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The General Solution of Bianchi Type III Vacuum Cosmology

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Abstract. The theory of symmetries of systems of coupled, ordinary differential equations (ODE's) is used to develop a concise algorithm for cartographing the space of solutions to vacuum Bianchi Einstein's Field Equations (EFE). The symmetries used are the well known automorphisms of the Lie algebra for the corresponding isometry group of each Bianchi Type, as well as the scaling and the time reparameterization symmetry. Application of the method to Type III results in: a) the recovery of all known solutions without prior assumption of any extra symmetry, b) the enclosure of the entire unknown part of the solution space into a single, second order ODE in terms of one dependent variable and c) a partial solution to this ODE. It is also worth-mentioning the fact that the solution space is seen to be naturally partitioned into three distinct, disconnected pieces: one consisting of the known Siklos (pp-wave) solution, another occupied by the Type III member of the known Ellis-MacCallum family and the third described by the aforementioned ODE. Lastly, preliminary results reported show that the unknown part of the solution space for other Bianchi Types is described by a strikingly similar ODE, pointing to a natural operational unification as far as the problem of solving the cosmological EFE's is concerned.

1. Introduction

Since the early times of cosmology, Automorphisms have been identified as possible key elements for a unified treatment of spatially homogeneous Bianchi Geometries [1]. Harvey has found the automorphisms of all 3-dimensional Lie Algebras [2], while the corresponding results for the 4-dimensional Lie Algebras have been reported in [3]. Jantzen's tangent space approach sees the automorphic matrices as the means for achieving a convenient parametrization of a full scale-factor matrix in terms of a, desired, diagonal matrix [4]. Samuel and Ashtekar were the first to look upon Automorphisms from a space viewpoint [5]. The notion of *Time-Dependent Automorphism Inducing Diffeomorphisms* (A.I.D.'s), i.e. coordinate transformations mixing space and time in the new spatial coordinates and inducing automorphic motions on the scale-factor matrix, the lapse and the shift has been developed in [6].

In this communication we revisit the problem of solving the EFE's for vacuum Bianchi Geometries. We begin with a full metric, i.e. we make no assumption for the lapse function N^2, the shift vector N^α and the spatial metric $\gamma_{\alpha\beta}$. Then we use the Time-Dependent A.I.D.'s to put the shift vector to zero. At this point the idea is to exploit, in a systematic way, the remaining symmetries of the field equations –sometimes called "rigid" [7]– to transform them to the most simple form possible, without loss of generality. These are the well known symmetries following from the constant Automorphism group within each Bianchi Type, as well as the scaling of the metric by a constant and the time reparameterization symmetry (see e.g. [8]). Applying this analysis to Bianchi Type III Vacuum Cosmology we produce an exhaustive cartography of the entire space of its solutions.

2. The Method

As it is well known, for spatially homogeneous spacetimes with a simply transitive action of the corresponding isometry group [9], [8], the line element, assumes the form

$$ds^2 = \left(N^\alpha N_\alpha - N^2\right) dt^2 + 2N_\alpha \sigma^\alpha_i dx^i dt + \gamma_{\alpha\beta}\sigma^\alpha_i \sigma^\beta_j dx^i dx^j \tag{1}$$

where the 1-forms σ^α_i, are defined from:

$$d\sigma^\alpha = C^\alpha_{\beta\gamma}\sigma^\beta \wedge \sigma^\gamma \Leftrightarrow \sigma^\alpha_{i,j} - \sigma^\alpha_{j,i} = 2C^\alpha_{\beta\gamma}\sigma^\gamma_i \sigma^\beta_j . \tag{2}$$

Then the field equations are (e.g. [6]):

$$E_o \doteq K^{\alpha\beta}K_{\alpha\beta} - K^2 - \mathbf{R} = 0 \tag{3}$$

$$E_\alpha \doteq K^\mu_\alpha C^\epsilon_{\mu\epsilon} - K^\mu_\epsilon C^\epsilon_{\alpha\mu} = 0 \tag{4}$$

$$E_{\alpha\beta} \doteq \dot{K}_{\alpha\beta} + N\left(2K^\tau_\alpha K_{\tau\beta} - KK_{\alpha\beta}\right) + 2N^\rho\left(K_{\alpha\nu}C^\nu_{\beta\rho} + K_{\beta\nu}C^\nu_{\alpha\rho}\right) - N\mathbf{R}_{\alpha\beta} = 0 \tag{5}$$

where

$$K_{\alpha\beta} = -\frac{1}{2N}\left(\dot{\gamma}_{\alpha\beta} + 2\gamma_{\alpha\nu}C^\nu_{\beta\rho}N^\rho + 2\gamma_{\beta\nu}C^\nu_{\alpha\rho}N^\rho\right) \tag{6}$$

is the extrinsic curvature and

$$\begin{aligned}\mathbf{R}_{\alpha\beta} = \; & C^\kappa_{\sigma\tau}C^\lambda_{\mu\nu}\gamma_{\alpha\kappa}\gamma_{\beta\lambda}\gamma^{\sigma\nu}\gamma^{\tau\mu} + 2C^\kappa_{\beta\lambda}C^\lambda_{\alpha\kappa} + 2C^\mu_{\alpha\kappa}C^\nu_{\beta\lambda}\gamma_{\mu\nu}\gamma^{\kappa\lambda} + \\ & 2C^\lambda_{\beta\kappa}C^\mu_{\mu\nu}\gamma_{\alpha\lambda}\gamma^{\kappa\nu} + 2C^\lambda_{\alpha\kappa}C^\mu_{\mu\nu}\gamma_{\beta\lambda}\gamma^{\kappa\nu}\end{aligned} \tag{7}$$

the Ricci tensor of the hyper-surface.

In [6] particular spacetime coordinate transformations have been found, which reveal as symmetries of (3), (4), (5) the following transformations of the dependent variables $N, N_\alpha, \gamma_{\alpha\beta}$:

$$\tilde{N} = N,\; \tilde{N}_\alpha = \Lambda^\rho_\alpha\left(N_\rho + \gamma_{\rho\sigma}P^\sigma\right),\; \tilde{\gamma}_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu \gamma_{\alpha\beta} \tag{8}$$

where the matrix Λ and the triplet P^α must satisfy:

$$\Lambda^\alpha_\rho C^\rho_{\beta\gamma} = C^\alpha_{\mu\nu}\Lambda^\mu_\beta \Lambda^\nu_\gamma, \quad 2P^\mu C^\alpha_{\mu\nu}\Lambda^\nu_\beta = \dot{\Lambda}^\alpha_\beta \tag{9}$$

For all Bianchi Types, this system of equations admits solutions which contain three arbitrary functions of time plus several constants depending on the Automorphism group of each type. The three functions of time, are distributed among Λ and P (which also contains derivatives of these functions). So one can use this freedom either to simplify the form of the

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scale factor matrix or to set the shift vector to zero. The second action can always be taken, since, for every Bianchi type, all three functions appear in P^α.

In this work we adopt the latter point of view. When the shift has been set to zero, there is still a remaining "gauge" freedom consisting of all constant Λ^α_β (Automorphism group matrices). Indeed the system (9) accepts the solution $\Lambda^\alpha_\beta = constant$, $P^\alpha = \mathbf{0}$. The generators of the corresponding motions, induced in the space of dependent variables spanned by $\gamma_{\alpha\beta}'s$ (the lapse is given in terms of $\gamma_{\alpha\beta}$, $\dot{\gamma}_{\alpha\beta}$ by algebraically solving the quadratic constraint equation) $\tilde{\gamma}_{\mu\nu} = \Lambda^\alpha_\mu \, \Lambda^\beta_\nu \, \gamma_{\alpha\beta}$ are [10] :

$$X_{(I)} = \lambda^\rho_{(I)\alpha} \, \gamma_{\rho\beta} \, \frac{\partial}{\partial \gamma_{\alpha\beta}} \tag{10}$$

with λ satisfying:

$$\lambda^\alpha_{(I)\rho} \, C^\rho_{\beta\gamma} = \lambda^\rho_{(I)\beta} \, C^\alpha_{\rho\gamma} + \lambda^\rho_{(I)\gamma} \, C^\alpha_{\beta\rho}. \tag{11}$$

Now, these generators define a Lie algebra and each one of them induces, through its integral curves, a transformation on the configuration space spanned by the $\gamma_{\alpha\beta}$'s. If a generator is brought to its normal form (e.g. $\frac{\partial}{\partial z_i}$), then the Einstein equations, written in terms of the new dependent variables, will not explicitly involve z_i. They thus become a *first order* system in the function $\dot{z}_i$ [11]. If the above Lie algebra happens to be abelian, then all generators can be brought, to their normal form simultaneously. If this is not the case, we can diagonalize in one step the generators corresponding to any eventual abelian subgroup. The rest of the generators (not brought in their normal form) continue to define a symmetry of the reduced system of EFE's if the algebra of the $X_{(I)}$'s is solvable [12]. One can thus repeat the previous step, by choosing one of these remaining generators. This choice will of course depend upon the simplifications brought to the system at the previous level. Finally if the algebra does not contain any abelian subgroup, one can always choose one of the generators, bring it to its normal form, reduce the system and search for its symmetries (if there are any). Lastly, two further symmetries of (3), (4), (5) are also present and can be used in conjunction with the constant automorphisms: The time reparameterization $t \to f(t) + \alpha$, owing to the non-explicit appearance of time in these equations, and the scaling by a constant $\gamma_{\alpha\beta} \to \mu\gamma_{\alpha\beta}$ as can be straightforwardly verified. Their corresponding generators are:

$$Y_1 = \frac{1}{\dot{f}} \frac{\partial}{\partial t}, \quad Y_2 = \gamma_{\alpha\beta} \, \frac{\partial}{\partial \gamma_{\alpha\beta}} \tag{12}$$

These generators commute among themselves, as well as with the $X_{(I)}$'s, as it can be easily checked.

3. Application to Bianchi Type III

We are now going to apply the Method, previously discussed, to the case of Bianchi Type III. For this type the structures constants are [13]

$$\begin{array}{ll} C^1_{13} = -C^1_{31} = 1 & \\ C^\alpha_{\beta\gamma} = 0 & \quad for\, all\, other\, values\, of\, \alpha\beta\gamma \end{array} \tag{13}$$

Using these values in the defining relation (2) of the 1-forms σ^α_i we obtain

$$\sigma^\alpha_i = \begin{pmatrix} 0 & e^{-x} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \tag{14}$$

The corresponding vector fields ξ^i_α (satisfying $[\xi_\alpha,\xi_\beta]=C^\gamma_{\alpha\beta}\xi_\gamma$) with respect to which the Lie Derivative of the above 1-forms is zero are:

$$\xi_1=\partial_y \quad \xi_2=\partial_z \quad \xi_3=\partial_x+y\partial_y \tag{15}$$

The Time Depended A.I.D.'s are described by

$$\Lambda^\alpha_\beta=\begin{pmatrix} e^{-2P(t)} & 0 & x(t) \\ 0 & c_{22} & c_{23} \\ 0 & 0 & 1 \end{pmatrix} \tag{16}$$

$$P^\alpha=\left(x(t)\dot{P}(t)+\frac{1}{2}\dot{x}(t), P^2(t), \dot{P}(t)\right) \tag{17}$$

where $P(t), x(t)$ and $P^2(t)$ are arbitrary functions of time. As we have already remarked the three arbitrary functions appear in P^α and thus can be used to set the shift vector to zero.

The remaining symmetry of the EFE's is, consequently, described by the constant matrix:

$$M=\begin{pmatrix} e^{s_1} & 0 & s_4 \\ 0 & e^{s_2} & s_3 \\ 0 & 0 & 1 \end{pmatrix} \tag{18}$$

where the parametrization has been chosen so that the matrix becomes identity for the zero value of all parameters.

Thus the induced transformation on the scale factor matrix is $\tilde{\gamma}_{\alpha\beta}=M^\mu_\alpha M^\nu_\beta\gamma_{\mu\nu}$, which define a group of transformations G_r of dimension $r=dim(Aut(III))=4$. The four generators of the group are:

$$X_1=2\gamma_{11}\frac{\partial}{\partial\gamma_{11}}+\gamma_{12}\frac{\partial}{\partial\gamma_{12}}+\gamma_{13}\frac{\partial}{\partial\gamma_{13}} \tag{19}$$

$$X_2=\gamma_{12}\frac{\partial}{\partial\gamma_{12}}+2\gamma_{22}\frac{\partial}{\partial\gamma_{22}}+\gamma_{23}\frac{\partial}{\partial\gamma_{23}} \tag{20}$$

$$X_3=\gamma_{12}\frac{\partial}{\partial\gamma_{13}}+\gamma_{22}\frac{\partial}{\partial\gamma_{23}}+2\gamma_{23}\frac{\partial}{\partial\gamma_{33}} \tag{21}$$

$$X_4=\gamma_{11}\frac{\partial}{\partial\gamma_{13}}+\gamma_{12}\frac{\partial}{\partial\gamma_{23}}+2\gamma_{13}\frac{\partial}{\partial\gamma_{33}} \tag{22}$$

The algebra g_r that corresponds to the group G_r has the following table of commutators:

$$\begin{array}{lll} [X_1,X_2]=0, & [X_1,X_3]=0, & [X_1,X_4]=X_4, \\ [X_2,X_3]=X_3, & [X_2,X_4]=0, & [X_3,X_4]=0 \end{array} \tag{23}$$

As it is evident from the above commutators (23) the group is non-abelian, so we cannot diagonalize at the same time all the generators, but as it can easily verified, the group is solvable. Furthermore X_3, X_4, Y_2 generate an Abelian subgroup, and we can, therefore, bring them to their normal form simultaneously. The appropriate transformation of the dependent variables is:

$$\begin{cases} \gamma_{11} = e^{u_1+2\,u_6} \\ \gamma_{12} = e^{u_1+u_2+u_4+u_6} \\ \gamma_{13} = e^{u_1+u_6}\,\left(e^{u_6}\,u_3 + e^{u_2+u_4}\,u_5\right) \\ \gamma_{22} = e^{u_1+2\,u_4} \\ \gamma_{23} = e^{u_1+u_4}\,\left(e^{u_2+u_6}\,u_3 + e^{u_4}\,u_5\right) \\ \gamma_{33} = e^{u_1}\,\left(1 + e^{2\,u_6}\,{u_3}^2 + 2\,e^{u_2+u_4+u_6}\,u_3\,u_5 + e^{2\,u_4}\,{u_5}^2\right) \end{cases} \tag{24}$$

In these coordinates the generators Y_2, X_A assume the form:

$$\begin{array}{lll} Y_2 = \frac{\partial}{\partial u_1} & X_3 = \frac{\partial}{\partial u_3} & X_4 = \frac{\partial}{\partial u_5} \\ X_2 = \frac{\partial}{\partial u_4} - u_5 \frac{\partial}{\partial u_5} & X_1 = \frac{\partial}{\partial u_6} - u_3 \frac{\partial}{\partial u_3} & \end{array} \tag{25}$$

Except of the parametrization (24) there is also another one achieving the same result (25), which simply attributes a - sign to γ_{12} and therefore any solution later described will remain valid under this change.

Evidently, a first look at (24) gives the feeling that it would be hopeless even to write down the Einstein equation. However, the simple form of the generators (25) ensures us that these equations will be of first order in the functions $\dot{u}_1$, $\dot{u}_3$ and $\dot{u}_5$.

3.1. Description of the Solution Space

Before we begin solving the Einstein equations, a few comments for the possible values of the functions $u_i, i = 1, \ldots, 6$ will prove very useful.

The determinant of $\gamma_{\alpha\beta}$, is

$$det[\gamma_{\alpha\beta}] = e^{3\,u_1+2\,(u_4+u_6)}\,\left(1 - e^{2\,u_2}\right) \tag{26}$$

so we must have $u_2 < 0$.

The transformation from the γ' s to the u' s, becomes singular when $\gamma_{12} = 0$, since the function u_2 equals to

$$u_2 = \ln(|\gamma_{12}|) - \frac{\ln(\gamma_{11}\,\gamma_{22})}{2}. \tag{27}$$

So two cases are naturally arising, according to whether γ_{12} is different or equal to zero. If $\gamma_{12} \neq 0$ the two linear constraint equations, written in the new variables (24), give

$$E_1 = 0 \Rightarrow -e^{u_6}\,\left(e^{u_6}\,\dot{u}_3 + e^{u_2+u_4}\,\dot{u}_5\right) = 0 \tag{28}$$

$$E_2 = 0 \Rightarrow -\frac{1}{2}\,e^{u_4}\,\left(e^{u_2+u_6}\,\dot{u}_3 + e^{u_4}\,\dot{u}_5\right) = 0 \tag{29}$$

This system admits only the trivial solution, since the determinant of the 2x2 matrix formed by the coefficients of $\dot{u}_3, \dot{u}_5$ becomes zero only for the forbidden value $u_2 = 0$. We thus have

$$u_3 = k_3, \quad u_5 = k_5 \tag{30}$$

Now, these values of u_3, u_5 make γ_{13}, γ_{23} functionally dependent upon $\gamma_{11}, \gamma_{12}, \gamma_{22}$ (see (24)). It is thus possible to set these two components to zero by means of an appropriate constant automorphism.
In the case $\gamma_{12} = 0$ we can again bring simultaneously into normal form the corresponding X_3, X_4, Y_2 . The appropriate change of dependent variables is given by:

$$\gamma_{\alpha\beta} = \begin{pmatrix} e^{u_1+2\,u_6} & 0 & e^{u_1+2\,u_6}\,u_3 \\ 0 & e^{u_1+2\,u_5} & e^{u_1-u_4+u_5} \\ e^{u_1+2\,u_6}\,u_3 & e^{u_1-u_4+u_5} & e^{u_1}\left(1+e^{-2\,u_4}+e^{2\,u_6}\,u_3^2\right) \end{pmatrix} \tag{31}$$

In these variables all three linear constraint equations can be integrated, yielding:

$$E_1 = 0 \Rightarrow -e^{2\,u_6}\,\dot{u}_3 = 0 \Rightarrow u_3 = k_3 \tag{32}$$

$$E_2 = 0 \Rightarrow -\frac{1}{2}\,e^{-u_4+u_5}\,(\dot{u}_4 + \dot{u}_5) = 0 \Rightarrow u_5 = k_5 - u_4 \tag{33}$$

$$E_3 = 0 \Rightarrow \; -2\,e^{2\,u_4+2\,u_6}\,u_3\dot{u}_3 + \dot{u}_4 + \dot{u}_5 + 2\,e^{2\,u_4}\,\dot{u}_6 = 0 \Rightarrow u_6 = k_6 \tag{34}$$

Again, these values imply that a constant automorphism suffices to set the (13) and (23) components of the scale-factor matrix to zero, i.e. to put it into diagonal form. We have thus reached a first important conclusion, that is:

Without loss of generality, we can start our investigation of the solution space for Type III vacuum Bianchi Cosmology from a block-diagonal form of the scale-factor matrix (and, of course, zero shift)

$$\gamma_{\alpha\beta} = \begin{pmatrix} \gamma_{11} & \gamma_{12} & 0 \\ \gamma_{12} & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{pmatrix} \tag{35}$$

Note that this conclusion could have not been reached off mass-shell, due to the fact that the time-dependent Automorphism (16) does not contain the necessary two arbitrary functions of time in the (13) and (23) components (besides the fact that all the freedom in arbitrary functions of time has been used to set the shift to zero). As we have earlier remarked, since the algebra (23) is solvable, the remaining (reduced) generators X_1, X_2 (corresponding to diagonal constant automorphisms) as well as Y_2 continue to define a Lie-Point symmetry of the reduced EFE's and can thus be used for further integration of this system of equations.

3.1.1. Case I: $\gamma_{12} = 0$ The remaining (reduced) automorphism generators are

$$X_1 = 2\gamma_{11}\frac{\partial}{\partial\gamma_{11}}, \; X_2 = 2\gamma_{22}\frac{\partial}{\partial\gamma_{22}} \tag{36}$$

The appropriate change of dependent variables which brings these generators -along with Y_2- into normal form, is described by the following scale-factor matrix :

$$\gamma_{\alpha\beta} = \begin{pmatrix} e^{u_1+u_3} & 0 & 0 \\ 0 & e^{u_2+u_3} & 0 \\ 0 & 0 & e^{u_3} \end{pmatrix} \tag{37}$$

In these variables the first two linear constraint equations are identically satisfied, while the third reads $E_3 = 0 \Rightarrow -2\,\dot{u}_1 = 0 \Rightarrow u_1 = k_1$. Substituting this value of u_1 into the quadratic constraint equation E_0 we obtain the lapse function

$$N^2 = \frac{1}{16}\, e^{u_3}\, \dot{u}_3(2\dot{u}_2 + 3\dot{u}_3) \tag{38}$$

Now, substitution of $u_1 = k_1$ and the above value for the lapse N^2 into the spatial EFE's results in the single, independent equation :

$$(\dot{u}_2 + \dot{u}_3)(2\dot{u}_3\ddot{u}_2 - 2\dot{u}_2\ddot{u}_3 + 2\dot{u}_2^2\dot{u}_3 + 3\dot{u}_3^2 + 5\dot{u}_2\dot{u}_3^2) \tag{39}$$

This equation is, as expected from the theory, of the first order in $\dot{u}_2, \dot{u}_3$. Notice that this result could have not been reached had we chosen any particular time gauge, such as $N^2 = F(u_2, u_3, t)$. Not only u_2, u_3, t would appear in the Spatial EFE's, but also the number of independent such equations would have been increased to 2. This remark should not be taken as a negative view for complete gauge fixing, but rather as pointing to the fact that keeping the gauge freedom into the game helps manifesting the symmetries of the system and eventually solving the equations. Equation (39) is readily integrated, leading to two different space-times according to which parenthesis is set to zero. If the first is made to vanish, i.e. $u_2 = k_2 - u_3$, the ensuing line-element is the known (Type III) cosmological disguise of Minkowski space-time ([14]):

$$ds^2 = -\frac{1}{16}e^{u_3}\dot{u}_3^2\, dt^2 + \frac{1}{4}e^{u_3}\, dx^2 + e^{k_1+u_3-2x}\, dy^2 + e^{k_2}\, dz^2 \tag{40}$$

the constants being of course absorbable by the constant automorphisms and a shift in u_3. If the second parenthesis of (39) is set to zero, i.e. $u_2 = k_3 - \frac{3u_3}{2} + \ln(1 + k_2\, e^{\frac{u_3}{2}})$, we obtain an equivalent form of the Type III member of the known Ellis-MacCallum family of solutions ([8],[14]):

$$ds^2 = \kappa^2 \left(-\frac{e^{\frac{3u_3}{2}}\dot{u}_3^2}{4(e^{\frac{u_3}{2}} - 1)}\, dt^2 + e^{u_3}\, dx^2 + e^{u_3-2x}\, dy^2 + e^{\frac{-u_3}{2}}(e^{\frac{u_3}{2}} - 1)\, dz^2 \right) \tag{41}$$

where again we have used constant automorphisms and a shift of u_3 to take outside of the metric an overall constant. The properties of this line element were investigated in [16]. The interesting thing is that the metric (41) admits, except of (15), a fourth Killing vector field acting on the surfaces of simultaneity, namely

$$\xi_4 = -y\,\partial_x + \frac{e^{2x} - y^2}{2}\,\partial_y \tag{42}$$

There is thus a G_4 symmetry group acting (of course, multiply transitively) on each V_3 of this metric, with an algebra having the following table of (non-vanishing) commutators:

$$[\xi_1, \xi_3] = \xi_1,\ [\xi_1, \xi_4] = -\xi_3,\ [\xi_3, \xi_4] = \xi_4 \tag{43}$$

However, it is interesting to note that we have not imposed the extra symmetry from the beginning, but rather it emerged as a result of the investigation process.

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3.1.2. Case II: $\gamma_{12} \neq 0$ The remaining (reduced) automorphism generators are

$$X_1 = 2\gamma_{11}\frac{\partial}{\partial\gamma_{11}} + \gamma_{12}\frac{\partial}{\partial\gamma_{12}},\ X_2 = \gamma_{12}\frac{\partial}{\partial\gamma_{12}} + 2\gamma_{22}\frac{\partial}{\partial\gamma_{22}} \tag{44}$$

The appropriate change of dependent variables which brings these generators -along with Y_2- into normal form, is now given by:

$$\gamma_{\alpha\beta} = \begin{pmatrix} e^{u_1+2\,u_4} & e^{u_1+u_2+u_4} & 0 \\ e^{u_1+u_2+u_4} & e^{u_1+2\,u_2}\,u_3 & 0 \\ 0 & 0 & e^{u_1} \end{pmatrix} \tag{45}$$

The generators are now reduced to

$$Y_2 = \frac{\partial}{\partial u_1},\ X_2 = \frac{\partial}{\partial u_2},\ X_1 = \frac{\partial}{\partial u_4} \tag{46}$$

indicating that the system will be of first order in the derivatives of these variables. The remaining variable u_3 will enter, (along with $\dot{u}_3$, $\ddot{u}_3$) explicitly in the system and is therefore advisable (if not mandatory) to be used as the time parameter, i.e. to effect the change of time coordinate

$$t \to u_3(t) = s,\ u_1(t) \to u_1(t(s)),\ u_2(t) \to u_2(t(s)),\ u_4(t) \to u_4(t(s)). \tag{47}$$

This choice of time will of course be valid only if u_3 is not a constant. We are thus led to consider two cases according to the constancy or non-constancy of this variable.

The case $u_3 = k_3$

In this case the integration of EFE's is yields (see [16]) the line element:

$$ds^2 = -\lambda^2\,d\xi^2 + \frac{\xi^2}{4}\,dx^2 + e^{-2x}\xi^{4\lambda}\,dy^2 + \frac{\lambda-1}{2\lambda-1}\,dz^2 + 2e^{-x}\xi^{2\lambda}\,dy\,dz \tag{48}$$

where $0 < \lambda < \frac{1}{2}$.

This metric is an equivalent form of a solution originally given by Siklos [15] and reproduced in [14]. An overall multiplicative constant has been omitted from (87) since it admits the following Homothetic Killing vector field ($\mathcal{L}_H\,g_{AB} = \mu\,g_{AB}$)

$$H^A = \xi\frac{\partial}{\partial\xi} + (1-2\lambda)y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

It also admits three more Killing vector fields (except (15)) acting on space-time, namely

$$\begin{aligned}
v_1 &= e^{-\frac{x}{2\lambda}}\,\partial_\xi + \frac{2\,\lambda}{\xi}\,e^{-\frac{x}{2\lambda}}\,\partial_x \\
v_2 &= e^{-\frac{x}{2\lambda}}\,y\,\partial_\xi + \frac{2\,\lambda\,y}{\xi}\,e^{-\frac{x}{2\lambda}}\,y\,\partial_x + \frac{\lambda(\lambda-1)}{4\lambda-1}\,e^{\frac{4\lambda-1}{2\lambda}\,x}\,\xi^{-4\lambda+1}\,\partial_y \\
&\quad -\lambda\,e^{\frac{2\lambda-1}{2\lambda}\,x}\,\xi^{-2\lambda+1}\partial_z \\
v_3 &= e^{-\frac{x}{2\lambda}}\,z\,\partial_\xi + \frac{2\,\lambda\,z}{\xi}\,e^{-\frac{x}{2\lambda}}\partial_x - \lambda\,e^{\frac{2\lambda-1}{2\lambda}\,x}\,\xi^{-2\lambda+1}\,\partial_y \\
&\quad -\lambda\,(2\lambda-1)\,e^{-\frac{x}{2\lambda}}\,\xi\,\partial_z
\end{aligned}$$

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The first of these is null $v_1^A\, v_1^B\, g_{AB} = 0$ and covariantly constant $v^A_{1;B} = 0$, signaling that the metric is a pp-wave.

The case $u_3 \neq k_3$

The function u_3 is now a valid choice of time and $det[\gamma_{\alpha\beta}] = e^{3\,u_1+2\,(u_2+u_6)}\,(-1+s)$ implies the range $(1,+\infty)$ for the new time s. The only non-vanishing linear constraint equation $E_3 = 0$ yields

$$u_4 = \int \frac{\dot{u}_2}{2s-1}\, ds + k_4 \tag{49}$$

while the quadratic constraint equation $E_0 = 0$ gives the lapse

$$\begin{aligned}(N)^2 \;&=\; \frac{e^{u_1}}{4\,(1-2\,s)^2\,(-3+4\,s)}\,\big[2\,(2\,s-1)^2\,\dot{u}_1 + 3\,(2\,s-1)^2\,(s-1)\,\dot{u}_1^2 \\ &\quad +(4\,s-2)\,\dot{u}_2 + 8\,s\,(s-1)\,(2\,s-1)\,\dot{u}_1\,\dot{u}_2 + 4\,s\,(s-1)\,\dot{u}_2^2\big]\end{aligned} \tag{50}$$

If we insert these values $(N)^2\,, u_4$ into the spatial EFE's they become the following polynomial system of first order in $\dot{u}_1$, $\dot{u}_2$

$$\ddot{u}_1 = \begin{pmatrix}1\;\dot{u}_1\;\dot{u}_1^2\;\dot{u}_1^3\end{pmatrix}\,A_1\begin{pmatrix}1\\ \dot{u}_2\\ \dot{u}_2^2\\ \dot{u}_2^3\end{pmatrix}\,,\; \ddot{u}_2 = \begin{pmatrix}1\;\dot{u}_1\;\dot{u}_1^2\;\dot{u}_1^3\end{pmatrix}\,A_2\begin{pmatrix}1\\ \dot{u}_2\\ \dot{u}_2^2\\ \dot{u}_2^3\end{pmatrix} \tag{51}$$

$$A_1 = \begin{pmatrix} 0 & \frac{2}{4\,s^2-7\,s+3} & \frac{4\,s}{8\,s^2-10\,s+3} & 0 \\ \frac{1}{4\,s^2-7\,s+3} & 4 & \frac{8\,s\,(2\,s-3)\,(s-1)}{8\,s^2+10\,s-3} & 0 \\ \frac{2\,s-3}{4\,s-3} & -\frac{16\,s^2\,(s-1)}{8\,s^2-10\,s+3} & 0 & 0 \\ -\frac{6\,s\,(s-1)}{4\,s-3} & 0 & 0 & 0 \end{pmatrix} \tag{52}$$

$$A_2 = \begin{pmatrix} 0 & \frac{-8\,s+5}{8\,s^3-18\,s^2+13\,s-3} & \frac{24\,s^2-50\,s+18}{8\,s^2-10\,s+3} & \frac{8\,s\,(2\,s-3)(s-1)}{8\,s^2-10\,s+3} \\ \frac{-4\,s+2}{4\,s^2-7\,s+3} & \frac{12\,s}{-2\,s+3} & -\frac{16\,s^2\,(s-1)}{8\,s^2+10\,s-3} & 0 \\ \frac{-6\,s+3}{4\,s-3} & -\frac{6\,s\,(s-1)}{4\,s-3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{53}$$

Due to the form of A_1, A_2 (their components are rational functions of the time s), system (51) can be partially integrated with the help of the following Lie-Bäklund transformation

$$\begin{aligned}\dot{u}_1(s) \;&=\; \frac{(2\,s-3)\,\tan r(s) - 2\,s\,(8\,s^2-10\,s+3)\,\dot{r}(s)}{4\,s\,\sqrt{s-1}\,(4\,s-3)} \\ \dot{u}_2(s) \;&=\; \frac{2\,s-1}{8\,s(4\,s-3)\,\sqrt{(s-1)^3}}\,\big(2\,(-4\,s+3)\,\sqrt{s-1} + 3\,(s-1)\,\tan r(s) \\ &\quad +2\,s\,(s-1)\,(4\,s-3)\,\dot{r}(s)\big)\end{aligned} \tag{54}$$

resulting in the single, second order ODE for the variable $r(s)$

$$\begin{aligned}\ddot{r} &= \left(\tan r - \frac{\sqrt{s-1}}{2}\right)\dot{r}^2 + \frac{(-16\,s+6)\,\sqrt{s-1}+(5\,s-3)\,\tan r}{2\,s\,(4\,s-3)\,\sqrt{(s-1)}}\,\dot{r} \\ &+ \frac{-9\,(s-1)^2\,\tan^2 r + 18\,(s-1)^{3/2}\,\tan r + 4\,s\,(4\,s-3)}{8\,s^2\,(4\,s-3)^2\,(s-1)^{3/2}} \end{aligned} \tag{55}$$

This equation contains all the information concerning the unknown part of the solution space of the Type III vacuum Cosmology. Unfortunately, it does not posses any Lie-point symmetries that can be used to reduce its order and ultimately solve it. However, its form can be substantially simplified through the use of new dependent and independent variable $(\rho, u(\rho))$ according to $r(s) = \pm\arcsin\frac{u(\rho)}{\sqrt{\rho^2-1}}$, $s = \frac{3\,(\rho-1)}{3\,\rho-5}$, $\rho > \frac{5}{3}$ thereby obtaining the equation

$$\ddot{u} = \pm\frac{1-\dot{u}^2}{\sqrt{(6\,\rho-10)\,(\rho^2-u^2-1)}} \Rightarrow \ddot{u}^2 = \frac{(1-\dot{u}^2)^2}{(6\,\rho-10)\,(\rho^2-u^2-1)} \tag{56}$$

with the corresponding lapse

$$(N)^2 = \frac{\dot{u}^2-1}{8\,(3\,\rho-5)\,(\rho^2-u^2-1)}\,e^{u_1} \tag{57}$$

($\dot{u} = \frac{du}{d\rho}$) and the scale-factor matrix is given by (45) after insertion of (49), $u_3 = s = \frac{3\,(\rho-1)}{3\,\rho-5}$ and the transformations of u_1, u_2 that led to u. Independently of the way we have reached this result, one can check (through an algebraic computing facility such as Mathematica) that the line element thus described is indeed a solution of all the EFE's, provided of course (56) is satisfied. One can also check that it does not admit any Homothetic or null, covariantly constant vector field. Therefore, the two independent constants of the general solution to (56) along with a multiplicative constant will comprise the expected three essential constants of the general Type III vacuum Cosmology.

In [16] a partial solution of (56) was presented and in [17] the general solution of this equation was obtained. The corresponding line element is

$$ds^2 = -N^2\,d\rho^2 + \gamma_{\alpha\beta}\,\sigma^{\alpha}_{\ i}\,\sigma^{\beta}_{\ j}\,dx^i\,dx^j \tag{58}$$

where the scale factor matrix $\gamma_{\alpha\beta}(\rho)$ and the lapse function $N(\rho)$ are given by the equations:

$$(N)^2 = \frac{{u'}^2-1}{8\,(3\,\rho-5)\,(\rho^2-u^2-1)}\,e^{u_1} \quad and \quad \gamma_{\alpha\beta} = \begin{pmatrix} e^{u_1+2\,u_4} & e^{u_1+u_2+u_4} & 0 \\ e^{u_1+u_2+u_4} & \frac{3\,\rho-3}{3\,\rho-5}\,e^{u_1+2\,u_2} & 0 \\ 0 & 0 & e^{u_1} \end{pmatrix} \tag{59}$$

The functions u_1, u_2, u_4 satisfy

$$u_1' = \frac{-3\,u + (3\,\rho-1)\,u'}{2\,\left({u'}^2-1\right)}\,u'' \tag{60}$$

$$u_2' = \frac{(3\,\rho-1)\,\left(-1+{u'}^2+(3\,\rho-5)^2\,u\,u''-(3\,\rho-5)^2\,(\rho-1)\,u'\,u''\right)}{4\,(3\,\rho-5)^2\,(\rho-1)\,\left({u'}^2-1\right)} \tag{61}$$

$$u_4' = \frac{3\,\rho-5}{3\,\rho-1}\,u_2' \tag{62}$$

and the function $u(\rho)$ obeys a second order differential equation, of the form:

$$\ddot{u}^2 = \frac{(-1+\dot{u}^2)^2}{(\kappa+\lambda\,\rho)\,(\rho^2-u^2-1)} \quad \kappa=-10,\ \lambda=6 \tag{63}$$

In order to solve (63), for arbitrary constants $(\kappa,\ \lambda)$, we apply the contact transformation:

$$\begin{aligned} u(\rho) &= -\tfrac{8}{\lambda}\,y(x)+\tfrac{4\,(2x-1)}{\lambda}\,y'(x) & \rho &= -\tfrac{\kappa}{\lambda}+\tfrac{4}{\lambda}\,y'(x) \\ u'(\rho) &= 2\,x-1 & u''(\rho) &= \tfrac{\lambda}{2\,y''(x)} \end{aligned} \tag{64}$$

which reduces it to

$$x^2\,(x-1)^2\,y''^2 = -4y'\,(x\,y'-y)^2+4\,y'^2\,(x\,y'-y)-\frac{\kappa}{2}\,y'^2+\frac{\kappa^2-\lambda^2}{16}\,y' \tag{65}$$

This equation is a special form of the equation SD-Ia, appearing in [18], where a classification of second order second degree ordinary differential equations was performed. The general solution of (65) is obtained with the help of the sixth Painlevé transcendent $P := \mathbf{P_{VI}}(\alpha,\beta,\gamma,\delta)$ and reads:

$$\begin{aligned} y \;=\; & \frac{x^2\,(x-1)^2}{4\,P\,(P-1)(P-x)}\left(P'-\frac{P\,(P-1)}{x\,(x-1)}\right)^2 \\ & +\frac{1}{8}\,(1\pm\sqrt{2\,\alpha})^2\,(1-2\,P)-\frac{\beta}{4}\left(1-\frac{2\,x}{P}\right) \\ & -\frac{\gamma}{4}\left(1-\frac{2\,(x-1)}{P-1}\right)+\left(\frac{1}{8}-\frac{\delta}{4}\right)\left(1-\frac{2\,x\,(P-1)}{P-x}\right) \end{aligned} \tag{66}$$

where the sixth Painlevé transcendent $P := \mathbf{P_{VI}}(\alpha,\beta,\gamma,\delta)$ is defined by the ODE:

$$\begin{aligned} P'' \;=\; & \frac{1}{2}\left(\frac{1}{-1+P}+\frac{1}{P}+\frac{1}{-x+P}\right)P'^2-\left(\frac{1}{-1+x}+\frac{1}{x}+\frac{1}{-x+P}\right)P' \\ & +\frac{(-1+P)\;P\;(-x+P)}{(-1+x)^2\,x^2}\left(\alpha+\frac{(-1+x)\;\gamma}{(-1+P)^2}+\frac{x\,\beta}{P^2}+\frac{(-1+x)\;x\,\delta}{(-x+P)^2}\right) \end{aligned} \tag{67}$$

The values of the parameters $(\alpha,\beta,\gamma,\delta)$ of the Painlevé transcendent, can be obtained from the solution of the following system:

$$\alpha-\beta+\gamma-\delta\pm\sqrt{2\,\alpha}+1 \;=\; -\frac{\kappa}{2} \tag{68}$$

$$(\beta+\gamma)\left(\alpha+\delta\pm\sqrt{2\,\alpha}\right) \;=\; 0 \tag{69}$$

$$(\gamma-\beta)\left(\alpha-\delta\pm\sqrt{2\,\alpha}+1\right)+\frac{1}{4}\left(\alpha-\beta-\gamma+\delta\pm\sqrt{2\,\alpha}\right)^2 \;=\; \frac{\kappa^2-\lambda^2}{16} \tag{70}$$

$$\frac{1}{4}\,(\gamma-\beta)\left(\alpha+\delta\pm\sqrt{2\,\alpha}\right)^2+\frac{1}{4}\,(\beta+\gamma)^2\left(\alpha-\delta\pm\sqrt{2\,\alpha}+1\right) \;=\; 0 \tag{71}$$

Plugging in (68) the values of $\kappa=-10, \lambda=6$ for Type III, we have twelve solutions, of this system. The eight of them correspond to the "$-\sqrt{2\,\alpha}$" case and the rest four to the "$+\sqrt{2\,\alpha}$" case.

Case I: $-\sqrt{2\,\alpha}$

$$\alpha = 0,\ \beta = 0,\ \gamma = 0,\ \delta = -4 \tag{72}$$
$$\alpha = 0,\ \beta = -2,\ \gamma = 2,\ \delta = 0 \tag{73}$$
$$\alpha = 2,\ \beta = 0,\ \gamma = 0,\ \delta = -4 \tag{74}$$
$$\alpha = 2,\ \beta = -2,\ \gamma = 2,\ \delta = 0 \tag{75}$$
$$\alpha = 8,\ \beta = 0,\ \gamma = 0,\ \delta = 0 \tag{76}$$
$$\alpha = \frac{1}{2},\ \beta = -\frac{9}{2},\ \gamma = \frac{1}{2},\ \delta = \frac{1}{2} \tag{77}$$
$$\alpha = \frac{1}{2},\ \beta = -\frac{1}{2},\ \gamma = \frac{9}{2},\ \delta = \frac{1}{2} \tag{78}$$
$$\alpha = \frac{9}{2},\ \beta = -\frac{1}{2},\ \gamma = \frac{1}{2},\ \delta = -\frac{3}{2} \tag{79}$$

Case II: $+\sqrt{2\,\alpha}$

$$\alpha = 0,\ \beta = 0,\ \gamma = 0,\ \delta = -4 \tag{80}$$
$$\alpha = 0,\ \beta = -2,\ \gamma = 2,\ \delta = 0 \tag{81}$$
$$\alpha = 2,\ \beta = 0,\ \gamma = 0,\ \delta = 0 \tag{82}$$
$$\alpha = \frac{1}{2},\ \beta = -\frac{1}{2},\ \gamma = \frac{1}{2},\ \delta = -\frac{3}{2} \tag{83}$$

Particular solutions of (67) give raise to Kinnersley vacuum solution [19] and to a line element with Euclidean signature [17].

3.2. Preview for other Bianchi Types

The method described in the previous sections can be applied to other Types as well. The general pattern is s similar to that of Type III: The pp-wave solutions (for Types admitting such geometries) occupy one part of the solution space, the other known solutions reside on another part, and the unknown part of the solution space is always described by the same ODE (63), with different parameters κ, λ for each Type. As indicative examples we give the form of the ODE for Types IV and VII_h:

Type IV

$$\ddot{u}^2 = \frac{(-1+\dot{u}^2)^2}{(\kappa+\lambda\,\rho)\,(\rho^2-u^2-1)} \quad \kappa = -6,\ \lambda = 6 \tag{84}$$

Type $\mathbf{VII_h}$

$$\ddot{u}^2 = \frac{(-1+\dot{u}^2)^2}{(\kappa+\lambda\,\rho)\,(\rho^2-u^2-1)} \quad \kappa = -6+\tfrac{4}{h^2},\ \lambda = 6 \tag{85}$$

and of course

Type III

$$\ddot{u}^2 = \frac{(-1+\dot{u}^2)^2}{(\kappa+\lambda\,\rho)\,(\rho^2-u^2-1)} \quad \kappa = -10,\ \lambda = 6 \tag{86}$$

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4. Discussion

We have seen how the Automorphisms of Bianchi Type Geometries can be used as symmetries of the corresponding EFE's, in order to reduce the degree of these equations, and ultimately integrate them in full. For the case of Type III the solution space is seen to be naturally partitioned in three disconnected components: One occupied by the Type III member of the known Ellis-MacCallum family, another described by the non-linear equation equation which is fully integrated and a piece occupied by the known Siklos solution, an equivalent form of which is

$$ds^2 = -\lambda^2\, d\xi^2 + \frac{\xi^2}{4}\, dx^2 + e^{-2x}\xi^{4\lambda}\, dy^2 + \frac{\lambda - 1}{2\lambda - 1}\, dz^2 + 2e^{-x}\xi^{2\lambda}\, dy\, dz \tag{87}$$

This line element obtains from the general case, for the particular value of $u_3 := \frac{\gamma_{11}\,\gamma_{22}}{\gamma_{12}^2} = cont$; u_3 in this paper is, by a choice of time "gauge", taken to be the term $\frac{3\,\rho-3}{3\,\rho-5}$ in $\gamma_{\alpha\beta}$.

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