u^\nu$  are two vectors parallel transported.

- When nonmetricity is present, the inner product of two vectors  $u^\mu$  and  $v^\nu$  parallel transported along a curve, namely  $u.v := g_{\mu\nu}u^\mu v^\nu$  is not conserved and the variation is related to the nonmetricity tensor via

$$D(u.v) = -Q_{\lambda\mu\nu}u^\mu v^\nu dx^\lambda \quad (20)$$

An important consequence of such a variation is that in a space with nonmetricity, the magnitude of a vector is not preserved and this has a direct link to the Weyl's scale theory.

## 4. Perspectives on the calculation of topological observables

The topological invariants in the formalism of fibre bundle in differential geometry describe the global (topological) structure of the manifold. One can know the global features of a manifold by using some local functionals in the geometry of such a manifold. We should notice that the metric properties described by the vielbein  $e^a$  and the affine ones described by the spin connection  $\omega^a_b$  in a metric-affine spacetime are independent.

The well known examples of topological invariants in four dimensions constructed by dint of the curvature 2-form,  $R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b$  are the Pontryagin and Euler classes which are defined by [25]

$$P_4 = \frac{1}{8\pi^2} \int_{M_4} R^{ab} \wedge R_{ab} \quad (21)$$

and

$$E_4 = \frac{1}{32\pi^2} \int_{M_4} \epsilon_{abcd} R^{ab} \wedge R^{cd} \quad (22)$$

with  $\epsilon_{abcd}$  is the totally antisymmetric Levi-Cevita symbol. The Nieh-Yan 4-form constructed by dint of the torsion 2-form,  $T^a = de^a + \omega^a_b \wedge e^b$  is defined as

$$N = T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b \quad (23)$$

The authors of [26] determined the BRST and anti-BRST transformations in gauge-affine gravity geometrically by using a superspace formalism (enlarging the metric-affine manifold to incorporate Grassmann coordinates and the result is a (4,2)-dimensional superspace). Besides, in [27] one can find the BRST algebra for topological 4D gravity (with torsion) using an  $ISO(4)$  ( $SO(5)$ )-superconnection. The bulk of our work as perspective will be to construct nontrivial topological observables in the case of the theory of metric-affine gravity. The authors of [27] had made contributions also to the calculation of topological observables in the case of a theory based on the two special orthogonal gauge groups  $ISO(4)$  and  $SO(5)$ . We will try to calculate the topological observables in the case of this theory, i.e. MAG, using the same procedure in [27, 28]; the ingredients of the calculus are the structure equations and the BRST transformations in the superconnection formalism.

## 5. Concluding remarks

In this paper we have reviewed the important notions in metric-affine gravity with a brief historical approach. In addition, we have derived the Lie derivative of the metric tensor in the presence of torsion and nonmetricity, starting with its very geometrical definition and some tensor algebra. We have arrived at an expression which depends on torsion as well as nonmetricity. The considered expression (12) has been already known and was derived in with direct calculation and without using of any non-Riemannian constraint. Furthermore, we have discussed the consequences of such an expression and we have found that for a particular considerations given by equation (18), the notion of torsion and nonmetricity are indistinguishable and can be treated on equal footing.

As perspectives, we have presented indications about our current calculations of the topological invariants in the case of metric-affine gravity viewed as a gauge theory of the general affine group  $GA(4, R) = T^4 \otimes GL(4, R)$ .

## 6. Forthcoming research

Using Dirac's formalism [29] of co-covariant calculus and completely relaxing of the Riemannian constraints, the authors of [30], using tensor calculus, derived the equations of motion related to the metric-affine scale-covariant gravity. By dint of the analytical calculations in [30], we will try to extend his work to the biggest and cumbersome case of MAG.

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## References

- [1] Le Verrier U 1859 Comptes rendus hebdomadaires des séances de l'Académie des sciences (Paris) **49** 379-83
- [2] Roseveare N T 1982 *Mercury's Perihelion from Le Verrier to Einstein* (Oxford: Oxford University Press)
- [3] Hulse R A and Taylor J H 1975 *Astrophys. J.* **195** L51-53
- [4] Christensen N 2015 *Science* **348** (6236) 766-66
- [5] Schwarzschild B 1993 *Physics Today* **46** (12) 17-19
- [6] Penrose R 2006 *Encyclopedia of Mathematical Physics* Vol. 2 ed J.-P. Francoise *et al.* (Oxford, Elsevier) pp 487-94
- [7] The collected papers of Albert Einstein (New Jersey: Princeton University Press) Vol. 8
- [8] Gronwald F and Hehl F W 1996 *Proc. of the 14th Course of the School of Cosmology and Gravitation on Quantum Gravity (Erice)* ed P G Bergmann *et al.* (Singapore: World Scientific) pp 148-98
- [9] Hehl F W, McCrea J D, Mielke E W and Ne'eman Y 1995 *Phys. Rep.* **258** 1-171
- [10] Damianos Iosifidis Metric-affine Gravity and Cosmology/ Aspects of Torsion and Non-metricity in Gravity Theories, (PhD thesis, Aristotle University of Thessaloniki, 2019) *Preprint* arXiv:1902.09643
- [11] O'raifeartaigh L 1997 *The Dawning of Gauge Theory* (Princeton: Princeton University Press)
- [12] Weyl H 1929 *Zeit. f. Physik* **56** 330-52
- [13] Yang C N and Mills R L 1954 *Phys. Rev.* **96** 191-95
- [14] Utiyama R 1956 *Phys. Rev.* **101** 1597-607
- [15] Hehl F W 2017 *Towards a Theory of Spacetime Theories*, ed D Lehmkuhl *et al.* (New York: Springer)
- [16] Kibble T W B 1961 *J. Math. Phys.* **2** 212-21
- [17] Blagojevic M and Hehl F W 2013 *Gauge Theories of Gravitation : A Reader with Commentaries* (London: Imperial College Press)
- [18] Sciama D W 1962 *Recent developments in General Relativity* Festschrift for Infeld (Oxford: Pergamon Press) pp 415-39 (*Reprinted in* )  
Sciama D W 1964 *Rev. Mod. Phys.* **36**, 463-69 ; **36**, 1103
- [19] Hehl F W, von der Heyde P, Kerlick G D and Nester J M 1976 *Rev. Mod. Phys.* **48** (3) 393-416
- [20] Stelle K S and West P C 1980 *Phys. Rev. D* **21** (3) 1466-88
- [21] Hammond T R 2002 *Rep. Prog. Phys. D* **65** 599-649
- [22] Pagels Heinz R 1984 *Phys. Rev. D* **29** 1690-98
- [23] Hehl F W, Kerlick G D and von der Heyde P 1976 *Phys. Lett. B* **63** 446-48  
Hehl F W, Kerlick G D and von der Heyde P 1976 *Z. Naturforsch* **31 A** 111-14 ; 524-27 ; 823-27  
Lord E A 1978 *Phys. Lett. A* **65** 1-4
- [24] Damianos Iosifidis 2019 *Class. Quantum Grav.* **36** 085001 (46pp)
- [25] Nieh H T and Yan M L 1982 *J. Math. Phys.* **23** 373-74  
Nieh H T and Yan M L 1982 *Ann. Phys.* **138** 237-59
- [26] Meziane A and Tahiri M 1996 *Mod. Phys. Lett. A* **11** (11) 885-90
- [27] Mena M and Tahiri M 1998 *Phys. Rev. D* **57** (12) 7312-17  
Aouane R, Mena M and Tahiri M 2007 *Class. Quantum Grav.* **24** 2445-51
- [28] Aouane R Nouvelles observables torsionnelles en gravitation topologique à 4 dimensions et formalisme de la superconnexion, (Thèse de Magister, Université d'Oran, 2006)
- [29] Dirac P A M 1973 *Proc. R. Soc. Lond. A* **333** 403-18
- [30] Poberii E A 1994 *Gen. Rel. Grav.* **26** (10) 1011-54