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Coulomb Scattering of an Electron by a Monopole*

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A classical Lagrangian formalism has recently been given[1] from which both the symmetrized set of Maxwell's equations and the equations of motion for both electrically and magnetically charged particles can be derived. This analysis uses two potentials, following Cabibbo and Ferrari,[2] and employs space-time algebra,#1 the Clifford algebra appropriate to four-dimensional Minkowski space-time.[4]

Four linearly independent vectors γ_μ are used as a basis set for space-time algebra. The (Clifford) products of these vectors yield 16 linearly independent quantities which partition into scalar, vector, tensor, axial vector (or pseudovector), and pseudoscalar objects, in complete analogy to the bilinear forms constructed using solutions to the Dirac equation. Noting this analogy, the purpose of this letter is to indicate the possibility of a naive extension of the classical results to quantum theory by simply incorporating the form of the interaction terms as given by the classical theory into a lowest order quantum mechanical calculation of the "Coulomb" scattering of an electron by a fixed monopole. The physical implications of this dual role for the γ matrices is not clear. Chisholm and Farwell,[5] who analyze these questions from a somewhat different perspective, developing a generalized gauge theory which includes magnetic monopoles, introduce the notion of "spin torsion" in an eight component spinor theory.

The interaction term of the Lagrangian mentioned above is the Clifford product of a generalized current

$$\mathcal{J} = j - \gamma_5 k \ , \tag{1}$$

times a generalized potential

$$\mathcal{A} = A - \gamma_5 M \ , \tag{2}$$

where A is the usual vector potential, associated with the electric current density vector j, and M is a "magnetic" vector potential associated with the magnetic current density k. The space-time algebraic expansions of these quantities are

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#1 The idea to employ space-time algebra (sometimes called Dirac algebra) to incorporate magnetic monopoles into classical electromagnetic theory was proposed by de Faria-Rosa *et al.*[3]

$j = j^\mu \gamma_\mu, A = A^\mu \gamma_\mu$, etc., where j^μ, A^μ, etc. are the usual tensor quantities. The pseudoscalar of space-time algebra, γ_5, is here[#2] defined by

$$\gamma_5 \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3 \ . \tag{3}$$

When written out, the generalized interaction term is

$$-\mathcal{J}\mathcal{A} = -(j - \gamma_5 k)(A - \gamma_5 M) = -(jA - j\gamma_5 M - \gamma_5 kA + \gamma_5 k \gamma_5 M) \ . \tag{4}$$

The usual interaction term, $-jA$, describes the interaction of an electric current with the usual vector potential. This is the interaction responsible for the usual Coulomb scattering. The term $-\gamma_5 k \gamma_5 M = -kM$ is the analogous interaction of the magnetic current with the magnetically generated vector potential. The cross terms, $j\gamma_5 M + \gamma_5 kA$, describe the forces of magnetically generated fields on electric currents and vice versa. The term $j\gamma_5 M$, then, is the one that is appropriate to the scattering of an electron by a fixed magnetic monopole. This form for the interaction term is analogous to that obtained by Chisholm and Farwell.[5]

The topic of Coulomb scattering of electrons is a well-studied problem and can be found in any book on quantum mechanics. In order to furnish a framework to calculate the Coulomb scattering of an electron by a magnetic monopole, we first record the Coulomb scattering of an electron by an infinitely heavy charge Q. We follow the derivation (and notation) of Bjorken and Drell[7] in which the electric current is given by $e\bar{\psi}_f \gamma^\mu \psi_i$, where

$$\psi_i = \sqrt{\frac{m}{E_i V}} u(p_i, s_i) e^{-ip_i \cdot x} \tag{5}$$

and

$$\bar{\psi}_f = \sqrt{\frac{m}{E_f V}} \bar{u}(p_f, s_f) e^{ip_f \cdot x} \tag{6}$$

are the initial and final wave functions, respectively, and $\hbar = c = 1$. The electron has charge e (<0), mass m, momentum p, and spin s. The wave functions ψ are normalized to unit probability in a box of volume V.

#2 There is a factor i difference between the definition of γ_5 by Eq. (3) and that by Bjorken and Drell.[6] Since a cross section (without interference terms) is being calculated, we can ignore this distinction.

For the usual electron Coulomb scattering by a point charge Q, one expands the interaction term jA and calculates the transition matrix element

$$S_{fi} = -ie \int d^4x \bar{\psi}_f A_\mu \gamma^\mu \psi_i \ , \tag{7}$$

where

$$A_0 = \frac{Q}{4\pi \mid \boldsymbol{x} \mid} \text{ and } \boldsymbol{A} = 0 \tag{8}$$

represent the Coulomb potential. Boldface type is used to represent 3-vectors in Euclidean 3-space.

The transition rate per particle into the final states in the interval d^3p_f is given by

$$\mid S_{fi} \mid^2 \frac{V d^3 p_f}{(2\pi)^3} \ , \tag{9}$$

which leads to the differential cross section (averaged over initial state spins and summed over final state spins) of

$$\frac{d\bar{\sigma}_{eQ}}{d\Omega} = \frac{Q^2 \alpha}{8\pi \mid \boldsymbol{q} \mid^4}(8E_iE_f - 4p_i \cdot p_f + 4m^2) \ , \tag{10}$$

where $\alpha = \frac{e^2}{4\pi} \cong 1/137$, and $\boldsymbol{q} = \boldsymbol{p}_f - \boldsymbol{p}_i$ is the 3-momentum transfer to the electron. In terms of the scattering energy $E = E_i = E_f$ and scattering angle θ, one obtains from Eq. (10) the Mott cross section

$$\frac{d\bar{\sigma}_{eQ}}{d\Omega} = \frac{Q^2 \alpha[1 - \beta^2 \sin^2(\theta/2)]}{16\pi \boldsymbol{p}^2 \beta^2 \sin^4(\theta/2)} \ , \tag{11}$$

where $\beta = v/c$ and $\boldsymbol{p}^2 = \mid \boldsymbol{q} \mid^2 /(2 \sin\theta/2)^2$.

At this point, to obtain the analogous cross section for the scattering of an electron by a monopole, we follow the form suggested by the classical expression of the generalized electromagnetic interaction given by Eq. (4), and make the substitution $A_\mu \gamma^\mu \to M_\mu \gamma^\mu \gamma^5$ in Eq. (7). This substitution is tantamount to an assumed definition for the interaction between electrons and monopoles. There are, of course, other possible assumptions for the electron-monopole interaction.[8,9] Since the Coulomb field is not quantized, we do not need to worry about the nature of the source of the magnetic field or the two-photon question.[1,2]

The relevant transition matrix element for electron-monopole scattering is then given by

$$S_{fi}^{eg} = -ie \int d^4x \bar{\psi}_f M_\mu \gamma^\mu \gamma_5 \psi_i \ , \tag{12}$$

where

$$M_0 = \frac{g}{4\pi \mid \boldsymbol{x} \mid} \text{ and } \boldsymbol{M} = 0 \tag{13}$$

represent the "Coulomb" potential of a magnetic monopole of pole strength g.

Carrying through the same analysis, except for the incorporation of the γ_5 into the spin sums, leads to

$$\frac{d\bar{\sigma}_{eg}}{d\Omega} = \frac{g^2\alpha}{8\pi \mid \boldsymbol{q} \mid^4}(8E_iE_f - 4p_i \cdot p_f - 4m^2) \ . \tag{14}$$

Equation (14) reduces to

$$\frac{d\bar{\sigma}_{eg}}{d\Omega} = \frac{g^2\alpha \cos^2(\theta/2)}{16\pi \boldsymbol{p}^2 \sin^4(\theta/2)} \ , \tag{15}$$

where we see that the β^2 factor in the denominator of the scattering of an electron by the charge Q, as given by Eq. (11), is not present in the monopole scattering. This is exactly what has been found in previous classical[10] and quantum mechanical[11,12] calculations and is what one would expect from an examination of the Lorentz force on an electric charge due to an electric field $\boldsymbol{E}$ versus that due to a magnetic field $\boldsymbol{H}$:

$$\boldsymbol{F} = e(\boldsymbol{E} + \boldsymbol{\beta} \times \boldsymbol{H}) \ . \tag{16}$$

Now, it is well known that the vector interaction of relativistic quantum electrodynamics tends to conserve helicity.[13] By making an elementary qualitative argument, one can easily see that the pseudovector interaction of Eq. (12) also tends to conserve helicity. The argument is essentially the same as that for the QED vector interaction. Consider the matrix element for a right-handed relativistic election to scatter to a left-handed electron via a $\gamma_\mu\gamma_5$ interaction. Such a matrix element is proportional to

$$\bar{u}_f(\frac{1+\gamma_5}{2})\gamma_\mu\gamma_5(\frac{1+\gamma_5}{2})u_i = \bar{u}_f\gamma_\mu(\frac{1-\gamma_5}{2})(\frac{1+\gamma_5}{2})\gamma_5 u_i = 0 \ . \tag{17}$$

This result is in agreement with previous quantum mechanical calculations, which show that the major contribution to the forward scattering cross section of electrons off of monopoles is the helicity non-flip scattering.[12] This result also furnishes some physical insight into the notion of spin torsion proposed by Chisholm and Farwell.[5]

The fact that these results show agreement with prior calculations gives justification for the assumption of a $\gamma_\mu \gamma_5$ form for the interaction between electrons and monopoles. Extension of this initial result to a complete perturbation theory with a consistent set of Feynman rules would be of considerable interest. This approach—using two potentials—does not contain any singular vector potentials, as does the original formulation of Dirac[14] or that of Schwinger.[15] In addition, it does not suffer from such complications as an arbitrary unit vector[8] which sacrifices manifest space-time isotropy, or postulated nonlocal interactions.[9] On the other hand, the two-photon question needs resolution. Another question also arises: one observes that the cross section in the backward direction (as $\theta \to \pi$) given by Eq. (15) does not agree with prior calculations. However, this is just the region where one would expect the higher order scattering terms to dominate. While higher order calculations using the $\gamma_\mu \gamma_5$ interaction would presumably ameliorate this deficiency, there would be a convergence problem if g is too large. But this is also the case for large Q. These questions are presently being studied.

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