Thesis

Topological conformal field theories from gauge-fixed topological gauge theories: a case study

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Abstract

This thesis is devoted to the study of a new construction of two-dimensional topological conformal field theories by gauge fixing two-dimensional topological gauge theories. We study in detail the Lorenz-gauged abelian and non-abelian BF theory, which are topological conformal classically and on the quantum level. We find that the abelian model corresponds to Witten's B-model with a parity shifted flat target space. It is therefore obtained by twisting a N = (2,2) supersymmetric model, while no such twist exists in the non-abelian case. Furthermore, we study an analogue of Gromov-Witten periods in the abelian model. Finally, we show that the non-abelian model allows non-trivial Jordan blocks of the Hamiltonian and thus defines a logarithmic conformal field theory. The existence of infinite-dimensional Jordan blocks allows an explicit construction of primary fields, whose conformal weights are subject to quantum corrections.

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Topological conformal field theories from gauge-fixed topological gauge theories: a case study

THÈSE

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«Topological Conformal Field Theories from Gauge-fixed Topological Gauge Theories: a Case Study»

La Faculté des sciences, sur le préavis de Monsieur A. ALEXEEV, professeur ordinaire et directeur de thèse (Section de mathématiques), Monsieur M. MARINO BEIRAS, professeur ordinaire (Section de mathématiques - Section de physique) et Monsieur P. MNEV, professeur (Department of Mathematics, University of Notre Dame, Indiana, USA), autorise l'impression de la présente thèse, sans exprimer d'opinion sur les propositions qui y sont énoncées.

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Le Doyen

N.B. - La thèse doit porter la déclaration précédente et remplir les conditions énumérées dans les "Informations relatives aux thèses de doctorat à l'Université de Genève".

“Es genügt nicht, keine Gedanken zu haben,
man muss auch unfähig sein, sie auszudrücken.”

Karl Kraus

An Piddboo, und an Bobo

Résumé

Cette thèse est consacrée à l'étude des théories conforme des champs topologique en deux dimension. On propose une nouvelle constructions des théories conforme des champs topologique en fixant la jauge des théories topologiques de jauge. En particulier, on analyse deux exemples en détail: la théorie BF abélien et non-abélien. On trouve que les deux théories en fixant la jauge de Lorenz sont calssiquement ainsi que quantiquement des théories conforme des champs topologique. En plus, on peut constater que le modèle abélien est relié au modèle B de Witten où ce dernier est défini par rapport d'un éspace vectoriel impair. En conséquence, il peut être obtenu par un twist topologique d'un théorie supermétrique $\mathcal{N} = (2, 2)$. Le parité impair de l'espace des coefficients nous permet d'étudier nouvelles deformations. Notamment, on trouve que la théorie non-abélien se révèle comme une deformation (dans l'éspace des théories conforme des champs topologique) du modèle abélien. Par contre, la deformation introduit des termes d'interaction qui à leur tour cassent les symétries internes (nommées "R-symmetries") qui sont necessaire pour un twist topologique. On peut en conclure que le modèle non-abélien n'est pas donné au moyen d'un twist d'une théorie supersymétrique. Autre résultats de cette thèse sont:

Modèle abélien:

- On étudie un toy model de la théorie de Gromov-Witten: On construit des formes differentielles fermées sur l'espace de modules $\mathcal{M}_{0,n}$ des surfaces de Riemann de genre zéro avec n points marqués par rapport des fonctions de correlations du modèle.

Modèle non-abélien:

- On montre que tous fonctions de corrélation sont donnée par des intégrales convergentes. En particulier, pour des algèbre de Lie réductive, les diagrammes de Feynman à une boucle s'annulent dans la somme. Il suit que seulement les diagrammes des arbres contribuent aux fonctions de corrélation des champs fondamentales.
- On montre que les fonctions de corrélation des champs fondamentales sont données par des intégrales convergentes des expressions polylogarithmiques.
- On montre que l'operateur L_0 a des blocs de Jordan non-triviale et on en déduit que le modèle définit une théorie conforme des champs logarithmique.
- On donne une construction explicite des champs primaires dont leur poids conformes montrent une correction quantique.

Summary

This thesis is devoted to the study of two-dimensional topological conformal field theories. In particular, we propose a new construction of topological conformal field theories by gauge fixing two-dimensional topological field theories. We study in detail two examples, namely abelian and non-abelian BF theory and prove that after implementing a Lorenz gauge fixing both models become topological conformal on the classical, as well as on the quantum level. We find that the abelian model can be related to Witten's B-model with a parity shifted flat target and is therefore obtained by twisting a $\mathcal{N} = (2, 2)$ supersymmetric theory. The parity shift allows us to study new deformations of the model which lead us to the Lorenz gauged non-abelian BF theory, which can be seen as a deformation (as a topological conformal field theory) of the abelian model by a second descent of a certain observable. As it turns out, the non-abelian theory cannot be (un)twisted to a $\mathcal{N} = (2, 2)$ supersymmetric theory. The reason for this is that the non-abelian model is an interacting field theory and the interaction breaks the residual symmetries (R-symmetries) which have been used to (un)twist the abelian model. Further results are:

Abelian model:

- We discuss a toy model of Gromov-Witten periods: We construct closed differential one-forms on the moduli space $\mathcal{M}_{0,n}$ of genus zero curves with n punctures from correlators of the model and study their periods.

Non-abelian model:

- We show that the model is solvable: all correlation functions are given by convergent integrals. In particular, we find that for reductive Lie algebras one-loop contributions vanish and hence, at least on the level of fundamental fields, only tree-diagrams contribute.

- We show that correlation functions of fundamental fields are given by convergent integrals of polylogarithmic type.

- We show that L_0 has a non-trivial Jordan block structure and argue that the model defines a logarithmic conformal field theory.

- We give an explicit construction of primary fields whose conformal weights are subject to quantum corrections.

Zusammenfassung

Die vorliegende Thesis gilt dem Studium zweidimensionaler topologischer, konformer Feldtheorien. Insbesondere soll eine neue Konstruktion topologischer, konformer Feldtheorien durch Eichung topologischer Eichfeldtheorien diskutiert werden. Es werden eingehend zwei Beispiele untersucht: die abelsche und die nicht-abelsche BF Theorie. Wir zeigen, dass beide Theorien unter Lorenzeichung Beispiele topologischer, konformer Feldtheorien sind, klassisch wie quantenmechanisch. Es stellt sich heraus, dass das abelsche Model äquivalent zu Wittens B-Model ist, definiert über einem Vektorraum mit ungerader Parität. Das abelsche Modell ist demnach durch einen topologischen Twist einer $\mathcal{N} = (2, 2)$ super-symmetrischen Theorie zu erhalten. Die umgekehrte Parität des Koeffizientenraums erlaubt es neue Störungen der Theorie zu studieren. Insbesondere finden wir, dass das nicht-abelsche Modell sich als eine solche Störung (innerhalb des Raumes der topologischen, konformen Feldtheorien) schreiben lässt. Die Störung erzeugt Wechselwirkungsterme, die die sogenannten R-Symmetrien brechen, welche es ermöglichten, das abelsche Modell als einen topologischen Twist einer supersymmetrischen Theorie zu beschreiben. Das nicht abelsche Modell kann demnach nicht von einer super-symmetrischen Theorie durch einen solchen Twist gewonnen werden. Weitere Ergebnisse dieser Thesis sind:

Abelsches Modell:

- Es wird ein Spielzeugmodel der Gromov-Witten Theorie untersucht: Wir studieren die Konstruktion geschlossener Differentialformen auf dem Modulraum $\mathcal{M}_{0,n}$ n-punktierter Riemannscher Flächen des Geschlechts null mit Hilfe von Korrelatoren.

Nicht-abelsches Modell:

- Wir zeigen, dass alle Korrelationsfunktionen durch konvergierende Integrale gegeben sind. Es zeigt sich, dass für reduktive Lie-Algebren Feynman-Diagramme der Loop-Ordnung eins in der Summe verschwinden. Insbesondere tragen nur tree-level-Diagramme zu Korrelatoren fundamentaler Felder bei.
- Wir zeigen, dass Korrelatoren fundamentaler Felder durch konvergierende Integrale polylogarithmischer Ausdrücke gegeben sind.
- Wir zeigen, dass der Operator L_0 nicht-triviale Jordanblöcke aufweist und folgern, dass das Modell eine logarithmische konforme Feldtheorie definiert.
- Wir geben eine explizite Konstruktion primärer Felder, deren konforme Gewichte eine Quantenkorrektur aufweisen.

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Contents

CHAPTER 1

Introduction

The quantization of general quantum field theories remains until today an immensely difficult problem. The situation, however, improves if the theory is constrained by symmetry. Symmetries, without doubt, have always been a guiding principle in the study of quantum field theories. For example, two-dimensional conformal field theories thrive from an infinite-dimensional (local) conformal symmetry algebra which leads to many exactly solvable models [2]. The infinite-dimensional symmetry constrains the correlation function to such an extent that they can often times be solved algebraically. This is known as the bootstrap approach.

In the study of string theory, it was observed that a special class of conformal field theories can be obtained from $\mathcal{N} = (2, 2)$ supersymmetric theories by a "topological twist" [11, 12, 16]. This "twist" amounts to a reinterpretation of the geometric nature of the fields. Equivalently, it shifts the stress-energy tensor of the model by a total differential and hence the (conformal) dimensions of the fields. At the same time, the twist deforms the supersymmetric charges among which one now finds a nilpotent operator Q inducing a fermionic symmetry, such that the space of physical observables coincides with the cohomology of Q.

In addition, the (Hilbert) stress-energy tensor $T_{\mu\nu}$, which measures the reaction of the action to a perturbation in the background geometry, is Q-exact: $T_{\mu\nu} = QG_{\mu\nu}$. In particular, the partition function of such theories does not depend on the geometry of the worldsheet. Indeed, the partition function varies by a correlator of an Q-exact term when one varies the background metric $\delta_g Z = \langle Q(\dots)\rangle$. Since Q generates a symmetry, and assuming that the path integral measure is Q-invariant, the correlator of Q-exact expressions vanishes.

The twisted models are known as *topological conformal field theories*. The arguably most popular examples are Witten's A- and B-model [15]. Topological conformal field theories enjoy diverse applications in both pure mathematics, e.g. in the study of Gromov-Witten invariants within enumerative problems of algebraic geometry, and theoretical physics, especially in the construction of two-dimensional quantum gravity [13, 14].

The exactness of the stress-energy tensor has far reaching consequences. Foremost, it implies that the partition function and correlation functions of observables, i.e. of Q-cocycles, are independent of the background geometry of the worldsheet [4, 9]. Notably, this independence is not due to averaging of all worldsheet geometries, i.e. due to an integration over all background metrics within the path integral, as is the case in string theory, but rather an intrinsic feature of the theory. Another consequence is that the central charge of any such conformal field theory must vanish.

Even though the stress-energy tensor vanishes on the cohomology of Q, it is remarkably fruitful to go beyond cohomology. Let us denote by $\overline{\mathcal{M}}_{g,n}$ the compactified moduli space of n-punctured compact Riemann surfaces of genus g. Analogously to the bosonic string, one can construct closed differential forms on $\overline{\mathcal{M}}_{g,n}$ by studying correlation functions involving the primitive $G_{\mu\nu}$ [4, 9]. Integrating these closed forms over cycles in $\overline{\mathcal{M}}_{g,n}$, one constructs solutions of the famous Witten-Dijkgraaf-Verlinde-Verlinde associativity equations [5, 13]. The study of periods of closed differential forms on the moduli spaces $\overline{\mathcal{M}}_{g,n}$ is known as coupling the theory to gravity. The resulting correlation functions appear in the study of topological string theories [15].

In addition to twisting $\mathcal{N} = (2, 2)$ supersymmetric theories, one can study yet another method to construct topological conformal field theories, namely by suitably gauge fixing topological field theories which is the main subject of study in this thesis. Whereas the twisting of supersymmetric theories relies on the existing of holomorphic $U(1)$ currents arising from residual symmetries of the action, so-called R-symmetries [4, 15], those currents may now be absent [8]. This means, in particular, that topological conformal field theories obtained by gauge fixings do not necessarily admit a supersymmetric partner theory.

As a special example, the zero area limit of two-dimensional Yang-Mills theory (also known as BF theory) is analyzed in Chapter 4 and 5. In particular, it is shown that the non-abelian model does not arise from twisting a $\mathcal{N} = (2, 2)$ supersymmetric theory and therefore belongs to a new class of topological conformal field theories.

Two-dimensional BF theory is an example of the Poisson sigma model, which, on one hand, has strong ties to two-dimensional gravity [6, 10] and is, on the other hand, the two-dimensional quantum field theory behind Kontsevich's deformation quantization of Poisson manifolds [3, 7]. The Poisson sigma model itself is an example of an even more general construction of (two-dimensional) classical topological field theories known as AKSZ sigma models [1].

In a broader sense, this thesis proposes a new mechanism for constructing two-dimensional topological conformal field theories by gauge fixing two-dimensional AKSZ sigma models. After coupling these to gravity, we intend to study in the future the resulting AKSZ topological string theories. In particular, new solutions of WDVV equations can be obtained by integrating closed differential forms on the moduli space of punctured compact Riemann surfaces against cycles. The present text is a first step in this direction, which initiates the study of linear two-dimensional AKSZ models, namely BF theories.

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CHAPTER 2

Background material

By a *topological quantum field theory* (TQFT) we shall mean a quantum field theory (QFT) whose defining action functional is classically invariant under diffeomorphisms. Such theories are called *generally covariant* in the physics literature.

The invariant action functional associates via path integral techniques to any (compact) closed manifold an object, the partition function, which depends solely on the topological properties of the manifold. The machinery of path integrals can therefore be used to calculate topological invariants of manifolds. This observation has triggered an ongoing blossoming interaction of theoretical physics and pure mathematics. The catch is that to define the partition function of a field theory one has to perform an integral over an infinite dimensional manifold (typically certain function spaces) whose integration theory is most of the time not rigorously defined. A mathematically accurate definition is therefore highly desirable.

One approach to the problem is to try to generalize the abstract features of the path integral formulation of quantum mechanics to the desired situation: consider a quantum mechanical system with Hilbert space (space of states) $\mathcal{H} = L^2(\mathbb{R})$ and Hamiltonian $\hat{H}$. A state $\psi \in \mathcal{H}$ evolves according to the action of the evolution operator

$$Z(t) = e^{-it\hat{H}/\hbar} \in \mathrm{End}(\mathcal{H})$$

whose action can be expressed in terms of an integral kernel

$$Z(t)\colon \psi(x) \mapsto \psi_t(x) = \int \mathcal{D}y(t)\ Z_t(x,y)\psi(y).$$

Here, the integral is taken over all paths $y\colon [0,t] \to \mathbb{R}$. Notably, the integral kernel $Z_t(x,y)$ satisfies the “gluing” property

$$Z_{t_1+t_2}(z,x) = \int \mathcal{D}y(s)\ Z_{t_2}(z,y)Z_{t_1}(y,x),$$

which results into a semi-group law of the evolution operator

$$Z(t_1+t_2) = Z(t_1)Z(t_2).$$

The partition function is defined as the trace of the evolution operator

$$Z = \mathrm{tr}_{\mathcal{H}}\, e^{-it\hat{H}/\hbar} = \int \mathcal{D}x\ Z_t(x,x).$$

The main idea of the path integral formulation of quantum mechanics is that the integral kernel $Z_t(x,y)$ can be represented as an integral over all paths $q\colon [0,t] \to \mathbb{R}$ satisfying the boundary conditions $q(0) = y$ and $q(t) = x$ weighted by the exponential of some action functional

$$Z_t(x,y) = C \int_{\substack{q(0)=y \\ q(t)=x}} \mathcal{D}q(s) e^{iS(q)/\hbar}.$$

This view point allows the interpretation of the integral kernel $Z_t(x,y)$ as a transition amplitude

$$\langle x | e^{-it\hat{H}/\hbar} | y \rangle$$

from an initial state $|y\rangle$ (so-called *in-state*) to a final state $|x\rangle$ (so-called *out-state*) in time t.

If we adopt the point of view that quantum mechanics is defined by a Hilbert space $\mathcal{H}$ and an evolution operator $Z(t)$ satisfying the aforementioned properties, then we can summarize the abstract properties of the theory as follows: quantum mechanics is a machinery Z that takes points and (oriented) intervals between points as inputs and associates

- to points $\bullet$ a vector space $Z(\bullet) = \mathcal{H}$,
- to an interval $\underset{t}{\bullet\!\!\rightarrow\!\!\bullet}$ a linear map $Z(t) = Z(\underset{t}{\bullet\!\!\rightarrow\!\!\bullet}) \in \operatorname{End}(\mathcal{H})$,
- to a circle $\circlearrowleft_t$ of length t a number $\operatorname{tr}_{\mathcal{H}} Z(t)$

such that gluing intervals corresponds to composition of the associated maps

$$Z(\underset{s+t}{\bullet\!\!\longrightarrow\!\!\bullet}) = Z(\underset{s}{\bullet\!\!\rightarrow\!\!\bullet} \sqcup \underset{t}{\bullet\!\!\rightarrow\!\!\bullet}) = Z(\underset{s}{\bullet\!\!\rightarrow\!\!\bullet}) \circ Z(\underset{t}{\bullet\!\!\rightarrow\!\!\bullet}).$$

In particular, the partition function is defined by gluing an interval to a circle

$$Z(\circlearrowleft_t) = Z(\overset{\bullet}{\underset{\bullet}{\vdots}}\!\circlearrowleft_t) = \operatorname{tr}_{\mathcal{H}} Z(\underset{t}{\bullet\!\!\rightarrow\!\!\bullet})$$

The above machinery defines quantum mechanics in terms of axioms and ultimately in terms of a functor between the category of one-dimensional cobordisms, whose objects are points and whose morphisms are intervals between these points, and the category of vector spaces.

The idea of a functorial definition of a QFT was made precise and put forward by Segal [30] and later by Atiyah [1]. In modern language, a QFT should be defined as a functor from the "space-time category" (the category of cobordisms) to the category of vector spaces (or more appropriately the category of Hilbert spaces). The different nature among QFTs comes from different geometric structures one allows on the cobordisms.

The aforementioned axiomatic definition comes with the drawback that it focuses on the structural properties of the QFT while postponing the question of its existence and its explicit construction. Heuristically, the partition function itself should realize the functor. However, the issue of a mathematically rigorous definition of the path integral remains. Recently, a lot of work was put into the perturbative quantization of QFTs on manifolds with boundaries using the Batalin-Vilkovisky (BV) formalism [7, 8] which focuses in particular on establishing the cutting and gluing properties of the partition function.

Nevertheless, it should be clear that the nature of the QFT is, to a large extent, governed by the symmetries of the partition function, which in turn is determined by the symmetries of the action functional. Indeed, if the cobordisms carries some geometric structure then the partition function, i.e. the functor defining the QFT, should be invariant under any deformation of the cobordism preserving this geometric structure. For example, if the cobordism carries no geometric structure whatsoever, the partition function should be invariant under diffeomorphisms; if the cobordism is endowed with a conformal structure, the partition function should be invariant under all conformal diffeomorphisms; and so on.

In the sequel, we shall therefore focus on the construction of action functionals which are invariant under certain symmetries. We will concentrate on two types of action functionals, those who are invariant under diffeomorphisms of the underlying manifold, and those who are invariant under conformal diffeomorphisms.

In order to be self-contained, we begin by spelling out the functorial definition of a QFT by Atiyah and Segal. Then we will give a brief and by no means exhaustive introduction to the theory of TQFTs and CFTs, in which we recall the main definition needed for our purposes. Finally, we will combine the two notions into theories known as *topological conformal field theories* (TCFTs). At first, this notion might seem contradicting. After all, if a theory is topological, i.e. invariant under all diffeomorphisms, it is in particular invariant under conformal diffeomorphisms and hence conformal. In Section 2.4, we will give a more precise definition.

Since the study of TQFTs and CFTs is so vast, an in-depth review of all techniques and connections to other areas of physics and mathematics is beyond the scope of this text and so we will focus in the sequel on the main ideas and techniques which are important for this thesis.

2.1. The Atiyah-Segal picture of quantum field theory

In this section we want to briefly outline the idea of a functorial definition of QFTs. Such a definition tries to formulate the idea of *locality* in a mathematical rigorous way. In layman's term, locality of a QFT is the idea that the model defined on some manifold M can be computed by a divide-and-conquer principle: one chops M into simpler pieces $M = \coprod_i M_i$, computes the theory on the components M_i and then glues the pieces back together to obtain the theory on the whole of M.

For example, in two-dimensions, it is known that any surface is glued from disks, and pair of pants, see Figure 1 for an example. Therefore, it

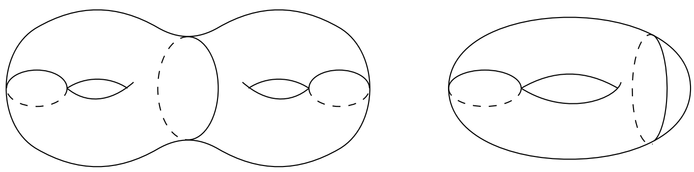
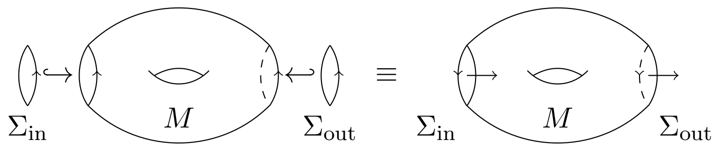


FIGURE 1. Examples of surfaces which are glued form disks and pairs of pants.

should be enough to compute a local two-dimensional QFT on the disk and a pair of pants and then glue the pieces back together to obtain the answer on any surface of any genus.

2.1.0.1. *Atiyah's axiomatic approach to TQFT.* The idea of cutting and gluing can be neatly packaged in the following definition due to Atiyah [1]: Firstly, by a *cobordism* between two compact closed oriented $(n-1)$-dimensional manifolds Σ_{in} and Σ_{out}, we shall mean a compact n-dimensional manifold M with boundary together with a decomposition $\partial M = \bar{\Sigma}_{\text{in}} \sqcup \Sigma_{\text{out}}$. By convention, $\bar{\Sigma}$ has its orientation reversed with respect to Σ. We can nicely represent a cobordism pictorially:



An *in-boundary* is defined by a boundary component whose orientation is opposite to the orientation induced by M. It is indicated by an arrow directed inwards. Likewise an *out-boundary* is defined by a boundary component whose orientation agrees with the orientation induced by M and is indicated by an arrow directed outwards. For a cobordism M between Σ_{in} and Σ_{out} we will often write

$$\Sigma_{\text{in}} \overset{M}{\rightsquigarrow} \Sigma_{\text{out}}.$$

The manifold M underlying the cobordism $\Sigma_{\text{in}} \overset{M}{\rightsquigarrow} \Sigma_{\text{out}}$ will play the role of the n-dimensional worldsheet.

We are now ready to state the following

DEFINITION 2.1.1 (Atiyah [1]). An *n-dimensional topological (quantum) field theory* (TQFT) Z is an assignment of the following form:

- to a compact closed oriented $(n-1)$-dimensional manifold Σ, one assigns a vector space $Z(\Sigma) = \mathcal{H}_\Sigma$ over $\mathbb{C}$ (or $\mathbb{R}$). This vector space plays the role of the space of states of the QFT.
- to a cobordism $\Sigma_{\text{in}} \overset{M}{\rightsquigarrow} \Sigma_{\text{out}}$, one assigns a $\mathbb{C}$-linear map
$$Z_M \colon \mathcal{H}_{\text{in}} \to \mathcal{H}_{\text{out}}.$$
The map Z_M will play the role of the partition function, possibly with prescribed boundary conditions.
- to a diffeomorphism $\varphi \colon \Sigma \to \Sigma$, one assigns an isomorphism
$$Z(\varphi) \colon \mathcal{H}_\Sigma \to \mathcal{H}_\Sigma.$$
This assignment is compatible with the composition of diffeomorphisms. If φ is orientation preserving, then $Z(\varphi)$ is $\mathbb{C}$-linear; if φ is orientation reversing, then $Z(\varphi)$ is $\mathbb{C}$-anti-linear

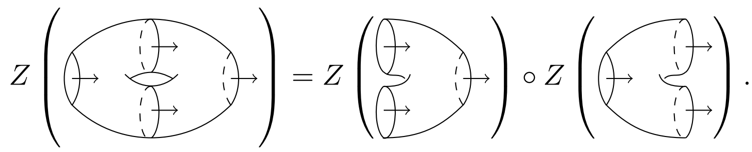
In addition, the above assignment is subject to the following axioms:

- *multiplicativity*: disjoint unions of $(n-1)$-dimensional manifolds are mapped to tensor products of the associated vector spaces:
$$\mathcal{H}_{\Sigma_1 \sqcup \Sigma_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}.$$
- *involutory*: $\mathcal{H}_{\bar{\Sigma}} = \mathcal{H}^*_\Sigma$ where $\mathcal{H}^*_\Sigma$ is the dual vector space of $\mathcal{H}_\Sigma$.

- *functorial/gluing*: given two cobordisms $\Sigma_1 \overset{M_{12}}{\rightsquigarrow} \Sigma_2$ and $\Sigma_2 \overset{M_{23}}{\rightsquigarrow} \Sigma_3$, where the out-boundary of M_{12} coincides with the in-boundary of M_{23} (here denoted by Σ_2), one can glue them into a cobordism $\Sigma_1 \overset{M_{13}}{\rightsquigarrow} \Sigma_3$, with $M_{13} = M_{12} \cup_{\Sigma_2} M_{23}$, such that $Z(M_{13})$ decomposes into the concatenation of $Z(M_{23})$ and $Z(M_{12})$:

  $$Z(M_{13}) = Z(M_{23}) \circ Z(M_{12})\colon \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}.$$

  Pictorially, this means

  If we denote the pairing of $\mathcal{H}_\Sigma$ and its dual $\mathcal{H}_\Sigma^*$ by $\langle \cdot, \cdot \rangle_\Sigma$, then the gluing property can be written as

  $$Z(M_{13}) = \langle Z(M_{23}), Z(M_{12}) \rangle_{\mathcal{H}_{\Sigma_2}}.$$

- *normalization*: the theory Z assigns the base field $\mathbb{C}$ (or $\mathbb{R}$) to the empty $(n-1)$-dimensional manifold $\varnothing$:

  $$Z(\varnothing) = \mathcal{H}_\varnothing = \mathbb{C}.$$

- *diffeomorphism equivariance*: Z commutes with diffeomorphisms in the sense that the following diagram commutes

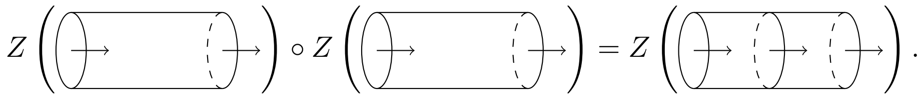
  $$\begin{array}{ccc} \mathcal{H}_{\Sigma_{1,\text{in}}} & \xrightarrow{Z_{M_1}} & \mathcal{H}_{\Sigma_{1,\text{out}}} \\ Z(\varphi_{\text{in}}) \big\downarrow & \circlearrowleft & \big\downarrow Z(\varphi_{\text{out}}) \\ \mathcal{H}_{\Sigma_{2,\text{in}}} & \xrightarrow[Z_{M_2}]{} & \mathcal{H}_{\Sigma_{2,\text{out}}} \end{array}$$

  Here, $\varphi\colon M_1 \to M_2$ is a diffeomorphism between two (possibly the same) cobordisms $\Sigma_{1,\text{in}} \overset{M_1}{\rightsquigarrow} \Sigma_{1,\text{out}}$ and $\Sigma_{2,\text{in}} \overset{M_2}{\rightsquigarrow} \Sigma_{2,\text{out}}$ and $\varphi_{\text{in/out}}$ its restriction to the in- respectively out-boundary.

Since a closed n-dimensional manifold M can be seen as a cobordism $\varnothing \overset{M}{\rightsquigarrow} \varnothing$ between the empty $(n-1)$-dimensional manifold and itself, the normalization axiom implies that a TQFT Z assigns a $\mathbb{C}$-linear function $Z(M)\colon \mathbb{C} \to \mathbb{C}$ to M, i.e. $Z(M)$ is a multiplication by some number $Z_M \in \mathbb{C}$. By abuse of notation, one identifies the map $Z(M)$ with the number Z_M. The interpretation of Z_M is the partition function of the TQFT.

Remark 2.1.1. In the jargon of physics, the axiom of diffeomorphism equivariance is equivalent to the requirement that the theory is covariant.

Let us now discuss some consequences of the above axioms. The multiplicativity axiom implies that for a cylinder $\Sigma \times I$, where I is some interval, the map $Z(\Sigma \times I) \in \mathrm{End}(\mathcal{H}_\Sigma)$ is idempotent. This can best be seen pictorially:



In [1], an additional axiom is imposed:

- *TFT:* $Z(\Sigma \times I) = id_{\mathcal{H}_\Sigma}$.

REMARK 2.1.2. This additional axiom links the axiomatic definition further with the physical intuition: If one assumes that the $(n-1)$-dimensional manifold Σ encodes all spatial directions, then the extra dimension of the cylinder $\Sigma \times I$ encodes a time direction. This is the situation which one usually encounters in Hamiltonian dynamics. The map $Z(\Sigma \times I)$ describes, therefore, the evolution of an in-state (an element of $\mathcal{H}_{\Sigma_{\mathrm{in}}}$) into an out-state (an element of $\mathcal{H}_{\Sigma_{\mathrm{out}}}$), that is, $U(T) = Z(\Sigma \times I)$ is the evolution operator of the system. Here T is the length of the interval. The axiom implies that the evolution operator is the identity. On the other hand, the quantum mechanical evolution operator is given by the exponential of the Hamiltonian. Therefore, the axiom implies that the Hamiltonian of the system is (classically) zero. This is the main feature of TQFTs.

Instead of the cylinder $\Sigma \times I$, we could equally well consider the torus $\Sigma \times S^1$. Due to the gluing axiom, we can think of the torus as being glued from the cylinder by identifying its in- and out-boundary, see Figure 2

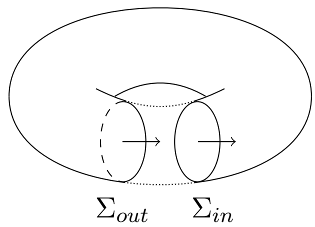


FIGURE 2. Gluing a cylinder into a torus.

Since

$$Z(\Sigma \times I) \in \mathcal{H}_\Sigma \otimes \mathcal{H}_\Sigma^*$$

it follows that if we glue the cylinder to a torus one has

$$Z(\Sigma \times S^1) = \mathrm{tr}\,(id_{\mathcal{H}_\Sigma}) = \dim \mathcal{H}_\Sigma.$$

Often times, the vector space associated to Σ is infinite dimensional. The trace has therefore to be regularized in a way consistent with the gluing axiom.

REMARK 2.1.3. In modern language, Atiyah's definition can be summarized as follows:

DEFINITION 2.1.2 (Atiyah modern language). An *n-dimensional TQFT* is a symmetric monoidal functor between the n-dimensional cobordism category $\mathrm{Cob}_n^{\sqcup}$ and the category of vector spaces $\mathrm{Vect}_{\mathbb{C}}^{\otimes}$.

Here the objects of the category Cob_n are $(n-1)$-dimensional compact closed oriented manifolds Σ and morphisms are cobordisms $\Sigma_1 \overset{M}{\rightsquigarrow} \Sigma_2$. The monoidal structure is the disjoint union.

2.1.0.2. *Segal's axiomatic approach to QFT.* Atiyah's axiomatic definition of a TQFT was largely inspired by Segal's axiomatic approach to two-dimensional CFT [30]. Segal's idea was a similar functorial definition with the difference that the cobordisms are allowed to carry local geometric structures, like conformal structures, complex structures, a volume form etc. The vector spaces and maps which are associated to cobordisms and their boundaries may now depend on these geometric structures. Of course, the above axioms have to be modified accordingly.

For example, cobordisms may carry a Riemannian metric. In order to be able to glue two n-cobordisms (M_1, g_1) and (M_2, g_2), one has to consider not only $(n-1)$-dimensional manifolds Σ but rather such manifolds with collars, i.e. endowed with germs of Riemannian metrics on the infinitesimal cylinder $\Sigma \times [-\varepsilon, \varepsilon]$ [26]. Gluing of the two Riemannian manifolds along a common boundary Σ is then possible if and only if Σ can be endowed with a Riemannian metric g which coincides with the restriction of g_1 on $\Sigma \times [-\varepsilon, 0]$ and with g_2 on $\Sigma \times [0, \varepsilon]$, c.f. Figure 3.

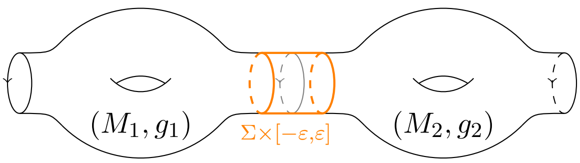


FIGURE 3. Schematically gluing two cobordisms endowed with a metric over a common boundary

In addition, the diffeomorphism equivariance axiom must be modified: If g denotes any geometric structure (not necessarily a Riemannian metric) then the following diagram commutes [26]:

$$\begin{array}{ccc} \mathcal{H}_{\Sigma_{1,\mathrm{in}}, g_{\mathrm{in}}} & \xrightarrow{Z_{M_1, g}} & \mathcal{H}_{\Sigma_{1,\mathrm{out}}, g_{\mathrm{out}}} \\ {\scriptstyle Z(\varphi_{\mathrm{in}})}\downarrow & \circlearrowleft & \downarrow{\scriptstyle Z(\varphi_{\mathrm{out}})} \\ \mathcal{H}_{\Sigma_{2,\mathrm{in}}, \varphi_* g_{\mathrm{in}}} & \xrightarrow[Z_{M_2, \varphi_* g}]{} & \mathcal{H}_{\Sigma_{2,\mathrm{out}}, \varphi_* g_{\mathrm{out}}} \end{array}$$

where $\varphi_* g$ is the pushforward of the geometric structure by φ.

2.1.0.3. *Examples of Atiyah-Segal QFTs.*

EXAMPLE 2.1.4 (Quantum mechanics). The first example, which has already been sketched in the introduction to this chapter, is quantum mechanics. This is a one-dimensional QFT in the sense of Atiyah and Segal. To the 0-dimensional manifolds, which are oriented points, one assigns the Hilbert space $\mathcal{H}$ of the quantum mechanical model. To the 1-dimensional cobordisms with boundary (intervals) one assigns the evolution operator $Z(t) = e^{-it\hat{H}/\hbar}$. The cobordisms are endowed with a Riemannian metric, which identifies the length of the cobordism with the length of the interval. The gluing axiom is equivalent to the semi-group law

$$Z(t_1 + t_2) = Z(t_1)Z(t_2),$$

and the partition function of the theory is given by the cobordism $\varnothing \overset{S^1}{\rightsquigarrow} \varnothing$:

$$Z(S^1) = \operatorname{tr}_{\mathcal{H}} Z(I),$$

where the circle S^1 is glued from the interval I.

Example 2.1.5 (Two-dimensional Yang-Mills). Let G be a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$ and Σ a surface (possibly with boundary). We denote its Killing form by tr. Usually, two-dimensional Yang-Mills theory is defined by a connection (gauge field) $A \in \Omega^1(\Sigma, \mathfrak{g})$ and the action functional

$$S = \frac{1}{2\varepsilon} \int_\Sigma \operatorname{tr}(F \wedge *F), \tag{2.1}$$

where F is the curvature of A and $*$ is the Hodge star operator on Σ. To phrase two-dimensional Yang-Mills theory as a QFT in the sense of Atiyah and Segal, one might therefore think that one has to endow the two-dimensional cobordisms (surfaces) with a Riemannian metric defining the Hodge star operator. However, in the first order formulation of (2.1)

$$S = \int_\Sigma \operatorname{tr} \beta F - \frac{\varepsilon}{2}\beta^2\mu. \tag{2.2}$$

Here, β is an auxiliary $\mathfrak{g}$-valued scalar field and μ is a volume form on Σ. By the equations of motion, $F = \beta\mu$, the two theories (2.1) and (2.2) are equivalent. Notably, this equivalence shows that it suffices to chose a volume form to define the theory. From Atiyah-Segal's point of view, we therefore need to endow the cobordisms only with a volume form.

In a two-dimensional QFT, the objects are compact closed oriented one-dimensional manifolds, i.e. a collection of circles. The two-dimensional cobordisms are surfaces. We denote the cobordisms[1] by Σ. To any circle, one associates the vector space (space of states) of complex-valued square-integrable G-invariant functions (class functions) on G:

$$\mathcal{H}_{S^1} = L^2(G)^G.$$

Endowed with the Hermitian inner product

$$\langle \psi, \varphi \rangle := \int_G dg \bar{\psi}(g)\varphi(g)$$

where dg is the Haar-measure on G, the vector space $\mathcal{H}_{S^1}$ becomes a Hilbert space. Let

$$g = P \ \exp\left(\int_{S^1} A\right) \in G$$

[1] Note that this differs from the notation of the previous discussion where the boundary components were denoted by Σ.

be the holonomy of the gauge field A around a boundary component. It turns out that the set of characters $\chi_R(g)$, where R labels an irreducible representations of G, forms an orthonormal basis of $\mathcal{H}_{S^1}$. It was shown in [9, 39] that the partition function of a cobordism with m in-boundaries and n out-boundaries is (see also [26])

$$Z_\Sigma(g_1^{\text{in}}, \dots, g_1^{\text{out}}, \dots) = \sum_R (\dim R)^{\chi(\Sigma)} e^{-aC_2(R)} \prod_{j=1}^m \bar{\chi}_R(g_j^{\text{in}}) \prod_{k=1}^n \chi_R(g_k^{\text{out}}).$$

Here, $\chi(\Sigma) = 2-2g-m-n$ denotes the Euler characteristic of the cobordism Σ, a stands for the total area $\int_\Sigma \mu$, $g_j^{\text{in/out}}$ are the holonomies of the gauge field around the appropriate in-/out-boundary circles and $C_2(R)$ denotes the quadratic Casimir element of the irreducible representation R. The partition function Z_Σ is to be understood as an integral kernel:

$$Z_\Sigma \colon \psi \mapsto (Z_\Sigma\psi)(g^{\text{out}}) = \int_{G^{\times m}} dg^{\text{in}} Z_\Sigma(g^{\text{out}}, g^{\text{in}})\psi(g^{\text{in}}).$$

Due to the orthogonality property of the characters $\chi_R(g)$, the partition function satisfies indeed the gluing axiom. For example, a direct computation shows that for a cylinder, $\Sigma = S^1 \times I$, one has

$$\int_G dg\, Z_\Sigma(g_1, g) Z_\Sigma(g, g_2) = Z_\Sigma(g_1, g_2).$$

2.2. Topological quantum field theory

Albeit the aforementioned difficulties of a universal definition of the path integral, there are essentially two approaches to a definition of a TQFT.

The first attempt is to build an action functional purely out of diffeomorphism invariant objects, such as differential forms. These theories are called *TQFTs of Schwarz type* and their thorough study was initiated in the seminal paper by A. Schwarz [29]. Prominent representatives of this type of TQFTs are the celebrated Chern-Simons theory and the so-called BF models.

The second attempt is by making use of the presence of a global odd symmetry: Suppose the space of fields carries a grading (one may think of bosonic (even) and fermionic (odd) degrees of freedom). By an *odd symmetry* we mean a cohomological vector field Q on the space of fields, i.e. an operator of odd[2] degree who squares to zero. The prime example of a cohomological vector field is the BRST operator present in the *Becchi-Rouet-Stora-Tyutin* (BRST) approach to quantization of gauge theories. Also supersymmetry charges, or rather their topological twists, are examples which one often encounters in the literature.

Given such an operator Q, one can consider an action which is Q-exact. While the primitive may carry any explicit dependence on geometric data, for example a metric, the partition function is independent thereof. Indeed, if one varies the geometric data, the partition function varies by a Q-exact

[2] In the case of a $\mathbb{Z}$-grading, Q is of degree 1.

term under the correlator. Since Q generates a global symmetry, the latter vanishes which proves the invariance of the partition function on the worldsheet geometry.

In the following we will give a brief introduction the two types of TQFTs and broad overview of their most important properties. We will follow closely [5, 29].

2.2.1. TQFT of Witten type. Let us assume that we are given a graded space of states endowed with a differential Q, i.e. a nilpotent operator Q which satisfies the Leibniz rule.

TQFTs of Witten type are models whose action functional can be written as a Q-commutator

$$S = \int_\Sigma \{Q, V(\phi)\}, \tag{2.3}$$

where ϕ stands for a generic field of the theory. In addition, one has to assume that the divergence of Q with respect to the (formal) path integral measure $\mathcal{D}\phi$ vanishes, i.e. that the path integral measure is invariant under Q:

$$\mathcal{L}_Q \mathcal{D}\phi = 0.$$

Since Q is nilpotent, $\{Q, Q\} = 0$, Q generates a symmetry of the action: Let us assume that the fields ϕ vary infinitesimally according to $\delta\phi = \{Q, \phi\}$. Moreover, since Q is a derivation, it satisfies the Leibniz rule and therefore

$$\delta V(\phi) = \{Q, V(\phi)\}$$

for any functional $V(\phi)$. It follows that

$$\delta S = \int_\Sigma \{Q, \delta V(\phi)\} = \int_\Sigma \{Q, \{Q, V(\phi)\}\} = \frac{1}{2}\int_\Sigma \{\underbrace{\{Q, Q\}}_{=0}, V(\phi)\} = 0.$$

By assumption the path integral measure is invariant under Q and hence invariant under the generated symmetry. Therefore, correlation functions of Q-exact expressions vanish. This can be seen as follows: Consider the following formal expression of a general correlation function

$$\langle \mathcal{O}_1(\phi) \dots \mathcal{O}_n(\phi) \rangle = \int \mathcal{D}\phi \, e^{-S(\phi)} \mathcal{O}_1(\phi) \dots \mathcal{O}_n(\phi).$$

After an infinitesimal change of the integration variable $\phi \to \tilde{\phi} = \phi + \varepsilon\{Q, \phi\}$, for which the Jacobian is trivial, due to the Q-invariance of the path integral measure, one has

$$\int \mathcal{D}\phi \, e^{-S(\phi)} \mathcal{O}_1(\phi) \dots \mathcal{O}_n(\phi) = \int \mathcal{D}\tilde{\phi} \, e^{-S(\tilde{\phi})} \mathcal{O}_1(\tilde{\phi}) \dots \mathcal{O}_n(\tilde{\phi}).$$

Expanding the right hand side one finds up to first order in ε

$$\begin{aligned}
&\int \mathcal{D}\tilde{\phi}\, e^{-S(\tilde{\phi})}\mathcal{O}_1(\tilde{\phi})\dots\mathcal{O}_n(\tilde{\phi}) = \\
&= \int \mathcal{D}\phi\, e^{-S(\phi)}\left(\mathcal{O}_1(\phi)+\varepsilon\{Q,\mathcal{O}_1(\phi)\}\right)\dots\left(\mathcal{O}_n(\phi)+\varepsilon\{Q,\mathcal{O}_n(\phi)\}\right) \\
&= \int \mathcal{D}\phi\, e^{-S(\phi)}\left(\mathcal{O}_1(\phi)\dots\mathcal{O}_n(\phi)+\varepsilon\left\{Q,\mathcal{O}_1(\phi)\dots\mathcal{O}_n(\phi)\right\}\right)+\mathcal{O}(\varepsilon^2) \\
&= \langle\mathcal{O}_1(\phi)\dots\mathcal{O}_n(\phi)\rangle+\varepsilon\left\langle\{Q,\mathcal{O}_1(\phi)\dots\mathcal{O}_n(\phi)\}\right\rangle.
\end{aligned}$$

Comparison yields

$$\langle\{Q,\mathcal{O}_1(\phi)\dots\mathcal{O}_n(\phi)\}\rangle = 0.$$

Returning to the situation (2.3), the primitive V may carry any dependence on the geometry of the worldsheet Σ. Indeed, if we denote the geometric data on Σ by g, then the partition function

$$Z = \int \mathcal{D}\phi\, e^{-S(\phi)}$$

is, by the above argument, independent of g:

$$\frac{\partial Z}{\partial g} = \int \mathcal{D}\phi\, e^{-S(\phi)}\left\{Q, -\tfrac{\partial V}{\partial g}\right\} = 0.$$

Suppose now that the geometry of the worldsheet is encoded in a choice of a metric $g_{\mu\nu}$. The stress-energy tensor, which measures the reaction of the theory to a variation of the underlying worldsheet metric

$$T_{\mu\nu} \propto \frac{\partial S}{\partial g^{\mu\nu}} \propto \{Q, G_{\mu\nu}\}, \qquad G_{\mu\nu} \propto \frac{\partial V}{\partial g^{\mu\nu}}$$

is Q-exact. It therefore vanishes on the cohomology of Q, i.e. at physical states. In particular, the Hamiltonian $H = T_{00}$ vanishes on the space of physical states and therefore the model has no (classical) dynamics.

It is often times the case, that the differential Q comes from BRST approach to gauge fixing. An action functional which is Q-exact indicates that one has quantized the trivial action $S = 0$ [5]. Of course, one has to fix a space of fields and define which gauge symmetry will be fixed, in order to arrive at a well-defined model.

Remark 2.2.1 (BRST formalism). Suppose we are given a gauge theory defined by a space of fields $\mathcal{F}_{\text{cl}}$ and an action functional S_{cl}. In the BRST approach to gauge fixing, one resolves the space of classical observables $C^\infty(\mathcal{F}_{\text{cl}}/\text{gauge})$ as follows: One first embeds $\mathcal{F}_{\text{cl}}$ into a $\mathbb{Z}$-graded space of fields $\mathcal{F}^\bullet_{\text{BRST}}$ whose algebra of functions $\mathcal{A}^\bullet_{\text{BRST}} = C^\infty(\mathcal{F}^\bullet_{\text{BRST}})$ is endowed with a differential (a nilpotent derivation) such that Q has degree one, i.e.

$$Q\colon \mathcal{A}^\bullet_{\text{BRST}} \to A^{\bullet+1}_{\text{BRST}},$$

and the zeroth cohomology of Q is isomorphic to $C^\infty(\mathcal{F}_{\text{cl}}/\text{gauge})$

$$H^0_Q(\mathcal{A}_{\text{BRST}}) \cong C^\infty(\mathcal{F}_{\text{cl}}/\text{gauge}).$$

The grading is usually referred to as the *ghost number*. In particular, classical fields and functionals, such as the action, have ghost number zero. Elements of ghost number one are Faddeev-Popov ghosts, elements of ghost number two are ghosts of ghosts, and so on. The gauge symmetry $\delta\phi$ of the classical fields, whuch we denote collectively by ϕ, is encoded in the action of the differential Q:

$$\delta\phi = \{Q, \phi\}.$$

The main observation is that the gauge invariance of the action can now be phrased as the statement that S_{cl} is Q-closed

$$\delta S_{\text{cl}}(\phi) = 0 \implies \{Q, S_{\text{cl}}(\phi)\} = 0.$$

Therefore, S_{cl} defines a Q-cohomology class $[S_{\text{cl}}] \in H_Q$ and gauge fixing is done by choosing an appropriate representative of this cohomology class:

$$S_{\text{gf}} = S_{\text{cl}} + \{Q, \Psi\}.$$

"Appropriate" means in this case that the gauged fixed action S_{gf} should make the partition function

$$Z = \int_{\mathcal{F}^\bullet_{\text{BRST}}} d\mu \, e^{-S_{\text{gf}}}$$

at least perturbatively well-defined. One then takes the perturbation expansion of the left hand side as a definition of the partition function. Moreover, since S_{gf} must have ghost number zero, the primitive Ψ must have ghost number one and is known as the *gauge fixing fermion*. In order that the path integral is independent of a choice of representative of $[S_{\text{cl}}]$, one has to assume the existence of an Q-invariant measure $d\mu$ on $\mathcal{F}^\bullet_{\text{BRST}}$. Indeed, the existence of such a measure ensures that Q-exact expressions vanish under the correlator which in turns guarantees that the partition function is independent of the choice of representative of $[S_{\text{cl}}]$:

$$\begin{aligned} Z &= \int_{\mathcal{F}^\bullet_{\text{BRST}}} d\mu \, e^{-S_{\text{cl}} + \{Q,\Psi\}} \\ &= \int_{\mathcal{F}^\bullet_{\text{BRST}}} d\mu \, e^{-S_{\text{cl}}} \left(1 + \{Q, \Psi e^{\{Q,\Psi\}}\}\right) \\ &= \int_{\mathcal{F}^\bullet_{\text{BRST}}} d\mu \, e^{-S_{\text{cl}}}. \end{aligned}$$

EXAMPLE 2.2.2 (The supersymmetric free massless particle). Following [5], we consider a one-dimensional QFT, i.e. quantum mechanics, whose field content is one scalar field $\phi \in C^\infty(S^1, \mathbb{R})$. The defining action functional is taken to be trivial: $S = 0$. We then declare arbitrary shifts in ϕ, $\delta_\alpha \phi = \alpha$ for any $\alpha \in C^\infty(S^1, \mathbb{R})$, as the gauge symmetry we seek to fix. The BRST procedure augments the gauge parameter α to a ghost ψ, i.e. a Grassmann-valued function from the circle to the real line. Furthermore, we have to specify a gauge condition. As a convenient choice, we shall take

$$d\phi = 0.$$

To ensure the above gauge condition, we adjoin an anti-ghost $\bar{\psi}$ and a new scalar field λ which will turn out to be an auxiliary field, i.e. it enters non-dynamically in the action. We extend the action of Q as follows:

$$\{Q, \bar{\psi}\} = \lambda, \qquad \{Q, \lambda\} = 0.$$

In principle, it is enough to choose a gauge fixing fermion which leads to an action linear in λ, such as

$$\Psi_0 = \oint_{S^1} i\bar{\psi} d\phi.$$

Then

$$S_{\mathrm{gf},0} = \{Q, \Psi_0\} = \oint_{S^1} i\lambda d\phi + i\bar{\psi} d\psi.$$

The field λ plays thus the role of a Lagrangian multiplier. However, it is computationally often more convenient to choose a gauge fixing fermion which leads to an action which is quadratic in λ. For example, let us choose the following:

$$\Psi = \oint_{S^1} \bar{\psi}\left(id\phi + *\tfrac{1}{2}\lambda\right),$$

where $*$ denotes the Hodge star operator for some chosen metric on S^1 and such that the gauge fixed action reads

$$S_{\mathrm{gf}} = \{Q, \Psi\} = \oint_{S^1} i\lambda d\phi + \tfrac{1}{2}\lambda * \lambda + i\bar{\psi} d\psi. \tag{2.4}$$

The field λ ceases to be a Lagrangian multiplier and becomes an auxiliary field instead.

Remark 2.2.3 (Homotopic gauge-fixing fermions). The actions obtained from Ψ_0 and Ψ describe the same physics. In fact, there exists a family Ψ_t of gauge fixing fermions which interpolates between Ψ_0 and $\Psi_1 \equiv \Psi$, namely

$$\Psi_t = \oint_{S^1} \bar{\psi}\left(id\phi + *\tfrac{t}{2}\lambda\right).$$

Notice that the action

$$S_{\mathrm{gf}}(t) = \{Q, \Psi_t\}$$

trivially depends on t only in a Q-exact manner. Therefore, the partition function is independent of t:

$$\frac{dZ}{dt} = \int d\mu \, \{Q, -\tfrac{d}{dt}\Psi_t\} e^{-S_{\mathrm{gf}}} = 0,$$

where $d\mu$ stands for the Q-invariant path integral measure. The theories defined by Ψ_0 and Ψ are therefore physically equivalent.

Let us return to the model defined by the action (2.4). Its Hamiltonian is indeed Q-exact:

$$H = \sum_{\varphi \in \{\phi, \psi, \bar{\psi}, \lambda\}} \pi_\varphi \dot{\varphi} - \mathcal{L} = \tfrac{1}{2}\pi_\phi^2 = \{Q, -\tfrac{1}{2}\pi_\psi \pi_\phi\}$$

where $\mathcal{L}$ denotes the Lagrangian and the only non-vanishing conjugate momenta π_φ are[3]

$$\pi_\phi = i\lambda, \qquad \pi_\psi = -i\bar{\psi}$$

which satisfy the operator equation

$$\{Q, \pi_\psi\} = -\pi_\phi.$$

[3] Following [16], the conjugated momenta for the fermion ψ is given by a right derivative $\pi_\psi = \frac{\overleftarrow{\partial} \mathcal{L}}{\partial \dot{\psi}}$.

On the other hand, since λ enters the action non-dynamically, one can integrate out λ. The Gaussian integration yields

$$S_{\text{gf}} = \oint_{S^1} \tfrac{1}{2} d\phi \wedge * d\phi + i\bar{\psi} d\psi = \oint_{S^1} dt \ \left(\tfrac{1}{2}\dot{\phi}^2 - i\bar{\psi}\dot{\psi} \right). \tag{2.5}$$

The BRST transformations become

$$\{Q, \phi\} = \psi, \qquad \{Q, \bar{\psi}\} = -i\dot{\phi}, \qquad \{Q, \psi\} = 0$$

which are realized to be supersymmetry transformations. Importantly, Q is no longer nilpotent off-shell

$$\tfrac{1}{2}\{\{Q, Q\}, \bar{\psi}\} = \{Q\{Q, \bar{\psi}\}\} = -i\dot{\bar{\psi}}.$$

Rather its square vanishes modulo the equations of motion for ψ: $\dot{\psi} = 0$.

Remark 2.2.4 (Supersymmetry). In fact, the action (2.5) is invariant under a second supersymmetry generated by

$$\{\bar{Q}, \phi\} = -\bar{\psi}, \qquad \{\bar{Q}, \psi\} = i\dot{\phi}, \qquad \{\bar{Q}, \bar{\psi}\} = 0.$$

As for Q, the operator $\bar{Q}$ nilpotent only on-shell:

$$\frac{1}{2}\{\{\bar{Q}, \bar{Q}\}, \psi\} = \{\bar{Q}, \{\bar{Q}, \psi\}\} = i\dot{\psi}.$$

One can now define an operator δ which acts on a general field φ as

$$\delta\varphi = \{Q + \bar{Q}, \varphi\}.$$

Then, one verifies

$$\delta^2 = 2i\tfrac{d}{dt}.$$

Intuitively, the supersymmetry charges Q and $\bar{Q}$ realize a square root of the time derivative operator d/dt.

Remark 2.2.5 (Hilbert stress-energy tensor). There is an interesting observation, which shows on the nose that the Hamiltonian is Q-exact. Recall that before integrating out λ the action reads

$$S_{\text{gf}} = \{Q, \Psi\}, \qquad \Psi = \oint_{S^1} \bar{\phi} \left(d\psi + * \tfrac{1}{2}\lambda \right).$$

Suppose that we have chosen a background metric $ds_a^2 = a^2 dt^2$ on S^1, where $a \in \mathbb{R}$ determines the length of the circle and $ds_0^2 = dt^2$ is the standard normalized metric on the circle. The metric ds_a^2 determines the Hodge star operator $*_a$ which acts by $*_a 1 = a dt, *_a dt = a^{-1}$. It follows that

$$\Psi(a) = \oint_{S^1} \bar{\phi} \left(d\psi + *_a \tfrac{1}{2}\lambda \right) = \oint_{S^1} dt \bar{\psi} \left(-\dot{\phi} + \tfrac{a}{2}\lambda \right).$$

The Hilbert stress-energy tensor is defined as the derivative of the action with respect to the background metric. In the case at hand, it is given by

$$T = -\frac{\partial S}{\partial a} = -\left\{ Q, \frac{\partial \Psi(a)}{\partial a} \right\} = -\{Q, \tfrac{1}{2}\bar{\psi}\lambda\}$$

which coincides with the Hamiltonian obtained form the Legendre transformation of the Lagrangian.

2.2.2. TQFTs of Schwarz type. We have seen in the previous section that TQFTs of Witten type can be obtained from gauge fixing trivial actions $S = 0$. We now turn to the discussion of TQFTs who admit a non-trivial action but nevertheless do not depend on the worldsheet geometry. This can be achieved, for example, by constructing the action out of differential forms. The arguably best known example is Chern-Simons theory which has been studied intensively in the last decades in relations to a vast variety of topics, such as link invariants [37], three-dimensional quantum gravity [17, 36], conformal field theory [32], string theory and matrix models [25, 41], and many more. It's perturbative quantization has been discussed in detail in [2, 3] for closed manifolds and in [7, 8] for manifolds with boundaries.

Since this thesis focuses on two-dimensional QFTs, we will, however, explain the ideas of TQFTs of Schwarz type by means of another important class of examples, namely so-called BF theories. Even if BF theories can be easily defined in any dimension, we will restrict ourselves to two-dimensions.

Before giving the abstract definition, let us spell out how two-dimensional BF theory arises as the zero-area limit of two-dimensional Yang-Mills theory; it is therefore sometimes called "topological Yang-Mills theory". For an in-depth introduction to two-dimensional Yang-Mills theory we refer the interested reader to the literature [9].

Let Σ be a Riemann surface and G a compact complex Lie group with Lie algebra $\mathfrak{g}$. We denote the Killing form on $\mathfrak{g}$ by tr. Moreover, let A be a connection of a principal G-bundle P over Σ. For simplicity, we only consider the case where P is trivial, such that A is a $\mathfrak{g}$-valued one-form on Σ with the following transformation behavior under a gauge-transformation:

$$A \mapsto A^g = g^{-1}Ag + g^{-1}dg, \qquad g\colon \Sigma \to G.$$

Suppose we have chosen a metric on Σ which defines a Hodge star operator $*$. We can then define the two-dimensional Yang-Mills action functional:

$$S_{YM} = \tfrac{1}{2}\int_\Sigma \operatorname{tr} F \wedge *F, \tag{2.6}$$

where $F = dA + \frac{1}{2}[A.A]$ is the curvature of A.

Notice that S_{YM} is second order. However, it admits an equivalent first order description:

$$S_{YM} = \int_\Sigma \operatorname{tr} BF - \tfrac{1}{2}\operatorname{tr} B * B. \tag{2.7}$$

In two dimensions, the Hodge star operator sends two-forms to scalars, and vice versa. The scalar field B enters the action non-dynamically (only B enters in S_{YM} but none of its derivatives) and up on substituting the equations of motion

$$F = *B$$

one recovers the original action (2.6). In addition, we observe that

$$\operatorname{tr} B * B = \operatorname{tr} B^2 d\mu$$

where $d\mu$ is an area form on Σ. Therefore

$$S_{YM} = \int_\Sigma \operatorname{tr} BF - \tfrac{1}{2}\operatorname{tr} B^2 d\mu.$$

In the zero area limit, we are thus left with the action

$$S = \int_\Sigma \operatorname{tr} BF \tag{2.8}$$

which is defined purely in terms of differential forms and is hence invariant under diffeomorphisms of the worldsheet Σ. In consequence, the model defined by the action (2.8) defines a TQFT in the sense that its partition function is independent of the specific worldsheet geometry. The zero area limit is therefore known as "topological Yang-Mills theory".

Abstractly, the action (2.8) does only depend on a $\mathfrak{g}$-valued one-form A and a $\mathfrak{g}$-valued zero-form B. More generally, we could consider the scenario where the zero-form B takes values in the dual Lie algebra $\mathfrak{g}^*$ rather than in $\mathfrak{g}$ itself. In this case, one can define an action

$$S_{BF} = \int_\Sigma \langle B, F \rangle \tag{2.9}$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. This action is indeed diffeomorphism invariant and thus defines a TQFT. This model is known as a *BF theory*. The field content is a gauge field $A \in \Omega^1(\Sigma, \mathfrak{g})$ and a scalar field $B \in \Omega^0(\Sigma, \mathfrak{g}^*)$.

REMARK 2.2.6. BF theories are defined for arbitrary dimensions. If the worldsheet Σ has dimension d, then the field content of the model is

$$(B, A) \in \Omega^{d-2}(\Sigma, \mathfrak{g}^*) + \Omega^1(\Sigma, \mathfrak{g}).$$

The action functional stays the same as in (2.9).

The classical equations of motions obtained from (2.9) are

$$\begin{aligned} F = dA + \tfrac{1}{2}[A, A] &= 0, \\ d_A B = dB + ad_A^* B &= 0. \end{aligned}$$

A model defined by the action (2.9) thus studies (classically) flat connections. In fact, a heuristic argument shows that in the path integral formalism, the scalar field B plays the role of a Lagrangian multiplier. Therefore, the partition function computes formally the volume of the moduli space of flat connections

$$Z = \int \mathcal{D}B\mathcal{D}A \, e^{-S_{BF}} = \int_{F=0} \mathcal{D}A \, .$$

The question of *which* volume of the moduli space of flat connection is computed by Z, was settled by E. Witten in [39], where he showed that Z computes the *symplectic* volume.

Notice, however, that the action (2.9) admits a gauge symmetry:

$$\delta_\alpha A = d\alpha, \qquad \alpha \in \Omega^0(\Sigma, \mathfrak{g}).$$

The path integral is therefore not well-defined. In his seminal paper [29], A. Schwarz computed the partition function and showed that it is given by the Ray-Singer torsion of Σ. Schwarz's idea was to introduce an auxiliary metric on Σ in order to implement the gauge condition $d^*A = 0$, where

$d^*\colon \Omega^k(\Sigma) \to \Omega^{k-1}(\Sigma)$ is the adjoint[4] of the deRham differential d with respect to the Hodge inner product

$$(\beta, \alpha) = \int_\Sigma \langle \beta \wedge *\alpha \rangle, \qquad \alpha \in \Omega(\Sigma, \mathfrak{g}), \beta \in \Omega(\Sigma, \mathfrak{g}^*). \tag{2.10}$$

Let us describe the above argument in slightly more detail. For simplicity, we consider the abelian case, $\mathfrak{g} = \mathbb{R}$. The gauge fixing $d^*A = 0$ is equivalent to the *Lorenz gauge condition* $d * A = 0$. The latter can be implemented via the Faddeev-Popov trick. The gauge fixed action is given by

$$S_{gf} = \int_\Sigma BdA + \lambda d * A + bd * dc \tag{2.11}$$

where (c, b) are the Faddeev-Popov ghost anti-ghost pair and λ is the Lagrangian multiplier implementing the Lorenz gauge.

Now, by Hodge theory, we can decompose the gauge field A in terms of an exact, co-exact and harmonic part

$$A = d\alpha + d^*\beta + \eta,$$

where $\alpha \in \Omega^0(\Sigma), \beta \in \Omega^2(\Sigma)$ and $\eta \in \Omega^1(\Sigma)$. For the time being, we will omit the zero-modes η. The gauge fixed action can then be written in terms of α and β as follows:

$$\begin{aligned} S_{gf} &= \int_\Sigma Bdd^*\beta + \lambda * d^*d\alpha + b * d^*dc \\ &= (*B, \Delta_2\beta) + (\lambda, \Delta_0\alpha) + (b, \Delta_0 c) \\ &= \big(\lambda + *B, (\Delta_0 + \Delta_2)(\alpha + \beta)\big) \end{aligned} \tag{2.12}$$

where $(\cdot, \cdot)$ denotes the Hodge inner product (2.10). Moreover, we denote the Hodge Laplacian acting on k-forms by $\Delta_k = d_k^* d_k + d_k d_k^*$. The change of variables $A = d\alpha + d^*\beta$ introduces a Jacobian in the path integral measure DA: Let us define the linear operator $T = d_0 + d_2^*$ where d_0 denotes the de Rham differential acting on zero-forms and d_2^* its dual acting on two-forms. The change of variables can then be written as $A = T\varphi$ where $\varphi \in \Omega^0(\Sigma) + \Omega^2(\Sigma)$. It follows that formally

$$DA = D\varphi \det \tfrac{\partial A}{\partial \varphi} = D\varphi \det T.$$

The determinant of the linear operator T is given by the square root of the determinant of T^*T, namely

$$\begin{aligned} \det T &= (\det T^*T)^{1/2} = (\det(d_0^* + d_2)(d_0 + d_2^*))^{1/2} \\ &= (\det(\Delta_0 + \Delta_2))^{1/2} = (\det \Delta_0 \det \Delta_2)^{1/2}. \end{aligned}$$

Since T has an infinite spectrum, the determinants are of course not well-defined and are understood in the zeta-regularized sense. It follows that the partition function is formally given by

$$Z = \int D\alpha D\beta DbDc \, (\det \Delta_0 \det \Delta_2)^{1/2} \, e^{iS_{gf}(B,\alpha,\beta,\lambda,b,c)}$$

[4]For a general n-dimensional Riemannian manifold, the action of d^* on k-form is defined by $d_k^* = (-1)^{n(k-1)+1} * d *$.

Integrating over the the ghost fields b and c gives $\det \Delta_0$, while the integration over B, α, β and λ is a bit more subtle: First, note that

$$\Big(\lambda + *B, (\Delta_0 + \Delta_2)(\alpha + \beta)\Big) = (\lambda + *B, T^*T(\alpha+\beta)) \equiv (\rho, T^*T\sigma),$$

where we have defined the inhomogeneous forms $\rho = \lambda + *B$, $\sigma = \alpha+\beta$. Since the Hodge star gives a duality between $\Omega^0(\Sigma)$ and $\Omega^2(\Sigma)$, integrating over all B is the same as integrating over all two-forms $*B$. This in turn implies that the integration over $B, \lambda, \alpha, \beta$ is formally the same as the integration over the superfields ρ and σ, which gives

$$(\det T^*T)^{-1} = (\det \Delta_0 \det \Delta_2)^{-1}.$$

The partition function is therefore given by

$$Z = \underbrace{(\det \Delta_0 \det \Delta_2)^{1/2}}_{\text{Jacobian}} \underbrace{\det \Delta_0}_{\text{ghosts}} \underbrace{(\det \Delta_0 \det \Delta_2)^{-1}}_{\text{bosonic fields}} = \left(\frac{\det \Delta_0}{\det \Delta_2}\right)^{1/2}. \tag{2.13}$$

The Ray-Singer torsion of a flat vector bundle $E \to M$ over an n-dimensional Riemannian manifold (M, g) is defined as

$$T_M(E) = \prod_{k=0}^{n} (\det \Delta_k)^{(-1)^k \frac{k}{2}}$$

which in the case of a two-dimensional manifold and a trivial vector bundle gives

$$T_\Sigma = (\det \Delta_1)^{-1/2} \det \Delta_2.$$

As before, by Hodge theory, the space of one-forms decomposes into exact, co-exact and harmonic forms

$$\Omega^1(\Sigma) = d\Omega^0(\Sigma) + d^*\Omega^2(\Sigma) + \text{harmonic}.$$

We point out that

$$\Delta_1(d\alpha + d^*\beta + \varepsilon) = d\Delta_0\alpha + d^*\Delta_2\beta.$$

Thus, if we neglect the harmonic forms, then Δ_1 is in one-to-one correspondence with $\Delta_0 + \Delta_2$. Therefore, the Ray-Singer torsion is

$$T_\Sigma = (\det \Delta_0 \det \Delta_2)^{-1/2} \det \Delta_2 = \left(\frac{\det \Delta_2}{\det \Delta_0}\right)^{1/2}.$$

It follows that the right hand side of (2.13) is the inverse of the Ray-Singer torsion of the trivial vector bundle $\Sigma \times \mathbb{R} \to \Sigma$ [29, 39]:

$$Z = T_\Sigma^{-1}.$$

It is known that the Ray-Singer torsion is an analytic analog of the Reidemeister Torsion. Their equivalence is known as the Cheeger-Müller theorem. In particular, the Ray-Singer torsion, and hence the partition function, is independent of the chosen metric. The BF theory defines therefore indeed a TQFT.

Remark 2.2.7. In fact, it is known that the Ray-Singer torsion is trivial for an even dimensional compact manifold. In the case at hand, this can be seen by the duality $*\colon \Omega^0(\Sigma) \xrightarrow{\cong} \Omega^2(\Sigma)$ which implies $\det \Delta_0 = \det \Delta_2$.

Remark 2.2.8 (The zero-mode problem). In the above discussion we have always neglected the zero-modes by discarding the harmonic part of the gauge field A. Indeed, by Hodge theory, a differential form η is harmonic, $\Delta\eta = 0$ if and only if it is closed and co-closed: $d\eta = d^*\eta = 0$. This is a consequence of the non-degeneracy of the Hodge inner product (2.10):

$$0 = (\eta, \Delta\eta) = (\eta, (d^*d + dd^*)\eta) = \|d\eta\|^2 + |d^*\eta|.$$

A harmonic form is therefore a zero-mode and does not appear in the action: Indeed, if $A = d\alpha + d^*\beta + \eta$, with η harmonic, then the bosonic part of the action reads

$$S = \int_\Sigma Bdd^*\beta + \lambda d * d\alpha,$$

and is therefore independent of η. Therefore, the gauged-fixed action (2.11) admits a residual gauge symmetry:

$$\delta A = \eta, \qquad \eta \text{ harmonic}.$$

Moreover, Hodge theory tells us that harmonic forms are in one-to-one correspondence with de Rham cohomology: $H^k_{\mathrm{dR}}(\Sigma) = \ker\Delta_k$. The residual gauge symmetry is therefore given by the first cohomology of Σ. To fix the gauge symmetry, one wants to project out the cohomology [5]: let e_i be an orthonormal basis of $H^1(\Sigma)$, i.e. $(e_i, e_j) = vol(\Sigma)\delta_{ij}$, where $vol(\Sigma) = \int_\Sigma *1$ is the Riemannian volume of Σ. Let η be the harmonic part of A. To project out $\eta = \eta^i e_i$, we add the condition $\eta^i = 0$ via Lagrange multiplier to the action:

$$\int_\Sigma \sigma * A = (\sigma, \eta),$$

where $\sigma = \sigma^i e_i \in H^1_{\mathrm{dR}}(\Sigma)$. As usual, the gauge condition is accompanied by a ghost term, which takes the form

$$\int \bar{\varsigma} * \varsigma = (\bar{\varsigma}, \varsigma)$$

where ς is the ghost field and $\bar{\varsigma}$ the anti-ghost. The fully gauged fixed action of abelian BF theory is therefore

$$\int_\Sigma BdA + \lambda d * A + bd * dc + \sigma * A + \bar{\varsigma} * \varsigma.$$

The integration over σ projects out the harmonic part of A: $\eta = 0$. On the other hand, integrating over the ghosts ς and $\bar{\varsigma}$, gives an overall factor of $vol(\Sigma)$. The partition function Z therefore depends on the Riemannian volume of Σ and hence on the chosen metric. However, the combination $Z_{\mathrm{reg}} = vol(\Sigma)^{-1}Z$ is independent and is taken to be the definition of the regularized partition function of the theory.

2.3. Conformal field theory in two dimensions

In this section we will give a very brief and by no means complete introduction to conformal field theories (CFTs). Since this thesis focuses on path integral techniques, we will mostly consider CFTs who admit a Lagrangian formulation. In particular, we will describe how operator product expansions (OPEs) can be seen as "partial" Wick contractions where one allows open half edges. On the other hand, we will neither discuss critical phenomena in two-dimensional statistical models, from where the theory originated, nor the operator formalism approach to CFT.

Given the importance of the field, the existing literature on CFT is enormous and the following results and ideas are by now considered to be standard. This section is based on [4, 6, 10, 27] and follows closely [14].

2.3.1. The (local) conformal algebra in two-dimensions. Consider the two-dimensional Euclidean plane with coordinates (x^1, x^2) and a general metric of the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

By definition, conformal transformations are coordinate transformations which preserve angles, i.e. which leave the metric $g_{\mu\nu}$ invariant up to a positive scale:

$$g_{\mu\nu}(x) \mapsto g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{2.14}$$

The function $\Omega(x)$ is known as the *conformal factor* of ds^2.

Besides conformal transformation, we will need the notion of *Weyl transformations*. A Weyl transformation is a point-wise scaling of the metric:

$$g_{\mu\nu}(x) \mapsto e^{2\omega(x)} g_{\mu\nu}(x). \tag{2.15}$$

Notice that even if at first Equations (2.14) and (2.15) seem to be the same, they are fundamentally different. While Equation (2.14) is a coordinate transformation, i.e. a diffeomorphism, Equation (2.15) is not. It honestly changes the metric

$$ds^2 \to e^{2\omega(x)} ds^2$$

while ds^2 is certainly invariant under any diffeomorphism.

Under an infinitesimal coordinate transformation, $x^\mu \to x^\mu + \varepsilon^\mu(x)$, the metric changes by

$$\delta ds^2 = (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) dx^\mu dx^\nu. \tag{2.16}$$

To satisfy (2.14), the right hand side of (2.16) must be proportional to $g_{\mu\nu}$:

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \alpha \cdot g_{\mu\nu}.$$

Taking the trace of both sides, one finds

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \partial_\lambda \varepsilon^\lambda g_{\mu\nu}.$$

Considering the standard Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$, one is therefore lead to the condition

$$\partial_1 \varepsilon_1 = \partial_2 \varepsilon_2, \qquad \partial_2 \varepsilon_1 = -\partial_1 \varepsilon_2,$$

which are nothing but the Cauchy-Riemann equations. It follows, that conformal transformations in two dimensions are generated by holomorphic functions. If we choose a complex coordinate system $z = x + iy$, conformal transformations are therefore given by

$$z \to f(z), \qquad \bar{z} \to \bar{f}(\bar{z}),$$

where f is any holomorphic function. We will see later, that the associated (local) conformal algebra factorizes into two independent algebras, c.f. [14]. It is convenient to regard the above transformations as independent from each other, i.e. z and $\bar{z}$ are considered to be two independent variables and f and $\bar{f}$ as two independent holomorphic and anti-holomorphic functions.

This means that one works in $\mathbb{C}^2$ coordinatized by $(z, \bar{z})$ rather than in $\mathbb{C}$. The physical model is then recovered by projecting to the real surface $\{z^* = \bar{z}\} \subset \mathbb{C}^2$, where z^* denotes the complex conjugate of z [14]. Therefore, at places where the discussion between holomorphic and anti-holomorphic coordinate transformations and their implications is parallel, we will, for the sake of brevity, restrict the discussion to the holomorphic case.

In complex coordinates, the standard (real) metric can be written as $ds^2 = \frac{i}{2}dzd\bar{z}$. For a holomorphic coordinate change $z \to f(z)$, the metric transforms as

$$ds^2 \to |\partial f|^2 ds^2,$$

which identifies the conformal factor $\Omega = |\partial f|^2$. Here and henceforth, if not stated otherwise, we will abbreviate $\partial_z = \partial$ and $\partial_{\bar{z}} = \bar{\partial}$.

Since locally, any holomorphic function admits a Laurent expansion, we may write an infinitesimal conformal transformation as

$$\delta z = \varepsilon(z) = \sum_{n \in \mathbb{Z}} \varepsilon_n z^{n+1}.$$

The unusual shift in the power is chosen for later convenience. Notably, an infinitesimal conformal transformation is generated by the vector field

$$\varepsilon(z) = -\sum_{n \in \mathbb{Z}} \varepsilon_n \ell_n$$

where

$$\ell_n = -z^{n+1}\partial. \tag{2.17}$$

These vector fields satisfy the Witt algebra

$$[\ell_m, \ell_n] = (m - n)\ell_{m+n}. \tag{2.18}$$

So far, we have only considered local conformal transformations. Ultimately, however, one is interested in models defined over arbitrary Riemann surfaces. However, not all local conformal transformations can be integrated to give a globally well-defined transformation. For example, for the Riemann sphere $S^2 = \mathbb{C}P^1$, only the generators $\{l_{-1}, l_0, l_1\}$ can be integrated to global transformations, which follows from demanding that the vector field $\varepsilon(z) = -\sum \varepsilon_n \ell_n$ is non-singular on $\mathbb{C}P^1$. From (2.17), and its complex conjugated version, one can conclude that $\ell_{-1}, \bar{\ell}_{-1}$ are the generators of translations, $\ell_0 + \bar{\ell}_0$ and $i(\ell_0 - \bar{\ell}_0)$ are the generators of dilatation (scaling) and rotations and finally one can check that ℓ_1 and $\bar{\ell}_1$ generate special conformal transformations. In total, the action of $\{\ell_{-1}, \ell_0, \ell_1\}$ integrates to

$$z \to \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{C}, ad - bc = 1.$$

This is, of course, nothing but the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{C}P^1$, acting by fractional linear transformations. The conformal group of $\mathbb{C}P^1$ is therefore $\mathrm{PSL}_2(\mathbb{C})$.

In the following, we will define a (two-dimensional) *conformal field theory* as a (two-dimensional) QFT, whose space of fields is a module of the (local) conformal algebra generated by the vector fields (2.17) and whose correlation functions are conformally invariant. It is standard to denote the eigenvalues of ℓ_0 and $\bar{\ell}_0$ by h and $\bar{h}$ respectively. We stress, that $\bar{h}$ is not necessarily the complex conjugate of h. The eigenvalues of the dilatation and rotation operators, $H = \ell_0 + \bar{\ell}_0$ and $P = i(\ell_0 - \bar{\ell}_0)$ are commonly denoted by $\Delta = h + \bar{h}$ and $s = h - \bar{h}$. In physical terms, Δ is known as the *conformal dimension* or *scaling dimension* of the field and s as its spin.

2.3.2. Two-dimensional conformal field theories. A *conformal field theory* (CFT) can be defined as a QFT whose partition function and correlators are invariant under conformal and Weyl transformations. It is therefore in particular scale independent. Theories which look the same at each scale are also called *self-similar*. Their origin of study was in particular in statistical models and critical phenomena (see [10] for a review).

As it turns out, the requirement of conformal invariance has far reaching implications. In particular, one finds that the spectrum of the theory is generated from a special class of fields called primary fields. In the following, we will spell out the main consequences of conformal invariance in two dimensions.

2.3.2.1. *Implications of conformal invariance.* The main object of study, in a CFT are, next to the partition function, correlation functions of a special class of fields which transform tensorially under *local* conformal transformations of the worldsheet:

$$\phi(z, \bar{z}) \to (\partial f)^h (\bar{\partial} \bar{f})^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})). \tag{2.19}$$

The field ϕ is known as a *primary field* and $(h, \bar{h})$ as its *conformal weight*.

In the case that $(h, \bar{h})$ are integers, there exists a nice geometrical interpretation: A field ϕ transforming according to (2.19), is a complex tensor of the form

$$\phi(z, \bar{z}) dz^h d\bar{z}^{\bar{h}},$$

where by convention $dz^{-1} \equiv \partial_z$.

For example, a (holomorphic) primary field ϕ of weight $(1, 0)$ corresponds to a holomorphic (1,0)-form. Indeed, since a one-form $\omega = \phi(z)dz$ is defined independently of any chart, it satisfies $\omega = f^*\omega$. It follows that

$$f^*\omega = \phi(f(z))df(z) = \phi(f(z))\partial f dz = \phi(z)dz,$$

from which we read off the transformation behavior of ϕ: $\phi(z) \to \partial f\ \phi(f(z))$. By (2.19), ϕ is therefore indeed a primary field of weight $(1, 0)$.

Remark 2.3.1. In fact, the geometric interpretation holds for rational weights $(h, \bar{h}) \in \mathbb{Q}^2$. For example, a (holomorphic) primary field ψ of weight $1/2$ can be interpreted as a spinor, i.e. as a section of the square root $K^{1/2}$ of the canonical bundle K over the worldsheet. For an intuition, one may think about the example $\Sigma = \mathbb{C}$ where sections of K are just $(1, 0)$-forms $\alpha(z)dz$. Hence a primary field of weight $(1/2, 0)$ is of the form $\psi(z)\sqrt{dz}$.

On the other hand, in some CFTs there exist primary fields of arbitrary real conformal weights $(h, \bar{h}) \in \mathbb{R}^2$. Since the geometrical interpretation of these fields is less obvious, they are also known as *fields of anomalous dimensions* or *vertex operators*. In general, a primary field of weight $(h, \bar{h})$ is mathematically speaking a $(h, \bar{h})$-density.

Next to primary fields, there exists another important class of fields, so-called *quasi primary fields*. These fields differ from primary fields in the sense that they only behave tensorially under a *global* conformal transformation.

Conformal invariance of the theory is quite restrictive, in particular for correlation functions. In what follows, we will consider mostly correlation functions of primary fields. The idea is that in a CFT, these correlation functions must be conformally invariant. By that we mean that if we consider primary fields $\phi_i(z,\bar{z})$ of weight $(h_i,\bar{h}_i)$, then the object

$$\big\langle \prod_i dz_1^h d\bar{z}_1^{\bar{h}} \phi_i(z_i,\bar{z}_i)\big\rangle$$

is invariant under conformal transformations. This on the other hand implies that under a local conformal transformation $z_i \to f(z_i)$, the correlation function $\big\langle \prod_i \phi_i(z_i,\bar{z}_i)\big\rangle$ must transform as

$$\big\langle \prod_i \phi_i(z_i,\bar{z}_i)\big\rangle = \prod_j (\partial f)^{h_j} (\bar{\partial}\bar{f})^{\bar{h}_j} \,\big\langle \prod_i \phi_i(f(z)_i,\bar{f}(z)_i)\big\rangle. \tag{2.20}$$

These constraints are so strong that they fix both the two- and three-point function up to a constant. Let us spell out the argument for the two-point function: Since the correlation functions must be translational invariant, they must be functions of the differences $z_{ij} = z_i - z_j$ only. Moreover, the transformation behavior (2.20) for a scaling and rotation $z \to \lambda z$, for some $\lambda \in \mathbb{C}$, implies that the two point function must be of the form

$$\langle \phi_1(z_1,\bar{z}_2)\phi_2(z_2,\bar{z}_2)\rangle = \frac{C_{12}}{z_{12}^{h_1+h_2}\bar{z}_{12}^{\bar{h}_1+\bar{h}_2}}.$$

Finally, the two-point function has to be invariant also under special conformal transformations which are combinations of inversions and translations. Since it is already invariant under translations, it suffices to check invariance under inversions. It is easy to check that the transformation behavior (2.20) is satisfied if and only if $(h_1,\bar{h}_1) = (h_2,\bar{h}_2)$. This forces the two-point function to be of the form

$$\langle \phi_1(z_1,\bar{z}_2)\phi_2(z_2,\bar{z}_2)\rangle = C_{12}\,\frac{\delta_{h_1,h_2}\delta_{\bar{h}_1,\bar{h}_2}}{z_{12}^{2h}\bar{z}_{12}^{2\bar{h}}}. \tag{2.21}$$

Likewise, one can show that the three-point function is also fixed up to a constant: (for brevity we write $\phi_i(z_i)$ for $\phi_i(z_i,\bar{z}_i)$)

$$\begin{aligned}\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = C_{123}\, &\frac{1}{z_{12}^{h_1+h_2-h_3} z_{13}^{h_1+h_3-h_2} z_{23}^{h_2+h_3-h_1}} \\ &\cdot \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{13}^{\bar{h}_1+\bar{h}_3-\bar{h}_2} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1}}.\end{aligned} \tag{2.22}$$

The first non-fixed correlator is the four-point function. Its most general form is [14]

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3,)\phi_4(z_4)\rangle = f(x,\bar{x}) \prod_{i<j} z_{ij}^{-(h_i+h_j)+h/3} \bar{z}_{ij}^{-(\bar{h}_i+\bar{h}_j)+\bar{h}/3},$$

where $h = \sum_i h_i$ and $\bar{h} = \sum_i \bar{h}_i$. The variable x denotes the cross ratio $x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$, while the function f remains undetermined.

2.3.2.2. *Radial quantization.* In many applications, it is convenient to define the theory on a cylinder. Let us therefore consider a CFT defined on a cylinder with longitudinal coordinate t and angular coordinate x, c.f. Figure 4.

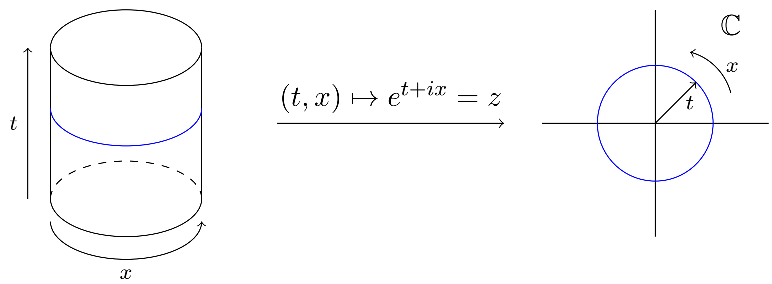


FIGURE 4. Conformal mapping between the cylinder and the complex plane.

Since the cylinder can be mapped conformally onto the complex plane and the theory is assumed to be invariant under conformal mappings, one can pass in its description back and forth between the cylinder and the complex plane. Of course, the idea is to exploit the power of complex analysis on $\mathbb{C}$ to describe various phenomena of the theory on the cylinder.

First note that the origin of the complex plane corresponds to $t = -\infty$ and that ∞ in $\mathbb{C}$ would correspond to $t = +\infty$. Moreover, circles in the complex plane of constant radius r correspond to constant time slices $t = r$ on the cylinder. Furthermore, time translations on the cylinder now correspond to radial translations, i.e. scaling on the complex plane. The Hamiltonian H on $\mathbb{C}$ is therefore given by $H = L_0 + \bar{L}_0$ which is indeed the generator of scaling. Likewise, a translation in space on the cylinder (that is a shift of the x-coordinate) corresponds to a rotation in $\mathbb{C}$. Therefore, the momentum operator P is given by $P = i(L_0 - \bar{L}_0)$, the generator of rotations. This picture of quantization is known in the literature as *radial quantization*.

Now, by Noether's theorem to every symmetry there exists an associated conserved current J, which in two dimensions is a one-form. To every conserved current, there exists furthermore a conserved charge q constructed by integrating the time-component over an equal-time slice

$$q_{\text{cyl}} = \int_{\text{equal-time slice}} J^0 dx. \tag{2.23}$$

Passing from the cylinder to the complex plane, an equal-time slice at time $t = r$ becomes a circle of radius r centered at the origin and the time-component of the current becomes its radial component. We are therefore ought to replace Equation (2.23) by

$$\oint_{S^1_r} J_r d\theta = \oint_{S^1} J_r r d\theta = \frac{1}{i} \oint_{S^1} J_z dz + \bar{J}_{\bar{z}} d\bar{z}.$$

Here S^1_r denotes the circle of fixed radius r, while S^1 denotes the unit circle. All circles are oriented counterclockwise. After a normalization, one therefore arrives at

$$q_{\text{plane}} = \frac{1}{2\pi i}\oint_{S^1} J_z dz + \bar{J}_{\bar{z}} d\bar{z} = \frac{1}{2\pi i}\oint J, \tag{2.24}$$

with $J = J_z dz + \bar{J}_{\bar{z}} d\bar{z}$. Since J is conserved, i.e. $dJ = 0$, the circular contour can be deformed to any other contour. Of course, on the quantum level, Equation (2.24) is only a formal expression, since the contour of integration might include another field. Therefore, by the conservation of J, we can shrink the contour to an infinitesimal circle surrounding all fields inside the contour, see Figure 5. In particular, this implies that a conserved Noether

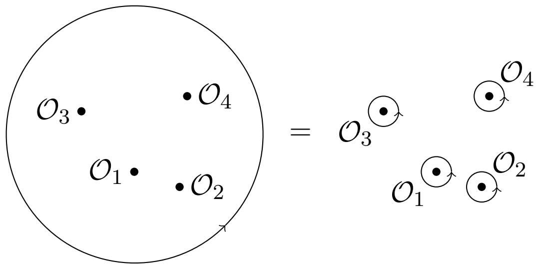


FIGURE 5. Action of the conserved Noether charge Q associated to a conserved Noether current J.

charge Q, seen as an operator on the space of fields, is a derivation, i.e. it satisfies the Leibniz identity

$$Q(\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)) = Q(\mathcal{O}_1(z_1))\mathcal{O}_2(z_2) \pm \mathcal{O}_1(z_1)Q(\mathcal{O}_2(z_2)).$$

Here the $\pm$ remembers the fact that the space of fields could be graded and one therefore has to introduce appropriate signs when passing fields through each other.

2.3.2.3. *The stress-energy tensor, OPEs and conformal transformations.* In any CFT, the most important symmetry is of course the conformal symmetry. General coordinate transformations are generated by the stress-energy tensor $T_{\mu\nu}$, which is a divergence free symmetric quadratic tensor. The conserved current associated to an infinitesimal coordinate transformation $\delta_\varepsilon = \varepsilon^\mu$ is given by

$$J_\mu = T_{\mu\nu}\varepsilon^\nu.$$

In particular, considering an infinitesimal scaling transformation $\varepsilon^\mu = \lambda x^\mu$, the conservation law of J_μ implies

$$0 = \partial^\mu J_\mu = \lambda \partial^\mu (T_{\mu\nu} x^\nu) = \lambda (x^\nu \partial^\mu T_{\mu\nu} + T^\mu_\mu) = \lambda T^\mu_\mu.$$

Here, $\partial^\mu T_{\mu\nu} = 0$ since $T_{\mu\nu}$ is divergence free. It follows that in a CFT, classically, the stress-energy tensor is traceless: $T^\mu_\mu = 0$. We will see later, c.f. Remark 2.3.5, that on the quantum level the stress-energy tensor is traceless only for a flat background.

In complex coordinates, this implies that $T_{z\bar{z}} = 0$. It is customary to denote the other components by

$$T_{zz} = T, \qquad T_{\bar{z}\bar{z}} = \bar{T}.$$

The vanishing of the divergence of $T_{\mu\nu}$ can then be interpreted as a holomorphicity and anti-holomorphicity condition for T and $\bar{T}$:

$$\bar{\partial} T = 0, \qquad \partial \bar{T} = 0.$$

One can then easily verify that a symmetry generated by a holomorphic vector field $v(z)$ induces the Noether current

$$T_v = T(z)v(z)dz$$

which is indeed conserved since T and v are both holomorphic. An analogous expression holds for the Noether current associated to a symmetry generated by an anti-holomorphic vector field $\bar{v}(\bar{z})$.

Let us take a step back and consider again a general (global) symmetry of the theory. Let J be the associated Noether current and Q the Noether charge. Classically, a field ϕ varies under the (infinitesimal) symmetry by its equal-time commutator with the charge [14], i.e.

$$\delta_\varepsilon \phi = \varepsilon\{Q, \phi\},$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket. After quantization, the Poisson bracket is replaced by the commutator bracket of operators, and passing from the cylinder to the complex plane, the action of the charge is replaced by integrating the current around a chosen contour.

According to the above discussion, a field ϕ varies under an infinitesimal conformal transformation generated by the vector fields $\delta z = \varepsilon(z)$ according to

$$\delta_{\varepsilon,\bar{\varepsilon}} \phi(w, \bar{w}) = \oint \frac{dz}{2\pi i} \big[\varepsilon(z)T(z), \phi(w, \bar{w})\big]. \tag{2.25}$$

On the quantum level, an expression like the above is ought to be considered under the correlator, that is in the presence of test fields φ_j:

$$\Big\langle \delta_{\varepsilon,\bar{\varepsilon}} \phi(w, \bar{w}) \varphi_1(z_1) \dots \Big\rangle = \Big\langle \oint \frac{dz}{2\pi i} \big[\varepsilon(z)T(z), \phi(w, \bar{w})\big] \varphi_1(z_1) \dots \Big\rangle. \tag{2.26}$$

As is standard, expressions under the correlation functions are considered to be time-ordered. In the radial quantization picture, time-ordered products become radial-ordered

$$R(\phi_1(z)\phi_2(w)) = \begin{cases} \phi_1(z)\phi_2(w) & |z| > |w|, \\ \phi_2(w)\phi_1(z) & |z| < |w|. \end{cases}$$

The integration has therefore to be performed once under the assumption that $|w| < |z|$ and once assuming $|w| > |z|$, with a relative sign coming from the commutator. Therefore, we can write (2.26) as

$$\Big\langle \delta_{\varepsilon,\bar{\varepsilon}} \phi(w, \bar{w}) \dots \Big\rangle = \Big\langle \Big(\oint\limits_{|z|>|w|} - \oint\limits_{|z|<|w|} \Big) \frac{dz}{2\pi i} \varepsilon(z)T(z), \phi(w, \bar{w}) \dots \Big\rangle.$$

Since T is holomorphic we can now deform the contours, c.f. Figure 6, such that

$$\Big\langle \delta_{\varepsilon,\bar{\varepsilon}} \phi(w, \bar{w}) \dots \Big\rangle = \Big\langle \oint_{C_w} \frac{dz}{2\pi i} \varepsilon(z)T(z)\phi(w, \bar{w}) \dots \Big\rangle, \tag{2.27}$$

where C_w is an arbitrary small contour surrounding ϕ, but none of the test fields. Since the integrand of (2.27) is holomorphic, the only contribution to

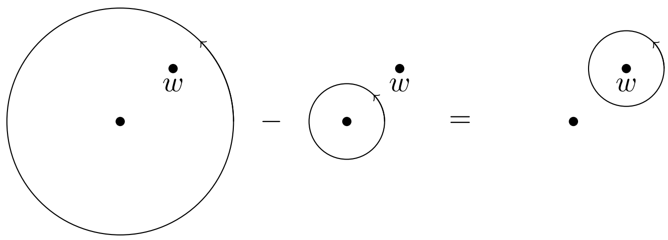


FIGURE 6. Contour integration in the equal-time commutator (2.25).

the integral comes from the singularities developed as T approaches ϕ. The singularities which occur this way are summarized in what is known as an *operator product expansion* (OPE).

In full generality, an OPE is a formal expression

$$\mathcal{O}_i(z)\mathcal{O}_j(w) \sim \sum_k \sigma_k(z-w)[\mathcal{O}_i\mathcal{O}_j]_k(w).$$

Here $\sim$ means "in the limit $z \to w$ and up to regular terms". The $\sigma_n(z-w)$ are single-valued functions which are possibly singular at $z = w$. The terms $[\mathcal{O}_i\mathcal{O}_j]_k$ are known as the coefficients of the OPE and can be any composite field, i.e. any product of fields (and their derivatives) which are allowed in the theory. Often one requires that the space of composite fields forms an algebra under OPEs, i.e.

$$\mathcal{O}_i(z)\mathcal{O}_j(w) \sim \sum_k c_{ij}^k(z-w)\mathcal{O}_k,$$

where the structure constants $c_{ij}^k(z-w)$, akin to the $\sigma_n(z-w)$ before, are single valued functions, with possible singularities as $z \to w$.

If $\mathcal{O}_i$ and $\mathcal{O}_j$ are holomorphic, then their OPE takes the form

$$\mathcal{O}_i(z)\mathcal{O}_j(w) \sim \sum_k (z-w)^{-k}[\mathcal{O}_i\mathcal{O}_j]_k(w). \tag{2.28}$$

The above expressions are formal in the sense that they should be considered under the correlator with test fields inserted far away

$$\langle \mathcal{O}_i(z)\mathcal{O}_j(w)\varphi_1(z_1)\dots\rangle = \sum_k \langle \sigma_k(z-w)[\mathcal{O}_i\mathcal{O}_j]_k(w)\varphi_1(z_1)\dots\rangle\,. \tag{2.29}$$

Of course, "far away" is meaningless in a scale invariant theory. What we mean is that the distances of $\mathcal{O}_i$ and $\mathcal{O}_j$ to the insertions of the test fields are large compared to the relative distance between $\mathcal{O}_i$ and $\mathcal{O}_j$. The OPE therefore encodes the singular behaviors of n-point functions as two of the fields collide.

REMARK 2.3.2 (OPEs and action functionals). We want to pause here to digress a bit further on OPEs. As it turns out, a CFT can be characterized completely by its primary fields, their OPEs and its central charge (to be defined later) [10]. Therefore, it is a priori not necessary to have a defining action functional. Indeed, many CFTs, which appear foremost as statistical models, admit no Lagrangian formulation. However, for this thesis, we are mainly interested in those CFTs who admit an action functional and who therefore can be quantized via the path integral formalism. In the path integral formalism, correlation functions are given by sums of Feynman diagrams: Let us consider a free theory, i.e. a theory whose action S_{free} is at most quadratic in the fields. Correlations functions are then defined as

$$\langle \mathcal{O}_1(z_1)\dots\rangle := \frac{1}{Z}\int \mathcal{D}\phi\, e^{S_{\text{free}}}\mathcal{O}_1(z_1)\dots \tag{2.30}$$

As it is written, (2.30) is not well-defined due to the integration over an infinite-dimensional space. However, it is defined combinatorially: For each composite field[5] $\mathcal{O}_i$ one lays down a vertex, with as many half-edges attached to it as there are constituents of the composite field. Each half-edge thus corresponds to a fundamental field (possibly part of a composite field). Two half-edges are combined to an edge by the propagator (the inverse of the operator appearing in the quadratic part of the action). We shall call this procedure a contraction. The (free) n-point function (2.30) is then given by a sum over all possible contractions, such that there is no half-edge left unpaired. This definition is known as Wick's lemma.

For an interacting theory, with coupling constant g,

$$S = S_{\text{free}} + gS_{int}$$

the picture is more complicated. One approach to a rigorous treatment of interacting QFTs is via perturbation theory, where one expands the exponential of the action in a formal power series in the coupling constant

$$\begin{aligned}\langle \mathcal{O}_1(z_1)\dots\rangle &= \frac{1}{Z}\int \mathcal{D}\phi\, e^{S_{\text{free}}+gS_{\text{int}}}\mathcal{O}_1(z_1)\dots \\ &\sim \sum_n \frac{g^n}{n!}\frac{1}{Z}\int \mathcal{D}\phi\, e^{S_{\text{free}}}(S_{\text{int}})^n\mathcal{O}_1(z_1)\dots\end{aligned}$$

In addition to the vertices corresponding to the field insertions $\mathcal{O}_i$, one now has to consider also internal vertices coming from $(S_{\text{int}})^n$. In general, one has to integrate over the position of those internal vertices. The resulting diagrams are known as *Feynman diagrams*.

Now, recall that the OPE (2.28) encodes the singular behavior of a correlator as two fields collide, c.f. (2.29). On the other hand, the correlator

$$\langle \mathcal{O}_i(z)\mathcal{O}_j(w)\dots\rangle$$

shows a singular behavior when $z \to w$ if and only if $\mathcal{O}_i$ and $\mathcal{O}_j$ are contracted at least once, while all the other possible contractions occur either between $\mathcal{O}_i$ or $\mathcal{O}_j$ with the test fields or solely among the test fields. In each such diagram, one has effectively replaced $\mathcal{O}_i(z)\mathcal{O}_j(w)$ with a new field $[\mathcal{O}_i\mathcal{O}_j]_k$

[5] Here and henceforth we shall mean by a "composite field" we shall mean a possibly renormalized product of fundamental fields and their derivatives, where by a "fundamental" field we mean any field appearing explicitly in the action.

which one can now safely Taylor expand around w. This situation is sketched in Figure 7. It follows that

$$\lim_{z\to w}\langle \mathcal{O}_i(z)\mathcal{O}_j(w)\dots\rangle = \lim_{z\to w}\langle \sum_k \sigma_k(z-w)[\mathcal{O}_i\mathcal{O}_j]_k(w)\dots\rangle,$$

This allows the interpretation of an OPE as a "partial" Wick contraction,

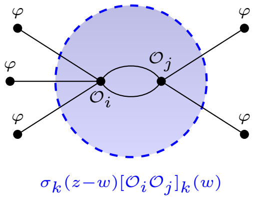
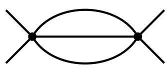
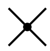


FIGURE 7. Graphical presentation of an OPE between $\mathcal{O}_i$ and $\mathcal{O}_j$ as a partial Wick contraction

namely, under the correlator one replaces $\mathcal{O}_i(z)\mathcal{O}_j(w)$ with all possible contractions between $\mathcal{O}_i$ and $\mathcal{O}_j$ allowing open (uncontracted) half-edges making up the field $[\mathcal{O}_i\mathcal{O}_j]_k$, and Taylor expands the resulting field around w. Figuratively,

$\mathcal{O}_i(z)\mathcal{O}_j(w) \sim \sum\ \mathcal{O}_i(z)$  $\mathcal{O}_j(w)$

$\sim \sum_k \sigma(z-w)_k$  $[\mathcal{O}_i\mathcal{O}_j]_k(w)$.

Now, let us return to the transformation behavior of a field ϕ under a local conformal transformation (2.27). Let us furthermore suppose that ϕ is primary field of weight $(h,0)$, that is under a local coordinate transformation ϕ transforms according to (2.19). Under an infinitesimal conformal transformation $\delta z = \varepsilon(z)$, the field thus transforms according to

$$\delta_\varepsilon\phi = h\partial\varepsilon\phi + \varepsilon\partial\phi.$$

By comparison with (2.27), we see that the OPE between T and a primary field ϕ must be of the form

$$T(z)\phi(w) \sim \frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w} + \text{reg}\,. \tag{2.31}$$

2.3.2.4. *Mode operators and Virasoro algebra.* In the previous section we have seen that the transformation behavior of a field under (local) conformal transformation is encoded in the OPE of the field with the stress-energy tensor:

$$\delta_\varepsilon\phi(w) = \oint \frac{dz}{2\pi i}\varepsilon(z)T(z)\phi(w).$$

In particular, under a translation $\delta z = \varepsilon$ with ε constant, any field transforms as

$$\delta_\varepsilon\phi(w) = \varepsilon\partial\phi(w),$$

the coefficient of the first order pole of the OPE between T and ϕ must therefore be the derivative of the field

$$T(z)\phi(w) \sim \cdots + \frac{\partial\phi(w)}{z-w} + \text{reg}\,. \tag{2.32}$$

Put differently,

$$\operatorname*{Res}_{z\to w}(T(z)\phi(w)) = \partial\phi(w).$$

Likewise, since any quasi-primary field ϕ_{qp} is invariant under global conformal transformations, its OPE with T must be of the form

$$T(z)\phi_{\text{qp}}(w) \sim \cdots + \frac{h\phi_{\text{qp}}(w)}{(z-w)^2} + \frac{\partial\phi_{\text{qp}}(w)}{z-w} + \text{reg},$$

where h is the conformal weight of ϕ_{qp}. Notably, unlike a primary field, a quasi-primary field can have finitely many poles of order greater than two.

For example, if ϕ is a primary field of weight $(h, \bar{h})$, then its derivative $\partial\phi$ is usually only quasi-primary:

$$\begin{aligned} T(z)\partial\phi(w) &\sim \partial_w\left(\frac{h\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}\right) + \text{reg} \\ &\sim \frac{2h\phi(w)}{(z-w)^3} + \frac{(h+1)\partial\phi}{(z-w)^2} + \frac{\partial^2\phi(w)}{z-w} + \text{reg}\,. \end{aligned}$$

Likewise, all higher derivatives of a primary field is a quasi-primary field.

Remark 2.3.3. There exists an exception: in the case that ϕ is a primary field of conformal weight $(h, \bar{h}) = (0, 0)$, the first derivative $\partial\phi$ is an honest primary field of weight $(1, 0)$. Similarly, $\bar{\partial}\phi$ is an honest primary field of weight $(0, 1)$.

Another example of a quasi-primary field is the stress-energy tensor itself. Indeed, since it is a (holomorphic) symmetric quadratic differential, it is invariant under global conformal transformations and has naturally conformal weight 2. On the quantum level, however, there can be quantum corrections. The most general form of the TT OPE is

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{2.33}$$

Notably, there cannot exist a third order pole, since the left hand side is symmetric in exchanging z and w. Moreover, neither can there exist any higher order pole, because under scaling $z, w \to \lambda z, \lambda w$ the left hand side transforms with an overall factor λ^{-4}, while in the presence of a pole of order 5 and higher, the right hand side would show an inconsistent transformation behavior. Likewise, c must be a number. This is known as the *central charge* and characterizes the CFT.

Remark 2.3.4. If the model admits a Lagrangian formulation, than the central charge arises typically from loop diagrams of the OPE. Thus, the central charge characterizes quantum corrections to the TT OPE.

Remark 2.3.5 (The Weyl anomaly). A non-zero central charge also implies that in a curved background, the stress-energy tensor ceases to be traceless. We will follow closely [31].

Suppose that the CFT is defined by an action functional S. Classically, the stress-energy tensor measures the reaction of the theory to a change in the geometry of the worldsheet. The classical stress-energy tensor can thus be define by the equation

$$\delta_g S = -\int_\Sigma d^2x \sqrt{\det g}\, T_{\mu\nu} \delta g^{\mu\nu}. \tag{2.34}$$

Let us now consider how the expectation value of the trace T^μ_μ of the stress-energy tensor varies if we vary the metric infinitesimally:

$$\begin{aligned}\delta_g \left\langle T^\mu_\mu(y)\right\rangle &= \int \mathcal{D}\phi\, e^{-S} T^\mu_\mu(y)(-\delta_g S) \\ &= \int \mathcal{D}\phi\, e^{-S} T^\mu_\mu(y) \int_\Sigma d^2x \sqrt{\det g}\, T_{\alpha\beta}(x)\delta g^{\alpha\beta}.\end{aligned}$$

In two dimensions, any metric is conformally flat. It follows that for an infinitesimal Weyl transformation to the flat metric $\delta g^{\alpha\beta} = -2\omega(x)\delta^{\alpha\beta}$,

$$\delta_\omega \left\langle T^\mu_\mu(y)\right\rangle = \int_\Sigma d^2x\, (-2\omega(x)) \left\langle T^\mu_\mu(y) T^\alpha_\alpha(x)\right\rangle. \tag{2.35}$$

Since the stress-energy tensor is divergence free, in complex coordinates, one has

$$\partial T_{z\bar z} = -\bar\partial T_{zz},$$

and from the general form of the TT OPE (2.33) one finds

$$\partial_z T_{z\bar z} \partial_w T_{w\bar w} = \bar\partial_{\bar z}\bar\partial_{\bar w} T_{zz} T_{ww} \sim \bar\partial_{\bar z}\bar\partial_{\bar w}\left(\frac{c/2}{(z-w)^4} + \dots\right).$$

Now, a careful analysis shows that

$$\bar\partial_{\bar z}\bar\partial_{\bar w}\frac{c/2}{(z-w)^4} = \frac{c\pi}{6}\partial_z^2 \partial_w \bar\partial_{\bar w}\delta^{(2)}(z-w).$$

Comparison, and keeping track of factors of 2 when passing between real and complex coordinates, gives the $T_{z\bar z}T_{w\bar w}$ OPE

$$T^\mu_\mu(y) T^\alpha_\alpha(x) = 16 T_{z\bar z} T_{w\bar w} \sim 16 \cdot \frac{c\pi}{6}\partial_z \bar\partial_{\bar w}\delta^{(2)}(z-w) = \frac{c\pi}{3}\partial_x\partial_y \delta^{(2)}(x-y)$$

where we used $8\partial_z\bar\partial_{\bar w}\delta^{(2)}(z-w) = \partial_x\partial_y\delta^{(2)}(x-y)$. Therefore, the left hand side of (2.35) gives

$$\int_\Sigma d^2x\, (-2\omega(x)) \left\langle T^\mu_\mu(y) T^\alpha_\alpha(x)\right\rangle = \int_\Sigma d^2x\, (-2\omega(x))\partial_x\partial_y\delta^{(2)}(x-y) = \frac{2c\pi}{3}\partial^2\omega.$$

Finally, in two dimensions, the Ricci curvature for a metric $g_{\mu\nu} = e^{2\omega(x)}\delta_{\mu\nu}$ is given by

$$R = -2e^{-2\omega}\partial^2\omega$$

such that up to first order in ω

$$\frac{1}{4\pi}\delta_\omega \left\langle T^\mu_\mu\right\rangle = -\frac{c}{12} R + \mathcal{O}(\omega^2).$$

The prefactor $1/4\pi$ is a normalization convention. Importantly, the above shows that the stress-energy tensor ceases to be traceless on the quantum level if both the central charge and the curvature of the worldsheet are non-vanishing. This is known as the *Weyl anomaly*. Since it is desirable to formulate the CFT on a generic worldsheet Σ, the conformal invariance is broken on the quantum level unless $c \neq 0$. In this way, the central charge measures an anomaly of the theory, namely the failure to be conformally invariant on the quantum level.

Returning to the general story, we want to study the stress-energy tensor a little bit more. Since the OPE with the stress-energy tensor encodes the transformation behavior of the field, one can deduce the transformation behavior of the stress-energy tensor on the quantum level. From (2.33), one can deduce that infinitesimally T varies as

$$\delta_\varepsilon T(z) = \varepsilon(z)\partial T(z) + 2\partial\varepsilon(z)T(z) + \frac{c}{12}\partial^3\varepsilon. \tag{2.36}$$

It follows that under a finite conformal transformation $z \to f(z)$, T transforms as

$$T(z) \to (\partial f)^2 T(f(z)) + \frac{c}{12}\{f, z\} \tag{2.37}$$

where

$$\{f, z\} = \frac{\partial^3 f}{\partial f} - \frac{3}{2}\left(\frac{\partial^2 f}{\partial f}\right)^2$$

is the Schwarzian derivative of f.

Remark 2.3.6. Equation (2.37) is the transformation property of a projective connection. Indeed, in [13] it is shown that the stress-energy tensor can be interpreted as a connection on the moduli space $\mathcal{M}_{g,n}$ of conformal structures on a genus g surface with n punctures.

Now, it is convenient to expand the stress-energy tensor T in a Laurent series

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n, \qquad L_n = \oint \frac{dz}{2\pi i} z^{n+1}T(z). \tag{2.38}$$

As in the discussion of the Noether charge (2.24), on the quantum level the expression of the Laurent modes L_n are only formal and one must allow the possibility that the contour surrounds several field insertions.

More generally, suppose that ϕ is a holomorphic primary field of conformal weight $(h, 0)$. Then its Laurent series

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-h}$$

defines so-called *mode operators* defined by

$$(\phi_n\mathcal{O})(w) := \oint \frac{dz}{2\pi i} z^{n+h-1}\phi(z)\mathcal{O}(w), \tag{2.39}$$

where $\mathcal{O}(w)$ is some (renormalized product) of fields.

A direct calculation shows that due to the form of the OPE (2.33), the mode operators L_n associated to the stress-energy tensor satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{2.40}$$

Likewise, the operator $\bar{T}$ gives rise to an independent second copy of the Virasoro algebra with generators $\bar{L}_n$ (Laurent modes of $\bar{T}$). In particular, according to Equation (2.32), L_{-1} is the holomorphic derivative

$$(L_{-1}\phi)(w) = \partial\phi(w).$$

Moreover, L_0 is diagonal in the space of (quasi)-primary fields

$$(L_0\phi)(w) = h\phi(w).$$

REMARK 2.3.7. Similar calculations using the OPE (2.31) show that if $\phi(z)$ is a holomorphic primary field of weight h with Laurent expansion

$$\phi(z) = \sum_n \phi_n z^{-n-h},$$

then

$$[L_m, \phi_n] = ((h-1)m - n)\,\phi_{n+m}.$$

REMARK 2.3.8 (Sugawara construction). Suppose that the CFT admits holomorphic primary fields j of weight $(h, \bar{h}) = (1, 0)$. Such fields are called *currents*. Given a holomorphic current j, suppose furthermore that it satisfies the OPE

$$j(z)j(w) \sim \frac{k}{(z-w)^2}.$$

In this case, one can define a stress-energy tensor (i.e. a holomorphic quasi primary field of weight $h = 2$ satisfying the TT OPE (2.33)) by

$$T(z) := \frac{1}{2}j^2(z),$$

where j^2 is defined by the normal ordering procedure

$$j^2(z) := \lim_{z\to w}\left(j(w)j(z) - \tfrac{k}{(z-w)^2}\right).$$

The stress energy tensor $T = \frac{1}{2}j^2$ defines then a CFT with central charge $c = k^2$.

More generally, suppose that the currents form a Kac-Moody algebra at level k

$$j_a(z)j_b(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + \sum_c \frac{f_{ab}^c j_c}{z-w},$$

where f_{ab}^c are structure constants of some Lie algebra $\mathfrak{g}$. Then one can define a stress-energy tensor by

$$T(z) := \frac{1}{2(k + C_{\mathfrak{g}})}\sum_a j^a j^a(z),$$

where $C_{\mathfrak{g}}$ denotes the Coxeter number of the Lie algebra $\mathfrak{g}$. In this case, the stress-energy tensor defines a CFT of central charge

$$c = \frac{k \dim \mathfrak{g}}{k + C_{\mathfrak{g}}}.$$

The above construction of the stress-energy tensor is known as the *Sugawara construction*.

Remark 2.3.9 (Operator formalism). In any CFT, there exists an one-to-one correspondence between operators (fields) and states. This is most intuitive in the Atiyah-Segal picture, c.f. Section 2.1: Recall that a CFT associates to each boundary of the surface a Hilbert space and to the surface itself a map between these Hilbert spaces. Given a collections of states on the incoming and outgoing boundaries, we can therefore interpret the surface with boundaries as the evolution of in-states into out-states. Now, we can form a closed surface from a surface with boundaries by gluing in punctured disks. The boundaries correspond to a small neighborhood of the punctures. On the other hand, a CFT associates to every closed surface a number. If instead of a state $|\phi\rangle$ supported on a boundary component, we consider an operator (field) ϕ supported at the puncture of the disk, then this number can be interpreted as the transition amplitude of the evolution of in-states to out-states.

In a CFT, in-states are motivated by the following example. Suppose that we have formulated the model on an infinite cylinder. We assume in addition that the Hilbert space admits a unique $\mathrm{PSL}_2(\mathbb{C})$-invariant vacuum $|0\rangle$. The invariance of the vacuum means that

$$L_n|0\rangle = 0, \qquad n = -1, 0, 1.$$

As we have seen before, the infinite past $t = -\infty$ is mapped conformally to the origin of the complex plane. An asymptotic in-state $|\phi\rangle$, created from acting with an operator ϕ on the vacuum $|0\rangle$ and which is supported at the circle at $t = -\infty$ should therefore be mapped to a state on the complex plane which is supported at a single point, namely the origin. This leads us to the definition (c.f. [6, 10, 14])

$$|\phi\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle. \tag{2.41}$$

Suppose that $\phi(z,\bar{z})$ was a primary operator of weight $(h,\bar{h})$. Then it admits a Laurent expansion

$$\phi(z,\bar{z}) = \sum_{n,\bar{m}} \phi_{n,\bar{m}}\, z^{-n-h}\bar{z}^{-\bar{h}-\bar{m}}.$$

Since the in-state (2.41) is required to be non-singular at $z,\bar{z} = 0$, one must impose the conditions

$$\phi_{n,\bar{m}}|0\rangle = 0, \qquad \forall n > -h, \bar{m} > -\bar{h}.$$

In particular, since the vacuum is $\mathrm{PSL}_2(\mathbb{C})$-invariant, one finds

$$L_n|0\rangle = 0, \qquad n > 1.$$

One can thus simplify (2.41), and write the in-state $|\phi\rangle$ purely in terms of the mode operators $\phi_{n,\bar{m}}$.

$$|\phi\rangle = \phi_{-h,-\bar{h}}|0\rangle. \tag{2.42}$$

This correspondence between states and local operators is known as the *operator-state correspondence*. Even if the correspondence exists in more general QFTs, the important observation for a CFT is that the correspondence is one-to-one, due to its scale invariance. The operator-state correspondence opens up the possibility to study CFTs in the operator formalism, i.e. in terms of representation theory of the Virasoro algebra. This approach was used with high efficiency in the past and continues to be of utmost importance. This representation theoretic approach will, however, not be used in this thesis. Rather we will focus on a CFT admitting a Lagrangian description and its perturbation theory.

2.3.3. Example: the free scalar field. In this section, we want to present an important example of two-dimensional CFTs - the free scalar field model. It shows a variety of properties which we will encounter later in Chapter 4 and 5, especially the existence of vertex operators. For an extensive review of the model we refer the interested reader to the existing literature [6, 10, 14].

2.3.3.1. *The free scalar field.* The free scalar field model is defined by the space of fields $\{\phi\colon \Sigma \to \mathbb{R}\}$ and the action

$$S = \frac{1}{2}\int_\Sigma d\phi \wedge *d\phi, \tag{2.43}$$

where $*$ is the Hodge star operator depending on a choice of metric on the Riemann surface Σ. The equations of motions are

$$\Delta\phi = 0,$$

where $\Delta = d^*d + d^*d$ is the Hodge Laplacian. Solutions are given by harmonic functions on Σ.

Let us fix $\Sigma = \mathbb{C}$. In complex coordinates z on $\mathbb{C}$, the action is written as

$$S = 2\int_{\mathbb{C}} d^2x\ \partial\phi\bar{\partial}\phi,$$

where $d^2x = \frac{i}{2}dzd\bar{z}$ is the real measure on $\mathbb{C}$. The action functional (2.43) is conformally invariant: Consider any vector field ξ on Σ. Then

$$\delta_\xi S = \int_\Sigma \mathcal{L}_\xi(d\phi \wedge *d\phi) = \int_\Sigma d\phi \wedge [\mathcal{L}_\xi, *]d\phi.$$

If ξ is a conformal vector field, i.e. ξ is a generator of a conformal coordinate transformation, then ξ commutes with the Hodge star operator, $[\mathcal{L}_\xi, *] = 0$, and therefore $\delta_\xi S = 0$. Hence the model defines classically a CFT.

Remark 2.3.10. Another way to show conformal invariance of the action (2.43) is by noticing that the Hodge star operator defined on an n-dimensional manifold and acting on a p-form changes under a conformal transformation $\tilde{g}_{\mu\nu} = e^{2f}g_{\mu\nu}$ according to

$$*_{\tilde{g}} = e^{(n-2p)f} *_g .$$

Now, in the case at hand, $\dim\Sigma = 2$ and the Hodge star acts only on one-forms. Therefore it does not change under a conformal transformation of the metric. The action is thus indeed conformally invariant.

Recall from Equation (2.34) that classically the stress-energy tensor is defined as the reaction of the theory with respect to a change in the metric, namely

$$\delta_g S = -\int_\Sigma d^2x\sqrt{\det g}\ T_{\mu\nu}\delta g^{\mu\nu}.$$

In complex coordinates, one finds

$$T \equiv T_{zz} = -\frac{1}{2}\partial\phi\partial\phi, \qquad T_{z\bar{z}} = 0, \qquad \bar{T} \equiv T_{\bar{z}\bar{z}} = -\frac{1}{2}\bar{\partial}\phi\bar{\partial}\phi.$$

Correlation functions are defined by

$$\langle \mathcal{O}_1(z_1)\dots\rangle := \frac{1}{Z}\int \mathcal{D}\phi\, e^{-S/4\pi}\mathcal{O}_1(z_1)\dots$$

This gives the propagator

$$\langle \phi(z)\phi(w)\rangle = -2\log|z-w| + C, \tag{2.44}$$

where C is some divergent constant.

Remark 2.3.11. Following the standard procedure, the propagator is defined as the Green's function of the Laplace operator Δ:

$$\partial\bar{\partial}G(z,w) = \delta(z-w),$$

where $G(z,w) := \langle\phi(z)\phi(w)\rangle$. In other words, the propagator $G(z,w)$ is the integral kernel of the inverse of the Laplacian. In order to invert the Laplacian, one has to fix its zero modes. Since the zero-modes are just constant functions[6], they are fixed by requiring that the field ϕ vanishes at ∞. Put differently, we are secretly considering the model on the compactified complex plane, i.e. the Riemann sphere $\mathbb{C}P^1$. Instead of demanding that the fields ϕ vanish at ∞, we can therefore equivalently replace $\mathbb{C} = \mathbb{C}P^1 - \{\infty\}$ by a disk D_R of some large radius $R >> 1$ centered at the origin and enforce Dirichlet boundary conditions on ϕ: $\phi(z) = 0$ whenever $|z| = R$. The radius R serves us therefore as an infrared regularization. The propagator (2.44) must then be calculated under the assumption of the Dirichlet boundary condition, which yields

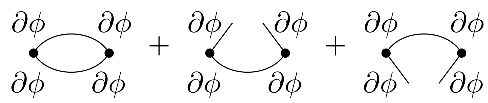
$$C = 2\log R.$$

Following the discussion of Remark (2.3.2), the OPE between two (composite) fields is given by all possible partial Wick contractions, i.e. Wick contractions where open half edges are allowed. For example, one has

$$\partial\phi(z)\partial\phi(w) = \partial\phi(z)\bullet\!\!-\!\!\bullet\partial\phi(w) = -2\partial_z\partial_w\log|z-w| = -\frac{1}{(z-w)^2}.$$

Likewise, one has

$$T(z)T(w) \sim \sum$$



$$\sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w},$$

where one Taylor expands fields at z around w after having performed all possible contractions. From the TT OPE we read of the central charge: $c = 1$. Likewise one finds

$$T(z)\partial\phi(w) \sim \frac{\partial\phi(w)}{(z-w)^2} + \frac{\partial^2\phi(w)}{z-w},$$

[6]More generally, by Hodge theory, zero-modes of the Laplacian are in one-to-one correspondence with the cohomology of the worldsheet.

that is $\partial\phi$ is a primary field of weight $h = 1$. Moreover, by the equations of motions, $\bar{\partial}\partial\phi = 0$, $j = \partial\phi$ is a holomorphic primary field of weight $(h, \bar{h}) = (1, 0)$, i.e. $j = \partial\phi$ is a holomorphic current. Indeed the stress-energy tensor is quadratic in those currents and thus given by the Sugawara construction, c.f. Remark 2.3.8:

$$T(z) = -\frac{1}{2}j^2(z).$$

Finally, let us point out that the free scalar model admits vertex operators – primary fields of arbitrary weights, i.e. of anomalous dimensions. Let

$$V_\alpha = e^{i\alpha\phi}, \qquad \alpha \in \mathbb{R}.$$

Then an elementary calculation shows that

$$T(z)V_\alpha(w) \sim \frac{\alpha^2/2V_\alpha(w)}{(z-w)^2} + \frac{\partial V_\alpha(w)}{z-w},$$

and similarly for the OPE with $\bar{T}$. It follows that V_α are primary fields of weight $(\frac{\alpha^2}{2}, \frac{\alpha^2}{2})$.

Remark 2.3.12 (Free scalar field theory on the Riemann sphere). Recall that so far we have considered the theory of a free scalar field on the complex plane, where we have fixed the zero-modes at ∞. On the other hand, the theory is equally well defined on the Riemann sphere $\mathbb{C}P^1$. The problem of the zero-mode remains and is fixed by enforcing the condition that the field ϕ vanishes at some point $p \in \mathbb{C}P^1$. Equivalently, we could cut a small disk $D(p)$ around p and impose Dirichlet boundary conditions $\phi|_{\partial D(p)} = 0$ at the boundary of the disk. We ought therefore introduce states which enforce the boundary conditions. Those may be thought of as Dirac delta-function-like operators $\delta(\phi)$ with the property that

$$\phi(z)\delta(\phi(z)) = 0.$$

Even though these operators tend to appear quite often, for example in string perturbation theory [42], there are many open questions tied to them, some of which are addressed in Chapter 5 in a slightly different setup. In the case of the scalar free particle, however, there exists a better way to deal with the zero-modes, namely one considers the field not to take values in $\mathbb{R}$ but in a circle S^1 of some radius R, that is

$$\phi(z) \sim \phi(z) + 2\pi R.$$

This solves the problem of the zero-modes insofar that the integration over the zero mods in the path integral yields a finite factor $2\pi R$. The catch is that now all correlation functions are defined in explicit dependence of R. One should think of a choice of R as regularization scheme. In particular, to define a theory where ϕ takes values in the real line $\mathbb{R}$ one ought to do all computations for ϕ taking values in a circle and at the end take the radius of the circle to infinity.

2.3.3.2. *Deforming the free scalar theory.* In this section, we want to discuss the deformation theory of the scalar field theory compactified on a circle S^1_R of radius R.

Let $\phi\colon \mathbb{C} \to S^1_R$, i.e.

$$\phi(z) \sim \phi(z) + 2\pi R.$$

The undeformed action is still

$$S_0 = 2\int_{\mathbb{C}} d^2x\ \partial\phi\bar{\partial}\phi.$$

Classically, a CFT can always be deformed by a primary operator $\phi^{(1,1)}$ of dimension $(1,1)$. Indeed, the two-form $\phi^{(1,1)}dzd\bar{z}$ is conformally invariant and hence the deformed action

$$S = S_0 + g\int_{\Sigma} d^2x\ \phi^{(1,1)}, \qquad d^2x = \tfrac{i}{2}dzd\bar{z},$$

is classically invariant under conformal transformations. Here g is a small coupling constant. Of course, things are more complicated on the quantum level. There, additional conditions have to be put on $\phi^{(1,1)}$. For example, the OPE between the deformation $\phi^{(1,1)}$ and itself cannot obtain singularities of the form $|z|^{-2}$.

For the scalar field, the Lagrangian is itself a $(1,1)$ primary field,

$$\phi^{(1,1)} = \partial\phi\bar{\partial}\phi.$$

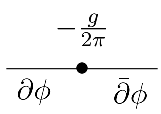
We can therefore deform the action by itself, so to speak:

$$S = 2(1+g)\int_{\mathbb{C}} d^2x\ \partial\phi\bar{\partial}\phi. \tag{2.45}$$

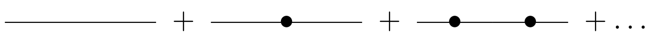
As before, correlation functions are computed according to

$$\langle \mathcal{O}_1(z_1)\dots\rangle = \frac{1}{Z}\int \mathcal{D}\phi\ e^{-S/4\pi}\mathcal{O}_1(z_1)\dots$$

The deformation introduces a bivalent interaction vertex

(2.46)


which leads to a dressed propagator

(2.47)


In formulas

$$\langle\phi(z)\phi(w)\rangle = \frac{-\log|z-w|^2}{1+g} = -\log|z-w|^2\sum_{n\geq 0}(-g)^n. \tag{2.48}$$

Finally, the stress-energy tensor becomes

$$T = -\frac{1+g}{2}\partial\phi\bar{\partial}\phi. \tag{2.49}$$

On the quantum level, OPEs in the deformed model must now be computed by using the dressed propagator. For example, one finds

$$
\begin{aligned}
T(z)V_\alpha(w) &\sim -\frac{1+g}{2}\left(\frac{-\alpha^2 V_\alpha(w)}{(1+g)^2(z-w)^2} + \frac{-\partial V_\alpha(w)}{(1+g)(z-w)}\right) \\
&\sim \frac{\frac{1}{2}\frac{\alpha^2}{1+g}V_\alpha(w)}{(z-w)^2} + \frac{\partial V_\alpha(w)}{z-w}.
\end{aligned} \tag{2.50}
$$

This determines a flow of conformal weights $h_\alpha(g)$ depending on the coupling constant g:

$$
h_\alpha(g) = \frac{1}{2}\frac{\alpha^2}{(1+g)} = \frac{\alpha^2}{2}\sum_{n\geq 0}(-g)^n = \frac{\alpha^2}{2}(1-g) + \mathcal{O}(g^2) \tag{2.51}
$$

Even if ϕ takes values in a circle of radius R, the vertex operator V_α is ought to be well-defined. This implies

$$
V_\alpha = e^{i\alpha\phi} = e^{i\alpha(\phi+2\pi R)}.
$$

Therefore, α only takes values of the form $\alpha = n/R$ for some $n \in \mathbb{Z}$. The flow of conformal weights $h_\alpha(g) = \alpha^2(g)/2$ then defines a new $\alpha(g)$

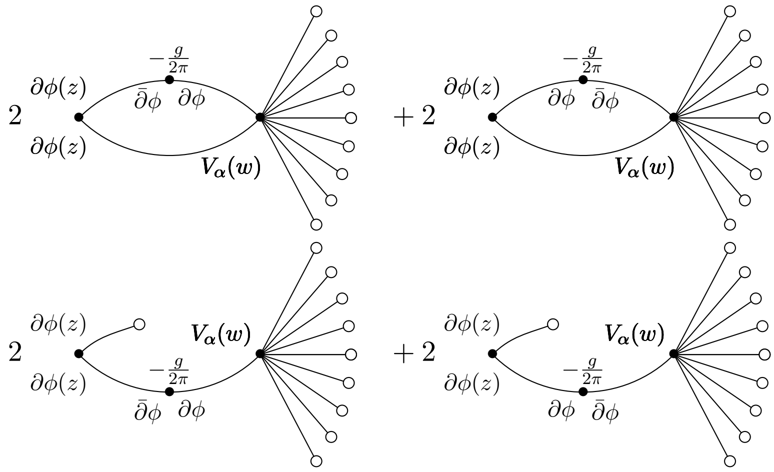
$$
\alpha(g) = \frac{\alpha}{\sqrt{1+g}} = \frac{n}{R\sqrt{1+g}} = \frac{n}{R(g)}, \qquad n \in \mathbb{Z}.
$$

Therefore, the deformation of the action according to Equation (2.45) can be interpreted as varying the radius of the compactifying circle.

Perturbation theory. The above study of the deformed model was exact, in the sense that the coupling constant g could a priori take any value. In practice, however, we want to treat the model perturbatively. Henceforth we shall thus assume that g is sufficiently small. We now show how to recover the flow of conformal weights (2.51) at first order in g when treating the deformation as a perturbation.

As mentioned before, the perturbation introduces the bivalent interaction vertex as shown in Equation (2.46). Following the standard procedure in perturbation theory, the $\mathcal{O}(g)$ correction of the $\partial\phi\partial\phi V$ OPE is calculated schematically as follows:

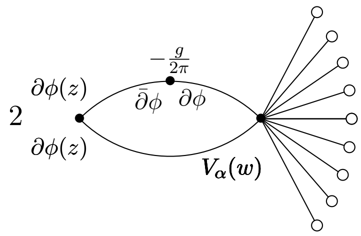
$$\partial\phi\partial\phi(z)V_\alpha(w)\left(-\frac{g}{2\pi}\right)\int d^2u\ \partial\phi\bar\partial\phi =$$



The first integral is easily computed by using the contact term

$$\partial\bar\partial\phi(z)\phi(w) \sim \pi\delta(z-w) \tag{2.52}$$

implied by the equations of motion. It follows that

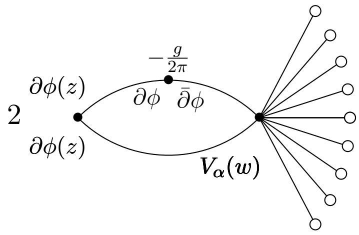


$$= 2\,(-g)\,\frac{-1}{z-w}\int\frac{d^2u}{2\pi}\ \pi\delta(z-u)\left(\frac{-1}{u-w}\right)\left(-\alpha^2V_\alpha(w)\right)$$
$$= \frac{g\alpha^2V_\alpha}{(z-w)^2}$$

The second diagram is a bit more involved: Let us define

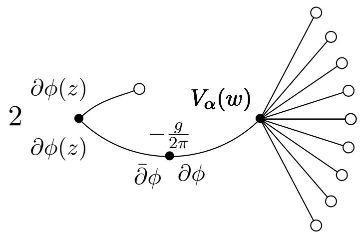
$$I(z|w) := \int_{D_\rho}\frac{du^2}{\pi}\ \frac{1}{z-u}\,\frac{1}{\bar u-\bar w} = -\log\left(\frac{\rho^2-z\bar w}{|z-w|^2}\right)\underset{\rho\to\infty}{\to}\log|z-w|^2 \tag{2.53}$$

where D_ρ is a disk of radius ρ (the infrared cut-off) which regularizes the theory on the plane. Then



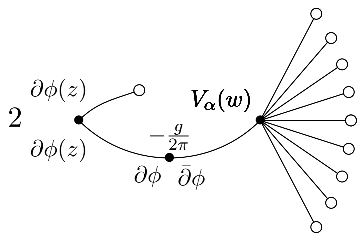
$$= 2\left(-\frac{g}{2}\right)\frac{-1}{z-w}\int\frac{du^2}{\pi}\,\frac{-1}{(z-u)^2}\,\frac{-1}{\bar{u}-\bar{w}}(-\alpha^2)V_\alpha(w)$$
$$= g\alpha^2 V_\alpha(w)\frac{\partial_z I(z|w)}{z-w}$$
$$= \frac{g\alpha^2 V_\alpha(w)}{(z-w)^2}$$

Similarly we calculate the contribution of the remaining two diagrams:

(2.54) 

$$= \frac{g\partial V_\alpha(w)}{z-w}$$

and

(2.55) 

$$= \frac{g\partial V_\alpha(w)}{z-w}.$$

We are now able to calculate the $\mathcal{O}(g)$ correction to the TV_α OPE:

$$\begin{aligned} T(z)V_\alpha(w) &\sim -\frac{1+g}{2}\partial\phi\partial\phi(z)V_\alpha(w) \\ &\sim -\frac{1+g}{2}\left(-\frac{(1-2g)\alpha^2 V_\alpha}{(z-w)^2} - \frac{2(1-g)\partial V_\alpha(w)}{z-w}\right) + \mathcal{O}(g^2) \\ &\sim \frac{(1-g)\frac{\alpha^2}{2}V_\alpha(w)}{(z-w)^2} + \frac{\partial V_\alpha(w)}{z-w} + \mathcal{O}(g^2). \end{aligned}$$

This determines the conformal weight

$$h_\alpha(g) = \frac{\alpha^2}{2}(1-g) + \mathcal{O}(g^2) \tag{2.56}$$

which coincides with the expansion up to first order of the non-perturbative result, c.f. Equation (2.51).

Remark 2.3.13. We end this section with yet another method of obtaining the flow of the conformal weights (2.51) (up to first order in g).

Let $h_\alpha(g)$ be the flow of conformal weight of the vertex operator V_α. Consider the (deformed) two point function on the real surface $\bar{z} = z^*$:

$$\langle V_\alpha(z)V_{-\alpha}(w)\rangle_g = \frac{1}{(z-w)^{2h_\alpha(g)}(\bar{z}-\bar{w})^{\bar{h}_\alpha(g)}} = \frac{1}{|z-w|^{2h_\alpha(g)}}. \tag{2.57}$$

Taking the derivative w.r.t. g, we have on one hand

$$\left.\frac{d}{dg}\right|_{g=0} \langle V_\alpha(z)V_{-\alpha}(w)\rangle_g = -2\frac{dh_\alpha(g)}{dg}\log|z-w|^2 \langle V_\alpha(z)V_{-\alpha}(w)\rangle_0 \tag{2.58}$$

On the other hand, the derivative w.r.t. g pulls down an insertion of the deformation, such that

$$\left.\frac{d}{dg}\right|_{g=0} \left\langle V_\alpha(z)V_{-\alpha(w)}\right\rangle_g = \left\langle V_\alpha(z)V_{-\alpha}(w)\left(-\frac{1}{2\pi}\right)\int d^2u\ \partial\phi\bar{\partial}\phi\right\rangle_0. \tag{2.59}$$

In the above correlator we have two possible contractions, both yielding the bulk term:

$$(i\alpha)(-i\alpha)\langle V_\alpha(z)V_{-\alpha}(w)\rangle_0 I(z|w) = \alpha^2\log|z-w|^2\langle V_\alpha(z)V_\alpha(w)\rangle_0. \tag{2.60}$$

Comparison yields the initial value problem

$$\begin{cases} \left.\frac{dh_\alpha(g)}{dg}\right|_{g=0} = -\frac{\alpha^2}{2} \\ h_\alpha(0) = \frac{\alpha^2}{2} \end{cases} \tag{2.61}$$

which, up to first order, integrates to

$$h_\alpha(g) = \frac{\alpha^2}{2}(1-g) + \mathcal{O}g^2, \tag{2.62}$$

which, at first order, indeed coincides with the non-perturbative result (2.51).

2.4. Topological conformal field theory

In this section we will introduce the notion of *topological conformal field theories* (TCFT). At first sight, the notion seems tautological. The idea is, that a TCFT is not an honest TQFT, in the sense that *all* correlators are independent under diffeomorphisms of the worldsheet. A TCFT is rather a CFT who is topological in the physical relevant sector. In particular, correlation functions of physical observables are topological. As we will see, such theories often carry a global odd symmetry generated by an odd nilpotent operator Q such that the stress-energy tensor is Q-exact. Such an odd symmetry arises for example from the BRST gauge fixing procedure. Moreover, physical observables live in the Q-cohomology. In particular, the stress-energy tensor vanishes on the space of physical observables and hence there is no dynamics, a characteristic feature of topological TQFTs. The important point is that going beyond Q-cohomology, i.e. beyond the sector relevant for pure physics, one can detect interesting mathematical structures which one would miss otherwise and thus one hopes to get a better understanding of the full theory. For example, the study of correlation functions of physical observables with some insertions of the Q-primitive of the stress-energy tensor leads to closed differential forms on the moduli space of punctured Riemann surfaces. The periods of those differential forms encode important information about the moduli space. This makes TCFTs very interesting also from a purely mathematical point of view.

In the following we will give a toy model for a TCFT, namely topological quantum mechanics seen as a one-dimensional QFT. Here, we will already see most of the consequences which follow from the existence of the nilpotent operator Q and the Q-exactness of the Hamiltonian. We will then generalize the constructions to higher dimensional QFTs, where the role of the Hamiltonian is replaced by the stress-energy tensor.

The question of an explicit construction of a TCFT still remains. One approach due to E. Witten [34, 35] is to twist $\mathcal{N} = (2,2)$ supersymmetric theories. We shall spend some time to explain these ideas at the end of this section. There exists a second construction, namely gauge fixing topological gauge theories. This thesis is devoted to the second construction which we spell out in detail in Chapter 4 and 5.

2.4.1. A warm up: topological quantum mechanics. A first study of quantum topological mechanics focused on the interplay of quantum mechanics and Morse theory initiated in [33] and later generalized to higher dimensional QFTs in [19]. We will, however, be more interested in algebraic structures which have to be satisfied by correlators and their differential geometric interpretation. In what follows, we closely follow [21, 22].

Consider a quantum mechanical model whose space of states is a Hermitian $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$. Let $H \in \mathrm{End}(V)$ be the Hamiltonian of the system. We denote the parity operator by $(-1)^F$. It acts on V_0 by the identity, and on V_1 by minus the identity. Elements of V_0 are called bosonic (even) and elements of V_1 fermionic (odd). An operator $\mathcal{O} \in \mathrm{End}(V)$ is called even/odd if it commutes/anti-commutes with the parity operator $(-1)^F$. For any two operators $\mathcal{O}_1, \mathcal{O}_2$ with parity $p(\mathcal{O}_1), p(\mathcal{O}_2)$, we define their super-commutator

$$\{\mathcal{O}_1, \mathcal{O}_2\} = \mathcal{O}_1\mathcal{O}_2 - (-1)^{p(\mathcal{O}_1)\cdot p(\mathcal{O}_2)}\mathcal{O}_2\mathcal{O}_1.$$

The Hamiltonian H is assumed to be an even operator with positive spectrum.

We assume that there exists a nilpotent odd operator $Q \in \mathrm{End}(V)$, $\{Q, Q\} = 0$, which commutes with the Hamiltonian: $\{Q, H\} = 0$. The operator Q thus generates a symmetry of the model. Moreover, since Q is odd, we say that the symmetry is odd.

Assume furthermore that the Hamiltonian can be written as a commutator of Q and its adjoint which, for later references, we will denote by G:

$$H = \{Q, G\}. \tag{2.63}$$

The partition function is defined to be the weighted trace of the evolution operator

$$Z(\beta) = \mathrm{tr}(-1)^F e^{-\beta H}.$$

In the Atiyah-Segal picture, β is the length (or volume) of the circle, which should be thought of as the underlying one-dimensional worldsheet c.f. Example 2.1.4. Equation (2.63) has a remarkable consequence: The partition function is in fact independent of β. Using (2.63), we find

$$\frac{d}{d\beta}Z(\beta) = -\operatorname{tr}(-1)^F\{Q,G\}e^{-\beta H} = \operatorname{tr}\{Q,(-1)^F G e^{-\beta H}\} = 0. \tag{2.64}$$

Put differently, the partition function is independent of the volume of the underlying worldsheet and hence of any metric on it. It depends purely on the topology of the worldsheet, hence the name *topological quantum mechanics*.

Notice that the independence of the worldsheet volume follows from the Q-exactness of the Hamiltonian. Mathematically, Q is a differential and since it commutes with the Hamiltonian, the only states contributing to the partition function are elements of the cohomology of Q. The cohomology of Q represents thus fully the space of physical states. Indeed, suppose that $|v_\lambda\rangle$ is an eigenstate of the Hamiltonian with eigenvalue λ: $H|v_\lambda\rangle = \lambda|v_\lambda\rangle$. Then $Q|v_\lambda\rangle \equiv |Qv_\lambda\rangle$ is also an eigenstate corresponding to the same eigenvalue. However, since Q is an odd operator, $|Qv_\lambda\rangle$ has opposite parity compared to $|v_\lambda\rangle$. It follows that if $\lambda \neq 0$, the eigenspaces E_λ of the Hamiltonian are two-fold degenerate. Since the two generators of E_λ have opposite parity, they therefore cancel in the partition function:

$$\begin{aligned} Z &= \sum_{\lambda\neq 0}\Big(\langle v_\lambda|(-1)^F e^{-\beta H}|v_\lambda\rangle + \langle Qv_\lambda|(-1)^F e^{-\beta H}|Qv_\lambda\rangle\Big) + \sum_{v\in E_0}\langle v|(-1)^F|v\rangle \\ &= \sum_{\lambda\neq 0}(-1)^F e^{-\beta\lambda}\Big(\langle v_\lambda|v_\lambda\rangle - \langle Qv_\lambda|Qv_\lambda\rangle\Big) + \sum_{v\in E_0}\langle v|(-1)^F|v\rangle \\ &= \sum_{v\in E_0}\langle v|(-1)^F|v\rangle. \end{aligned}$$

Since V is a Hermitian vector space, a state $|v\rangle$ which is annihilated by H must be annihilated by both, Q and G, as it follows from

$$0 = \langle v|H|v\rangle = \langle v|QG + GQ|v\rangle = \|G|v\rangle\|^2 + \|Q|v\rangle\|^2.$$

Now, consider the $\mathbb{Z}_2$-graded complex

$$V_1 \xrightarrow{Q} V_0 \xrightarrow{Q} V_1 \xrightarrow{Q} V_0, \tag{2.65}$$

and its cohomology

$$H_Q^{(0)} := \frac{\ker\, Q\colon V_0 \to V_1}{\operatorname{im}\, Q\colon V_1 \to V_0}, \qquad H_Q^{(1)} := \frac{\ker\, Q\colon V_1 \to V_0}{\operatorname{im}\, Q\colon V_0 \to V_1}.$$

This complex decomposes into energy levels [16] and for each excited level $\lambda \neq 0$ the cohomology vanishes: if $|v_\lambda\rangle$ is an eigenstate with eigenvalue λ and Q-closed, then

$$|v_\lambda\rangle = \lambda^{-1}H|v_\lambda\rangle = Q\Big(\lambda^{-1}G|v_\lambda\rangle\Big)$$

is Q-exact and hence zero in cohomology. On the other hand, Q vanishes identically on zero-energy states and hence the only contribution to the cohomology comes from the ground states of the system [16]. The partition function is therefore given by

$$Z = \sum_{v\in E_0}\langle v|(-1)^F|v\rangle = \sum_{v\in H_Q}\langle v|(-1)^F|v\rangle = \dim H_Q^{(0)} - \dim H_Q^{(1)}, \tag{2.66}$$

and is called the *Witten index*. In particular, the partition function is a number and hence indeed independent of β, as has been already observed before.

REMARK 2.4.1. In many examples it happens that the Hilbert space is $\mathbb{Z}$-graded: $V = \bigoplus_{n\in\mathbb{Z}} V_n$. The $\mathbb{Z}_2$-grading can be recovered by setting $V_{\bar{0}} = \bigoplus_{n\in\mathbb{Z}} V_{2n}$ and $V_{\bar{1}} = \bigoplus_{n\in\mathbb{Z}} V_{2n+1}$. The parity operator $(-1)^F$ now acts by

$$(-1)^F|v_n\rangle = (-1)^n|v_n\rangle, \qquad |v_n\rangle \in V_n.$$

The $\mathbb{Z}_2$-graded complex (2.65) becomes a $\mathbb{Z}$-graded complex

$$\dots \xrightarrow{Q} V_{n-1} \xrightarrow{Q} V_n \xrightarrow{Q} V_{n+1} \xrightarrow{Q} \dots$$

with cohomology

$$H^n_Q := \frac{\ker Q\colon V_n \to V_{n+1}}{\operatorname{im} Q\colon V_{n-1} \to V_n}.$$

In this case, the Witten index is the Euler characteristic of the complex [16]:

$$Z(\beta) = \operatorname{tr}(-1)^F e^{-\beta H} = \sum_{n\in\mathbb{Z}} (-1)^n \dim H^n_Q.$$

The fact that the partition function is independent of the volume of the one-dimensional worldsheet rises the question if more general correlation functions share a similar feature.

We recall that in the Heisenberg picture operators evolve in time (along the one-dimensional worldsheet) according to

$$\mathcal{O}(t) = e^{tH}\mathcal{O}e^{-tH} = U^{-1}(t)\mathcal{O}U(t),$$

where $U(t) = \exp(-tH)$ is the Euclidean evolution operator. Transition amplitudes between an in- and an out-states are defined by

(2.67)
$$\langle out|\mathcal{O}_1(t_1)\dots\mathcal{O}_n(t_n)|in\rangle = \langle out|e^{t_1H}\mathcal{O}_1U(t_{12})\dots U(t_{n-1\,n})\mathcal{O}_n e^{-t_nH}|in\rangle,$$

where t_{ij} denotes the distance $t_i - t_j$ and we assume that $t_1 < \dots < t_n$ such that the product of operators is time ordered.

Now, a *topological correlator* should be a correlation function which is independent of the relative distances of the inserted operators and should solely depend on their order.

Such correlators can be constructed as follows: Suppose that we start with an operator $\mathcal{O}^{(0)}$ which commutes with Q. Such operators are known as *zero-observables* [21]. We define its *first descent* $\mathcal{O}^{(1)}$ by

(2.68)
$$\mathcal{O}^{(1)}(t) = -dt\{G, \mathcal{O}^{(0)}\}.$$

It follows that

$$\begin{aligned}
\{Q, \mathcal{O}^{(1)}(t)\} &= dt\ \{Q, \{G, \mathcal{O}^{(0)}(t)\}\} \\
&= dt\ \{\{Q, G\}, \mathcal{O}^{(0)}(t)\} - dt\ \{G, \underbrace{\{Q, \mathcal{O}^{(0)}(t)\}}_{=0}\} \\
&= dt\ \underbrace{\{H, \mathcal{O}^{(0)}(t)\}}_{\dot{\mathcal{O}}^{(0)}} \\
&= d\mathcal{O}(t),
\end{aligned}$$

i.e. the first descent $\mathcal{O}^{(1)}$ commutes with Q up to an d-exact term.

REMARK 2.4.2 (Non-local Q-closed operators). If C is any one-cycle in the homology of the worldsheet, then

$$\mathcal{O}_C := \int_C \mathcal{O}^{(1)}(t)$$

is a non-local operator supported on C, which commutes with Q:

$$\{Q, \mathcal{O}_C\} = \int_C \{Q, \mathcal{O}^{(1)}(t)\} = \int_C d\mathcal{O}^{(0)}(t) = \int_{\partial C=\emptyset} \mathcal{O}^{(0)}(t) = 0$$

The descent equation

$$\{Q, \mathcal{O}^{(1)}\} = d\mathcal{O}^{(0)} \tag{2.69}$$

has much more consequences.

For example, if the boundary conditions $|in\rangle$ and $|out\rangle$ are both annihilated by Q, i.e. they are zero energy states, then the descent equations (2.69) imply that correlation function of zero-observables $\mathcal{O}_i$ is topological, in the above sense:

$$\begin{aligned} d\langle out|\mathcal{O}_1(t_1)\dots\mathcal{O}_n(t_n)|in\rangle &= \sum_i \langle out|\mathcal{O}_1(t_1)\dots d\mathcal{O}_i(t_i)\dots\mathcal{O}_n(t_n)|in\rangle \\ &= \sum_i \langle out|\mathcal{O}_1(t_1)\dots\{Q,\mathcal{O}_i^{(1)}(t_i)\}\dots\mathcal{O}_n(t_n)|in\rangle \\ &= \sum_i \langle out|\{Q,\mathcal{O}_1(t_1)\dots\mathcal{O}_i^{(1)}(t_i)\dots\mathcal{O}_n(t_n)\}|in\rangle \\ &= 0. \end{aligned}$$

Since the above transition amplitudes are independent of the relative distances t_{ii+1}, they are defined by their limit $t_{ii+1} \to \infty$. In this limit, the evolution operator $U(t_{ii+1}) = \exp(-t_{ii+1}H)$ becomes the projector to zero-energy states, i.e. the projector to the cohomology of Q

$$pr_{H_Q} = \sum_c |c\rangle\langle c|.$$

Here, $|c\rangle$ denotes a basis element of H_Q. Therefore, the transition amplitude satisfies the following factorization property [21, 22]

$$\begin{aligned} \langle out|\mathcal{O}_1(t_1)\dots\mathcal{O}_n(t_n)|in\rangle &= \langle out|\mathcal{O}_1 U(t_{12})\dots U(t_{n-1\,n})\mathcal{O}_n|in\rangle \\ &= \lim_{t_{ii+1}\to\infty} \langle out|\mathcal{O}_1 U(t_{12})\dots U(t_{n-1\,n})\mathcal{O}_n|in\rangle \\ &= \sum_{i_1,\dots,i_{n-1}} \langle out|\mathcal{O}_1|c_{i_1}\rangle\langle c_{i_1}|\dots|c_{i_{n-1}}\rangle\langle c_{i_{n-1}}|\mathcal{O}_n|in\rangle. \end{aligned}$$

In particular, if the Q cohomology is trivial, that is Q annihilates only the non-degenerate vacuum state $|0\rangle$, then the vacuum expectation value of a product is given by the product of the vacuum expectation values [22]

$$\langle 0|\mathcal{O}_1(t_1)\dots\mathcal{O}_n(t_n)|0\rangle = \langle 0|\mathcal{O}_1|0\rangle\dots\langle 0|\mathcal{O}_n|0\rangle.$$

Recall that the transition amplitudes (2.67), with Q-closed boundary states, are independent of the length of the interval. Therefore, we may regard them as functions on $\mathbb{R}^n_+$. Since they are clearly invariant under simultaneous translation of the t_i's, they descent to (constant) functions on the the infinite open cube $M_n = \mathbb{R}_+^n/\mathbb{R}_+ = (0,\infty)^{n-1}$ parameterized by the relative distances $(t_{12}, t_{23}, \ldots, t_{n-1\,n})$.

In fact, we can define more general closed (inhomogeneous) differential forms on M_n as follows: let $\mathcal{O}^\bullet = \mathcal{O}^{(0)} + \mathcal{O}^{(1)}$ be the *total descent* of a Q-cohomology class $\mathcal{O}^{(0)}$. Notice that on total descents, d and Q are equivalent:

$$\{Q, \mathcal{O}^\bullet\} = \{Q, \mathcal{O}^{(0)} + \mathcal{O}^{(1)}\} = d\mathcal{O}^{(0)} = d(\mathcal{O}^{(0)} + \mathcal{O}^{(1)}) = d\mathcal{O}^\bullet.$$

Here we used $d\mathcal{O}^{(1)} = 0$ since it is a two-form. Then, for two Q-closed states $|a\rangle, |b\rangle$

$$\omega_{ab} = \langle a|\mathcal{O}_1^\bullet(t_1)\ldots\mathcal{O}_n^\bullet(t_n)|b\rangle \in \Omega^\bullet_{\mathrm{cl}}(M_n) \tag{2.70}$$

defines an inhomogeneous closed differential form on M_n. As before, closedness follows from the descent equation (2.69)

$$\begin{aligned} d\omega_{ab} &= \sum_i \langle a|\mathcal{O}_1^\bullet(t_1)\ldots d\mathcal{O}_i^\bullet(t_i)\ldots\mathcal{O}_n^\bullet(t_n)|b\rangle \\ &= \sum_i \langle a|\mathcal{O}_1^\bullet(t_1)\ldots\{Q,\mathcal{O}_i^\bullet(t_i)\}\ldots\mathcal{O}_n^\bullet(t_n)|b\rangle \\ &= \langle a|\{Q,\ldots\}|b\rangle = 0. \end{aligned}$$

Following [21], we can reinterpret the above construction as coupling the theory to "topological gravity". In order to do so, we promote the coordinates $t_{ij} = t_i - t_j$ on M_n to super-coordinates $(t_{ij}, \psi_{ij} = dt_{ij})$. Correlators, which were previously functions of the t_{ij} now become functions of the pairs (t_{ij}, ψ_{ij}), i.e. they become differential forms. Furthermore, we define a differential

$$Q^{\mathrm{tot}} = Q - \psi_{ij}\frac{\partial}{\partial t_{ij}} = Q - d, \qquad \{Q^{\mathrm{tot}}, Q^{\mathrm{tot}}\} = 0,$$

where d denotes the de Rham differential on the moduli space of n-punctured intervals M_n. The evolution operator $U(t) = \exp(-tH)$ is then promoted to a super-evolution operator

$$U(t, dt) = \exp\left(-\{Q^{\mathrm{tot}}, tG\}\right) = \exp\left(-tH + dtG\right),$$

which is Q^{tot} closed

$$\{Q^{\mathrm{tot}}, U(t,dt)\} = \{Q - d, U(t,dt)\} = 0.$$

Therefore, Q^{tot} defines an odd symmetry of the theory coupled to gravity. Now, for any operator $\mathcal{O}$, one has

$$U^{-1}(t,dt)\mathcal{O}U(t,dt) = e^{-tH}\mathcal{O}^\bullet e^{-tH} = \mathcal{O}^\bullet(t).$$

The differential forms (2.70) are nothing but correlation functions of the theory coupled to gravity:

$$\omega_{ab} = \langle a|\mathcal{O}_1^\bullet(t_1)\ldots\mathcal{O}_n^\bullet(t_n)|b\rangle = \langle a|\mathcal{O}_1 U(t_{12}, dt_{12})\ldots\mathcal{O}_n|b\rangle. \tag{2.71}$$

Recall that moduli spaces M_n are isomorphic to the open cubes $(0,\infty)^{n-1}$, coordinatized by relative distances. They can be compactified simply by gluing in the missing points.

For example, the space $\overline{M}_1$ is a single point, while $\overline{M}_2$ is isomorphic to the two point compactification $[0, \infty]$ of the interval $(0, \infty)$.The boundary $\partial\overline{M}_2$ corresponds simply to the boundary of the interval $[0, \infty]$. Figuratively, 0 corresponds to a collision of the two punctures, and ∞ to an infinite stretching of their relative distance.

Now, the differential forms ω_{ab} extend to the compactification $\overline{M}_n$: When two neighboring punctures collide the evolution operator $U(t_{i\,i+1}, dt_{i\,i+1})$ becomes the identity; when they are infinitely far apart, the evolution operator becomes the projector pr_{H_Q} to cohomology. Suppose now that the operators $\mathcal{O}_i$ form an algebra

$$\mathcal{O}_i\mathcal{O}_j = c_{ij}^k\mathcal{O}_k. \tag{2.72}$$

In this case, we get various interesting relations among the integrals of the transition amplitudes

$$\omega^a_{i_1\ldots i_n;\,b} = \langle a|\mathcal{O}_1^\bullet(t_1)\ldots\mathcal{O}_n^\bullet(t_n)|b\rangle \in \Omega^\bullet_{\mathrm{cl}}(\overline{M}_n) \tag{2.73}$$

by integrating them over cycles in $\overline{M}_n$ [21]. To fix notation, let

$$\tau^a_{i_1\ldots i_n;\,b} := \int_{\overline{M}_n}\omega^a_{i_1\ldots i_n;\,b}. \tag{2.74}$$

Note that since M_1 is a point, we have

$$\omega^a_{i;\,b} = \tau^a_{i;\,b} = \langle a|\mathcal{O}_i|b\rangle. \tag{2.75}$$

In order to get a feeling of how these relation arise, let us consider the two-point correlation function

$$\omega^a_{ij;\,b} = \langle a|\mathcal{O}_iU(t_{ij}, dt_{ij})\mathcal{O}_j|b\rangle$$

The boundary components 0 and ∞ of $\overline{M}_2 \cong [0, \infty]$ can be seen as zero-cycles. By Stokes' theorem, we have

$$\int_{\{0\}\sqcup\{\infty\}}\omega^a_{ij;\,b} = \int_{\partial\overline{M}_2}\omega^a_{ij;\,b} = \int_{\overline{M}_2}d\omega^a_{ij;\,b} = 0.$$

As stated above, when $t_{ij} = 0$, the evolution operator $U(t_{ij}, dt_{ij})$ becomes the identity, while if $t_{ij} = \infty$, it becomes the projector pr_{H_Q} onto Q-cohomology. It follows that

$$\int_{\{0\}\sqcup\{\infty\}}\omega^a_{ij;\,b} = -\langle a|\mathcal{O}_i\mathcal{O}_j|b\rangle + \sum_c\langle a|\mathcal{O}_i|c\rangle\langle c|\mathcal{O}_j|b\rangle = 0,$$

where the sum runs over a basis of H_Q. Since by assumption the $\mathcal{O}_i$ form an algebra, $\mathcal{O}_i \cdot \mathcal{O}_j = c^k_{ij}\mathcal{O}_k$, we find

$$\langle a|\mathcal{O}_i\mathcal{O}_j|b\rangle = c^k_{ij}\langle a|\mathcal{O}_k|b\rangle$$

and therefore, by (2.75), the relation

$$c^k_{ij}\tau^a_{k;\,b} = \tau^a_{i;\,c}\tau^c_{j;\,b}.$$

Now, if we consider more general moduli spaces M_n, the boundaries get more complicated. For example, let us consider the integral of

$$\omega^a_{ijk;\,b} = \langle a|\mathcal{O}_iU(t_{ij}, dt_{ij})\mathcal{O}_jU(t_{jk}, dt_{jk})\mathcal{O}_k|b\rangle$$

over $\partial\overline{M}_3$. Note that

$$\partial\overline{M}_3 = \{0\} \times M_2 \sqcup \{\infty\} \times M_2 \sqcup M_2 \times \{0\} \sqcup M_2 \times \{\infty\},$$

where the first factor represents the domain of the coordinate t_{12} and the second factor the domain of the coordinate t_{23}. It follows that

$$\begin{aligned} 0 &= \int_{\partial\overline{M}_3} \omega^a_{ijk;\,b} \\ &= -\int_{M_2} \langle a|\mathcal{O}_i\mathcal{O}_j U(t,dt)\mathcal{O}_k|b\rangle + \sum_c \int_{M_2} \langle a|\mathcal{O}_i|c\rangle\langle c|\mathcal{O}_j U(t,dt)\mathcal{O}_k|b\rangle \\ &\quad - \int_{M_2} \langle a|\mathcal{O}_i U(t,dt)\mathcal{O}_j\mathcal{O}_k|b\rangle + \sum_c \int_{M_2} \langle a|\mathcal{O}_i U(t,dt)\mathcal{O}_j|c\rangle\langle c|\mathcal{O}_k|b\rangle \\ &= -c^\ell_{ij}\tau^a_{\ell k;\,b} + \tau^a_{i;\,c}\tau^c_{jk;\,b} - c^\ell_{jk}\tau^a_{i\ell;\,b} + \tau^a_{ij;\,c}\tau^c_{k;\,b}. \end{aligned}$$

We have hence obtained the quadratic relation

$$c^\ell_{ij}\tau^a_{\ell k;\,b} + c^\ell_{jk}\tau^a_{i\ell;\,b} = \tau^a_{i;\,c}\tau^c_{jk;\,b} + \tau^a_{ij;\,c}\tau^c_{k;\,b} \tag{2.76}$$

The n-point generalization of the above is known in the mathematical literature as an A_∞-module of the algebra of zero observables (2.72) [21].

To end this subsection, we want to consider another construction involving zero-observables, namely deformations of the theory. Suppose that we deform the differential Q by a zero-observable $\mathcal{O}^{(0)}$:

$$Q(u) = Q + u\mathcal{O}^{(0)},$$

where u is some parameter. We are interested in the situations when $Q(s)$ squares to zero. This gives the condition

$$\{Q(u), Q(u)\} = u^2\{\mathcal{O}^{(0)}, \mathcal{O}^{(0)}\} = 0. \tag{2.77}$$

For such zero-observables $\mathcal{O}^{(0)}$, we can define a deformed Hamiltonian by

$$H(u) = \{Q(u), G\}$$

which defines a new model.

2.4.1.1. *Example: BRST quantization of the relativistic particle.* As an example of topological quantum mechanics, let us discuss the BRST quantization of the free relativistic (massive) particle. We will follow the closely [27, 43].

We will formulate the theory of a free relativistic particle as a one-dimensional QFT, that is within the path integral formalism. The model will be determined in terms of a space of fields, for which we will take maps $\phi\colon M \to \mathbb{R}^n$, where M is either a circle or an interval, and an action functional.

Remark 2.4.3. More generally, one can consider maps $\phi\colon M \to X$ from some compact one-dimensional worldsheet into some target manifold X. QFTs, whose space of fields are maps from one manifold to another, are generally known as *sigma models.*

Additionally to the maps ϕ, we will consider an auxiliary field, namely an einbein $\boldsymbol{e} = e(x)dx$ (a parallelization of M). The defining action functional of the model is given by

$$S(e,\phi) = \frac{1}{2}\int_M dx\ \left(e^{-1}\phi'^\mu\phi'^\nu g_{\mu\nu} - m^2 e\right), \tag{2.78}$$

where $g_{\mu\nu}$ is a (constant) metric on $\mathbb{R}^n$ and ϕ' denotes the derivative of ϕ. This action is invariant under reparameterization of M: As $d\phi$ and $\boldsymbol{e}$ are one-forms on M, their coefficients ϕ'^μ and e transform accordingly:

$$e \to e(f(x))f', \qquad \phi'^\mu(x) \to \phi'^\mu(f(x))f'.$$

It follows that $e^{-1}\phi'^\mu\phi'^\nu$ transforms as the coefficient of a one-form and hence $e^{-1}\phi'^\mu\phi'^\nu dx$ is invariant (as is $\boldsymbol{e}$). Therefore the action (2.78) is indeed invariant under reparameterizations of M.

Since the einbein enters the action (2.78) non-dynamically, integrating over e is the same as imposing its equation of motion

$$e^2 = -\frac{1}{m^2}\phi'^\mu\phi'^\nu g_{\mu\nu}.$$

Doing so, one obtains the more familiar action of the massive free relativistic particle

$$S(\phi) = -m\int_M dx\ \left(\phi'^\mu\phi'^\nu g_{\mu\nu}\right)^{1/2}.$$

In the path integral formalism, the reparameterization invariance plays the role of a gauge symmetry. Under an infinitesimal reparameterization $x \to x + \varepsilon(x)$, the fields transform according to

$$\delta_\varepsilon\phi^\mu = \varepsilon\phi'^\mu, \qquad \delta_\varepsilon e = \left(\varepsilon e\right)'.$$

We recall that in the BRST formalism, one replaces the gauge parameter ε by a ghost c and adjoins it to the space of fields: $(\phi, \boldsymbol{e}) \to (\phi, \boldsymbol{e}, c)$, (see Remark 2.2.1). In particular, the space of fields is now graded. The grading, also known as the ghost number, assigns 0 to the original fields ϕ and e, and 1 to the ghost field c. In fact, the graded space of fields can be equipped with a differential, the BRST operator s, which we define as follows: Firstly, its action on ϕ and e encodes their gauge transformations

$$s\colon \qquad \phi \mapsto \phi = c\phi', \qquad e \mapsto (ce)'.$$

It is convenient to express the BRST operator s by the BRST charge Q, such that

$$s(\varphi) = \{Q, \varphi\}.$$

Here and for the rest of this section $\{\cdot,\cdot\}$ denotes the canonical Poisson bracket. It is clear, that the BRST charge has ghost number $gh(Q) = 1$, i.e. $gh(\{Q,\varphi\}) = 1 + gh(\varphi)$ for any field φ of the augmented complex. If we want $s^2 = 0$, i.e. $\{Q,Q\} = 0$, we need to impose an additional transformation rule for the ghost field c. For example, we have

$$s^2(\phi) = s(c\phi') = s(c)\phi' - c(c\phi')' = (s(c) - cc')\phi' \stackrel{!}{=} 0$$

where we have used $c^2 = 0$ due to the fact that c has ghost number 1, i.e. c is a Grassmann odd field. Therefore,

$$s(c) = \{Q, c\} = cc'.$$

It is now an easy check, that $s^2 e = s^2 c = 0$ and hence $s^2 = 0$. Notice in particular that the gauge invariance of the action (2.78) is equivalent to the statement that S is Q-closed, i.e. Q generates an odd symmetry. Let us recall that in the path integral approach to quantization, correlation functions of operators $\mathcal{O}_i$ are defined by

$$\langle \mathcal{O}_{i_1}(t_1)\dots\mathcal{O}_{i_n}(t_n)\rangle := \int_{\text{fields}} d\mu e^{iS/\hbar}\Big(\mathcal{O}_{i_1}(t_1)\dots\mathcal{O}_{i_n}(t_n)\Big),$$

where we suppose that $d\mu$ is a suitable measure on the space of fields which in most cases does not exists mathematically. If we assume that $d\mu$ is invariant, i.e. $\mathcal{L}_Q d\mu = 0$, then the correlation function of a Q-closed expression vanishes

$$\langle\{Q,\dots\}\rangle = 0.$$

Therefore, we are free to change the action by any Q-closed expression:

$$\langle\dots\rangle = \int_{\text{fields}} d\mu e^{iS/\hbar}\Big(\dots\Big) = \int_{\text{fields}} d\mu e^{i(S+\{Q,\Psi\})/\hbar}\Big(\dots\Big).$$

The idea of BRST gauge fixing is to choose Ψ such that the path integral is "well-defined".

Returning to the action (2.78), the reparameterization invariance is strong enough to fix $e = 1$ [27]. It is convenient to implement this condition directly in the action by adjoining a trivial pair (b, λ) to the space of fields, with ghost numbers $gh(b) = -1$ and $gh(\lambda) = 0$.

By "trivial" we mean that the action of Q extends to an action on the augmented space of fields in such a way that H_Q remains unaltered. This can be achieved by imposing the following

$$\{Q, b\} = \lambda, \qquad \{Q, \lambda\} = 0.$$

Hence, adding the fields b and λ does not change the cohomology of Q, since we essentially added only an exact generator. The field b is known as an *anti-ghost*, while we will see below that λ will play the role of a Lagrangian multiplier. We can now choose a gauge-fixing fermion, which implements the gauge condition $e = 1$ as follows:

$$\Psi = \int_M dx\; b(e-1).$$

It follows that

$$\{Q,\Psi\} = \int_M dx\; \{Q, b(e-1)\} = \int_M dx\; \lambda(e-1) - b(ce)'.$$

Now, it is convenient to perform a integration by parts on the second summand. Note, however, that b is an odd field and hence we have to be careful with signs. In fact, one finds

$$\begin{aligned}\int_M dx\; b(ce)' &= -\int_M bd(ce) = \int_M \Big(d(bce) - (db)ce\Big)\\ &= -\int_{\partial M} bce - \int_M dx\; b'ce.\end{aligned}$$

Since M is closed, $\partial M = 0$ and hence the full gauged fixed action reads

$$\begin{aligned} S_{gf} &= S + \{Q, \Psi\} \\ &= \int_M dx \left(\tfrac{1}{2} \left(e^{-1} \phi'^\mu \phi'^\nu g_{\mu\nu} - m^2 e \right) + \lambda(e-1) + b' c e \right), \end{aligned} \tag{2.79}$$

where the BRST transformation are given by

$$\begin{aligned} \{Q, \phi^\mu\} &= c\phi'^\mu \\ \{Q, e\} &= (ce)' \\ \{Q, c\} &= cc' \\ \{Q, b\} &= \lambda \\ \{Q, \lambda\} &= 0. \end{aligned} \tag{2.80}$$

We will find it convenient to proceed with the discussion in Hamiltonian formalism. From the action (2.79), one finds the following canonical momenta:

$$\begin{aligned} (\pi_\phi)_\mu \equiv p_\mu &= g_{\mu\nu} \phi'^\nu e^{-1} \\ \pi_b &= ce \\ \pi_e &= \pi_\lambda = \pi_c = 0, \end{aligned} \tag{2.81}$$

with canonical commutation relations[7] $\{\varphi, \pi_\varphi\} = 1$. Notably, the last equation, $\pi_e = \pi_\lambda = \pi_c = 0$, defines three constraint. The extended Hamiltonian of the system is then

$$H = H_0 e + \pi_\lambda \lambda' + \pi_e e' + \pi_c c' - \lambda(e-1),$$

where $H_0 = \frac{1}{2}(p^2 + m^2)$ with $p^2 = g^{\mu\nu} p_\mu p_\nu$. Following the general theory of constraint Hamiltonian mechanics (see e.g. [15]), we are ought to think of λ', e' and c' as yet undetermined functions and we shall write them as u_1, u_2 and u_3 instead. It is instructive to think of u_i as Lagrangian multipliers which impose the constraints. To continue, we must determine new constraints by demanding that the Poisson bracket of the Hamiltonian with the constraints vanish. For example, we find

$$\{H, \pi_c\} = 0, \qquad \{H, \pi_\lambda\} = e - 1 \stackrel{!}{=} 0, \qquad \{H, \pi_e\} = -\lambda + H_0 \stackrel{!}{=} 0.$$

Therefore we find two new constraints $e - 1 = 0$ and $-\lambda + H_0 = 0$, which we add to the Hamiltonian with two new Lagrangian multipliers u_4, u_5. At this point, the augmented Hamiltonian looks like

$$H = H_0 e + \pi_\lambda u_1 + \pi_e u_2 + \pi_c u_3 - \lambda(e-1) + u_4(e-1) + u_5(-\lambda + H_0).$$

Now, all constraints should again vanish in the Poisson bracket with the augmented Hamiltonian. A priori, calculating the conditions

$$\{H, \text{constraint}\} = 0$$

yields new constraints. However, in our case, it rather determines the Lagrangian multipliers u_i. For example,

$$0 \stackrel{!}{=} \{H, e-1\} = u_2, \qquad 0 \stackrel{!}{=} \{H, -\lambda + H_0\} = u_1$$

[7] For the rest of this section, $\{\cdot, \cdot\}$ denotes the Poisson bracket.

determines $u_1 = u_2 = 0$. Thus

$$H = H_0 e + \pi_c u_3 - \lambda(e-1) + u_4(e-1) + u_5(-\lambda + H_0).$$

Proceeding with this new Hamiltonian we find

$$0 \stackrel{!}{=} \{H, \pi_\lambda\} = -u_5 - (e-1)$$

and thus

$$\begin{aligned} H &= H_0 e + \pi_c u_3 - \lambda(e-1) + u_4(e-1) - (e-1)(-\lambda + H_0) \\ &= H_0 + \pi_c u_3 + u_4(e-1). \end{aligned}$$

Finally,

$$0 \stackrel{!}{=} \{H, \pi_e\} = u_4,$$

which determines $u_4 = 0$. The correct Hamiltonian to consider when computing the equations of motion is therefore

$$H = H_0 + \pi_c c' = \frac{1}{2}(p^2 + m^2) + \pi_c c', \tag{2.82}$$

where we have set $u_3 = c'$. The equations of motion are now given by

$$\varphi' = \{H, \varphi\}\big|_{\text{constraints}}$$

and the constraints are

$$\begin{aligned} e - 1 &= 0 \\ H_0 - \lambda &= 0 \\ \pi_\lambda = \pi_e = \pi_c &= 0. \end{aligned} \tag{2.83}$$

Note that If we label the constraints

$$\varphi_0 = \pi_c,\ \varphi_1 = \pi_\lambda,\ \varphi_2 = H_0 - \lambda,\ \varphi_3 = e - 1,\ \varphi_4 = \pi_e,$$

such that the constraints (2.83) are equivalent to $\varphi_i = 0$, then we observe that the Poisson brackets $\{\varphi_0, \varphi_i\}$ vanish for all i. Therefor, φ_0 defines a first class constraint. On the other hand, one has $\{\varphi_1, \varphi_2\} = \{\varphi_3, \varphi_4\} = 1$, such that those constraints are second class. The Poisson bracket has therefore to be replaced by the Dirac bracket:

$$\{f, g\}_{DB} := \{f, g\} - \sum_{i,j=1}^{4} \{f, \varphi_i\} M^{ij} \{\varphi_j, g\}, \tag{2.84}$$

where M^{ij} is the inverse of the matrix

$$(M)_{i,j} = \{\varphi_i, \varphi_j\} = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}.$$

If one were to pass to canonical quantization, then the canonical commutation relations have to be defined with respect to the Dirac bracket (2.84).

The crucial observation is now the following: Modulo the constraints (2.83), the BRST charge, c.f. (2.80), can be expressed in terms of the Hamiltonian:

$$Q \approx cH. \tag{2.85}$$

For example, one finds

$$\{cH, \phi\}_{DB} = c\{H, \phi\}_{DB} = c\phi' = \{Q, \phi\}.$$

Similarly,

$$\{cH, c\}_{DB} = c\{H, c\}_{DB} = cc' = \{Q, c\}.$$

More importantly,

$$\{Q, b\} = \{cH, b\}_{DB} = H,$$

where we point out that modulo the constraints $\pi_b = c$. The above equation expresses the fact, that the Hamiltonian of the system is Q-exact. The role of the primitive G of section 2.4.1 is played by the anti-ghost b. The BRST gauge fixed relativistic particle is therefore an example of topological quantum mechanics.

Remark 2.4.4 (Hilbert stress-energy tensor and variations of gauge-fixings). It is interesting to study also the Hilbert stress-energy tensor. Let us recall that it is defined by the variation of the action with respect to a chosen background metric. Suppose that we choose a more general gauge-fixing condition $e = e_0$ for some fixed background metric e_0. Using the BRST formalism, we can implement the above gauge fixing by choosing the gauge-fixing fermion to be

$$\Psi = \int_M dx\, b(e - e_0),$$

which gives the gauge-fixed action

$$S_{\text{gf}} = S + \{Q, \Psi\}.$$

Now, as discussed before, the Hilbert stress-energy tensor of the gauge-fixed model is given by a variation with respect to e_0. In particular, from this point of view, one can interpret the stress-energy tensor as the reaction of the theory to a variation of the gauge fixing. One finds

$$T = -\frac{\partial S_{\text{gf}}}{\partial e_0} = -\left\{Q, \frac{\partial \Psi}{\partial e_0}\right\} = \{Q, b\}$$

which coincides with the result we obtained in the Hamiltonian formalism.

2.4.2. Topological conformal field theories. Much of the aforementioned algebraic structures we encounter in topological quantum mechanics can be generalized to higher dimensional QFTs. The most important feature of topological quantum mechanics was the presence of an odd symmetry, generated by a nilpotent operator Q, and the fact that the Hamiltonian was Q-exact. In higher dimensional QFTs, the Hamiltonian is replaced by the stress-energy tensor $T_{\mu\nu}$.

Definition 2.4.1. A conformal field theory (CFT), admitting a Lagrangian description, is called *topological conformal field theory* (TCFT), if it admits the following data [11, 12]

- a graded space of states $\mathcal{H}^\bullet$;
- a nilpotent odd operator $Q \in \text{End}(\mathcal{H}^\bullet)$;
- an even action functional S which is annihilated by Q: $QS = 0$;
- a stress-energy tensor which is Q-exact: $T_{\mu\nu} = \{Q, G_{\mu\nu}\}$.

Remark 2.4.5. Since $QS = 0$, Q generates a fermionic symmetry of the theory. Moreover, the stress-energy tensor is even with respect to the grading, and hence its primitive $G_{\mu\nu}$ is odd. In particular, in the case that the space of states is $\mathbb{Z}$-graded, Q will be taken to be of degree 1, while $G_{\mu\nu}$ will have degree -1.

Analogously to the case of topological quantum mechanics, the fact that the stress-energy tensor is Q-exact, has far reaching consequences. For example, any TCFT has vanishing central charge. Indeed, as is customary, we will set

$$T_{zz} = T, \qquad T_{\bar{z}\bar{z}} = \bar{T}, \qquad G_{zz} = G, \qquad G_{\bar{z}\bar{z}} = \bar{G},$$

where T, G and $\bar{T}, \bar{G}$ are holomorphic and anti-holomorphic fields respectively. Recall from section 2.3 that the central charge c of a CFT is given by the coefficient of the fourth order pole of the TT OPE:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}.$$

Now, since T is Q-exact, it is in particular Q-closed and thus the TT OPE is Q-exact:

$$T(z)T(w) = \{Q, G(z)T(w)\}.$$

Since a number cannot be Q-exact, it follows that the central charge must be zero.

Likewise, a primary field ϕ which is Q-closed but has non-zero conformal weight is Q-exact:

$$h\phi = L_0\phi = \{Q, G_0\}\phi = \{Q, G_0\phi\} \implies \phi = h^{-1}\{Q, G_0\phi\}.$$

It follows that the cohomology of Q is fully contained in the ground states of the CFT, i.e. if a primary field ϕ represents a Q-cohomology class, then its conformal weight must be zero. This phenomena is akin to the one we have observed in topological quantum mechanics.

Let $\phi^{(0)}$ be a zero-observable, i.e. a state which is annihilated by Q. Then we define the *p-th descent* of ϕ as a solution of the descent equations [34, 35]

$$\begin{aligned} Q\phi^{(0)} &= 0, \\ Q\phi^{(p)} &= d\phi^{(p-1)}, \qquad \phi^{(p)} \in \Omega^p(\Sigma). \end{aligned} \tag{2.86}$$

Of course, in a two-dimensional theory, p ranges from 0 to 2.

The primitive $G_{\mu\nu}$ gives an elegant way to solve the descent equations: Recall that in a CFT, the stress-energy tensor is traceless (in the absence of a Weyl anomaly).

In Chapter 4, we will define a descent operator Γ which acts on operators by

$$\Gamma\phi = -dz(G_{-1}\phi) - d\bar{z}(\bar{G}_{-1}\phi).$$

It is then an easy exercise to check that

$$\{Q, \Gamma\} = d$$

and thus that

$$\phi^{(p)} := \frac{1}{p!}\Gamma^p\phi$$

satisfy the descent equations (2.86).

REMARK 2.4.6. More generally, since the momentum operator $P_\mu = T_{\mu 0}$ is Q-exact,

$$P_\mu = \{Q, G_\mu\}, \qquad G_\mu = G_{\mu 0},$$

the operators

$$\phi^{(p)}(x) = \frac{(-1)^p}{p!} dx^{\mu_1} G_{\mu_1} \dots dx^{\mu_p} G_{\mu_p} \phi^{(0)}$$

satisfy the descent equations

$$Q\phi^{(p)} = dx^\mu P_\mu \phi^{(p-1)}.$$

The usual form of the descent equations (2.86) are recovered when passing to canonical quantization, i.e. by replacing P_μ by $-i\partial_\mu$.

As in the example of topological quantum mechanics, one can arrange zero-observables and their descents nicely into superfields (inhomogeneous differential forms) called *total descents*

$$\phi^\bullet = \phi^{(0)} + \phi^{(1)} + \phi^{(2)} = e^\Gamma \phi^{(0)}.$$

Notably, the descent equations (2.86) are equivalent to

$$(Q - d)\phi^\bullet = 0.$$

In particular, Q and the de Rham differential coincide on the space of total descents.

The descent equations imply that correlation functions of zero-observables are topological, i.e. that they do not depend on the points of insertions of the operators:

$$\begin{aligned} d_{z_j}\Big\langle \phi_{i_1}(z_1)\dots\phi_{i_n}(z_n)\Big\rangle &= \Big\langle \phi_{i_1}(z_1)\dots d_{z_j}\phi_{i_j}(z_j)\dots\phi_{i_n}(z_n)\Big\rangle \\ &= \Big\langle \phi_{i_1}(z_1)\dots Q\phi^{(1)}_{i_j}(z_j)\dots\phi_{i_n}(z_n)\Big\rangle \\ &= \Big\langle Q\Big(\phi_{i_1}(z_1)\dots\phi^{(1)}_{i_j}(z_j)\dots\phi_{i_n}(z_n)\Big)\Big\rangle = 0. \end{aligned}$$

Here we used that Q generates a symmetry and hence the correlator of a Q-exact operator vanishes. Moreover, integrating a p-th descent $\phi^{(p)}$ over a p-cycle $C \in H_p(\Sigma)$,

$$\mathcal{O}_C(\phi) = \int_C \phi^{(p)},$$

one obtains cohomology classes, i.e. physical observables, which are supported on C, i.e. they are *non-local*. Indeed, by the descent equations and Stokes' theorem one has

$$Q\mathcal{O}_C(\phi) = \int_C Q\phi^{(p)} = \int_C d\phi^{(p-1)} = \int_{\partial C} \phi^{(p-1)} = 0.$$

Furthermore, akin to the situation of topological quantum mechanics, TCFTs can be coupled to topological gravity [40], which eventually boils down to a study of periods of closed differential forms over certain moduli spaces. The moduli spaces in questions are the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of complex structures on n-punctured compact Riemann surfaces of genus g. The closed differential forms on $\overline{\mathcal{M}}_{g,n}$ are, analogously to the case in topological quantum mechanics, constructed form correlation functions containing the primitive $G_{\mu\nu}$ of the stress-energy tensor. The general construction is as follows: Let H_Q denote the cohomology of Q and let $\mathcal{O}_j$ represent a class in H_Q. One then defines closed differential k-forms $\omega \in \Omega^k(\overline{\mathcal{M}}_{g,n})$ depending on the cocycles $\mathcal{O}_j$ by its action on k vector fields $\mu_j \in \Gamma T\overline{\mathcal{M}}_{g,n}$ (so-called Beltrami differentials) as follows [23]:

$$\omega_{\Sigma_g}^{\mathcal{O}_1,\dots,\mathcal{O}_n}(\mu_1,\dots,\mu_k) = \left\langle \prod_{j=1}^{k} (\mu_j, G)\ \mathcal{O}_1(z_1)\dots\mathcal{O}_n(z_n) \right\rangle_{\Sigma_g}. \tag{2.87}$$

Here $\langle\dots\rangle_{\Sigma_g}$ stands for the path integral on the worldsheet Σ_g and

$$(\mu_j, G) = \int_{\Sigma_g} \left(G_{zz}\mu_j{}^z{}_{\bar{z}} + G_{\bar{z}\bar{z}}\mu_j{}^{\bar{z}}{}_z \right) d^2z.$$

For genus zero, studying periods of these differential forms, i.e. integrals of over cycles in $\overline{\mathcal{M}}_{0,4}$, leads to new solutions of the WDVV associativity equations [12, 38].

Remark 2.4.7. Studying integrals over the moduli spaces $\overline{\mathcal{M}}_{g,n}$ is equivalent to integrating over all worldsheet geometries. They can therefore be identified with certain amplitudes in string theory. In fact, the periods of the correlation functions (2.87) are known as *topological string amplitudes*.

Remark 2.4.8. In Witten's A-model (see Section 2.4.3.4) this construction leads to the celebrated Gromov-Witten theory.

Finally, second descents can be used to deform the action

$$S \mapsto S(g) = S + g \int_\Sigma \phi^{(2)},$$

where g is a small coupling constant. This deformation defines a family of theories. We stress, however, that a priori the deformed theory is *not* a TCFT again. It is interesting to study under which conditions this happens.

Example 2.4.9 (The bosonic string). One of the prime examples of a TCFT is the bosonic string [11, 27] which generalizes the example of the free relativistic particle.

The defining action functional of the (Euclidean) bosonic string is

$$S[X,g] = \frac{1}{4\pi\alpha'} \int_\Sigma d^2x \sqrt{g}\, g^{\mu\nu} \partial_\mu X^i \partial_\nu X_i$$

where α' is related to the string tension, X is a scalar field taking values in $\mathbb{C}^n$ and $d^2x = dx^1dx^2$ denotes the real measure on Σ. In particular, the worldsheet metric g is a field of the theory and not a fixed background. The bosonic string action is invariant under conformal transformation, which now play the role of a gauge symmetry. In particular, in two-dimensions, it is always possible to relate the worldsheet metric g to the flat metric up to a Weyl transformation, i.e. up to a conformal factor:

$$g_{\mu\nu}(x) \to e^{2\omega(x)}\delta_{\mu\nu}.$$

We can therefore use the gauge freedom to fix $g_{\mu\nu} = e^{2\omega(x)}\delta_{\mu\nu}$, which is called the *conformal gauge* [28]. According to usual gauge fixing procedures, the price to pay for fixing the gauge comes in terms of Faddeev-Popov ghosts. In complex coordinates, the gauge fixed action is given by

$$S_{gf}(X,b,c) = \frac{1}{\pi\alpha'}\int_\Sigma d^2x\, \partial X^i\bar{\partial}X_i + \frac{1}{\pi}\int_\Sigma d^2x\, b\bar{\partial}c + \bar{b}\partial\bar{c}, \tag{2.88}$$

where $b, \bar{b}$ are quadratic differentials and $c, \bar{c}$ are vector fields on Σ. By the equations of motion, b and c are holomorphic, while $\bar{b}$ and $\bar{c}$ are anti-holomorphic. The stress-energy tensor splits into a matter and a ghost part: $T = T_{\mathrm{m}} + T_{\mathrm{gh}}$ with (see e.g. [10])

$$T_{\mathrm{m}} = -\frac{1}{\alpha'}\partial X_i \partial X^i, \qquad T_{\mathrm{gh}} = -2b\partial c - \partial b c = -b\partial c - \partial(bc).$$

Remark 2.4.10 (Beltrami differential). A quick way to find the stress-energy tensor is to parameterize the complex structure on the worldsheet Σ by a Beltrami differential $\hat{\mu} = \mu^z_{\bar{z}} d\bar{z} \otimes \partial_z$ [18]. In the following, we will abbreviate $\mu \equiv \mu^z_{\bar{z}}$. Given a Beltrami differential μ, a complex structure on Σ is described with respect to a reference complex structure: Let z be a complex coordinate, then any other choice of complex structure gives a complex coordinate Z such that

$$dZ = \lambda(dz + \mu d\bar{z}), \tag{2.89}$$

where one defines $\lambda = \partial Z$. The factor λ is an integrating factor and is subject to the integrability condition [18]

$$(\bar{\boldsymbol{\partial}}_{\bar{Z}})^2 = 0 \iff (\bar{\partial} - \mu\partial)\log\lambda = \partial\mu,$$

where $\bar{\boldsymbol{\partial}}_{\bar{Z}}$ denotes the Dolbeault operator with respect to the coordinate $\bar{Z}$. Locally, the integrability condition is equivalent to the Beltrami equation (see e.g. [20])

$$(\bar{\partial} - \mu\partial)Z = 0.$$

From (2.89), it follows that

$$\bar{\partial}_{\bar{Z}} = \frac{1}{\bar{\lambda}}\frac{\bar{\partial} - \mu\partial}{1 - |\mu|^2}.$$

General $(j, \bar{\jmath})$ differentials can be parameterized as follows:

$$\Phi(dZ)^j(d\bar{Z})^{\bar{\jmath}} = \Phi\lambda^j\bar{\lambda}^{\bar{\jmath}}(dz+\mu d\bar{z})^j(d\bar{z}+\bar{\mu}dz)^{\bar{\jmath}} = \phi(z,\bar{z})(dz+\mu d\bar{z})^j(d\bar{z}+\bar{\mu}dz)^{\bar{\jmath}}.$$

Moreover, one finds

$$\tfrac{i}{2}dZ\wedge d\bar{Z} = d^2x|\lambda|^2(1-|\mu|^2)$$

where as usual $d^2x = \frac{i}{2}dz\wedge d\bar{z}$.

To parameterize gauged fixed Polyakov action (2.88) by Beltrami differentials, let us set

$$\chi = X, \qquad \beta = \lambda^{-2}b, \qquad \varsigma = \lambda c.$$

Then, the Polyakov action reads

$$S(\mu) = \frac{1}{\pi\alpha'}\int_\Sigma \tfrac{i}{2}dZ\wedge d\bar{Z}\; \partial_Z\chi^i\bar{\partial}_{\bar{Z}}\chi_i + \frac{1}{\pi}\int_\Sigma \tfrac{i}{2}dZ\wedge d\bar{Z}\; \beta\bar{\partial}_{\bar{Z}}\varsigma + \bar{\beta}\partial_Z\bar{\varsigma}.$$

Expanding the matter sector yields

$$\begin{aligned}
&\frac{1}{\pi\alpha'}\int_\Sigma d^2x\,|\lambda|^2(1-|\mu|^2)\frac{(\partial-\bar\mu\bar\partial)X^i(\bar\partial-\mu\partial)X_i}{|\lambda|^2(1-|\mu|^2)^2}=\\
=&\frac{1}{\pi\alpha'}\int_\Sigma d^2x\,\frac{(\partial-\bar\mu\bar\partial)X^i(\bar\partial-\mu\partial)X_i}{(1-|\mu|^2)}\\
=&\frac{1}{\pi\alpha'}\int_\Sigma d^2x\,\frac{1+|\mu|^2}{1-|\mu|^2}\partial X^i\bar\partial X_i-\frac{\mu\partial X^i\partial X_i+\bar\mu\bar\partial X^i\bar\partial X_i}{1-|\mu|^2}\\
=&\frac{1}{\pi\alpha'}\int_\Sigma d^2x\,\frac{1+|\mu|^2}{1-|\mu|^2}\partial X^i\bar\partial X_i+\frac{1}{\pi}\int_\Sigma d^2x\frac{\mu T_{\mathrm{m}}+\bar\mu\bar T_{\mathrm{m}}}{1-|\mu|^2}\\
=&\frac{1}{\pi\alpha'}\int_\Sigma d^2x\,\partial X^i\bar\partial X_i+\frac{1}{\pi}\int_\Sigma d^2x\,\left(\mu T_{\mathrm{m}}+\bar\mu\bar T_{\mathrm{m}}\right)+\mathcal{O}(|\mu|^2)
\end{aligned}$$

while expanding the ghost sector yields

$$\begin{aligned}
&\frac{1}{\pi}\int_\Sigma d^2x\,|\lambda|^2(1-|\mu|^2)\left(\lambda^{-2}b\frac{\bar\partial-\mu\partial}{\bar\lambda(1-|\mu|^2)}(\lambda c)\right)+\text{c.c.}\\
=&\frac{1}{\pi}\int_\Sigma d^2x\,\lambda^{-1}b\left(c(\bar\partial-\mu\partial)\lambda+\lambda(\bar\partial-\mu\partial)c\right)+\text{c.c.}\\
=&\frac{1}{\pi}\int_\Sigma d^2x\,b\left(\bar\partial-\mu\partial+\partial\mu\right)c+\text{c.c.}\\
=&\frac{1}{\pi}\int_\Sigma d^2x\,\left(b\bar\partial c-\mu b\partial c-\mu\partial(bc)\right)+\text{c.c.}\\
=&\frac{1}{\pi}\int_\Sigma d^2x\,\left(b\bar\partial c+\mu T_{\mathrm{gh}}\right)+\text{c.c.}
\end{aligned}$$

where c.c. is shorthand for "complex conjugate". Putting the terms together, we find up to first order in μ

$$S(\mu)=S_{gf}(X,b,c)+\frac{1}{\pi}\int_\Sigma d^2x\,\left(\mu T+\bar\mu\bar T\right)+\mathcal{O}(|\mu|^2),$$

where $T=T_{\mathrm{m}}+T_{\mathrm{gh}}$ is the (full) stress-energy tensor of the model. The Beltrami differential μ serves therefore as a source for the stress energy tensor:

$$T=\pi\frac{\delta S}{\delta\mu},\qquad \bar T=\pi\frac{\delta S}{\delta\bar\mu}.$$

The crucial observation is that the model admits a BRST symmetry [27]:

$$\begin{aligned}
\{Q,X^\mu\}&=(c\partial+\bar c\bar\partial)X^\mu,\\
\{Q,b\}&=T_{\mathrm{m}}+T_{\mathrm{gh}},\\
\{Q,\bar b\}&=\bar T_{\mathrm{m}}+\bar T_{\mathrm{gh}},\\
\{Q,c\}&=c\partial c,\\
\{Q,\bar c\}&=\bar c\bar\partial\bar c,
\end{aligned}$$

whose current J can be written as the Q-commutator with the ghost current $I=bc$ [11]. One finds

$$J=c(T_{\mathrm{m}}+\tfrac{1}{2}T_{\mathrm{gh}}).$$

Even more importantly, the stress-energy tensor is Q-exact

$$T(z)=\{Q,b(z)\}.$$

As for the point particle, its Q-primitive is the anti-ghost b.

We have spent quite a bit of time to sketch the implications of the presence of an odd symmetry under which the stress-energy tensor is exact. Until now, however, we have postponed the question of a general construction of such models. There are essentially two approaches: The first one is called *topological twisting* of $\mathcal{N} = (2, 2)$ supersymmetric theories, whose main ideas we shall outline in the rest of this chapter. The second approach is gauge fixing topological gauge theories which is, to the best of the author's knowledge, a new approach. It will be discussed in detail in Chapters 4 and 5.

2.4.3. Topological twisting of $\mathcal{N} = (2, 2)$ supersymmetric theories. Twisting supersymmetric theories in such a way that the supersymmetry charges transform into a nilpotent operator was initiated by E. Witten in [34, 35]. The idea of twisting is essentially to modify the spins of the fields in a way that changes neither the action functional nor the equations of motion. After a small recap of the construction of $\mathcal{N} = (2, 2)$ supersymmetric theories, we will look at an examples of topological twisted theories.

To familiarize ourselves with the techniques, we will study two-dimensional abelian Yang-Mills theory. It turns out that the twisted model is the complex scalar field theory plus a second order ghost system and is hence conformal. We stress that two-dimensional Yang-Mills theory depends on the choice of an area-form of the worldsheet and is therefore *not* conformally invariant. However, the complex scalar field theory, even in the presence of a second order ghost system, is not a TCFT.

The second example we will investigate are Witten's A- and B-model. Here, we indeed obtain a TCFT in the sense of Definition 2.4.1.

2.4.3.1. $\mathcal{N} = (2, 2)$ *supersymmetric theories from superfield formalism.* This subsection follows [16, 25] and serves as a reminder of the construction of $\mathcal{N} = (2, 2)$ supersymmetric theories and their topological twisting.

$\mathcal{N} = (2, 2)$ supersymmetric theories are conveniently described as theories on superspace. Working in $\mathbb{R}^2$, endowed with the standard Euclidean metric[8], one adjoins to the usual bosonic coordinates x^1, x^2 four fermionic coordinates $\theta^\pm$ and $\bar{\theta}^\pm$. The four fermionic coordinates make up for two Dirac spinors $\theta = (\theta^+, \theta^-)$ and $\bar{\theta} = (\bar{\theta}^+, \bar{\theta}^-)$. Working in complex coordinates $z = x^1 + ix^2$, the Euclidean Lorentz transformations form the group $U(1)$ which acts as

$$e^{i\alpha}\colon z \mapsto e^{i\alpha} z, \qquad \theta^\pm \mapsto e^{\pm i\alpha/2}\theta^\pm, \qquad \bar{\theta}^\pm \mapsto e^{\pm i\alpha/2}\bar{\theta}^\pm. \tag{2.90}$$

Remark 2.4.11. It is instructive to think of θ^+ and $\bar{\theta}^+$ locally as the two square roots of dz:

$$\theta^+ = +\sqrt{dz}, \qquad \bar{\theta}^+ = -\sqrt{dz}.$$

The remaining two fermionic coordinates are given by their complex conjugate: $(\theta^\pm)^* = \theta^\mp$ and $(\bar{\theta}^\pm)^* = \bar{\theta}^\mp$. In particular, the superscript $\pm$ of θ and $\bar{\theta}$ indicates their holomorphic and anti-holomorphic nature, i.e. $\theta^+, \bar{\theta}^+ \propto \sqrt{dz}$ and $\theta^-, \bar{\theta}^- \propto \sqrt{d\bar{z}}$.

[8] For an extensive treatment of Minkowskian spacetime see [16]

The coordinates $z, \theta^+, \bar{\theta}^+$ make up the superspace $\mathbb{C}^{1|2}$. A *superfield* is simply a function on superspace and defined by its Taylor expansion

$$f(x^\mu, \theta^\pm, \bar{\theta}^\pm) = f_0(x^\mu) + \theta^\alpha f_\alpha(x^\mu) + \bar{\theta}^\alpha \bar{f}_\alpha(x^\mu) + \theta^+\theta^- f_{+-}(x^\mu) \dots \tag{2.91}$$

The Lorentz group acts on superfields according to

$$e^{i\alpha\Lambda}\colon f(x^\mu, \theta^\pm, \bar{\theta}^\pm) \mapsto q^{i\alpha q_\Lambda} f(e^{-i\alpha\Lambda}x^\mu, e^{\mp i\alpha/2}\theta^\pm, e^{\mp i\alpha/2}\bar{\theta}^\pm), \tag{2.92}$$

where q_Λ denotes the corresponding Lorentz charge. Assigning an overall Lorentz charge q_Λ to the superfield f induces an assignment of Lorentz charges for the constituent fields. For example, if we suppose that the superfield f is a scalar, i.e. $q_\Lambda = 0$, then

$$f(x^\mu, \theta^\pm, \bar{\theta}^\pm) = f(e^{-i\alpha\Lambda}x^\mu, e^{\mp i\alpha/2}\theta^\pm, e^{\mp i\alpha/2}\bar{\theta}^\pm)$$

and the expansion (2.91) imply

$$e^{i\alpha\Lambda}\colon \phi(x^\mu) \mapsto \phi(e^{-i\alpha\Lambda}x^\mu), \qquad f_\pm \mapsto e^{\mp i\alpha/2} f_\pm(e^{-i\alpha\Lambda}x^\mu), \qquad \text{etc.}$$

that is

$$q_\Lambda(f_0) = 0, \qquad q_\Lambda(f_\pm) = \mp\frac{1}{2}, \qquad \text{etc.}$$

Next to the Lorentz group, which acts by (2.90), there exists so-called (vector and axial) *R-rotations* [16],

$$\begin{aligned} &\text{vector:} && \theta^\pm \mapsto e^{i\alpha}\theta^\pm, && \bar{\theta}^\pm \mapsto e^{-i\alpha}\bar{\theta}^\pm \\ &\text{axial:} && \theta^\pm \mapsto e^{\pm i\beta}\theta^\pm, && \bar{\theta}^\pm \mapsto e^{\mp i\beta}\bar{\theta}^\pm, \end{aligned} \tag{2.93}$$

which act on superfields via

$$\begin{aligned} e^{i\alpha F_V}\colon f(x^\mu, \theta^\pm, \bar{\theta}^\pm) &\mapsto e^{i\alpha q_V} f(x^\mu, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar{\theta}^\pm), \\ e^{i\beta F_A}\colon f(x^\mu, \theta^\pm, \bar{\theta}^\pm) &\mapsto e^{i\beta q_A} f(x^\mu, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\bar{\theta}^\pm). \end{aligned} \tag{2.94}$$

The numbers q_V, q_A are called vector R-charge and axial R-charge respectively. The R-rotations induce transformations of the constituents of the superfield which will play a crucial role in the definition of a *topological twist* of the theory.

REMARK 2.4.12. The vector R-rotation is a U(1) rotation of the full Dirac spinors $\theta = (\theta^+, \theta^-$ and $\bar{\theta} = (\bar{\theta}^+, \bar{\theta}^-)$

$$\theta \mapsto e^{i\alpha}\theta, \qquad \bar{\theta} \mapsto e^{-i\alpha}\bar{\theta}$$

while the axial R-rotation is a U(1) rotation where the two Weyl components $\theta^\pm$ (and $\bar{\theta}^\pm$ respectively) transform oppositely.

It is convenient to define the differential operators

$$\mathcal{Q}_+ = \frac{\partial}{\partial\theta^+} - \bar{\theta}^+\partial, \qquad \overline{\mathcal{Q}}_+ = \frac{\partial}{\partial\bar{\theta}^+} - \theta^+\partial, \tag{2.95a}$$

$$D_+ = \frac{\partial}{\partial\theta^+} + \bar{\theta}^+\partial, \qquad \overline{D}_+ = \frac{\partial}{\partial\bar{\theta}^+} + \theta^+\partial, \tag{2.95b}$$

together with their complex conjugates

$$\mathcal{Q}_- = \mathcal{Q}_+^*, \quad \overline{\mathcal{Q}}_- = \overline{\mathcal{Q}}_+^*, \quad D_- = D_+^*, \quad \bar{D}_- = \bar{D}_+^*.$$

The $\mathcal{Q}$ are known as *supercharges*, the D as covariant derivatives in super space. One finds the following commutation relations

$$\{\mathcal{Q}_+, \overline{\mathcal{Q}}_+\} = -2\partial, \qquad \{\mathcal{Q}_-, \overline{\mathcal{Q}}_-\} = -2\bar{\partial}$$
$$\{D_+, \overline{D}_+\} = 2\partial, \qquad \{D_-, \overline{D}_-\} = 2\bar{\partial},$$

and all other commutators are vanishing.

REMARK 2.4.13. Note that the differentials $D_\pm$ and $\overline{D}_\pm$ can be seen as the square root of the complex derivatives ∂ and $\bar{\partial}$.

Now, a *chiral superfield* Φ is a superfield which satisfy the condition

$$\overline{D}_\pm \Phi = 0.$$

Respectively, we call $\bar{\Phi}$ *anti-chiral* if

$$D_\pm \bar{\Phi} = 0.$$

One can show, that a chiral superfield is generally of the form

$$\Phi(x^\mu, \theta^\pm, \bar{\theta}^\pm) = \phi(y, \bar{y}) + \theta^\alpha \psi_\alpha(y, \bar{y}) + \theta^+\theta^- F(y, \bar{y})$$

with

$$y = z + \theta^+\bar{\theta}^+, \qquad y^* = \bar{z} + \theta^-\bar{\theta}^-.$$

With the notion of a chiral superfield at our disposal, one can now define action functionals which are invariant under (super)coordinate transformation generated by

$$\delta = \varepsilon_\alpha \mathcal{Q}_\alpha + \bar{\varepsilon}_\alpha \overline{\mathcal{Q}}_\alpha \tag{2.96}$$

generated by the $\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm$. Such an action functional is called supersymmetric.

For example, consider the model

$$S(\Phi) = \int d^2x d^4\theta\ \Phi\bar{\Phi} + \int d^2x d^2\theta\ W(\Phi) + \text{c.c.} = S_{kin} + S_W \tag{2.97}$$

defined by a single chiral superfield Φ. Here we denote $d^4\theta = d\theta^+ d\bar{\theta}^+ d\theta^- d\bar{\theta}^-$ and $d^2\theta = \sum_{\alpha=\pm} d\theta^\alpha d\bar{\theta}^\alpha$ and c.c is shorthand for complex conjugate. The holomorphic function $W(\Phi)$ is known as the *superpotential* of the model. Expanding the chiral superfield Φ as

$$\begin{aligned}\Phi = {} & \phi + \theta^+\bar{\theta}^+\partial\phi + \theta^-\bar{\theta}^-\bar{\partial}\phi + \theta^+\theta^-\bar{\theta}^+\bar{\theta}^-\partial\bar{\partial}\phi \\ & + \theta^+\psi_+ + \theta^+\theta^-\bar{\theta}^-\bar{\partial}\psi_+ + \theta^-\psi_- + \theta^-\theta^+\bar{\theta}^+\partial\psi_- + \theta^-\theta^+ F\end{aligned}$$

one can show that the action (2.97) can be written as[9]

$$\begin{aligned}S = \int d^2x\ \Big(& |\partial_\mu\phi\partial^\mu\phi|^2 + |W'(\phi)|^2 + \bar{\psi}_-\partial\psi_- + \bar{\psi}_+\bar{\partial}\psi_+ \\ & + W''(\phi)\psi_+\bar{\psi}_+ + \overline{W}''(\phi)\psi_-\bar{\psi}_- + |F + \overline{W}'(\bar{\phi})|^2\Big).\end{aligned}$$

[9]Due to our definition of the derivations $\mathcal{Q}_\pm$ and $D_\pm$, and since we are working in Euclidean signature, our formulae differ by signs and factor of i from the formulae derived in [16].

Notice that the field F is non-dynamical, i.e. it appears in the action without any derivative. It can therefore be integrated out by means of a Gaussian integration, which enforces the equation of motion

$$F = -\overline{W}'(\bar{\phi}).$$

Setting F to this value we get a slightly simplified action

$$S = \int d^2x \, \Big(\partial\phi\bar{\partial}\bar{\phi} + \bar{\partial}\phi\partial\bar{\phi} + |W'(\phi)|^2 + \bar{\psi}_-\partial\psi_- + \bar{\psi}_+\bar{\partial}\psi_+ + W''(\phi)\psi_+\bar{\psi}_+ + \overline{W}''(\phi)\psi_-\bar{\psi}_- \Big). \tag{2.98}$$

The fields of the theory are a scalar field ϕ moving in a potential $|W'(\phi)|^2$ and two Dirac fermions[10] $\psi_\pm, \bar{\psi}_\pm$ which are subject to a Yukawa-type interaction $W''(\phi)\psi_+\psi_-$.

By construction, the action (2.98) is supersymmetric, i.e. invariant under the coordinate transformation (2.96). One can show [25] that these coordinate transformations are equivalent to the following transformation among the fields (before integrating out the auxiliary field F):

$$\begin{aligned}
\delta\phi &= \varepsilon_+\psi_+ + \varepsilon_-\psi_-, & \delta\bar{\phi} &= \bar{\varepsilon}_+\bar{\psi}_+ + \bar{\varepsilon}_-\bar{\psi}_-,\\
\delta\psi_+ &= 2\bar{\varepsilon}_+\partial\phi + \varepsilon_-F, & \delta\psi_- &= 2\bar{\varepsilon}_-\bar{\partial}\phi - \varepsilon_+F,\\
\delta\bar{\psi}_+ &= 2\varepsilon_+\partial\bar{\phi} - \bar{\varepsilon}_-\bar{F}, & \delta\bar{\psi}_- &= 2\varepsilon_-\bar{\partial}\bar{\phi} + \bar{\varepsilon}_+\bar{F},\\
\delta F &= 2\bar{\varepsilon}_-\bar{\partial}\psi_+ - 2\bar{\varepsilon}_+\partial\psi_-, & \delta\bar{F} &= 2\varepsilon_+\partial\bar{\psi}_- - 2\varepsilon_-\bar{\partial}\bar{\psi}_+.
\end{aligned} \tag{2.99}$$

Finally, the supercharges transform as spinors under the Euclidean Lorentz group action (2.90):

$$Q_\pm \mapsto e^{\mp i\alpha/2}Q_\pm, \qquad \bar{Q}_\pm \mapsto e^{\mp i\alpha/2}\bar{Q}_\pm \tag{2.100}$$

2.4.3.2. *Topological twisting.* Next to the supersymmetry (2.99), the action (2.98) admits a residual global U(1) symmetry

$$\phi \mapsto \phi, \qquad \psi_\pm \mapsto e^{\mp i\alpha}\psi_\pm, \qquad \bar{\psi}_\pm \mapsto e^{\pm i\alpha}\bar{\psi}_\pm. \tag{2.101}$$

This is the so-called *axial R-symmetry*. In certain cases, namely when the superpotential $W(\Phi) = c\Phi^k$ is a monomial, the action (2.98) admits a second global U(1) symmetry, called a *vector R-symmetry*. The corresponding field transformations are

$$\phi \mapsto e^{2i\alpha/k}\phi, \qquad \psi_\pm \mapsto e^{(2/k-1)i\alpha}\psi_\pm. \tag{2.102}$$

Remark 2.4.14. The transformations (2.101) express how the constituent fields of the chiral super field Φ transform under an axial R-rotation of the coordinates, c.f. Equation (2.94), when one assigns axial charge $q_A = 0$ to Φ. Likewise, the transformations (2.102) express how the constituent fields transform under vector R-rotations, when one assigns the vector charge $q_V = 2/k$ to Φ.

[10] By a *fermion* we mean a Grassmann odd spinor, i.e. a section $\psi \in \Gamma K^{1/2}$ of a square root of the canonical bundle over the worldsheet Σ which has odd parity.

The conserved currents of the two residual U(1) symmetries, which are collectively known as R-symmetries, are

$$\begin{aligned} J_A &= \bar{\psi}_+\psi_+, \\ \bar{J}_A &= -\bar{\psi}_-\psi_-, \\ J_V &= \tfrac{2}{k}(\phi\partial\bar{\phi} - \bar{\phi}\partial\phi) - \left(\tfrac{2}{k} - 1\right)\bar{\psi}_+\psi_+, \\ \bar{J}_V &= \tfrac{2}{k}(\phi\bar{\partial}\bar{\phi} - \bar{\phi}\bar{\partial}\phi) - \left(\tfrac{2}{k} - 1\right)\bar{\psi}_+\psi_+. \end{aligned} \tag{2.103}$$

Importantly, the supercharges (2.95) transform under the R-symmetries as follows:

$$\begin{aligned} \text{axial:} \quad & Q_\pm \mapsto e^{\mp i\alpha}Q_\pm, \quad \bar{Q}_\pm \mapsto e^{\pm i\alpha}\bar{Q}_\pm, \\ \text{vector:} \quad & Q_\pm \mapsto e^{-i\alpha}Q_\pm, \quad \bar{Q}_\pm \mapsto e^{i\alpha}\bar{Q}_\pm. \end{aligned} \tag{2.104}$$

The idea of a *topological twist* is to use the R-symmetry of a model to change the spins of the fields. This is done as follows: imagine a supersymmetric theory which admits an R-symmetry $\mathrm{U}(1)_R$ with generator R. As an example, one may take the model (2.98), or an even easier model, where the superpotential vanishes $W \equiv 0$. Let us consider the Euclidean rotation group $\mathrm{SO}(2) = \mathrm{U}(1)_E$ with generator M_E. Recall that it acts on the supercharges $Q_\pm, \bar{Q}_\pm$ by

$$[M_E, Q_\pm] = \mp\tfrac{1}{2}Q_\pm, \qquad [M_E, \bar{Q}_\pm] = \mp\tfrac{1}{2}\bar{Q}_\pm.$$

By *twisting* [16] one understands the replacement of the Euclidean rotation group $\mathrm{U}(1)_E$ by the diagonal subgroup $\mathrm{U}'(1)_E$ of $\mathrm{U}(1)_E + \mathrm{U}(1)_R$ generated by

$$M' = M + \tfrac{1}{2}R. \tag{2.105}$$

The twist is called an *A-twist* or *B-twist* if the R-symmetry is a vector or an axial R-symmetry respectively.

Remark 2.4.15. On a curved worldsheet, one has to gauge the new rotation group $\mathrm{U}'(1)_E$ using the spin connection.

Let us show, that this procedure indeed changes the spins of the (constituent) fields of the chiral superfield: let us consider a B-twist, i.e. we suppose that the R-symmetry used to perform the twist comes from an axial R-symmetry of the model. In this case, the chiral superfield Φ has R-charge $q_A = 0$. By (2.101), ϕ and $\psi_\pm$ have R-charge $q_A(\phi) = 0$ and $q_A(\psi_\pm) = \mp 1$. According to the consideration following (2.92), ϕ is a scalar and hence its Lorentz charge is $q_\Lambda(\phi) = 0$. On the other hand, $\psi_\pm$ is a spinor[11] and thus has Lorentz charge $q_\Lambda(\psi_\pm) = \mp 1/2$. Therefore, the charge of ϕ and $\psi_\pm$ under the new rotation group $\mathrm{U}'(1)_E$ is

$$\begin{aligned} q'(\phi) &= q_\Lambda(\phi) + \tfrac{1}{2}q_A(\phi) = 0, \\ q'(\psi_\pm) &= q_\Lambda(\psi_\pm) + \tfrac{1}{2}q_A(\psi_\pm) = \mp 1. \end{aligned} \tag{2.106}$$

We thus see that while $\psi_\pm$ was a spinor with respect to the Lorentz rotations $\mathrm{U}(1)_E$, after the twist, it becomes a one-form or a vector respectively.

[11] i.e. a section of the square root $\sqrt{K}$ of the canonical bundle K. If the worldsheet is flat, one can think of $\psi_\pm$ as an element $\psi_\pm(z)\sqrt{dz}$.

Note in particular that the twist does affect neither the action nor the equations of motion. But it changes the geometric nature of the fields. Notably, in two-dimensions, the geometric nature of fields are defined by their conformal weight, which is determined by the OPE of the field with the stress-energy tensor. Since the field itself hasn't changed, this suggest that the twist changes the stress-energy tensor. However, since the action hasn't changed, the stress-energy tensor can change merely by a total derivative. Indeed, suppose that the Noether current I^{tot} of the R-symmetry splits into a holomorphic and anti-holomorphic part:

$$I^{\text{tot}}(z) = I(z)dz + \bar{I}(z)d\bar{z}, \qquad \bar{\partial} I = \partial \bar{I} = 0.$$

Then, the topological twist is equivalently given by shifting the stress-energy tensor by the derivative of I^{tot}:

$$T \to T + \tfrac{1}{2}\partial I, \qquad \bar{T} \to \bar{T} + \tfrac{1}{2}\bar{\partial}\bar{I}.$$

Equivalently, [11]

$$L_0 \to L_0 + \tfrac{1}{2} I_0, \qquad \bar{L}_0 \to \bar{L}_0 + \tfrac{1}{2}\bar{I}_0.$$

Remark 2.4.16. More generally, one can consider twists

$$T \to T + \kappa\partial I, \qquad \bar{T} \to \bar{T} + \kappa\bar{\partial}\bar{I},$$

where κ is a constant which determines the shift of the conformal weights of the fields.

This view point on topological twisting will be adopted in Chapter 4 to establish a link between the gauged fixed abelian BF theory and Witten's B-model.

2.4.3.3. *Twisting two-dimensional Yang-Mills theory.* As an example of how the twisting procedure works in detail and how it can change the very nature of the theory, we will consider two-dimensional Yang-Mills theory. Of course, two-dimensional Yang-Mills theory is not conformally invariant. However, we will show that after twisting the model with respect to a global U(1) symmetry (which plays the role of the aforementioned R-symmetries), we can obtain the complex free scalar CFT. More interestingly, as an intermediate result, we show that two-dimensional Yang-Mills theory twists to a gauge fixed abelian BF model. The gauge-fixing differs slightly from the pure Lorenz gauge fixing considered in Chapter 4 and 5.

We start from the abelian Yang-Mills action functional in two-dimensions: Let Σ be a Riemann surface endowed with a metric. The metric defines a Hodge star operator, which we denote by $*$. The action functional is given by

$$S_{YM} = \frac{1}{2\varepsilon}\int_\Sigma F * F. \tag{2.107}$$

Here, $F = d\omega$ denotes the curvature of an abelian connection ω over a trivial U(1) bundle on Σ. Via the equations of motion, the above action functional is equivalent to

$$S_0 = \int_\Sigma \beta F - \frac{\varepsilon}{2}\beta * \beta \tag{2.108}$$

whose equation of motion for β is

$$F = *\beta.$$

The model (2.108) admits a gauge symmetry

$$\delta\omega = d\alpha, \qquad \alpha \in \Omega^0_\Sigma(\mathfrak{g})$$

which we fix via the BRST procedure. The BRST operator acts as

$$Q\colon \qquad \omega \mapsto dc, \qquad b \mapsto \lambda. \tag{2.109}$$

Let us chose the following gauge fixing fermion

$$\Psi = \int_\Sigma bd * \omega - \frac{\varepsilon}{2} b * \lambda. \tag{2.110}$$

The gauge fixed action then reads

$$S = S_0 + Q\Psi = \int_\Sigma \beta F + \lambda d * \omega - \frac{\varepsilon}{2}(\beta * \beta + \lambda * \lambda) + bd * dc. \tag{2.111}$$

It will be convenient to express the action in terms of complex coordinates fields. Let z be a local complex coordinate on Σ and define complex valued fields $\phi, \bar{\phi}, p, \bar{p}$ by

$$\omega = dz\phi + d\bar{z}\bar{\phi}, \qquad p = \tfrac{1}{2}(\lambda + i\beta), \qquad \bar{p} = \tfrac{1}{2}(\lambda - i\beta). \tag{2.112}$$

In terms of the complex fields the action (2.111) becomes

$$S = 4\int_\Sigma \left(p\bar{\partial}\phi + \bar{p}\partial\bar{\phi} - \frac{\varepsilon p\bar{p}}{2} + b\partial\bar{\partial}c\right) d^2x, \tag{2.113}$$

where d^2x denotes the real measure on Σ. The corresponding equations of motion are

$$\begin{aligned} \bar{\partial}\phi - \varepsilon\bar{p}/2 &= 0, & \partial\bar{\phi} - \varepsilon p/2 &= 0, \\ \bar{\partial}p &= 0, & \partial\bar{p} &= 0, \\ \bar{\partial}\partial b, &= 0 & \bar{\partial}\partial c &= 0. \end{aligned} \tag{2.114}$$

The stress-energy tensor $T_{\alpha\beta}$ of the gauged fixed model is defined by varying the action (2.111) with respect to the metric:

$$\delta_g S = -\int \sqrt{g} d^2x\, T_{\alpha\beta}\delta g^{\alpha\beta}.$$

Using the identities

$$\delta_g(g^{\mu\nu}\sqrt{g}) = \left(g^\mu_{\ (\alpha}g^\nu_{\ \beta)} - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\right)\delta g^{\alpha\beta}, \qquad \delta_g(\sqrt{g}) = -\frac{1}{2}g_{\alpha\beta}\delta g^{\alpha\beta}$$

one finds

$$T_{\alpha\beta} = QG_{\alpha\beta} - \frac{\varepsilon}{4}g_{\alpha\beta}(\beta^2 + \lambda^2), \qquad G_{\alpha\beta} = \omega_{(\alpha}\partial_{\beta)}b - \frac{1}{2}g_{\alpha\beta}\omega_\mu\partial^\mu b.$$

In particular, $T_{\alpha\beta}$ is almost Q-exact. However, we will see that we can "twist" the theory in such a way, that the stress-energy tensor becomes indeed Q-exact. Let us define

$$T \equiv T_{zz} = \phi\partial\lambda + \partial b\partial c, \qquad \bar{T} \equiv T_{\bar{z}\bar{z}} = \bar{\phi}\bar{\partial}\lambda + \bar{\partial}b\bar{\partial}c, \qquad T_{z\bar{z}} = -\frac{\varepsilon p\bar{p}}{2}.$$

Note that the conservation law

$$\bar{\partial}T + \partial T_{z\bar{z}} \sim 0. \tag{2.115}$$

is indeed satisfied. Here and in the following, $\sim$ is treated as a synonym for "equal modulo the equations of motion".

In order to see how the fundamental fields ϕ, p and their complex conjugates change their geometric interpretation after twisting the theory, we first remark that ϕ is the coefficient function of a one form, while p is a scalar field. Hence, their conformal weights (which can easily be verified by a direct calculation) are

$$h(\phi) = 1, \qquad h(p) = 0. \tag{2.116}$$

As necessary for the twisting procedure, the model (2.113) admits a global $U(1)$ symmetry, under which the fields transform according to

$$\delta p = e^{i\theta} p,\ \delta\phi = e^{-i\theta}\phi, \qquad \delta\bar{p} = e^{-i\theta}\bar{p},\ \delta\bar{\phi} = e^{i\theta}\bar{\phi}. \tag{2.117}$$

The associated Noether current is given by

$$I^{\text{tot}} = dzI + d\bar{z}\bar{I} \qquad I = p\phi, \quad \bar{I} = \bar{p}\bar{\phi}. \tag{2.118}$$

Importantly, the current is only totally conserved

$$dI^{\text{tot}} = 0, \qquad \bar{\partial} I \neq 0, \partial\bar{I} \neq 0, \tag{2.119}$$

i.e. it does not split into a holomorphic and anti-holomorphic part. While this is different from the situation we outlined in the previous section, it is still possible to define a (full) twist by

$$\begin{aligned} \tilde{T} &= T - \partial I = \partial b \partial c - p\partial\phi, \\ \tilde{T}_{z\bar{z}} &= T_{z\bar{z}} + \bar{\partial} I \sim 0 \end{aligned} \tag{2.120}$$

In particular, twisting the $T_{z\bar{z}}$ component of the stress-energy tensor is necessary to satisfy conservation law: if $\tilde{T}_{z\bar{z}} = T_{z\bar{z}} + A$ then using (2.115) one has

$$\bar{\partial}\tilde{T}_{zz} + \partial\tilde{T}_{z\bar{z}} = -\bar{\partial}\partial I + \partial A = 0.$$

We conclude that $A = \bar{\partial} I$ up to a anti-holomorphic function. Now,

$$\bar{\partial}\tilde{T}_{zz} \propto \bar{\partial}(p\partial\phi) \sim 0.$$

and

$$\tilde{T}_{z\bar{z}} = T_{z\bar{z}} + \bar{\partial} I \sim p\left(-\frac{\varepsilon\bar{p}}{2} + \bar{\partial}\phi\right) \sim 0.$$

Therefore, $\tilde{T}^{\text{tot}}$ is traceless and $\tilde{T}$ holomorphic

$$\bar{\partial}\tilde{T} \propto \bar{\partial}\partial\phi \sim 0. \tag{2.121}$$

Since the stress-energy tensor has changed, so did the conformal weights of ϕ and p. A direct calculation yields

$$\tilde{h}(\phi) = 0, \qquad \tilde{h}(p) = 1. \tag{2.122}$$

The fact that the new stress-energy tensor is traceless, $\tilde{T}_{z\bar{z}} = 0$, and holomorphic, $\bar{\partial}\tilde{T} = 0$, suggests that the twisted model is a CFT. Further evidence comes from the $\tilde{T}\tilde{T}$ OPE which takes the from of

$$\tilde{T}_z\tilde{T}_w = \frac{2\tilde{T}_w}{(z-w)^2} + \frac{\partial\tilde{T}}{z-w}. \tag{2.123}$$

Notably, the central charge of the CFT vanishes.

In fact, after integrating out $p, \bar{p}$ in the path integral governed by the gauged fixed action (2.111), the theory defined by $\tilde{T}$ is realized to be nothing else than the complex free scalar theory plus a second order ghost system:

$$S \to S = \frac{1}{\varepsilon} \int d^2x \, \left(\partial\bar{\phi}\bar{\partial}\phi + b\partial\bar{\partial}c\right).$$

This action is indeed conformally invariant and the central charge of the model is zero: The contributions from the scalar field sector and the ghost sector cancel each other.

It is instructive to express the action (2.111) in terms of real fields. Let $\omega = \xi dx + \eta dy$, then

$$\begin{aligned}\int \beta d\omega + \lambda d * \omega &= \int (-d\beta\omega - d\lambda * \omega)\\ &= \int -d\beta(\xi dx + \eta dy) - d\lambda(\xi dy - \eta dx)\\ &= \int \xi d(\beta dx + \lambda dy) + \eta d * (\beta dx + \lambda dy)\end{aligned}$$

Setting $\pi = \beta dx + \lambda dy$, the action of the twisted theory, $\tilde{S}$, can be written as

$$\begin{aligned}\tilde{S} &= \int_\Sigma \beta d\omega + \lambda d * \omega + \frac{\varepsilon}{2}(\beta * \beta + \lambda * \lambda)\\ &= \int_\Sigma \xi d\pi + \eta d * \pi + bd * dc + \frac{\varepsilon}{2}\pi * \pi\end{aligned} \tag{2.124}$$

Thus, in the twisted theory, the (real) components of ω play the role of the Lagrangian multipliers. In particular, the action (2.124) can be seen as a deformation of the abelian BF theory in Lorenz gauge

$$S_{\text{ab BF}} = \int_\Sigma BdA + \lambda d * A + bd * dc + \frac{\varepsilon}{2}A * A,$$

which we will study in-depth in Chapter 4.

Remark 2.4.17. The Lagrangian multipliers ξ and η enter the action (2.124) in a symmetric way, if at the same time one sends $\pi \to *\pi$.

Remark 2.4.18. Even if the twisted theory is a CFT, it is not a TCFT, since its stress-energy tensor is not Q-exact.

To end this section, we will sketch the construction of vertex operators. Let us fix $\Sigma = \mathbb{C}$. We define correlation functions perturbatively by a Feynman diagram expansion of

$$\langle \mathcal{O}_{i_1}(z_1) \dots \mathcal{O}_{i_n}(z_n) \rangle = \int \mathcal{O}_{i_1}(z_1) \dots \mathcal{O}_{i_n}(z_n) e^{-\tilde{S}/4\pi}. \tag{2.125}$$

From (2.113), one derives the propagators:

$$\langle \phi_z p_w \rangle = \frac{1}{z - w}, \qquad \langle c_z b_w \rangle = \log|z - w|^2. \tag{2.126}$$

Due to the interaction term proportional to $p\bar{p}$, however, one finds in addition a non-trivial propagator between ϕ and $\bar{\phi}$:

$$\begin{aligned}\langle \phi_z \bar{\phi}_w \rangle &= \left\langle \phi_z \bar{\phi}_w \frac{\varepsilon}{2\pi} \int d^2u \, p\bar{p} \right\rangle_0 \\ &= \frac{\varepsilon}{2\pi} \int \frac{d^2u}{(u-z)(\bar{w}-\bar{u})} = -\varepsilon \log|z-w| + C\end{aligned} \tag{2.127}$$

where the constant C is proportional to the logarithm of an infrared cutoff ρ: $C \sim \log \rho$.The propagators give the following OPEs between fundamental fields:

$$\phi_z p_w \sim (z-w)^{-1}, \qquad \phi_z \bar{\phi}_w \sim -\varepsilon \log|z-w|.$$

Vertex operators can now be defined as exponential of the scalar fields:

$$V_{\alpha,\beta} = \exp(\alpha\phi + \beta\bar{\phi}).$$

Indeed, by a straight forward calculation, their OPE with the stress-energy tensor can be computed:

$$\tilde{T}(z) V_{\alpha,\beta}(w) \sim \frac{-\frac{\varepsilon\alpha\beta}{2} V_{\alpha,\beta}}{(z-w)^2} + \frac{\partial V_{\alpha,\beta}}{z-w}$$

where in the calculation on uses the equation of motion $\partial\bar{\phi} = \varepsilon p/2$. Hence, $V_{\alpha,\beta}$ are primaries of weight $-\varepsilon\alpha\beta/2$.

2.4.3.4. *Witten's A- and B-model.* In Section 2.4.3.3 we have seen an example of how twisting can produce a CFT from a non-CFT. However, the twisted CFT was not a TCFT in the sense of Definition 2.4.1. In the following we want to present the arguably two most important examples of how twisting a $\mathcal{N} = (2,2)$ supersymmetric theory produces a TCFT. The construction is originally due to Witten [35, 40]. Nowadays, the construction is well-known and there are many excellent reviews of the topic, see e.g. [11, 16, 24].

The construction starts by a slight generalization of the supersymmetric model (2.97) of one chiral superfield. To start, let us recall the defining action of the Example (2.97):

$$S(\Phi) = \int d^2x d^4\theta \, \Phi\bar{\Phi} + \int d^2x d^2\theta \, W(\Phi) + c.c. = S_{kin},$$

where we have set the superpotential $W(\Phi)$ to zero. Notice that the chiral superfield can be seen as a map from superspace to $\mathbb{C}$. More generally, we can easily define a multi-component version of (2.97) by passing from a scalar superfield Φ to a vector-valued superfield with components Φ^i. We can modify the kinetic part of the action by specifying a function K and replace the naive product $\Phi\bar{\Phi}$ by a more general term $K(\Phi, \bar{\Phi})$:

$$S_{\text{kin}}(\Phi) = \int d^2x d^4\theta \, K_{i\bar{\jmath}} \Phi^i \bar{\Phi}^{\bar{\jmath}}. \tag{2.128}$$

We may view $\mathbb{C}$ as a local chart of a more general curved manifold, i.e. we think of the superfield Φ as a map from superspace to a curved target manifold X.

As it turns out, it is possible to glue these local charts together: consider the case where the target manifold X is a Kähler manifold with Kähler with coordinates $\phi^i, \bar{\phi}^{\bar{\jmath}}$. The Kähler metric g is a $(1,1)$ symmetric tensor whose non-vanishing components we denote by $g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K(\phi, \bar{\phi})$. The function $K(\phi, \bar{\phi})$ is known as a Kähler potential. The only non-vanishing Christoffel symbols are $\Gamma^i_{jk} = g^{i\bar{\ell}} \partial_j g_{k\bar{\ell}}$ and its complex conjugate. This leads to a Riemann tensor of the form $R^i_{jk\bar{\ell}} = -\partial_{\bar{\ell}} \Gamma^i_{jk}$. We now choose K in (2.128) to be the Kähler potential of the target manifold X, and expand it in terms of the constituent fields ϕ^i, ψ^i, F^i. After the integration over $d^4\theta$ and integrating out F^i which again enters as an auxiliary field, i.e. as a non-dynamical field, we are left with (we follow the notation and conventions[12] of [40])

$$S_{\text{kin}}(\phi, \psi, F) = \int d^2x \Big(g_{i\bar{\jmath}} \partial^\mu \phi^i \partial_\mu \bar{\phi}^{\bar{\jmath}} + i g_{i\bar{\jmath}} \bar{\psi}^{\bar{\jmath}}_- D_z \psi^i_- + i g_{i\bar{\jmath}} \bar{\psi}^{\bar{\jmath}}_+ D_{\bar{z}} \psi^i_+ + R_{i\bar{\jmath}k\bar{\ell}} \psi^i_+ \psi^k_- \bar{\psi}^{\bar{\jmath}}_- \bar{\psi}^{\bar{\ell}}_+ \Big). \tag{2.129}$$

Here we defined the covariant derivatives

$$D_z \psi^i_\pm := \partial_z \psi^i_\pm + \partial_z \phi^j \Gamma^i_{jk} \psi^k_\pm, \qquad D_{\bar{z}} \psi^i_\pm := \partial_{\bar{z}} \psi^i_\pm + \partial_{\bar{z}} \phi^j \Gamma^i_{jk} \psi^k_\pm.$$

REMARK 2.4.19. The action (2.129) is known as a *Landau-Ginzburg model* with zero superpotential.

The action (2.129) is covariant under holomorphic changes of local coordinates z^i and under so-called *Kähler transformations*

$$K(\Phi, \bar{\Phi}) \to K(\Phi, \bar{\Phi}) + f(\Phi) + \bar{f}(\bar{\Phi})$$

where f is any holomorphic and $\bar{f}$ any anti-holomorphic function [16]. The invariance under these transformations ensures that the local charts patch up correctly such that the model is well-defined over Σ and X.

Before we recall the supersymmetry transformations of the model, let us briefly point out the geometric natures of the fields involved. The complex scalar field $\phi = (\phi^i, \bar{\phi}^{\bar{\imath}})$ plays the role of a local parametrization of the worldsheet Σ, embedded in the target manifold X. The fermions $\psi^i_\pm, \bar{\psi}^{\bar{\imath}}_\pm$ are sections of the square root $K^{1/2}$ respectively $\bar{K}^{1/2}$ of the canonical and anti-canonical bundles over Σ valued in the pullback $\phi^*(T^{1,0}X)$ (respectively $\phi^*(T^{0,1}X)$) of the tangent bundle of the target manifold. Now, the supersymmetry transformation in questions are given by [40]

$$\begin{aligned}
\delta\phi^i &= i\varepsilon_- \psi^i_+ + i\varepsilon_+ \psi^i_-, & \delta\phi^{\bar{\imath}} &= i\tilde{\varepsilon}_- \bar{\psi}^{\bar{\imath}}_+ + i\tilde{\varepsilon}_+ \bar{\psi}^{\bar{\imath}}_-, \\
\delta\psi^i_+ &= -\tilde{\varepsilon}_- \partial_z \phi^i - i\varepsilon_+ \psi^j_- \Gamma^i_{jk} \psi^k_+, & \delta\bar{\psi}^{\bar{\imath}}_+ &= -\varepsilon_- \partial_z \bar{\phi}^{\bar{\imath}} - i\tilde{\varepsilon}_+ \bar{\psi}^{\bar{\jmath}} \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} \bar{\psi}^{\bar{k}}_+, \\
\delta\psi^i_- &= -\tilde{\varepsilon}_+ \partial_{\bar{z}} \phi^i - i\varepsilon_- \psi^j_+ \Gamma^i_{jk} \psi^k_-, & \delta\bar{\psi}^{\bar{\imath}}_- &= -\varepsilon_+ \partial_{\bar{z}} \bar{\phi}^{\bar{\imath}} - i\tilde{\varepsilon}_- \bar{\psi}^{\bar{\jmath}}_+ \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} \bar{\psi}^{\bar{k}}_+.
\end{aligned} \tag{2.130}$$

Note that in general, the parameters $\varepsilon, \tilde{\varepsilon}$ are sections of certain line bundles. Indeed, the supersymmetry transformation changes only the statistic of the fields (bosons get send to fermions and vice versa). It does not change the geometric nature. For example, the bosonic scalar field ϕ^i is send to a fermionic scalar field $\delta\phi^i = i\varepsilon_- \psi^i_+ + i\varepsilon_+ \psi^i_-$. But since ψ^i_+ is a section of $K^{1/2} \otimes \phi^*(T^{1,0}X)$, the parameter ε_- must be a section of $K^{-1/2}$. Similar considerations apply to the other parameters $\varepsilon_+, \tilde{\varepsilon}_+$ and $\tilde{\varepsilon}_-$.

[12] In particular, regarding factors of i.

The action (2.129) has several global U(1) symmetries, among them

$$\text{(2.131)} \qquad \mathrm{U}(1)_A\colon \begin{cases} \psi^i_+ \mapsto e^{-i\alpha}\psi^i_+, & \bar{\psi}^{\bar{\imath}}_+ \mapsto e^{i\alpha}\bar{\psi}^{\bar{\imath}}_+, \\ \psi^i_- \mapsto e^{i\alpha}\psi^i_-, & \bar{\psi}^{\bar{\imath}}_- \mapsto e^{-i\beta}\bar{\psi}^{\bar{\imath}}_-, \end{cases}$$

and

$$\text{(2.132)} \qquad \mathrm{U}(1)_B\colon \begin{cases} \psi^i_+ \mapsto e^{i\beta}\psi^i_+, & \bar{\psi}^{\bar{\imath}}_+ \mapsto e^{-i\beta}\bar{\psi}^{\bar{\imath}}_+, \\ \psi^i_- \mapsto e^{i\beta}\psi^i_-, & \bar{\psi}^{\bar{\imath}}_- \mapsto e^{-i\beta}\bar{\psi}^{\bar{\imath}}_-. \end{cases}$$

The corresponding charges of $\psi^i_\pm, \bar{\psi}^{\bar{\imath}}_\pm$ are given in Table 1.

	ψ^i_+	$\bar{\psi}^{\bar{\imath}}_+$	ψ^i_-	$\bar{\psi}^{\bar{\imath}}_-$
q_A	-1	1	1	-1
q_B	1	-1	1	-1

Table 1. Charges corresponding to the R-symmetries (2.132).

Let us denote the generators of the R-symmetries $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_B$ by R_A and R_B respectively. Following the discussion in Section 2.4.3.1, one defines the A and B-twisted theory by considering the diagonal subgroup $\mathrm{U}'(1)_A$ of $\mathrm{U}(1)_E + \mathrm{U}(1)_A$ or $\mathrm{U}'(1)_B$ of $\mathrm{U}(1)_E + \mathrm{U}(1)_B$ as the new global rotation group of the theory. These groups are generated by

$$M_A = M + \tfrac{1}{2}R_A,$$
$$M_B = M + \tfrac{1}{2}R_B,$$

where M denotes the generator of the Euclidean Lorentz group $\mathrm{U}(1)_E$. The A-twisted theory is known as the A-model, the A-twisted theory as B-model [35, 40]. Under the new rotation groups, the fermions change their conformal weights according to Table 2. In particular, we have the following geometric

	ψ^i_+	$\bar{\psi}^{\bar{\imath}}_+$	ψ^i_-	$\bar{\psi}^{\bar{\imath}}_-$
$(h_A, \bar{h}_A)$	(0,0)	(1,0)	(0,1)	(0,0)
$(h_B, \bar{h}_B)$	(1,0)	(0,0)	(0,1)	(0,0)

Table 2. Charges corresponding to the R-symmetries (2.132).

interpretation of the fields:

$$\text{A-model:} \begin{cases} \psi^i_+ \in \phi^*(T^{1,0}X), & \bar{\psi}^{\bar{\imath}}_- \in \phi^*(T^{0,1}X) \\ \bar{\psi}^{\bar{\imath}}_+ \in K \otimes \phi^*(T^{0,1}X), & \psi^i_- \in \bar{K} \otimes \phi^*(T^{1,0}X) \end{cases}$$

$$\text{B-model:} \begin{cases} \psi^i_+ \in K \otimes \phi^*(T^{1,0}X), & \bar{\psi}^{\bar{\imath}}_- \in \phi^*(T^{0,1}X) \\ \bar{\psi}^{\bar{\imath}}_+ \in \phi^*(T^{0,1}X), & \psi^i_- \in \bar{K} \otimes \phi^*(T^{1,0}X) \end{cases}$$

Therefore, after twisting the fermions are either functions or one-forms.

Remark 2.4.20. Even if the geometric nature of the fields have changed, the $\psi_\pm, \bar{\psi}_\pm$ are still Grassmann odd variables.

The A-model. The new geometric interpretation of the fields also changes the geometric nature of the coefficients $\varepsilon_\pm, \tilde{\varepsilon}_\pm$ of (2.130). For example, in the A-model, ε_- and $\tilde{\varepsilon}_+$ become functions, whereas ε_+ and $\tilde{\varepsilon}_-$ become sections of $\bar{K}^{-1}$ and K^{-1} respectively [40]. Choosing ε_-, $\tilde{\varepsilon}_+$ to be constants and ε_+, $\tilde{\varepsilon}_-$ to vanish, yields a nilpotent fermionic symmetry Q: For simplicity of notation, let

$$\begin{array}{lll} \text{(functions)} & \chi^i = \psi^i_+, & \bar{\chi}^{\bar{\imath}} = \bar{\psi}^{\bar{\imath}}_-, \\ \text{(one-forms)} & \bar{\psi}^{\bar{\imath}}_z = \bar{\psi}^{\imath}_+, & \psi^i_{\bar{z}} = \psi^i_-. \end{array}$$

and define Q_A by $\delta_A \Phi = -i\varepsilon\{Q, \phi\}$, where $\delta_A\Phi$ denotes any of the transformation (2.130) after performing the A-twist and setting $\varepsilon = \varepsilon_+ = \tilde{\varepsilon}_-$. Then

$$\begin{aligned} &\{Q_A, \phi^i\} = -\chi^i, && \{Q_A, \bar{\phi}^{\bar{\imath}}\} = -\bar{\chi}^{\bar{\imath}} \\ &\{Q_A, \chi^i\} = 0, && \{Q_A, \bar{\chi}^{\bar{\imath}}\} = 0 \\ &\{Q_A, \bar{\psi}^{\imath}_z\} = -i\partial_z \bar{\phi}^{\bar{\imath}} + \bar{\chi}^{\bar{\jmath}} \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} \bar{\psi}^{\bar{k}}_z, && \{Q_A, \psi^i_{\bar{z}}\} = -i\partial_{\bar{z}} \phi^i + \chi^j \Gamma^i_{jk} \psi^k_{\bar{z}}. \end{aligned} \tag{2.133}$$

It turns out that Q_A is nilpotent, $\{Q_A, Q_A\} = 0$, only modulo the equations of motion.

Remark 2.4.21. By adding auxiliary fields, it is possible to extend Q_A to an honest nilpotent operator, i.e. such that $\{Q_A, Q_A\} = 0$ holds also off-shell [35].

Now, modulo terms which vanish after imposing the equations of motion, the action splits into a topological term and a term which is Q-exact [35, 40]:

$$\begin{aligned} S &= i\{Q, V\} + \int_\Sigma \phi^* \mathcal{K}, \\ V &= \int_\Sigma d^2x \; g_{i\bar{\jmath}} \left(\bar{\psi}^{\bar{\jmath}}_z \partial_{\bar{z}} \phi^i + \partial_z \bar{\phi}^{\bar{\jmath}} \psi^i_{\bar{z}} \right), \\ \mathcal{K} &= g_{i\bar{\jmath}} dz^i d\bar{z}^{\bar{\jmath}}. \end{aligned} \tag{2.134}$$

Here, $\mathcal{K}$ is the Kähler form of the target manifold X. To see that the A-model is indeed a TCFT in the sense of Definition 2.4.1, notice first that we can write the primitive V purely in terms of the Dolbeault operators $\partial, \bar{\partial}$ and the one-forms $\bar{\psi} = \bar{\psi}_z dz$ and $\psi = \psi_{\bar{z}} d\bar{z}$:

$$V = \int_\Sigma g(\bar{\psi}, \bar{\partial}\phi) + g(\partial\bar{\phi}, \psi)$$

From this expression the conformal invariance of the action (2.134) is clear. Moreover, $\int \phi^* \mathcal{K}$ depends only on the homotopy class of ϕ and the cohomology class of $\mathcal{K}$ and is therefore independent of the worldsheet metric, c.f. [40]. Thus, the only place where the metric enters the action explicitly, namely in form of a complex structure on Σ needed to define the Dolbeault operators, is in the primitive V. One therefore easily sees that the stress-energy tensor is Q_A-exact: If we define the stress-energy tensor by

$$\delta_g S = -\int_\Sigma d^2x \; T_{\mu\nu} \delta g^{\mu\nu}, \tag{2.135}$$

we have

$$T_{\mu\nu} = \{Q, G_{\mu\nu}\}$$

where the primitive $G_{\mu\nu}$ is analogously defined by

$$\delta_g V = -\int_\Sigma d^2x\, G_{\mu\nu}\delta g^{\mu\nu}. \tag{2.136}$$

The B-model. Considering a B-twist, this time one can set $\varepsilon_\pm$, which are now sections of $\bar{K}^{-1}$ and K^{-1} respectively, to zero and the functions $\tilde{\varepsilon}_\pm$ to a constant, say $\tilde{\varepsilon}_+ = \tilde{\varepsilon}_- = \varepsilon$. It is customary to define

$$\eta^{\bar{\imath}} = \bar{\psi}^{\bar{\imath}}_+ + \bar{\psi}^{\bar{\imath}}_-, \qquad \theta_i = g_{i\bar{\jmath}}\left(\bar{\psi}^{\bar{\jmath}}_+ - \bar{\psi}^{\bar{\jmath}}_-\right),$$

as well as the one-form $\rho \in \Omega^1(\Sigma, \phi^*(T^{1,0}X))$ whose components are given by

$$\rho^i_z = \psi^i_+, \qquad \rho^i_{\bar{z}} = \psi^i_-.$$

In the notation above, the fermionic symmetry of the B-model is generated by the nilpotent operator Q_B which acts by

$$\begin{aligned} &\{Q_B, \phi^i\} = 0, && \{Q_B, \bar{\phi}^{\bar{\imath}}\} = -\eta^{\bar{\imath}} \\ &\{Q_B, \eta^{\bar{\imath}}\} = 0, && \{Q_B, \theta_i\} = 0 \\ &\{Q_B, \rho^i_z\} = -i\partial_z\bar{\phi}^i, && \{Q_B, \rho^i_{\bar{z}}\} = -i\partial_{\bar{z}}\phi^i. \end{aligned} \tag{2.137}$$

Again, Q_B squares to zero, $\{Q_B, Q_B\} = 0$, modulo the equations of motion.

As for the A-model (c.f. Equation (2.134)) we can write the action of the B-model as a topological plus a Q_B-exact term [35, 40]:

$$\begin{aligned} S &= i\{Q, V\} + \int_\Sigma W, \\ V &= \int_\Sigma d^2x\, g_{i\bar{\jmath}}\left(\rho^i_z\partial_{\bar{z}}\bar{\phi}^{\bar{\jmath}} + \rho^i_{\bar{z}}\partial_z\bar{\phi}^{\bar{\jmath}}\right), \\ W &= -\theta_i D\rho^i - \tfrac{i}{2}R_{i\bar{\imath}j\bar{\jmath}}\rho^i\rho^j\eta^{\bar{\imath}}\theta_k g^{k\bar{\jmath}}. \end{aligned} \tag{2.138}$$

Here, D denotes the exterior derivative on the worldsheet which one has to extend (by using the pullback of the Levi-Civita connection on the target manifold X) in order to act on forms valued in $\phi^*(T^{1,0}X)$.

Let us rewrite V as follows:

$$V = \int_\Sigma g(\rho, *d\bar{\phi}),$$

where $*$ stands for the Hodge star operator on Σ. Since in two dimensions $*$ is conformally invariant when acting on one-forms, V is conformally invariant. Furthermore, noticing that W is written completely in terms of differential forms, one easily sees that the action (2.138) is conformally invariant and hence defines a CFT. It is less obvious, however, that the B-model actually defines a TCFT, i.e. that the stress-energy tensor is Q_B-exact. In the special case where the target manifold X is flat, thus $R_{i\bar{\imath}j\bar{\jmath}} = 0$ and $D = d$, the only place where the metric enters is in the primitive V. The stress-energy tensor, as defined in (2.135) is therefore Q_B-exact, and its primitive is given by the variation of V with respect to the metric, c.f. (2.136).

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CHAPTER 3

Description of results

This thesis is devoted to the study of a new mechanism of constructing TCFTs, namely by gauge fixing topological gauge theories. The example studied in this thesis is two-dimensional BF theory. The gauge freedom will be fixed by imposing the Lorenz gauge condition $d * A = 0$ via the BRST formalism. We will therefore pick a metric on the worldsheet, which in the following we will always take to be either the complex plane $\mathbb{C}$ or the Riemann sphere $\mathbb{C}P^1$. The metric dependence of the theory enters only in the gauge fixing condition, which in the BRST formalism enters by a Q-exact term, where Q is the nilpotent BRST operator. It is therefore to be expected, and indeed verified in the next two chapters, that the stress-energy tensor is Q-exact. Furthermore, we prove that the theory is a CFT, both on the classical and on the quantum level. Therefore, we will proof that the Lorenz gauge fixed two-dimensional BF theory is indeed a TCFT.

We shall now give an extended summary of the results.

3.1. Abelian BF theory

In Chapter 4 we study thoroughly the abelian model. We start with the discussion of the classical aspects of two-dimensional BF theory and introduce the Lorenz gauge fixing condition via the BRST formalism. It is then realized, that the gauged fixed action is the sum of three free CFTs: two copies of a $\beta\gamma$-system (one chiral and one anti-chiral copy) and a second order ghost system. For the quantization, we restrict ourselves to the complex plane. After a careful discussion of the notion of *composite fields*, it is shown that the Q-cohomology, which we term *space of observables*, is given by polynomial functions of the B field and the ghost c.[1] Geometrically, one can interpret the Q-cohomology as the space of polyvector fields on an odd vector space coordinatized by the ghost fields c. It is later shown that the space of observables admits an action of the homology of the framed little disk operad, which endows it with the structure of a BV algebra. The B field and the ghost c are shown to be conjugate variables with respect to the BV bracket. In fact, this BV algebra is recognized to be the standard BV algebra of the space of polyvector fields. This is an instant of a more general result by E. Getzler, which states that the Q-cohomology of any TCFT admits a BV algebra structure stemming from an action of the homology of the framed little disk operad. In the case at hand, the construction of the

[1]Fields of two-dimensional abelian BF theory take values in an abelian Lie algebra (and its dual), known as the parameter space. We often consider the parameter space to be simply the real line $\mathbb{R}$.

bracket and the BV Laplacian is explicitly given and uses intrinsically the mode operators of the primitive of the stress-energy tensor.

Furthermore, the descent equations are solved explicitly, by defining a differential operator acting on the space of composite fields Γ, which we call the *descent operator*. It has the property that its commutator with the BRST differential Q is the exterior derivative of the worldsheet. For any observable $\mathcal{O}$, its descents are defined by

$$\mathcal{O}^{(1)} := \Gamma\mathcal{O}, \qquad \mathcal{O}^{(2)} := \frac{1}{2}\Gamma^2\mathcal{O}.$$

The descents $\mathcal{O}^{(k)}$ have a nice interpretation in the AKSZ[2] formulation of BF theory. To any observable $\mathcal{O} \in H_Q$, let us denote its total descent by

$$\mathcal{O}^\bullet = \mathcal{O}^{(0)} + \mathcal{O}^{(1)} + \mathcal{O}^{(2)},$$

which is an inhomogeneous differential form on the worldsheet. The observation is that the total descent coincides with the corresponding AKSZ superfield restricted to the gauge fixing Lagrangian $\mathcal{L}$ corresponding to the Lorenz gauge. For example, one finds

$$c^\bullet = c + A = \mathcal{A}|_\mathcal{L}, \qquad B^\bullet = B - *db = \mathcal{B}|_\mathcal{L},$$

where c denotes the ghost and b the anti-ghost.

Having solved the descent equations, applications of the general theory of TCFTs are considered. In particular, a toy model of a Gromov-Witten-type theory is constructed: from the general theory we know that correlation functions of the form

$$\langle G_{\text{tot}}(w_1)\dots G_{\text{tot}}(w_p)\Phi_1(z_1)\dots\Phi_n(z_n)\rangle \tag{3.1}$$

define closed p-forms on the moduli space of of conformal structures $\mathcal{M}_{g,n}$ of a genus g surface Σ with n punctures. Since the forms (3.1) are closed, one can study their periods, namely their integration along p-cycles of $M_{g,n}$. These integrals are an analogue of Gromov-Witten periods which one encounters in the study of Witten's A-model.

In Chapter 4, we considering the easiest non-trivial example, $\mathcal{M}_{0,4}$. It is shown that

$$\rho = \langle\Gamma\left(c(z_0)B(z_1)\Theta(z_2)c(z_3)\right)\rangle_{\mathbb{C}P^1} = 2d\arg(z_0,z_3;z_1,z_2),$$

where $(z_0,z_3;z_1,z_2) = (z_0-z_1)(z_3-z_2)(z_0-z_2)^{-1}(z_3-z_2)^{-1}$ is the cross ratio of four points in $\mathbb{C}P^1$. Here, $\Theta = b\delta(\gamma)\delta(\bar\gamma)$ is a field which fixes the zero-modes for b,γ and $\bar\gamma$. The problem of zero-modes and how to pass from the model defined on the complex plane to the model defined on the Riemann sphere $\mathbb{C}P^1$ is discussed in Chapter 5.

Clearly,

$$\rho \in \Omega^1_{\text{cl}}(\text{Conf}_4(\mathbb{C}P^1)^{\text{PSL}_2(\mathbb{C})} = \Omega^1_{\text{cl}}(M_{0,4}).$$

and we can therefore study it's period. In fact, it is shown that if one integrates ρ over z_0 along a simple closed curve $\sigma \subset M_{0,4}$, one obtains the

[2]Acronym for a construction of TFTs from supergeometry due to M. Alexandrov, M. Kontsevich, A. S. Schwarz and O. Zaboronsky.

"linking number" of σ and the 0-cycle $[z_1]-[z_2]$, that is the difference of winding numbers of σ around the point z_1 and around the point z_2:

$$\int_{\sigma\ni z_0}\rho = 4\pi\,\mathrm{lk}(\sigma,[z_1]-[z_2]).$$

The example generalizes to an arbitrary number of insertions of the form $e^{\alpha_i B(z_i)}$

$$\rho = \left\langle \Gamma\left(c(z_0)e^{\alpha_1 B(z_1)}\dots e^{\alpha_n B(z_n)}\Theta(z_{n+1})c(z_{n+2})\right)\right\rangle .$$

In this case, the period $\int_\sigma\rho$ computes the linking number of σ with the 0-cycle $\sum_{k=1}^n \alpha_k[z_k]-(\sum_{k=1}^n\alpha_k)[z_{n+1}]$.

Finally it is observed that the abelian model exhibits a residual global U(1)-symmetry. This extra symmetry plays the role of an R-symmetry, c.f. the discussion in Section 2.4.3. Untwisting[3] the theory by the U(1)-symmetry, yields a $\mathcal{N}=(2,2)$ supersymmetric theory. In fact, the resulting theory turns out to be a superconformal theory with central charge -3 (in the N-component model we find $c=-3N$).

More surprisingly, however, is the observation that this $\mathcal{N}=(2,2)$ supersymmetric theory is in one-to-one correspondence with the B-model, c.f. Section 2.4.3.4, with an odd linear target space coordinatized by the ghost field c. The BRST operator Q of the gauged fixed abelian BF model coincides with the nilpotent operator Q_B (defined in (2.137)).

The unexpected relation between the gauged fixed abelian BF theory and the B-model opens the possibility of studying non-trivial deformations of both theories. Standard deformations of the B-model are deformations in the complex structure of the target space, adding a holomorphic superpotential or deformations by a bivector field. In all cases, the deforming tensor has even tensor weight. The novelty in the abelian BF theory is that the target space is odd. In particular, this allows us to study deformations by tensors which have an odd tensor weight, such as, for example, a vector field. In Chapter 5, we will show how the deformation by a vector field leads to the Lorenz gauged non-abelian BF theory.

3.2. Non-abelian BF theory

Chapter 5 discusses in-depth the Lorenz gauged two-dimensional non-abelian BF theory, where fields are now taking values in an arbitrary Lie algebra $\mathfrak{g}$ or its dual $\mathfrak{g}^*$. As for the abelian model, we fix the gauge freedom by imposing the Lorenz gauge. The gauged fixed action is classically invariant and parallel to the discussion in Chapter 4, the choice of a worldsheet metric (required by the Lorenz gauge condition) enters the action only via a Q-exact term which implies that the stress-energy tensor is Q-exact. The gauged fixed model is therefore classically a TCFT.

We show that the action of the non-abelian model can be obtained as a deformation of the abelian model by a second descent

$$S_{\mathrm{non-ab}} = S_{\mathrm{ab}} + g\int_\Sigma \mathcal{O}^{(2)}$$

[3] The "twist" is in the opposite direction compared to the discussion in Section 2.4.3 since we start from a TCFT and obtain a $\mathcal{N}=(2,2)$ supersymmetric theory. Hence, we call the procedure "untwisting" as opposed to twisting.

where g is a coupling constant and one chooses $\mathcal{O}^{(0)} = \frac{1}{2}\langle B, [c,c]\rangle$. In fact, this deformation is an example of the aforementioned new deformation by a vector field. We discuss carefully how the BRST differential, the stress-energy tensor and its primitive of the non-abelian theory can be seen as deformations of the corresponding objects in the abelian model. Notably, the primitive G does not deform.

One novelty of the gauged fixed non-abelian BF theory is the loss of holomorphic currents. In particular, the BRST current J fails to split into a holomorphic $J^{(1,0)}$ and anti-holomorphic $J^{(0,1)}$ part. However, since the BRST operator Q generates a symmetry, the total current J remains conserved as is required by Noether's theorem. Indeed, due to the interaction terms, which are proportional to the structure constants of the Lie algebra $\mathfrak{g}$ and therefore absent in the abelian model, chiral and anti-chiral sectors are mixed. In particular, one observes a loss of the residual U(1) symmetries which were used to untwist the abelian theory. This strongly indicates that the gauged fixed non-abelian BF theory is not obtained by twisting a $\mathcal{N} = (2,2)$ supersymmetric theory and therefore serves as a new example of a TCFT.

Since the non-abelian model is an interacting theory, it is treated perturbatively, i.e. correlation functions are defined via Feynman diagrams. A characteristic feature of any BF theory is that its perturbation theory admits only tree level and one-loop Feynman diagrams, at least if one considers correlation functions of fundamental fields (linear composite fields). Surprisingly, we find that for a reductive Lie algebra $\mathfrak{g}$, the one-loop contributions from gauge fields and ghost fields cancel. Therefore, at least if one restricts oneself to fundamental fields, the theory is classical (the only non-trivial contributions come from tree-level Feynman diagrams). Moreover, we show that all correlation functions are monomials in the coupling constant and can be computed by a finite sum of convergent integrals. The theory is therefore finite.

To our surprise, however, even relatively simple correlation functions admit logarithmic singularities. For example we find

$$\langle \gamma(z_1)\bar{\gamma}(z_2)a^c(z_3)\rangle = \frac{g f_{ab}^c}{z_1 - z_3}\log\left|\frac{z_1 - z_2}{z_3 - z_2}\right|.$$

More generally, higher point correlation functions admit higher logarithmic singularities. This observation hints to the fact that the non-abelian BF theory is actually a logarithmic CFT, which is confirmed later in the chapter.

The logarithmic singularities of the correlation functions also show in the OPEs. Even more exotic singularities of the form $\bar{z}z^{-1}$ show up in OPEs. For example we find

$$\begin{aligned} a^a(z)\bar{\gamma}_b(w) &\sim -g f_{bc}^a a^c(w)\log|z-w|, \\ a^a(z)\gamma_b(w) &\sim \frac{\delta_b^a}{z-w} + \frac{g}{2} f_{bc}^a \bar{a}^c(w)\frac{\bar{z}-\bar{w}}{z-w}, \end{aligned}$$

where "$\sim$" stands for "up to regular terms". Furthermore, it is proven that also OPEs have only a finite number of contributing Feynman diagrams. We stress that this is a stronger statement than for correlation functions.

The unusual singularities lead us to a refined definition of composite fields. While in the abelian theory it was enough to subtract only the leading singularity as two (or more fields) merge into one, now we need to subtract all singularities:

$$(\Phi_1\Phi_2)(z) := \lim_{w\to z}\left(\Phi_1(w)\Phi_2(z) - \text{all singularities}\right).$$

Interestingly, the definition of a composite field by consecutive merging of fundamental fields, depends on the order of merging. For example, one finds that the difference of merging a^a into the composite field $(\gamma_b\gamma_c)$ and merging $(\gamma_b\gamma_c)$ into a^a, while subtracting all singular terms, is given by

$$(\lim_{z\to w} - \lim_{w\to z})(a^a(z)(\gamma_b\gamma_c)(w)) = \delta^a_b\partial\gamma_c + \delta^a_c\partial\gamma_b.$$

This definition of a composite field could be taken verbatim to define a renormalized product on the space of composite fields:

$$(\Phi_1 *_R \Phi_2)(z) := \lim_{w\to z}(\Phi_1(w)\Phi_2(z) - \text{ all singularities}).$$

Here an interesting question arises: Does the product $*_R$ define a pre-Lie algebra, i.e. does the following hold?

$$\Phi_1 *_R (\Phi_2 *_R \Phi_3) \pm \Phi_2 *_R (\Phi_1 *_R \Phi_3) = (\Phi_1 *_R \Phi_2 \Phi_2 *_R \Phi_1) *_R \Phi_3.$$

where $\pm = (-1)^{|\Phi_1||\Phi_2|+1}$. We plan on exploring this question in more detail elsewhere.

Since the stress-energy tensor T, its primitive G and the BRST current J are all composite fields, it is natural to ask how these fields depend on the order of merging. Surprisingly, we find that possible singular subtractions cancel each other and that they are independent of the order of merging. In addition, we proof the conservation of these fields and the usual form of the TT OPE for vanishing central charge. Since on the quantum level the action of the BRST operator is replaced by the action of its current,

$$(Q\Phi)(z) \to (Q_q\Phi)(z) = \frac{1}{4\pi}\oint J(w)\Phi(z),$$

it is a priori not clear that the relation $T = \{Q, G\}$ holds also on the quantum level, i.e. that

$$T(z) = (Q_qG)(z).$$

However, we proof this relation by explicit calculation. We therefore conclude that the Lorenz gauged fixed non-abelian BF theory is a TCFT also on the quantum level.

As we have mentioned before, OPEs and correlation functions show (poly)logarithmic singularities, which are characteristic features of logarithmic CFTs. A more careful analysis shows that these logarithmic singularities are due a non-trivial Jordan decomposition of the Hamiltonian $L_0 + \bar{L}_0$. Interestingly, the Hamiltonian of the model admits infinite dimensional Jordan cells. In fact, we construct fields $V^{(n)}$ such that

$$L_0V^{(n)} = V^{(n)} + gV^{(n-1)}, \qquad \bar{L}_0V^{(n)} = gV^{(n-1)},$$

where g is the coupling constant. Those fields can be summed into vertex operators, i.e. fields of anomalous dimensions

$$V = \sum_n V^{(n)}, \qquad (h, \bar{h}) = (1 + g, g).$$

The construction uses intrinsically the Lie algebra structure. In particular, the anomalous dimensions depend on the coupling constants. Finally, we compare the vertex operators of gauged fixed non-abelian BF model with the standard vertex operators in the scalar field theory. In both cases, the singularities which one has to subtract in the definition of a composite fields depend on local coordinates. The anomalous dimensions can be understood as a consequence of this coordinate dependence.

The next two chapters are the result of joint effort of A. S. Losev, P. Mnev and the author. They consist of the following two publications:

- Chapter 4: A. S. Losev, P. Mnev, and D. R. Youmans. Two-dimensional abelian BF theory in Lorenz gauge as a twisted N=(2,2) superconformal field theory. *J. Geom. Phys.*, 131:122âĂŞ137, 2018.
- Chapter 5: A. S. Losev, P. Mnev, and D. R. Youmans. Two-dimensional non-abelian BF theory in Lorenz gauge as a solvable logarithmic TCFT. *Comm. Math. Phys.*, 2019.

CHAPTER 4

Two-dimensional abelian BF theory in Lorenz gauge as a twisted N=(2,2) superconformal field theory

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Foreword

The note is designed as a self-contained exposition, with relevant background on (super)conformal field theory included in the text for the reader's convenience. We refer to the sources [6, 8, 4] for details.

For the readers who are well acquainted with superconformal field theory, we suggest to look at the formula (4.7) and then read the subsections 4.1.2.1 and 4.2.7.1 for the quick gist of the story.

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4.1. Classical theory

We consider the abelian BF theory on a two-dimensional oriented surface Σ defined classically by the action functional

$$S_0 = \int_\Sigma B\, dA \tag{4.1}$$

where the fields are a 1-form A and a 0-form B on Σ; d is the de Rham operator. The equations of motion read $dA = 0$, $dB = 0$ and the theory has gauge symmetry $A \mapsto A + d\alpha, B \mapsto B$ with the 0-form α being the generator of the gauge transformation.

4.1.1. Gauge-fixed model in BRST formalism. We impose the Lorenz gauge condition $d^*A = 0$ where $d^* = - * d*$ is the Hodge dual of the de Rham operator associated to a choice of a Riemannian metric g on Σ, with $*$ the Hodge star. The corresponding gauge-fixed action (the Faddeev-Popov action) is:

$$S = \int_\Sigma B\, dA + \lambda\, d * A + b\, d * dc \tag{4.2}$$

with scalar field λ the Lagrange multiplier imposing the Lorenz gauge condition and b, c the Faddev-Popov ghosts – the odd scalar fields. Thus, the space of BRST fields of the model is:

$$\mathcal{F} = \underbrace{\Omega^1}_{A} \oplus \underbrace{\Omega^0}_{B} \oplus \underbrace{\Omega^0}_{\lambda} \oplus \underbrace{\Pi\Omega^0}_{b} \oplus \underbrace{\Pi\Omega^0}_{c} \tag{4.3}$$

with Π the parity reversal symbol. Equivalently, it is the space of sections $\mathcal{F} = \Gamma(\Sigma, \underline{\mathsf{F}})$ of the super vector bundle

$$\underline{\mathsf{F}} = \underbrace{T^*\Sigma}_{A} \oplus \underbrace{\mathbb{R}^2}_{B,\lambda} \oplus \underbrace{\Pi\mathbb{R}^2}_{b,c} \tag{4.4}$$

over Σ, with last two terms the trivial even and odd rank 2 bundles.[1] The BRST operator acts as

$$Q: \qquad A \mapsto dc, \quad b \mapsto \lambda, \quad B, c, \lambda \mapsto 0$$

One clearly has $Q^2 = 0$ and $Q(S) = 0$ – the gauge-invariance of the action. Also, the gauge-fixed action differs from S_0 by a Q-exact term:

$$S = S_0 + Q(\Psi) \tag{4.5}$$

with

$$\Psi = \int_\Sigma b\, d * A \tag{4.6}$$

the gauge-fixing fermion.

Equations of motion for the gauge-fixed action (4.2) read

$$dA = 0, \quad d * A = 0, \quad dB - *d\lambda = 0, \quad \Delta b = 0, \quad \Delta c = 0$$

with $\Delta = *d * d$ the Laplacian acting on functions on Σ.

4.1.2. Rewriting the action in terms of complex fields, conformal invariance. It is convenient to split the 1-form field A into its $(1,0)$ and $(0,1)$-components $A = \mathsf{a} + \bar{\mathsf{a}}$, where the splitting $\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$ is inferred from the complex structure on Σ compatible with the chosen metric (in particular, $\Omega^{1,0}$ and $\Omega^{0,1}$ are the $-i$ and i-eigenspaces for the Hodge star). We also combine the real scalar fields B, λ into a complex scalar field $\gamma := \frac{1}{2}(\lambda + iB)$ with conjugate $\bar{\gamma} = \frac{1}{2}(\lambda - iB)$. Action (4.2) can then be written as

$$S = 2i \int_\Sigma -\gamma\, \bar{\boldsymbol{\partial}}\mathsf{a} + \bar{\gamma}\, \boldsymbol{\partial}\bar{\mathsf{a}} + b\, \boldsymbol{\partial}\bar{\boldsymbol{\partial}}\, c \tag{4.7}$$

with $\boldsymbol{\partial}, \bar{\boldsymbol{\partial}}$ the holomorphic and anti-holomorphic Dolbeault operators (written as $\boldsymbol{\partial} = dz\frac{\partial}{\partial z}, \bar{\boldsymbol{\partial}} = d\bar{z}\frac{\partial}{\partial \bar{z}}$ in local complex coordinates $z, \bar{z}$; we reserve the non-boldface symbols $\partial = \frac{\partial}{\partial z}, \bar{\partial} = \frac{\partial}{\partial \bar{z}}$ for the partial derivatives themselves). Written in this form, the action is manifestly dependent only on the complex structure induced by the metric g, i.e. only on the conformal class of the metric g modulo Weyl transformations $g \sim \Omega \cdot g$ with Ω any positive function on Σ. Thus the gauge-fixed abelian BF theory is conformal. The BRST operator Q written in terms of the new fields reads:

$$Q: \qquad \mathsf{a} \mapsto \boldsymbol{\partial} c, \quad \bar{\mathsf{a}} \mapsto \bar{\boldsymbol{\partial}} c, \quad b \mapsto \gamma + \bar{\gamma}, \qquad \gamma, \bar{\gamma}, c \mapsto 0 \tag{4.8}$$

[1] One can enhance the $\mathbb{Z}_2$-grading on fields to a $\mathbb{Z}$-grading by the "ghost number", where b, c are assigned degrees -1 and $+1$, respectively, and A, B, λ are assigned degree 0.

and the equations of motion are:

$$\bar{\boldsymbol{\partial}}\mathsf{a}=0,\quad \bar{\boldsymbol{\partial}}\gamma=0,\quad \boldsymbol{\partial}\bar{\mathsf{a}}=0,\quad \boldsymbol{\partial}\bar{\gamma}=0,\quad \boldsymbol{\partial}\bar{\boldsymbol{\partial}}\,b=0,\quad \boldsymbol{\partial}\bar{\boldsymbol{\partial}}\,c=0 \tag{4.9}$$

Note that fields $\gamma,\bar{\gamma}$ are more adapted to the action and the equations of motion, whereas fields B,λ are more adapted to the BRST operator.

Remark 4.1.1. Using local coordinates x^1,x^2 on Σ (such that g is in the conformal class of $(dx^1)^2+(dx^2)^2$) and the corresponding complex coordinates $z=x^1+ix^2$, $\bar{z}=x^1-ix^2$, we can write $\mathsf{a}=dz\,a$, $\bar{\mathsf{a}}=d\bar{z}\,\bar{a}$ with $a,\bar{a}$ scalars. Then the action (4.7) reads

$$S=4\int d^2x\;(\gamma\bar{\partial}a+\bar{\gamma}\partial\bar{a}+b\,\partial\bar{\partial}c) \tag{4.10}$$

with $d^2x=dx^1dx^2=\frac{i}{2}dz\,d\bar{z}$ the coordinate area form. In our conventions, for the fields $a,\bar{a}$ one gets a sign in BRST transformations

$$Q:\ a\mapsto-\partial c,\ \bar{a}\mapsto-\bar{\partial}c.$$

The BRST symmetry defines, via the Noether theorem, a current

$$J_{\mathrm{tot}}=2i(\gamma\,\boldsymbol{\partial}c-\bar{\gamma}\,\bar{\boldsymbol{\partial}}c)=2i\Big(dz\,\underbrace{\gamma\,\partial c}_{=:J}-d\bar{z}\,\underbrace{\bar{\gamma}\,\bar{\partial}c}_{=:\bar{J}}\Big) \tag{4.11}$$

It is conserved modulo equations of motion: $dJ_{\mathrm{tot}}\underset{\text{e.o.m.}}{\sim}0$. In fact, one has a stronger statement that both chiral parts of the current are conserved independently: $\bar{\partial}J\underset{\text{e.o.m.}}{\sim}0$ and $\partial\bar{J}\underset{\text{e.o.m.}}{\sim}0$.

4.1.2.1. *Abelian BF theory as a twisted superconformal field theory: an anticipation.* It is remarkable – see section 4.2.7 for details – that action (4.7) is a free type B twisted $\mathcal{N}=(2,2)$ superconformal theory where the parity of the fields is changed (so scalars are fermions while the first order systems are constructed with bosons). The "holomorphic field" is just the Faddev-Popov ghost. As we will show later the Q_{BRST} becomes a sum of two scalar charges, as usual in the twisted theory; their currents have changed the dimension from 3/2 to 1 in both holomorphic and antiholomorphic sectors. This unexpected property allows to pose questions in BF theory that where prohibited by the naive understanding of allowed correlators – in particular, it is possible to study correlators of some gauge non-invariant observables, like the superpartner (BRST-primitive) of the energy-momentum tensor (see below). On the other hand non-abelian BF theory in Lorenz gauge may serve as an example of a new conformal field theory (this question is clear classically and we will return to this important question on the quantum level in subsequent work).

4.1.3. Stress-energy tensor and its BRST-primitive. The stress-energy tensor is defined via the variational derivative of the action (4.2) with respect to metric. Explicitly, in a coordinate chart on Σ, one defines $T_{\mu\nu}$ via

$$\delta_g S=-\int_\Sigma\sqrt{\det g}\,d^2x\,T_{\mu\nu}\delta g^{\mu\nu} \tag{4.12}$$

where the left hand side is the variation w.r.t. the metric g; indices μ,ν take values in $\{1,2\}$ or $\{z,\bar{z}\}$. The total stress-energy tensor $T_{\mathrm{tot}}=T_{\mu\nu}\,dx^\mu\cdot dx^\nu$ is a section of the symmetric square of the cotangent bundle of Σ; the dot stands for the symmetric tensor product in $\mathrm{Sym}^\bullet T^*\Sigma$. Note that S depends on the metric only via the dependence of the gauge-fixing fermion Ψ on the metric, entering via the Hodge star. Thus, from (4.5) we have that the components of the stress-energy tensor are exact w.r.t. the BRST operator,

$$T_{\mu\nu} = QG_{\mu\nu}$$

where $G_{\mu\nu}$ is defined, similarly to (4.12), via

$$\delta_g \Psi = -\int_\Sigma \sqrt{\det g}\, d^2x\, G_{\mu\nu} \delta g^{\mu\nu}$$

Explicitly, in holomorphic coordinates $z, \bar{z}$, one obtains[2]

$$G_{\mathrm{tot}} = (dz)^2 \underbrace{a\,\partial b}_{G_{zz}=:G} + (d\bar{z})^2 \underbrace{\bar{a}\,\bar{\partial} b}_{G_{\bar{z}\bar{z}}=:\bar{G}} = \mathsf{a}\cdot \boldsymbol{\partial} b + \bar{\mathsf{a}} \cdot \bar{\boldsymbol{\partial}} b \tag{4.13}$$

and

$$T_{\mathrm{tot}} = QG_{\mathrm{tot}} = (dz)^2 \underbrace{(-\partial c\, \partial b + a\, \partial \lambda)}_{T_{zz}=:T} + (d\bar{z})^2 \underbrace{(-\bar{\partial} c\, \bar{\partial} b + \bar{a}\, \bar{\partial}\lambda)}_{T_{\bar{z}\bar{z}}=:\bar{T}}$$
$$= (\boldsymbol{\partial} c \cdot \boldsymbol{\partial} b + \mathsf{a}\cdot \boldsymbol{\partial}\lambda) + (\bar{\boldsymbol{\partial}} c \cdot \bar{\boldsymbol{\partial}} b + \bar{\mathsf{a}} \cdot \bar{\boldsymbol{\partial}} \lambda)$$

The components $G_{z\bar{z}}$, $T_{z\bar{z}}$ vanish (equivalently, the traces $G^\mu{}_\mu$, $T^\mu{}_\mu$ vanish), which is a manifestation of the conformal invariance of the theory. The stress-energy tensor and its primitive are conserved modulo equations of motion:

$$\bar{\partial} G \underset{\text{e.o.m.}}{\sim} 0, \quad \partial \bar{G} \underset{\text{e.o.m.}}{\sim} 0; \qquad \bar{\partial} T \underset{\text{e.o.m.}}{\sim} 0, \quad \partial \bar{T} \underset{\text{e.o.m.}}{\sim} 0$$

4.2. Quantum abelian BF theory as a conformal field theory

From now on we specialize to the case of the surface Σ being the plane $\mathbb{R}^2 = \mathbb{C}$ with coordinates x^1, x^2 (or the complex coordinate $z = x^1 + ix^2$ and its conjugate $\bar{z} = x^1 - ix^2$), endowed with the standard Euclidean metric $g = (dx^1)^2 + (dx^2)^2 = dz \cdot d\bar{z}$.

4.2.1. Correlation functions. We are interested in studying the normalized correlation functions

$$\langle \Phi_1(z_1) \cdots \Phi_n(z_n) \rangle := \frac{1}{Z} \int_{\mathcal{F}} e^{-\frac{1}{4\pi} S}\, \Phi_1(z_1) \cdots \Phi_n(z_n) \tag{4.14}$$

Here:

[2] Here is the computation in local coordinates: Hodge star acts on 1-forms via $*dx^\mu = \sqrt{g}\, g^{\mu\nu} \epsilon_{\nu\rho} dx^\rho$ with $\epsilon_{\nu\rho}$ the Levi-Civita symbol and $\sqrt{g}$ the shorthand notation for $\sqrt{\det g}$. Variation w.r.t. the metric is thus $\delta_g * dx^\mu = \sqrt{g}(-\frac{1}{2} g_{\alpha\beta} \delta g^{\alpha\beta}\, g^{\mu\nu} + \delta g^{\mu\nu}) \epsilon_{\nu\rho} dx^\rho$. Next, we use this to compute the variation of the gauge-fixing fermion: $\delta_g \Psi = \int db \wedge \delta_g * dx^\mu\, A_\mu = \int \sqrt{g}\, db \wedge (-\frac{1}{2} g_{\alpha\beta} g^{\mu\nu} + \delta^\mu_\alpha \delta^\nu_\beta) \epsilon_{\nu\rho} dx^\rho A_\mu \delta g^{\alpha\beta}$. Note that the coefficient of $\sqrt{g}\, \delta g^{\alpha\beta}$ in the integrand is manifestly invariant under Weyl transformations $g \mapsto \Omega \cdot g$. In particular, we can compute this coefficient in holomorphic coordinates $z, \bar{z}$, using the standard metric $g = dz \cdot d\bar{z}$. One obtains $\delta_g \Psi = -\int \frac{i}{2} dz \wedge d\bar{z}\, (\partial b\, a\, \delta g^{zz} + \bar{\partial} b\, \bar{a}\, \delta g^{\bar{z}\bar{z}})$; reading off the coefficients of the variation of the metric, we obtain (4.13).

- $\Phi_1(z_1), \cdots, \Phi_n(z_n)$ are composite fields[3] – polynomials in the fields $a, \bar{a}, B, \lambda, b, c$ and their derivatives of arbitrary order, evaluated at pairwise distinct points $z_1, \ldots, z_n \in \mathbb{C}$.
- S is the action (4.10).
- The normalization factor $Z := \int_{\mathcal{F}} e^{-\frac{1}{4\pi}S}$ is the partition function. The r.h.s. of (4.14) is a ratio of path integrals over $\mathcal{F}$ which individually need a regularization (both ultraviolet and infrared) to be defined. However the ratio is independent of the regularization.
- The normalization factor $\frac{1}{4\pi}$ in the exponential in the r.h.s. of (4.14) is introduced to have a convenient normalization of propagators.

Since the action S is quadratic, the theory is free and the correlators are given by Wick's lemma with the following basic propagators[4]:

$$\begin{aligned} \langle c(w)\, b(z)\rangle &= 2\log|w-z| + C, \\ \langle a(w)\, \gamma(z)\rangle &= \frac{1}{w-z}, \\ \langle \bar{a}(w)\, \bar{\gamma}(z)\rangle &= \frac{1}{\bar{w}-\bar{z}}. \end{aligned} \tag{4.15}$$

Propagators for all other pairs of fields from the set $\{a, \bar{a}, \gamma, \bar{\gamma}, b, c\}$ vanish. In terms of fields A, B, λ this implies

$$\langle A(w)\, B(z)\rangle = 2\, d_w \arg(w-z), \quad \langle A(w)\, \lambda(z)\rangle = 2\, d_w \log|w-z| \tag{4.16}$$

When constructing the correlators (4.14) for composite fields by Wick's lemma, we do not allow matchings of two basic fields in the same composite field Φ_i (which would have led to an ill-defined expression) – this corresponds to the assumption that the composite fields Φ_i are normally ordered.

Note also that if one of the fields Φ_i vanishes modulo equations of motion (4.9), the correlator (4.14) vanishes identically.

4.2.2. The space of composite fields. One can formalize the notion of a composite field by considering the symmetric powers of the jet bundle of $\underline{F}^*_{\mathbb{C}} = \mathbb{C} \otimes \underline{F}^*$ – the complexified bundle dual to the bundle of BRST fields, and then taking a quotient by the ideal generated by the equations of motion (4.9) and their derivatives:

$$\underline{\mathbb{F}} := \mathrm{Sym}^\bullet \mathrm{Jet}\, \underline{F}^*_{\mathbb{C}}/\text{e.o.m.}$$

---

[3] We are using this terminology to emphasize the distinction between the "basic" BRST fields (4.3) and the objects that can be used as decorations of punctures on Σ when calculating correlation functions. We call the latter the composite fields. Another possible term is "observables", or "0-observables" (though often one reserves the word "observable" only for the Q-closed expressions in fields).

[4] One constructs the propagator $\langle c(z)\, b(w)\rangle$ from the action (4.10) as the Green's function (the integral kernel of the inverse operator) for the operator $\frac{1}{4\pi}\Delta$. Similarly, $\langle a(w)\, \gamma(z)\rangle$ is the Green's function for $\frac{1}{\pi}\bar{\partial}$. We implicitly fix the zero-mode for the Dolbeault operator by requiring that the fields $a, \gamma, \bar{a}, \bar{\gamma}$ vanish at infinity; in other words we are considering the theory on the compactified plane $\bar{\mathbb{C}} = \mathbb{CP}^1$ relative to the point $\{\infty\}$. To fix the zero-mode of the Laplacian we do an infrared regularization by replacing the pair $(\mathbb{CP}^1, \{\infty\})$ with a disk of large radius R relative to the boundary, i.e., we impose the Dirichlet boundary conditions on b, c. The constant shift C in the propagator $\langle c(w)\, b(z)\rangle$ depends on the infrared cut-off, $C = -2\log R$.

It is a graded complex vector bundle over Σ with $\mathbb{Z}$-grading given by the ghost number (by assigning degree $+1$ to c and degree -1 to b and zero to all other basic fields). Thus, a composite field $\Phi(z)$, regarded modulo equations of motion, is an element of the fiber $\mathbb{F}_z$. Fibers of $\underline{\mathbb{F}}$ are differential graded commutative algebras, with the differential given by the BRST operator Q. The n-point correlator (4.14) can then be regarded as bundle morphism from $\iota^*(\underline{\mathbb{F}}\boxtimes\cdots\boxtimes\underline{\mathbb{F}})$ to the trivial line bundle over $\mathrm{Conf}_n(\Sigma)$ – the (open) configuration space of n pairwise distinct ordered points on Σ; here $\iota:\mathrm{Conf}_n(\Sigma)\hookrightarrow\Sigma^{\times n}$ is the tautological inclusion.

Explicitly, the space of composite fields $\mathbb{F}_z$ is a free graded commutative algebra generated by the fields

$$b,\ c;\ \underbrace{\{\partial^k b\}_{k\geq 1},\ \{\partial^k c\}_{k\geq 1},\ \{\partial^k a\}_{k\geq 0},\ \{\partial^k\gamma\}_{k\geq 0}}_{\text{holomorphic sector}};\quad \underbrace{\{\bar\partial^k b\}_{k\geq 1},\ \{\bar\partial^k c\}_{k\geq 1},\ \{\bar\partial^k\bar a\}_{k\geq 0},\ \{\bar\partial^k\bar\gamma\}_{k\geq 0}}_{\text{anti}-\text{holomorphic sector}} \tag{4.17}$$

The BRST differential Q is a derivation defined on the generators by (4.8) together with the rule $Q(\mathcal{D}\phi)=\mathcal{D}Q(\phi)$ for any differential operator $\mathcal{D}=\partial^k\bar\partial^l$ and ϕ a basic field, and then extended to the whole $\mathbb{F}_z$ by Leibniz identity.

The cohomology of Q acting on $\mathbb{F}_z$ is calculated straightforwardly and yields the subalgebra of $\mathbb{F}_z$ generated by fields $B=\frac{\gamma-\bar\gamma}{i}$ and c (but not their derivatives):[5]

$$\mathbb{O}_z:=H^\bullet_Q(\mathbb{F}_z)=\mathbb{C}[B,c] \tag{4.18}$$

We denote the Q-cohomology by $\mathbb{O}_z$ (for "observables"). Note that it is concentrated in degrees 0 and 1 only, since polynomials in c have degree at most 1. However, one can consider an N-component abelian BF theory, i.e., N non-interacting copies of the theory; in other words, one replaces the fiber (4.4) of the bundle of BRST fields with

$$\underline{\mathsf{F}}\mapsto\underline{\mathsf{F}}^{[N]}:=\underline{\mathsf{F}}\otimes\mathbb{R}^N \tag{4.19}$$

(we will use the superscript $[N]$ when we want to emphasize that we work with N-component theory). In this case, there are N odd generators c^j and the polynomials in them can have degree up to N and thus the cohomology $\mathbb{O}_z^{[N]}:=H^\bullet_Q(\mathbb{F}_z^{[N]})=\mathbb{C}[B_1,\ldots,B_N,c^1,\ldots,c^N]$ is spread across degrees $0,1,\ldots,N$.

One can geometrically interpret the space of observables as the space of polyvector fields

$$\mathbb{O}_z^{[N]}=T_{\mathrm{poly}}(\Pi V) \tag{4.20}$$

[5] Here is the calculation: denote by Y the linear span of the generators (4.17). Note that, by freeness of the theory, Q acts on Y as a differential and Q on $\mathbb{F}_z$ is the extension of this action by Leibniz identity. Thus $H^\bullet_Q(\mathbb{F}_z)=H^\bullet(\mathrm{Sym}\,Y)=\mathrm{Sym}\,H^\bullet(Y)$. To compute $H^\bullet(Y)$, notice that Q maps $\partial^{k-1}a\mapsto-\partial^k c$, $\bar\partial^{k-1}\bar a\mapsto-\bar\partial^k c$, $\partial^k b\mapsto\partial^k\gamma$, $\bar\partial^k b\mapsto\bar\partial^k\bar\gamma$ for all $k\geq 1$. Thus we can remove the acyclic subcomplex spanned by these generators out of Y and we have $H^\bullet(Y)=H^\bullet\mathrm{Span}(b,c,\gamma,\bar\gamma)=H^\bullet\mathrm{Span}(b,c,B,\lambda)$. Finally, since Q maps $b\mapsto\lambda$ and vanishes on B,c, we have $H^\bullet(Y)=\mathrm{Span}(B,c)$ and hence $H^\bullet_Q(\mathbb{F}_z)=\mathrm{Sym}\,\mathrm{Span}(B,c)$.

on the odd space ΠV where $V = \mathbb{R}^N$ is the space of coefficents in the N-component theory. Here the ghosts c^j are interpreted as coordinates on the base ΠV and $B_j = \frac{\partial}{\partial c^j} \in \Pi T_c(\Pi V)$ are interpreted as tangent vectors.[6]

4.2.3. Operator product expansions. We are interested in analyzing the singular part of the asymptotics of the correlator

$$\langle \Phi_1(w)\Phi_2(z)\phi_1(z_1)\cdots\phi_n(z_n)\rangle \tag{4.21}$$

as the point w approaches z, with $\phi_1(z_1),\ldots,\phi_n(z_n)$ being the "test fields". This asymptotics is controlled by the operator product expansion (OPE) of the fields Φ_1 and Φ_2 which is an expression of the form

$$\Phi_1(w)\,\Phi_2(z) \sim \sum_{j=1}^{p} f_j(w-z)\widetilde{\Phi}_j(z) + \text{reg.} \tag{4.22}$$

with $\widetilde{\Phi}_j$ some fields and f_j some singular coefficient functions, typically a product of negative powers of $(w-z)$ and $(\bar{w}-\bar{z})$ and can also contain $\log(w-z)$ and $\log(\bar{w}-\bar{z})$; reg. stands for terms which are regular as $w\to z$; the number p of singular terms depends on Φ_1,Φ_2. Thus, (4.22) means that in the correlator (4.21) one can replace the first two fields with the expression on the r.h.s. of (4.22), reducing the number of points by one.

Example 4.2.1. For instance, we have

$$\begin{aligned} a(w)\,\gamma(z) &\sim \frac{1}{w-z} + (a\gamma)_z + (w-z)\cdot(\partial a\,\gamma)_z + \frac{1}{2}(w-z)^2\cdot(\partial^2 a\,\gamma)_z + \ldots \\ &\sim \frac{1}{w-z} + \text{reg.} \end{aligned}$$

and

$$a(w)\,\bar{\gamma}(z) \sim (a\bar{\gamma})_z + (w-z)\cdot(\partial a\,\bar{\gamma})_z + \frac{1}{2}(w-z)^2\cdot(\partial^2 a\,\bar{\gamma})_z + \cdots \quad \sim \text{reg.}$$

For brevity we put the point where the field is evaluated as a subscript (i.e. $\Phi(z) = \Phi_z$). Note that the OPE $a(w)\bar{\gamma}(z)$ is purely regular: correlators containing this pair of fields have a well-defined limit as $w\to z$, whereas a correlator containing $a(w)\gamma(z)$ will have a first order pole as $w\to z$.

Generally, since we are dealing with a free theory, the OPE $\Phi_1(w)\,\Phi_2(z)$ for two generic composite fields is constructed by Wick's lemma. In particular, for Φ_1,Φ_2 two monomials in (derivatives of) basic fields, the recipe for OPE is as follows. We consider all partial matchings of basic fields in Φ_1 against basic fields in Φ_2. For each matching, we replace matched pairs of basic fields by their propagators (acted on by respective derivatives in $w,\bar{w},z,\bar{z}$ that were acting on those basic fields). We multiply the result with the unmatched basic fields (acted on by respective derivatives), while also replacing basic fields evaluated at w with their Taylor expansion around z. Finally, we sum these contributions over all partial matchings.

Using this recipe one obtains the following OPEs of distinguished fields – the stress-energy tensor $T = \partial b\,\partial c + a\,\partial\lambda$, its BRST primitive $G = a\,\partial b$

[6] In terms of $\mathbb{Z}$-grading: $\mathbb{O}_z^{[N]} = T_{\text{poly}}(V[1])$ – sections of the symmetric powers of the shifted tangent bundle $T[-1](V[1])$ over $V[1]$, or, equivalently, functions on the graded space $T^*[1]V[1]$.

and the BRST current $J = \gamma\,\partial c$ (and their anti-holomorphic counterparts $\bar{T}, \bar{G}, \bar{J}$):

$$T(w)\,T(z) \sim \frac{2T(z)}{(w-z)^2} + \frac{\partial T(z)}{w-z} + \text{reg} \tag{4.23}$$

$$J(w)\,G(z) \sim -\frac{1}{(w-z)^3} + \frac{(\gamma\,a)_z}{(w-z)^2} + \frac{T(z)}{w-z} + \text{reg} \tag{4.24}$$

$$T(w)\,G(z) \sim \frac{2G(z)}{(w-z)^2} + \frac{\partial G(z)}{w-z} + \text{reg}$$

$$T(w)\,J(z) \sim \frac{J(z)}{(w-z)^2} + \frac{\partial J(z)}{w-z} + \text{reg}$$

and their complex conjugates. Also, one has $G(w)\,G(z) \sim \text{reg}$ and likewise $J(w)\,J(z) \sim \text{reg}$ All OPEs between holomorphic objects T, G, J and anti-holomorphic objects $\bar{T}, \bar{G}, \bar{J}$ are regular. In particular, from the absence of 4-th order pole in (4.23), we see that the central charge of the theory vanishes $\mathsf{c} = 0$.[7] Also, we see that G and J are primary fields with conformal weights $(2, 0)$ and $(1, 0)$ respectively.

Example 4.2.2. For example, here is the computation of the OPE (4.24). By (4.15), we have the propagators $\langle \gamma_w\, a_z \rangle = -\frac{1}{w-z}$ and $\langle (\partial c)_w (\partial b)_z \rangle = \partial_w \partial_z \langle c_w\, b_z \rangle = \partial_w \partial_z 2 \log |w - z| = \frac{1}{(w-z)^2}$. In the OPE $(\gamma\partial c)_w (a \partial b)_z$ one gets three singular terms from the Wick contractions of either γ_w with a_z or $(\partial c)_w$ with $(\partial b)_z$ or both. Thus,

$$\begin{aligned} &(\gamma\partial c)_w (a\partial b)_z \sim \\ &\sim \frac{-1}{w-z} \cdot \frac{1}{(w-z)^2} + \frac{1}{(w-z)^2} :\gamma_w a_z: + \frac{-1}{w-z} :(\partial c)_w (\partial b)_z: + \text{reg} \\ &\sim \frac{-1}{(w-z)^3} + \frac{(\gamma a)_z}{(w-z)^2} + \frac{1}{w-z} (\partial\gamma\, a + \partial b\, \partial c)_z + \text{reg} \end{aligned} \tag{4.25}$$

Here in the last step we replaced fields at w with their Taylor expansions centered at z. Note that the products of fields at w and at z occurring at the intermediate stage are normally ordered, i.e. Wick contractions inside them are prohibited. Finally, notice that $\partial\gamma\, a + \partial b\, \partial c$ is equivalent to T modulo equations of motion. Thus we obtain (4.24).

For $\Phi(z) = \Phi(B, c)_z$ a Q-closed field (4.18), we obtain

$$T(w)\Phi(z) \sim \frac{\partial\Phi(z)}{w-z} + \text{reg}, \qquad \bar{T}(w)\Phi(z) \sim \frac{\bar{\partial}\Phi(z)}{\bar{w}-\bar{z}} + \text{reg}$$

Thus all polynomials in B, c are primary fields of weight $(0, 0)$.

---

[7] Recall that a conformal field theory with central charge c is characterized by the following OPE of the stress-energy tensor with itself: $T(w)\,T(z) \sim \frac{\mathsf{c}/2}{(w-z)^4} + \frac{2T(z)}{(w-z)^2} + \frac{\partial T(z)}{w-z} + \text{reg}$ (plus the conjugate expression for $\bar{T}\bar{T}$, plus $T\bar{T} \sim \text{reg}$). Also recall that a field Φ is *primary*, of conformal weight $(h, \bar{h})$ iff its OPEs with the stress-energy tensor are: $T(w)\,\Phi(z) \sim \frac{h\,\Phi(z)}{(w-z)^2} + \frac{\partial\Phi(z)}{w-z} + \text{reg}$ and the conjugate $\bar{T}(w)\,\Phi(z) \sim \frac{\bar{h}\,\Phi(z)}{(\bar{w}-\bar{z})^2} + \frac{\bar{\partial}\Phi(z)}{\bar{w}-\bar{z}} + \text{reg}$

4.2.4. Extended Virasoro algebra. Every field α which is holomorphic, i.e. satisfies $\bar\partial\alpha \underset{\text{e.o.m.}}{\sim} 0$, and has conformal weight $(h,0)$, determines *mode operators*

$$\alpha_n^{(z)}: \quad \Phi(z) \mapsto \oint_{\mathcal{C}_z} \frac{dw}{2\pi i}\,(w-z)^{n+h-1}\,\alpha(w)\Phi(z) \tag{4.26}$$

on the space of fields $\mathbb{F}_z$ where on the r.h.s. one has the integral in variable w over a contour $\mathcal{C}_z$ going once counterclockwise around z.[8] In other words, $\alpha_n^{(z)}$ acts on a field $\Phi(z)$ by taking the coefficient of $(w-z)^{-n-h}$ in the OPE $\alpha(w)\,\Phi(z)$. I.e. one has the mode expansion – the equality

$$\alpha(w) = \sum_{n\in\mathbb{Z}+h} (w-z)^{-n-h}\alpha_n^{(z)} \tag{4.27}$$

of w-dependent operators on $\mathbb{F}_z$. Here the left hand side acts on a field $\Phi(z)$ by sending it to the OPE $\alpha(w)\Phi(z)$. Similarly, an anti-holomorphic field $\bar\alpha$ (i.e. satisfying $\partial\bar\alpha \underset{\text{e.o.m.}}{\sim} 0$), of weight $(0,\bar h)$, determines mode operators $\bar\alpha_n^{(z)}: \quad \Phi(z) \mapsto \oint_{\mathcal{C}_z} \frac{d\bar w}{-2\pi i}\,(\bar w-\bar z)^{n+\bar h-1}\bar\alpha(w)\Phi(z)$.

An important case of this construction is for α a conserved (holomorphic) Noether current – a primary field of conformal weight $(1,0)$ satisfying $\bar\partial\alpha \underset{\text{e.o.m.}}{\sim} 0$. Then

$$\hat\alpha^{(z)} := \alpha_0^{(z)}: \quad \Phi(z) \mapsto \oint_{\mathcal{C}_z} \frac{dw}{2\pi i}\,\alpha(w)\Phi(z)$$

is the corresponding quantum Noether charge acting on $\mathbb{F}_z$.

In particular, it is a straightforward check that the operator $\hat J_{\text{tot}} := \hat J + \hat{\bar J}$ associated to the total BRST current[9] $J_{\text{tot}} = 2i(dz\,J - d\bar z\,\bar J)$ coincides with the classical BRST operator Q acting on $\mathbb{F}_z$.[10]

One defines the Virasoro generators $L_n^{(z)} := T_n^{(z)}$ with $n\in\mathbb{Z}$, as the mode operators for the stress-energy tensor T, defined by (4.26) with $h=2$. Similarly, the anti-holomorphic Virasoro generators $\bar L_n^{(z)}$ are the mode operators for $\bar T$. We will also need the mode operators $G_n^{(z)}$ of the BRST-primitive G (which also has weight $h=2$) and their conjugate counterparts $\bar G_n^{(z)}$ associated to $\bar G$.

[8]The contour is supposed to be a boundary of a small neighborhood (e.g. a disk) of z, where "small" means that all the other field insertions in the correlators we are considering happen outside the neighborhood. Note that the holomorphic property $\bar\partial\alpha \underset{\text{e.o.m.}}{\sim} 0$ implies that one can deform the contour as long as it does not intersect with field insertions.

[9] Our normalization convention here is as follows: $\hat J_{\text{tot}}\Phi(z) := -\frac{1}{4\pi}\oint_{\mathcal{C}_z}(J_{\text{tot}})_w\Phi_z = \oint_{\mathcal{C}_z}(\frac{dw}{2\pi i}J_w + \frac{d\bar w}{-2\pi i}\bar J_w)\Phi_z$. Here the factor $-\frac{1}{4\pi}$ is the same as the factor accompanying the action in the path integral (4.14).

[10] For example: $(\gamma\partial c)_w b_z \sim \frac{\gamma}{w-z} + \text{reg}$ and $(\bar\gamma\bar\partial c)_w b_z \sim \frac{\bar\gamma}{\bar w-\bar z} + \text{reg}$, hence $\hat J b = \gamma, \hat{\bar J} b = \bar\gamma$, and thus $(\hat J + \hat{\bar J})b = \gamma + \bar\gamma = \lambda = Q(b)$. Likewise, $(\gamma\partial c)_w a_z \sim \frac{-\partial c}{w-z} + \text{reg}$ and $(\bar\gamma\bar\partial c)_w a_z \sim \text{reg}$, hence $\hat J a = -\partial c, \hat{\bar J} a = 0$ and thus $(\hat J + \hat{\bar J})a = -\partial c = Q(a)$. Another example is $\hat J_{\text{tot}} G$ which is given by the residue in the OPE (4.24), thus $\hat J_{\text{tot}} G = T$ which is a confirmation that in the quantum setting the classical relation $T = Q(G)$ still holds.

EXAMPLE 4.2.3. Operators $L_{-1}^{(z)}$ and $\bar{L}_{-1}^{(z)}$ act on $\mathbb{F}_z$ as partial derivatives:

$$L_{-1}^{(z)} = \partial_z, \qquad \bar{L}_{-1}^{(z)} = \partial_{\bar{z}} \tag{4.28}$$

From the OPEs between T, G, J and their conjugates, one obtains the following super commutation relations (Lie brackets) for the graded Lie algebra linearly generated by the operators $Q, \{L_n\}, \{G_n\}, \{\bar{L}_n\}, \{\bar{G}_n\}$:[11]

$$\begin{gathered}[Q,Q]=0, \quad [L_n, L_m] = (n-m)L_{n+m}, \\ [Q, L_n] = 0, \quad [Q, G_n] = L_n, \quad [L_n, G_m] = (n-m)G_{n+m}, \quad [G_n, G_m] = 0\end{gathered} \tag{4.29}$$

plus the conjugate relations. Commutators involving a holomorphic generator $\in \{L_n, G_n\}$ and an anti-holomorphic generator $\in \{\bar{L}_n, \bar{G}_n\}$ vanish. The degrees (ghost numbers) of the generators are:

Q	$+1$
$L_n, \bar{L}_n$	0
$G_n, \bar{G}_n$	-1

In particular, this is an extension of the direct sum of two copies (coming from holomorphic and anti-holomorphic sectors) of Virasoro algebra with central charge $\mathsf{c} = 0$.

REMARK 4.2.4. The theory contains a "logarithmic field" $(cb)_z$ whose OPE with T is: $T(w)(cb)_z \sim \frac{1}{(w-z)^2} + \frac{\partial(cb)_z}{w-z} + \text{reg}$ Its presence implies that the Hamiltonian of the theory $\hat{H} = L_0 + \bar{L}_0$ is not diagonalizable and has a Jordan block (with eigenvalue 0) consisting of the eigenvector 1 and a generalized eigenvector $\frac{1}{2}cb$.

REMARK 4.2.5 (Sugawara construction). Consider the holomorphic fields $a, \gamma, \partial b, \partial c$ and consider their Fourier modes around z defined by (4.27). Note that the stress-energy tensor $T = a\,\partial\gamma + \partial b\,\partial c$, BRST current $J = \gamma\,\partial c$ and the primitive $G = a\,\partial b$ are explicitly written as quadratic expressions in the four fields $a, \gamma, \partial b, \partial c$. Thus, for the Fourier modes we have

$$\begin{aligned} L_n &= \sum_m -(n-m)a_m\gamma_{n-m} + (\partial b)_m(\partial c)_{n-m}, \\ J_n &= \sum_m \gamma_m(\partial c)_{n-m}, \\ G_n &= \sum_m a_m(\partial b)_{n-m}. \end{aligned} \tag{4.30}$$

Commutation relations between the modes of $a, \gamma, \partial b, \partial c$ follow from OPEs $a_w\gamma_z \sim \frac{1}{w-z} + \text{reg}$, $(\partial b)_w(\partial c)_z \sim \frac{-1}{(w-z)^2} + \text{reg}$ between these fields. Explicitly, one has the commutation relations

$$[a_n, \gamma_m] = \delta_{n,-m}, \quad [(\partial b)_n, (\partial c)_m] = m\,\delta_{n,-m} \tag{4.31}$$

and all the other Lie brackets vanish. Formulae (4.30) can be seen as an analog of the Sugawara construction in Wess-Zumino-Witten theory, expressing

[11] We omit for brevity the superscript (z), understanding that all operators here act on $\mathbb{F}_z$ for a fixed point $z \in \Sigma$.

Virasoro generators as quadratic combinations of generators of the current algebra.

4.2.5. Witten's descent of observables. We are interested in constructing "p-observables" – composite fields $\mathcal{O}^{(p)}$ with values in p-forms on Σ with the property that

$$Q\mathcal{O}^{(p)} = d\mathcal{O}^{(p-1)} \tag{4.32}$$

for some $\mathcal{O}^{(p-1)}$ and d the de Rham operator. This would imply that, for $\gamma \subset \Sigma$ any p-cycle, the integral $\int_\gamma \mathcal{O}^{(p)}$ is Q-closed; in particular, a correlator of several such expressions is a gauge-independent quantity. Equation (4.32) is known as Witten's descent equation for observables [9].

One can solve equation (4.32) using operators $G_{-1}, \bar{G}_{-1}$. Namely, we introduce the operator

$$\Gamma = -dz\, G_{-1} - d\bar{z}\, \bar{G}_{-1} : \qquad \mathbb{F}_z \otimes \wedge^p T^*_z\Sigma \to \mathbb{F}_z \otimes \wedge^{p+1} T^*_z\Sigma \tag{4.33}$$

It can be viewed as the contraction of the de Rham operator $d = dz\,\partial + d\bar{z}\,\bar{\partial}$ with Fourier mode -1 of G_{tot}. By virtue of (4.29) and (4.28), we have

$$[Q, \Gamma] = dz\, L_{-1} + d\bar{z}\, \bar{L}_{-1} = d \tag{4.34}$$

We fix a Q-closed 0-observable $\mathcal{O}^{(0)} \in \mathbb{O}_z$ – a polynomial in B and c, cf. (4.18), and construct

$$\mathcal{O}^{(1)} := \Gamma\mathcal{O}^{(0)} = -(dz\, G_{-1} + d\bar{z}\, \bar{G}_{-1})\, \mathcal{O}^{(0)}, \tag{4.35}$$

$$\mathcal{O}^{(2)} := \frac{1}{2}\Gamma^2\mathcal{O}^{(0)} = -dz\, d\bar{z}\, G_{-1}\, \bar{G}_{-1}\, \mathcal{O}^{(0)} \tag{4.36}$$

Observables $\mathcal{O}^{(0)}, \mathcal{O}^{(1)}, \mathcal{O}^{(2)}$ satisfy the descent equation (4.32) for $p = 0, 1, 2$.[12]

Explicitly, G_{-1} and $\bar{G}_{-1}$ act on $\mathbb{F}_z$ as derivations defined on generators by

$$G_{-1}: \qquad c \mapsto -a,\ \gamma \mapsto \partial b, \quad \bar{\gamma}, b, a, \bar{a} \mapsto 0 \tag{4.37}$$

$$\bar{G}_{-1}: \qquad c \mapsto -\bar{a},\ \bar{\gamma} \mapsto \bar{\partial} b, \quad \gamma, b, a, \bar{a} \mapsto 0 \tag{4.38}$$

Consider the case of an N-component theory (4.19). For $\mathcal{O}^{(0)} \in \mathbb{O}_z^{[N]}$ a polynomial in $B_1, \ldots, B_N, c^1, \ldots c^N$, the descended observables defined by (4.35,4.36) are:

$$\begin{aligned} \mathcal{O}^{(1)} &= \left((dz\, a^j + d\bar{z}\, \bar{a}^j) \frac{\partial}{\partial c^j} + (i\, dz\, \partial b_j - i\, d\bar{z}\, \bar{\partial} b_j) \frac{\partial}{\partial B_j} \right) \mathcal{O}^{(0)} \\ &= \left(A^j \frac{\partial}{\partial c^j} - {*db_j} \frac{\partial}{\partial B_j} \right) \mathcal{O}^{(0)} \end{aligned} \tag{4.39}$$

---

[12] Indeed, the descent equation for $p = 0$ reads $Q\mathcal{O}^{(0)} = 0$ which is satisfied by assumption. Next, for $p = 1$, we have $Q\mathcal{O}^{(1)} = [Q, \Gamma]\mathcal{O}^{(0)} = d\mathcal{O}^{(0)}$ by (4.34). Finally, for $p = 2$, we have $Q\mathcal{O}^{(2)} = \frac{1}{2}Q\Gamma\mathcal{O}^{(1)} = \frac{1}{2}([Q, \Gamma]\mathcal{O}^{(1)} + \Gamma Q\mathcal{O}^{(1)}) = \frac{1}{2}(d\mathcal{O}^{(1)} + \Gamma d\mathcal{O}^{(0)}) = \frac{1}{2}(d\mathcal{O}^{(1)} + d\mathcal{O}^{(1)}) = d\mathcal{O}^{(1)}$. Here we used that Γ commutes with $d = dz\, L_{-1} + d\bar{z}\, \bar{L}_{-1}$.

and

(4.40)

$$\begin{aligned}\mathcal{O}^{(2)} &= \\ &= -dz\,d\bar{z}\left(a^j\bar{a}^k\frac{\partial^2}{\partial c^j\partial c^k}+i(a^j\bar\partial b_k+\bar{a}^j\partial b_k)\frac{\partial^2}{\partial c^j\partial B_k}+\partial b_j\bar\partial b_k\frac{\partial^2}{\partial B_j\partial B_k}\right)\mathcal{O}^{(0)} \\ &= \left(-\frac{1}{2}A^j\wedge A^k\frac{\partial^2}{\partial c^j\partial c^k}-A^j\wedge *db_k\frac{\partial^2}{\partial c^j\partial B_k}+\frac{1}{2}*db_j\wedge *db_k\frac{\partial^2}{\partial B_j\partial B_k}\right)\mathcal{O}^{(0)}\end{aligned}$$

Example 4.2.6. Taking $\mathcal{O}^{(0)}=c$ (in 1-component theory), we get $\mathcal{O}^{(1)}=dz\,a+d\bar{z}\,\bar{a}=A$ and $\mathcal{O}^{(2)}=0$. In particular, we can integrate this 1-observable along a closed oriented curve $\gamma\subset\Sigma$, obtaining a Q-closed expression $\oint_\gamma A$. Then one can, e.g., consider a correlator $\left\langle B(z)\oint_\gamma A\right\rangle$. The expression in the correlator is Q-closed and thus the correlator is topological – invariant under isotopy. Using the propagator $\langle B(z)A(w)\rangle=2d_w\arg(w-z)$, we can compute this correlator:

$$\left\langle B(z)\oint_\gamma A\right\rangle=4\pi\,\mathrm{lk}(\gamma,z) \tag{4.41}$$

where $\mathrm{lk}(\gamma,z)$ is the "linking number" – the winding number of the curve γ around the point z.[13]

Example 4.2.7. In N-component theory, consider

$$\mathcal{O}^{(0)}=W(c)$$

a polynomial in variables c^j containing only monomials of even degree. Then we have

$$\mathcal{O}^{(1)}=A^j\frac{\partial}{\partial c^j}W(c),\qquad \mathcal{O}^{(2)}=-\frac{1}{2}A^j\wedge A^k\frac{\partial^2}{\partial c^j\partial c^k}W(c)$$

This 2-observable determines a deformation of the abelian theory analogous to the deformation of the Landau-Ginzburg model by a superpotential.

Example 4.2.8. In N-component theory, consider the cubic observable

$$\mathcal{O}^{(0)}=\frac{1}{2}f^i_{jk}B_i c^j c^k$$

with f^i_{jk} arbitrary constant coefficients with $f^i_{jk}=-f^i_{kj}$. Note that, from the viewpoint of interpretation (4.20) of 0-observables as polyvectors, this is a quadratic vector field on $\Pi\mathbb{R}^N$. Then we have

$$\mathcal{O}^{(1)}=f^i_{jk}B_iA^jc^k-\frac{1}{2}f^i_{jk}*db_i\,c^jc^k,\qquad \mathcal{O}^{(2)}=\frac{1}{2}f^i_{jk}B_iA^jA^k-f^i_{jk}*db_i\,A^jc^k$$

This 2-observable, in the case when f^i_{jk} are the structure constants of a Lie algebra, determines the deformation of the abelian BF theory into the non-abelian BF theory.

[13] Here we are implicitly assuming that z is not on γ. If $z\in\gamma$, the correlator also makes sense: the linking number in (4.41) then takes a half-integer value – the half-sum of the values obtained by displacing z normally to γ in two possible directions.

4.2.5.1. *Descent vs. AKSZ construction.* Within Batalin-Vilkovisky formalism, the N-component abelian BF theory is defined by the master action coming from the AKSZ construction [1],

$$S^{\mathrm{BV}} = \int B_k dA^k + A_k^* dc^k + \lambda_k b^{*k} = \underbrace{\int \mathcal{B}_k d\mathcal{A}^k}_{S^{\mathrm{AKSZ}}} + \underbrace{\int \lambda_k b^{*k}}_{S^{\mathrm{aux}}} \tag{4.42}$$

– a function on the space of BV fields

$$\begin{aligned} \mathcal{F}^{\mathrm{BV}} &= \mathcal{F}^{\mathrm{AKSZ}} \times \mathcal{F}^{\mathrm{aux}} \\ &= \mathrm{Map}(T[1]\Sigma, T^*[1]V[1]) \times \Big(\underbrace{\Omega^0}_{\lambda^k} \oplus \underbrace{\Omega^0[-1]}_{b_k} \oplus \underbrace{\Omega^2[-1]}_{\lambda^{*k}} \oplus \underbrace{\Omega^2}_{b^{*k}} \Big) \end{aligned} \tag{4.43}$$

Here $V = \mathbb{R}^N$ is the coefficient space of the theory; Ω^p is a shorthand for $\Omega^p(\Sigma) \otimes V$. The first factor above – the AKSZ mapping space is parameterized by the fields c^k, A^k, B_k and the respective anti-fields c_k^*, A_k^*, B^{*k} which assemble into two *AKSZ superfields* – nonhomogeneous forms on Σ,

$$\mathcal{A}^k = c^k + A^k + B^{*k}, \qquad \mathcal{B}_k = B_k + A_k^* + c_k^*$$

parameterizing the first and second term in $\mathcal{F}^{\mathrm{AKSZ}} = \Omega^\bullet(\Sigma, V[1]) \oplus \Omega^\bullet(\Sigma, V^*)$. Thus the entire field content in BV setting is:

field/antifield	c^k	A^k	B^{*k}	B_k	A_k^*	c_k^*	λ_k	b_k	λ^{*k}	b^{*k}
form degree on Σ	0	1	2	0	1	2	0	0	2	2
ghost number	1	0	-1	0	-1	-2	0	-1	-1	0

Here objects without stars are the BRST fields and objects with stars are the corresponding anti-fields. $\mathcal{F}^{\mathrm{BV}}$ carries the symplectic form of ghost number -1, $\omega^{\mathrm{BV}} = \sum_\phi \int \delta\phi \wedge \delta\phi^*$ where the sum is over all species of BRST fields, $\phi \in \{c^k, A^k, B_k, \lambda_k, b_k\}$. The action (4.42) satisfies the classical master equation

$$(S^{\mathrm{BV}}, S^{\mathrm{BV}}) = 0$$

with $(-,-)$ the Poisson bracket (the *BV anti-bracket*) on functions on $\mathcal{F}^{\mathrm{BV}}$ associated to the symplectic structure ω^{BV}.

Imposing the Lorenz gauge-fixing corresponds in the BV language to restricting from the whole space of BV fields to a Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}^{\mathrm{BV}} = T^*[-1]\mathcal{F}$ defined as the graph of the exact 1-form $\delta\Psi$ on the space of BRST fields, with Ψ the gauge-fixing fermion (4.6). Explicitly, $\mathcal{L}$ is given by

$$\mathcal{L}: \quad \begin{cases} c^k, A^k, B_k, \lambda_k, b_k \quad \text{are free} \\ A_k^* = - * db_k, \quad b^{*k} = d * A^k, \quad c_k^* = B^{*k} = \lambda^{*k} = 0 \end{cases} \tag{4.44}$$

In particular the restriction $S^{\mathrm{BV}}\big|_{\mathcal{L}}$ is exactly the gauge-fixed action (4.2).

Denote by $\mathcal{X} = T^*[1]V[1] = V[1] \oplus V^*$ the target of the AKSZ mapping space, appearing in (4.43). Let $\mathrm{ev}: \mathcal{F}^{\mathrm{AKSZ}} \times T[1]\Sigma \to \mathcal{X}$ be the evaluation map for the AKSZ mapping space. Looking at (4.20) and our computation of the descent (4.39,4.40), we make the following observations:

(i) The space of 0-observables $\mathbb{O}_z$ (4.20) coincides with the space of functions on the AKSZ target $\mathcal{X}$.

(ii) For any 0-observable, $\mathcal{O}^{(0)} \in \mathbb{O}_z$, adding to it its first and second descent, we obtain the pullback of $\mathcal{O}^{(0)}$, regarded as a function on the AKSZ target, by the evaluation map:

$$\mathcal{O}^{(0)} + \mathcal{O}^{(1)} + \mathcal{O}^{(2)} = \mathrm{ev}^* \mathcal{O}^{(0)}\Big|_{\mathcal{L}} \tag{4.45}$$

For example, for $\mathcal{O}^{(0)} = c^k$, (4.45) yields $c^k + A^k = \mathcal{A}^k|_{\mathcal{L}}$ – the restriction of the AKSZ superfield to the Lagrangian (4.44). Likewise, for $\mathcal{O}^{(0)} = B_k$, we get $B_k - *db_k = \mathcal{B}_k|_{\mathcal{L}}$.

(iii) As immediately implied by the previous point, a deformation $S \to S + g\int \mathcal{O}^{(2)}$ of the abelian BF action by a 2-observable is the same as turning on the target Hamiltonian $g\mathcal{O}^{(0)}$ in AKSZ construction, i.e., adding to the BV action the term $g\int \mathrm{ev}^*\mathcal{O}^{(0)}$ and imposing the Lorenz gauge by restricting to the Lagrangian (4.44).

Remark 4.2.9. To any vector field ξ on Σ one can associate the following function of BV fields of ghost number -2:

$$\mathbb{G}_\xi := \int c_k^*\, \iota_\xi A^k + A_k^*\, \iota_\xi B^{*k} + db_k\, \iota_\xi \lambda^{*k}$$

The object $\mathbb{G}_\xi$ is the generator of the action of the vector field ξ, regarded as an infinitesimal diffeomorphism of Σ, on BV fields as a BV gauge transformation. I.e., one has

$$((S^{\mathrm{BV}}, \mathbb{G}_\xi), -) = \sum_\psi \int L_\xi \psi \frac{\delta}{\delta \psi}$$

– the lifting of the Lie derivative L_ξ operating on the BV fields to a vector field on $\mathcal{F}^{\mathrm{BV}}$; here ψ runs over all species of BV fields. This is an adaptation of the construction of [?] to the model in question. One has the following relation between $\mathbb{G}_\xi$ and the descent operator Γ (4.33):

$$(\mathbb{G}_\xi, \phi_z)\Big|_{\mathcal{L}} = (\iota_\xi \Gamma) \circ \phi_z \tag{4.46}$$

for $\phi \in \{A^k, B_k, c^k, \lambda_k, b_k\}$ any BRST field.

Reformulation with auxiliary fields extended to AKSZ superfields. Note that in (4.42,4.43) the BV system is presented as a sum of an AKSZ system and an auxiliary system which is not of AKSZ form. One can in fact cast the auxiliary system in AKSZ form, too, by extending the four auxiliary fields $b_k, \lambda_k, b^{*k}, \lambda^{*k}$ to a quadruple of AKSZ superfields:

$$\begin{aligned}
\widehat{\lambda_k} &= \boxed{\lambda_k} + \mu_k + \nu_k \\
\widehat{\lambda^{*k}} &= \nu^{*k} + \mu^{*k} + \boxed{\lambda^{*k}} \\
\widehat{b_k} &= \boxed{b_k} + f_k^* + e_k^* \\
\widehat{b^{*k}} &= e^k + f^k + \boxed{b^{*k}}
\end{aligned}$$

The form degrees and ghost numbers of the field components here are as follows.

field/antifield	λ_k	μ_k	ν_k	ν^{*k}	μ^{*k}	λ^{*k}	b_k	f^*_k	e^*_k	e^k	f^k	b^*_k
form degree on Σ	0	1	2	0	1	2	0	1	2	0	1	2
ghost number	0	-1	-2	1	0	-1	-1	-2	-3	2	1	0

The BV action (4.42) in this setting is replaced with the full AKSZ action[14]

$$\widetilde{S} = \int \mathcal{B}_k\, d\mathcal{A}^k + \widehat{\lambda^{*k}}\, d\widehat{\lambda_k} + \widehat{b^{*k}}\, d\widehat{b_k} + \widehat{\lambda_k}\, \widehat{b^{*k}} \tag{4.47}$$

– a function on the full AKSZ mapping space

$$\widetilde{\mathcal{F}} = \mathrm{Map}(T[1]\Sigma\ ,\ \underbrace{T^*[1]V[1]}_{\mathcal{X}} \times \underbrace{T^*[1](V^*[-1]\oplus V^*)}_{\mathcal{X}^{\mathrm{aux}}})$$

The target $\mathcal{X}^{\mathrm{full}} = \mathcal{X}\times\mathcal{X}^{\mathrm{aux}} = T^*[1](V[1]\oplus V^*[-1]\oplus V^*)$ is a shifted cotangent bundle with base coordinates $\underline{c}^k, \underline{b}_k, \underline{\lambda}_k$ (corresponding to the superfields $\mathcal{A}^k$, $\widehat{b_k}$, $\widehat{\lambda_k}$) and fiber coordinates $\underline{B}_k, \underline{e}^k, \underline{\nu}^{*k}$ (corresponding to superfields $\mathcal{B}_k$, $\widehat{b^{*k}}$, $\widehat{\lambda^{*k}}$). Kinetic term of (4.47) corresponds to the standard canonical 1-form on the target (as a cotangent bundle); term $\widehat{\lambda_k}\,\widehat{b^{*k}}$ corresponds to the target Hamiltonian $\Theta = \underline{\lambda}_k\, \underline{e}^k$.[15] The gauge-fixing Lagrangian is $\widetilde{\mathcal{L}} = \mathrm{graph}(\delta\Psi)$ in $\widetilde{\mathcal{F}}$, regarded as the cotangent bundle to the space of non-starred fields, with Ψ as before (4.6) (viewed as a constant function in fields μ_k, ν_k, e^k, f^k):

$$\widetilde{\mathcal{L}}: \quad \begin{cases} c^k, A^k, B_k, \lambda_k, \mu_k, \nu_k, b_k, e^k, f^k \quad \text{are free}, \\ A^*_k = -*\, db_k, \quad b^{*k} = d*A^k, \\ c^*_k = B^{*k} = \lambda^{*k} = \mu^{*k} = \nu^{*k} = f^*_k = e^*_k = 0 \end{cases}$$

Restriction of the action (4.47) to $\widetilde{\mathcal{L}}$ yields

$$\widetilde{S}\Big|_{\widetilde{\mathcal{L}}} = S + \int (\mu_k f^k + \nu_k e^k)$$

with S the BRST action (4.2). Integrating out the fields μ_k, ν_k, e^k, f^k, we obtain the action S.

In this setting for abelian *BF* theory, with auxiliary AKSZ superfields, the generator of an infinitesimal diffeomeorphism (cf. Remark 4.2.9) is:

$$\begin{aligned}\widetilde{\mathbb{G}}_\xi &= \int \mathcal{B}_k\, \iota_\xi \mathcal{A}^k + \widehat{\lambda^{*k}}\, \iota_\xi \widehat{\lambda_k} + \widehat{b^{*k}}\, \iota_\xi \widehat{b_k} \\ &= \int c^*_k\, \iota_\xi A^k + A^*_k\, \iota_\xi B^{*k} + \lambda^{*k}\, \iota_\xi \mu_k + \mu^{*k}\, \iota_\xi \nu_k + b^{*k}\, \iota_\xi f^*_k + f^k\, \iota_\xi e^*_k\end{aligned}$$

Relation (4.46) holds again in this setting, modulo equations of motion.

[14] In fact, one can write a simpler action $\bar{S} = \int \mathcal{B}_k\, d\mathcal{A}^k + \widehat{\lambda_k}\,\widehat{b^{*k}}$, which is BV canonically equivalent to $\widetilde{S}$ by $\widetilde{S} = \bar{S} + \left(\bar{S}, \int \widehat{\lambda^{*k}}\, d\widehat{b_k}\right)$.

[15] Note that passing to the cohomology of the cohomological vector field $Q_{\mathrm{target}} = (\Theta, -)$ acting on functions on $\mathcal{X}^{\mathrm{full}}$ contracts the auxiliary part of the target and yields functions on $\mathcal{X}$, or the space of 0-observables $\mathbb{O}_z$.

4.2.5.2. *Towards Gromov-Witten invariants: a toy example.* Correlators of the form

$$\big\langle G_{\mathrm{tot}}(w_1)\cdots G_{\mathrm{tot}}(w_p)\;\Phi_1(z_1)\cdots\Phi_n(z_n)\big\rangle$$

with Q-closed fields $\Phi_1,\ldots,\Phi_n$ define closed p-forms on $\mathcal{M}_{\Sigma,n}$ – the moduli space of conformal structures on Σ with n marked points $z_1,\ldots,z_n$ (here Σ can be any surface). Integrating such a form over a p-cycle on $\mathcal{M}_{\Sigma,n}$, one obtains interesting periods – a version of Gromov-Witten invariants. Example 4.2.6 above leads to a simple example of such a period.

Example 4.2.10. [16] For $\Sigma=\mathbb{C}P^1$, consider the correlator

$$\rho=\langle\,\Gamma\big(c(z_0)B(z_1)\widetilde{\Theta}(z_2)c(z_3)\big)\,\rangle_{\mathbb{C}P^1} \tag{4.48}$$

where we understand that the descent operator Γ (4.33) acts on a product of fields as a derivation. This is a correlator on the sphere rather than on a plane, with $\widetilde{\Theta}(z_2)=\delta(b)\delta(\gamma)\delta(\bar\gamma)\big|_{z_2}$ and $c(z_3)=\delta(c)|_{z_3}$ the "soaking operators" for zero-modes of the kinetic operators $\partial,\bar{\boldsymbol{\partial}},\partial\bar{\boldsymbol{\partial}}$ in the action. We refer the reader to section 2.4 in [5] for further details, and to section 10 of [**?**] for the general technology of correlators involving delta-functions of fields in $\beta\gamma$-systems. Explicit evaluation of ρ yields

$$\rho=2\,d\,\arg\frac{(z_0-z_1)(z_2-z_3)}{(z_0-z_2)(z_1-z_3)}\qquad\in\Omega^1(\mathrm{Conf}_4(\mathbb{C}P^1))$$

Note that here the four points on $\mathbb{C}P^1$ enter via their cross-ratio. In particular, ρ is a closed $PSL_2(\mathbb{C})$-*basic* 1-form on the open configuration space of 4 points on $\mathbb{C}P^1$ and thus descends to a closed 1-form on the (non-compactified) moduli space $\mathcal{M}_{0,4}$ of conformal structures on $\mathbb{C}P^1$ with 4 marked points.

As an example of a period: integrating ρ over z_0 in a contour $\boldsymbol{\gamma}\subset\mathbb{C}P^1-\{z_1,z_2,z_3\}$, we get the "linking number" of $\boldsymbol{\gamma}$ and the 0-cycle $[z_1]-[z_2]$ – the difference of winding numbers of $\boldsymbol{\gamma}$ around z_1 and around z_2:[17]

$$\oint_{\boldsymbol{\gamma}\ni z_0}\rho=4\pi\;\mathrm{lk}(\boldsymbol{\gamma},[z_1]-[z_2])$$

One can see this computation as an integral of a closed 1-form on $\mathcal{M}_{0,4}$ over a 1-cycle $\boldsymbol{\gamma}$ in the fiber of the projection $\mathcal{M}_{0,4}\to\mathcal{M}_{0,3}=*$ forgetting the point z_0, yielding the linking number as a simplest Gromov-Witten period.

As a small variation on this example, instead of a single insertion of B in (4.48), we can consider several insertions of $e^{\alpha B}$, with α some coupling

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[16] This example is a corrected version of the one given in the previous version of the paper. The old example contained a mistake: the descent was applied only to one field in the correlator, which led to a 1-form on the configuration space which was $PSL_2(\mathbb{C})$-invariant but not horizontal, and hence did not descend to the moduli space.

[17] Note that the winding number of a closed curve $\boldsymbol{\gamma}$ around a point z_k is well-defined on a plane but not on a sphere. However, the difference is well-defined on a sphere. I.e., mapping the sphere onto the plane by stereographic projection with "North pole" at $p\in\mathbb{C}P^1-\boldsymbol{\gamma}\cup\{z_1,z_2,z_3\}$, we prescribe values to winding numbers which jump when p crosses $\boldsymbol{\gamma}$, but the difference of winding numbers does not jump. More generally, on the sphere one has a well-defined linking number $\mathrm{lk}(\boldsymbol{\gamma},\xi)$ of a closed curve and a 0-cycle $\xi=\sum_{k=1}^n\alpha_k[z_k]$ if and only if the sum of coefficients vanishes, $\sum_k\alpha_k=0$.

constants, which gives the period

$$\int_{\gamma\ni z_0} \Big\langle\ \Gamma\Big(c(z_0)e^{\alpha_1 B(z_1)}\cdots e^{\alpha_n B(z_n)}\widetilde{\Theta}(z_{n+1})c(z_{n+2})\Big)\ \Big\rangle$$
$$= 4\pi\ \mathrm{lk}\Big(\gamma, \sum_{k=1}^n \alpha_k[z_k] - \big(\sum_{k=1}^n \alpha_k\big)[z_{n+1}]\Big)$$

Here on the l.h.s. we integrate a closed 1-form on $\mathcal{M}_{0,n+3}$ (corresponding to the $PSL_2(\mathbb{C})$-basic 1-form $2\sum_{k=1}^n \alpha_k d\ \arg\frac{(z_0-z_k)(z_{n+1}-z_{n+2})}{(z_0-z_{n+1})(z_k-z_{n+2})}$ on the open configuration space of $n+3$ points on $\mathbb{C}P^1$) over a 1-cycle γ in the fiber of $\mathcal{M}_{0,n+3}\to\mathcal{M}_{0,n+2}$.

4.2.6. BV algebra structure on the space of 0-observables. [18]

The space of 0-observables $\mathbb{O}_z$ in addition to being a graded commutative algebra has a degree -1 Poisson bracket defined by

$$\{\mathcal{O}_1,\mathcal{O}_2\} := \frac{(-1)^{|\mathcal{O}_1|}}{4\pi}\oint_{\mathcal{C}_z}(\Gamma\mathcal{O}_1)_w(\mathcal{O}_2)_z \tag{4.49}$$

I.e. one descends the first 0-observable to a 1-observable and integrates over a contour encircling the second 0-observable.

EXAMPLE 4.2.11. For $\mathcal{O}_1 = c$, $\mathcal{O}_2 = B$, we have $\Gamma(c) = A$ and we obtain

$$\{c,B\} = -\frac{1}{4\pi}\oint_{\mathcal{C}_z}\underbrace{A_w B_z}_{\sim 2d_w\,\arg(w-z)+\mathrm{reg}} = -1$$

cf. example 4.2.6. In particular, c and B are conjugate variables for the Poisson bracket.

Explicitly, the Poisson bracket (4.49) is:

$$\{\mathcal{O}_1,\mathcal{O}_2\} = \frac{\partial}{\partial B}\mathcal{O}_1\ \frac{\partial}{\partial c}\mathcal{O}_2 + (-1)^{|\mathcal{O}_1|}\frac{\partial}{\partial c}\mathcal{O}_1\ \frac{\partial}{\partial B}\mathcal{O}_2$$

Commutative multiplication together with this bracket comprise the structure of a "P_2 algebra" on $\mathbb{O}_z$ (the algebra over the homology of the operad E_2 of little 2-disks[19]).

In addition to the bracket $\{-,-\}$, one has the operator

$$G_0^- := \frac{1}{2i}(G_0 - \bar{G}_0) \tag{4.50}$$

[18] We refer the reader to [3] for the details on emergence of the BV structure in the context of twisted superconformal theory.

[19] Recall that E_2 is a topological operad with the space of n-ary operations $E_2(n)$ being the space of configurations $\mathbf{o}$ of n ordered disjoint disks inside the unit disk in $\mathbb{R}^2\simeq\mathbb{C}$; an i-th composition $\mathbf{o}_1\circ_i\mathbf{o}_2$ of two operations corresponds to fitting a rescaled disk configuration $\mathbf{o}_2$ instead of i-th disk in $\mathbf{o}_1$. It is instructive to think of a disk configuration $\mathbf{o}$, considered modulo rescalings, as a 2-dimensional cobordism from n in-circles (boundaries of the inner disks) to the single out-circle, with composition in E_2 corresponding to composition (gluing) of cobordisms. The *framed* operad of little 2-disks, E_2^{fr} is defined in the same way where additionally one marks a point on the boundary of each disk, and the outer unit disk comes with the standard marked point $1\in\mathbb{C}$. Then in the composition, in addition to rescaling a disk configuration, one does a rotation so as to fit the marked point on the out-disk of $\mathbf{o_2}$ with the marked point on the i-th disk of $\mathbf{o_1}$.

– the contraction of G_{tot} with the vector field corresponding to rotation about the point z, acting on $\mathbb{F}_z$ and in particular on $\mathbb{O}_z$.

EXAMPLE 4.2.12. E.g. acting on $\mathcal{O} = Bc$, one has

$$\begin{aligned} G_w \mathcal{O}_z &= (a\partial b)_w (Bc)_z \\ &\sim \frac{-i}{w-z}\frac{-1}{w-z} + \frac{-i}{w-z}(\partial b)_w\, c_z + \frac{-1}{w-z} a_w B_z + \text{reg} \\ &\sim \frac{i}{(w-z)^2} + \frac{(-i\partial b\, c - aB)_z}{w-z} + \text{reg} \end{aligned}$$

Thus, $G_0(Bc) = i$ – the coefficient of the second order pole in the OPE above, and similarly one obtains $\bar{G}_0(Bc) = -i$. Therefore, $G_0^-(Bc) = 1$.

Explicitly, the operator G_0^- acts on 0-observables by

$$G_0^- : \quad \mathcal{O} \mapsto \frac{\partial^2}{\partial B\, \partial c}\mathcal{O}$$

Thus, one recognizes in G_0^- the Batalin-Vilkovisky Laplacian (of degree -1) and hence $(\mathbb{O}_z, \cdot, \{-,-\}, \Delta)$ is a BV algebra with the bracket and the Laplacian of degree -1. In other words, it is an algebra over the homology of the operad E_2^{fr} of *framed* little 2-disks.

Note that, from the standpoint of identification of 0-observables with polyvectors (4.20), this is the standard BV algebra structure on polyvectors.

REMARK 4.2.13. In the example 4.2.8, we expect $\mathcal{O}^{(2)}$ to give a classically consistent deformation of the action $S \mapsto S + g\int_\Sigma \mathcal{O}^{(2)}$ (with g the deformation parameter) if and only if f^i_{jk} satisfy Jacobi identity, i.e., define a Lie algebra on the space of coefficients $\mathbb{R}^N$ and we expect the deformation to be consistent on the quantum level if additionally the unimodularity property $f^i_{ij} = 0$ holds. Note that these two cases correspond to, respectively, classical and quantum BV master equation holding for $\mathcal{O}^{(0)} = \frac{1}{2} f^i_{jk} B_i c^j c^k$:

$$\{\mathcal{O}^{(0)}, \mathcal{O}^{(0)}\} = 0, \qquad G_0^- \mathcal{O}^{(0)} = 0$$

REMARK 4.2.14. One can consider S^1-equivariant version of BRST cohomology (4.18) – cohomology of the equivariant extension of the BRST operator

$$Q_{S^1} := Q + \epsilon G_0^-$$

acting on the kernel of $L_0^- \propto Q_{S^1}^2$ in $\mathbb{F}_z[\epsilon]$ (rotationally-invariant fields valued in polynomials in ϵ), with ϵ the degree 2 equivariant parameter. This equivariant cohomology evaluates, in the context of N-component theory, to

$$H_{S^1}(\mathbb{F}_z) \simeq H_{\epsilon G_0^-}(\mathbb{O}_z[\epsilon]) = T_{\text{poly}}^{\text{div-free}}(\Pi\mathbb{R}^N) \oplus c^1 \ldots c^N \cdot \epsilon\mathbb{C}[\epsilon]$$

– the space of divergence-free (or "unimodular") polyvectors on $\Pi\mathbb{R}^N$, plus the $\mathbb{C}[\epsilon]$-linear span of the products of all ghosts times ϵ.

4.2.6.1. *Structure of an algebra over the framed E_2 operad on the space of composite fields.* The space of composite fields itself $\mathbb{F}_z$ has the structure of an algebra over the operad E_2^{fr} of framed little 2-disks. Namely, given a configuration $\mathbf{o} \in E_2^{\mathrm{fr}}(n)$ of n framed disks with centers at $z_1, \ldots, z_n$, radii $r_1, \ldots, r_n$ and rotation angles $\theta_1, \ldots, \theta_n$, one constructs a map

$$\mathbf{o} : \mathbb{F}_{z_1} \otimes \cdots \otimes \mathbb{F}_{z_n} \to \mathbb{F}_0 \tag{4.51}$$

which sends an n-tuple of composite fields $\Phi_1(z_1), \ldots, \Phi_n(z_n)$ to a field $\Psi \in \mathbb{F}_0$ characterized by the property that

$$\left\langle \left(\prod_{j=1}^{n} r_j^{\hat{H}^{(z_j)}} e^{i\theta_j \hat{P}^{(z_j)}} \Phi_j(z_j) \right) \phi_1(y_1) \cdots \phi_m(y_m) \right\rangle = \langle \Psi(0) \phi_1(y_1) \cdots \phi_m(y_m) \rangle \tag{4.52}$$

for any test fields $\phi_1, \ldots, \phi_m$ inserted at points $y_1, \ldots, y_m$ outside the unit disk on $\mathbb{C}$. Here $\hat{H}^{(z)} := L_0^{(z)} + \bar{L}_0^{(z)}$ and $\hat{P}^{(z)} := L_0^{(z)} - \bar{L}_0^{(z)}$ are the energy and momentum operators acting on fields at z; in particular, for a field $\Phi(z)$ of conformal weights $(h, \bar{h})$, the rescaling factor in the l.h.s. of (4.52) is

$$r^{h+\bar{h}} e^{i\theta(h-\bar{h})} \tag{4.53}$$

Thus, operation (4.51) is an n-point version of an operator product expansion, rescaled appropriately to account for the size and orientation of the disks.

One calculates operations (4.51) explicitly using Wick's lemma: one considers all partial contractions between basic fields in $\Phi_1, \ldots, \Phi_n$, replaces those with the appropriate propagators and replaces all the remaining fields with their Taylor expansion at zero. Finally, one rescales the result with the factors (4.53) (we are assuming for simplicity that fields Φ_j, with $1 \leq j \leq n$, have well-defined conformal weights, i.e., are eigenvectors for the operators $L_0, \bar{L}_0$).

Example 4.2.15. Let $\mathbf{o}$ be a configuration of two disks centered at z_1, z_2 with radii r_1, r_2 and rotation angles θ_1, θ_2, and let $\Phi_1 = J = \gamma\, \partial c$ and $\Phi_2 = G = a\, \partial b$. Recall that the conformal weights are $(h, \bar{h}) = (1, 0)$ for J and $(h, \bar{h}) = (2, 0)$ for G. We obtain from Wick's lemma

$$\begin{aligned} &\mathbf{o}(J \otimes G) = \\ &= r_1 e^{i\theta_1} (r_2 e^{i\theta_2})^2 \Bigg(-\frac{1}{(z_1 - z_2)^3} + \frac{:\gamma_{z_1} a_{z_2}:}{(z_1 - z_2)^2} - \\ &\qquad\qquad - \frac{:(\partial c)_{z_1} (\partial b)_{z_2}:}{z_1 - z_2} + :(\gamma \partial c)_{z_1} (a \partial b)_{z_2}: \Bigg) \\ &= r_1 e^{i\theta_1} (r_2 e^{i\theta_2})^2 \Bigg(-\frac{1}{(z_1 - z_2)^3} + \\ &\qquad + \sum_{k,l \geq 0} \frac{z_1^k z_2^l}{k!\, l!} \left(\frac{\partial^k \gamma\, \partial^l a}{(z_1 - z_2)^2} - \frac{\partial^{k+1} c\, \partial^{l+1} b}{z_1 - z_2} + \partial^k (\gamma \partial c)\, \partial^l (a \partial b) \right) \Bigg) \end{aligned}$$

Here in the last expression all fields are evaluated at $z = 0$. Note that the Taylor series in k, l converges under the correlator with test fields inserted at points outside the unit disk, using that z_1, z_2 are inside the unit disk.

This way one constructs the E_2^{fr}-algebra structure on the space of composite fields $\mathbb{F}_z$. Extending it by linearity, one gets the action of singular 0-chains of E_2^{fr} on $\mathbb{F}_z$. In a similar way one constructs the action of all chains $C_\bullet(E_2^{\mathrm{fr}})$ on $\mathbb{F}_z \otimes \wedge^\bullet T_z^*\Sigma$ – composite fields with values in differential forms: one constructs the following differential form on $E_2^{\mathrm{fr}}(n)$ with values in products of fields:

$$\prod_{j=1}^{n} \underbrace{\zeta_j^{L_0}\left(1-\frac{d\zeta_j}{\zeta_j}G_0\right)\bar{\zeta}_j^{\bar{L}_0}\left(1-\frac{d\bar{\zeta}_j}{\bar{\zeta}_j}\bar{G}_0\right)}_{=\exp\left[Q-d_\zeta\,,\,G_0\log\zeta_j+\bar{G}_0\log\bar{\zeta}_j\right]}\mathbf{\Phi}_j(z_j) \tag{4.54}$$

and integrates it over the chain in E_2^{fr}. This construction is considered under the correlator with an arbitrary collection of test fields outside the unit disk, as in (4.52). Here $\zeta_j = r_j e^{i\theta_j}$ and $\mathbf{\Phi}_j(z_j)$ are composite fields with values in differential forms on Σ; we suppressed the superscripts (z_j) for the operators L_0, G_0 and their conjugates.

Further, one can restrict the construction above (4.54) to fields of form

$$\mathbf{\Phi}(z) = \Phi(z) + \Gamma\Phi(z) + \frac{1}{2}\Gamma^2\Phi(z) = e^\Gamma\Phi(z) \tag{4.55}$$

with Γ as in (4.33) – i.e. sums of an ordinary (not form-valued) composite field and its first and second descents. This way we obtain a representation of E_2^{fr} as a *differential graded* operad on the space of composite fields $\mathbb{F}_z$ (not form-valued), viewed as a cochain complex with BRST differential Q.[20]

Passing to (co)homology, we get the action of the homology $H_\bullet(E_2^{\mathrm{fr}})$ on $H_Q^\bullet(\mathbb{F}_z) = \mathbb{O}_z$ – the BV algebra structure $(\mathbb{O}_z, \cdot, \{-,-\}, G_0^-)$ described above.

REMARK 4.2.16. In the discussion of the E_2^{fr}-action on composite fields and BV algebra structure on 0-observables, we used only a part of the extended Virasoro symmetry of the space of composite fields – only modes $n=-1$ and $n=0$ (which displace and rotate/dilate the disks). Using the rest of the modes, one can infinitesimally reparameterize and deform the disks and thus, integrating the Virasoro action, one can extend the E_2^{fr}-action to the action of (the chains of) a larger operad of general genus zero conformal cobordisms $S^1 \sqcup \cdots \sqcup S^1 \to S^1$ with parameterized boundaries (more precisely, the operad of Riemannian spheres with $n+1$ disjoint conformally embedded disks – Segal's genus zero operad). In particular, in Segal's picture of conformal field theory [7], the complex $(\mathbb{F}, Q)$ is the non-reduced space of states associated to a circle and $\mathbb{O}$ is the reduced space of states.

[20] Indeed, denote the form (4.54) evaluated on fields of form (4.55) by $\Xi(\Phi_1,\ldots,\Phi_n)$. Then we have, by construction, $(d-Q)\Xi(\Phi_1,\ldots,\Phi_n) = \sum_{j=1}^n \pm\Xi(\Phi_1,\ldots,Q\Phi_j,\ldots,\Phi_n)$. With $d=\sum_j d_{\zeta_j}+d_{z_j}$ the de Rham differential on E_2^{fr} and $\pm$ the Koszul signs. Therefore, for $C \subset E_2^{\mathrm{fr}}$ a chain, one has $\int_{\partial C}\Xi(\Phi_1,\ldots,\Phi_n) = \int_C d\Xi = \int_C \big(Q\Xi(\Phi_1,\ldots,\Phi_n) - \sum_j \pm\Xi(\Phi_1,\ldots,Q\Phi_j,\ldots,\Phi_n)\big)$. Thus the map $C_\bullet(E_2^{\mathrm{fr}}(n)) \mapsto \mathrm{Hom}(\mathbb{F}^{\otimes n},\mathbb{F})$ sending a chain C to the multilinear map $\Phi_1\otimes\cdots\otimes\Phi_n \mapsto \int_C \Xi(\Phi_1,\ldots,\Phi_n)$ is a chain map.

REMARK 4.2.17. Note that the normally-ordered version of the expression (4.54) evaluated on fields (4.55) can be rewritten as follows:

$$:\Xi(\Phi_1,\ldots,\Phi_n) := \prod_{j=1}^{n}\left(e^{[Q-d\,,\log\zeta_j\,G_0+z_j\,G_{-1}+\log\bar\zeta_j\,\bar G_0+\bar z_j\,\bar G_{-1}]}\right)\circ\Phi_j(0)$$

where d is the de Rham operator on E_2^{fr}, i.e. the total de Rham operator in variables z_j, ζ_j (and conjugates).

4.2.7. The $U(1)$-current and twisting back to a superconformal field theory. Consider the field

$$j = \gamma a \tag{4.56}$$

which we have encountered as a coefficient of the second order pole in the $J(w)G(z)$ OPE (4.24). It is the Noether current for the $U(1)$-symmetry of the action, which rotates the phases of the fields a, γ in the opposite directions: $a \mapsto e^{i\theta}a$, $\gamma \mapsto e^{-i\theta}\gamma$ and does not touch the fields $b, c, \bar a, \bar\gamma$. This current is conserved modulo equations of motion, $\bar\partial j \underset{\text{e.o.m.}}{\sim} 0$, and satisfies the following OPEs:

$$\begin{aligned}
T_w j_z &\sim \frac{1}{(w-z)^3}+\frac{j_z}{(w-z)^2}+\frac{\partial j_z}{w-z}+\mathrm{reg} && (4.57)\\
j_w J_z &\sim \frac{J_z}{w-z}+\mathrm{reg}\\
j_w G_z &\sim \frac{-G_z}{w-z}+\mathrm{reg}\\
j_w j_z &\sim \frac{-1}{(w-z)^2}+\mathrm{reg} && (4.58)
\end{aligned}$$

In particular, the fields J and G have charges $+1$ and -1 respectively w.r.t. the operator $\hat{j}$. Similarly to j, we have its anti-holomorphic counterpart $\bar j = \bar\gamma\bar a$ which satisfies the same properties in the anti-holomorphic sector.

We can consider a new "untwisted" theory[21] with same field content as before and the deformed stress-energy tensor

$$\widetilde{T} := T - \frac{1}{2}\partial j$$

With respect to the new stress-energy tensor, the fields change their (holomorphic) conformal weights as follows:

field	weight w.r.t. T	weight w.r.t. $\widetilde{T}$
J	1	3/2
G	2	3/2
j	1 (not primary)	1
γ	0	1/2
a	1	1/2
$\bar\gamma, \bar a, b, c$	0	0

[21] We call it "untwisted", since the theory we started with is obtained from it by Witten's topological twist of type B, cf. [8, 4, 2].

Thus, fields $\gamma, dz\, a$ become (even) Weyl spinors $(dz)^{\frac{1}{2}}\gamma, (dz)^{\frac{1}{2}}a$ in the untwisted theory. Similarly, $\bar\gamma, d\bar z\, a$ become even spinors $(d\bar z)^{\frac{1}{2}}\bar\gamma, (d\bar z)^{\frac{1}{2}}\bar a$. The fields b, c are unchanged. Thus, the bundle of basic fields, replacing (4.4) in the untwisted theory, is

$$\underline{\widetilde{\mathsf{F}}} = \underbrace{K^{\otimes\frac{1}{2}}}_{a} \oplus \underbrace{K^{\otimes\frac{1}{2}}}_{\gamma} \oplus \underbrace{\bar K^{\otimes\frac{1}{2}}}_{\bar a} \oplus \underbrace{\bar K^{\otimes\frac{1}{2}}}_{\bar\gamma} \oplus \underbrace{\Pi\mathbb{R}^2}_{b,c}$$

with $K = (T^{1,0})^*\Sigma$ and $\bar K = (T^{0,1})^*\Sigma$ the canonical and anti-canonical line bundles on Σ.

Note that j was not a primary field w.r.t. T, due to the 3-rd order pole in (4.57). However, j is a primary field of weight $(1,0)$ in the untwisted theory. It is a conserved $U(1)$-current and its Fourier modes generate a Heisenberg Lie algebra due to (4.58).

One obtains the following OPE of the untwisted stress-energy tensor with itself:

$$\widetilde{T}(w)\,\widetilde{T}(z) \sim \frac{-3/2}{(w-z)^4} + \frac{2\,\widetilde{T}_z}{(w-z)^2} + \frac{(\partial\widetilde{T})_z}{w-z} + \text{reg}$$

Thus, the untwisted theory has central charge $\mathsf{c} = -3$.[22] In N-component theory, the central charge becomes $\mathsf{c} = -3N$.

Consider the Fourier modes of the fields $\widetilde{T}, J, G, j$, defined via

$$\widetilde{T}_w = \sum_n (w-z)^{-n-2}\widetilde{L}_n,$$
$$J_w = \sum_r (w-z)^{-r-\frac{3}{2}} J_r,\quad G_w = \sum_s (w-z)^{-s-\frac{3}{2}} G_s,\quad j_p = \sum_p (w-z)^{-p-1} j_p$$

with $n, p \in \mathbb{Z}$ and $r, s \in \mathbb{Z} + \frac{1}{2}$ for periodic (Neveu-Schwarz) boundary conditions on fermions and $r, s \in \mathbb{Z}$ for anti-periodic (Ramond) boundary conditions. These Fourier modes satisfy the relations of $\mathcal{N} = 2$ superconformal

[22] Forgetting about the BRST structure, we can regard the action (4.10) as a superposition of three non-interacting theories: the second-order ghost system (the bc system) with holomorphic/anti-holomorphic central charges $\mathsf{c}_{bc} = \bar{\mathsf{c}}_{bc} = -2$ and two first order chiral systems with Lagrangians $\gamma\bar\partial a$ and $\bar\gamma\partial\bar a$ with central charges $\mathsf{c}_{\gamma a} = -1, \bar{\mathsf{c}}_{\gamma a} = 0$ and $\mathsf{c}_{\bar\gamma\bar a} = 0, \bar{\mathsf{c}}_{\bar\gamma\bar a} = -1$ respectively (in the *untwisted model*, where $a, \gamma, \bar a, \bar\gamma$ are even spinors). Thus the total central charge of the system is $\mathsf{c} = \bar{\mathsf{c}} = (-2) + (-1) + (0) = -3$. In the topological (twisted) model, central charges of the first-order systems change to $\mathsf{c}_{\gamma a} = 2, \bar{\mathsf{c}}_{\gamma a} = 0$ and $\mathsf{c}_{\bar\gamma\bar a} = 0, \bar{\mathsf{c}}_{\bar\gamma\bar a} = 2$, while the central charge of the ghost system remains -2. Thus, $\mathsf{c}^{\mathrm{top}} = \bar{\mathsf{c}}^{\mathrm{top}} = (-2) + (2) + (0) = 0$.

algebra with central charge $\mathsf{c} = -3$:

$$\begin{aligned}
[\widetilde{L}_n, \widetilde{L}_m] &= (n-m)\,\widetilde{L}_{n+m} - \frac{1}{4}(n^3 - n)\,\delta_{n,-m}, \\
[j_p, j_q] &= -p\,\delta_{p,-q}, \\
[\widetilde{L}_n, J_r] &= \left(\frac{n}{2} - r\right) J_{n+r}, \\
[\widetilde{L}_n, G_s] &= \left(\frac{n}{2} - s\right) G_{n+s}, \\
[\widetilde{L}_n, j_p] &= -p\, j_{n+p}, \\
[J_r, G_s] &= \widetilde{L}_{r+s} + \frac{r-s}{2}\, j_{r+s} - \frac{1}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}, \\
[j_p, J_r] &= J_{p+r}, \quad [j_p, G_s] = -G_{p+s}, \\
[J_r, J_s] &= 0, \quad [G_r, G_s] = 0
\end{aligned} \tag{4.59}$$

plus the conjugate relation for the Fourier modes of $\bar{\widetilde{T}}, \bar{J}, \bar{G}, \bar{j}$. Lie brackets, involving one generator from holomorphic sector and one from anti-holomorphic sector, vanish. In the case of N-component theory, the three central extension terms in the commutation relations (4.59) – those proportional to $\delta_{\bullet,-\bullet} \cdot 1$ – get multiplied by N.

Thus, the fields $\widetilde{T}, J, G, j$ together with their anti-holomorphic counterparts define on the untwisted abelian BF theory the structure of an $\mathcal{N} = (2,2)$ superconformal theory with supersymmetry currents J, G and with j the R-symmetry current (plus the conjugates).

4.2.7.1. *Dictionary between abelian BF theory and the B model.* Recall (see e.g. [8, 4]) that the free $\mathcal{N} = (2,2)$ supersymmetric sigma model (or Landau-Ginzburg model with zero superpotential) with target $\mathbb{C}^N$ is defined by the action

$$S = 2t \int_\Sigma d^2x \left(\bar{\phi}_k\, \partial\bar{\partial}\phi^k - i\,\bar{\psi}_{+k}\,\bar{\partial}\psi^k_+ - i\,\bar{\psi}_{-k}\,\partial\psi^k_-\right) \tag{4.60}$$

with scalar fields $\phi^k, \bar{\phi}_k$ corresponding to the holomorphic and anti-holomorphic coordinates on the target $\mathbb{C}^N$, respectively, and with $\psi^k_\pm, \bar{\psi}_{\pm k}$ fermions of spin 1/2; t is a coupling constant (which is irrelevant in the free theory as it can be absorbed into the normalization of fields). Here, bar/no bar on fields corresponds to anti-holomorphic/holomorphic directions on the target and $\pm$ corresponds to holomorphic/anti-holomorphic directions on the source Σ. In the B-twisted sigma model the action is the same, however fields $\psi^k_\pm$ attain spin 1 and $\bar{\psi}_{\pm k}$ attain spin 0.

Comparing (4.60) with (4.10), we see that the N-component abelian BF theory (i.e. with coefficient space $\mathbb{R}^N$) is the B-twisted supersymmetric sigma model with *odd* target $\Pi\mathbb{C}^N$:

$$\begin{array}{ccc}
\text{abelian } BF \text{ theory} & & \text{B model} \\
\Big\uparrow{\scriptstyle\text{twist}} & & \Big\uparrow{\scriptstyle\text{twist}} \\
\begin{array}{c}\text{untwisted}\\ \text{abelian } BF \text{ theory}\end{array} & \xRightarrow{\text{parity reversal} + \text{complexification}} & \begin{array}{c}\text{supersymmetric}\\ \text{sigma model}\end{array}
\end{array}$$

We have the following dictionary (we use the notations of [8, 4] for the B side).

BF theory	B model
coefficient space $V=\mathbb{R}^N$	target $X=\mathbb{C}^N$
c^k	ϕ^k
b_k	$\bar\phi_k$
a^k	ψ_+^k
$i\gamma_k$	$\bar\psi_{+k}$
$\bar a^k$	ψ_-^k
$i\bar\gamma_k$	$\bar\psi_{-k}$
$-\frac{1}{2}B_k=\frac{i}{2}(\gamma_k-\bar\gamma_k)$	$\theta_k=\frac{1}{2}(\bar\psi_{+k}-\bar\psi_{-k})$
$i\lambda_k=i(\gamma_k+\bar\gamma_k)$	$\bar\eta_k=\bar\psi_{+k}+\bar\psi_{-k}\quad=\text{“}d\bar\phi_k\text{”}$
$A^k=dz\,a^k+d\bar z\,\bar a^k$	$\rho^k=dz\,\psi_+^k+d\bar z\,\psi_-^k$
term $\lambda_k\frac{\partial}{\partial b_k}$ in Q	Dolbeault differential on the target $\bar\eta_k\frac{\partial}{\partial\bar\phi_k}$
0-observables	
$\mathbb{O}_z=T_{\mathrm{poly}}(\Pi V)=\mathbb{C}[c^k,B_k]$	$\oplus_{p,q}H^{0,p}(X,\wedge^q T^{1,0}X)=\mathbb{C}[\phi^k,\theta_k]$
Supercurrents (in untwisted models)	
$J,\ \bar J$	$\bar G_+,\ \bar G_-$
$G,\ \bar G$	$G_+,\ G_-$
total $U(1)$-current $j_{\mathrm{tot}}=dz\,j+d\bar z\,\bar j$	axial R-symmetry current J_A

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CHAPTER 5

Two-dimensional non-abelian BF theory in Lorenz gauge as a solvable logarithmic TCFT

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Introduction and outline

A two-dimensional conformal field theory is called topological if it contains an odd symmetry Q satisfying $Q^2 = 0$ and such that the stress-energy tensor is Q-exact:

$$T = Q(G),\ \bar{T} = Q(\bar{G})$$

Given a topological conformal field theory, one can consider so-called "coupling to topological gravity." This amounts to considering correlators of fields G and $\bar{G}$ (in the presence of vertex operators) as differential forms on moduli spaces of complex structures on surfaces with marked points. Periods of these differential forms are called the (generalized) amplitudes in topological string theory (generalized Gromov-Witten invariants), see [11].

In the study of TCFTs, mostly A-twisted and B-twisted (2,2)-superconformal field theories were considered in the literature. In the end of 1980s another class of topological theories were studied, coming from gauge-fixing of a gauge theory with topological (diffeomorphism-invariant) action – for instance, Chern-Simons theory. One may expect that in two dimensions, in a proper gauge-fixing (like Lorenz gauge), these topological theories would be also conformal. In fact, in our previous work [7] we showed that the abelian BF theory in Lorenz gauge is a type B-twisted (2,2)-superconformal theory with target being an odd complex plane ($\Pi\mathbb{C}$ or $\mathbb{C}[1]$).

This work is devoted to the study of the two-dimensional non-abelian BF theory in Lorenz gauge as a topological confromal field theory.

In Section 5.1 we consider the two-dimensional BF theory for an arbitrary Lie algebra $\mathfrak{g}$, with fields being a $\mathfrak{g}$-valued one-form A with curvature F and a $\mathfrak{g}^*$-valued function B. We start by considering the classical action that appears after imposing Lorenz gauge. In the gauge-fixed theory in dimension two we have conformal invariance on the classical level. Since the metric enters the action only through gauge-fixing, we find that the stress-energy tensor is classically Q-exact. As it is clear from the form of the action, the non-abelian deformation violates accidental symmetries of the abelian theory. The only conserved currents in the deformed theory are: the holomorphic piece of the stress-energy tensor T, its complex conjugate $\bar{T}$, the superpartner G of T and its complex conjugate $\bar{G}$ and the total BRST current J (that is a sum of $J^{(1,0)}$ and $J^{(0,1)}$ pieces that are not conserved separately).

The characteristic feature of BF theories in any dimension is the upper-triangular structure of the interaction. Thus, we expect to get only tree level and one loop contributions in Feynman diagrams. In Section 5.2 we show that this property is preserved by Lorenz gauge-fixing. To our surprise, we find that for reductive Lie algebras (the exact condition is written in Subsection 5.2.1) the one-loop contribution vanishes due to cancellation between ghost and gauge fields. So, unexpectedly, on the level of correlators of fundamental fields, the theory is classical and hence finite.[1] Thus we conclude that the theory is conformal (since it does not need to have ultraviolet regularization and renormalization). We proceed by computing simplest correlators on the complex plane. Here we meet another surprise – the correlators involve logarithms, dilogarithms and so on. Thus, in this section we start to get evidence that the theory is logarithmic – this will be confirmed in Section 5.6. It would be interesting to compare this with the logarithmic theories arising as instantonic theories in [5]. We conclude this section by describing soaking observables (delta-functions of scalar fields) that allow one to pass from the plane to the sphere. Note that Witten in [10] had a different way to deal with the zero-modes of the field B. The insertion of delta-functions of scalar fields can be interpreted in terms of a modification of the moduli space of flat connections – we are planning to return to this question in the nearest future.

In Section 5.3 we compute OPEs of fundamental fields and observe unusual coefficient functions like $\log|z-w|$ and $\frac{\bar{z}-\bar{w}}{z-w}$. We think that such coefficient functions are characteristic features of a logarithmic conformal theory. We also find that not only correlators have finitely contributing diagrams, but also OPEs, which is a much stronger statement.

Since we would like to study correlators of T, G and J and they are composite fields, we extend our considerations to composite fields in Section 5.4. We start by defining the composite field as a result of consecutive mergings of fundamental fields accompanied by subtraction of singular parts. In this way the composite field depends on the order of mergings. Moreover, we can define in a similar way the bilinear product of composite fields – the result of merging of two composite fields accompanied by subtraction of the singular piece. Here we have an open question – does this product satisfy the pre-Lie algebra identity (5.96)? Proceeding to the fields T, G, J, we are surprised to find that these fields are independent of the order of merging and have zero singular subtractions. It would be interesting to understand this more conceptually, not merely as a result of a long computation. In this section we also present examples of correlators containing higher powers of logs.

Despite the fact that T, G, J are independent of the order of merging and contain no singular subtractions, it is a priori not clear why they are conserved and why their OPE is the standard one. To leave no doubt, we prove

[1] Therefore, it would be interesting to relate this theory to the instantonic theory (in the sense of [5]) for instantonic equations $dA + A^2 = 0, \;\; d^*A = 0$ but we will leave this for further studies.

these properties directly in Section 5.5. It would be interesting to understand if these properties of T, G, J could be deduced from the cancellations of singular subtractions we found in Section 5.4.

In Section 5.6 we compute conformal dimensions of some composite fields. We find that there are fields $V^{(n)}$ (5.162), for $n \geq 0$, with logarithmic anomalous dimensions in the sense that

$$L_0 V^{(n)} = V^{(n)} + gV^{(n-1)} \quad , \quad \bar{L}_0 V^{(n)} = gV^{(n-1)}$$

where g is the non-abelian coupling constant. This confirms the logarithmic nature of the theory. Moreover, we can build a vertex operator

$$V = \sum_n V^{(n)}$$

with anomalous dimension $(1+g, g)$. Like in the case of free scalar theory, the origin of the anomalous dimension may be explained as arising from the dependence of the singular subtraction on the local coordinate.

In conclusion we should mention that we have constructed and studied a novel class of topological logarithmic conformal field theories. Our next step would be the construction of topological string amplitudes in such theories. We plan to do that in the nearest future.

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5.1. Classical non-abelian theory

In this section we discuss the classical two-dimensional non-abelian BF theory, paralleling the treatment of the abelian case in [7].

Fix a finite-dimensional Lie algebra $\mathfrak{g}$.[2]

We consider the non-abelian BF theory on the complex plane $\mathbb{C}$,[3] defined classically by the action

$$S^{\mathrm{cl}} = \int_{\mathbb{C}} \left\langle B, dA + \frac{g}{2}[A, A] \right\rangle \tag{5.1}$$

Here the classical fields are: a $\mathfrak{g}$-valued 1-form A on $\mathbb{C}$ and a $\mathfrak{g}^*$-valued 0-form B; d is the de Rham operator; $\langle -, - \rangle$ is the canonical pairing between the coefficient Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$; g is a coupling constant (deformation parameter corresponding to the deformation of the abelian theory into the non-abelian one).

The equations of motion are:

$$dA + \frac{g}{2}[A, A] = 0 \text{ (flatness of } A \text{ as a connection)}, \qquad dB + g[A, B] = 0$$

Here $[A, B] = \mathrm{ad}^*_A(B)$ is a notation for the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$; it is consistent with the case when $\mathfrak{g}^*$ is identified with $\mathfrak{g}$ via non-degenerate

[2] When discussing quantization, we will need to assume that $\mathfrak{g}$ is *strongly unimodular*, see (5.34). In particular, this assumption holds for all semisimple and nilpotent Lie algebras, or sums of those.

[3] Throughout this section we can everywhere replace $\mathbb{C}$ by any surface Σ equipped with a metric (needed for the gauge-fixing). We specialize to $\mathbb{C}$ right away, as it will be the case of relevance in the discussion of quantization.

Killing form. The action S^{cl} is invariant under infinitesimal gauge transformations

$$A \mapsto A + d\alpha + g[A,\alpha], \qquad B \mapsto B + g[B,\alpha]$$

with generator α a $\mathfrak{g}$-valued 0-form.

5.1.1. Gauge-fixing in BRST formalism. We consider the non-abelian BF theory in *Lorenz gauge* $d*A=0$, with $*$ the Hodge star on $\mathbb{C}$. The corresponding Faddeev-Popov gauge-fixed action is:

$$S = \int_{\mathbb{C}} \left\langle B, dA + \frac{g}{2}[A,A]\right\rangle + \langle \lambda, d*A\rangle + \langle b, d*d_A c\rangle \tag{5.2}$$

Here λ is the Lagrange multiplier imposing the gauge condition and b,c are Faddeev-Popov ghosts (anti-commuting scalar fields); $d_A = d + g[A,-]$ is the de Rham operator twisted by A. Action S is a function on the space of BRST fields:

$$\mathcal{F} = \underbrace{\Omega^1(\mathfrak{g})}_{A} \oplus \underbrace{\Omega^0(\mathfrak{g}^*)}_{B} \oplus \underbrace{\Omega^0(\mathfrak{g}^*)}_{\lambda} \oplus \underbrace{\Omega^0(\mathfrak{g}^*)[-1]}_{b} \oplus \underbrace{\Omega^0(\mathfrak{g})[1]}_{c}$$

where $\Omega^p(\cdots)$ stands for the space of p-forms on $\mathbb{C}$ with coefficients in $\cdots$; $[\pm 1]$ are homological degree shifts and correspond to assigning ghost degree -1 to b and $+1$ to c. The BRST operator acts as follows:

$$Q: \quad A \mapsto d_A c, \quad B \mapsto g[c,B], \quad c \mapsto \frac{g}{2}[c,c], \quad b \mapsto \lambda, \quad \lambda \mapsto 0 \tag{5.3}$$

Action (5.2) is a shift of the classical action (5.1) by a Q-coboundary:

$$S = S^{\mathrm{cl}} + Q(\Psi) \tag{5.4}$$

with $\Psi = \int_{\mathbb{C}} \langle b, d*A\rangle$ the gauge-fixing fermion. Euler-Lagrange equations for the action (5.2) read:

$$dA + \frac{g}{2}[A,A] = 0, \quad d*A = 0, \quad d*d_A c = 0, \quad d_A * db = 0, \\ d_A B - *d\lambda - g[c, *db] = 0 \tag{5.5}$$

Remark 5.1.1 (Superfields). One can combine the fields A,B,c,b into two *superfields* (or, more precisely, "gauge-fixed AKSZ superfields") valued in non-homogeneous forms:

$$\mathcal{A} = c + A, \qquad \mathcal{B} = B - *db \tag{5.6}$$

Written in terms of superfields $\mathcal{A},\mathcal{B}$ and the Lagrange multiplier λ, the action (5.2) is:

$$S = \int_{\mathbb{C}} \left\langle \mathcal{B}, d\mathcal{A} + \frac{g}{2}[\mathcal{A},\mathcal{A}]\right\rangle + \langle \lambda, d*\mathcal{A}\rangle \tag{5.7}$$

Here we are integrating the 2-form component of the integrand. The integrand above differs from the integrand of (5.2) by a total derivative $d(\cdots)$ which is inconsequential.

5.1.2. Complex fields. Let x^1, x^2 be the real coordinates on $\mathbb{C} \sim \mathbb{R}^2$ and $z = x^1 + ix^2$ the complex coordinate.

We split the 1-form field A into $(1,0)$ and $(0,1)$-form components: $A = dz\, a + d\bar{z}\, \bar{a}$ where $a, \bar{a}$ are $\mathfrak{g}$-valued scalars. Also, we combine the field B and the Lagrange multiplier λ into a $\mathfrak{g}^*$-valued complex scalar field $\gamma = \frac{1}{2}(\lambda + iB)$ and its complex conjugate $\bar{\gamma} = \frac{1}{2}(\lambda - iB)$.

Written in terms of fields $(a, \bar{a}, \gamma, \bar{\gamma}, b, c)$, the action (5.2) takes the following form:

$$S = 4 \int_{\mathbb{C}} d^2x \Big(\left\langle \gamma, \bar{\partial} a \right\rangle + \langle \bar{\gamma}, \partial \bar{a} \rangle + \left\langle b, \partial \bar{\partial} c \right\rangle - \\ - \frac{g}{2} \langle \gamma - \bar{\gamma}, [a, \bar{a}] \rangle - \frac{g}{2} \langle \partial b, [\bar{a}, c] \rangle - \frac{g}{2} \left\langle \bar{\partial} b, [a, c] \right\rangle \Big) \tag{5.8}$$

Here $d^2x = dx^1 dx^2 = \frac{i}{2} dz\, d\bar{z}$ is the standard area form on $\mathbb{C}$ and $\partial = \frac{\partial}{\partial z}$, $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$ are the partial derivatives (not the holomorphic/anti-holomorphic Dolbeaux operators: we do not include $dz, d\bar{z}$ in $\partial, \bar{\partial}$ in our notations).

Equations of motion (5.5) written in complex fields take the form

$$\begin{aligned} \bar{\partial} a - \frac{g}{2}[a, \bar{a}] &= 0, & \partial \bar{a} + \frac{g}{2}[a, \bar{a}] &= 0, \\ \bar{\partial} \gamma + \frac{g}{2}[\bar{a}, \gamma - \bar{\gamma}] - \frac{g}{2}[c, \bar{\partial} b] &= 0, & \partial \bar{\gamma} - \frac{g}{2}[a, \gamma - \bar{\gamma}] - \frac{g}{2}[c, \partial b] &= 0, \\ \partial \bar{\partial} b + \frac{g}{2}[a, \bar{\partial} b] + \frac{g}{2}[\bar{a}, \partial b] &= 0, & \partial \bar{\partial} c + \frac{g}{2} \bar{\partial}[a, c] + \frac{g}{2} \partial [\bar{a}, c] &= 0 \end{aligned} \tag{5.9}$$

Finally, the BRST operator Q becomes the following:

$$Q: \quad \begin{aligned} a &\mapsto -\partial c - g[a, c], & \bar{a} &\mapsto -\bar{\partial} c - g[\bar{a}, c], \\ \gamma &\mapsto \frac{g}{2}[c, \gamma - \bar{\gamma}], & \bar{\gamma} &\mapsto -\frac{g}{2}[c, \gamma - \bar{\gamma}], \\ b &\mapsto \gamma + \bar{\gamma}, & c &\mapsto \frac{g}{2}[c, c] \end{aligned} \tag{5.10}$$

5.1.3. BRST current. The Noether current associated to BRST symmetry is

$$J^{\text{tot}} = -2i(dz\, J - d\bar{z}\, \bar{J}) \tag{5.11}$$

where

$$\begin{aligned} J &= \langle \gamma, \partial c \rangle + g \langle \gamma, [a, c] \rangle - \frac{g}{4} \langle \partial b, [c, c] \rangle, \\ \bar{J} &= \left\langle \bar{\gamma}, \bar{\partial} c \right\rangle + g \langle \bar{\gamma}, [\bar{a}, c] \rangle - \frac{g}{4} \left\langle \bar{\partial} b, [c, c] \right\rangle \end{aligned} \tag{5.12}$$

The current J^{tot} is conserved:

$$dJ^{\text{tot}} \underset{\text{e.o.m.}}{\sim} 0 \tag{5.13}$$

(where $\underset{\text{e.o.m.}}{\sim}$ means equivalence *modulo equations of motion*).

Warning: The $(1,0)$- and $(0,1)$-form components $J, \bar{J}$ of the current are not conserved separately (unlike in abelian BF theory): $\bar{\partial} J \not\sim 0$, $\partial \bar{J} \not\sim 0$.

In terms of real fields, the BRST current spells

$$J^{\text{tot}} = \langle B, d_A c \rangle + \langle \lambda, * d_A c \rangle - \frac{g}{2} \langle * db, [c, c] \rangle$$

5.1.4. Classical conformal invariance and the Q-exact stress-energy tensor. Action (5.2) can be considered on any surface Σ endowed with a Riemannian metric ξ, which enters the integrand via the Hodge star. Let us denote the action on (Σ, ξ) by $S_{\Sigma,\xi}$. Since Hodge star acts in (5.2) only on 1-forms, the action $S_{\Sigma,\xi}$ is invariant under Weyl transformations – rescaling of ξ by a positive function on Σ. Thus, the action $S_{\Sigma,\xi}$ depends only on the conformal class of the metric.

One defines the stress-energy tensor $T^{\text{tot}} = T_{\mu\nu} dx^\mu dx^\nu$ (a field-dependent section of the symmetric square of the cotangent bundle $\text{Sym}^2 T^*\Sigma$) as the reaction of $S_{\Sigma,\xi}$ to an infinitesimal change of metric ξ. More precisely, T^{tot} is defined via

$$\delta_\xi S_{\Sigma,\xi} = -\int_\Sigma \sqrt{\det \xi}\, d^2x\, T_{\mu\nu}\, \delta\xi^{\mu\nu}$$

Here x^1, x^2 are local coordinates on Σ, $\xi^{\mu\nu}$ are the components of the inverse metric ξ^{-1}; δ_ξ stands for the variation w.r.t. a change of metric.

Since the action can be written as $S^{\text{cl}} + Q(\Psi_\xi)$ where S^{cl} and Q are manifestly independent of the metric and only the gauge-fixing fermion $\Psi_\xi = \int \langle b, d *_\xi A\rangle$ is metric-dependent (via the Hodge star), the stress-energy tensor is Q-exact:

$$T^{\text{tot}} = Q(G^{\text{tot}}) \tag{5.14}$$

The primitive $G^{\text{tot}} = G_{\mu\nu} dx^\mu dx^\nu$ is defined in terms of the variation of the gauge-fixing fermion w.r.t. the metric: $\delta_\xi \Psi_\xi = -\int_\Sigma \sqrt{\det \xi}\, d^2x\, G_{\mu\nu}\, \delta\xi^{\mu\nu}$. The explicit calculation is the same as in the abelian theory [7] (since Ψ does not depend on the deformation parameter g) and yields the result

$$G^{\text{tot}} = (dz)^2 \underbrace{\langle a, \partial b\rangle}_{G} + (d\bar z)^2 \underbrace{\langle \bar a, \bar\partial b\rangle}_{\bar G} \tag{5.15}$$

– This result is valid for an arbitrary surface Σ, with $z, \bar z$ local complex coordinates (compatible with the conformal class of a given metric ξ), and, as a special case, for $\Sigma = \mathbb{C}$ with standard metric and z the global complex coordinate.

Next, we calculate the stress-energy tensor from (5.14) and the explicit formula (5.15) for G^{tot}:

$$T^{\text{tot}} = (dz)^2 T + (d\bar z)^2 \bar T \tag{5.16}$$

where the holomorphic component is:

$$T = Q(G) = \langle \partial b, \partial c + g[a, c]\rangle + \langle a, \partial(\gamma + \bar\gamma)\rangle \tag{5.17}$$

Modulo equations of motion, one can simplify it to an equivalent form

$$T \underset{\text{e.o.m.}}{\sim} \langle \partial b, \partial c\rangle + \langle a, \partial\gamma\rangle + \frac{g}{2}\langle \partial b, [a, c]\rangle \tag{5.18}$$

The anti-holomorphic component $\bar T$ of the stress-energy tensor is given by the complex conjugate of (5.17), (5.18). Note that the stress-energy tensor T^{tot} does not have a $dz\, d\bar z$ component, which is tantamount to conformal invariance of the action.

The components of the stress-energy tensor and its primitive are holomorphic/anti-holomorphic modulo equations of motion:

$$\bar{\partial} G \underset{\text{e.o.m.}}{\sim} 0, \quad \partial \bar{G} \underset{\text{e.o.m.}}{\sim} 0, \qquad \bar{\partial} T \underset{\text{e.o.m.}}{\sim} 0, \quad \partial \bar{T} \underset{\text{e.o.m.}}{\sim} 0 \tag{5.19}$$

5.1.5. Non-abelian theory as a deformation of abelian theory by a 2-observable. Setting $g = 0$ in all the formulae, we get the *abelian* BF theory (with coefficients in $\mathfrak{g}$ viewed as a vector space, or, equivalently, $\dim \mathfrak{g}$ non-interacting copies of abelian BF theory with coefficients in $\mathbb{R}$). We will indicate objects corresponding to the abelian theory by a subscript "0": we have the action S_0, BRST operator Q_0, stress-energy tensor T_0 etc. In particular, the abelian action

$$S_0 = 4 \int d^2x \left(\langle a, \bar{\partial}\gamma \rangle + \langle \bar{a}, \partial\bar{\gamma} \rangle + \langle b, \partial\bar{\partial} c \rangle \right) \tag{5.20}$$

is a sum of three conformal field theories: a holomorphic and an anti-holomorphic first-order $\beta\gamma$-system and a second-order ghost system. The three constituent conformal theories are free (quadratic) and do not interact between each other on the level of the action but are intertwined by the BRST operator Q_0.

We say that a sequence $\mathcal{O}^{(0)}, \mathcal{O}^{(1)}, \mathcal{O}^{(2)}$ of composite fields (local expressions in terms of fundamental fields of the theory and their derivatives of finite order at a given point), such that $\mathcal{O}^{(p)}$ is valued in p-forms, forms a Witten's hierarchy of observables if Witten's descent equation

$$Q\mathcal{O}^{(p)} \underset{\text{e.o.m.}}{\sim} d\mathcal{O}^{(p-1)}$$

is satisfied (modulo equations of motion) for $p = 0, 1, 2$; we understand the equation for $p = 0$ as $Q\mathcal{O}^{(0)} = 0$. We then say that $\mathcal{O}^{(p)}$ is a "p-observable" (in the sense that its integral over a p-cycle is a gauge-invariant expression), and we say that $\mathcal{O}^{(p)}$ is obtained from $\mathcal{O}^{(p-1)}$ via descent. Starting from a given 0-observable $\mathcal{O}^{(0)}$, one can solve the descent equation for $\mathcal{O}^{(1)}$ and then for $\mathcal{O}^{(2)}$ directly: one constructs $\mathcal{O}^{(p)}$ by using the operator product expansion of $\mathcal{O}^{(p-1)}$ with fields $G, \bar{G}$ - components of the BRST primitive of the stress-energy tensor. Explicitly (see [7] for details):

$$\mathcal{O}^{(p)}_w = -\frac{1}{p}\left(dw \oint_{C_w \ni z} \frac{dz}{2\pi i} G_z \mathcal{O}^{(p-1)}_w - d\bar{w} \oint_{C_w \ni z} \frac{d\bar{z}}{2\pi i} \bar{G}_z \mathcal{O}^{(p-1)}_w \right)$$

where the integration is over a simple closed contour C_w, going around w once counterclockwise; subscripts z, w are the points where the fields are inserted. The equality is understood as an equality under the correlator with any number of test fields inserted outside the integration contour C_w.

Within the abelian theory, starting from a Q_0-closed observable $\mathcal{O}^{(0)}$, one is interested in constructing the corresponding descents $\mathcal{O}^{(1)}$ and $\mathcal{O}^{(2)}$. Then one can deform the action of the abelian theory by

$$S_0 \mapsto S_0 + g \int_{\mathbb{C}} \mathcal{O}^{(2)} \tag{5.21}$$

with g a deformation parameter.

The non-abelian deformation of the abelian theory corresponds to choosing

$$\mathcal{O}^{(0)} = \frac{1}{2}\langle B, [c,c]\rangle$$

The corresponding first and second descent are:

$$\mathcal{O}^{(1)} = \langle B, [A,c]\rangle - \frac{1}{2}\langle *db, [c,c]\rangle \tag{5.22}$$

$$\mathcal{O}^{(2)} = \frac{1}{2}\langle B, [A,A]\rangle - \langle *db, [A,c]\rangle \tag{5.23}$$

– see the calculation in [7]. Thus, the deformed action is

(5.24)

$$S = \underbrace{\int_{\mathbb{C}} \langle B, dA\rangle + \langle \lambda, d*A\rangle + \langle b, d*dc\rangle}_{S_0} + \underbrace{\frac{g}{2}\langle B, [A,A]\rangle - g\langle *db, [A,c]\rangle}_{g\,\mathcal{O}^{(2)}}$$

It coincides with the gauge-fixed non-abelian action (5.2).

We also note that the hierarchy of observables $\mathcal{O}^{(0)}, \mathcal{O}^{(1)}, \mathcal{O}^{(2)}$ are in fact homogeneous components of form degree $0,1,2$ of the expression $\frac{1}{2}\langle \mathcal{B}, [\mathcal{A},\mathcal{A}]\rangle$ written in terms of superfields (5.6).

Non-abelian action (5.24) is a deformation of the free conformal field theory defined by the abelian action S_0 (5.20) by a 2-observable, which is in fact an *exact marginal operator*, i.e.:

- $\mathcal{O}^{(2)}$ is a primary field of conformal dimension $(1,1)$,
- the operator product expansion $\mathcal{O}^{(2)}_z\mathcal{O}^{(2)}_w$ does not contain the singularity $\frac{1}{|z-w|^2}$ (which would destroy the conformal invariance of the deformed theory on the quantum level).

See Section 5.3.3 below for the check of these properties; in particular, the first property relies on unimodularity of $\mathfrak{g}$.

Alongside the deformation (5.21) of the action, the other relevant objects of the theory deform:

$$Q_0 \mapsto Q = Q_0 + gQ_1, \quad T_0 \mapsto T = T_0 + gT_1, \quad J_0 \mapsto J = J_0 + gJ_1 \tag{5.25}$$

We read off the abelian part $(\cdots)_0$ and the deformation $(\cdots)_1$ (the subscript corresponds to the order in Taylor expansion in g of the object) as constant in g and linear in g terms in formulae (5.3), (5.17), (5.12). Note that the BRST primitive of the stress-energy tensor is the one object which does not deform: $G_0 = G = \langle a, \partial b\rangle$.

Remark 5.1.2. In the context of a general deformation of any n-dimensional gauge theory by an n-observable, $S_0 \mapsto S = S_0 + g\int \mathcal{O}^{(n)}$, Noether theorem gives the deformation of the BRST current in the form $J_0^{\text{tot}} \mapsto J^{\text{tot}} = J_0^{\text{tot}} + gJ_1^{\text{tot}} + O(g^2)$ with

$$J_1^{\text{tot}} = \mathcal{O}^{(n-1)} - \iota_{Q_0}\alpha_1 \tag{5.26}$$

Here α_1 is the deformation of the Noether 1-form $\alpha = \alpha_0 + g\alpha_1$, viewed as a 1-form on the space of fields valued in $(n-1)$-forms on the spacetime and defined from $\delta L = \sum_i EL_i\delta\phi^i + d\alpha$. Here L is the Lagrangian density of the action, the summation is over species of fields ϕ^i, EL_i is the Euler-Lagrange equation arising from variation of the field ϕ^i and δ the de Rham operator on the space of fields (as opposed to d – the de Rham operator on the spacetime). In (5.26) the first term is the $(n-1)$-observable linked to the n-observable deforming the action by descent (in non-deformed theory): $Q_0\mathcal{O}^{(n)} = d\mathcal{O}^{(n-1)}$. One finds (5.26) from the expression for the BRST current given by Noether theorem,

$$J^{\text{tot}} = \rho - \iota_Q \alpha$$

where ρ is defined by $QL = d\rho$. Restricting to $O(g^1)$ terms in this formula, one finds $\rho_1 = \mathcal{O}^{(n-1)} + \iota_{Q_1}\alpha_0$ which leads to (5.26). In our case – non-abelian deformation (5.24) of 2D abelian BF theory – we have

$$\alpha = \underbrace{-\langle B, \delta A\rangle - \langle \lambda, *\delta A\rangle + \langle *db, \delta c\rangle + \langle b, *d\delta c\rangle}_{\alpha_0} \underbrace{-g\langle \delta b, [*A, c]\rangle}_{g\alpha_1}$$

and formula (5.26) yields the $O(g^1)$ part of the current (5.11).

5.1.5.1. *Symmetries not surviving in the deformed theory.* A part of symmetries/conservation laws of the abelian theory gets destroyed by the non-abelian deformation. Most importantly, the left/right components $J, \bar{J}$ of the BRST current are conserved in the abelian theory but not in the deformed theory. In the abelian theory, conservation of $J, \bar{J}$ ultimately leads to the realization of abelian theory as a twisted $\mathcal{N} = (2,2)$ superconformal field theory [7], with $G, J, \bar{G}, \bar{J}$ corresponding via type B-twist to the two pairs of supercurrents. This picture does not carry over to the deformed theory. In particular, abelian theory has the conserved R-symmetry current $\langle \gamma, a\rangle$; in non-abelian theory this expression is not conserved (and does not correspond to a symmetry of the action).

In summary, we have the following table of conserved quantities on abelian vs. non-abelian side:

	abelian	non-abelian	(notes)
stress-energy tensor	$T_0, \bar{T}_0$	$T, \bar{T}$	
BRST primitive for stress-energy	$G_0, \bar{G}_0$	$G, \bar{G}$	$G = G_0$
BRST current	$J_0, \bar{J}_0$	only J^{tot}	
R-symmetry current	$\langle \gamma, a\rangle$, $\langle \bar{\gamma}, \bar{a}\rangle$	—	

Also, fields $a, \gamma, \partial b, \partial c$ are holomorphic in abelian theory (i.e. satisfy $\bar{\partial}(\cdots) \sim 0$ modulo equations of motion) but this property also does not carry over to the deformed theory (as one sees immediately from the equations of motion (5.9)).

5.2. Correlators

Let $\{t_a\}$ be a basis of generators in $\mathfrak{g}$ and $\{t^a\}$ the dual basis in $\mathfrak{g}^*$. Then the fundamental fields of the theory can be decomposed in components as $a = a^a t_a$, $\gamma = \gamma_a t^a$ (and similarly for complex conjugates), $c = c^a t_a$, $b = b_a t^a$. We denote $f^c_{ab} = \langle t^c, [t_a, t_b]\rangle$ the structure constants of the Lie algebra.

In quantum theory, we are interested in the correlation functions of local fields $\Phi_1, \ldots, \Phi_n$ placed at points $z_1, \ldots, z_n \in \mathbb{C}$. We assume the points to be pairwise distinct, $z_i \neq z_j$ for $i \neq j$. Such a correlator is formally defined by the path integral

$$\langle \Phi_1(z_1) \cdots \Phi_n(z_n)\rangle = \frac{1}{Z}\int_{\mathcal{F}} e^{-\frac{1}{4\pi}S}\Phi_1(z_1)\cdots\Phi_n(z_n) \tag{5.27}$$

where Z is the normalization (partition function), such that $\langle 1\rangle = 1$. Fields Φ_j are the fundamental fields of the theory $\gamma, \bar\gamma, a, \bar a, b, c$ or their derivatives of arbitrary order. More generally, one can allow Φ_j to be a product of such objects – a *composite field*. For example, one can have $\Phi(z) = (\phi\psi)(z)$ with ϕ, ψ linear in fundamental fields. Such a product is understood as "*renormalized*",[4]i.e., as a limit

$$\Phi(z) = \lim_{z'\to z} \Big(\phi(z')\psi(z) - [\phi(z')\psi(z)]_{\mathrm{sing}}\Big) \tag{5.28}$$

under the correlator with other fields; here the last term stands for the singular part, as z' approaches z, of the operator product expansion $\phi(z')\psi(z)$, see Section 5.3 below. Furthermore, formula (5.28) can be applied to (renormalized) composite fields ϕ, ψ, to construct their renormalized product. Thus, correlators of composite fields can be obtained from correlators of fundamental fields (or their derivatives), by merging some of the points z_i (and subtracting the singular terms). We defer the detailed discussion of composite fields and the procedure of building them as renormalized products until Section 5.4.1.

In abelian BF theory, one has correlators $\langle\Phi(z_1)\cdots\Phi(z_n)\rangle_0$ defined as in (5.27), but with S replaced by the free action S_0. These free theory correlators are given by Wick's lemma with propagators

$$\begin{gathered} \langle c^a(w) b_b(z)\rangle_0 = \delta^a_b\,\big(2\log|w-z| + C\big), \\ \langle a^a(w)\gamma_b(z)\rangle_0 = \frac{\delta^a_b}{w-z}, \qquad \langle \bar a^a(w)\bar\gamma_b(z)\rangle_0 = \frac{\delta^a_b}{\bar w - \bar z} \end{gathered} \tag{5.29}$$

Here C is an undetermined constant.[5] Propagators for other pairs of fields from the list $\{a, \bar a, \gamma, \bar\gamma, b, c\}$ are zero. Propagators (5.29) are obtained as Green's functions for the operators $\partial\bar\partial, \bar\partial, \partial$ in quadratic action (5.20). Parameterizing the space of fields by the superfields (5.6) and the Lagrange multiplier λ, one has the following propagators:

$$\langle \mathcal{A}^a(w)\,\mathcal{B}_b(z)\rangle_0 = \delta^a_b\, 2\,d\arg(w-z), \qquad \langle \mathcal{A}^a(w)\lambda_b(z)\rangle_0 = \delta^a_b\, 2\,d_w \log|w-z| \tag{5.30}$$

Here in the first formula, the propagator is understood as a 1-form on the configuration space $\mathrm{Conf}_2(\mathbb{C}) \subset \mathbb{C}\times\mathbb{C}$ of two distinct points (w,z) on $\mathbb{C}$; $d = d_w + d_z$ is the total de Rham operator on the configuration space, where d_w, d_z are the de Rham operators on the first and second copy of $\mathbb{C}$.

Using $S = S_0 + g\int \mathcal{O}^{(2)}$ in (5.27), the correlator of the deformed theory is expressed in terms of correlators in the abelian theory with insertions of $N \geq 0$ copies of the deforming observable $\mathcal{O}^{(2)}$ at points $u_1, \ldots, u_N$ integrated over

[4] Another possible term is the "*normally ordered*" product. We do not use this term here as it is somewhat ambiguous in a non-free theory (our prescription has nothing to do with creation/annihilation operators).

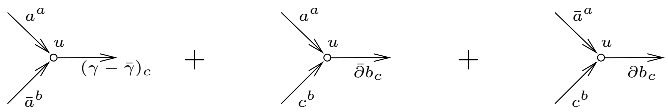
[5] If the theory is regularized by an infrared cut-off, by imposing a Dirichlet boundary condition on b, c on a circle of large radius R, then $C = -2\log R$.

$\mathbb{C}$:

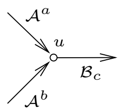
(5.31) $$\langle \Phi_1(z_1)\cdots\Phi_n(z_n)\rangle = \Big\langle \Phi_1(z_1)\cdots\Phi_n(z_n)\ \ e^{-\frac{g}{4\pi}\int_{\mathbb{C}\ni u}\mathcal{O}^{(2)}(u)}\Big\rangle_0$$
$$= \sum_{N\geq 0}\frac{1}{N!}\left(-\frac{g}{4\pi}\right)^N\int_{\mathbb{C}^N\ni(u_1,\ldots,u_N)}\Big\langle \Phi_1(z_1)\cdots\Phi_n(z_n)\ \ \mathcal{O}^{(2)}(u_1)\cdots\mathcal{O}^{(2)}(u_N)\Big\rangle_0$$

Free theory correlator on the r.h.s. is evaluated using Wick's lemma, as a sum of Wick's contractions of fields $\Phi_1(z_1),\ldots,\Phi_n(z_n)$ and N copies of $\mathcal{O}^{(2)}$ and yields a sum of Feynman graphs with N internal vertices decorated by $\mathcal{O}^{(2)}$ (their positions u_i are integrated over $\mathbb{C}$) and n vertices decorated by $\Phi_1,\ldots,\Phi_n$ at fixed points $z_1,\ldots,z_n$. In the case when Φ_i are fundamental fields or their derivatives, the corresponding fixed vertices are uni-valent.

For Feynman graphs, we adopt the convention where the graphs are oriented, with half-edges decorated by fields $a,\bar{a},c$ (or their derivatives) oriented towards the incident vertex and $\gamma,\bar{\gamma},b$ (or derivatives) oriented away from the vertex. In particular, the interaction vertex (cf. the cubic part of (5.8)) is:

(5.32) 

here we list the possible decorations of half-edges by fields. The vertex is decorated by the expression $g\, f^c_{ab}\int_{\mathbb{C}\ni u}\frac{d^2u}{2\pi}$. In terms of the superfields $\mathcal{A},\mathcal{B}$, the vertex is simply

(5.33) 

and is decorated by $\left(-\frac{g}{4\pi}\right)\frac{1}{2}f^c_{ab}\int_{\mathbb{C}\ni u}$.

In our convention for Feynman graphs for the correlator (5.31), we have two types of vertices:

- black (fixed) vertices corresponding to fields $\Phi_1(z_1),\ldots,\Phi_n(z_n)$ we calculate the correlator of,
- white (integrated) vertices corresponding to $\mathcal{O}^{(2)}(u)$ where the point u is integrated over.

5.2.1. General correlators of fundamental fields: admissible Feynman diagrams. Let us introduce a grading on fields – the "$\mathcal{AB}$-charge" – by assigning charge $+1$ to fields $\gamma,\bar{\gamma},b$ and charge -1 to fields $a,\bar{a},c$. The convention is that the charge in unchanged when taking derivatives of a field and is additive under multiplication.[6] Note that:

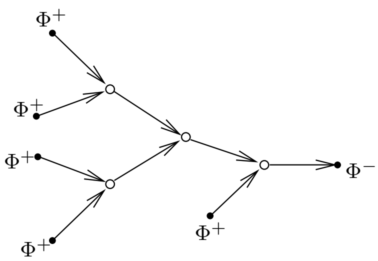
- A free theory correlator $\langle\cdots\rangle_0$ of a collection of fields can only be nonzero if the total charge of the fields vanishes. This is due to the form of propagators (5.29) which only pair $+1$-fields to -1-fields (or, in other words, due to the fact that the abelian action (5.20) has total charge zero).
- The charge of the deforming observable $\mathcal{O}^{(2)}$ is -1.

[6]The name $\mathcal{AB}$-charge is due to the fact that all components of the superfield $\mathcal{A}$ have charge -1 and all components of $\mathcal{B}$ (plus the Lagrange multiplier λ) have charge $+1$.

In particular, Feynman graphs contributing to the non-abelian correlator $\langle \Phi_1 \cdots \Phi_n \rangle$ must have exactly N internal vertices, with N the total charge of fields Φ_i. Thus, the correlator is proportional to g^N with no other powers of g present.

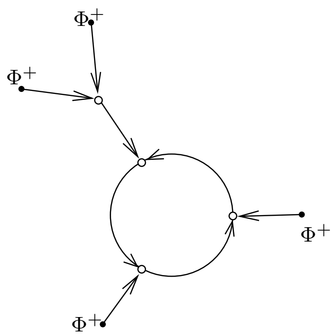
As follows from the form of the interaction vertex (5.32), (5.33), with two incoming half-edges and one outgoing half-edge, Feynman graphs contributing to a correlation function $\langle \Phi_1 \cdots \Phi_n \rangle$ of fundamental fields (or their derivatives) can have connected components of the following two types:

(i) Binary rooted trees with leaves (uni-valent vertices with outward orientation of the adjacent half-edge) decorated with fields $\gamma, \bar{\gamma}, b$ or their derivatives from the list $\{\Phi_i\}$ and the root (uni-valent vertex with inward orientation of the half-edge) decorated by $a, \bar{a}, c$ or derivatives. For example:



Here the superscript $\pm$ refers to fields of $\mathcal{AB}$-charge ± 1.

(ii) One-loop graphs, having the form of an oriented cycle with several binary trees rooted on the cycle (with leaves decorated by $\gamma, \bar{\gamma}, b$ or derivatives). For example:



We assume that Lie algebra $\mathfrak{g}$ is such that one has the identity

$$\mathrm{tr}_{\mathfrak{g}} \left(\mathrm{ad}_{X_1} \cdots \mathrm{ad}_{X_k}\right) = (-1)^k \mathrm{tr}_{\mathfrak{g}} \left(\mathrm{ad}_{X_k} \cdots \mathrm{ad}_{X_1}\right) \tag{5.34}$$

for $X_1, \ldots, X_k \in \mathfrak{g}$ arbitrary elements, for any $k \geq 1$. This identity holds for the following classes of Lie algebras:

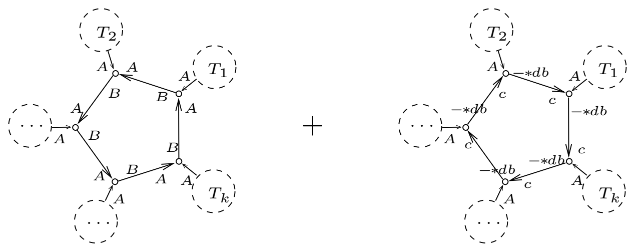
- any semisimple $\mathfrak{g}$,[7]
- any nilpotent $\mathfrak{g}$ (in a trivial way: the traces are zero),
- a direct sum of a semisimple and a nilpotent Lie algebras, e.g., any reductive $\mathfrak{g}$.

[7] Indeed, using the Killing form (which is nondegenerate due to semisimplicity) to identify $\mathfrak{g}^* \simeq \mathfrak{g}$, in the l.h.s. of (5.34) we have a trace of a product of k *anti-symmetric* matrices. Applying transposition under the trace, we get the r.h.s.

We call an algebra satisfying (5.34) *strongly unimodular*, since $k = 1$ case is equivalent to the usual unimodularity condition $\mathrm{tr}_{\mathfrak{g}}[X, -] = 0$.[8]

LEMMA 5.2.1 (Boson-fermion cancellation in the loop). Under assumption (5.34), graphs of type (ii) vanish, when summed over admissible decorations in the loop.

PROOF. Given a one-loop graph Γ, there are two possible decorations of the half-edges in the loop – by alternating fields A and B vs. by alternating c and $-*db$ (for this argument, it is convenient to switch to real fields) – and they give identical contributions of opposite sign:

(5.35)  $= 0$

Here $T_1, \ldots, T_k$ are arbitrary trees rooted on the cycle; note that the orientation of the cycle is switched between the two summands.

To see the cancellation (5.35) explicitly, we observe that the first graph contains the expression

$$\langle B, [A, A] \rangle_{u_k} \cdots \langle B, [A, A] \rangle_{u_2} \langle B, [A, A] \rangle_{u_1} = -2^k \mathrm{tr}_{\mathfrak{g}} \Big(d_1 \varphi_{1k} \; \mathrm{ad}_{A(u_k)} \; d_k \varphi_{k\,k-1} \; \mathrm{ad}_{A(u_{k-1})} \cdots d_2 \varphi_{21} \; \mathrm{ad}_{A(u_1)} \Big) \tag{5.36}$$

where remaining fields $A(u_i)$ are Wick-contracted with trees $T_1, \ldots, T_k$. Here $\varphi_{ij} = \arg(u_i - u_j)$ and d_i is the de Rham differential in u_i. Likewise, the second graph in (5.35) contains the expression

$$\langle -*db, [A, c] \rangle_{u_1} \langle -*db, [A, c] \rangle_{u_2} \cdots \langle -*db, [A, c] \rangle_{u_k} = -2^k \mathrm{tr}_{\mathfrak{g}} \Big(d_1 \varphi_{1k} \; \mathrm{ad}_{A(u_1)} \; d_2 \varphi_{21} \; \mathrm{ad}_{A(u_2)} \cdots \mathrm{ad}_{A(u_k)} \; d_k \varphi_{k\,k-1} \Big) \tag{5.37}$$

Using Lie algebra identity (5.34), one can see that expressions (5.36) and (5.37) are the same, up to a minus sign. □

PROPOSITION 5.2.2 (*Properties of correlators of fundamental fields*). A correlator $\langle \Phi_1 \cdots \Phi_n \rangle$ of fundamental fields or their derivatives satisfies the following properties:

(1) It is given by finitely many diagrams Γ which are unions of binary rooted trees $\Gamma = \sqcup_{j=1}^p T_j$.
(2) The number N of interaction vertices (and thus the order of $\langle \cdots \rangle$ in g) equals the total $\mathcal{AB}$-charge of fields Φ_i.
(3) The number p of trees equals the number of -1-charged fields among $\{\Phi_i\}$.

[8] This condition appeared in [2] in the context of Kashiwara-Vergne problem.

(4) The contribution of each diagram is given by an integral over $\mathbb{C}^N$ which is convergent if the field b is always hit by derivatives in $\{\Phi_i\}$.
(5) If the bare field b occurs among $\{\Phi_i\}$, an infrared regularization may be necessary (see Remark 5.2.6 below).

PROOF. Properties (1–3) summarize the discussion above. We proceed to show the convergence properties (4–5).

First, consider the correlator $\langle\Phi_1(z_1)\cdots\Phi_n(z_n)\rangle$ where all the fields Φ_i are from the list $\{a,\bar{a},\gamma,\bar{\gamma},c,\partial b,\bar{\partial}b\}$ – with no additional derivatives and no bare b ghost. Corresponding Feynman diagrams are given by integrals of the type

$$\int_{\mathbb{C}^N} d^2u_1\cdots d^2u_N \prod_{\{x,y\}\subset\{z_1,\ldots,z_n,u_1,\ldots,u_N\}} P(x,y) \tag{5.38}$$

where the product is over pairs of points corresponding to the edges of the Feynman graph Γ and the propagator $P(x,y)$ is either $\frac{1}{x-y}$ or $\frac{1}{\bar{x}-\bar{y}}$, depending on which pair of fields are connected by the edge. We need to analyze the potential obstructions to convergence arising from a collision of 2 or more points (ultraviolet problems) or from one or more points u_i going to infinity (infrared problems). We have the following possibilities.

(a) Collision of $r\geq 2$ interaction vertices. More precisely, consider the situation when points $u_{i_1},\ldots,u_{i_r}$ are at the distance between $C_1\epsilon$ and $C_2\epsilon$ from each other, with $C_1<C_2$ some constants and ϵ arbitrarily small. The integrand of (5.38) behaves at $\epsilon\to 0$ as $O(\epsilon^{-(r-1)})$, since there are at most $(r-1)$ propagators connecting a pair of points from the set of r colliding points, since the Feynman graph Γ is a tree. Thus, fixing u_{i_1} we have an integrable singularity for integration over $(u_{i_2},\ldots,u_{i_r})$ (which is a $2(r-1)$-fold integral). Therefore, there is no ultraviolet divergence in this case.
(b) Collision of $r\geq 1$ interaction vertices at z_j – the place of insertion of Φ_j. I.e. we consider the situation where points $u_{i_1},\ldots,u_{i_r},z_j$ are at distance between $C_1\epsilon$ and $C_2\epsilon$ from each other. For the same reason as in (a), there are at most r propagators connecting pairs of points from the colliding set. Therefore, the integrand in (5.38) behaves as $O(\epsilon^{-r})$ and we have an integrable singularity for integration over $u_{i_1},\ldots,u_{i_r}$. Thus, again we have no ultraviolet divergence.
(c) Situation where $r\geq 1$ points $u_{i_1},\ldots,u_{i_r}$ go to infinity. – If these points are at a distance $>C_1R$ from z's, the rest of u's, and from each other, the integrand of (5.38) behaves as $O(\frac{1}{R^{2r+1}})$ at $R\to\infty$, since there are $k\leq(r-1)$ propagators connecting pairs points in the set $\{u_{i_1},\ldots,u_{i_r}\}$ and $3r-2k$ propagators connecting points in this set to other points in the "finite" region (recall that the interaction vertices are trivalent), and thus overall $(3r-2k)+k\geq 2r+1$ propagators involving points $u_{i_1},\ldots,u_{i_r}$. Thus, the $2r$-fold integral over $u_{i_1},\ldots,u_{i_r}$ is convergent and there is no infrared divergence.
(d) One could have a potential mixed infrared/ultraviolet problem when several u_i's collide in an ϵ-neighborhood a large distance R away from z's and rest of u's. This situation is treated by a combination of the arguments of (a) and (c) – it also does not lead to a divergence.

In the case of a correlator involving higher derivatives of fundamental fields, we simply take respective derivatives of the correlator of the fundamental fields, given by convergent integrals.

In the case when bare ghost b is present among Φ's, the potential ultraviolet problems become even milder (as the ghost propagator (5.29) has just a log singularity, as opposed to a pole). However, the power counting in the case (c) can fail, see an example in Remark 5.2.6, thus such correlators may require an infrared regularization. □

Corollary 5.2.3. Since the theory is ultraviolet-finite and since the Lagrangian contains no dimensionful parameters, the theory is conformal.

Remark 5.2.4. In the case when fields $\Phi_1, \ldots, \Phi_n$ belong to the subset of real fields $\{A, B, c, *db\}$ (but no λ and no bare b field), the product of propagators (which are proportional to $d\arg(u_i - u_j)$) extends to a smooth form on the compactified configuration space of n points, so the integral is automatically convergent, as in the case of perturbative Chern-Simons theory [3] and Poisson sigma model [6]. The argument we gave above is more general: it allows the λ field (or equivalently, allows γ and $\bar{\gamma}$ independently, not just in the combination $B = -i(\gamma - \bar{\gamma})$).

5.2.1.1. *Weights (naive conformal dimensions) of fields.* We assign the holomorphic/anti-holomorphic *weight* $(h, \bar{h})$ to fields as follows: for a, we set $(h, \bar{h}) = (1, 0)$. For $\bar{a}$, we set $(h, \bar{h}) = (0, 1)$; for the remaining fundamental fields, $\gamma, \bar{\gamma}, b, c$, we set $(h, \bar{h}) = (0, 0)$.[9] Weight is additive with respect to multiplication of fields; applying ∂ to a field increases h by 1, while applying $\bar{\partial}$ increases $\bar{h}$ by 1. These rules define the weight for any composite field.

These weights could be understood as the "naive" conformal dimensions of the fields. Later we will show that weights of fundamental fields coincide with their actual conformal dimensions – see Section 5.5.2.3. However, for composite fields there will be an interesting difference (see Section 5.6).

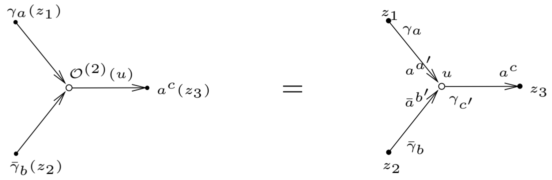
To summarize the various degrees of fields we introduced, we have the ghost degree, the $\mathcal{AB}$-charge and the weight. For fundamental fields they are as follows.

	a	$\bar{a}$	γ	$\bar{\gamma}$	b	c
ghost degree	0	0	0	0	-1	1
$\mathcal{AB}$-charge	-1	-1	1	1	1	-1
weight $(h, \bar{h})$	$(1,0)$	$(0,1)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$

5.2.2. Example: 3-point function of fundamental fields. Consider the 3-point correlation function

$$\langle \gamma_a(z_1)\bar{\gamma}_b(z_2)a^c(z_3)\rangle \tag{5.39}$$

We have the following diagram:



[9] This assignment corresponds to $a\,dz$ being classically a $(1,0)$-form, $\bar{a}\,d\bar{z}$ being a $(0,1)$-form and $\gamma, \bar{\gamma}, b, c$ being scalars.

The corresponding contribution to the correlator is the integral over u (the place of insertion of the deforming observable $\mathcal{O}^{(2)}$) of the product of three propagators:

$$\left\langle \gamma_a(z_1)\bar{\gamma}_b(z_2)a^c(z_3)\; g\, f^{c'}_{a'b'} \int \frac{d^2u}{2\pi} a^{a'}(u)\bar{a}^{b'}(u)\gamma_{c'}(u)\right\rangle_0 = g\, f^{c'}_{a'b'}\delta^{a'}_a \delta^{b'}_b \delta^c_{c'} \int_{\mathbb{C}\ni u} \frac{d^2u}{2\pi}\frac{1}{(u-z_1)(\bar{u}-\bar{z}_2)(z_3-u)} \tag{5.40}$$

In fact, the diagram above is the only contribution to the correlator (5.39) – the total $\mathcal{AB}$-charge of the fields $\gamma, \bar{\gamma}, a$ is $+1$ and thus a contributing diagram has to be a tree with a single interaction vertex. Thus, evaluating the integral above (see (5.178)) we get the explicit result for the correlator:

$$\langle \gamma_a(z_1)\bar{\gamma}_b(z_2)a^c(z_3)\rangle = g\, f^c_{ab}\frac{1}{z_1-z_3}\log\left|\frac{z_1-z_2}{z_3-z_2}\right| \tag{5.41}$$

Remark 5.2.5. The appearance of logs in correlators indicate that we are dealing with a logarithmic CFT [8], see Section 5.6.

By a similar calculation to (5.41), one finds

$$\langle \gamma_a(z_1)\partial b_b(z_2)c^c(z_3)\rangle = -\, g\, f^c_{ab}\frac{1}{z_1-z_2}\log\left|\frac{z_1-z_3}{z_2-z_3}\right|$$
$$\langle \bar{\gamma}_a(z_1)\partial b_b(z_2)c^c(z_3)\rangle = -\, g\, f^c_{ab}\frac{1}{z_2-z_3}\log\left|\frac{z_2-z_1}{z_3-z_1}\right|$$

These 3-point functions and their complex conjugates exhaust the non-vanishing 3-point functions $\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle$, with fields ϕ in the list $\{a, \bar{a}, \gamma, \bar{\gamma}, c, \partial b, \bar{\partial} b\}$. By taking derivatives of these answers, one obtains 3-point functions of arbitrary dervatives of the fundamental fields.

Remark 5.2.6. Note that here we did not consider 3-point functions involving the ghost b not hit by derivatives – such correlators are given by more involved integrals of dilogarithmic type, which contain an infrared divergence at $u \to \infty$. For instance:

$$\langle \gamma_a(z_1)b_b(z_2)c^c(z_3)\rangle = g\, f^c_{ab}\int_{\mathbb{C}\ni u}\frac{d^2u}{2\pi}\,\frac{2\log|u-z_2|+C}{(u-z_1)(\bar{u}-\bar{z}_3)} \tag{5.42}$$

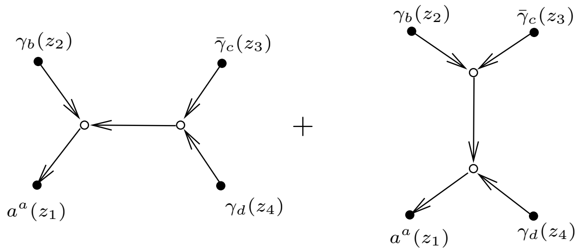
One needs an infrared regularization, e.g. by restricting the integration domain to a disk of large radius R, to have a convergent integral. Note that the constant C in the bc propagator (5.29) also depends on the infrared regularization (see footnote 5). At $R \to \infty$, the correlator (5.42) behaves as $\sim -g\, f^c_{ab}\log^2 R$.

5.2.3. Example: 4-point functions and dilogarithm. Consider the 4-point function

$$\langle a^a(z_1)\gamma_b(z_2)\bar{\gamma}_c(z_3)\gamma_d(z_4)\rangle \tag{5.43}$$

There are two contributing diagrams:

(5.44)



The first diagram yields

(5.45)

$$\Big\langle a^a(z_1)\gamma_b(z_2)\bar{\gamma}_c(z_3)\gamma_d(z_4)\cdot g f^{\widetilde{a}}_{\widetilde{b}\widetilde{c}}\int_{\mathbb{C}\ni u}\frac{d^2u}{2\pi}(\gamma_{\widetilde{a}}a^{\widetilde{b}}\bar{a}^{\widetilde{c}})(u)\cdot g f^{a'}_{b'c'}\int_{\mathbb{C}\ni u'}\frac{d^2u'}{2\pi}(-\bar{\gamma}_{a'}a^{b'}\bar{a}^{c'})(u')\Big\rangle$$

$$= g^2 f^a_{be}f^e_{cd}\int_{\mathbb{C}^2\ni(u,u')}\frac{d^2u}{2\pi}\frac{d^2u'}{2\pi}\frac{1}{(z_1-u)(u-z_2)(\bar{u}-\bar{u}')(\bar{u}'-\bar{z}_3)(u'-z_4)}$$

$$= -g^2 f^a_{be}f^e_{cd}\int_{\mathbb{C}\ni u}\frac{d^2u}{2\pi}\frac{\log\left|\frac{z_3-z_4}{u-z_4}\right|}{(z_1-u)(u-z_2)(\bar{u}-\bar{z}_3)} = g^2 f^a_{be}f^e_{cd}\,\mathbb{I}(z_1,z_2,z_3,z_4)$$

Here we introduced the notation

(5.46) $\mathbb{I}(z_1,z_2,z_3,z_4) =$

$$= \frac{1}{2z_{12}}\left(i\mathsf{D}\left(\frac{z_{34}}{z_{14}}\right) - i\mathsf{D}\left(\frac{z_{34}}{z_{24}}\right) + \log\left|\frac{z_{34}}{z_{14}}\right|\cdot\log\left|\frac{z_{23}}{z_{13}}\right| + \log\left|\frac{z_{14}}{z_{24}}\right|\cdot\log\left|\frac{z_{23}}{z_{34}}\right|\right)$$

where $z_{ij} = z_i - z_j$ and $\mathsf{D}(-)$ is the Bloch-Wigner dilogarithm function [13] (see Appendix 5.A.1 for a quick recap of the relevant properties). The integral over u' in (5.45) is evaluated using (5.178) and the remaining integral over u is evaluated using (5.178), (5.181). The integral (5.45) was considered in the literature, see [9] (Section 5).

The full result for the 4-point function (5.43) is:

(5.47)

$$\langle a^a(z_1)\gamma_b(z_2)\bar{\gamma}_c(z_3)\gamma_d(z_4)\rangle = g^2\Big(f^a_{be}f^e_{cd}\,\mathbb{I}(z_1,z_2,z_3,z_4) + f^a_{de}f^e_{cb}\,\mathbb{I}(z_1,z_4,z_3,z_2)\Big)$$

The two terms here corresponds to the two diagrams (5.44). Note that they are obtained from one another by interchanging points z_2 and z_4 and indices b and d.

By a similar computation, one finds the 4-point function

(5.48)

$$\langle a^a(z_1)\bar{\gamma}_b(z_2)\gamma_c(z_3)\bar{\gamma}_d(z_4)\rangle = g^2\Big(f^a_{be}f^e_{cd}\,\mathbb{J}(z_1,z_2,z_3,z_4) + f^a_{de}f^e_{cb}\,\mathbb{J}(z_1,z_4,z_3,z_2)\Big)$$

where

(5.49)

$$\mathbb{J}(z_1,z_2,z_3,z_4) = -\int_{\mathbb{C}^2}\frac{d^2u}{2\pi}\frac{d^2u'}{2\pi}\frac{1}{(z_1-u)(\bar{u}-\bar{z}_2)(u-u')(u'-z_3)(\bar{u}'-\bar{z}_4)}$$

$$= \frac{1}{2z_{13}}\left(i\mathsf{D}\left(\frac{z_{14}}{z_{24}}\right) - i\mathsf{D}\left(\frac{z_{34}}{z_{24}}\right) + \log\left|\frac{z_{34}}{z_{24}}\right|\cdot\log\left|\frac{z_{12}}{z_{23}}\right| + \log\left|\frac{z_{34}}{z_{14}}\right|\cdot\log\left|\frac{z_{12}}{z_{24}}\right|\right)$$

We also have 4-point functions involving ghosts which are computed similarly and are also expressed in terms of functions $\mathbb{I}, \mathbb{J}$:

$$\langle c^a(z_1)\gamma_b(z_2)\partial b_c(z_3)\gamma_d(z_4)\rangle = -g^2(f^a_{be}f^e_{cd}\mathbb{I}_{3412} + f^a_{de}f^e_{cb}\mathbb{I}_{3214}), \tag{5.50}$$

$$\langle c^a(z_1)\bar\gamma_b(z_2)\partial b_c(z_3)\bar\gamma_d(z_4)\rangle = -g^2(f^a_{be}f^e_{cd}\mathbb{J}_{1234} + f^a_{de}f^e_{cb}\mathbb{J}_{1432}), \tag{5.51}$$

$$\langle c^a(z_1)\bar\gamma_b(z_2)\partial b_c(z_3)\gamma_d(z_4)\rangle = -g^2\Big(f^a_{ce}f^e_{bd}(\mathbb{I}_{1324} + \mathbb{J}_{3142}) + \\ + f^a_{be}f^e_{cd}\mathbb{I}_{3421} + f^a_{de}f^e_{cb}\mathbb{J}_{4132}\Big) \tag{5.52}$$

where for brevity we denoted $\mathbb{I}_{ijkl} = \mathbb{I}(z_i, z_j, z_k, z_l)$ and $\mathbb{J}_{ijkl} = \mathbb{J}(z_i, z_j, z_k, z_l)$. Note that (5.51) is simply minus the correlator (5.48).

We remark that $\mathbb{I}$ and $\mathbb{J}$, our building blocks for 4-point functions, have the following symmetries:

$$\mathbb{I}_{2134} = \mathbb{I}_{1234}, \quad \mathbb{I}_{1243} = \frac{\bar z_{12}}{z_{12}}\,\overline{\mathbb{I}_{1234}}, \quad \mathbb{J}_{3412} = -\mathbb{J}_{1234}$$

Formulae (5.43, 5.48, 5.50, 5.51, 5.52) and their complex conjugates exhaust all nonzero 4-point function of fields from the set $\{a, \bar a, \gamma, \bar\gamma, c, \partial b, \bar\partial b\}$ with total $\mathcal{AB}$-charge of fields under the correlator equal to $+2$. The other possibility is to have total $\mathcal{AB}$-charge zero; in this case the correlator coincides with the abelian one and is the sum of products of propagators, e.g.

$$\Big\langle a^a(z_1)a^b(z_2)\gamma_c(z_3)\gamma_d(z_4)\Big\rangle = \frac{\delta^a_c\delta^b_d}{z_{13}z_{24}} + \frac{\delta^a_d\delta^b_c}{z_{14}z_{23}}$$

5.2.4. Aside: from correlators on the plane to correlators on the sphere. Restoring Möbius-invariance. Consider the two-point function

$$\langle a^a(z_1)\gamma_b(z_2)\rangle = \frac{\delta^a_b}{z_1 - z_2} \tag{5.53}$$

– it coincides with the abelian propagator (5.29), as there are no admissible Feynman graphs apart from the edge connecting z_1, z_2. As we will see (Section 5.5.2.3, Proposition 5.5.6), field a^a is primary, of conformal dimension $(\Delta, \bar\Delta) = (1, 0)$ and γ is primary of dimension $(0, 0)$. Global conformal invariance implies that two-point functions of primary fields of non-matching dimensions must vanish (see e.g. [4]). Thus, (5.53) seems to be in contradiction with conformal invariance.

The explanation to the apparent paradox is that (5.53) is indeed not compatible with the global conformal symmetry of the sphere $\mathbb{C}P^1$ (the group of Möbius transformations), but is compatible with the global conformal symmetry of the plane $\mathbb{C}$ (translations, rotations and scaling). Indeed, on a sphere the kinetic operators $\partial, \bar\partial, \partial\bar\partial$ appearing in (5.20) have zero-modes, which we have killed when constructing propagators (5.29) by imposing conditions on fields at $z = \infty$. In other words, a correlator (5.27) on $\mathbb{C}$ can be written as a correlator on the sphere with an additional field[10]

$$\Theta = \delta(\gamma)\delta(\bar\gamma)\delta(b)\delta(c)$$

[10] We refer the reader to Witten [12] (chapter 10) for details on soaking zero-modes and working with delta-functions of fields in $\beta\gamma$ systems.

inserted at $z = \infty$, which effectively imposes the necessary conditions on fields at infinity:

$$\begin{aligned}\langle \Phi_1(z_1)\cdots\Phi_n(z_n)\rangle_{\mathbb{C}} =& \frac{1}{Z}\int e^{-\frac{1}{4\pi}S_{\mathbb{C}P^1}}\Phi_1(z_1)\cdots\Phi_n(z_n)\Theta(\infty)\\ =& \langle \Phi_1(z_1)\cdots\Phi_n(z_n)\Theta(\infty)\rangle_{\mathbb{C}P^1}\end{aligned}$$

Here $\delta(\gamma) = \prod_a \delta(\gamma_a)$ and similarly for the other delta-functions. Moreover, we have $\delta(c) = \prod_a c^a$, $\delta(b) = \prod_a b_a$ since b, c are odd.

The version of the two-point function (5.53) on the sphere is the 3-point function

$$\begin{aligned}\langle dz_1\, a^a(z_1)\gamma_b(z_2)\Theta(z_0)\rangle_{\mathbb{C}P^1} =& dz_1\,\delta^a_b\left(\frac{1}{z_1-z_2}-\frac{1}{z_1-z_0}\right)\\ =& \delta^a_b\, dz_1\,\frac{z_2-z_0}{(z_1-z_2)(z_1-z_0)}\end{aligned} \tag{5.54}$$

Here we included the factor dz_1 with $a^a(z_1)$ for convenience of tracking invariance properties. This answer on the sphere is has the following properties:

- It reduces to (5.53) in the limit $z_0 \to \infty$ and is invariant under the Möbius group $PSL_2(\mathbb{C})$. (In fact, this property fully characterizes the answer.)
- Asymptotic behavior at $z_2 \to z_1$ (with z_0 fixed) is given by the pole (5.53).
- At $z_2 \to z_0$ the result vanishes, which is consistent with $(\gamma_b\delta(\gamma))(z_0) = 0$, cf. [12].

One can also express (5.54) in terms of the *Szegö kernel*

$$\mu_{wz} = \frac{(dw)^{\frac{1}{2}}\,(dz)^{\frac{1}{2}}}{w-z}$$

– a Möbius-invariant holomorphic half-differential on the configuration space of two points on $\mathbb{C}P^1$. Indeed, one has

$$\langle dz_1\, a^a(z_1)\gamma_b(z_2)\Theta(z_0)\rangle_{\mathbb{C}P^1} = \delta^a_b\,\frac{\mu_{z_1z_2}\mu_{z_1z_0}}{\mu_{z_2z_0}} \tag{5.55}$$

The benefit of this form of the answer is that it is manifestly Möbius-invariant.

Likewise, for instance, the 3-point function (5.41) on the plane arises as the limit $z_0 \to \infty$ of a Möbius-invariant 4-point function on the sphere:

$$\langle \gamma_a(z_1)\bar{\gamma}_b(z_2)\, dz_3\, a^c(z_3)\Theta(z_0)\rangle_{\mathbb{C}P^1} = -g\, f^c_{ab}\,\frac{\mu_{z_3z_1}\,\mu_{z_3z_0}}{\mu_{z_1z_0}}\log\left|\frac{(z_1-z_2)(z_3-z_0)}{(z_3-z_2)(z_1-z_0)}\right|$$

We have again included the factor dz_3 with $a^c(z_3)$ for convenience. Note that the expression in $\log|\cdots|$ is the cross-ratio of the quadruple of points $(z_1, z_3; z_2, z_0)$ – an invariant of the Möbius group. Also, note that the the first factor in the r.h.s. vanishes at $z_1 = z_0$ and the factor $\log|\cdots|$ vanishes at $z_2 = z_0$.

Conformally invariant version of the two-point function $\langle c^a(z_1)b_b(z_2)\rangle$ (5.29) is the following 4-point function on the sphere:

$$\left\langle c^a(z_1)b_b(z_2)\,\widetilde{\Theta}(z_0)\,\delta(c(z_0'))\right\rangle_{\mathbb{C}P^1} = 2\delta^a_b\log\left|\frac{(z_1-z_2)(z_0'-z_0)}{(z_1-z_0)(z_0'-z_2)}\right| \tag{5.56}$$

Here we have split the field Θ into $\widetilde{\Theta}=\delta(\gamma)\delta(\bar{\gamma})\delta(b)$ and $\delta(c)$.[11] The splitting of Θ at a point z_0 into $\widetilde{\Theta}$ at z_0 and $\delta(c)$ at a different "nearby" point z_0' is a version of the "infrared regularization" that we needed to define the bc propagator on the plane (cf. footnote 5).

REMARK 5.2.7. Note that in (5.55) we could also split $\Theta(z_0)$ as $\widetilde{\Theta}(z_0)\delta(c(z_0'))$. The resulting 4-point function on the sphere can be written in the form

$$\left\langle dz_1\, a^a(z_1)\gamma_b(z_2)\widetilde{\Theta}(z_0)\delta(c(z_0'))\right\rangle_{\mathbb{C}P^1}=dz_1\partial_{z_1}\left(2\delta^a_b\log\left|\frac{(z_1-z_2)(z_0'-z_0)}{(z_1-z_0)(z_0'-z_2)}\right|\right) \tag{5.57}$$

It does not depend on z_0' and coincides with (5.55).

As another example, 3-point function (5.42) becomes the following 5-point function on the sphere, with added insertions of $\widetilde{\Delta}$ at z_0 and $\delta(c)$ at $z_0'\neq z_0$:

$$\begin{aligned}\left\langle \gamma_a(z_1)b_b(z_2)c^c(z_3)\;\widetilde{\Theta}(z_0)\;\delta(c(z_0'))\right\rangle_{\mathbb{C}P^1}&=\\ =g\,f^c_{ab}\,\frac{i}{4\pi}\int_{\mathbb{C}\ni u}&\frac{\mu_{uz_1}\mu_{uz_0}}{\mu_{z_1z_0}}\cdot\frac{\bar{\mu}_{uz_3}\bar{\mu}_{uz_0'}}{\bar{\mu}_{z_3z_0'}}\cdot 2\log\left|\frac{(u-z_2)(z_0'-z_0)}{(u-z_0)(z_0'-z_2)}\right|\end{aligned}$$

Here the three factors under the integral are the conformally invariant replacements of the three propagators constituting the integrand of (5.42). Note that the integral above is convergent; it can be computed explicitly in terms of dilogarithms, using (5.180).

In summary: every n-point correlator $\langle\Phi_1\cdots\Phi_n\rangle$ on the plane not containing an infrared divergence (no bare b field among $\Phi_1,\ldots,\Phi_n$) has a unique Möbius-invariant extension as an $(n+1)$-point function on $\mathbb{C}P^1$, with an added insertion $\Theta(z_0)$. This extension is written in terms of Szegö kernels and cross-ratios. In the case of a plane n-point correlator requiring infrared regularization (case when bare field b occurs among $\Phi_1,\ldots,\Phi_n$), the Möbius-invariant extension on $\mathbb{C}P^1$ is an $(n+2)$-point function with added insertions of $\widetilde{\Theta}(z_0)$ and $\delta(c)$ at $z_0'\neq z_0$. It is also written in terms of Szegö kernels and cross-ratios, via replacing the propagators in the Feynman diagram expansion of the plane correlator with their $\mathbb{C}P^1$ counterparts (5.55), (5.56).

REMARK 5.2.8. Fields $\widetilde{\Theta}=\delta(\gamma)\delta(\bar{\gamma})\delta(b)$ and $\delta(c)$ which we use to "soak" the zero-modes satisfy the following:

- Both $\widetilde{\Theta}$ and $\delta(c)$ are Q-closed. Indeed:

$$\begin{aligned}Q\widetilde{\Theta}=-\frac{g}{2}f^a_{bc}c^b(\gamma-\bar{\gamma})_a\frac{\partial}{\partial\gamma_c}\delta(\gamma)\delta(\bar{\gamma})\delta(b)+\frac{g}{2}f^a_{bc}c^b(\gamma-\bar{\gamma})_a\delta(\gamma)\frac{\partial}{\partial\bar{\gamma}_c}\delta(\bar{\gamma})\delta(b)+\\ +(\gamma+\bar{\gamma})_a\delta(\gamma)\delta(\bar{\gamma})\frac{\partial}{\partial b_a}\delta(b)\quad=gf^a_{ba}c^b\widetilde{\Theta}=0\end{aligned}$$

[11] The insertion of $\delta(c)$ at a point corresponds to requiring the gauge transformations to be trivial at that point.

Here we use that $\gamma_a\delta(\gamma) = \bar{\gamma}_a\delta(\bar{\gamma}) = 0$. In the last step, we use unimodularity of the Lie algebra $\mathfrak{g}$. Also,

$$Q\delta(c) = \frac{g}{2} f^a_{bc} c^b c^c \frac{\partial}{\partial c^a}\delta(c) = 0$$

– vanishes as a product of $\dim\mathfrak{g}+1$ ghosts at a point (recall that $\delta(c) = c^{\dim\mathfrak{g}}\cdots c^2 c^1$ is the product of all components of the c-ghost) or in other words because $\wedge^{\dim\mathfrak{g}+1}\mathfrak{g}^* = 0$.[12] We further note that $\widetilde{\Theta}$ can be split further into Q-cocylces:

$$\widetilde{\Theta} = \prod_{a=1}^{\dim\mathfrak{g}} \Big(\delta(\lambda_a) b_a\Big)\cdot\delta(B) \tag{5.58}$$

(with appropriately normalized delta-functions). Here fields $\delta(\lambda_a)b^a$ are Q-closed for each a and $\delta(B)$ is Q-closed due to unimodularity of $\mathfrak{g}$.

- The operator product expansions $\mathcal{O}^{(2)}(u)\,\widetilde{\Theta}(z)$ and $\mathcal{O}^{(2)}(u)\,\delta(c(z))$ both have an integrable singularity in u at $u=z$ (see Section 5.3.3).
- The operator product expansion between the fields $\delta(c)$ and $\widetilde{\Theta}$ in the abelian theory (i.e. at $g=0$) has the form

$$\delta(c(z))\,\widetilde{\Theta}(w) \sim \delta(\gamma)\delta(\bar{\gamma})\sum_{p=0}^{\dim\mathfrak{g}-1}\frac{1}{p!}\Big(2\log|z-w|\Big)^{\dim\mathfrak{g}-p}\cdot\langle c,b\rangle^p + \mathrm{reg} \tag{5.59}$$

where all fields on the r.h.s. are at w and composite field $\langle c,b\rangle^p = (c^a b_a)^p$ is understood as *renormalized*, cf. (5.28). In the non-abelian theory, there are additional terms of order ≥ 1 in g, which also come with powers of $\log|z-w|$.
- Note that our way of soaking zero-modes is different from the way proposed by Witten in [10], by using $\exp(-g_0^2\int\mu\,\mathrm{tr}B^2)$ with μ an area form and g_0 the standard coupling constant in two-dimensional Yang-Mills theory. We will explain the geometrical meaning of our soaking operators in terms of the moduli space of flat connections elsewhere.

5.3. Operator product expansions

Given two fields Φ_1, Φ_2, we are interested in the singularity of the correlator

$$\langle\Phi_1(z)\Phi_2(w)\,\phi_1(x_1)\cdots\phi_n(x_n)\rangle \tag{5.60}$$

in the asymptotics $z\to w$; here $\phi_1,\ldots,\phi_n$ are arbitrary test fields inserted at finite distance away from z,w. Operator product expansion (OPE) is an expression of the form

$$\Phi_1(z)\Phi_2(w) \sim \sum_{i=1}^{s}\sigma_i(z-w)\widetilde{\Phi}_i(w) + \mathrm{reg}. \tag{5.61}$$

[12] Note that Q-closedness would fail if we would have split Θ instead into $\delta(\gamma)\delta(\bar{\gamma})\delta(c)$ and $\delta(b)$.

with $\widetilde{\Phi}_i$ some fields and $\sigma_i(z-w)$ some singular coefficient functions, typically of form $(z-w)^{-p}(\bar{z}-\bar{w})^{-q}\log^r|z-w|$ with $p+q\geq 0, r\geq 0$; reg. stands for terms which are regular (continuous) at $z\to w$. The number s of singular terms on the r.h.s. depends on fields Φ_1,Φ_2. Expression (5.61) means that one can replace the product $\Phi_1(z)\Phi_2(w)$ with the right hand side in a correlator (5.60) with arbitrary test fields $\phi_1,\ldots,\phi_n$ inserted away from z,w, reproducing the correct behavior of the correlator at $z\to w$, modulo terms having a well-defined limit at $z\to w$.

Test fields $\phi_1,\ldots,\phi_n$ in (5.60) can be assumed to be fundamental fields without loss of generality.

Let Φ_1,Φ_2 be two (possibly, composite) fields. The OPE is given by a sum of Feynman graphs $\boldsymbol{\gamma}$ with loose half-edges decorated by respective fundamental fields (or derivatives) – their product over the loose half-edges yields the composite field $\widetilde{\Phi}(w)$ in the term of the OPE corresponding to $\boldsymbol{\gamma}$. Graphs $\boldsymbol{\gamma}$ contributing to the OPE have the following properties:

(i) Graph $\boldsymbol{\gamma}$ contains one vertex decorated by $\Phi_1(z)$, one vertex decorated by $\Phi_2(w)$ and $k\geq 0$ interaction vertices decorated by $\mathcal{O}^{(2)}(u_1),\ldots,\mathcal{O}^{(2)}(u_k)$, with $u_1,\ldots,u_k$ integrated over $\mathbb{C}$.

(ii) Cutting any single edge in $\boldsymbol{\gamma}$, we do not create a connected component which contains neither vertex $\Phi_1(z)$, nor vertex $\Phi_2(w)$. This is an analog of the one-particle irreducibility; by an abuse of terminology, we will call graphs with this property 1PI graphs.

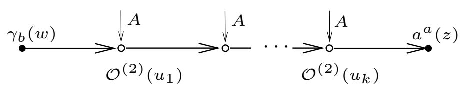
Graphs $\boldsymbol{\gamma}$ arise as subgraphs of Feynman graphs Γ contributing to the correlator (5.60). Loose half-edges correspond to edges of Γ that are severed when cutting out the subgraph. 1PI requirement for $\boldsymbol{\gamma}$ is imposed in order to avoid overcounting: for a graph Γ contributing to (5.61), there is a unique way to single out the OPE subgraph $\boldsymbol{\gamma}$ satisfying (i), (ii) above. The contribution of the quotient graph $\Gamma/\boldsymbol{\gamma}$ (i.e. Γ with the subgraph $\boldsymbol{\gamma}$ collapsed into a single vertex) to the correlator of the term $i=\boldsymbol{\gamma}$ in the r.h.s. of (5.61) with the test fields $\phi_1,\ldots,\phi_n$ is the same as the contribution of Γ to (5.60) (up to regular terms at $z\to w$).

5.3.1. OPEs of fundamental fields. For example, consider the OPE

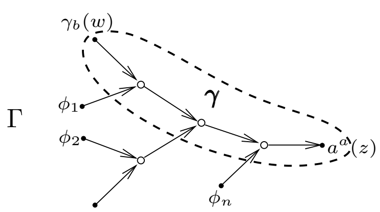
$$a^a(z)\,\gamma_b(w) \tag{5.62}$$

The only potentially contributing Feynman graphs $\boldsymbol{\gamma}$ are graphs of "branch" type

(5.63)

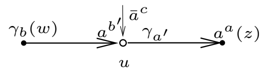


with $k\geq 0$ interaction vertices. They arise as subgraphs of trees (or disjoint unions of trees) Γ contributing to a correlator $\langle a^a(z)\,\gamma_b(w)\;\phi_1(x_1)\cdots\phi_n(x_n)\rangle$:



For instance, for $k = 0$, the branch (5.63) is a single edge connecting $\gamma_b(w)$ and $a^a(z)$, its contribution to the OPE (5.62) is simply the propagator $\frac{\delta^a_b}{z-w}$ (times the identity field suppressed in the notation).

For $k = 1$, the contribution of the branch graph



to the OPE is:

$$g\, f^a_{bc} \int_{\mathbb{C}} \frac{d^2u}{2\pi} \frac{\bar{a}^c(u)}{(z-u)(u-w)} \tag{5.64}$$

In this expression, we replace the field $\bar{a}^c(u)$ with its Taylor expansion around w,

$$\bar{a}^c(u) = \sum_{i,j\geq 0} \frac{1}{i!j!} \partial^i \bar{\partial}^j \bar{a}^c(w)\ (u-w)^i (\bar{u}-\bar{w})^j \quad = \bar{a}^c(w) + \mathcal{R}^c(u,w) \tag{5.65}$$

where split the Taylor expansion into the zeroth term and the remainder (error term) $\mathcal{R}^c(u,w)$ behaving as $O(|u-w|)$. Under the correlator with test fields, the term in (5.64) with $\bar{a}^c$ replaced by $\mathcal{R}^c(u,w)$ is continuous as $z \to w$: setting $z = w$, we get an integrable singularity of the integrand. Thus, up to a regular term, (5.64) is equivalent to

$$g\, f^a_{bc} \int_{\mathbb{C}} \frac{d^2u}{2\pi} \frac{\bar{a}^c(w)}{(z-u)(u-w)} = \frac{g}{2}\, f^a_{bc} \frac{\bar{z}-\bar{w}}{z-w}\, \bar{a}^c(w) \tag{5.66}$$

– cf. (5.179) for the evaluation of the integral.

Finally, for $k \geq 2$, branch graphs (5.63) give regular contributions to the OPE: setting $z = w$, we get an integrable singularity of the integrand as any subset of $u_1, \ldots, u_k$ approaches $z = w$ (by power counting arguments of the proof of Proposition 5.2.2).

Thus, we have a complete result for the OPE (5.62):

$$a^a(z)\, \gamma_b(w) \sim \frac{\delta^a_b}{z-w} + \frac{g}{2}\, f^a_{bc} \frac{\bar{z}-\bar{w}}{z-w}\, \bar{a}^c(w)\ + \text{reg.} \tag{5.67}$$

As a check of this result, we can take the correlator of left and right side of (5.67) with the test field $\bar{\gamma}_d(x)$. We obtain

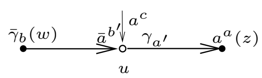
$$\langle a^a(z)\gamma_b(w)\bar{\gamma}_d(x)\rangle \overset{?}{\sim} \frac{g}{2} f^a_{bc} \frac{\bar{z}-\bar{w}}{z-w}\ \langle \bar{a}^c(w)\bar{\gamma}_d(x)\rangle + \text{reg.}$$

The the 3-point function on the left, known from (5.41), can be written as $\frac{g}{2} f^a_{bd} \frac{1}{z-w} \log|1 + \frac{z-w}{w-x}|^2$ and is indeed equivalent to r.h.s., $\frac{g}{2} f^a_{bd} \frac{\bar{z}-\bar{w}}{z-w} \frac{1}{\bar{w}-\bar{x}}$, as $z \to w$.

As another example, consider the OPE

$$a^a(z)\, \bar{\gamma}_b(w)$$

As in the previous case, we have branch graphs similar to (5.63), and graphs with $k \geq 2$ don't contribute to the singular part of the OPE by a power counting argument. Case $k = 0$ is now also absent: propagator between a and $\bar{\gamma}$ is zero. Thus, we only have the contribution of the $k = 1$ graph



This results in the following OPE:

$$\begin{aligned} a^a(z)\,\bar\gamma_b(w) &\sim -\,g f^a_{bc}\int_{\mathbb{C}}\frac{d^2u}{2\pi}\frac{a^c(u)}{(z-u)(\bar u-\bar w)}+\text{reg.}\\ &\sim -\,g f^a_{bc}\int_{\mathbb{C}}\frac{d^2u}{2\pi}\frac{a^c(w)}{(z-u)(\bar u-\bar w)}+\text{reg.}\\ &\sim -\,g f^a_{bc}\log|z-w|\;a^c(w)+\text{reg.} \end{aligned} \tag{5.68}$$

Here we use the same argument as above to replace $a(u)$ with $a(w)$. The resulting integral over u is logarithmically divergent at $u\to\infty$ and needs an infrared regularization $|u|<R$.[13] The regularized integral is given by (5.176). Changing the cutoff R does not affect the singular part of the result.

By similar computations, we have the following OPEs:

(5.69)

$$\gamma_a(z)\,\bar\gamma_b(w)\sim -\,g\,f^c_{ab}\,\log|z-w|\;(\gamma-\bar\gamma)_c(w)+\text{reg}$$

(5.70)

$$c^a(z)\,\partial b_b(w)\sim -\frac{\delta^a_b}{z-w}-\frac{g}{2}\,f^a_{bc}\,\frac{\bar z-\bar w}{z-w}\;\bar a^c(w)-g\,f^a_{bc}\,\log|z-w|\;a^c(w)+\text{reg}$$

(5.71)

$$c^a(z)\,\gamma_b(w)\sim -\,g\,f^a_{bc}\,\log|z-w|\;c^c(w)+\text{reg}$$

(5.72)

$$\partial b_a(z)\gamma_b(w)\sim -\frac{g}{2}\,f^c_{ab}\,\frac{\bar z-\bar w}{z-w}\;\bar\partial b_c(w)+\text{reg}$$

(5.73)

$$\partial b_a(z)\bar\gamma_b(w)\sim -\,g\,f^c_{ab}\,\log|z-w|\;\partial b_c(w)+\text{reg}$$

For each OPE, there is also the complex conjugate one.

Let us denote $\text{reg}^{(p)}$ a remainder term in an OPE which has continuous derivatives of order $\leq p$ at $z=w$. In particular, by default we write OPEs up to $\text{reg}=\text{reg}^{(0)}$ terms.

For the OPE $c(z)\,b(w)$ only the branch graph with $k=0$ (i.e. just a single edge) contributes:

$$c^a(z)\,b_b(w)\sim 2\,\delta^a_b\log|z-w|+\text{reg} \tag{5.74}$$

Note that the "regular" part here is just continuous but not differentiable at $z=w$, as implied by presence of $O(g)$ terms in $c(z)\partial b(w)$ OPE (5.70). The latter imply that (5.74) can be refined to

$$c^a(z)\,b_b(w)\sim\log|z-w|\Big(2\,\delta^a_b+g\,f^a_{bc}\,(z-w)\;a^c(w)+g\,f^a_{bc}\,(\bar z-\bar w)\;\bar a^c(w)\Big)+\text{reg}^{(1)}$$

where the remainder term is differentiable (but not twice differentiable) at $z=w$. The $O(g)$ contribution here can be seen as coming from $k=1$ branch graph for cb OPE which is continuous but not differentiable.

[13] More precisely: we split $a(u)=a(w)+\mathcal{R}(u,w)$ as in (5.65). Then we have $\int d^2u\frac{a(u)}{(z-u)(\bar u-\bar w)}=\int d^2u\frac{a(w)}{(z-u)(\bar u-\bar w)}+\int d^2u\frac{\mathcal{R}(u,w)}{(z-u)(\bar u-\bar w)}$. Here the integral on the left is convergent at $u\to\infty$ when placed under a correlator with a test field. On the right, it is split into two integrals which are both infrared-divergent, but their behavior (after imposing a cutoff $|u|<R$) at $z\to w$ is easier to analyze.

OPEs which are trivial due to Feynman diagram combinatorics. The OPEs of the following pairs of fundamental fields are purely regular:

(5.75)
$$\begin{aligned} a^a(z)\, a^b(w) \sim \mathrm{reg}^{(\infty)}, \quad a^a(z)\, \bar{a}^b(w) \sim \mathrm{reg}^{(\infty)}, \quad a^a(z)\, c^b(w) \sim \mathrm{reg}^{(\infty)}, \\ c^a(z)\, c^b(w) \sim \mathrm{reg}^{(\infty)}, \quad b_a(z)\, b_b(w) \sim \mathrm{reg}^{(\infty)}, \quad a^a(z)\, b_b(w) \sim \mathrm{reg}^{(\infty)} \end{aligned}$$

In each of these cases one can also take arbitrary derivatives of the first and second field and the OPE is still regular – there are no contributing Feynman graphs (cases bb and ab is slightly more subtle: there is an admissible orientation of branch graphs but no admissible decoration of half-edges by fields). In other words, right hand sides in OPEs (5.75) are infinitely-differentiable in $z, \bar{z}, w, \bar{w}$ at $z = w$.

On the other hand, we have

$$\gamma_a(z)\, \gamma_b(w) \sim \mathrm{reg}, \quad b_a(z)\, \gamma_b(w) \sim \mathrm{reg}$$

– with continuous but non-differentiable r.h.s. at $z = w$. In fact, OPEs (5.72), (5.73) imply that

$$b_a(z)\, \gamma_b(w) \sim -g\, f^c_{ab}(\bar{z} - \bar{w})\, \log|z - w|\, \bar{\partial} b_c(w) + \mathrm{reg}^{(1)}$$

with a remainder term which is differentiable but not twice differentiable at $z = w$. Similarly, $\gamma(z)\gamma(w)$ is non-differentiable because there are contributions of $k = 2$ branch diagrams to OPEs $\partial\gamma(z)\gamma(w)$, $\bar{\partial}\gamma(z)\gamma(w)$.

5.3.2. OPEs of derivatives of fundamental fields. For an OPE of general derivatives of fundamental fields, branch graphs with $k > 1$ can contribute – but only finitely many of them: for k sufficiently large, the limit $z = w$ of the integral over $u_1, \ldots, u_k$ is convergent. One can also find a bound on k from a weight counting argument (here "weight" is understood as in Section 5.2.1.1), as follows.

As an example, consider the OPE

$$\partial^p a^a(z)\, \partial^q \gamma_b(w) \tag{5.76}$$

with some $p, q \geq 0$. The weight of this expression is $(h, \bar{h}) = (p + q + 1, 0)$. A contribution of a branch graph with k interaction vertices to the OPE is a sum of terms of form

$$\prod_{i=1}^{r} (\partial^{\mu_i} \bar{\partial}^{\nu_i} a)(w) \cdot \prod_{j=1}^{s} (\partial^{\rho_j} \bar{\partial}^{\sigma_j} \bar{a})(w) \cdot (z - w)^l (\bar{z} - \bar{w})^m \log^\alpha |z - w| \tag{5.77}$$

with $r + s = k$ (fields a, $\bar{a}$ should be appropriately contracted via structure constants); note that derivatives of $a, \bar{a}$ arise from expanding a field inserted at u in a Taylor series centered at w. The weight of this expression is $(h, \bar{h}) = (r + \sum \mu_i + \sum \rho_j - l, s + \sum \nu_i + \sum \sigma_j - m)$. It has to coincide with the weight of (5.76). In particular, for the total weight $h + \bar{h}$, we have

$$\underbrace{r + s}_{k} + \underbrace{\sum \mu_i + \sum \nu_i + \sum \rho_j + \sum \sigma_j}_{\geq 0} - l - m = p + q + 1$$

In particular, we have $l + m \geq k - (p + q + 1)$. If $l + m \geq 1$, the term (5.77) is non-singular (continuous). Thus, one can only have singular terms in the

OPE if

$$k \leq p+q+1 \tag{5.78}$$

Similarly, for the OPE $\partial^p\bar\partial^{p'}a^a(z)\,\partial^q\bar\partial^{q'}\gamma_b(w)$, only branch graphs with $k\leq p+p'+q+q'+1$ can contribute to the singular part.

As another example, for the OPE $\partial^p\bar\partial^{p'}\gamma_a(z)\,\partial^q\bar\partial^{q'}\gamma_b(w)$, there are terms containing c, a derivative of b and $(k-2)$ fields $a,\bar a$ (each field can come with more derivatives) – for these terms one obtains the estimate $k\leq p+p'+q+q'+1$. There are also terms containing $\gamma-\bar\gamma$ and $(k-1)$ fields $a,\bar a$ – these yield the same estimate for k.

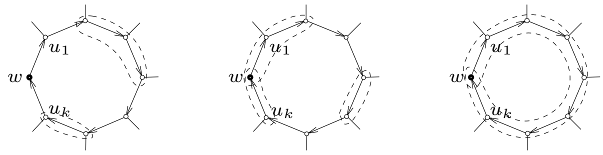
Convergence argument. As we mentioned above, alternatively to going the route of weight counting, one can prove that branch graphs with large k do not contribute to the singular part of the OPE by checking convergence of the integral over $u_1,\ldots,u_k$ in the limit $z=w$. For instance, consider the OPE (5.76). At $z=w$, the corresponding contribution contains an integral of the form

$$\int_{u_1,\ldots,u_k}\partial_w^p P_{wu_1}\cdot P_{u_1u_2}\cdot\ \cdots\ \cdot P_{u_{k-1}u_k}\cdot\partial_w^q P_{u_kw} \tag{5.79}$$

where each propagator P_{xy} is either $\frac{1}{x-y}$ or $\frac{1}{\bar x-\bar y}$. We should analyze the potential ultraviolet problems:

(1) If some of the u_i's collapse together (but not at w), we have an integrable singularity in (5.79) by the argument of (a) of the proof of Proposition 5.2.2.
(2) If a proper subset of u_i's, with indices $i\in S\subset\{1,\ldots,k\}$, collapses on w, we consider the integral over $\{u_i\}_{i\in S}$ in a disk $D_{w,\epsilon}$ centered at w of small radius ϵ, with the non-collapsing points u_i fixed outside of the disk and regarded as parameters. This gives a product of convergent integrals (by (b) of the proof of Proposition 5.2.2), one integral per each string of consecutive integers in S;[14] two of these integrals can be equipped with derivatives ∂_w^p and ∂_w^q which does not affect convergence.
(3) If *all* u_i's collapse at w, the integrand of (5.79) behaves as $O(\frac{1}{\epsilon^{p+q+k+1}})$ when all u_i's are at the distance of order ϵ from w and from each other. Thus, the $2k$-fold integral (5.79) may be divergent if $p+q+k+1\geq 2k$ (i.e. when $k\leq p+q+1$) but is convergent otherwise.

Here are the typical collapsing subgraphs of $\boldsymbol{\gamma}$ corresponding to these three cases (we draw them on the graph $\boldsymbol{\gamma}$ with vertices z and w identified).



(1) (2) (3)

[14]In other words, we have one integral per connected component of the collapsing subgraph of the branch graph $\boldsymbol{\gamma}$. Here the "collapsing subgraph" is the full subgraph of $\boldsymbol{\gamma}$ with vertices $\{u_i\}_{i\in S}\cup\{w,z\}$

Thus, only the situation (3) can lead to an ultraviolet problem at $z = w$, which gives us an upper bound for which k can contribute to the singular part of the OPE. Note that the bound is the same as the one we obtained above from weight counting (5.78).

EXAMPLE 5.3.1. For instance, the following OPEs contain the contributions of branch diagrams with $k \leq 2$:

$$a^a(z)\partial\gamma_b(w) \sim \tag{5.80}$$
$$\sim \frac{\delta^a_b}{(z-w)^2} + \frac{g}{2} f^a_{bc}\Big(\frac{\bar z - \bar w}{(z-w)^2}\bar a^c + \frac{\bar z - \bar w}{z-w}\partial \bar a^c + \frac{1}{2}\frac{(\bar z-\bar w)^2}{(z-w)^2}\bar\partial \bar a^c\Big) +$$
$$+ \frac{g^2}{4} f^a_{ce} f^e_{db}\Big(- \frac{\bar z - \bar w}{z-w} a^c \bar a^d + \frac{1}{2}\frac{(\bar z-\bar w)^2}{(z-w)^2} \bar a^c \bar a^d\Big) + \mathrm{reg}$$

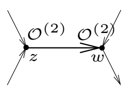
$$\partial c^a(z)\partial b_b(w) \sim \tag{5.81}$$
$$\sim \frac{\delta^a_b}{(z-w)^2} + \frac{g}{2} f^a_{bc}\Big(- \frac{1}{z-w} a^c - \frac{\bar z - \bar w}{z-w}\bar\partial a^c + \frac{\bar z - \bar w}{(z-w)^2}\bar a^c + \frac{1}{2}\frac{(\bar z-\bar w)^2}{(z-w)^2}\bar\partial \bar a^c\Big) +$$
$$+ \frac{g^2}{4} f^a_{ce} f^e_{db}\Big(-2\log|z-w| a^c a^d - \frac{\bar z - \bar w}{z-w} a^c \bar a^d - \frac{\bar z - \bar w}{z-w} \bar a^c a^d + \frac{1}{2}\frac{(\bar z-\bar w)^2}{(z-w)^2} \bar a^c \bar a^d\Big) + \mathrm{reg}$$

$$\partial c^a(z)\gamma_b(w) \sim \tag{5.82}$$
$$\sim -\frac{g}{2} f^a_{bc}\Big(\frac{1}{z-w} c^c + \frac{\bar z - \bar w}{z-w}\bar\partial c^c\Big) - \frac{g^2}{4} f^a_{ce} f^e_{db}\Big(2\log|z-w| a^c c^d + \frac{\bar z - \bar w}{z-w}\bar a^c c^d\Big) + \mathrm{reg}$$

Here all the fields on the r.h.s. are at w. These particular OPEs will be important when studying the stress-energy tensor T and BRST current J as composite fields in Section 5.4.

5.3.3. Some important OPEs in abelian theory involving $\mathcal{O}^{(2)}$. Limit $g = 0$ of an OPE between two composite fields Φ_1, Φ_2 is given by a sum of Wick contractions of some of the constituent fundamental fields of Φ_1 with some of the constituent fundamental fields of Φ_2, i.e., by Feynman graphs with two vertices corresponding to Φ_1 and Φ_2, with no interaction vertices and with loose half-edges allowed. Short loops are not allowed (which corresponds to the assumption that Φ_1, Φ_2 are renormalized/normally ordered[15]). Such OPEs in abelian BF theory were studied in [7]. Here, for the study of non-abelian theory as a deformation of the abelian one, we are interested in several OPEs involving the deforming observable $\mathcal{O}^{(2)}$.

As an example, consider the OPE $\mathcal{O}^{(2)}(z)\,\mathcal{O}^{(2)}(w)$ in the free theory. We have the following diagrams:



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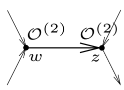

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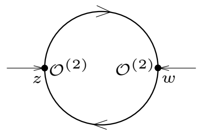


[15] An implicit assumption here is that the order in which fundamental fields (or their derivatives) are merged when building Φ_1, Φ_2 is such that $g = 0$ limit coincides with the usual normal ordering prescription in a free theory; one can always choose such an order, see Section 5.4.1.

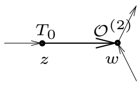
The singular contribution of the first one, taking into account all possible decorations of half-edges is:

$$\Big(\frac{1}{2}\langle B,[A,[A,A]]\rangle + \Big\langle -*db,[A,[A,c]]+\frac{1}{2}[c,[A,A]]\Big\rangle\Big)\, 2\, d\arg(z-w) = 0$$

– vanishes due to Jacobi identity in $\mathfrak{g}$. The second diagram is similar. Contribution of the third diagram vanishes by boson-fermion cancellation in a loop mechanism. Thus, the free theory OPE is trivial:

$$\mathcal{O}^{(2)}(z)\,\mathcal{O}^{(2)}(w) \underset{g=0}{\sim} \text{reg} \tag{5.83}$$

Next, consider the free theory OPE of the (abelian) stress-energy tensor $T_0 = \langle a, \partial\gamma\rangle + \langle \partial b, \partial c\rangle$ with $\mathcal{O}^{(2)}$. We have the following contributing diagrams:



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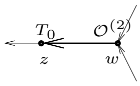

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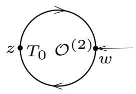


Here the last diagram gives

$$-4\, d^2w\, \frac{f^a_{ab}\bar{a}^b(w)}{(z-w)^3}$$

– a potential third order pole contribution to the OPE, which vanishes due to unimodularity. Thus, the OPE is given by the first two graphs, which yield

$$T_0(z)\,\mathcal{O}^{(2)}(w) \underset{g=0}{\sim} \frac{\mathcal{O}^{(2)}(w)}{(z-w)^2} + \frac{\partial\mathcal{O}^{(2)}(w)}{z-w} + \text{reg} \tag{5.84}$$

Together with the complex conjugate OPE, this implies that $\mathcal{O}^{(2)}$ is a primary field of conformal dimension $(1,1)$ in the abelian theory.[16]

As another example, consider the OPE of $\mathcal{O}^{(2)}$ with the "soaking field" $\widetilde{\Theta} = \delta(\gamma)\delta(\bar{\gamma})\delta(b)$ which appeared in Section 5.2.4. We have:

$$\begin{aligned}\mathcal{O}^{(2)}(z)\widetilde{\Theta}(w) \underset{g=0}{\sim} & -2d^2z\, f^a_{bc}\Big(\frac{(\gamma_a-\bar{\gamma}_a)(z)}{|z-w|^2}\,\big(\frac{\partial^2}{\partial\gamma_b\partial\bar{\gamma}_c}\widetilde{\Theta}\big)(w)+ \\ & + 2\frac{\log|z-w|}{z-w}\bar{\partial}b_a(z)\,\big(\frac{\partial^2}{\partial\gamma_b\partial b_c}\widetilde{\Theta}\big)(w) + c.c.\Big) + O\big(\frac{1}{|z-w|}\big)\end{aligned} \tag{5.85}$$

Here $c.c.$ stands for the complex conjugate of the second term. Note that the first term contains $\frac{1}{|z-w|^2}$ (coming from Wick contractions of $a, \bar{a}$ from $\mathcal{O}^{(2)}$ with $\gamma,\bar{\gamma}$ from $\widetilde{\Theta}$) times a sum of two expressions vanishing as $(z-w)$ and as $(\bar{z}-\bar{w})$ respectively – these zeroes arise from $\gamma(z)\delta(\gamma(w))$ and $\bar{\gamma}(z)\delta(\bar{\gamma}(w))$. Therefore the worst singularities in this OPE are in fact $\frac{\log|z-w|}{z-w}$ and $\frac{\log|z-w|}{\bar{z}-\bar{w}}$ coming from the second term and its complex conjugate – these terms arise from the pair of Wick contractions of a, c from $\mathcal{O}^{(2)}$ with γ, b from $\widetilde{\Theta}$ and the conjugate situation. In particular, this OPE has an integrable singularity in z at $z = w$.

[16] If z and w are allowed to collide, one must include an additional contact term in the OPE (5.84), see (5.129) below.

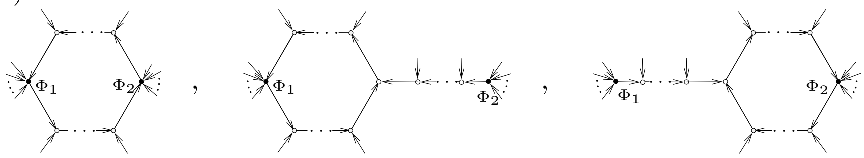
By a similar argument, the OPE of $\mathcal{O}^{(2)}$ with the second "soaking field" $\delta(c)$ behaves as $O(1)$:

$$\mathcal{O}^{(2)}(z)\delta(c(w)) \underset{g=0}{\sim} -2d^2 z f^a_{bc}\Big(\frac{\bar z-\bar w}{z-w}\,\big(\bar a^b\bar\partial c^c\frac{\partial}{\partial c^a}\delta(c)\big)(w)+c.c.\Big)+\text{reg} \tag{5.86}$$

5.3.4. A remark on OPEs of composite fields. Consider the OPE $\Phi_1(z)\Phi_2(w)$ for Φ_1,Φ_2 two composite fields. It is given, according to the general principle, as a sum of Feynman graphs $\boldsymbol{\gamma}$ with leaves satisfying properties (i), (ii) above. Part of the contributions come from branch graphs connecting one constituent fundamental field (or derivative of a fundamental field) ϕ_1 from Φ_1 and one (derivative of) fundamental field ϕ_2 from Φ_2 (one can think of such a contribution as a "dressed Wick contraction" of $\phi_1(z)$ and $\phi_2(z)$). Let us call the sum of these diagrams the "tree part" of the OPE, $[\Phi_1(z)\Phi_2(w)]_{\text{tree}}$ – it is readily calculated from OPEs of (derivatives of) fundamental fields.

Generally, in addition to tree diagrams there are loop diagrams with $l\geq 1$ loops. Let us focus on the case when Φ_1 and Φ_2 are at most linear in fields $b,\gamma,\bar\gamma$ or their derivatives (fields of $\mathcal{AB}$-charge $+1$). This case is of particular relevance, since several important composite fields in the theory – G, T, J, $\mathcal{O}^{(2)}$ – have this property. Under this assumption, OPE $\Phi_1(z)\Phi_2(w)$ cannot contain diagrams with ≥ 2 loops but can contain 1-loop diagrams of form

(5.87)



Note that diagrams where all vertices in the loop are $\mathcal{O}^{(2)}$ cancel out by Lemma 5.2.1.

In case when Φ_1 is linear in fields $b,\gamma,\bar\gamma$ or derivatives while Φ_2 does not contain them, we have

$$\Phi_1(z)\Phi_2(w)=[\Phi_1(z)\Phi_2(w)]_{\text{tree}} \tag{5.88}$$

5.4. Composite fields

5.4.1. Building composite fields via renormalized products. Order-of-merging ambiguity. Given two fields Φ_1,Φ_2, we define their renormalized product by the prescription (5.28):

(5.89)

$$(\Phi_1\Phi_2)(z)=\lim_{z'\to z}\Big(\Phi_1(z')\Phi_2(z)-\big[\Phi_1(z')\Phi_2(z)\big]_{\text{sing}}\Big)=\widetilde{\lim_{z'\to z}}\Phi_1(z')\Phi_2(z)$$

where we introduced the notation $\widetilde{\lim}$ meaning "subtract the singularity, then take the limit."

Generally, fields Φ_1, Φ_2 have an OPE of the form

$$\Phi_1(z')\Phi_2(z)\sim\sum_{p,q,r}\sigma_{pqr}(z'-z)\,\widetilde{\Phi}_{pqr}(z)+\text{reg} \tag{5.90}$$

with

$$\sigma_{pqr}(z'-z) = (z'-z)^{-p}(\bar{z}'-\bar{z})^{-q}\log^r|z'-z| \tag{5.91}$$

where the sum is over p,q,r with $p+q \geq 0, r \geq 0$ and $(p,q,r) \neq (0,0,0)$. The singular subtraction $\left[\Phi_1(z')\Phi_2(z)\right]_{\text{sing}}$ in (5.89) is defined uniquely as the r.h.s. of (5.90) without "reg" term.

REMARK 5.4.1. Note that the singular subtraction in (5.89) is defined with respect to a local coordinate z, using the explicit basis (5.91). In other words, renormalized product is not a diffeomorphism-invariant operation. In Section 5.6 we will see how this coordinate-dependence of the subtraction may lead to a nontrivial scaling behavior of composite fields.

If instead of having Φ_1 approach Φ_2 in (5.89), we do the opposite and make Φ_2 approach Φ_1, we can get a different finite part! For instance, if

$$\Phi_1(z')\Phi_2(z) \sim \frac{\widetilde{\Phi}(z)}{z'-z} + \Psi(z) + o(1)_{z'\to z} = \frac{\widetilde{\Phi}(z')}{z'-z} - \partial\widetilde{\Phi}(z') + \Psi(z') + o(1)_{z\to z'}$$

Then merging Φ_1 with Φ_2 yields Ψ while merging Φ_2 with Φ_1 yields a different field $\Psi - \partial\widetilde{\Phi}$. That is, we have an *order-of-merging ambiguity*

$$\widetilde{\lim_{z'\to z}}\Phi_1(z')\Phi_2(z) - \widetilde{\lim_{z'\to z}}\Phi_1(z)\Phi_2(z') = \partial\widetilde{\Phi}(z) \tag{5.92}$$

given by the derivative of the residue in the OPE between the constituent fields Φ_1 and Φ_2.

As an explicit example of this phenomenon, already in free (abelian) theory, at $g = 0$, we have

$$\widetilde{\lim_{z'\to z}} a^a(z')(\gamma_b\gamma_c)(z) - \widetilde{\lim_{z'\to z}} a^a(z)(\gamma_b\gamma_c)(z') = \delta^a_b\partial\gamma_c(z) + \delta^a_c\partial\gamma_b(z) \tag{5.93}$$

Note that the first term on the left corresponds to the standard normal ordering prescription in free theory – the field $:a^a\gamma_b\gamma_c:$ – e.g., its correlator with a test field $a^d(x)$ is vanishing, while it is nonvanishing for the second term on the l.h.s. Note that in non-abelian theory, for $g \neq 0$, the result (5.93) still holds: although the OPE $a^a(z')(\gamma_b\gamma_c)(z)$ acquires an additional term $O(\frac{\bar{z}'-\bar{z}}{z'-z})$, it does not contribute to the ambiguity.

As another example, we have

$$\widetilde{\lim_{z'\to z}}\partial a^a(z')\bar{\gamma}_b(z) - \widetilde{\lim_{z'\to z}}\partial a^a(z)\bar{\gamma}_b(z') = -\frac{g}{2}f^a_{bc}\partial a^c(z) \tag{5.94}$$

One has the following generalization of (5.92) for a general pair Φ_1, Φ_2, obtained by the same logic.

LEMMA 5.4.2. *For Φ_1, Φ_2 any pair of composite fields with OPE given by (5.90), the order-of-merging ambiguity in the product $\Phi_1\Phi_2$ is:*

$$\widetilde{\lim_{z'\to z}}\Phi_1(z')\Phi_2(z) - \widetilde{\lim_{z'\to z}}\Phi_1(z)\Phi_2(z') = \sum_{p,q\geq 0,\,(p,q)\neq(0,0)} \frac{(-1)^{p+q-1}}{p!q!}\partial^p\bar{\partial}^q\widetilde{\Phi}_{pq0}(z) \tag{5.95}$$

Note that terms in the OPE involving logarithms or involving positive powers, like $\frac{\bar{z}'-\bar{z}}{z'-z}$, do not contribute to the ambiguity.

Open question. Does the *pre-Lie algebra* identity hold

$$\Phi_1 *_R (\Phi_2 *_R \Phi_3) - (-1)^{|\Phi_1|\cdot|\Phi_2|}\Phi_2 *_R (\Phi_1 *_R \Phi_3) \stackrel{?}{=} (\Phi_1 *_R \Phi_2 - (-1)^{|\Phi_1|\cdot|\Phi_2|}\Phi_2 *_R \Phi_1) *_R \Phi_3 \tag{5.96}$$

for any triple Φ_1, Φ_2, Φ_3? Here we denoted $(\phi *_R \psi)(z) = \widetilde{\lim_{z'\to z}} \phi(z')\psi(z)$ the renormalized product merging the left factor onto the right factor; $|\phi|$ is the ghost number of the field ϕ. Note that the field 1 serves as left- and right-unit for the product $*_R$. Identity (5.96) holds in any *chiral* CFT, see Appendix 6.C in [4]; of course, our case of non-abelian BF is non-chiral and we cannot use that result.

The following is a special case of (5.96) which easy to prove independently; we will need it for our analysis of conservation laws under the correlator in Section 5.5.2.

LEMMA 5.4.3. Let $\Phi_1, \ldots, \Phi_n$ be a collection of fundamental fields (or their derivatives) of $\mathcal{AB}$-charge -1 and Ψ a fundamental field (or derivative) of $\mathcal{AB}$-charge $+1$. Then

$$\widetilde{\lim_{z_1\to z}} \cdots \widetilde{\lim_{z_n\to z}} \Phi_1(z_1)\cdots\Phi_n(z_n)\Psi(z) = \widetilde{\lim_{z'\to z}} \big(\Phi_1\cdots\Phi_n\big)(z')\Psi(z) \tag{5.97}$$

In particular, the resulting composite field is independent of the order in which one merges fields Φ_i onto Ψ. Field $(\Phi_1\cdots\Phi_n)$ appearing in the r.h.s. is independent of the order of merging, since fields Φ_i have regular OPE with each other.

PROOF. We give a proof for the case $n = 2$; the case of general n is similar. Consider the correlator

$$F(z_1, z_2, z; x_1, \ldots, x_m) = \langle \Phi_1(z_1)\Phi_2(z_2)\Psi(z)\phi(x_1)\ldots\phi(x_m)\rangle$$

with $\{\phi_i\}$ an arbitrary collection of test fields. The correlator is a sum of

(1) diagrams where Φ_1 and Ψ belong to the same tree and Φ_2 belongs to another tree,
(2) diagrams where Φ_2 and Ψ belong to the same tree and Φ_1 belongs to another one,
(3) diagrams where Φ_1, Φ_2 and Ψ belong to 3 different trees.

Thus, the correlator has the following structure:

$$F(z_1, z_2, z) = \sum_k G_k(z_1, z)H_k(z_2) + \sum_l \widetilde{G}_l(z_2, z)\widetilde{H}_l(z_1) + K(z_1, z_2, z) \tag{5.98}$$

where K has no singularities when any pair among z_1, z_2, z collides; we are suppressing the dependence on $x_1, \ldots, x_m$ in the notation. Merging first z_2 onto z and then z_1 onto z, we obtain

$$\widetilde{\lim_{z_1\to z}}\,\widetilde{\lim_{z_2\to z}} F(z_1, z_2, z) = \sum_k \widetilde{\lim_{z_1\to z}} G_k(z_1, z)\, H_k(z) + \sum_l \widetilde{\lim_{z_2\to z}} \widetilde{G}_l(z_2, z)\, \widetilde{H}_l(z) + K(z, z, z)$$

Setting $z_1 = z_2 = z'$ in (5.98) and then evaluating $\widetilde{\lim_{z'\to z}}$, we obtain the same result. Thus, we checked (5.97) for $n = 2$ by probing both sides by a correlator with a collection of test fields. □

The derivative of a renormalized product is defined in the natural way:

$$\partial\left(\widetilde{\lim_{z'\to z}}\Phi_1(z')\Phi_2(z)\right)=\widetilde{\lim_{z'\to z}}\Big(\partial\Phi_1(z')\Phi_2(z)+\Phi_1(z')\partial\Phi_2(z)\Big) \tag{5.99}$$

and similarly for $\bar\partial$ of a product. Here it is crucial that the terms on the right, arising from Leibnitz rule, respect the order of merging in the product $\Phi_1\Phi_2$ we take the derivative of. The following property is immediate from this definition.

LEMMA 5.4.4. Given two fields Φ_1, Φ_2, we have

$$\begin{aligned}&\widetilde{\lim_{z'\to z}}\big(\partial\Phi_1(z')\Phi_2(z)+\Phi_1(z')\partial\Phi_2(z)\big)-\widetilde{\lim_{z'\to z}}\big(\partial\Phi_1(z)\Phi_2(z')+\Phi_1(z)\partial\Phi_2(z')\big)\\ &\qquad=\partial\left(\widetilde{\lim_{z'\to z}}\Phi_1(z')\Phi_2(z)-\widetilde{\lim_{z'\to z}}\Phi_1(z)\Phi_2(z')\right)\end{aligned} \tag{5.100}$$

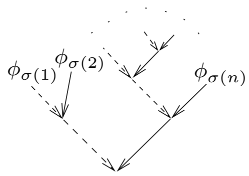
i.e., the ambiguity in the derivative $\partial(\Phi_1\Phi_2)=\partial\Phi_1\,\Phi_2+\Phi_1\,\partial\Phi_2$ is the derivative of the ambiguity of the product $\Phi_1\Phi_2$. The same holds if we replace ∂ with $\bar\partial$.

In summary, we have the following.

- A composite field built as a renormalized product of several fundamental fields (or their derivatives)

$$(\phi_1\cdots\phi_n)_\mu$$

  must be decorated with order-of-merging data μ, prescribing in which fields merge onto which and in what order. Generally, such data can be given by a planar binary rooted tree with n leaves decorated by some permutation σ of $\phi_1,\ldots,\phi_n$, where at each vertex the left incoming field merges onto the right one (as a possible convention).



  Here a solid incoming edge at a vertex represents the field onto which the merging occurs.
- As a special case of order-of-merging data, one may pick one of ϕ_k's as a "base" and consecutively merge other fields onto it. The limit $g=0$ of such a renormalized product coincides with the normally ordered product $:\phi_1\cdots\phi_n:$ of the free theory (and in particular is independent of the order in which fields are merged onto the "base"; at $g\neq 0$ the result can depend on the order).
- There are many examples of composite fields which turn out to be independent of the order of merging. For instance:
  - The product of any two fields from the list $\{a,\bar a,\gamma,\bar\gamma,c,b,\partial b,\bar\partial b\}$. (However, taking further derivatives can create a dependence on the order, as in (5.94)).

– Fields $\mathcal{O}^{(2)}$, J, G, T and complex conjugates, see Proposition 5.4.5 below.
– Expressions vanishing by equations of motion – left hand sides of (5.9).
– If a field Φ is independent of the order of merging, then any derivative $\partial^p\bar{\partial}^q\Phi$ is independent too, by Lemma 5.4.4, *as long as the order of merging is the same in all terms of $\partial^p\bar{\partial}^q\Phi$ produced by Leibnitz rule.*[17]

5.4.2. G, T, J as composite fields. When we consider fields G, T, J as composite fields, the corresponding singular subtractions miraculously vanish.

Indeed, consider the regularization of the stress-energy tensor by splitting the constituent fields:

$$T^{\mathrm{split}}(z',z) = \partial\gamma_a(z')a^a(z) + \partial b_a(z')\partial c^a(z) + \frac{g}{2}f^a_{bc}\partial b_a(z')(a^b c^c)(z) \tag{5.101}$$

Note that, since the OPE between a and c is regular, we can put them in the same point. The singular part of (5.101) at $z' \to z$, as calculated using the OPEs (5.80), (5.81), is:

$$\begin{aligned}\left[T^{\mathrm{split}}(z',z)\right]_{\mathrm{sing}} &= \frac{\dim\mathfrak{g}}{(z'-z)^2} + \frac{g^2}{4}K_{ab}\Big(-\frac{\bar{z}'-\bar{z}}{z'-z}a^a\bar{a}^b + \frac{1}{2}\frac{(\bar{z}'-\bar{z})^2}{(z'-z)^2}\bar{a}^a\bar{a}^b\Big) - \\ &- \frac{\dim\mathfrak{g}}{(z'-z)^2} + \frac{g^2}{4}K_{ab}\Big(2\log|z'-z|a^a a^b + 2\frac{\bar{z}'-\bar{z}}{z'-z}a^a\bar{a}^b - \frac{1}{2}\frac{(\bar{z}'-\bar{z})^2}{(z'-z)^2}\bar{a}^a\bar{a}^b\Big) - \\ &- \frac{g^2}{4}K_{ab}\Big(\frac{\bar{z}'-\bar{z}}{z'-z}a^a\bar{a}^b + 2\log|z'-z|a^a a^b\Big) = 0\end{aligned} \tag{5.102}$$

Here $K_{ab} = f^c_{ad}f^d_{bc}$ is the matrix of the Killing form; all $O(g)$ terms vanish by unimodularity. All fields on the right are at z. Thus, the *total singular subtraction in (5.101) vanishes* and the renormalized stress-energy tensor is simply

$$T(z) = \lim_{z'\to z} T^{\mathrm{split}}(z',z)$$

Likewise, we regularize J as

$$J^{\mathrm{split}}(z',z) = \gamma_a(z')\partial c^a(z) + g f^a_{bc}\gamma_a(z')(a^b c^c)(z) - \frac{g}{4}f^a_{bc}\partial b_a(z')(c^b c^c)(z) \tag{5.103}$$

Here the singular subtraction is calculated using (5.82):

$$\begin{aligned}\left[J^{\mathrm{split}}(z',z)\right]_{\mathrm{sing}} &= -\frac{g^2}{4}K_{ab}\Big(2\log|z'-z|a^a c^b + \frac{\bar{z}'-\bar{z}}{z'-z}\bar{a}^a c^b\Big) + \\ &+ \frac{g^2}{4}K_{ab}\Big(2\frac{\bar{z}'-\bar{z}}{z'-z}\bar{a}^a c^b + 4\log|z'-z|a^a c^b\Big) - \frac{g^2}{4}K_{ab}\Big(2\log|z'-z|a^a c^b + \frac{\bar{z}'-\bar{z}}{z'-z}\bar{a}^a c^b\Big) \\ &= 0\end{aligned} \tag{5.104}$$

[17] To illustrate the importance of the last condition, consider the derivative $\partial(a^a\bar{\gamma}_b)$ of an ordering-independent field $a^a\bar{\gamma}_b$. If we choose an inconsistent order of merging between the two terms, $\widetilde{\lim_{z'\to z}}\big(\partial a^a(z')\bar{\gamma}_b(z) - a^a(z)\partial\bar{\gamma}_b(z')\big)$, then it differs by a defect $-\frac{g}{2}f^a_{bc}\partial a^c$ from the consistent ordering and by twice that defect from the opposite inconsistent one.

The total singular subtraction vanishes again and thus the renormalized J field is just

$$J(z) = \lim_{z'\to z} J^{\mathrm{split}}(z',z)$$

The case of the field

$$G(z) = a^a(z)\partial b_a(z)$$

is trivial: the OPE between a and ∂b is regular, so we can safely put the fields at the same point. I.e., again we have a vanishing singular subtraction, but in the case of T, J the vanishing was a nontrivial cancellation between subtractions for different terms in the composite field, while for G it vanishes on the nose.

Furthermore, consider the field $\mathcal{O}^{(2)}$. We regularize it as

$$\mathcal{O}^{(2)\mathrm{split}}(z',z) = -2d^2z f^a_{bc}\Big((\gamma-\bar\gamma)_a(z')(a^b\bar a^c)(z)+\partial b_a(z')(\bar a^b c^c)(z)+\bar\partial b_a(z')(a^b c^c)(z)\Big)$$

One finds the singular subtraction to be

$$\begin{aligned}\big[\mathcal{O}^{(2)\mathrm{split}}(z',z)\big]_{\mathrm{sing}} &= -gd^2z\,K_{ab}\Big(\big(\frac{z'-z}{\bar z'-\bar z}a^a a^b+4\log|z'-z|a^a\bar a^b+\frac{\bar z'-\bar z}{z'-z}\bar a^a\bar a^b\big)-\\ &-\big(2\log|z'-z|a^a\bar a^b+\frac{\bar z'-\bar z}{z'-z}\bar a^a\bar a^b\big)-\big(\frac{z'-z}{\bar z'-\bar z}a^a a^b+2\log|z'-z|a^a\bar a^b\big)\Big) \qquad = 0\end{aligned}$$

Finally, consider the equations of motion – left hand sides in (5.9) – as composite fields. They all have zero singular subtractions on the nose except for the field $\bar\partial\gamma+\cdots$ and its complex conjugate. In this case, we have

$$\begin{aligned}\Big[\bar\partial\gamma_a(z') - \frac{g}{2}f^b_{ca}(\gamma_b-\bar\gamma_b)(z')\bar a^c(z) - \frac{g}{2}f^b_{ca}\bar\partial b_b(z')c^c(z)\Big]_{\mathrm{sing}} &= \\ = \frac{g^2}{4}K_{ab}\Big(\frac{z'-z}{\bar z'-\bar z}a^b+2\log|z'-z|\bar a^b\Big)-\frac{g^2}{4}K_{ab}\Big(\frac{z'-z}{\bar z'-\bar z}a^b+2\log|z'-z|\bar a^b\Big) &= 0\end{aligned} \tag{5.105}$$

– and again we have a cancellation for the singular subtraction. Thus, left hand sides in (5.9) all have zero singular subtractions as composite fields.

Note that in all the cases we considered here we did not encounter terms of form $\frac{\widetilde\Phi}{(z'-z)^p(\bar z'-\bar z)^q}$ among the terms in the singular subtractions, with $p,q\geq 0$ and $\widetilde\Phi$ a non-constant field. This implies that all these composite fields are independent of the order of merging.

In summary, we have proved the following.

Proposition 5.4.5. Fields G, T, J (and their complex conjugates) viewed as composite fields have the following properties:

(a) They are independent of the order of merging of the constituent fundamental fields.
(b) The total singular subtraction vanishes.

The same applies to $\mathcal{O}^{(2)}$ and to equations of motion – left hand sides of (5.9).

5.4.3. Examples of correlators and OPEs of composite fields. As a first example, consider the 2-point correlation function

$$\langle (a^a \bar{\gamma}_b)(z)\ \gamma_c(w)\rangle \tag{5.106}$$

The composite field $a^a\bar{\gamma}_b$ is defined by the prescription (5.28) – by placing the two constituent fundamental fields into distinct nearby points and subtracting the singular part of their OPE (5.68):

$$\begin{aligned}\big(a^a\bar{\gamma}_b\big)(z) &= \lim_{z'\to z}\Big(a^a(z')\bar{\gamma}_b(z) - \big[a^a(z')\bar{\gamma}_b(z)\big]_{\text{sing}}\Big)\\ &= \lim_{z'\to z}\Big(a^a(z')\bar{\gamma}_b(z) + g f^a_{bd}\log|z'-z|\, a^d(z)\Big)\end{aligned}\tag{5.107}$$

Thus, the correlator (5.106) is:

(5.108)

$$\begin{aligned}&\langle (a^a\bar{\gamma}_b)(z)\ \gamma_c(w)\rangle =\\ &= \lim_{z'\to z}\Big(\big\langle a^a(z')\bar{\gamma}_b(z)\gamma_c(w)\big\rangle + g f^a_{bd}\log|z'-z|\left\langle a^d(z)\gamma_c(w)\right\rangle\Big)\\ &= \lim_{z'\to z}\left(g f^a_{bc}\frac{1}{z'-w}\log\left|\frac{w-z}{z'-z}\right| + g f^a_{bc}\log|z'-z|\frac{1}{z-w}\right)\\ &= g f^a_{bc}\frac{\log|z-w|}{z-w}\end{aligned}$$

Here we used the result (5.41) for the 3-point function of fundamental fields.

Similarly, for the correlator $\langle (a^a\gamma_b)(z)\bar{\gamma}_c(w)\rangle$ we find

(5.109)

$$\begin{aligned}&\langle (a^a\gamma_b)(z)\ \bar{\gamma}_c(w)\rangle =\\ &= \lim_{z'\to z}\left\langle\left(a^a(z')\gamma_b(z) - \frac{\delta^a_b}{z'-z} - \frac{g}{2}f^a_{bd}\frac{\bar{z}'-\bar{z}}{z'-z}\bar{a}^d(z)\right)\ \bar{\gamma}_c(w)\right\rangle\\ &= \lim_{z'\to z}\left(g f^a_{bc}\frac{1}{z-z'}\log\left|\frac{z-w}{z'-w}\right| - \frac{g}{2}f^a_{bc}\frac{\bar{z}'-\bar{z}}{z'-z}\frac{1}{\bar{z}-\bar{w}}\right)\\ &= \frac{g}{2}f^a_{bc}\frac{1}{z-w}\end{aligned}$$

As the next example, consider the following correlator of two composite fields:

$$\langle (a^a\bar{\gamma}_b)(z)\,(\gamma_c\bar{\gamma}_d)(w)\rangle$$

We can obtain it from the 4-point function (5.48) by collapsing the first pair of points and the last pair of points (and subtracting the singularities). Collapsing a and $\bar{\gamma}$, we get the 3-point function

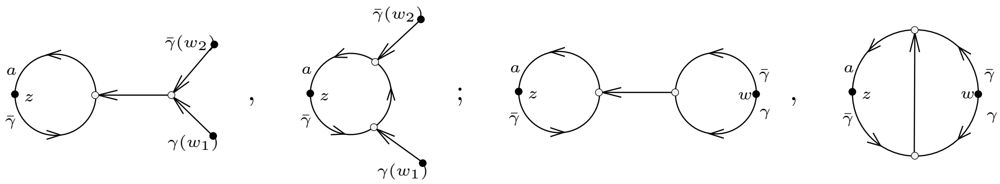
(5.110)

$$\begin{aligned}&\langle (a^a\bar{\gamma}_b)(z)\gamma_c(w_1)\bar{\gamma}_d(w_2)\rangle =\\ &= \lim_{z'\to z}\left\langle\big(a^a(z')\bar{\gamma}_b(z) + g f^a_{be}\log|z'-z|\,a^e(z)\big)\,\gamma_c(w_1)\bar{\gamma}_d(w_2)\right\rangle\\ &= \frac{g^2 f^a_{be}f^e_{cd}}{2(z-w_1)}\left(-i\mathsf{D}\left(\frac{w_1-w_2}{z-w_2}\right) - \log\left|\frac{w_1-w_2}{z-w_2}\right|\cdot\log|(z-w_1)(z-w_2)|\right)\\ &\quad + \frac{g^2 f^a_{de}f^e_{cb}}{2(z-w_1)}\left(i\mathsf{D}\left(\frac{w_1-w_2}{z-w_2}\right) - \log\left|\frac{w_1-w_2}{z-w_2}\right|\cdot\log\left|\frac{z-w_1}{z-w_2}\right|\right)\end{aligned}$$

Then, collapsing w_1 and w_2, we get

$$\begin{aligned}&\langle (a^a\bar\gamma_b)(z)\,(\gamma_c\bar\gamma_d)(w)\rangle = \\ (5.111)\quad &= \lim_{w'\to w}\Big\langle (a^a\bar\gamma_b)(z)\,\Big(\gamma_c(w')\bar\gamma_d(w) + g f^f_{cd}\log|w'-w|\,(\gamma-\bar\gamma)_f(w)\Big)\Big\rangle \\ &= g^2 f^a_{be}f^e_{cd}\frac{\log^2|z-w|}{z-w}\end{aligned}$$

Feynman diagrams corresponding to (5.110) are (5.111):

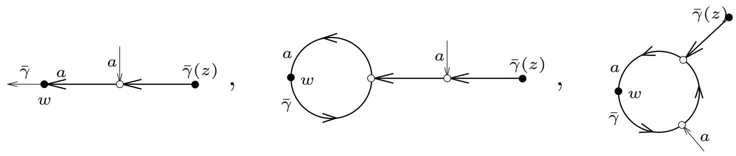


The last diagram here is, in fact, vanishing. Note that (5.111) corresponds to a *two-loop* diagram.

Next, consider the OPE

$$(a^a\bar\gamma_b)(w)\bar\gamma_c(z)$$

There are the following contributing Feynman diagrams



They give the following result:

$$(5.112)\quad (a^a\bar\gamma_b)(w)\bar\gamma_c(z) \sim$$
$$\sim -g f^a_{ce}\log|z-w|\,(a^e\bar\gamma_b)(z) - \frac{g^2}{2}(f^a_{be}f^e_{cf} - f^a_{ce}f^e_{bf})\log^2|w-z|\,a^f(z) + \mathrm{reg}$$

This OPE gives a singular subtraction needed to define the composite field with three constituent fundamental fields

$$(5.113)\qquad (a^a\bar\gamma_b\bar\gamma_c)(z) = \lim_{z'\to z}\Big((a^a\bar\gamma_b)(z')\bar\gamma_c(z) - \big[(a^a\bar\gamma_b)(z')\bar\gamma_c(z)\big]_{\mathrm{sing}}\Big)$$

Its correlator with γ_d is obtained by collapsing z with w_2 in (5.110):

$$(5.114)\qquad \langle (a^a\bar\gamma_b\bar\gamma_c)(z)\,\gamma_d(w)\rangle = \frac{g^2}{2}(f^a_{be}f^e_{cd} + f^a_{ce}f^e_{bd})\frac{\log^2|z-w|}{z-w}$$

5.4.4. Correlators involving the field $\bar\gamma\cdots\bar\gamma$. Here we give some examples of correlators containing an arbitrary power of log. These results will be the starting point for the construction of "vertex operators" – composite fields with a quantum correction to conformal dimension – in Section 5.6.

Lemma 5.4.6. The 3-point correlation function of the composite field $\bar\gamma\cdots\bar\gamma$ with a and γ is:

$$(5.115)\quad \langle a^a(w_1)(\bar\gamma_{b_1}\cdots\bar\gamma_{b_n})(z)\,\gamma_c(w_2)\rangle =$$
$$= \frac{g^n}{n!}\left(\sum_{\sigma\in S_n} f^a_{b_{\sigma(1)}e_1}f^{e_1}_{b_{\sigma(2)}e_2}\cdots f^{e_{n-1}}_{b_{\sigma(n)}c}\right)\frac{1}{w_1-w_2}\log^n\left|\frac{z-w_2}{z-w_1}\right|$$

where the sum on the right goes over permutations σ.

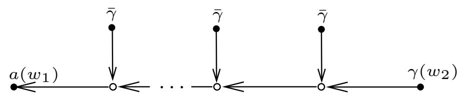
Proof. One proves this by first considering the correlator

$$\langle a^a(w_1)\bar{\gamma}_{b_1}(z_1)\cdots\bar{\gamma}_{b_n}(z_n)\gamma_c(w_2)\rangle = \tag{5.116}$$
$$= g^n\left(\sum_{\sigma\in S_n} f^a_{b_{\sigma(1)}e_1} f^{e_1}_{b_{\sigma(2)}e_2}\cdots f^{e_{n-1}}_{b_{\sigma(n)}c}\right)\mathbb{F}_n(w_1,z_1,\ldots,z_n,w_2)$$

where

(5.117)
$$\mathbb{F}_n(w_1,z_1,\ldots,z_n,w_2) = \int\frac{du_1}{2\pi}\cdots\frac{du_n}{2\pi}\frac{(-1)^n}{\prod_{k=1}^n(u_{k-1}-u_k)(\bar{u}_k-\bar{z}_k)\cdot(u_n-w_2)}$$

where we set $u_0 := w_1$. Here the contributing diagrams are:



where we need to sum over orders in which $\bar{\gamma}$'s are connected (hence the sum over $\sigma\in S_n$ above). Next, we set $z_1=\cdots=z_n=z$ (note that the integral is convergent in this limit – there are no singularities to be subtracted when merging z_i's):

$$\mathbb{F}_n^{\text{merged}}(w_1,z,w_2) = \mathbb{F}_n(w_1,z,\ldots,z,w_2)$$

We note that functions $\mathbb{F}_n^{\text{merged}}$ satisfy a recursion in n:

$$\mathbb{F}_n^{\text{merged}}(w_1,z,w_2) = -\int\frac{d^2u}{2\pi}\frac{\mathbb{F}_{n-1}^{\text{merged}}(w_1,z,u)}{(\bar{u}-\bar{z})(u-w_2)}$$

as follows from the form of the integrals (5.117). This allows us to check by induction in n that

$$\mathbb{F}_n^{\text{merged}}(w_1,z,w_2) = \frac{1}{n!}\frac{1}{w_1-w_2}\log^n\left|\frac{z-w_2}{z-w_1}\right| \tag{5.118}$$

□

Merging the field $\bar{\gamma}\cdots\bar{\gamma}$ with either a or γ in (5.115) and subtracting the singularity results in following the 2-point functions:

$$\langle(a^a\bar{\gamma}_{b_1}\cdots\bar{\gamma}_{b_n})(z)\,\gamma_c(w)\rangle = \tag{5.119}$$
$$= \frac{g^n}{n!}\left(\sum_{\sigma\in S_n} f^a_{b_{\sigma(1)}e_1} f^{e_1}_{b_{\sigma(2)}e_2}\cdots f^{e_{n-1}}_{b_{\sigma(n)}c}\right)\frac{\log^n|z-w|}{z-w},$$

$$\langle a^a(z)\,(\bar{\gamma}_{b_1}\cdots\bar{\gamma}_{b_n}\gamma_c)(w)\rangle = \tag{5.120}$$
$$= (-1)^n\frac{g^n}{n!}\left(\sum_{\sigma\in S_n} f^a_{b_{\sigma(1)}e_1} f^{e_1}_{b_{\sigma(2)}e_2}\cdots f^{e_{n-1}}_{b_{\sigma(n)}c}\right)\frac{\log^n|z-w|}{z-w}$$

Correlator (5.119) is a generalization of the results (5.108), (5.114).

5.5. Conformal and Q-invariance on the quantum level

5.5.1. Equations of motion under the correlator and contact terms. Consider the correlator

$$\left\langle \bar\partial a^a(z)\phi_1(x_1)\cdots\phi_n(x_n)\right\rangle$$

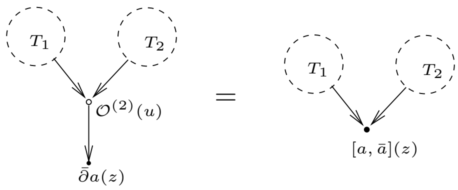
with $\phi_1,\ldots,\phi_n$ some test fields (assumed to be fundamental) inserted at points $x_1,\ldots,x_n$ distinct from z. Contributing Feynman graphs are binary trees with $\bar\partial a^a(z)$ at the root and $\phi_1(x_1),\ldots,\phi_n(x_n)$ decorating the leaves. The edge connecting the root with $\gamma_{a'}$ from the interaction vertex $\mathcal{O}^{(2)}(u)$ gets assigned

$$\bar\partial_z \frac{1}{z-u} = \pi\delta(z-u)$$

times the Kronecker symbol $\delta^a_{a'}$. This implies that

$$\begin{aligned}(5.121)\quad &\left\langle \bar\partial a^a(z)\phi_1(x_1)\cdots\phi_n(x_n)\right\rangle = \\ &= \sum_{N\geq 0}\frac{(-g/4\pi)^N}{(N-1)!}\left\langle \bar\partial \overbrace{a^a(z)\Big(\int_u \mathcal{O}^{(2)}}(u)\Big)\prod_{i=1}^{N-1}\Big(\int_{u_i}\mathcal{O}^{(2)}(u_i)\Big)\phi_1(x_1)\cdots\phi_n(x_n)\right\rangle_0 \\ &= \frac{g}{2}\left\langle [a,\bar a]^a(z)\ \phi_1(x_1)\cdots\phi_n(x_n)\right\rangle\end{aligned}$$

Graphically:



with T_1, T_2 arbitrary trees with leaves decorated by the test fields. Thus, graphically, integrating over u the delta-function arising in $\bar\partial$ of the propagator, results in chopping off the root of the tree. So, we obtained the identity

$$\left\langle \left(\bar\partial a - \frac{g}{2}[a,\bar a]\right)(z)\ \ \phi_1(x_1)\cdots\phi_n(x_n)\right\rangle = 0$$

Here the field in the brackets vanishes by classical equations of motion (5.9). Our result here is that it holds in the quantum world: correlators of this field with any collection of test fields vanish. This graphic argument for equations of motion under the correlator appeared in [1].

A point related to this calculation is that the free theory OPE

$$(5.122)\qquad \mathcal{O}^{(2)}(u)\,\bar\partial a^a(z) \underset{g=0}{\sim} -2\pi d^2u\ [a,\bar a]^a(u)\,\delta(u-z) + \mathrm{reg}$$

contains a *contact term*[18] singularity. Normally when considering OPEs we require the fields to be at non-coinciding points. However, non-abelian theory is constructed as abelian theory with arbitrarily many insertions of $\mathcal{O}^{(2)}$ which can hit other observables. Therefore, when talking about OPEs involving $\mathcal{O}^{(2)}$ we should allow it to hit the other field, and we should care about contact terms.

[18]By contact terms we generally mean terms containing delta-functions (or derivatives of delta-functions) in positions of fields

Generally, we say that a composite field Ξ is a *quantum equation of motion* if it vanishes under the correlator with an arbitrary collection of test fields inserted away from Ξ.

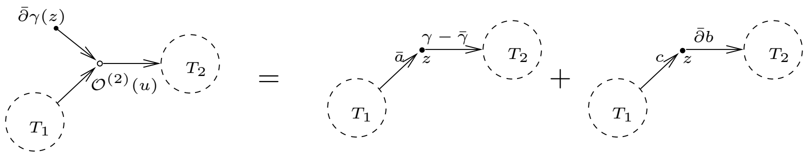
$$\langle \Xi(z)\ \phi_1(x_1)\cdots\phi_n(x_n)\rangle = 0$$

Thus, we just showed that $\Xi = \bar\partial a - \frac{g}{2}[a,\bar a]$ is a quantum equation of motion.

Similarly to (5.121), for the correlator of $\bar\partial\gamma_a(z)$ with test fields we find

(5.123)
$$\left\langle \bar\partial\gamma_a(z)\ \phi_1(x_1)\cdots\phi_n(x_n)\right\rangle = \frac{g}{2}\left\langle\left(-[\bar a,\gamma-\bar\gamma]+[c,\bar\partial b]\right)_a\ \phi_1(x_1)\cdots\phi_n(x_n)\right\rangle$$

Graphically:



Therefore, the classically vanishing expression

$$\bar\partial\gamma + \frac{g}{2}[\bar a,\gamma-\bar\gamma] - \frac{g}{2}[c,\bar\partial b]$$

vanishes under the correlator.

Likewise, we obtain

(5.124)
$$\left\langle \partial\bar\partial c^a(z)\ \phi_1(x_1)\cdots\phi_n(x_n)\right\rangle = -\frac{g}{2}\left\langle\left(\bar\partial[a,c]+\partial[\bar a,c]\right)^a\ \phi_1(x_1)\cdots\phi_n(x_n)\right\rangle$$

and

(5.125)
$$\left\langle \partial\bar\partial b_a(z)\ \phi_1(x_1)\cdots\phi_n(x_n)\right\rangle = -\frac{g}{2}\left\langle\left([a,\bar\partial b]+[\bar a,\partial b]\right)_a\ \phi_1(x_1)\cdots\phi_n(x_n)\right\rangle$$

Ultimately, we see that all the expressions (5.9) vanishing by classical equations of motion also vanish under the correlator. We further note that if a field $\Xi(z)$ vanishes under the correlator, then its product with any other field $\Phi(z)$ (the product is understood as renormalized in the sense of (5.28)) also vanishes under the correlator, since

$$\begin{aligned}\left\langle(\Phi\Xi)(z)\ \cdots\right\rangle &= \lim_{z'\to z}\left\langle\left(\Phi(z')\,\Xi(z) - [\Phi(z')\,\Xi(z)]_{\mathrm{sing}}\right)\ \cdots\right\rangle \\ &= \lim_{z'\to z}\Big(\underbrace{\langle\Phi(z')\,\Xi(z)\ \cdots\rangle}_{\mathrm{I}} - \underbrace{\langle[\Phi(z')\,\Xi(z)]_{\mathrm{sing}}\ \cdots\rangle}_{\mathrm{II}}\Big) = 0\end{aligned}$$

Here $\cdots$ are test fields inserted away from z. Term I vanishes as a correlator of $\Xi(z)$ with insertions away from z and II is, by definition of OPE, the singular part of I at $z'\to z$ and thus also vanishes. The same argument applies if choose the opposite order of merging in $\Phi\Xi$, i.e., if we merge Ξ onto Φ.

In summary, we have the following

Lemma 5.5.1. (a) Expressions (5.9) viewed as composite fields are quantum equations of motion.

(b) If Ξ is a quantum equation of motion, then any derivative $\partial^p\bar\partial^q\Xi$ is also a quantum equation of motion.[19]

(c) If Ξ is a quantum equation of motion and and Φ is any composite field, then the renormalized products $\widetilde{\lim_{z'\to z}}\Phi(z')\Xi(z)$, $\widetilde{\lim_{z'\to z}}\Phi(z)\Xi(z')$ are also quantum equations of motion.

Counterparts of the free theory OPE (5.122) corresponding to (5.123), (5.124), (5.125) are:

(5.126)
$$\begin{aligned}\mathcal{O}^{(2)}(u)\,\bar\partial\gamma_a(z) &\underset{g=0}{\sim} 2\pi d^2u\,\big([\bar a,\gamma-\bar\gamma]-[c,\bar\partial b]\big)_a(u)\,\delta(u-z)+\mathrm{reg}\\ \mathcal{O}^{(2)}(u)\,\partial\bar\partial c^a(z) &\underset{g=0}{\sim} 2\pi d^2u\,\big(\bar\partial[a,c]+\partial[\bar a,c]\big)^a(u)\,\delta(u-z)+\mathrm{reg}\\ \mathcal{O}^{(2)}(u)\,\partial\bar\partial b_a(z) &\underset{g=0}{\sim} 2\pi d^2u\,\big([a,\bar\partial b]+[\bar a,\partial b]\big)_a(u)\,\delta(u-z)+\mathrm{reg}\end{aligned}$$

Thus, for each $\xi\in\{\bar\partial a,\partial\bar a,\bar\partial\gamma,\partial\bar\gamma,\partial\bar\partial c,\partial\bar\partial b\}$ a derivative of a fundamental field vanishing by equations of motion in free theory, we have an OPE similar to (5.122), of form

$$\mathcal{O}^{(2)}(u)\,\xi(z)\underset{g=0}{\sim}4\pi d^2u\,\delta(u-z)\,r_\xi(u)+\mathrm{reg} \tag{5.127}$$

with r_ξ some composite field. Then the expression

$$\Xi=\xi+g\,r_\xi \tag{5.128}$$

vanishes under the correlator in the deformed theory. Thus, the deformation of equations of motion $\xi\to\Xi$ from abelian to non-abelian theory is given by the coefficient r_ξ of the contact term in the OPE of ξ with the deforming 2-observable $\mathcal{O}^{(2)}$.

Remark 5.5.2. The OPE $T_0(z)\mathcal{O}^{(2)}(u)$, see (5.84), in fact contains a contact term:

(5.129)
$$T_0(z)\mathcal{O}^{(2)}(u)\underset{g=0}{\sim}\frac{\mathcal{O}^{(2)}(u)}{(z-u)^2}+\frac{\partial\mathcal{O}^{(2)}(u)}{z-u}+4\pi\,d^2u\,\delta(z-u)\underbrace{\frac{1}{2}\,\langle\partial b,[a,c]\rangle}_{T_1}(u)+\mathrm{reg}$$

Observe that the composite field arising as the coefficient of the delta-function in the contact term is precisely T_1, the deformation of the stress-energy tensor induced by the non-abelian deformation of the theory, cf. (5.17), (5.25).

5.5.2. Quantum conservation laws: holomorphicity of G and T. Quantum BRST operator. Using Lemma 5.5.1, we can prove the following quantum counterpart of the classical conservation laws (5.13), (5.19).

Proposition 5.5.3. We have

$$\big\langle\bar\partial G(z)\cdots\big\rangle=0,\quad\big\langle\partial\bar G(z)\cdots\big\rangle=0, \tag{5.130}$$

$$\big\langle\bar\partial T(z)\cdots\big\rangle=0,\quad\big\langle\partial\bar T(z)\cdots\big\rangle=0, \tag{5.131}$$

$$\big\langle dJ^{\mathrm{tot}}(z)\cdots\big\rangle=0 \tag{5.132}$$

[19] Here we understand that the order of merging for the derivative is inferred from the order of merging for Ξ via (5.99).

with $\cdots$ any collection of test fields.

PROOF. We start by considering $\bar\partial T$. Classically, we have

$$\bar\partial T = \langle \partial(\underline{\bar\partial\gamma+\cdots}), a\rangle - \langle \underline{\partial\bar\gamma+\cdots}, \partial\bar a\rangle + \langle \underline{\partial\bar\partial b+\cdots}, \partial c\rangle + \langle \partial b, \underline{\partial\bar\partial c+\cdots}\rangle + \langle \partial\gamma, \underline{\bar\partial a+\cdots}\rangle + \langle \partial\bar\gamma, \underline{\partial\bar a+\cdots}\rangle \tag{5.133}$$

Here the underlined terms are the expressions (5.9) – the classical equations of motion. For brevity, we write explicitly only the top derivative term in each equation, thus $\bar\partial a+\cdots$ stands for $\bar\partial a - \frac{g}{2}[a,\bar a]$ and similarly for other equations.[20]

In the quantum setting, we split T, placing fields $\gamma,\bar\gamma,b$ or derivatives at z and fields $a,\bar a,c$ or derivatives at a point $z'\to z$. Then, taking the derivative, we will have the same splitting rule in (5.133). Then, using Lemma 5.4.3, we can equivalently re-assign fields $a,\bar a,c$ or derivatives in underlined terms in (5.133) to a point z'', so that we have $\widetilde{\lim_{z'\to z}}\bar\partial T^{\mathrm{split}}(z,z') = \widetilde{\lim_{z'\to z}}\,\widetilde{\lim_{z''\to z}}\bar\partial T^{\mathrm{split}}(z,z',z'')$. The latter expression vanishes under the correlator by Lemma 5.5.1.

One proves vanishing of $\bar\partial G$ and dJ^{tot} under the correlator by the same reasoning. In particular, one has

$$\bar\partial G = \langle \underline{\bar\partial a+\cdots}, \partial b\rangle + \langle a, \underline{\partial\bar\partial b+\cdots}\rangle$$

and[21]

$$dJ^{\mathrm{tot}} = 4d^2z\Big(\langle \gamma+\bar\gamma, \underline{\partial\bar\partial c+\cdots}\rangle + \langle \underline{\bar\partial\gamma+\cdots}, \partial c + g[a,c]\rangle + \langle \underline{\partial\bar\gamma+\cdots}, \bar\partial c + g[\bar a,c]\rangle - \frac{g}{2}\langle \underline{\partial\bar\partial b+\cdots}, [c,c]\rangle + \frac{g}{2}\langle [c,\gamma-\bar\gamma], \underline{\bar\partial a - \partial\bar a+\cdots}\rangle\Big) \tag{5.134}$$

□

5.5.2.1. *Quantum BRST operator.* We define the quantum BRST operator Q_q acting on a composite field $\Phi(z)$ as

$$Q_q : \Phi(z) \mapsto \frac{1}{4\pi}\oint_{C_z\ni w} J^{\mathrm{tot}}(w)\Phi(z) \tag{5.135}$$

– this is understood as an equality under a correlator with test fields. Here C_z is a simple closed contour going around z in positive direction and not enclosing any of the test fields. Note that the conservation law (5.132) implies that the result is independent under deformations of the contour.

It turns out that quantum BRST operator essentially coincides with the classical BRST operator. More precisely, taking care of the order-of-merging issue for the composite fields, one has the following.

PROPOSITION 5.5.4. Quantum BRST operator Q_q satisfies the following properties.

[20] Expansion (5.133) arises when we write the stress-energy as $T = \langle\partial\gamma, a\rangle + \langle\partial b,\partial c\rangle + \frac{g}{2}\langle\partial b,[a,c]\rangle$. If instead we use the classically equivalent expression (5.17), we should add to (5.133) the term $\bar\partial\langle\underline{\partial\bar\gamma+\cdots}, a\rangle$.

[21] We remark that formula (5.134) has the structure $dJ^{\mathrm{tot}} = \sum_i \pm\frac{\delta S}{\delta\phi_i}Q(\phi_i)$ where the sum runs over the species of fundamental fields $\phi_i\in\{a,\bar a,\gamma,\bar\gamma,b,c\}$. Similarly, (5.133) has the structure $\bar\partial T = \sum_i \pm\frac{\delta S}{\delta\phi_i}\partial\phi_i + \partial(\cdots)$.

(a) For Φ a fundamental field, quantum and classical BRST operators agree:

$$Q_q\Phi(z) = Q\Phi(z) \tag{5.136}$$

(b) Q_q commutes with derivatives:

$$Q_q(\partial\Phi) = \partial(Q_q\Phi) \quad , \quad Q_q(\bar\partial\Phi) = \bar\partial(Q_q\Phi)$$

for any composite field Φ.

(c) Q_q acts as an odd derivation on renormalized products:

$$Q_q \widetilde{\lim_{z'\to z}} \Phi_1(z')\Phi_2(z) = \widetilde{\lim_{z'\to z}} (Q_q\Phi_1)(z')\Phi_2(z) + \widetilde{\lim_{z'\to z}} (-1)^{|\Phi_1|}\Phi_1(z')(Q_q\Phi_2)(z) \tag{5.137}$$

for any composite fields Φ_1, Φ_2.

(d) For a general composite field $\Phi = (\phi_1\cdots\phi_n)_\mu$, with ϕ_i fundamental fields (or their derivatives) and μ the order-of-merging data (see Section 5.4.1), we have

$$Q_q\Phi = \sum_{i=1}^n \pm(\phi_1\cdots(Q\phi_i)\cdots\phi_n)_\mu \tag{5.138}$$

with $\pm = (-1)^{\sum_{j=1}^{i-1}|\phi_j|}$ the Koszul sign.

(e) Q_q squares to zero:

$$Q_q^2\Phi = 0 \tag{5.139}$$

for any composite field Φ.

Proof. Property (b) is immediate from the definition of Q_q (5.135), by applying a derivative to both sides.

Derivation property (c) is proven as follows. We have

$$\oint_{C_{12}\ni w} J^{\mathrm{tot}}(w)\Phi_1(z')\Phi_2(z) = \oint_{C_1} J^{\mathrm{tot}}(w)\Phi_1(z')\Phi_2(z) + \oint_{C_2} J^{\mathrm{tot}}(w)\Phi_1(z')\Phi_2(z) \tag{5.140}$$

under a correlator with test fields away from z, z'. Here C_{12} is a contour enclosing z and z', C_1 encloses only z' and C_2 encloses only z. The equality corresponds to splitting an integral over C_{12} into integrals over C_1 and C_2. Taking the limit $z' \to z$ while subtracting singular terms as $z' \to z$, yields the left and right sides of (5.137).[22]

Property (d) is an immediate consequence of properties (a), (b), (c). Property (e) follows from (d) by applying Q_q twice to a composite field $(\phi_1\cdots\phi_n)_\mu$ and using that the classical BRST operator Q squares to zero.

[22] More explicitly, let $\sum_i \sigma_i(z'-z)\widetilde{\Phi}_i(z)$ be the singular part of the OPE $\Phi_1(z')\Phi_2(z)$ with $\sigma_i(z'-z)$ the basis singular coefficient functions (5.91) and with $\widetilde{\Phi}_i$ some composite fields (only finitely many of them nonzero). Then the singular part of the l.h.s. of (5.140) is $\sum_i \sigma_i(z'-z)Q_q\widetilde{\Phi}_i(z)$. Two integrals on the r.h.s. of (5.140) have singular parts $\sum_i \sigma_i(z'-z)\widetilde{\Phi}_i^{(\alpha)}(z)$ with $\alpha = 1,2$ and $\widetilde{\Phi}_i^{(\alpha)}$ are some composite fields. Since l.h.s. and r.h.s. of (5.140) are equal under the correlator, the singular parts (and thus, coefficients of σ_i) must be equal: $Q_q\widetilde{\Phi}_i = \widetilde{\Phi}_i^{(1)} + \widetilde{\Phi}_i^{(2)}$. In particular, expressions which we need to subtract from l.h.s. and r.h.s. of (5.140) to obtain l.h.s. and r.h.s. of (5.137) are the same, which proves (5.137).

Lastly, consider property (a). For a fundamental field Φ, we prove (5.136) by a direct computation of the OPEs $J^{\mathrm{tot}}\Phi$. For instance, we compute

$$(5.141)\quad J^{\mathrm{tot}}(w)a^a(z)\sim$$
$$\sim \underbrace{-(\partial c+g[a,c])^a}_{Qa^a}\frac{2idw}{w-z} - 2ig[\bar\partial c+g[\bar a,c],a]^a d_w\Big((\bar w-\bar z)\log|w-z|\Big)+\mathrm{reg}$$

where fields on the r.h.s. are at z. The first term is a pole and gives a contribution Qa^a to the contour integral (5.135); the second term is a milder (logarithmic) singularity and vanishes under the contour integral (as do regular terms). Likewise, we find

$$J^{\mathrm{tot}}(w)c^a(z)\sim \frac{g}{4}[c,c]^a\Big(\frac{-2idw}{w-z}+\frac{2id\bar w}{\bar w-\bar z}\Big)+\cdots \tag{5.142}$$

$$J^{\mathrm{tot}}(w)b_a(z)\sim \gamma_a\frac{-2idw}{w-z}+\bar\gamma_a\frac{2id\bar w}{\bar w-\bar z}+\cdots \tag{5.143}$$

$$J^{\mathrm{tot}}(w)\gamma_a(z)\sim \frac{ig^2}{2}K_{ab}c^a(z)d_w\log^2|w-z|+ + \frac{g}{2}[c,\gamma]_a\frac{2idw}{w-z}+\frac{g}{2}[c,\bar\gamma]_a\frac{2id\bar w}{\bar w-\bar z}+\cdots \tag{5.144}$$

where $\cdots$ stands for milder singular (e.g. logarithmic and $O(\frac{\bar w-\bar z}{w-z})$) terms,[23] not contributing to the contour integral. In fact, (5.141) and (5.142) are computed easily from (5.88), using the results of Section 5.3.1. For (5.143) one could have 1-loop diagrams, but they vanish/are non-singular. OPE (5.144) is more complicated (see Remark 5.5.5 below for a shortcut to computing $Q_q\gamma$); the first term on the l.h.s. comes from 1-loop diagrams which contain potentially dangerous terms proportional to $g^2\frac{\log|w-z|}{w-z}K_{ab}c^a(z)$ (or the conjugate), with $K_{ab}=f^c_{ad}f^d_{bc}$ the matrix of the Killing form; these terms add up to a d_w-exact term when summed in (5.144). Ultimately, the first term on the l.h.s. of (5.144) vanishes under the contour integral over w and does not contribute to $Q_q\gamma_a$.

This finishes the proof of (5.136). □

Remark 5.5.5. The following trick allows one to simplify the computation of $Q_q\Phi$ for fundamental fields Φ, and in particular provides an alternative way to calculate $Q_q\gamma$, avoiding the direct computation of the OPE (5.144). Writing

$$Q_q\Phi=\sum_{k\geq 0}g^k\Phi_k \tag{5.145}$$

one can restrict the form of possible composite fields Φ_k appearing on the right by analyzing various degrees on the left and right side of (5.145) – the ghost number, $\mathcal{AB}$-charge, weight $(h,\bar h)$ and the number of constituent fundamental fields of the composite field Φ_k. We see that Φ_k must have the following properties

[23]Individually, a logarithmic or an $O(\frac{\bar w-\bar z}{w-z})$ term could contribute to an integral over a finite contour, but, due to (5.132), such terms always combine into d_w-closed expressions (cf. the second term in (5.141)), thus one can take the contour to be very small – in this limit, singular terms milder than a first order pole clearly vanish when integrated.

ghost degree	$\mathcal{AB}$-charge	weight $(h,\bar{h})$	# fund. fields
$\lvert\Phi\rvert+1$	$\mathcal{AB}(\Phi)-k$	$(h_\Phi,\bar{h}_\Phi)$	$k+1-2\#\text{loops}$

Here #loops is the number of loops in the Feynman diagram giving the contribution to OPE. These properties immediately imply that, writing the r.h.s. of $Q_q\phi$ schematically, up to numeric factors and indices, we have

$$Q_q a\approx \partial c+gac,\;\; Q_q c\approx gcc,\;\; Q_q b\approx \gamma+\bar{\gamma}+gbc+g\kappa,\;\; Q_q\gamma\approx g\gamma c+g^2 c$$

Here κ in $Q_q b$ is a component of constant vector in $\mathfrak{g}^*$; it must be zero due to global $\mathfrak{g}$-invariance (on the level of Feynman diagrams, it vanishes due to unimodularity of $\mathfrak{g}$). Furthermore, one can exclude the bc structure from $Q_q b$, since bare b ghost cannot appear on a leaf of a Feynman diagram for $J^{\text{tot}}b$ OPE. Finally, once $Q_q b$ is known, one can prove that $Q_q\gamma$ does not contain the 1-loop correction term g^2c, by probing it with the correlator with a test field $\partial b(x)$:

(5.146)
$$\langle (Q_q\gamma)(z)\;\partial b(x)\rangle=\langle\frac{1}{4\pi}\oint_{C_{zx}\ni w}J^{\text{tot}}(w)\gamma(z)\partial b(x)\rangle+\underbrace{\langle\gamma(z)\;Q_q(\partial b(x))\rangle}_{0}$$

Here on the left Q_q acts by a contour integral of J^{tot} around z – we present it on the right as an integral over a large contour C_{zx} encircling both z and x, minus a term with J^{tot} encircling only x. The second term is the correlator of already known $Q_q(\partial b)=\partial(\gamma+\bar{\gamma})$ with $\gamma(z)$ and vanishes trivially, since 2-point functions $\langle\gamma(x)\gamma(z)\rangle$, $\langle\bar{\gamma}(x)\gamma(z)\rangle$ are zero. If $Q_q\gamma$ would contain a g^2c term, the correlator (5.146) would behave as $O\big(\frac{1}{z-x}\big)$. Considering the asymptotics $z\to x$, we see that the r.h.s. does not behave this way, since the OPE $\gamma\partial b$ (5.72) does not – it behaves as $O\big(\frac{\bar{z}-\bar{x}}{z-x}\big)$. Hence, the coefficient of g^2c in $Q_q\gamma$ must be zero.

5.5.2.2. *OPEs of G with fundamental fields.* Since field G is holomorphic under the correlator, its OPE with any composite field Φ must have the form

$$G(w)\Phi(z)\sim\sum_{k=1}^{p}(w-z)^{-k}\Phi_k(z)+\text{reg}^{(\infty)} \tag{5.147}$$

with Φ_k some composite fields and some $p\geq 0$. For instance, such an OPE cannot contain terms like $\log|w-z|$ or $\frac{\bar{w}-\bar{z}}{w-z}$ which we have seen in other OPEs. The remainder in (5.147) is holomorphic at $w\to z$; in particular this OPE can be differentiated arbitrarily many times. Similarly, one has that the singular part $\bar{G}(w)\Phi(z)$ is a Laurent polynomial in $\bar{w}-\bar{z}$. Since the stress-energy tensor T is also holomorphic, same observation applies to $T(w)\Phi(z)$: one has

$$T(w)\Phi(z)\sim\sum_{k=1}^{q}(w-z)^{-k}\widetilde{\Phi}_k(z)+\text{reg}^{(\infty)} \tag{5.148}$$

and similarly for $\bar{T}(w)\Phi(z)$.

Another observation is that for Φ a fundamental field, the OPE $G(w)\Phi(z)$ does not have admissible decorations for 1-loop diagrams, and hence this

OPE satisfies (5.88). Explicitly, we obtain:

$$\begin{aligned} &G(w)a^a(z) \sim \mathrm{reg}^{(\infty)}, && G(w)\bar{a}^a(z) \sim \mathrm{reg}^{(\infty)}, \\ &G(w)\gamma_a(z) \sim \frac{\partial b_a(z)}{w-z} + \mathrm{reg}^{(\infty)}, && G(w)\bar{\gamma}_a(z) \sim \mathrm{reg}^{(\infty)}, \\ &G(w)c^a(z) \sim -\frac{a^a(z)}{w-z} + \mathrm{reg}^{(\infty)}, && G(w)b_a(z) \sim \mathrm{reg}^{(\infty)} \end{aligned} \tag{5.149}$$

By complex conjugation, one obtains OPEs of $\bar{G}$ with fundamental fields. Also, the OPE of G with itself is trivial:

$$G(w)G(z) \sim \mathrm{reg}^{(\infty)} \quad , \quad G(w)\bar{G}(z) \sim \mathrm{reg}^{(\infty)} \tag{5.150}$$

Computing the OPE between G and the BRST current, one gets

$$\begin{aligned} G(w)J(z) &\sim \frac{-\dim \mathfrak{g}}{(w-z)^3} + \frac{-\langle \gamma, a\rangle}{(w-z)^2} + \frac{T - \partial \langle \gamma, a\rangle}{w-z} + \mathrm{reg}^{(\infty)}, \\ G(w)\bar{J}(z) &\sim \frac{\bar{\partial}\langle \gamma, a\rangle}{w-z} + \pi\delta(w-z)\langle \bar{\gamma}, a\rangle + \mathrm{reg}^{(\infty)} \end{aligned} \tag{5.151}$$

All the fields on the right are at z. Here the cubic pole comes from a 1-loop diagram. Thus, for GJ^{tot}, one has

$$G(w)J^{\mathrm{tot}}(z) \sim -\frac{2iT dz}{w-z} + 2i d_z \left(\frac{\frac{1}{2}\dim \mathfrak{g}}{(w-z)^2} + \frac{\langle \gamma, a\rangle}{w-z} \right) + (\text{contact term}) + \mathrm{reg}^{(\infty)} \tag{5.152}$$

Integrating J^{tot} around G in (5.152), we find

$$Q_q G = T \tag{5.153}$$

– the quantum counterpart of (5.14).

5.5.2.3. *Quantum stress-energy tensor. Examples of primary fields. TT OPE.* Recall that a field Φ is called *primary*, of conformal dimension $(\Delta, \bar{\Delta})$, if its OPEs with $T, \bar{T}$ are of the form

$$\begin{aligned} T(w)\Phi(z) &\sim \frac{\Delta \Phi(z)}{(w-z)^2} + \frac{\partial \Phi(z)}{w-z} + \mathrm{reg}^{(\infty)}, \\ \bar{T}(w)\Phi(z) &\sim \frac{\bar{\Delta}\Phi(z)}{(\bar{w}-\bar{z})^2} + \frac{\partial \Phi(z)}{\bar{w}-\bar{z}} + \mathrm{reg}^{(\infty)} \end{aligned} \tag{5.154}$$

Proposition 5.5.6.

(a) Fundamental fields $a, \bar{a}, \gamma, \bar{\gamma}, b, c$ are all primary (each component of these fields), of conformal dimension $(1,0)$ for a, $(0,1)$ for $\bar{a}$ and $(0,0)$ for the rest.

(b) Fields $G, \bar{G}$ are primary, of conformal dimension $(2,0)$ and $(0,2)$, respectively.

(c) Stress-energy tensor satisfies the OPE

$$T(w)T(z) \sim \frac{2T(z)}{(w-z)^2} + \frac{\partial T(z)}{w-z} + \mathrm{reg}^{(\infty)} \quad , \quad T(w)\bar{T}(z) \sim \mathrm{reg}^{(\infty)} \tag{5.155}$$

Thus, T has the standard OPE of a conformal field theory with central charge $\mathsf{c} = \bar{\mathsf{c}} = 0$ (since we do not have a 4-th order pole in TT and $\bar{T}\bar{T}$

OPEs).[24] Put another way, T and $\bar{T}$ themselves are primary fields of dimensions $(2,0)$ and $(0,2)$, respectively.

PROOF. First note that for any field Φ we have

$$T(w)\Phi(z) = Q_q G(w)\Phi(z) = Q_q\big(G(w)\Phi(z)\big) + G(w)\, Q_q\Phi(z) \tag{5.156}$$

– by combining the BRST-exactness of the stress-energy tensor (5.153) with the contour-switching argument (5.140); Q_q in the first term on the r.h.s. means "integrate J^{tot} over a contour enclosing both w and z." Thus, computing the OPE $T\Phi$ reduces to computing OPEs of G with Φ or $Q_q\Phi$; computing of Q_q on any field is straightforward (reduces to computing the classical BRST operator) by Proposition 5.5.4.

Next, we make the following remark: if Φ is at most linear in fundamental fields of $\mathcal{AB}$-charge $+1$, then

$$G(w)\Phi(z) = [G(w)\Phi(z)]_{\mathrm{tree}} \qquad \text{unless } \Phi \text{ contains } c \text{ and } \gamma \text{ or } \bar{\gamma} \tag{5.157}$$

This is a special case of the remark of Section 5.3.4: 1-loop graphs (5.87) involving G and Φ have no admissible decorations unless Φ contains c and γ or $\bar{\gamma}$ (possibly with derivatives).

For Φ a fundamental field, we calculate $G(w)\, Q\Phi(z)$ (recall that $Q_q = Q$ on fundamental fields) using (5.157):

$$G(w)\, Qa^a(z) \sim \frac{a^a}{(w-z)^2} + \frac{\partial a^a}{w-z} + \mathrm{reg}^{(\infty)}, \quad G(w)\, Q\bar{a}^a(z) \sim \frac{\partial \bar{a}^a}{w-z} + \mathrm{reg}^{(\infty)}$$
$$G(w)\, Qc^a(z) \sim -g\frac{[a,c]^a}{w-z} + \mathrm{reg}^{(\infty)}, \qquad G(w)\, Qb_a(z) \sim \frac{\partial b_a}{w-z} + \mathrm{reg}^{(\infty)}$$
$$G(w)\, Q\gamma_a(z) \sim -\frac{\partial\bar{\gamma}_a}{w-z} + \mathrm{reg}^{(\infty)}, \qquad G(w)\, Q\bar{\gamma}_a(z) \sim \frac{\partial\bar{\gamma}_a}{w-z} + \mathrm{reg}^{(\infty)}$$

where all fields in the r.h.s. are at z.[25] Combining these OPEs with Q_q applied to OPEs (5.149), as in (5.156), we obtain the OPEs of the standard primary form (5.154) between T and any fundamental field; OPEs between $\bar{T}$ and fundamental fields are complex conjugates of the ones we already found. This proves item (a).

[24] Recall that in a conformal theory with (holomorphic) central charge c, the stress-energy tensor satisfies the OPE $T(w)T(z) \sim \frac{\mathsf{c}/2}{(w-z)^4} + \frac{2T(z)}{(w-z)^2} + \frac{\partial T(z)}{w-z} + \mathrm{reg}^{(\infty)}$ and similarly for $\bar{T}\bar{T}$ and anti-holomorphic central charge $\bar{\mathsf{c}}$.

[25] Note that the field $Q\gamma_a = \frac{g}{2}[c, \gamma - \bar{\gamma}]_a$ does contain both c and γ or $\bar{\gamma}$, so there is a possibility of a 1-loop correction to the OPE $G(w)Q\gamma_a(z)$. However, this 1-loop diagram vanishes by unimodularity. The same applies to $G(w)Q\bar{\gamma}_a(z)$. Another remark is that these OPEs are written modulo equations of motion and modulo contact terms. – In fact, there is a contact term arising in $G(w)Q\bar{a}^a(z)$. It corresponds to a contraction between $\partial\bar{\gamma}(w)$ and $\bar{a}(z)$ in $T\bar{a}$ OPE when the stress-energy tensor is written in the form (5.17). Similarly, $G\bar{T}$ OPE (5.158) below would contain a contact term if $\bar{T}$ were written as complex conjugate of (5.17) instead of (5.18).

Next, we calculate $G(w)T(z)$, which is straightforward using (5.157) and (5.149):

$$G(w)\underbrace{\big(\langle\partial\gamma,a\rangle+\langle\partial b,\partial c\rangle+\frac{g}{2}\langle\partial b,[a,c]\rangle\big)(z)}_{T(z)}\sim$$

$$\sim\frac{\langle\partial b,a\rangle(z)}{(w-z)^2}+\frac{\langle\partial^2 b,a\rangle(z)}{w-z}+\frac{\langle\partial b,a\rangle(z)}{(w-z)^2}+\frac{\langle\partial b,\partial a\rangle(z)}{w-z}+\mathrm{reg}^{(\infty)}\sim$$
$$\sim\frac{2\langle\partial b,a\rangle(z)}{(w-z)^2}+\frac{\partial\langle\partial b,a\rangle(z)}{w-z}+\mathrm{reg}^{(\infty)}\sim\frac{2G(w)}{(z-w)^2}+\frac{\partial G(w)}{z-w}+\mathrm{reg}^{(\infty)}$$

where in the last step we re-expanded the fields in the r.h.s. at w instead of z. Similarly, one finds

$$G(w)\underbrace{\big(\langle\bar\partial\gamma,\bar a\rangle+\langle\bar\partial b,\bar\partial c\rangle+\frac{g}{2}\langle\bar\partial b,[\bar a,c]\rangle\big)(z)}_{\bar T(z)}\sim$$
$$\sim\frac{\langle\bar\partial b,\bar\partial a\rangle(z)}{w-z}+\frac{g}{2}\frac{\langle\bar\partial b,[\bar a,a]\rangle(z)}{w-z}+\mathrm{reg}^{(\infty)}\quad\sim\quad\mathrm{reg}^{(\infty)} \tag{5.158}$$

Thus, G is indeed a primary field of dimension $(2,0)$ (note that we do not see the pole $\frac{1}{\bar z-\bar w}$ in $\bar T G$ OPE, since its coefficient $\bar\partial G$ vanishes under the correlator). By complex conjugation, we get that $\bar G$ is $(0,2)$-primary. This proves item (b).

Finally, item (c) follows immediately from (b) by applying Q_q to the GT, $G\bar T$ OPEs and using the fact that $T,\bar T$ are Q_q-closed:

$$T(w)T(z)=Q_q(G(w)T(z))+G(w)\underbrace{Q_qT(z)}_{0}\sim$$
$$\sim Q_q\Big(\frac{2G(z)}{(w-z)^2}+\frac{\partial G(z)}{w-z}+\mathrm{reg}^{(\infty)}\Big)\sim\frac{2T(z)}{(w-z)^2}+\frac{\partial T(z)}{w-z}+\mathrm{reg}^{(\infty)}$$

and

$$T(w)\bar T(z)=Q_q(\underbrace{G(w)\bar T(z)}_{\mathrm{reg}^{(\infty)}})+G(w)\underbrace{Q_q\bar T(z)}_{0}\sim\mathrm{reg}^{(\infty)}$$

□

Example 5.5.7. If a field Φ is primary of conformal dimension $(0,\bar\Delta)$, then $\partial\Phi$ is also primary, of dimension $(1,\bar\Delta)$. This follows from (5.154) with $\Delta=0$, by applying ∂_z. Similarly, for Φ primary of dimension $(\Delta,0)$, $\bar\partial\Phi$ is primary of dimension $(\Delta,1)$. This implies in particular that derivatives of fundamental fields

$$\bar\partial a,\partial\gamma,\bar\partial\gamma,\partial\bar\partial\gamma,\partial b,\partial\bar\partial b,\partial c,\partial\bar\partial c$$

and complex conjugates are primary (but higher derivatives are non-primary).

5.6. Examples of fields with a quantum correction to dimension ("vertex operators")

In this section we will present vertex operators with anomalous dimensions, i.e., with the actual conformal dimension different from the naive one, defined in Section 5.2.1.1. We obtain these dimensions in two ways: first, by considering OPEs with T in Subsection 5.6.1. The second way is due to singular subtractions in renormalized products in Subsection 5.6.2.

These two ways in the standard case of the free scalar field are:

(1) OPE of the vertex operator $V_\alpha =: e^{i\alpha\phi}:$ with the energy-momentum tensor $T = -\frac{1}{2} :\partial\phi\partial\phi:$ is

$$T(w)V_\alpha(z) \sim \frac{\frac{\alpha^2}{2} V_\alpha(z)}{(w-z)^2} + \frac{\partial V_\alpha(z)}{w-z} + \text{reg}$$

Together with the similar OPE $\bar{T}(w)V_\alpha(z)$, this implies that V_α is primary, of conformal dimension $(\Delta = \frac{\alpha^2}{2}, \bar{\Delta} = \frac{\alpha^2}{2})$.

(2) Recall that the renormalized (normally ordered) field depends on the choice of local coordinate. In particular, under the infinitesimal change of local coordinate $z \to z' = (1+\epsilon)z$, the renormalized field $:\phi^k(0):$ transforms as

$$:\phi^k:_z \to \ :\phi^k:_{z'} = \ :\phi^k:_z + \epsilon\, k(k-1) :\phi^{k-2}:_z \tag{5.159}$$

up to $O(\epsilon^2)$ terms, as proven by induction in k using

$$:\phi^{k+1}(0):_z = \lim_{p\to 0} \Big(\phi(p) :\phi^k(0):_z + 2k :\phi^{k-1}(0):_z \log|z(p)| \Big)$$

in local coordinate z. Here p is a point of insertion of an observable and here we take care to distinguish between a point p and its coordinate $z(p)$. Summing (5.159) over k with coefficients $\frac{(i\alpha)^k}{k!}$, we obtain the transformation law for the vertex operator:

$$(V_\alpha)_z \to (V_\alpha)_{z'} = \big(1 - \epsilon\alpha^2\big)\,(V_\alpha)_z \tag{5.160}$$

This is consistent with scaling dimension $\Delta + \bar{\Delta} = \alpha^2$.

5.6.1. New vertex operators V and W and their conformal dimensions. Let us fix $X \in \mathfrak{g}$ a Lie algebra element, fix $Y \in \mathfrak{g}$ an eigenvector of ad_X with eigenvalue α and fix $\rho \in \mathfrak{g}^*$ an eigenvector of the coadjoint action ad^*_X with eigenvalue $-\alpha$, i.e.:

$$\mathrm{ad}_X\, Y = \alpha Y, \qquad \mathrm{ad}^*_X\, \rho = -\alpha\rho$$

Consider the following composite fields ("vertex operators"):[26]

$$V_{X,\rho} = \langle \rho, a\rangle\, e^{\langle X, \bar{\gamma}\rangle}, \qquad W_{X,Y} = \langle \gamma - \bar{\gamma}, Y\rangle\, e^{\langle X,\bar{\gamma}\rangle} \tag{5.161}$$

Note that V depends only on the vectors X, ρ and W depends only on X, Y.

PROPOSITION 5.6.1. Fields $V_{X,\rho}$, $W_{X,Y}$ are primary, of conformal dimensions

$$(\Delta = 1 - \frac{\alpha g}{2}, \bar{\Delta} = -\frac{\alpha g}{2}) \text{ for } V, \qquad (\Delta = \frac{\alpha g}{2}, \bar{\Delta} = \frac{\alpha g}{2}) \text{ for } W$$

[26] We understand $V_{X,\rho}$ as $\sum_{n\geq 0} \frac{1}{n!} \langle \rho, a\rangle\, \langle X, \bar{\gamma}\rangle^n$. Each term in the sum is a composite field understood as the renormalized product $\widetilde{\lim_{z'\to z}} \frac{1}{n!} \langle \rho, a(z')\rangle\, \langle X, \bar{\gamma}(z)\rangle^n$. Here we can safely put all $\bar{\gamma}$'s into the same point as they are regular with each other. In fact, since $a(z')\bar{\gamma}^n(z)$ OPE contains only powers of logs (cf. (5.115)), there is no order-of-merging ambiguity in the renormalized product above. Similar remarks apply to $W_{X,Y}$.

Proof. First, note that for any composite field Φ, the residue of the first order pole in the OPE $T(w)\Phi(z)$ (commonly denoted $L_{-1}\Phi = \frac{1}{2\pi i}\oint T(w)\Phi(z)$) is $\partial\Phi$. This follows from the fact that $L_{-1} = \partial$ on fundamental fields (from Proposition 5.5.6), and hence for derivatives of fundamental fields (as L_{-1} commutes with derivatives, by the same logic as (b) of Proposition 5.5.4). Finally, L_{-1} is a derivation of the renormalized product, by the same logic as (c) of Proposition 5.5.4 (contour switching argument).

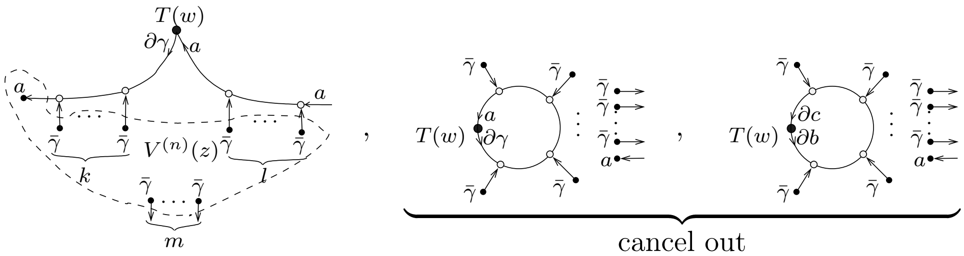
Consider the field

$$V^{(n)} = \frac{1}{n!}\langle\rho, a\rangle\langle\bar{\gamma}, X\rangle^n \tag{5.162}$$

Its OPE with T must be of the form

$$T(w)V^{(n)}(z) \sim \frac{\cdots}{(w-z)^3} + \frac{\cdots}{(w-z)^2} + \frac{\partial V^{(n)}(z)}{w-z} + \mathrm{reg}^{(\infty)} \tag{5.163}$$

Here we cannot get a pole higher than third order, because l.h.s. has weight $(3,0)$ (and we don't have fields of negative weight to accompany a pole of order > 3). The coefficient of the third order pole must be of weight $(0,0)$ and in fact there are no such contributing diagrams.[27] Looking for the second order pole, we look for diagrams producing a field of weight $(1,0)$. There are three families of such diagrams:



Diagrams of second and third type cancel each other by the mechanism of Lemma 5.2.1. Diagrams of the first type, evaluated using the computations of Section 5.4.4, jointly give the following contribution to the OPE (5.163):

$$\sum_{k,l,m\geq 0, k+l+m=n} \frac{1}{n!}\begin{pmatrix} n \\ k,l,m\end{pmatrix} g^k \left\langle \rho, \mathrm{ad}_X^k\, t_a\right\rangle \partial_w \frac{\log^k|z-w|}{z-w}\cdot$$

$$\cdot (-g)^l \left\langle t^a, \mathrm{ad}_X^l\, t_b\right\rangle a^b \log^l|z-w| \cdot \langle\bar{\gamma}, X\rangle^m$$

$$= \frac{1}{(z-w)^2}\left(\frac{1}{n!}\langle\rho,a\rangle\langle\bar{\gamma},X\rangle^n - \frac{g\alpha}{2(n-1)!}\langle\rho,a\rangle\langle\bar{\gamma},X\rangle^{n-1}\right)$$

Here all fields are at z; $\begin{pmatrix} n \\ k,l,m\end{pmatrix}$ is the multinomial coefficient. Note that all the terms involving positive powers of log have cancelled out (as expected)

[27] Indeed, in such a diagram external half-edges would need to be decorated by either γ or $\bar{\gamma}$ (since $a, \bar{a}, \partial b, \bar{\partial} b$ have nonzero weight and presence of c on an external half-edge would require, by conservation of ghost number, another external half-edge decorated by ∂b or $\bar{\partial} b$). Thus, the diagram must consist of ≥ 2 trees rooted at a from $V^{(n)}$, at a or c from T and at external half-edges. Leaves of the trees are decorated jointly by n fields $\bar{\gamma}$ and one $\partial\gamma$ or ∂b (from T). Therefore, there must be a tree whose leaves are decorated only by $\bar{\gamma}$'s. Such a tree vanishes.

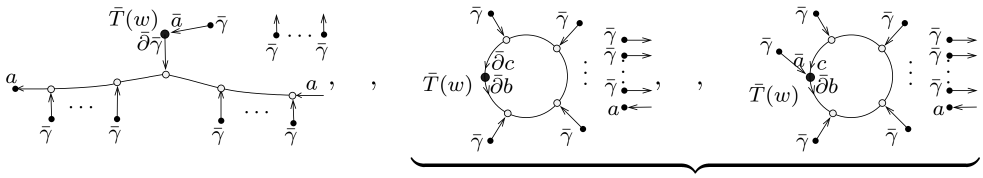
– in fact, all diagrams except ones with $k+l\leq 1$ cancel out when summed with $k+l$ fixed. Thus, we obtained the explicit form of the OPE (5.163):[28]

$$T(w)V^{(n)}(z)\sim\frac{\left(V^{(n)}-\frac{g\alpha}{2}V^{(n-1)}\right)(z)}{(w-z)^2}+\frac{\partial V^{(n)}(z)}{w-z}+\mathrm{reg}^{(\infty)} \tag{5.164}$$

Here by convention $V^{(-1)}=0$. Summing over $n\geq 0$, we find that our field $V_{X,\rho}=\sum_{n\geq 0}V^{(n)}$ satisfies the standard primary OPE with T, with holomorphic conformal dimension $\Delta=1-\frac{\alpha g}{2}$. A similar computation yields the OPE

$$\bar{T}(w)V^{(n)}(z)\sim\frac{-\frac{g\alpha}{2}V^{(n-1)}(z)}{(\bar{w}-\bar{z})^2}+\frac{\bar{\partial}V^{(n)}(z)}{\bar{w}-\bar{z}}+\mathrm{reg}^{(\infty)}$$

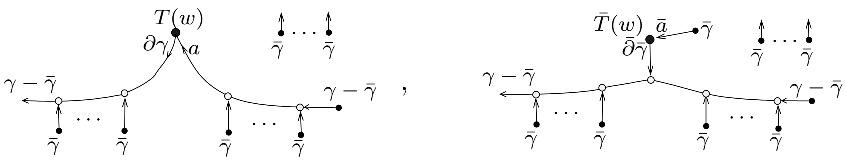
where the relevant diagrams are



cancel out

Thus, the anti-holomorphic dimension of the field $V_{X,\rho}$ is $\bar{\Delta}=-\frac{\alpha g}{2}$.

Computation of the OPEs TW, $\bar{T}W$ is similar, with the following relevant diagrams (we omit the families cancelling by boson-fermion cancellation in the loop):



□

5.6.2. Another view on conformal dimensions and examples of correlators. Ultimately, the source of the shift of the conformal dimension is in singular subtractions – powers of logs – needed in the renormalized products when we build the vertex operators from fundamental fields. These subtractions depend on the local coordinate and ultimately lead to the anomalous scaling behavior.

Explicitly: consider the field (5.162) viewed as renormalized product

$$V^{(n)}(0)=\lim_{p\to 0}\left(V^{(n)}_{\mathrm{split}}(p,0)-\left[V^{(n)}_{\mathrm{split}}(p,0)\right]_{\mathrm{sing}}\right) \tag{5.165}$$

where

$$V^{(n)}_{\mathrm{split}}(p,0)=\frac{1}{n!}\left\langle\rho,a(p)\right\rangle\left\langle\bar{\gamma}(0),X\right\rangle^n$$

If we make an infinitesimal change of local coordinate $z\to z'=(1+\epsilon)z$, we have the following:

[28] This OPE implies that fields $\left(-\frac{g\alpha}{2}\right)^{-n}V^{(n)}$ comprise a Jordan cell of infinite rank of the Virasoro operator L_0, see Section 5.6.3.

(a) The split field transforms as

$$V^{(n)}_{\mathrm{split}}(p,0)_z \to V^{(n)}_{\mathrm{split}}(p,0)_{z'} = (1-\epsilon)V^{(n)}_{\mathrm{split}}(p,0)_z$$

up to $O(\epsilon^2)$ terms; the subscript z, z' refers to the coordinate system. Here the ϵ-correction comes from the transformation of a.

(b) The singular subtraction in (5.165) is transformed as

$$\left[V^{(n)}_{\mathrm{split}}(p,0)\right]^{\mathrm{sing}}_z \to \left[V^{(n)}_{\mathrm{split}}(p,0)\right]^{\mathrm{sing}}_{z'} = (1-\epsilon)\left[V^{(n)}_{\mathrm{split}}(p,0)\right]^{\mathrm{sing}}_z - \epsilon\,\alpha g V^{(n-1)}_z(0)$$

(c) The field $V^{(n)}$ is transformed as

$$V^{(n)}_z \to V^{(n)}_{z'} = (1-\epsilon)V^{(n)}_z + \epsilon\,\alpha g V^{(n-1)}_z \tag{5.166}$$

where all the fields are at the origin.

One proves this by induction in n: (a) is staightforward,[29] (c) follows from (a) and (b) immediately. In turn, (b) follows from the (c) for smaller n and from the OPE

$$V^{(n)}_{\mathrm{split}}(p,0) \sim \sum_{k=0}^{n} \frac{(-\alpha g)^k}{k!} \log^k|z(p)|\, V^{(n-k)}(0) + o(1)_{p\to 0} \tag{5.167}$$

obtained similarly to (5.115). Here terms $1 \le k \le n$ give the singular part of the OPE and $k=0$ is the regular part (modulo terms which are continuous and vanishing at $p\to 0$). To see (b) explicitly, we compute from (5.167):

$$\left[V^{(n)}_{\mathrm{split}}(p,0)\right]^{\mathrm{sing}}_{z'} = \sum_{k=1}^{n} \frac{(-\alpha g)^k}{k!}\log^k|z'(p)|\, V^{(n-k)}_{z'} = (1-\epsilon)\left[V^{(n)}_{\mathrm{split}}(p,0)\right]^{\mathrm{sing}}_z +$$

$$+\epsilon\underbrace{\left(\sum_{k=1}^{n}\frac{(-\alpha g)^k}{k!} k\log^{k-1}|z(p)|\,V^{(n-k)}_z + \sum_{k=1}^{n-1}\frac{(-\alpha g)^k}{k!}\alpha g \log^k|z(p)|\,V^{(n-k-1)}_z\right)}_{-\alpha g V^{(n-1)}_z}$$

Here k-th term in the first sum on the r.h.s., for $k\neq 1$, is cancelled by $(k-1)$-st term in the second sum.

Summing (5.166) over n, we obtain the transformation property for the vertex operator:

$$(V_{X,\rho})_z \to (V_{X,\rho})_{z'} = (1-\epsilon(1-\alpha g))(V_{X,\rho})_z$$

confirming the scaling dimension $\Delta + \bar{\Delta} = 1-\alpha g$ we obtained in Proposition 5.6.1. The case of the second vertex operator, $W_{X,Y}$, is treated similarly.

Example 5.6.2. Starting with correlators (5.119), (5.120), contracting all $\bar{\gamma}$'s with X, contracting a with ρ and γ with Y, and summing over $n\ge 0$, we obtain the following 2-point functions

$$\langle V_{X,\rho}(z)\ \langle(\gamma-\bar{\gamma})(w), Y\rangle\rangle = \langle \rho, Y\rangle\,\frac{|z-w|^{\alpha g}}{z-w} \tag{5.168}$$

$$\langle\langle \rho, a(z)\rangle\ W_{X,Y}(w)\rangle = \langle \rho, Y\rangle\,\frac{|z-w|^{-\alpha g}}{z-w} \tag{5.169}$$

– power laws consistent with the dimensions of the primary fields involved.

[29] Note that in the composite field $\langle\bar{\gamma}, X\rangle^n$ there are no singular subtractions (logarithmic or otherwise), thus there is no anomalous dimension.

REMARK 5.6.3. Soaking fields $\widetilde{\Theta}$ and $\delta(c)$ (cf. Section 5.2.4) are primary, of dimension $(0,0)$.[30]Thus, correlators (5.168), (5.169) can be extended to 4-point functions of primary fields on a sphere, e.g.

$$\left\langle V_{X,\rho}(z_1)\ \langle(\gamma-\bar{\gamma})(z_2),Y\rangle\ \widetilde{\Theta}(z_3)\ \delta(c(z_4))\right\rangle_{\mathbb{C}P^1} = \langle\rho,Y\rangle\left|\frac{z_{23}}{z_{12}z_{13}}\right|^{-\alpha g}\frac{z_{23}}{z_{12}z_{13}}$$

where $z_{ij}=z_i-z_j$ (note that the r.h.s. does not depend on z_4). This result is consistent with the ansatz for 4-point functions of primary fields implied by global conformal invariance on $\mathbb{C}P^1$.

Furthermore, we can introduce the field

$$H_X = e^{\langle\bar{\gamma},X\rangle}$$

It is primary, of conformal dimension $(0,0)$ (as proven by the same technology as in the proof of Proposition 5.6.1 above; cf. also footnote 29). From (5.115), we find the 3-point function

$$\left\langle\ \langle\rho,a(w_1)\rangle\ H_X(z)\ \langle(\gamma-\bar{\gamma})(w_2),Y\rangle\ \right\rangle = \frac{\langle\rho,Y\rangle}{w_1-w_2}\left|\frac{z-w_2}{z-w_1}\right|^{\alpha g} \tag{5.170}$$

OPEs of H_X with either a or $\gamma-\bar{\gamma}$ yield our two vertex operators:

$$\langle\rho,a(w)\rangle\ H_X(z)\sim V_{X,\rho}(z)|w-z|^{-\alpha g}+o(1)_{w\to z} \tag{5.171}$$

$$\langle(\gamma-\bar{\gamma})(w),Y\rangle\ H_X(z)\sim W_{X,Y}(z)|w-z|^{\alpha g}+o(1)_{w\to z} \tag{5.172}$$

Here (5.171) follows immediately from the OPE (5.167).

We make the following remarks.

- Our construction of "vertex operators" is based on exact summation of perturbation theory in all orders in g. E.g., non-trivial exponents in the correlators (5.168), (5.169), (5.170) arise from the summation of powers of logs appearing in the correlators of Section 5.4.4.
- Vertex operators are not differential polynomials (of finite order) in fundamental fields – we need to add infinitely many monomials to produce a field of non-trivial dimension.

5.6.3. A remark on logarithmic phenomena. Usually, primary fields are defined by their OPE with the energy-momentum tensor (5.154). However, exploring phenomena like in (5.164), we recall the refinement of this definition [8]. A field is called "pseudo-primary" if it has at most a second-order pole in its OPE with $T,\bar{T}$.[31] Then pseudo-primary fields form a closed subspace w.r.t. $L_0,\bar{L}_0$. If $L_0,\bar{L}_0$ acting on the space of pseudo-primary fields are jointly diagonalizable, we get the standard definition of primary fields. If not, we have the Jordan cell structure where only the lowest component is

[30]The idea of proof is as follows. Since the fields $\delta(c)$ and $\widetilde{\Theta}$ are Q-closed, one can recover their OPE with T from their OPE with G using (5.156). For the OPE with G, there can be only poles of orders 1 and 2, due to weight counting. Second order pole in fact has no contributing diagrams (as follows from Feynman diagram combinatorics and weight restrictions). Coefficient of the first order pole is easily found as $G_{-1}\Phi(z)=\frac{1}{2\pi i}\oint G(w)\Phi(z)$, for Φ a soaking field, from the fact that G_{-1} is a derivation (as proven by the contour switching argument similar to (5.140)). Interestingly, the renormalized product $\Theta=\delta(c)\widetilde{\Theta}$ is non-primary – already in the abelian case – due to logarithmic singular subtractions (cf. (5.59)).

[31] Another way to say it is: Φ is pseudo-primary if $L_n\Phi=0$, $\bar{L}_n=0$ for $n\geq 1$.

a primary field and all the rest are only pseudo-primary (but not primary). More precisely, the space of pseudo-primary fields splits into a direct sum of filtered subspaces[32] $\mathrm{Span}\{\Phi_0,\ldots,\Phi_r\}$ – Jordan cells – satisfying the OPEs:

$$T(w)\Phi_k(z) \sim \frac{\Delta\Phi_k(z)+\Phi_{k-1}(z)}{(w-z)^2}+\frac{\partial\Phi_k(z)}{w-z}+\mathrm{reg}, \tag{5.173}$$

$$\bar{T}(w)\Phi_k(z) \sim \frac{\bar{\Delta}\Phi_k(z)+\Phi_{k-1}(z)}{(\bar{w}-\bar{z})^2}+\frac{\bar{\partial}\Phi_k(z)}{\bar{w}-\bar{z}}+\mathrm{reg} \tag{5.174}$$

where by convention $\Phi_{-1}=0$. Here $(\Delta,\bar{\Delta})$ are called conformal dimensions. Actually, the condition that the infinitesimal rotation operator $L_0-\bar{L}_0$ integrates to a representation of the group $U(1)$ is tantamount to requiring that $L_0-\bar{L}_0$ is diagonalizable, with integer eigenvalues. Thus, we must have $\Delta-\bar{\Delta}\in\mathbb{Z}$ and the upper-triangular parts of $L_0,\bar{L}_0$ must be the same (in other words, we have the same Φ_{k-1} appearing in the OPE of Φ_k with T and with $\bar{T}$).

OPEs (5.173), (5.174) imply the following behavior of fields Φ_k under a change of coordinates $z\to z'=\Lambda z$ with $\Lambda\in\mathbb{C}-\{0\}$ a scaling factor:

$$(\Phi_k)_z \to \quad (\Phi_k)_{z'} = \Lambda^{-L_0}\bar{\Lambda}^{-\bar{L}_0}(\Phi_k)_z = \Lambda^{-\Delta}\bar{\Lambda}^{-\bar{\Delta}}\sum_{j=0}^{k}\frac{(-2\log|\Lambda|)^j}{j!}(\Phi_{k-j})_z \tag{5.175}$$

where the fields are at zero.

Example 5.6.4. Consider the theory with Lagrangian $b\partial\bar{\partial}c$ (this is the ghost sector of the abelian BF theory). Field 1 is primary. Field $:cb:$ is pseudo-primary, with $L_0:cb:=1$ (see [7]). In this case, the rank of Jordan cell is 2.

Example 5.6.5. In the free scalar field theory, fields 1 and ϕ are primary. Field $:\phi^2:$ is pseudo-primary, in the same Jordan cell as 1, with $L_0:\phi^2:=-1$. Pseudo-primary field $:\phi^3:$ is in the same Jordan cell as ϕ, with $L_0:\phi^3:=-3\phi$. Actually, due to infinite-dimensionality of the space of pseudo-primary fields, these Jordan cells are of infinite rank.

Example 5.6.6. Consider the fields $V^{(k)}$ defined in (5.162). For $k=0$, $V^{(0)}=\langle\rho,a\rangle$ is a primary field. Fields $V^{(k)}$ for $k\geq 1$ are pseudo-primary and they are in the same Jordan cell, see (5.164). Setting $\Phi_k=\left(-\frac{g\alpha}{2}\right)^{-k}V^{(k)}$, we have a standard basis for the Jordan cell.

Remark 5.6.7. Every time when we have a Jordan cell of infinite rank, we can form a family of vertex operators

$$\mathsf{V}_{\varkappa}=\sum_{k=0}^{\infty}\varkappa^k\Phi_k$$

[32] I.e., a summand of the space of pseudo-primary fields is a filtered subspace $\mathbb{F}_0\subset\cdots\subset\mathbb{F}_r$, with $\dim\mathbb{F}_k=k+1$ and with L_0 preserving the filtration (while $L_0-\bar{L}_0$ acts on $\mathbb{F}_r$ as a multiple of identity). For each k we choose a vector $\Phi_k\in\mathbb{F}_k$ with nonzero image in the quotient $\mathbb{F}_k/\mathbb{F}_{k-1}$. We can make this sequence of choices in such a way that L_0 has the standard Jordan cell form in the basis $\{\Phi_0,\ldots,\Phi_r\}$.

parameterized by $\varkappa \in \mathbb{R}$. Each $\mathsf{V}_\varkappa$ is a primary field of conformal dimension $(\Delta + \varkappa, \bar{\Delta} + \varkappa)$, as follows from (5.173), (5.174). Note that by this mechanism the two infinite Jordan cells of Example 5.6.5 give rise to the vertex operators $:\cos(\alpha\phi):$, $:\sin(\alpha\phi):$ – linear combinations of the standard vertex operators $:e^{\pm i\alpha\phi}:$. Likewise, when applied to the infinite Jordan cell of Example 5.6.6 with $\varkappa = -\frac{g\alpha}{2}$, this mechanism produces the new vertex operator $V_{X,\rho}$ defined in (5.161).

5.A. Some useful plane integrals

Let $D_R = \{u \in \mathbb{C} \mid |u| \leq R\}$ be a disk of radius R in $\mathbb{C}$ centered at zero. Then we have

$$\int_{D_R} \frac{d^2u}{\pi} \frac{1}{(u-z)(\bar{u}-\bar{w})} = \log\left(\frac{R^2 - z\bar{w}}{|z-w|^2}\right) \tag{5.176}$$

for $z \neq w$ two points inside D_R. One finds this by writing the integrand as $\frac{\partial}{\partial \bar{u}} \frac{\log(\bar{u}-\bar{w})}{u-z}$, replacing the integration domain with D_R with a cut from w to the boundary of D_R and with a small disk around z removed, and applying Stokes' theorem. Explicitly, denoting the l.h.s. of (5.176) by $I_R(z,w)$ and denoting the new integration domain $\mathcal{D}$, we have:

$$\begin{aligned} I_R(z,w) &= \int_{\mathcal{D}} d\bar{u} \frac{\partial}{\partial \bar{u}} \frac{du}{2\pi i} \frac{\log(\bar{u}-\bar{w})}{u-z} = \int_{\partial\mathcal{D}} \frac{du}{2\pi i} \frac{\log(\bar{u}-\bar{w})}{u-z} \\ &= \underbrace{-\int_{-R}^{w} \frac{du}{u-z}}_{I} \underbrace{-\log(\bar{z}-\bar{w})}_{II} + \underbrace{\int_{-\pi}^{\pi} \frac{d\phi}{2\pi} Re^{i\phi} \frac{\log(Re^{-i\phi}-\bar{w})}{Re^{i\phi}-z}}_{III} \end{aligned}$$

Three terms here come from components of the contour ∂D. Term I comes from the jump of the integrand on the cut between $u = w$ and $u = -R$ and evaluates to $\log \frac{R+z}{z-w}$. Term II is the contribution of the small circle around $u = z$. Term III is the contribution of the big circle ∂D_R; it evaluates to

$$\begin{aligned} &\int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \frac{R - i\phi + \log(1 - \frac{\bar{w}}{R}e^{i\phi})}{1 - \frac{z}{R}e^{-i\phi}} \\ &= \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \Big(\log R \sum_{p\geq 0} \left(\frac{z}{R}\right)^p e^{-ip\phi} - i\phi \sum_{p\geq 0} \left(\frac{z}{R}\right)^p e^{-ip\phi} - \underbrace{\sum_{p\geq 0} \left(\frac{z}{R}\right)^p e^{-ip\phi} \cdot \sum_{q\geq 1} \frac{1}{q} \left(\frac{\bar{w}}{R}\right)^q e^{iq\phi}}_{\text{only } p=q \text{ contributes}} \Big) \\ &= \log R + \sum_{p\geq 1} \frac{1}{p} \left(-\frac{z}{R}\right)^p - \sum_{p\geq 0} \frac{1}{p} \left(\frac{z\bar{w}}{R^2}\right)^p = \log R - \log(1+\frac{z}{R}) + \log(1-\frac{z\bar{w}}{R^2}) = \log \frac{R^2 - z\bar{w}}{R+z} \end{aligned}$$

Collecting all the terms, we get the result (5.176).

Similarly, one can treat the cases when one or both points z, w are outside D_R:

$$\begin{aligned} I_R(z,w) &= -\log\left(1 - \tfrac{R^2}{z\bar{w}}\right) && \text{if } |z|,|w| > R \\ I_R(z,w) &= -\log\left(1 - \tfrac{w}{z}\right) && \text{if } |w| < R < |z| \\ I_R(z,w) &= -\log\left(1 - \tfrac{\bar{z}}{\bar{w}}\right) && \text{if } |z| < R < |w| \end{aligned} \tag{5.177}$$

One can use (5.176) to evaluate integrals over $\mathbb{C}$ of products of expressions $\frac{1}{u-z_i}$ and $\frac{1}{\bar{u}-\bar{z}_i}$. For example, for z, w, x three distinct points in $\mathbb{C}$ we have

$$\int_{\mathbb{C}} \frac{d^2u}{\pi} \frac{1}{(u-z)(u-x)(\bar{u}-\bar{w})} = \frac{1}{z-x} \lim_{R\to\infty} (I_R(z,w) - I_R(x,w)) = \frac{2}{z-x} \log \left| \frac{x-w}{z-w} \right| \tag{5.178}$$

where we used the expansion $\frac{1}{(u-z)(u-x)} = \frac{1}{z-x}(\frac{1}{u-z} - \frac{1}{u-w})$ to reduce the integral to (5.176). Integral (5.178) is crucial for the computation of 3-point functions.

Another useful integral of this type is

$$\int_{\mathbb{C}} \frac{d^2u}{\pi} \frac{1}{(u-z)(u-x)} = -\frac{\bar{z}-\bar{x}}{z-x} \tag{5.179}$$

One obtains it by presenting the integrand as $\frac{\partial}{\partial \bar{u}} \frac{\bar{u}}{(u-z)(u-x)}$ and using Stokes' theorem on the plane with two small disks around $u=z$ and $u=x$ removed.

5.A.1. The dilogarithm integral. The following integral over a disk is useful for evaluating 4-point functions and can be evaluated in terms of the dilogarithm function:

$$\int_{D_R} \frac{d^2u}{\pi} \frac{\log|u|}{(u-z)(\bar{u}-\bar{w})} = \tag{5.180}$$
$$= \log^2 R + i\mathsf{D}\left(\frac{z}{w}\right) - \log|zw| \cdot \log|z-w| + \log|z| \cdot \log|w| + O\left(\frac{\log R}{R^2}\right)$$

Here $\mathsf{D}(z) = \operatorname{Im}\operatorname{Li}_2(z) + \arg(1-z)\log|z|$ is the *Bloch-Wigner dilogarithm*, see [13]. It is the monodromy-free variant of the standard dilogarithm $\operatorname{Li}_2(z) = -\int_0^z dt \frac{\log(1-t)}{t}$ – the analytic continuation of the sum $\sum_{n\geq 1} \frac{z^n}{n^2}$ convergent on the disk $|z| \leq 1$. In particular, $\mathsf{D}(z)$ is a real-analytic function everywhere on $\mathbb{C}P^1$ except at $z = 0, 1, \infty$ where it is continuous (and vanishes) but is not differentiable. Function $\mathsf{D}(z)$ satisfies the identity $\mathsf{D}(1/z) = -\mathsf{D}(z)$,[33] thus it is clear that the r.h.s. of (5.180) conjugates when z and w are interchanged. The $O\left(\frac{\log R}{R^2}\right)$ remainder term in (5.180) can be written explicitly as $\log R \log(1 - \frac{z\bar{w}}{R^2}) - \frac{1}{2}\operatorname{Li}_2(\frac{z\bar{w}}{R^2})$.

Starting from (5.180), similarly to (5.178), one obtains

$$\int_{\mathbb{C}} \frac{d^2u}{\pi} \frac{\log|u|}{(u-z_1)(u-z_2)(\bar{u}-\bar{z}_3)} = \tag{5.181}$$
$$= \frac{1}{z_1-z_2}\left(i\mathsf{D}\left(\frac{z_1}{z_3}\right) - \log|z_1 z_3| \cdot \log|z_1 - z_3| + \log|z_1| \cdot \log|z_3| - \quad (z_1 \leftrightarrow z_2) \right)$$

The last term in the brackets stands for the previous terms with z_1 replaced by z_2.

[33] The more general identity is that, under a Möbius transformation permuting points $0, 1, \infty$, $\mathsf{D}(z)$ changes by the sign of the permutation: $\mathsf{D}(z) = \mathsf{D}(\frac{1}{1-z}) = \mathsf{D}(1-\frac{1}{z}) = -\mathsf{D}(\frac{1}{z}) = -\mathsf{D}(1-z) = -\mathsf{D}(\frac{z}{z-1})$.

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