Solution of XXZ and XYZ spin chains with boundaries by separation of variables

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THÈSE

Pour obtenir le grade de

Docteur de l'Université de Bourgogne

Discipline :

Mathématiques

par

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Solutions de chaînes de spin XXZ et XYZ avec bords par la séparation des variables

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Soutenue le 11 Decembre 2014 devant le jury composé de :

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Solutions de chaînes de spin XXZ et XYZ avec bords par la séparation des variables

Résumé

Dans cette thèse nous donnons une solution des chaînes quantiques de spin-$1/2$ XXZ et XYZ ouvertes avec les termes de bord intégrables les plus généraux. En utilisant la méthode de la Séparation des Variable (SoV), à la Sklyanin, on est capable, dans le cas inhomogène, de construire l'ensemble complet des états propres et des valeurs propres associés. La caractérisation de ces quantités est faite par un système maximal de N équations quadratiques, où N est la taille du système. Des méthodes differentes, comme l'ansatz de Bethe algébrique (ABA) ou autres généralisations de l'ansatz de Bethe, ont été utilisés dans le passé pour résoudre ces problèmes. Aucune méthode a pu effectivement reproduire l'ensemble complet des états propres et valeur propres dans le cas de conditions au bord les plus génériques. Une expression, sous forme d'un déterminant à la Vandermonde, pour les produits scalaires entre les états en représentation de SoV est aussi obtenue. La formule pour les produits scalaires représente la première étape pour approcher le problème relié au calcul des facteurs de forme et fonctions de corrélations.

Abstract

In this thesis we give accounts on the solution of the open XXZ and XYZ quantum spin-$1/2$ chains with the most generic integrable boundary terms. By using the the Separation of Variables method (SoV), due to Sklyanin, we are able, in the inhomogeneous case, to build the complete set of eigenstates and the associated eigenvalues. The characterization of these quantities is made through a maximal system of N quadratic equations, where N is the size of the chain. Different methods, like the Algebraic Bethe ansatz (ABA) or other generalized Bethe ansatz techniques, have been used, in the past, in order to tackle these problems. None of them resulted effective in the reproduction of the full set of eigenstates and eigenvalues in the case of most general boundary conditions. A Vandermonde determinant formula for the scalar products of SoV states is obtained as well. The scalar product formula represents a first step towards the calculation of form factors and correlation functions.

REMERCIEMENTS

La première personne que je voudrai remercier c'est surement mon directeur de thèse, Prof. Nikolai Kitanine (ou simplement Kolya). Le calme et la patience qu'il a montrés, avec son grand professionalisme, sa passion et son expérience, ont surement été des éléments fondamentaux pendant ces trois ans. Grâce à lui, j'ai pu recevoir mon initiation à la recherche et d'une certaine façon j'ai appris un " métier ", qui sera surement utile dans mon future professionnel et, pourquoi pas, dans ma vie.

De la même façon, je considère fondamental le précieux travail de collaboration auquel j'ai eu le plaisir et l'honneur de participer avec Kolya et Dr. Giuliano Niccoli. Ce dernier peut pleinement être considéré un co-encadrant de mes recherches et il m'a permis de contribuer à une partie de ses projets et dans le même temps m'a donné une occasion pour apprendre des méthodes et des techniques très importantes pour mon domaine. Je peux sans doutes dire que sans Kolya et Giuliano ce projet n'aurait jamais pu aboutir.

Ensuite je voudrai remercier tout le labo de Maths de Dijon (IMB) et ses membres. Dès que j'ai eu la possibilité de discuter et de travailler avec plusieurs d'entre eux, j'ai toujours relevé beaucoup de disponibilité et engagement dans les domaines respectifs d'expertise, soit scientifique, soit pédagogique, soit administratif et humain. Autant que membres du Labo, un particulier merci à tout le staff du secrétariat, en particulier Anissa, Caroline, Ibtissam et Magali et aux personnels techniques.

Une autre composante de l'IMB qui a été très importante pour moi, c'est surement le corps des doctorants et la DMD. J'ai passé beaucoup des bons temps avec plusieurs d'entre eux et j'ai pu partager des moments précieux sous un point de vue professionnel et humain aussi. En particulier mes remerciements vont aux plus anciens : Martin (ainsi que mon maitre d'escalade), Sébastien (ainsi que mon enseignant de français et Verlan), Amhed (le sage), Charlie (le dealer de Chablis), Bruno, Johan (le maitre de conf avec l'âme de doctorant). Mais il ne faut pas oublier les " nouveaux " non plus, en particulier Ben, Bruno (l'autre), Taron, Michael et Adriana (Andres aussi eh). Du coté non professionnel je voudrai remercier mes parents qui m'ont toujours supporté dans mes choix et parcours d'études et de vie. De même je voudrai remercier aussi le reste de ma famille, grands parents en tête, et j'espère qu'ils peuvent être fier de ce que j'ai accompli jusqu'à maintenant.

Après le support reçu en famille il faut sans doute reconnaître le support reçu par les amis, les vieux et les nouveaux, qui ont été, sont et seront une partie très important de ma vie : Ale, Ale Lupo, Andre, Dario, Chiara&Manu (sans votre miel je n'aurai pas eu la force de rédiger jusqu'à la fin), Lore, Luca, Vincio&Nina, Helbert et tous les autres que j'ai oublié.

La dernière personne que je voudrai remercier avec tout mon cœur est Anett pour tout bon moment passé ensemble et pour la grande dose de patience qu'elle a démontré dans ces derniers mois de stress pré-soutenance et pour m'avoir aidé à la préparation du grand jour.

Cette thèse a été possible grâce à un financement de la Région Bourgogne, je suis très reconnaissant pour l'opportunité reçue.

Dijon, 05/12/2014
Simone Faldella

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INTRODUCTION

" . . . The ripplings are seen in calm weather approaching from a distance, and in the night their noise is heard a considerable time before they come near. They beat against the sides of a ship with great violence, and pass on, the spray sometimes coming on deck; and by carrying out oceanographic measurements from a ship, a small boat could not always resist the turbulence of these remarkable ripplings . . . "

Matthew Fontaine Maury

The Andaman sea is a basin of the Indian ocean delimited by the Malay Peninsula at the west and the Nicobar and the Andaman Islands at the east. On the south-east, between Malaysia and the Indonesian island of Sumatra, there is the Strait of Malacca. This strait constitutes a very important passage for commercial expeditions, being a link between the Indian and the Pacific oceans and the relative economies. For centuries seafarers and adventurers of all-time have crossed this corridor of water on their journeys between India and the Far East and they used to note on their wet sea-dogs logbooks all the observations and interesting information about the travel, the ship, the ocean, the weather and other aspects of the life on the sea. A peculiar phenomenon happened to be recorded in the area: very special patterns of wave trains and superficial roughnesses joint with unexpected tremblings on the deck of the ships. This kind of wave phenomena were addressed in different ways during time, in particular, they became somehow famous, to sailors used to travel in the area, as bands of choppy water or "ripplings", after the description given by Matthew Fontaine Maury in his book *The physical geography of the sea, and its meteorology (1861)* [94]. Maury was a Commodore officer of the U.S. navy and

Figure 1: Detail from *Isole Dell' Indie. . .* by *V.Coronelli*, 1861, Venice. Image source: `http://www.bergbook.com`

much more, he was as well an astronomer, oceanographer, cartographer, educator and he had many other passions and hobbies both in the scientific and humanistic direction . He was frequently nicknamed as the *Pathfinder of the Seas* and he is considered as one of the fathers of modern oceanography, being the book cited above among the first treaties of oceanography ever published. The epigraph opening this introductory chapter reproduces a little passage of Maury's book, where he describes this unusual event observed in the Strait of Malacca. In order to get a more rigorous scientific analysis and, possibly, explanation for the phenomenon accounted by Maury we have to wait a little bit more than a century and some giant steps in technological development.

Indeed, a big contribution to the description of these *ripplings* in the Andaman Sea came from another type of navigation, a navigation not at all tied to the ground and neither to the tropical waters of that south eastern sea: the space program of the 60's and 70's and in particular the joint U.S.- Soviet space flight known as Apollo-Soyuz, conducted in July 1975 [126]. Aside from the other research projects carried in the ASTP (Apollo-Soyuz Test Project), the vigilant eye of the astronaut Vance Brand caught on his 35-mm camera an important snapshot, reproduced in Fig. 2, from the Apollo capsule after undocking from the Soyuz. The photograph depicts the wave patterns that were first described by Maury 114 years before. With the words of the spacecraft crew

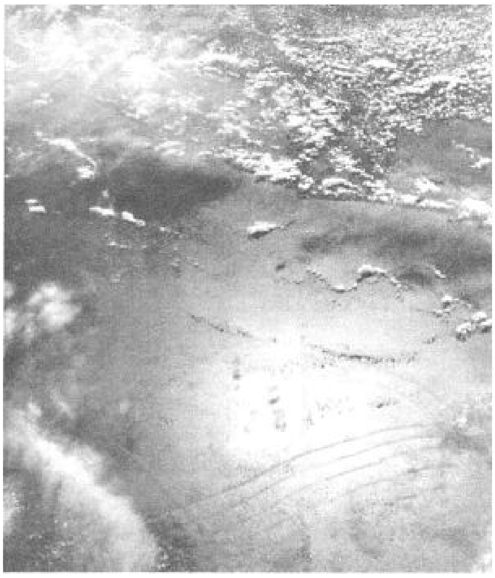

Figure 2: Sea-surface expressions of solitons in the Andaman Sea, observed and photographed by V.Brand, 23 July 1975, from the Apollo spacecraft, after undocking from the USSR's Soyuz.

> As we orbited west of the Andaman Sea, the Sun was just right to give a good glint from the ocean surface. I saw what looked like huge internal waves and clicked off three shots with the 35-mm camera. We were all out of 70-mm film by then.

When this photograph came to the attention of Dr. A.R. Osborne and collaborators, with the fact that internal waves * had already been associated to the superficial ripplings by oceanographic measurements [107], it gave him the clue to explain the observations and data collected from the Exxon exploratory drilling platforms in the Andaman Sea. In his classic paper [106] he showed and explained how the superficial effects observed were in fact related to traveling internal *solitons* with propagation rates around 2.2 m/s and amplitudes of 60 m. This discovery was quite a blast for the experts of the time and turned out to be a very precious information for the Exxon company which used to have floating oil-drilling platforms in the Andaman Sea. These huge platforms were disconnected from their moorings and carried for several kilometers before the waves went past them and they were left floating far away from where they started [65].

This interesting story of a peculiar scientific discovery in the field of oceanography turns out to be related to what can be considered as our extended domain. This connection is evident after a closer look

∗. An internal wave is a gravity wave phenomenon occurring under the water surface, between layers having different densities, caused, for example, by different salinities and/or temperatures.

to the research work carried by Osborne and collaborators, which was heavily based on the most recent mathematical developments of the XXth century in the theory of nonlinear PDEs and more generically in the theory of classical integrable systems. The word solitons above is a hint of how such a beautiful and purely theoretical construction can have found, among others, a direct practical application to ocean waves dynamics. In order to better appreciate the whole concept of integrability and in order to give a more complete historical review we need to take some steps backwards and take the time to cite and homage the works of the Gotha of our interesting domain.

The study of integrability has been a long and difficult journey in the history of mathematical physics and it has not yet come to an end. In classical mechanics the concept of integrable system has been largely investigated and reasonably understood. The definition of the integrability *á la Liouville* can be considered as the accepted standardized form of this concept and its core-idea can be summarized as

A Hamiltonian system is said integrable when there exist a maximal set of algebraically independent, globally defined Poisson commuting invariant quantities..

The definition above can be considered the more *physical* one and the term "invariants" is translated in different ways according to the degree of math deepness one is seeking for, i.e. physicists would be more acquainted to refer to them as integrals, or constants, of motion. The Liouville's statement is very clear and powerful, implying that the differential equations, describing the time evolution of the system, can be solved by quadratures. In order to get to this conclusion one has to consider the whole Hamilton-Jacobi construction and the concept of *action-angle* variables, see for example [6], method that, anyway, heavily relies on the statement by Liouville. From a practical point of view, this definition plays a further important role since it catalogs the different type of models in two big classes with manifestly different physical behaviors.

Classical mechanics, more generally, can be considered as the prototype for all physics' theories, since it is a very strong and powerful description of reality and it is tailored in such a beautiful, compact and elegant formalism. Maybe for this reason or other motivations, the scientific community, after the *heroic age* of Liouville,Hamilton and Jacobi, somehow put its efforts elsewhere and the classical integrability had not been investigated or extended for a while. For a genuine renewed interest to arise we have to wait for the 20th century. Although, the source for this scientific revival can be rooted to the first half of the 19th century and to the works of the scottish civil engineer John Scott Russell. In 1834, Russell was riding on horseback by the side of the Union Canal near Edinburgh when he made his greatest discovery. The results of the observations of his *wave of translations* were reported in [111] and we reproduce here an essential passage

> " I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon.

The work of Russell paved the way for the birth of *soliton theory* and the remarkable extension of integrability through the so-called inverse scattering method. Anyway, this discovery remained unnoticed on some dusty proceeding publication of nautical engineering for almost sixty years. In 1895, two Dutch mathematicians Diederik Kortweg and Gustav de Vries published their historic paper [78], in which they defined an evolution equation governing long * one-dimensional, small amplitude, surface gravity waves propagating in a shallow channel of water

$$u_t + 6uu_x + u_{xxx} = 0,$$

now known as *KdV equation*. The solitary wave of Russell, and somehow of Maury as well, finally found a mathematically rigorous background. Indeed, one of the solutions to this differential equation happened to coincide, because of its $sech^2$-shaped profile and further properties, with the standing wave observed by the scottish engineer.

The general interest on the KdV equation and, in general, on nonlinear equations was rediscovered only in the mid 50's of the 20th century. An important work, which paved the way to further studies on the subject, was due to Enrico Fermi, John Pasta and Stan Ulam (FPU), [50]. Their study was about a *thermalization* problem in solid-state physics; a simple model for a solid material is a chain of N identical particles of mass m connected to nearest neighbours by identical springs. Briefly, by inserting generic perturbations of the equilibrium state they couldn't observe any equipartition of energy among the whole set of accessible levels on the long time scale, as one would have expected, but rather a sort of recurrence of the energy. This was cured by the insertion of a non-linear Hooke's term in the Newton's law governing their model which provided the link with the KdV equation. Later in 1965, Zabusky and Kruskal tried to explain this phenomenon studying a continuum version of the FPU-problem [136]. They essentially reduced the continuum FPU problem, under certain conditions, to the KdV equation, which, then, implied the presence of the solitary waves solution, now re-baptized *solitons*, established by Kortweg and de Vries and observed for first by Russell. Their solitons had remarkable non-linear properties such as the complete elasticity of scattering processes and the subsequent preservation of shape and speed after collisions. This work did stimulate the search of an analytic explanation for what Kruskal and Zabusky discovered by means of numerical techniques.

This huge wave of renewed enthusiasm led physicists and mathematicians to develop new analytical methods to solve nonlinear PDEs given certain initial-time data. The first solid result was due to Gardner, Greene, Kruskal and Miura (GGKM) who developed a mathematical machinery to solve generically the KdV equation [56]. In fact, even if the KdV equation and its solutions were already known by the end of 19th century, there was no exact method to find them. GGKM created the first realization of what is now known as *inverse scattering method* (ISM). A tool that has then been generalized and applied to a huge number of nonlinear problems. We know that many physical problems are modeled by nonlinear partial differential equations, but, unfortunately, the Fourier method, a true math's cornerstone for the study of linear systems, cannot be used to solve them. In fact, as mentioned above, before the work of GGKM, there were no general methods for solving a nonlinear PDE with smooth and well-behaved initial conditions. In this perspective, the nature of the ISM can be understood as a "non-linear Fourier transform" [2], definition that seems quite appropriate once the main steps of the method are investigated, see Fig. 3. The technique developed by GGKM got further generalized by the work of Peter Lax [83], who, more inherently, introduced the concept of the, now known as, *Lax pairs* and auxiliary problem. His results can be summarized in the following theorem.

Theorem (Lax '68). *Given an evolution equation*

$$u_t = N(u)$$

*. with long/short waves it is meant that the ratio between wave length and depth of channel is greater/smaller than 1.

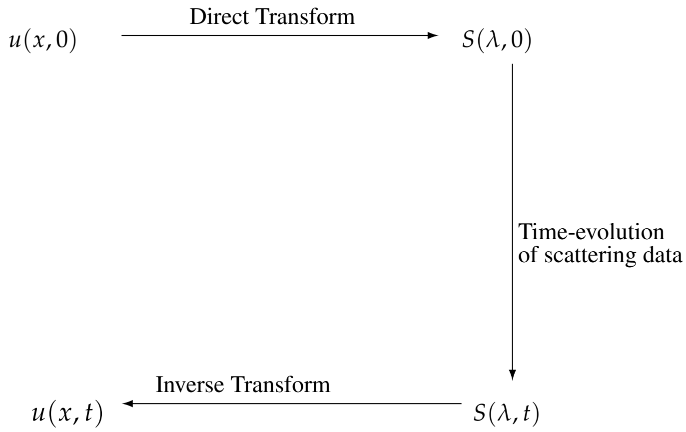

Figure 3: Inverse Scattering Transform Method.

that can be expressed in Lax representation

$$L_t + LM - ML = L_t + [L, M] = 0,$$

where L and M, the Lax pair, are self-adjoint matrices and $[L, M]$ *their commutator, and if*

$$L\psi = \lambda\psi,$$

then for real λ*,* $\lambda_t = 0$ *and the vector-valued function* ψ *evolves according to equation*

$$\psi_t = M\psi.$$

The amazing work of Lax stimulated even more an already excited community and led people to investigate other non-linear models, out of the Lax-KdV hierarchy, in order to show that the method was general enough and it was not just a KdV fluke. Indeed, few years later Zakharov and Shabat (ZS) proved that the method could be used to solve the following equation [112]

$$iu_t + u_{xx} + \hat{k}u^2u^* = 0,$$

the well-known *non-linear Schrödinger equation*. The door was then wide opened, and people started developing the technique for a huge set of non-linear PDE's. Wadati [131] gave the solution for the modified KdV equation (mKdV)

$$u_t - 6u^2u_x + u_{xxx} = 0,$$

and Ablowitz, Kaup, Newell and Segur (AKNS) [1] solved the *Sine-Gordon equation*

$$u_{xt} = \sin u.$$

The methods elaborated by Lax, ZS and AKNS led to a far more standardized formalism and the conditions of applicability boiled down to the eventual identification of a very simple representation for the problem under study. Indeed, it is sufficient to reproduce a certain non-linear PDE as a consistency condition of the linear system

$$\begin{cases} \Psi_x = U(x,t,\lambda)\Psi \\ \Psi_t = V(x,t,\lambda)\Psi \end{cases},$$

in order to label it solvable, in the context of ISM. The λ-dependency appearing above is usually called *spectral parameter* since its role is basically the same played by the lambda eigenvalue in the Lax theorem. The consistency condition, or more geometrically *zero curvature condition*, is a constraint on the crossed derivation which explicitly reads

$$U_t - V_x + [U, V] = 0.$$

All these developments cleared the view of what people was actually dealing with. From these works and the recognition of a true Hamiltonian structure behind such non-linear differential equations [137] led to a new understanding and re-evaluation of the concept of integrability. The definition of Liouville now could be implemented in the ISM language. An actual correspondence was proved to be valid between ISM approach and the usual construction of Hamiltonian mechanics. The introduction of the classical r-matrices, solutions of the *classical Yang-Baxter equation* (CYBE), in the formalism, provided all the tools to actually reproduce the Poisson structure of Hamiltonian system, see [44] for a complete review.

These big advances in the theory, the huge enthusiasm and the amount of nice results obtained in the study of such classical models proved to be a fertile ground to make a step further, to leave the well-studied classical world to enter the far more obscure quantum domain. But, before discussing the beauty and the problematics of what now it is known as, roughly speaking, *quantum integrability*, we need to take some steps back.

The beginning of this story is generally set in 1931, when Hans Bethe developed a method to solve the isotropic Heisenberg spin chain [21]. He managed to retrieve the spectrum of the model by setting up a technique that goes, nowadays, under the name of *coordinate Bethe ansatz* (CBA). For the first time the concepts of Bethe ansatz equations (BAE) and Bethe roots were actively exploited in order to define the spectrum of a quantum model. The impact of this work in the later history of theoretical physics and quantum mechanics was huge, and still nowadays the essentials of the method are far to be outdated. The work of Bethe on a one-dimensional quantum model stimulated the community to investigate other systems such as, Bose gas [88, 87] (i.e. the quantum variant of the non-linear Schrödinger equation), Hubbard model [90], XXZ spin chain [105] etc. Most importantly, from a physicist's perspective, Bethe ansatz technique found numerous physical applications : the Kondo problem [4, 130] (metals with magnetic impurities, quantum dots), the already cited Hubbard model (superconductivity), in nonlinear optics [110] as well. Bethe's method was then developed, extended and applied in many situations and the main actors of the scene can be recognized in: Hulthen, Yang and Yang, Lieb, Sutherland, Baxter Gaudin and others, see [57], [89] and [133]. In particular, it emerged the link between one-dimensional quantum problems and two-dimensional models of statistical mechanics, of which Baxter can be considered the biggest pioneer and expert, see for example [17]. The idea of quantum integrability was starting to emerge from the huge pile of works at hand, but a first consecration of its actual importance definitely came with the set up of the machinery known as *quantum inverse scattering method* (QISM) at the end of the 70's. The QISM formalism, as pointed out in the introduction of [117], can be considered as a confluent stream of different traditions: the study of one-dimensional quantum systems and two-dimensional statistical mechanical models, introduced few lines above, and the fresh dicovery of the ISM formalism for classical systems that we had the chance to briefly discuss in the previous part of this introduction. To be fair one should also recognize the contribution coming from the *factorizable S-matrices theory* pioneered by A.B. Zamolodchikov and Al.B. Zamolodchikov [138]. The QISM framework reached a new level of maturity when an algebraic approach was developed by the so-called *Leningrad school*, i.e., most remarkably, by Faddeev, Takhtadzan, Sklyanin, Kulish, Izergin, Korepin etc. Its roots were established in the seminal papers [42], [119] and [120]. Important contributions came as well from the american school of the Fermilab, headed by Thacker, Creamer and Wilkinson. The theory developed by these researchers brought a whole new level of rigor and beauty in the Bethe ansatz approach and permitted to clearly define which structure could identify certain systems as integrable. Briefly speaking, the Yang-Baxter equation, as already known for some models, turned out to be a central object in the definition of the QISM algebra and a good candidate as an integrability discriminant.

The actual situation about a shared definition of quantum integrability is still foggy, since it doesn't seem to exist yet a universal definition that could please and satisfy all the various sectors of the quantum theory: fundamental models from QISM, free theories, QFT's, long-range interacting models, Richardson-Gaudin type models etc. Although the subject doesn't seem to be actively investigated as a primary research topic, everybody in the community talk about it and is, of course, highly appealed by the possibility to give a general answer. In a recent paper, Caux and Mossel [27] (and Wiegert before them [132]) try to address the problem by reviewing all the definitions' attempts that have been formulated up to now and trying to generate a more general one. We don't want to review or comment their definition but they touch some interesting points. We learn that a good definition should have some basic features like: unambiguity, discrimination of all models into distinct classes with distinct physical behaviours. For example, given the Liouville definition of classical integrability, one would be tempted to extend such a definition to the quantum world by labeling a quantum system integrable if it possesses a maximal set of independent observables $\mathcal{Q}_\alpha$ with $\alpha = 1, \ldots, \dim(\mathcal{H})$. They tag this definition as *naive* as all quantum systems associated with a finite-dimensional Hilbert space would fall in this category since it is always possible to build a maximal set of commuting observables out of the projectors on each eigenstate. Then, the definition wouldn't be effective at all for cataloguing purposes. An other unsatisfying possibility would be to consider a quantum system integrable if it is exactly solvable, i.e. one can construct its full set of eigenstates explicitly. But also this definition turns out to be quite naive and inefficient and so not eligible to be a good candidate. Among other definitions, one is based on the diffractive nature of the scattering processes, another on the energy-level statistics and other ones on other aspects that would allegedly look like consequences of integrability rather than definitions. The lack of a clear and concise explanation of the concept of quantum integrability was one of the reasons which moved Sklyanin to try to tackle the problem and develop a method, always classifiable inside the QISM framework but alternative to *algebraic Bethe ansatz* (ABA), now known as quantum *separation of variables method* (SoV) [117, 121]. The idea behind it was very simple, i.e. since in classical mechanics the concept of Liouville integrability and the application of the Hamilton-Jacobi algorithm, leading to the classical separation of variables, were interchangeable, Sklyanin investigated the possibility to extend this concept, rather the Liouville's one, to the quantum side. Indeed, the separation of variables had already been extended to classical systems treatable with the ISM machinery [69, 118] and a direct implementation in the quantum world had revealed possible as well [64, 75]. The natural idea then was to extend the method to the QISM formalism and investigate its limits, advantages and applicability conditions. More details will be given in chapter **2**. The idea seems appealing but, at the moment, some work is still needed. On the other hands the method permits to solve exactly models that would have been untreatable otherwise by conventional techniques, in particular ABA. Among such models we find the very well-known and studied fundamental systems, also known as spin chains with *exotic* boundary conditions such as non-diagonal twisted boundary conditions in the case of closed chains and non-diagonal boundaries in the case of open chains. Spin chains have been for years the natural laboratory, where developing new techniques and methods, because of their relative simplicity. And these efforts largely payed back since very important and useful results were established, at least for the simplest periodic chains, such as spectrum, eigenstates, form factors, correlation functions (see [77] for an extensive review and reference therein, and [72, 73] for the modern results on correlators in periodic models and [70, 71] for the open chains with parallel boundary magnetic fields).

The original work in the thesis at hand is about the study and solution of the open XXZ and XYZ quantum spin-$1/2$ chains with the most generic boundary conditions by separation of variables method. These systems, which are important *per se*, being particular realizations of fundamental models, recently revealed themselves to be useful tool, as well, for the study of physical phenomena in the out-of-equilibrium case. The open XXZ model with non-diagonal boundary conditions is a simple example that proved to be essential to tackle some problems as the relaxation bahaviour of some classical stochastic models, i.e. the *asymmetric exclusion problems* (ASEP)[124, 91, 35, 36, 31, 32], to the transport properties of the quantum spin systems [109, 113]. We should then mention that the usual algebraic Bethe ansatz technique for open systems [116] (derived from the original work of Cherednik [28] in S-matrix theory) fails once non-diagonal boundary terms are considered. A lot of brilliant researchers started then to elaborate new techniques in order to tackle the problem and some good results followed even if not in the most generic scenario. The first successful attempt to study the open XXZ model with unparallel boundary terms came from Nepomechie [97, 98], who applied the baxter's TQ-equation method. This approach works just for some particular realizations of the model, such as the *roots of unity* points and just if the boundary parameters satisfied some particular constraint. Similar constraints were obtained in an other work [23] within the framework of the algebraic Bethe ansatz. The authors introduced some gauge transformations, inspired by the original work of Baxter [13, 14, 15] and Faddeev-Takhtadzan [43], and then exploited the consequent gauge freedom in order to circumvent direct constraints on the boundary terms. For the first time the set of eigenstates of the XXZ chain with unparallel boundary terms were constructed not in the roots of unity case. With a more algebraically solid approach , in [51] the authors used a different version of this technique, i.e. the face-vertex transformation, but in a more restrictive situation. Although this approach seems to be the more natural ground in order to build correlation functions, it turns out that scalar products of Bethe states remains an open problem and unfortunately it seems impossible to get rid of the constraints on the boundary parameters.

Other approaches were developed in order to deal with the most generic setting of the problem. In [55] the eigenvalue characterization was obtained in the XXZ case through a new functional approach leading to a sort of nested Bethe ansatz type equations similar to the one introduced in [95].

By using techniques that are usually referred as generalized algebraic Bethe ansatz, a quantity of results, in particular the construction of eigenstates, were obtained in the partial non-diagonal case [30, 19], and more recently [18], for the XXZ model and XXX as well [29, 20]. The q-Onsager method was developed in [9, 7, 8] leading to the characterization of the eigenstates of the transfer matrix through the roots of some characteristic polynomials. An *off-diagonal* Bethe ansatz technique was set up in [26, 25] for both the open XXZ and XYZ models *. This technique fully exploits some functional relation satisfied by the elements of the QISM algebra in order to characterize the spectrum.

An alternative way to study models which stand out of the ABA solution capability is given by the quantum version of the separation of variable method elaborated by Sklyanin [115] for the quantum Toda chain. besides the spectrum, the method permitted to compute the scalar products and led to the construction of manageable expressions for matrix elements of local operators of different models, such as cyclic Sine-Gordon model, antiperiodic spin chains, SOS models and others [104, 61, 62, 102, 103, 102]. Successively the method was applied to the open XXZ chain [101], resulting in a successful construction of eigenstates, eigenvalues and scalar products under the condition that one of the boundary matrices K is triangular. Indeed, this was not yet a general solution of the model with generic boundary terms but tightened the slip-knot around the solution of the problem, since the constraint is just on one matrix, instead of being a condition relating *left* and *right* parameters. This kind of result already appeared in [53, 52] for the open XXX model were it is possible to get such a condition by exploiting the $SU(2)$ symmetry of the bulk monodromy matrix.

$*$. For other models solved by using the same method see [24, 25].

In the thesis at hand we investigate, analyze and solve the spectral problems related to the inhomogeneous, most general, spin-$1/2$ representations of the 6-vertex and 8-vertex reflection algebras by means of the SoV method [45, 46]. These systems coincide, via homogeneous limit, to the open XXZ and XYZ quantum spin-$1/2$ chains with the most generic boundary terms. The main steps in order to retrieve the spectrum of these two models consists in applying the gauge transformations, in the fashion of [23], and then using the SoV algorithm. The gauge freedom introduced permits to impose a triangularity condition on one of the boundary matrices K, thus reconciling with the result of [101] for the XXZ model, without imposing any constraint on the boundary parameters. The combination of these two steps leads to the description of the spectrum and eigenstates and, furthermore, to a compact determinant formula for the scalar products. Most remarkably, the spectrum built through SoV method result to be complete by construction, meaning that a set of $\dim(\mathcal{H}) = 2^N$ independent eigenstates and relative distinct eigenvalues is a built-in feature of the method, in opposition to the *completeness problem* of ABA.

The characterization of the transfer matrix eigenvalues is given through a system of quadratic "Baxter-like" equations in the *separated variables*. However, it must be stressed that the study and classification of the possible solutions to these quadratic equations represent a new open problem in the field of quantum integrability and then it would deserve extensive further analysis. We have to mention that an analogous characterization, for both XXZ and XYZ models, was built in [26, 25] by means of the, above mentioned, off-diagonal Bethe ansatz procedure. Although, the method used by the authors do not lead to any characterization of eigenstates and then scalar products.

After the publication of [45], a true Baxter TQ-equation was built [74] for the open XXZ model treated in this thesis. The authors used as a starting point the SoV characterization of the transfer matrix spectrum discussed in the thesis at hand and could formally build a polynomial Q-operator satisfying an inhomogeneous Baxter equation with the transfer matrix. A conjectured form for the inhomogeneous TQ-relations for both the XXZ and XYZ problems was given in [25].

The main advantages of our study relies in the simplicity of the final representation for the eigenstates in the SoV basis and the determinant Vandermonde formulae for the scalar products. Another remarkable point is the absolute equivalence between the representations of the 6-vertex and 8-vertex reflection algebras in the SoV language. It is in fact possible to proceed completely in a parallel way for the two models taking just into account the different nature of the functions, trigonometric or elliptic, appearing in the construction. On the other hand the physical information relative to the XXZ and XYZ spin chain lies behind an eventual homogeneous limit, which, up so far, can't be taken easily. The thermodynamic limit as well doesn't seem to be trivially established. However we think those problem can reasonably be solved within the framework of our approach.

Organization

The following text has to be considered as divided into two distinct parts. Ch.1 and Ch.2 are meant as a brief introduction on related works for review purposes, notation fixing and the sake of completeness. Ch.4 and Ch.5 reproduce the original research work carried out by the author and collaborators and contained in publications [45] and [46].

- **Chapter 1.** The main idea behind the algebraic formulation of quantum integrable models has been here reviewed. Both the cases of closed systems and open systems are considered, i.e. the Yang-Baxter algebra and the reflection algebra are discussed in general and later built for some specific model. The XXZ model has been taken as a working example to explain the general construction of QISM machinery. Basic properties of the two algebras are discussed and proven in details such as: trace identities, quantum determinants and other properties. The ABA method is briefly described, in both cases, for a purely sake of completeness.

– **Chapter 2.** The Sklyanin's separation of variables method is here reviewed. Some ideas of classical Hamiltonian methods are considered in order to introduce the concept of separation of variables. An example is also discussed and then the idea of separation of variables in the quantum theory is introduced. Afterwards we reproduced the step-by-step construction made by Sklyanin in [117] of the SoV method for the Yangian $Y[sl(2)]$ in order to introduce all the elements and ingredients that would have appeared in the two following chapters.

– **Chapter 3.** We address the eigenproblem related to the most generic representation of the 6-vertex spin-$1/2$ reflection algebra. Further notation on the open spin-$1/2$ XXZ quantum spin chain is given. Then we introduce the gauge transformations that will make possible the solution of the model for the most generic boundary conditions. After applying the gauges to all the useful operators of the algenra the SoV program is built step-by-step: construction of SoV representation space, characterization of the conjugated momenta, definition of the B's eigenvalues and then the separated variables, solution of the transfer matrix eigenproblem via the exact definition of eigenstates and a system of quadratic equation for the eigenvalues. Some existence conditions are then discussed. Finally the scalar products formulae for the SoV states are established.

– **Chapter 4.** This chapter can be considered as the mirror of Ch.3 for the 8-vertex reflection algebra. The equivalent notation introduced for the trigonometric algebra in Ch. 1 and Ch. 3 is here considered and discussed for the elliptic algebra. Then the gauge transformations are defined and applied to the main operators of the algebra in order to establish some of the gauge equivalent properties discussed in the first part. Then the SoV algorithm is analyzed in detail, pointing out subtleties, difficulties and advantages. The spectrum of the transfer matrix is characterized through a system of quadratic Baxter-like TQ-equations and the eigenstates built explicitly. Finally, after a discussion about applicability conditions, the scalar products for SoV states is considered.

CHAPTER 1

THE ALGEBRAIC FORMULATION OF THE QUANTUM INVERSE SCATTERING TRANSFORM

The machinery of QISM [41, 42] can be set up by purely algebraic arguments [39, 81] which find their roots in the seminal Heisenberg picture of quantum mechanics. This approach provided a matrix formulation of the theory and led to remarkable results such as the algebraic characterization of the quantum harmonic oscillator or the hydrogen atom. We know that the basic procedure to solve the eigenproblem associated to a standard quantum Hamiltonian consists in describing such a system in terms of a complete set of commuting observables (CSCO)

$$[A_i, A_j] = 0, \quad \forall i, j \in \{1, \dots, d\},$$

acting on some d-dimensional Hilbert space $\mathcal{H}$. This involution implies that the observables can be simultaneously diagonalized and the Hamiltonian as well. Once these *quantum integrals of motions* have been spotted, it is sometimes useful to envelop them in some bigger algebra, as it is the case for the two examples cited above. Afterwards, one should treat the quantum space as the representation space of this bigger algebra and find a way to diagonalize the CSCO by algebraic means. In the case of the quantum harmonic oscillator, for example, this would be accomplished using the *Heisenberg Lie algebra* generated by the only conserved charge of the system, the number of particles N, and by three additional elements: the creation/annihilation operators $a/a^\dagger$ and the *Planck* constant h as a *Casimir* . The commutation relations of the algebra read

$$\begin{gathered} [h, a] = [h, a^\dagger] = [h, N] = 0, \\ [a, a^\dagger] = h, \\ [N, a] = -a \quad , \quad [N, a^\dagger] = a^\dagger \end{gathered}$$

and their use permits to correctly solve the eigenproblem of the oscillator. In this example, it turns out that the enveloping algebra is a Lie algebra, but usually in the QISM framework one has to consider more generic ones. These new algebras are typically not finite-dimensional either and they are labelled simply as *associative*.

This chapter is divided in two sections, the first dealing with the algebraic formalism of the closed (periodic) chains (§1.1, §1.2) and the second with the open ones. The XXZ model in its spin-$1/2$ representation will be taken as explicit example for both the closed and open boundary conditions. The chapter itself is to be intended as a review of the set-up of QISM in order to fix the notation and provide the reader with proves and details of some important past results which will be largely used in the following parts.

1.1 The Yang-Baxter algebra

The algebra of QISM $\mathcal{M}$ is generated by a set of operators $M_{\alpha,\beta} \in \mathrm{End}(\mathcal{H})$ which are usually considered as operator entries of the so-called *monodromy matrix* $M_0(\lambda) \in \mathrm{End}(\mathcal{V}_0 \otimes \mathcal{H})$ depending on a *spectral parameter* λ, with $\mathcal{H}$ the physical Hilbert space and $\mathcal{V}_0$ some linear auxiliary space. The commutation relations among the elements of the algebra are generated by the well celebrated *Yang-Baxter* (YB) relation

$$R_{12}(\lambda-\mu)M_1(\lambda)M_2(\mu) = M_2(\mu)M_1(\lambda)R_{12}(\lambda-\mu) \tag{1.1.1}$$

defined to act on the tensor product $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{H}$. Here, we used the following matrix notation

$$M_1(\lambda) = \underset{1}{M}(\lambda) \otimes \underset{2}{\mathbb{1}} \quad M_2(\mu) = \underset{1}{\mathbb{1}} \otimes \underset{2}{M}(\mu)$$

and λ and μ are the spectral parameters *associated* respectively to the auxiliary spaces $\mathcal{V}_1$ and $\mathcal{V}_2$. The *intertwiner* appearing in relation (1.1.1) is a *structure constants tensor*, known as *R-matrix*, and it has to satisfy the consistency condition, defined in $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3$,

$$R_{12}(\lambda)R_{13}(\lambda+\mu)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda+\mu)R_{12}(\lambda), \tag{1.1.2}$$

i.e. the *Yang-Baxter equation* (YBE). The key point of this construction consists in the fact that having an operator algebra $\mathcal{M}$ with elements $\{M_{\alpha\beta}\}$ satisfying the YB relation leads quite simply to the definition of the opportune generator of a family of involutive quantum charges: the *transfer matrix* $t(\lambda)$, which reads as

$$t(\lambda) = \mathrm{tr}_0\left(M_0(\lambda)\right) \tag{1.1.3}$$

and, by virtue of (1.1.1), satisfies

$$[t(\lambda), t(\mu)] = 0, \quad \forall(\alpha, \mu) \in \mathbb{C}^2. \tag{1.1.4}$$

By diagonalizing the transfer matrix (1.1.4) one diagonalises in the same moment all the family of conserved charges that it can generate. Among them, the quantum Hamiltonian will be of particular interest and the method brings eventually to the solution of the eigenproblem associated to it. This solution process is typically approached by pinpointing a certain *reference state* or *pseudo-vacuum* $|0\rangle$ from which, by acting with the opportune ladder operators of the algebra, it is possible to build the whole eigenbasis.

As mentioned in the introduction, the QISM framework has a component strongly influenced by S-matrix theory. Indeed, the R-matrix can be understood as a scattering matrix and the YBE (1.1.2) as a factorization equivalence for a three-particle scattering process. Here, we would like to introduce some diagrammatic representation that are completely equivalent to the objects and relations introduced up to now. The YBE is represented in Fig. 1.1 where each line (particle world-line) labelled by a spectral parameter λ_i (momentum) represents an auxiliary space and the crossing of two lines an R-matrix (scattering process). The monodromy matrix $M_0(\lambda)$ has a *scattering* interpretation as well, it should

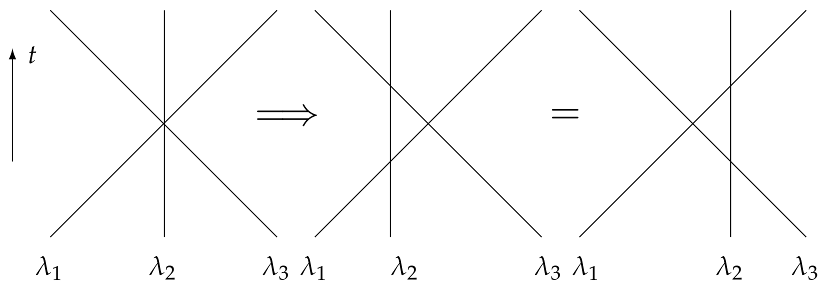

Figure 1.1: Three-particle scattering process and YB equation.

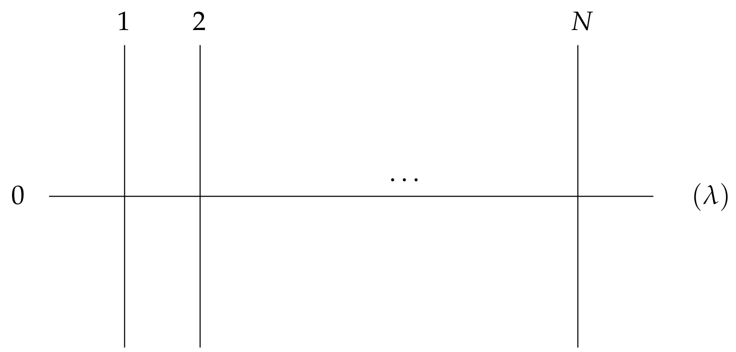

Figure 1.2: The monodromy matrix $M_0(\lambda)$.

be considered as a scattering matrix for all type of excitation in the system. Its graph can be found in Fig. 1.2 where the "0" labels the auxiliary space while the other indices stand for the N quantum spaces (systems). By combining this two graphs it is then possible to reproduce a representation for the YB relation (1.1.1), as pictured in Fig. 1.3. Finally the realization of the trace in the auxiliary space which appears in the construction of the transfer matrix (1.1.3) is given in Fig. 1.4

1.2 The periodic XXZ model

In this section, we will give a concrete example of how QISM works and provide an explicit representation of its algebraic formulation. The XXZ quantum spin-$1/2$ chain [105] has been a very popular and well-studied system over the years [42, 80, 81]. Let us start by defining the Hilbert space where the system lives

$$\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i, \quad \text{with} \quad \mathcal{H}_i = \mathbb{C}^2. \tag{1.2.1}$$

As we read in (1.2.1), this space is the tensor product of N local quantum spin-$1/2$ spaces $\mathbb{C}^2$. The main interest is to study the spin dynamics along the chain given few local interaction rules encoded by the Hamiltonian of the model: a self-adjoint operator acting on the above defined Hilbert space. It reads

$$H_{\text{XXZ}} = \sum_{n=1}^{N} \left[\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cosh(\eta) \left(\sigma_j^z \sigma_{j+1}^z - 1 \right) \right] \in \text{End}(\mathcal{H}) \tag{1.2.2}$$

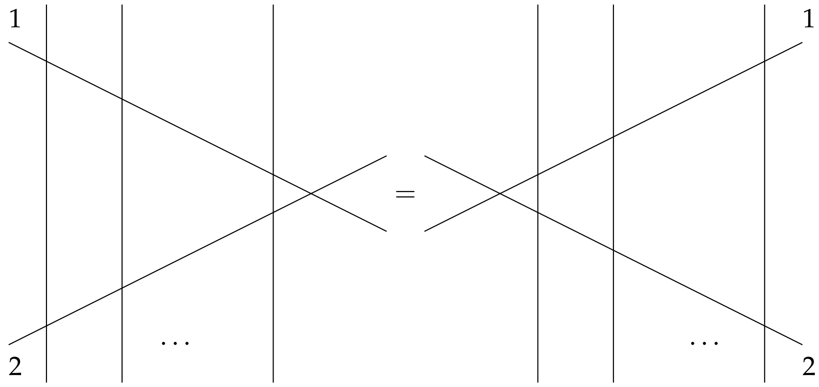

Figure 1.3: The YB relation.

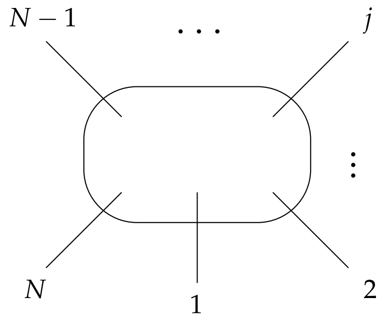

Figure 1.4: The transfer matrix $t(\lambda)$.

where

$$\sigma_i^\alpha = \underset{1}{\mathbb{1}} \otimes \cdots \otimes \underset{i}{\sigma^\alpha} \otimes \cdots \otimes \underset{N}{\mathbb{1}} \quad \text{with } \alpha \in \{x, y, z\}$$

and with the periodicity conditions $\sigma_{i+N}^\alpha = \sigma_i^\alpha$. The parameter η is called *anisotropy* since it selects a preferential magnetization axis and produces a different coupling on it. The sigma matrices are the usual Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2.3}$$

In the following we will define the proper representations for all the objects introduced in the last paragraph in order to set some notation for the rest of this thesis and present the historically important facts and results related to the construction. The particular technique and QISM realization that will be described goes under the name of *algebraic Bethe ansatz* (ABA).

1.2.1 The R-matrix

The R-matrix that suits the study of the periodic XXZ spin-1/2 chain by QISM is given by the so-called *trigonometric* or *6-vertex* solution of the YBE (1.1.2). The first name refers to the nature of functions appearing in it, which are in general expressed in their elliptic trigonometric form. The second, instead, is related to the study of the statistical mechanical ice-type model characterized by 6 possible vertex configurations, which was solved by Lieb and Sutherland [84, 85, 86, 127]. It turns out that the algebraic QISM related to this model is equivalent to the spin chain that we are treating in this short review. Let the auxiliary spaces' dimensions be $\dim(\mathcal{V}_{1,2}) = 2$ then, in the natural $\mathbb{C}^2 \otimes \mathbb{C}^2$ basis, the R-matrix reads [17]

$$R_{12}^{6v}(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}, \qquad \begin{aligned} a(\lambda) &= \sinh(\lambda + \eta), \\ b(\lambda) &= \sinh \lambda, \\ c(\lambda) &= \sinh \eta. \end{aligned} \tag{1.2.4}$$

The matrix (1.2.4) has the following properties:

$$\textit{Permutation op. point} \qquad R_{12}^{6v}(0) = \sinh \eta \cdot \mathbb{P}_{12} \tag{1.2.5a}$$

$$\textit{Antisymetrizer op. point} \qquad R_{12}^{6v}(-\eta) = (-\sinh \eta) \cdot P_{12}^{-} \tag{1.2.5b}$$

$$\textit{Unitarity} \qquad R_{12}^{6v}(\lambda) R_{12}^{6v}(-\lambda) = -\sinh(\lambda + \eta) \sinh(\lambda - \eta) \mathbb{1} \tag{1.2.5c}$$

$$\textit{Crossing Unitarity} \qquad \sigma_1^y R_{12}^{6v}(\lambda) \sigma_1^y = -(R_{12}^{6v})^{t_2}(-\lambda - \eta) \tag{1.2.5d}$$

$$\textit{PT-Symmetry} \qquad (R_{12}^{6v})^{t_1 t_2}(\lambda) = \mathbb{P}_{12} R_{12}^{6v} \mathbb{P}_{12} = R_{12}^{6v}(\lambda) \tag{1.2.5e}$$

$$\mathbb{Z}_2\textit{-Symmetry} \qquad \sigma_1^j \sigma_2^j R_{12}^{6v}(\lambda) \sigma_1^j \sigma_2^j = R_{12}^{6v}(\lambda) \text{ , with } j = x, y, z \tag{1.2.5f}$$

where the symbol $\mathbb{P}_{12}$ is the permutation operator $\mathbb{P}_{12}(a \otimes b) = b \otimes a$ and $P_{12}^{-} = (1 - \mathbb{P}_{12})/2$ the antisymmetrizer operator. The properties (1.2.5) can be easily proven by direct computation.

1.2.2 The monodromy matrix, the transfer matrix and the trace identity

The XXZ spin chain is one of the fundamental integrable models, which means that the associative algebra $\mathcal{M}_{\text{xxz}}$ is strictly connected to the R-matrix definition (1.2.4). Let us introduce the following *quantum Lax operators*

$$L_a(\lambda) = R_{0a}(\lambda - \xi_a - \eta/2) \in \text{End}(\mathcal{V}_0 \otimes \mathcal{H}_a), \quad \text{for } a = 1, \dots, N; \tag{1.2.6}$$

where each of them acts on the tensor product between the auxiliary space $\mathcal{V}_0$ and one of the local quantum spaces $\mathcal{H}_a$; moreover it satisfies the YB relation [*] (1.1.1) by definition. A new set of parameters $(\xi_1, \dots, \xi_N) \in \mathbb{C}^N$ has been added for convenience, since they will be strictly needed in the following. Please note, that for the sake of ABA, the algebra could also be constructed without them which would lead to an equivalent analysis. Indeed we'll see that the connection to the physics of the quantum model is realized by putting these *inhomogeneities* to zero. At this point we are ready to define the proper monodromy matrix:

$$M_0(\lambda) = L_N(\lambda) L_{N-1}(\lambda) \dots L_1(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{[\mathcal{V}_0]} \in \text{End}(\mathcal{V}_0 \otimes \mathcal{H}). \tag{1.2.7}$$

It is trivial to show how this monodromy matrix, by construction, satisfies the YB relation (1.1.1); it is sufficient to exploit the commutativity of L-operators acting on different spaces and the YB relation satisfied by them multiple times. Note that the YB relation (1.1.1) is very important in order to completely characterize the commutation relations of the algebra generated by the four operators $A(\lambda)$,

∗. It is important to carefully note that the index displayed in (1.2.6) is referred to the local quantum space and not the auxiliary one.

$B(\lambda)$, $C(\lambda)$ and $D(\lambda)$, which acts in Hilbert space $\mathcal{H}$. It is indeed a simple exercise to prove, for example, the following subset of relations:

$$[T_{jk}(\lambda), T_{jk}(\mu)] = 0, \quad \text{for} \quad j,k = 1,2 \tag{1.2.8a}$$

$$[B(\lambda), B(\mu)] = 0 \tag{1.2.8b}$$

$$A(\mu)B(\lambda) = f(\mu,\lambda)B(\lambda)A(\mu) + g(\mu,\lambda)B(\mu)A(\lambda) \tag{1.2.8c}$$

$$D(\mu)B(\lambda) = f(\mu,\lambda)B(\lambda)D(\mu) + g(\lambda,\mu)B(\mu)D(\lambda) \tag{1.2.8d}$$

$$\begin{aligned}[C(\mu), B(\lambda)] &= g(\lambda,\mu)\left(A(\mu)D(\lambda) - A(\lambda)D(\mu)\right)\\ &= g(\lambda,\mu)\left(D(\lambda)A(\mu) - D(\mu)A(\lambda)\right)\end{aligned} \tag{1.2.8e}$$

$$\begin{aligned}[D(\mu), A(\lambda)] &= g(\lambda,\mu)\left(B(\mu)C(\lambda) - B(\lambda)C(\mu)\right)\\ &= g(\lambda,\mu)\left(C(\lambda)B(\mu) - C(\mu)B(\lambda)\right)\end{aligned} \tag{1.2.8f}$$

where

$$f(\lambda,\mu) = \frac{\sinh(\lambda - \mu + \eta)}{\sinh(\lambda - \mu)} \qquad \text{and} \qquad g(\lambda,\mu) = \frac{\sinh\eta}{\sinh(\lambda-\mu)}.$$

Finally, the expression for the transfer matrix follows directly from (1.1.3) and takes form

$$t(\lambda) = \mathrm{tr}_0\{M_0(\lambda)\} = A(\lambda) + D(\lambda) \in \mathrm{End}(\mathcal{H}) \tag{1.2.9}$$

The connection between the transfer matrix (1.2.9) and the quantum model, also known as *trace identity*, was shown in [81] and we will reproduce it in the following proposition:

Proposition 1.2.1 (Kulish-Sklyanin-Faddeev-Takhtadzhan). *Given the YB algebra defined by the R-matrix* (1.2.4)*, its property* (1.2.5a) *and the monodromy matrix* (1.2.7)*, the Hamiltonian for the periodic XXZ quantum spin-*$1/2$ *chain is given by the following trace identity:*

$$H_{xxz} = 2\sinh\eta \frac{d}{d\lambda}\ln(t(\lambda))\Big|_{\substack{\lambda=\eta/2\\ \xi_1,\dots,\xi_N=0}} - 2N\cosh\eta. \tag{1.2.10}$$

Proof. As stated above the property (1.2.5a) is fundamental to build the desired link with the quantum system. Consider the following derivative

$$\frac{d}{d\lambda}\ln(t(\lambda))\Big|_{\substack{\lambda=\eta/2\\ \xi_1,\dots,\xi_N=0}} = \left[\frac{1}{t(\lambda)}\frac{d}{d\lambda}t(\lambda)\right]\Big|_{\substack{\lambda=\eta/2\\ \xi_1,\dots,\xi_N=0}}$$

and let's start by computing the prefactor $t(\eta/2)$

$$\begin{aligned} t(\eta/2) &= \mathrm{tr}_0\{R_{0N}(0)\dots R_{01}(0)\} = (\sinh\eta)^N \mathrm{tr}_0\{\mathbb{P}_{0N}\dots\mathbb{P}_{01}\}\\ &= (\sinh\eta)^N\mathrm{tr}_0\{\mathbb{P}_{1N}\dots\mathbb{P}_{12}\mathbb{P}_{01}\} = (\sinh\eta)^N\mathbb{P}_{N-1N}\dots\mathbb{P}_{12}\end{aligned} \tag{1.2.11}$$

where we have used the fact that $\mathbb{P}_{0i}\mathbb{P}_{0j} = \mathbb{P}_{ij}\mathbb{P}_{0i}$, $\mathrm{tr}_0\{\mathbb{P}_{01}\} = \mathbb{1}_1$ and the cycle permutation decomposition $(1N)\dots(12) = (12\dots N) = (N-1N)\dots(12)$. Moreover we have put all the inhomogeneities to zero as required by (1.2.10), and it won't be recalled again in next formulas of this proof.

We can now proceed with the calculation

$$\begin{aligned}
\frac{d}{d\lambda}t(\lambda)|_{\lambda=\eta/2} &= \frac{d}{d\lambda}\mathrm{tr}_0\{M_0(\lambda)\}|_{\lambda=\eta/2} = \mathrm{tr}_0\{\frac{d}{d\lambda}M_0(\lambda)\}|_{\lambda=\eta/2} \\
&= (\sinh\eta)^{N-1}\sum_{n=1}^{N}\mathrm{tr}_0\{\mathbb{P}_{0N}\dots\mathbb{P}_{0n-1}L'_{0n}(\eta/2)\mathbb{P}_{0n+1}\dots\mathbb{P}_{01}\} \\
&= (\sinh\eta)^{N-1}\sum_{n=1}^{N}\mathrm{tr}_0\{L'_{n+1n}(\eta/2)\mathbb{P}_{0N}\dots\mathbb{P}_{0n-1}\mathbb{P}_{0n+1}\dots\mathbb{P}_{01}\} \\
&= (\sinh\eta)^{N-1}\sum_{n=1}^{N}L'_{nn+1}(\eta/2)\mathbb{P}_{N-1N}\dots\mathbb{P}_{n-1n+1}\dots\mathbb{P}_{12}
\end{aligned} \tag{1.2.12}$$

and by multiplying it times $[t(\eta/2)]^{-1}$, we get

$$\frac{d}{d\lambda}\ln(t(\lambda))|_{\lambda=\eta/2} = (\sinh\eta)^{-1}\sum_{n=1}^{N}L'_{nn+1}(\eta/2)\mathbb{P}_{nn+1}. \tag{1.2.13}$$

Now by explicit calculation

$$\begin{aligned}
L'_{nn+1}(\eta/2)\mathbb{P}_{nn+1} &= \begin{pmatrix} \cosh\eta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cosh\eta \end{pmatrix} \\
&= \frac{1+\sigma_n^z\sigma_{n+1}^z}{2}\cosh\eta + \frac{\sigma_n^x\sigma_{n+1}^x}{2} + \frac{\sigma_n^y\sigma_{n+1}^y}{2}
\end{aligned}$$

and the result (1.2.10) follows. □

1.2.3 The quantum determinant

Here, we would like to give account about an important object that appears in the algebraic analysis of integrable models: the *quantum determinant*. It is essentially a *c-number* or *Casimir* operator of the YB algebra generated by the operator-valued entries of the monodromy matrix $M_0(\lambda)$. Its definition and the basic properties were established for first in [68], further developed in [81], and can be simply organized in the following proposition.

Proposition 1.2.2 (Kulish-Sklyanin). *The Yang-Baxter algebra related to the quantum spin-$1/2$ chain possesses the following Casimir operator known as quantum determinant*

$$\begin{aligned}
q\text{-}det\,(M_0(\lambda)) &= tr_{12}\{P_{12}^{-}M_1(\lambda-\eta/2)M_2(\lambda+\eta/2)\} \\
&= tr_{12}\{M_2(\lambda+\eta/2)M_1(\lambda-\eta/2)P_{12}^{-}\} \in End(\mathcal{H}),
\end{aligned} \tag{1.2.14}$$

which, being a central element of the algebra, satisfies

$$[q\text{-}det\,(M_0(\lambda)), M_0(\mu)] = 0\ \forall\lambda,\mu. \tag{1.2.15}$$

The formula (1.2.14) *can be expressed explicitly in terms of YB generators*

$$\begin{aligned}
q\text{-}det\,(M_0(\lambda)) &= A(\lambda+\eta/2)D(\lambda-\eta/2) - B(\lambda+\eta/2)C(\lambda-\eta/2) \\
&= A(\lambda-\eta/2)D(\lambda+\eta/2) - C(\lambda-\eta/2)B(\lambda+\eta/2) \\
&= D(\lambda+\eta/2)A(\lambda-\eta/2) - C(\lambda+\eta/2)B(\lambda-\eta/2) \\
&= D(\lambda-\eta/2)A(\lambda+\eta/2) - B(\lambda-\eta/2)C(\lambda+\eta/2)
\end{aligned} \tag{1.2.16}$$

or in its basic functional representation as well

$$q\text{-}det\,(M_0(\lambda)) = a(\lambda+\eta/2)d(\lambda-\eta/2)\times \mathbb{1}_{\mathcal{H}},$$
$$\text{where}\quad a(\lambda)=\prod_{j=1}^{N}\sinh(\lambda-\xi_j+\eta/2),\quad d(\lambda)=a(\lambda-\eta). \tag{1.2.17}$$

The quantum determinant respects the co-multiplication: given

$$M_0(\lambda) = M_0(\lambda;1)M_0(\lambda;2)$$

then

$$q\text{-}det\,\left(M_0(\lambda;1)M_0(\lambda;2)\right) = q\text{-}det\,(M_0(\lambda;1))q\text{-}det\,(M_0(\lambda;2)); \tag{1.2.18}$$

provided that all matrix elements of $M_0(\lambda;1)$ commute with all matrix elements of $M_0(\lambda;2)$. The name of this operator comes from the fact that it is a quantum generalization of the ordinary matrix determinant, as it plays a role in the inversion of the monodromy matrix. This aspect can be seen from the following expression

$$M_0^{-1}(\lambda+\eta/2) = \frac{(-1)^N}{q\text{-}det\,(M_0(\lambda))}\hat{M}_0(-\lambda+\eta/2), \tag{1.2.19}$$

where

$$\hat{M}_0(\lambda) = (-1)^N\sigma_0^y[M_0(-\lambda)]^{t_0}\sigma_0^y \tag{1.2.20}$$

satisfies the YB relation (1.1.1) *as well for $\lambda \to -\lambda$.*

For the sake of completeness we would like to give a sketch of the proof of proposition 1.2.2, basically reproducing the major arguments from [81].

Proof. We can first of all prove the fact that the quantum determinant commutes with the transfer matrix of our problem and consequently with the Hamiltonian under study. This fact can be proven by considering that the following object

$$M_{(12)}(\lambda) = P_{12}^{-}M_1(\lambda-\eta/2)M_2(\lambda+\eta/2) \tag{1.2.21}$$

satisfies a YB relation with our monodromy matrix and an appropriate R-matrix intertwiner. The correct definition of such a matrix should be researched in papers [81, 79] where the building process of higher rank representations for R-matrices has been considered. In this case, by using the S-matrix theory language already introduced in §1.1, one should look for the R-matrix modeling the scattering process between a spin-1/2 particle and a couple of spin-1/2 particles which add up to a spin-1 representation. It reads * [79]

$$R_{(12),3}^{\pm} = P_{12}^{\pm}R_{13}(\lambda-\eta/2)R_{23}(\lambda+\eta/2)P_{12}^{\pm} \tag{1.2.22}$$

where the antisymmetrized (symmetrized) version $R_{(12),3}^{-}$ ($R_{(12),3}^{+}$) corresponds to the spin-0 (spin-1) sector of the representation obtained as a sum of the two spin-1/2 representations given by particles *1* and *2*. By choosing, in particular, $R_{(12),3}^{-}$ as the R-matrix for our task, it is simple to prove that the relation

$$R_{(12),3}^{-}(\lambda-\mu-\eta/2)M_{(12)}(\lambda)M_3(\mu) = M_3(\mu)M_{(12)}(\lambda)R_{(12),3}^{-}(\lambda-\mu-\eta/2) \tag{1.2.23}$$

*. We have already defined the antisymmetrizer $P_{12}^{-} = 1/2(\mathbb{1}-\mathbb{P}_{12})$ in section 1.2.1. We can equally define the symmetrizer as $P_{12}^{+} = 1/2(\mathbb{1}+\mathbb{P}_{12})$.

is satisfied and it follows directly by considering that the definition of $M_{(12)}(\lambda)$ coincides with the lhs of (1.1.1), see property (1.2.5b), and by acting multiple times with (1.1.1) itself. Once (1.2.23) has been established, by taking the trace in the auxiliary spaces $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ the researched commutation rule is obtained. But in order to prove the result (1.2.15) one should investigate the effective explicit expression (1.2.17) of the quantum determinant, which will be proven in the next lines. Expressions (1.2.16) can be established by a direct computation of the matrix products and trace operation in (1.2.15), by using the basic tensor product representation

$$M_1(\lambda) = M(\lambda) \otimes \mathbb{1} = \begin{pmatrix} A(\lambda) & 0 & B(\lambda) & 0 \\ 0 & A(\lambda) & 0 & B(\lambda) \\ C(\lambda) & 0 & D(\lambda) & 0 \\ 0 & C(\lambda) & 0 & D(\lambda) \end{pmatrix}$$

$$M_2(\lambda) = \mathbb{1} \otimes M(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) & 0 & 0 \\ C(\lambda) & D(\lambda) & 0 & 0 \\ 0 & 0 & A(\lambda) & B(\lambda) \\ 0 & 0 & C(\lambda) & D(\lambda) \end{pmatrix}$$

and the opportune commutation relations from (1.2.8). The equivalence between each line of (1.2.16) follows from the ciclicity property of the trace or, once more, they can be proven by using again the commutation relations (1.2.8). The very same result, in terms of algebra operators, can be obtained by the explicit matrix product (1.2.19) and again by using relations (1.2.8). The explicit form of the quantum determinant given by (1.2.17) follows from (1.2.19) too, once we have established the following equivalence

$$\hat{M}_0(\lambda) = R_{01}(\lambda + \xi_1 - \eta/2) \dots R_{0N}(\lambda + \xi_N - \eta/2)$$

which implies

$$M_0(\lambda + \eta/2)\hat{M}_0(-\lambda + \eta/2) = R_{0N}(\lambda - \xi_N) \dots R_{01}(\lambda - \xi_1) \times R_{01}(-\lambda + \xi_1) \dots R_{0N}(-\lambda + \xi_N).$$

Then, by exploiting the property (1.2.5c) of the R-matrix, the result is proven. The last bit we lack is the co-multiplicativity property of quantum determinants (1.2.18). It suffices to exploit the commutativity of the various objects defined on different spaces in order to show that

$$\begin{aligned} \text{q-det}\,(M_0(\lambda)) &= \text{tr}_{12}\{P^-_{12} M_1(\lambda + \eta/2; 1) M_2(\lambda - \eta/2; 1) \\ &\qquad \times M_2(\lambda - \eta/2; 2) M_1(\lambda + \eta/2; 2)\} \\ &= \text{tr}_{12}\{P^-_{12} M_1(\lambda + \eta/2; 1) M_2(\lambda - \eta/2; 1) \\ &\qquad \times M_2(\lambda - \eta/2; 2) M_1(\lambda + \eta/2; 2) P^-_{12}\}, \end{aligned}$$

where we have used that $(P^-_{12})^2 = P^-_{12}$. By finally exploiting the YB relation and the following identity

$$M_{(12)}(\lambda) = \text{q-det}\,(M_0(\lambda)) P^-_{12} \tag{1.2.24}$$

one can show the validity of the property. □

1.2.4 The Algebraic Bethe ansatz

In this short review of how the algebraic formalism of QISM works, we would like to report some important results about a very well studied technique, constructed for the first time in [41, 128], in order to tackle the eigenproblem related to quantum integrable models. We won't give detailed proofs of statements and theorems as did before since the exposition will be for purely historical relevance and completeness rather than being useful for the results of this thesis.

The main idea consists in building eigenstates of the transfer matrix via creation and annihilation operators acting on a pseudovacuum. These operators can be easily found among the YB generators contained in $M_0(\lambda)$. Consider first of all what has been called pseudovacuum or reference state,

$$|0\rangle = |\uparrow\rangle_1 \otimes \cdots \otimes |\uparrow\rangle_N , \quad \text{where} \begin{cases} S^z_j \, |\uparrow\rangle_j = \dfrac{1}{2} \, |\uparrow\rangle_j , \\ |\uparrow\rangle_j = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in (\mathcal{H}_j = \mathbb{C}^2), \end{cases} \tag{1.2.25}$$

where $S^z_j = (1/2)\sigma^z_j$ is the z-component of the spin operator in the jth quantum space. The existence of such a reference state can be stated differently. Indeed the Hamiltonian (1.2.2) is symmetric for the action of the total z-component of the spin operator: $S^z = \sum_{j=1}^N \frac{1}{2}\sigma^z_j$. It's evident that (1.2.25) is an highest weight eigenvector of S^z.

Let us notice that the action of the L-operator (1.2.6) on a local pseudovacuum $|\uparrow\rangle$ makes it triangular

$$L_n(\lambda) \, |\uparrow\rangle_n = \begin{pmatrix} \sinh(\lambda - \xi_n + \eta/2) & \star \\ 0 & \sinh(\lambda - \xi_n - \eta/2) \end{pmatrix} |\uparrow\rangle_n$$

where "$\star$" is some non-trivial expression. Once convinced of this, It is simple to generalize to the action of the monodromy matrix $M_0(\lambda)$ on the global pseudovacuum (1.2.25)

$$M_0(\lambda) \, |0\rangle = \begin{pmatrix} a(\lambda) & \star \\ 0 & d(\lambda) \end{pmatrix} |0\rangle ,$$

where $a(\lambda)$ and $d(\lambda)$, already introduced in (1.2.17), are, respectively, the eigenvalues of the operators $A(\lambda)$ and $D(\lambda)$ associated to the common eigenstate $|0\rangle$. Now, keeping in mind the commutation relations (1.2.8b), (1.2.8c) and (1.2.8d) (along with the functions appearing in them), the following theorem holds true [*] :

Theorem 1.2.1 (Faddeev-Sklyanin-Takhtadzan). *The following vector of the Hilbert space $\mathcal{H}$*

$$\big|\Psi_M(\{\lambda_k\}_{k=1,\ldots,M})\big\rangle = \prod_{n=1}^{M} B(\lambda_n) \, |0\rangle , \quad \forall M \in (1, \ldots, N) \tag{1.2.26}$$

is an eigenstate of the transfer matrix (1.2.9) $t(\lambda) = A(\lambda) + D(\lambda)$ *with eigenvalue*

$$\tau(\lambda) = a(\lambda) \prod_{j=1}^{M} f(\lambda; \lambda_j) + d(\lambda) \prod_{j=1}^{M} f(\lambda_j; \lambda) \tag{1.2.27}$$

provided that the set of ***Bethe roots*** $\{\lambda_k\}_{k=1,\ldots,M}$ *satisfy the* ***Bethe ansatz equations*** *(BAE):*

$$\forall k \in (1, \ldots, M) \quad \frac{d(\lambda_k)}{a(\lambda_k)} \prod_{\substack{j=1 \\ j \neq k}}^{M} \frac{f(\lambda_k; \lambda_j)}{f(\lambda_j; \lambda_k)} = 1. \tag{1.2.28}$$

Remark 1.2.1. One should pay attention to the fact that the theorem 1.2.1 doesn't imply completeness for the spectrum of the Hamiltonian. This is a delicate issue and It is possible to find different publications dedicated to this problem, concerning spin chains and integrable models in general. The interested reader should refer for example to [129], [38],[37],[96] and [16]. The problem of completeness will be faced and solved, for the inhomogenous case exclusively, in the following of this thesis.

∗. The proof of this theorem can be found in numerous publications and, in particular, numerous review papers [40, 54, 108] and the book [77].

1.3 The reflection algebra

An interesting direction, in order to extend the theory of quantum integral models, consists in the change of boundary conditions. The whole construction presented in sections §1.1 and §1.2 holds for periodic, twisted or untwisted, or, at an algebra only level, for anti-periodic boundaries *. In the current section we will present how to characterize mathematically quantum models with *open boundaries*, with a particular focus on the XXZ chain once more. The key point of the theory, as we learnt for the closed systems, is to find an appropriate algebra where to envelope our quantum Hamiltonian and its CSCO. The solution to this issue comes again from the scattering theory and the first one to deal with such a problem was Cherednik [28], concerned to describe scattering processes on the half-infinite line including interactions with a boundary "wall". By interpreting the R-matrix as a factorizing two-particles scattering matrix, solution to YB equation, and introducing a K-matrix, encoding the interaction with the wall, Cherednik arrived to define the following *reflection equation*

$$R_{12}(\lambda-\mu)K_1(\lambda)R_{21}(\lambda+\mu)K_2(\mu) = K_2(\mu)R_{21}(\lambda+\mu)K_1(\lambda)R_{12}(\lambda-\mu) \tag{1.3.1}$$

defined on $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3$, which is graphically depicted in Fig. 1.5. This seminal work inspired and

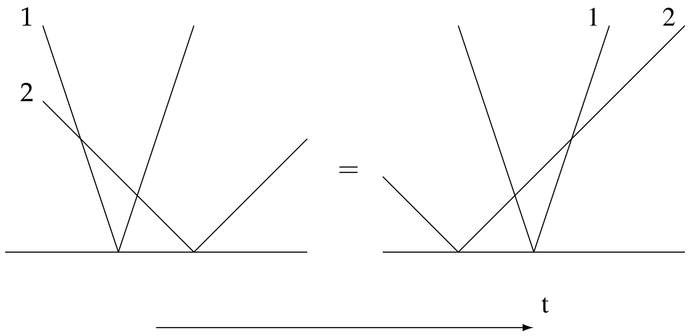

Figure 1.5: Scattering on the half-infinite line and the reflection Equation

led Sklyanin [116] to develop a theory well suited for integrable models with open boundaries. The important passage consists in pinpointing the right definition of a new monodromy matrix, which will be this time solution to a reflection equation (1.3.1), by solely using the ingredients, already introduced above, from the algebraic framework of QISM. In particular, he introduced the family of isomorphic boundary matrices $K_\pm(\lambda)$, both solutions to (1.3.1), encoding the interaction with a *left wall* and *right wall*, and resulting, in the end, in a couple of isomorphic reflection algebras $\mathcal{U}^\pm$. The QISM algebra is generated by the operators entries $(\mathcal{U}_-(\lambda))_{\alpha\beta} \in \mathrm{End}(\mathcal{H})$ $((\mathcal{U}_+(\lambda))_{\alpha\beta})$ of a *double-row monodromy matrix* $\mathcal{U}_-(\lambda) \in \mathrm{End}(\mathcal{V}_0 \otimes \mathcal{H})$ $(\mathcal{U}_+^{t_0}(\lambda))$ and they depend on a spectral parameter λ. The algebra's structure is defined by the relations among operators that result from these two equivalent reflection

$*$. With twisted or anti-periodic boundary conditions we mean, respectively, the insertion of a matrix of the type

$$Q_{\text{twist}} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad Q_{\text{anti}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{with } (\alpha,\beta) \in \mathbb{C}^2$$

in the transfer matrix. See, for example, [10, 135, 55, 102].

equations *

$$\begin{aligned}
&R_{12}(\lambda-\mu)\overset{1}{\mathcal{U}}_-(\lambda)R_{21}(\lambda+\mu-\eta)\overset{2}{\mathcal{U}}_-(\mu) \\
&\qquad\qquad = \overset{2}{\mathcal{U}}_-(\mu)R_{21}(\lambda+\mu-\eta)\overset{1}{\mathcal{U}}_-(\lambda)R_{12}(\lambda-\mu)
\end{aligned} \tag{1.3.2}$$

and

$$\begin{aligned}
&R_{12}(-\lambda+\mu)\overset{1}{\mathcal{U}}{}_+^{t_1}(\lambda)R_{21}(-\lambda-\mu-\eta)\overset{2}{\mathcal{U}}{}_+^{t_2}(\mu) \\
&\qquad\qquad = \overset{2}{\mathcal{U}}{}_+^{t_2}(\mu)R_{21}(-\lambda-\mu-\eta)\overset{1}{\mathcal{U}}{}_+^{t_1}(\lambda)R_{12}(-\lambda+\mu)
\end{aligned} \tag{1.3.3}$$

where this particular shift in "η" has been chosen for notational convenience in view of next section.

Proposition 1.3.1 (Sklyanin). *The algebras $\mathcal{U}^-$ and $\mathcal{U}^+$ are isomorphic.*

Proof. Consider the map $X : \mathcal{U}^- \to \mathcal{U}^+$

$$X(\mathcal{U}_-(\lambda)) = \mathcal{U}_-^{t_0}(-\lambda),$$

that once introduced in (1.3.2), with the help of properties (1.2.5), proves the proposition. □

This *boundary monodromy matrix* $\mathcal{U}_-(\lambda)$, and equivalently $\mathcal{U}_+^{t_0}(\lambda)$, are defined, in terms of the *bulk* monodromy matrix $M_0(\lambda)$ and the boundary matrices $K_\pm(\lambda)$ as

$$\mathcal{U}_-(\lambda) = M_0(\lambda)K_-(\lambda)M_0^{-1}(-\lambda) \quad \text{and} \quad \mathcal{U}_+^{t_0}(\lambda) = M_0^{t_0}(\lambda)K_+^{t_0}(\lambda)[M_0^{-1}(-\lambda)]^{t_0}. \tag{1.3.4}$$

It is not difficult to show that these two expressions satisfy the reflection equations (1.3.2) and (1.3.3) respectively, by virtue of the YB relation (1.1.1) satisfied by M and the reflection equation (1.3.1) satisfied by $K_\pm$; the graphic representation is given in Fig. 1.6. We can then introduce the "reflection-

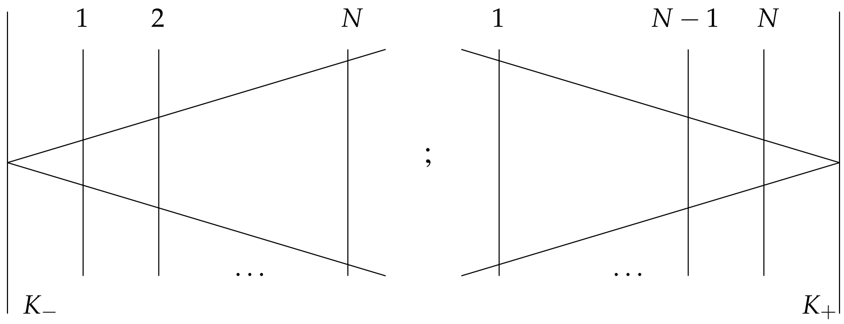

Figure 1.6: The double-row monodromy matrices $\mathcal{U}_-(\lambda)$,on the left, and $\mathcal{U}_+^{t_0}(\lambda)$ on the right.

equivalent" transfer matrix. For open systems such an object reads † [116]

$$t(\lambda) = \mathrm{tr}_0\{\mathcal{U}_-(\lambda)K_+(\lambda)\} = \mathrm{tr}_0\{\mathcal{U}_+(\lambda)K_-(\lambda)\} \tag{1.3.5}$$

or, for the graph, see Fig. 1.7. In this case too, the transfer matrix is a good generator of the conserved charges since it holds

$$[t(\lambda), t(\mu)] = 0, \quad \forall(\alpha, \mu) \in \mathbb{C}^2$$

as it follows from the reflection equation (1.3.2) after a trace extraction.

*. Notice that the boundary matrices $K_\pm(\lambda)$ are particular solutions of the reflection equation as stated in (1.3.1). In particular they are often referred as *scalar* or *numerical* solutions. Indeed, the internal representation space (the Hilbert space in the case of the double-row monodromy matrix) coincides with $\mathbb{C}$, hence the name.

†. We used the same notation for the two transfer matrices (1.1.3) and (1.3.5) even if they do not coincide and shouldn't be interchanged.

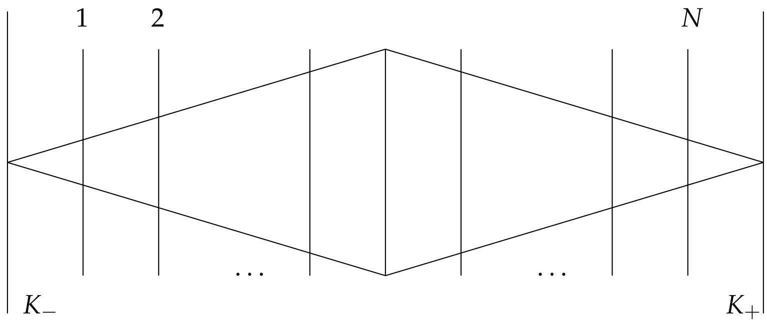

Figure 1.7: The boundary transfer matrix

1.4 The open XXZ model

We will now introduce the open boundary version of the quantum system treated in §1.2. This section will be very important since it will give the opportunity to fix notation once more and to review some basics about the open XXZ spin chain. Here, we would like to describe the so-called *diagonal-boundary* XXZ model since it will be useful to us and, for completeness, it permits to present the equivalent version of the ABA, already outlined in section 1.2.4. The Hamiltonian of the model is defined to act on the same Hilbert space defined in (1.2.1) $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ and it reads [116, 3]

$$\begin{aligned} H^{\text{D.B.}}_{\text{XXZ}} = \sum_{n=1}^{N} \Big[\sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1} + \cosh(\eta) \left(\sigma^z_j \sigma^z_{j+1} - 1 \right) \Big] \\ + \sinh\eta\, (\sigma^z_1 \coth\zeta_- + \sigma^z_N \coth\zeta_+) \in \text{End}(\mathcal{H}) \end{aligned} \tag{1.4.1}$$

where the usual tensor notation holds. The parameters $\zeta_\pm$ are the boundary parameters which tune the interactions with the border regions. We should notice that the Hamiltonian (1.4.1) still satisfies the commutation relation

$$[\mathcal{H}, S^z] = 0 \tag{1.4.2}$$

implying an existence of a suitable reference state and then it permits to use the ABA technique in order to solve its eigenproblem.

1.4.1 The XXZ reflection algebra

As we learned in the past sections, the first ingredient to be considered in the QISM construction is the R-matrix. For the XXZ spin-$1/2$ chain the good choice is always the trigonometric solution (1.2.4) to YB equation, equipped with all its properties (1.2.5). Then we can directly give the scalar representation for the diagonal solution to the reflection equation (1.3.1) found in the original paper by Cherednik [28]

$$K^{\text{d}}(\lambda;\zeta) = \frac{1}{\sinh\xi} \begin{pmatrix} \sinh(\lambda+\xi) & 0 \\ 0 & \sinh(-\lambda+\xi) \end{pmatrix} \in \text{End}(\mathcal{V}_0 \simeq \mathbb{C}^2). \tag{1.4.3}$$

In order to build the algebras $\mathcal{U}^\pm$, as sketched before, we can define the two left and right boundary matrices

$$K^{\text{d}}_-(\lambda) = K^{\text{d}}(\lambda - \eta/2; \zeta_-, \kappa_-, \tau_-), \quad K^{\text{d}}_+(\lambda) = K^{\text{d}}(\lambda + \eta/2; \zeta_+, \kappa_+, \tau_+). \tag{1.4.4}$$

Consider the bulk monodromy matrix $M_0(\lambda)$ (1.2.7) and its dual $\hat{M}_0(\lambda)$ (1.2.20) defined above, then the explicit form of the double row monodromy matrices is

$$\mathcal{U}_-(\lambda) = M(\lambda)K_-^{\mathrm{d}}(\lambda)\hat{M}(\lambda) = \begin{pmatrix} \mathcal{A}_-(\lambda) & \mathcal{B}_-(\lambda) \\ \mathcal{C}_-(\lambda) & \mathcal{D}_-(\lambda) \end{pmatrix} \in \mathrm{End}(\mathcal{V}_0 \otimes \mathcal{H}) \tag{1.4.5}$$

$$\mathcal{U}_+^{t_0}(\lambda) = M^{t_0}(\lambda)(K_+^{\mathrm{d}})^{t_0}(\lambda)\hat{M}^{t_0}(\lambda) = \begin{pmatrix} \mathcal{A}_+(\lambda) & \mathcal{C}_+(\lambda) \\ \mathcal{B}_+(\lambda) & \mathcal{D}_+(\lambda) \end{pmatrix} \in \mathrm{End}(\mathcal{V}_0 \otimes \mathcal{H}) \tag{1.4.6}$$

where the subscript 0 appearing in (1.2.7) and (1.2.20) has been omitted since the notation should be clear enough. As it was done for the periodic chain, we can now present some important commutation relations that follow from the reflection equation (1.3.2) and the definition of (1.4.5)

$$\begin{aligned} \mathcal{A}_-(\lambda)\mathcal{B}_-(\mu) =& \frac{\sinh(\lambda-\mu-\eta)\sinh(\lambda+\mu-\eta)}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)}\mathcal{B}_-(\mu)\mathcal{A}_-(\lambda) \\ &+ \frac{\sinh(\eta)\sinh(2\mu-\eta)}{\sinh(\lambda-\mu)\sinh(2\mu)}\mathcal{B}_-(\lambda)\mathcal{A}_-(\mu) \\ &- \frac{\sinh\eta}{\sinh(\lambda+\mu)\sinh 2\mu}\mathcal{B}_-(\lambda)\tilde{\mathcal{D}}_-(\mu) \end{aligned} \tag{1.4.7a}$$

$$\begin{aligned} \tilde{\mathcal{D}}_-(\lambda)\mathcal{B}_-(\mu) =& \frac{\sinh(\lambda-\mu-\eta)\sinh(\lambda+\mu-\eta)}{\sinh(\lambda-\mu)\sinh(\lambda+\mu)}\mathcal{B}_-(\mu)\tilde{\mathcal{D}}_-(\lambda) \\ &+ \frac{\sinh(\eta)\sinh(2\lambda+\eta)\sinh(2\mu-\eta)}{\sinh(\lambda+\mu)\sinh(2\mu)}\mathcal{B}_-(\lambda)\mathcal{A}_-(\mu) \\ &- \frac{\sinh(\eta)\sinh(2\lambda+\eta)}{\sinh(\lambda-\mu)\sinh(2\mu)}\mathcal{B}_-(\lambda)\tilde{\mathcal{D}}_-(\mu) \end{aligned} \tag{1.4.7b}$$

where $\tilde{\mathcal{D}}_-(\lambda) = \mathcal{D}_-(\lambda)\sinh 2\lambda - \mathcal{A}_-(\lambda)\sinh\eta$. Similar relations can be proven for the algebra $\mathcal{U}^+$ as well.

Finally, by considering the arguments followed in the previous section, we arrive to the define the transfer matrix

$$\begin{aligned} \mathcal{T}(\lambda) &= \mathrm{tr}_0\left\{K_+^{\mathrm{d}}(\lambda)M(\lambda)K_-^{\mathrm{d}}(\lambda)\hat{M}(\lambda)\right\} \\ &= \mathrm{tr}_0\left\{K_+^{\mathrm{d}}(\lambda)\mathcal{U}_-(\lambda)\right\} = \mathrm{tr}_0\left\{\mathcal{U}_+(\lambda)K_-^{\mathrm{d}}(\lambda)\right\}. \end{aligned} \tag{1.4.8}$$

Also in this case it is possible to prove a *trace identity* in order to link the QISM machinery to the Hamiltonian (1.4.1), this bridge has been built in the following proposition:

Proposition 1.4.1 (Kulish-Sklyanin). *Given the reflection algebra defined by the R-matrix* (1.2.4)*, its property* (1.2.5a) *and the double-row monodromy matrices* (1.4.5) *and* (1.4.6)*, the Hamiltonian of the diagonal boundary open XXZ quantum spin-*$1/2$ *chain is given by the following trace identity:*

$$H_{XXZ}^{D.B.} = \frac{2(\sinh\eta)^{(1-2N)}}{tr\left\{K_+^d(\eta/2)\right\} tr\left\{K_-^d(\eta/2)\right\}} \frac{d}{d\lambda}\ln(t(\lambda))\Big|_{\substack{\lambda=\eta/2 \\ \xi_1,\ldots,\xi_N=0}} - c(\eta), \tag{1.4.9}$$

with $c(\eta) = (2N-1)\cosh\eta + \tanh\eta\sinh\eta$.

Proof. The result can be showed explicitly in the same way of the other trace identity 1.2.1. Let us just remind (1.2.5a): the R-matrix for $\lambda = 0$ is proportional to the permutation operator

$$R(0) = \sinh\eta \cdot \mathbb{P}_{12}$$

and the fact that

$$K_-(\eta/2) = \mathbb{1}.$$

First of all, consider the explicit expression of the transfer matrix

$$\mathcal{T}(\lambda) = \mathrm{tr}_0\{K_+^{\mathrm{d}}(\lambda)R_{0N}(\lambda - \xi_N - \eta/2)\dots R_{01}(\lambda - \xi_1 - \eta/2) \\ \times K_-^{\mathrm{d}}(\lambda)R_{01}(\lambda + \xi_1 - \eta/2)\dots R_{0N}(\lambda + \xi_N - \eta/2)\}$$

and by setting $\{\xi_j\}_{j=1,\dots,N} = 0$, it follows by explicit computation

$$\frac{d}{d\lambda}\mathcal{T}(\lambda)|_{\lambda=\eta/2} = s^{2N}\mathrm{tr}_0\{\hat{K_+^{\mathrm{d}}}(\eta/2)\}+ \\ + s^{2N}\mathrm{tr}_0\{K_+^{\mathrm{d}}(\eta/2)\mathbb{P}_{0N}\dots\mathbb{P}_{01}\hat{K_-^{\mathrm{d}}}(\eta/2)\mathbb{P}_{01}\dots\mathbb{P}_{0N}\} \\ + 2s^{2N-1}\sum_{j=1}^{N}\left[\mathrm{tr}_0\{K_+^{\mathrm{d}}(\eta/2)\mathbb{P}_{0N}\dots R'_{0j}(0)\mathbb{P}_{0j}\dots\mathbb{P}_{0N}\}\right]$$

where $\hat{K_\pm^{\mathrm{d}}}(\eta/2) = \frac{d}{d\lambda}K_\pm^{\mathrm{d}}(\lambda)|_{\lambda=\eta/2}$ and $s = \sinh\eta$. Now considering that $\mathbb{P}_{0j}\underset{0}{X}\mathbb{P}_{0j} = \underset{1}{X}$, it follows that

$$\mathrm{tr}_0\{K_+^{\mathrm{d}}(\eta/2)\mathbb{P}_{0N}\dots\mathbb{P}_{01}\hat{K^{\mathrm{d}}}_-(\eta/2)\mathbb{P}_{01}\dots\mathbb{P}_{0N}\} = \mathrm{tr}_0\{K_+^{\mathrm{d}}(\eta/2)\}\cdot\underset{1}{\hat{K_-^{\mathrm{d}}}}(\eta/2) \\ = \mathrm{tr}_0\{K_+^{\mathrm{d}}(\eta/2)\}\cdot\begin{pmatrix}\frac{\cosh\zeta_-}{\sinh\zeta_-} & 0\\ 0 & -\frac{\cosh\zeta_-}{\sinh\zeta_-}\end{pmatrix}_{[1]} = \mathrm{tr}_0\{K_+^{\mathrm{d}}(\eta/2)\}\cdot\coth\zeta_-\sigma_1^z.$$

Moreover the following equality holds

$$2\sum_{j=1}^{N}\left[\mathrm{tr}_0\{K_+^{\mathrm{d}}(\eta/2)\mathbb{P}_{0N}\dots R'_{0j}(0)\mathbb{P}_{0j}\dots\mathbb{P}_{0N}\}\right] \\ = 2\sum_{j=1}^{N-1}\mathrm{tr}_0\{K_+^{\mathrm{d}}(\eta/2)\}R'_{jj+1}(0)\mathbb{P}_{jj+1} + 2\mathrm{tr}_0\{K_+^{\mathrm{d}}(\eta/2)R'_{0N}(0)\mathbb{P}_{0N}\},$$

where we can recognize in the first term of the *rhs* the same structure that appeared in 1.2.1

$$R'_{jj+1}(0)\mathbb{P}_{jj+1} = \frac{1+\sigma_n^z\sigma_{n+1}^z}{2}\cosh\eta + \frac{\sigma_n^x\sigma_{n+1}^x}{2} + \frac{\sigma_n^y\sigma_{n+1}^y}{2}.$$

The second term results in

$$\mathrm{tr}_0\{K_+^{\mathrm{d}}(\eta/2)R'_{0N}(0)\mathbb{P}_{0N}\} = \mathrm{tr}_0\{\mathbb{P}_{0N}K_+^{\mathrm{d}}(\eta/2)\mathbb{P}_{0N}\mathbb{P}_{0N}R'_{0N}(0)\} \\ = \underset{N}{K_+^{\mathrm{d}}}(\eta/2)\mathrm{tr}_0\{R'_{0N}(0)\mathbb{P}_{0N}\} = \underset{N}{K_+^{\mathrm{d}}}(\eta/2)\cosh\eta \\ = \begin{pmatrix}\frac{\sinh(\eta+\zeta_+)}{\sinh\zeta_+} & 0\\ 0 & \frac{\sinh(-\eta+\zeta_+)}{\sinh\zeta_+}\end{pmatrix}_{[N]}\cosh\eta = (\sinh\eta\coth\zeta_+\sigma_N^z + \cosh\eta\mathbb{1})\cosh\eta.$$

Finally, considering that $\mathrm{tr}_0\{K_+^{\mathrm{d}}(\eta/2)\} = 2\cosh\eta$ one gets the result (1.4.9). □

1.4.2 Quantum determinants and some basic properties

Here, we want to reproduce some results which are equivalent to the ones established for the closed chain in section §1.2.3. Some of the explicit consequences of the properties we are going to present will be left for a later chapter to avoid repetitions and for a better organization of the current dissertation. By following once more Sklyanin's prescriptions, we can resume some important properties of the XXZ reflection algebra $\mathcal{U}^-$ in the following proposition.

Proposition 1.4.2 (Sklyanin). *The quantity*

$$q\text{-}det\ (\mathcal{U}_-(\lambda)) = tr_{12}\{P_{12}^- \underset{1}{\mathcal{U}_-}(\lambda-\eta/2)R_{21}(2\lambda-\eta)\underset{2}{\mathcal{U}_-}(\lambda+\eta/2)\} \tag{1.4.10}$$

is a central element of the reflection algebra $\mathcal{U}^-$

$$[q\text{-}det\ (\mathcal{U}_-(\lambda)), \mathcal{U}_-(\mu)] = 0, \quad \forall(\alpha,\mu) \in \mathbb{C}^2. \tag{1.4.11}$$

This object plays a role in the inversion of the monodromy matrix since

$$\mathcal{U}_-^{-1}(\lambda+\eta/2) = \frac{\sinh(2\lambda-2\eta)}{q\text{-}det\ (\mathcal{U}_-(\lambda))}\mathcal{U}_-(-\lambda+\eta/2). \tag{1.4.12}$$

The expression in terms of algebra operators is given by

$$\begin{aligned} q\text{-}det\ (\mathcal{U}_-(\lambda)) =& \tilde{\mathcal{D}}_-(\lambda-\eta/2)\mathcal{A}_-(\lambda+\eta/2) - \tilde{\mathcal{B}}_-(\lambda-\eta/2)\mathcal{C}_-(\lambda+\eta/2) \\ =& \tilde{\mathcal{A}}_-(\lambda-\eta/2)\mathcal{D}_-(\lambda+\eta/2) - \tilde{\mathcal{C}}_-(\lambda-\eta/2)\mathcal{B}_-(\lambda+\eta/2) \\ =& \mathcal{A}_-(\lambda+\eta/2)\tilde{\mathcal{D}}_-(\lambda-\eta/2) - \mathcal{C}_-(\lambda+\eta/2)\tilde{\mathcal{B}}_-(\lambda-\eta/2) \\ =& \mathcal{D}_-(\lambda+\eta/2)\tilde{\mathcal{A}}_-(\lambda-\eta/2) - \mathcal{B}_-(\lambda+\eta/2)\tilde{\mathcal{C}}_-(\lambda-\eta/2) \end{aligned} \tag{1.4.13}$$

where we have used the operator entries of the algebraic adjunct $\tilde{\mathcal{U}}_-(\lambda)$ *defined as*

$$\begin{aligned} \underset{1}{\tilde{\mathcal{U}}_-}(\lambda) = 2tr_2\{P_{12}^- \underset{2}{\mathcal{U}_-}(\lambda)R_{12}(\lambda)\} = \begin{pmatrix} \tilde{\mathcal{D}}_-(\lambda) & -\tilde{\mathcal{B}}_-(\lambda) \\ -\tilde{\mathcal{C}}_-(\lambda) & \tilde{\mathcal{A}}_-(\lambda) \end{pmatrix} \\ = \begin{pmatrix} -\sinh\eta\mathcal{A}_-(\lambda)+\sinh(2\lambda)\mathcal{D}_-(\lambda) & -\sinh(2\lambda+\eta)\mathcal{B}_-(\lambda) \\ -\sinh(2\lambda+\eta)\mathcal{C}_-(\lambda) & -\sinh\eta\mathcal{D}_-(\lambda)+\sinh(2\lambda)\mathcal{A}_-(\lambda) \end{pmatrix}. \end{aligned} \tag{1.4.14}$$

It's then clear that we can express

$$q\text{-}det\ (\mathcal{U}_-(\lambda)) = \tilde{\mathcal{U}}_-(\lambda-\eta/2)\mathcal{U}_-(\lambda+\eta/2) = \mathcal{U}_-(\lambda+\eta/2)\tilde{\mathcal{U}}_-(\lambda-\eta/2). \tag{1.4.15}$$

The explicit functional expression of the quantum determinant is given by

$$\begin{aligned} q\text{-}det\ (\mathcal{U}_-(\lambda)) =& q\text{-}det\ (M_0(\lambda))q\text{-}det\ (M_0(-\lambda))q\text{-}det\ (K_-^d(\lambda)) \\ =& \Big(-\sinh(2\lambda-\eta)\frac{\sinh(\lambda+\zeta_-)\sinh(\lambda-\zeta_-)}{\sinh^2\zeta_-}\times \\ & \prod_{j=1}^N \sinh(\lambda-\xi_j+\eta)\sinh(\lambda-\xi_j-\eta)\sinh(\lambda+\xi_j-\eta)\sinh(\lambda+\xi_j+\eta)\Big) \end{aligned} \tag{1.4.16}$$

Proof. The fact that (1.4.10) is indeed a good definition for a central element of the algebra comes from a slight generalization of what it was shown in proposition 1.2.2 and more extensively in [81]. Let us start to prove the expression (1.4.16), by considering the following chain of relations

$$\begin{aligned}\text{q-det}\,(\mathcal{U}_-(\lambda)) &= \mathrm{tr}_{12}\{P^-_{12}\underset{1}{\mathcal{U}_-}(\lambda-\eta/2)R_{21}(2\lambda-\eta)\underset{2}{\mathcal{U}_-}(\lambda+\eta/2)\} \\ &= \mathrm{tr}_{12}\{P^-_{12}M_1(\lambda-\eta/2)\underset{1}{K^{\mathrm{d}}_-}(\lambda-\eta/2)\hat{M}_1(\lambda-\eta/2)R_{21}(2\lambda-\eta) \\ &\qquad \times M_2(\lambda+\eta/2)\underset{2}{K_-}(\lambda+\eta/2)\hat{M}_2(\lambda+\eta/2)\}\end{aligned}$$

then by using the YB relation

$$\hat{M}_1(\lambda-\eta/2)R_{21}(2\lambda-\eta)M_2(\lambda+\eta/2) = M_2(\lambda+\eta/2)R_{21}(2\lambda-\eta)\hat{M}_1(\lambda-\eta/2)$$

and exploiting the commutativity of objects defined on different spaces we arrive at the result

$$\begin{aligned}&\mathrm{tr}_{12}\{P^-_{12}M_1(\lambda-\eta/2)M_2(\lambda+\eta/2)P^-_{12}\underset{1}{K^{\mathrm{d}}_-}(\lambda-\eta/2)R_{21}(2\lambda-\eta) \\ &\qquad \times \underset{2}{K^{\mathrm{d}}_-}(\lambda+\eta/2)\hat{M}_1(\lambda-\eta/2)\hat{M}_2(\lambda+\eta/2)P^-_{12}\} \\ &\qquad\qquad = \text{q-det}\,(M_1(\lambda))\text{q-det}\,(K^{\mathrm{d}}_-(\lambda))\text{q-det}\,(M_2(-\lambda)).\end{aligned}$$

For the last equality property (1.2.24) and $(P^-_{12})^2 = P^-_{12}$ have been used. The quantum determinant relative to the K_- matrix can be computed explicitly by using the definition and it reads

$$\text{q-det}\,(K^{\mathrm{d}}_-(\lambda)) = \sinh(2\lambda-2\eta)\frac{\sinh(\lambda+\zeta_-)\sinh(-\lambda+\zeta_-)}{\sinh^2\zeta_-}. \tag{1.4.17}$$

Representations (1.4.13) are simply obtained by direct computation keeping in mind the standard tensor representation, used in Prop. 1.2.2, and commutation relations (1.4.7). The very same expressions can be obtained provided we define the algebraic adjunct (1.4.14) and perform a simple matrix product in the auxiliary space (1.4.15). Finally in order to prove (1.4.12), consider the following equality

$$[K^d_-(\lambda+\eta/2)]^{-1} = \frac{\sinh(2\lambda-2\eta)}{\text{q-det}\,(K^{\mathrm{d}}_-(\lambda))}K_-(-\lambda+\eta/2). \tag{1.4.18}$$

where the q-det (K^{d}_-) has already been calculated above. Now, by direct computation

$$\begin{aligned}\mathcal{U}_-(\lambda+\eta/2)\mathcal{U}_-(-\lambda+\eta/2) &= M_0(\lambda+\eta/2)K^{\mathrm{d}}_-(\lambda+\eta/2)\hat{M}_0(\lambda+\eta/2) \\ &\quad \times M_0(-\lambda+\eta/2)K^{\mathrm{d}}_-(-\lambda+\eta/2)\hat{M}_0(-\lambda+\eta/2) \\ &\qquad = \text{q-det}\,(M_1(\lambda))\text{q-det}\,(K^{\mathrm{d}}_-(\lambda))\text{q-det}\,(M_2(-\lambda))\end{aligned}$$

where property (1.2.19) has been used. □

Proposition 1.4.3. *The following parity relations among generator families hold true*

$$\mathcal{A}_-(\lambda) = \frac{\sinh(2\lambda-\eta)}{\sinh 2\lambda}\mathcal{D}_-(-\lambda) + \frac{\sinh\eta}{\sinh 2\lambda}\mathcal{D}_-(\lambda), \tag{1.4.19a}$$

$$\mathcal{D}_-(\lambda) = \frac{\sinh(2\lambda-\eta)}{\sinh 2\lambda}\mathcal{A}_-(-\lambda) + \frac{\sinh\eta}{\sinh 2\lambda}\mathcal{A}_-(\lambda), \tag{1.4.19b}$$

$$\mathcal{B}_-(-\lambda) = -\frac{\sinh(2\lambda+\eta)}{\sinh(2\lambda-\eta)}\mathcal{B}_-(\lambda), \tag{1.4.19c}$$

$$\mathcal{C}_-(-\lambda) = -\frac{\sinh(2\lambda+\eta)}{\sinh(2\lambda-\eta)}\mathcal{C}_-(\lambda). \tag{1.4.19d}$$

Proof. By comparing formulas (1.4.12) and (1.4.15) is immediate to show that

$$\tilde{\mathcal{U}}_-(\lambda) = \sinh(2\lambda - \eta)\mathcal{U}_-(-\lambda), \tag{1.4.20}$$

and proposition is then proved by equating term to term the two matrices and simple algebraic manipulations. □

A set of properties, equivalent to the one organized in propositions 1.4.2 and 1.4.3, can be established for the $\mathcal{U}^+$ algebra as well.

1.4.3 The ABA for the open chain

We would like now to give a brief sketch of results for what concerns the algebraic Bethe ansatz technique applied to the diagonal boundary open XXZ spin-1/2 chain.

The technique works mostly just as in the periodic case. Indeed, thanks to the S^z property (1.4.2) of the Hamiltonian one can use the very same magnetic pseudovacuum (1.2.25) $|0\rangle = \bigotimes_j |\uparrow\rangle_j$. We can explicitly see that it is a good reference state since

$$\begin{aligned} &\mathcal{C}_\pm(\lambda)\,|0\rangle = 0, \quad \forall\lambda \\ &\mathcal{A}_\pm(\lambda) = \alpha(\lambda)\,|0\rangle\,, \quad \alpha(\lambda) \in \mathbb{C} \\ &\mathcal{D}_\pm(\lambda) = \delta(\lambda)\,|0\rangle\,, \quad \delta(\lambda) \in \mathbb{C} \end{aligned}$$

and the commutation relations

$$[\mathcal{B}_\pm(\lambda), \mathcal{B}_\pm(\mu)] = [\mathcal{C}_\pm(\lambda), \mathcal{C}_\pm(\mu)] = 0$$

still hold. At this point we can give the theorem

Theorem 1.4.1. *Consider the vector*

$$\left|\Psi^\pm_M(\{\lambda_k\}_{k=1,\dots,M})\right\rangle = \prod_{n=1}^{M} \mathcal{B}_\pm(\lambda_n)\,|0\rangle\,, \quad \forall M \in (1,\dots,N). \tag{1.4.21}$$

It is an eigenvector of the transfer matrix (1.4.8)

$$\mathcal{T}(\lambda) = (K_\pm(\lambda))_{11}\mathcal{A}_\mp(\lambda) + (K_\pm(\lambda))_{22}\mathcal{D}_\mp(\lambda)$$

for any λ with eigenvalue $\tau(\lambda)$

$$\begin{aligned} \tau(\lambda;\Psi_M) =& a(\lambda-\eta/2)d(-\lambda-\eta/2)\frac{\sinh(2\lambda+\eta)}{\sinh 2\lambda} \\ & \frac{\sinh(\lambda+\zeta_+-\eta/2)\sinh(\lambda+\zeta_--\eta/2)}{\sinh\zeta_+\sinh\zeta_-} \\ & \times\prod_{j=1}^{M}\frac{\sinh(\lambda-\lambda_j-\eta)\sinh(\lambda+\lambda_j-\eta)}{\sinh(\lambda-\lambda_j)\sinh(\lambda+\lambda_j)} \\ & + a(-\lambda-\eta/2)d(\lambda-\eta/2)\frac{\sinh(2\lambda-\eta)}{\sinh 2\lambda} \\ & \frac{\sinh(\lambda-\zeta_++\eta/2)\sinh(\lambda+\zeta_-+\eta/2)}{\sinh\zeta_+\sinh\zeta_-} \\ & \times\prod_{j=1}^{M}\frac{\sinh(\lambda-\lambda_j+\eta)\sinh(\lambda+\lambda_j+\eta)}{\sinh(\lambda-\lambda_j)\sinh(\lambda+\lambda_j)} \end{aligned} \tag{1.4.22}$$

provided the following BAE hold

$$\frac{a(\lambda_k - \eta/2)d(-\lambda_k - \eta/2)}{a(-\lambda_k - \eta/2)d(\lambda_k - \eta/2)} \frac{\sinh(\lambda_k + \zeta_+ - \eta/2)\sinh(\lambda_k + \zeta_- - \eta/2)}{\sinh(\lambda_k - \zeta_+ + \eta/2)\sinh(\lambda_k - \zeta_- + \eta/2)} = \prod_{j=1, j\neq k}^{M} \frac{\sinh(\lambda_k - \lambda_j + \eta)\sinh(\lambda_k + \lambda_j + \eta)}{\sinh(\lambda_k - \lambda_j - \eta)\sinh(\lambda_k + \lambda_j - \eta)}$$

Remark 1.4.1. The ABA technique is a useful tool for open chains just in some particular situation. We saw it works for the diagonal case, but in order to retrieve informations about the spectrum of such a transfer matrix with less strict constraints on the boundaries, different methods and approaches have been developed so far (consider for example: [97, 98, 23, 55, 9]. One of this approaches, the *separation of variables method*, will be applied in the future chapters of this thesis in order to deal with completely generic boundary conditions.

CHAPTER 2

THE *SEPARATION OF VARIABLES METHOD*

The quantum *separation of variables* (SoV) method was developed by E.K. Sklyanin between the end of the 80's and beginning of the 90's. In this section we would like to present the method in its, as far as possible, general set-up and philosophy. The core reference for the following text is the review papers [117, 121] and the exposition has to be meant as merely introductory, bibliographic and, hopefully, useful to the reader in order to understand the motivations for next chapters to come.

The SoV method, *functional Bethe ansatz* (FBA) in [117], can be considered essentially as the meeting point for two traditions in the study of integrable models: the classical separation of variables method (CSoV) and the quantum inverse scattering method, which we described in the previous chapter. The CSoV [59, 6] is a very old idea, which was successfully applied in the study of Hamiltonian systems and can be briefly defined as a *reduction of a multi-dimensional problem to a set of one-dimensional ones*. It was originated from the works of D'Alembert and Fourier in wave theory and Jacobi in Hamiltonian mechanics. For long time the CSoV had been the only known method in order to solve "exactly" problems of mathematical physics.

The Hamilton-Jacobi method: an overview

Consider a mechanical system with f degrees of freedom described by the canonical variables $(q_1, \dots, q_f)$ and the conjugated momenta $(p_1, \dots, p_f)$. The system will be defined by a Hamiltonian functional which depends on the set of variables just defined, i.e. $H = H(p_1, \dots, p_f, q_1, \dots, q_n)$ *. The time evolution of the system is regulated by the *Hamilton's equations*

$$\begin{cases} \dot{p}_n = -\frac{\partial H}{\partial q_n}, \\ \dot{q}_n = \frac{\partial H}{\partial p_n}, \end{cases} \qquad \forall n \in \{1, \dots, f\} \tag{2.0.1}$$

*. For this short review, we decided to give a hint of what CSoV is just in the case of a stationary Hamiltonian, i.e. a functional which has no explicit temporal dependence. The theory is well defined as well in the case of a Hamiltonian which depends explicitly on time, see [6, 59].

where $(\dot{p}_n, \dot{q}_n) \equiv (\frac{dp_n}{dt}, \frac{dq_n}{dt})$. Equations (2.0.1) are often rewritten in different notations; for example by introducing the compact notation $\boldsymbol{x} = (p_1, \ldots, p_f, q_1, \ldots, q_f)$, the gradient operator $\nabla_{\boldsymbol{x}} = (\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_f}, \frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_f})$ and the symplectic matrix

$$\Omega = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{2.0.2}$$

where $\mathbb{1} = \prod_{n=1}^{f} \mathbb{1}_n$, we get the following expression for the Hamilton's equations

$$\dot{\boldsymbol{x}} = \Omega \nabla_{\boldsymbol{x}} H. \tag{2.0.3}$$

A useful object that can be defined in order to symmetrize equations (2.0.1) and (2.0.3) is the Poisson brackets.

Definition 2.0.1. Given two functions $f(p_1, \ldots, p_f, q_1, \ldots, q_f)$ and $g(p_1, \ldots, p_f, q_1, \ldots, q_f)$, their Poisson brackets are defined as

$$\{f, g\} = \sum_{i=1}^{f} \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right] = \nabla_{\boldsymbol{x}} f \cdot \Omega \cdot \nabla_{\boldsymbol{x}} g. \tag{2.0.4}$$

Indeed, Hamilton's equations can now be rewritten in the more symmetrical form

$$\begin{cases} \dot{p}_i = \{p_i, H\}, \\ \dot{q}_i = \{q_i, H\}. \end{cases} \tag{2.0.5}$$

The Poisson brackets are also useful to define the time evolution of a function $f(p_1, \ldots, p_f, q_1, \ldots, q_f)$, since it follows that

$$\frac{df}{dt} = \{f, H\} \tag{2.0.6}$$

as it can be easily established by direct computation and taking into account equations (2.0.1). This gives the possibility to characterize special quantity known as *integrals of motion*, or ,in other words, objects which stay constant during the time evolution of the system. These quantities possess then the property, after (2.0.6)

$$\{f, H\} = 0. \tag{2.0.7}$$

As mentioned in the Introduction, a Hamiltonian system will be said integrable, *à la Liouville*, if it possess a number f of integrals of motion $\{H_1, \ldots, H_f\}$ which are in involution, i.e. they satisfy the following Poisson commutation relations

$$\{H_i, H_j\} = 0, \qquad \forall i, j = 1, \ldots, f. \tag{2.0.8}$$

A powerful technique that was developed in order to solve the Hamilton's equations is called *Hamilton-Jacobi method*. The main idea consists in searching a particular coordinate transformation, a canonical transformation, which maintains unchanged the symplectic structure of (2.0.1), and, in the same time, maps (2.0.1) in other equations associated to a constant Hamiltonian. This new set of coordinates are commonly known as *action-angle* variables $(I_1, \ldots, I_f, \varphi_1, \ldots, \varphi_N)$. Following [6], in order to define a canonical transformation, one has to introduce a further object: the *generating function* $S(\boldsymbol{h}(\boldsymbol{I}), \boldsymbol{q})$, such that

$$h_j = h_j(I_1, \ldots, I_f), \quad p_j = \frac{\partial S(\boldsymbol{h}(\boldsymbol{I}), \boldsymbol{q})}{\partial q_j}, \quad \varphi_j = \frac{\partial S(\boldsymbol{h}(\boldsymbol{I}), \boldsymbol{q})}{\partial I_j}, \quad \forall j \in \{1, \ldots, f\}, \tag{2.0.9}$$

where the $\boldsymbol{h}$ are some value fixing of the constant quantities $\boldsymbol{H}$. The generating function has to be solution of the *Hamilton-Jacobi equation*

$$H(\frac{\partial S(\boldsymbol{h}(\boldsymbol{I}),\boldsymbol{q})}{\partial q_j},\boldsymbol{q}) = E, \tag{2.0.10}$$

where E is the constant value taken by the Hamiltonian in the new coordinates. All the other integrals of motion will satisfy a similar equation

$$H_j(\frac{\partial S(\boldsymbol{h}(\boldsymbol{I}),\boldsymbol{q})}{\partial q_j},\boldsymbol{q}) = h_j(\boldsymbol{I}). \tag{2.0.11}$$

The separation of variables method consists in the research of conditions on the structure of the Hamiltonian that permit a step-by-step elimination of variables through factorization. In term of Hamilton-Jacobi equation, it translates in looking for a solution of the type

$$S(\boldsymbol{h},q_1,\ldots,q_f) = S_1(\boldsymbol{h},q_1) + S_2(\boldsymbol{h},q_2) + \cdots + S_f(\boldsymbol{h},q_f), \tag{2.0.12}$$

where every jth term depends uniquely on the jth coordinate q_j, being the $\boldsymbol{h}$s fixed parameters. This program leads to a solution of the Hamilton's equation by quadrature, which is exactly the reduction of a multi-dimensional problem to a set of one-dimensional ones.

An example: the point particle in a central potential

Consider, for example, the problem related to the motion of a point particle in the central Newtonian potential. In spherical coordinates $\{r,\theta,\phi\}$, the Hamiltonian reads

$$H_{\scriptscriptstyle Newton} = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + \frac{p_\phi^2}{2r^2\sin^2(\theta)} - \frac{M}{r}, \tag{2.0.13}$$

where we set the gravitational constant $G = 1$ and the mass of the point particle $m = 1$. It turns out that the Hamilton-Jacobi equation of this problem is the following

$$\frac{1}{2}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{2r^2\sin 2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2 - \frac{M}{r} - E = 0. \tag{2.0.14}$$

Let us introduce, as a solution attempt, the following separated form

$$S(r,\theta,\phi) = S_\phi(\phi) + S_r(r) + S_\theta(\theta), \tag{2.0.15}$$

After some easy manipulation one arrives at

$$r^2\sin^2\theta\frac{1}{2}\left(\frac{dS_r}{dr}\right)^2 + sin^2\theta\frac{1}{2r^2}\left(\frac{dS_\theta}{d\theta}\right)^2 + -2r^2\sin 2\theta\left(\frac{M}{r} - E\right) = -\left(\frac{dS_\phi}{d\phi}\right)^2. \tag{2.0.16}$$

Now, we see that the rhs depends just on ϕ while the lhs is independent of it; the only possibility is that both sides have a constant value then it holds

$$S_\phi(\phi) = p_\phi\phi. \tag{2.0.17}$$

By similar arguments one can arrive to a set of separated equations for the components S_r and S_θ

$$\begin{aligned} \frac{dS_r}{dr} &= \sqrt{2\left(\frac{M}{r} + E - \frac{L^2}{2r^2}\right)}, \\ \frac{dS_\theta}{d\theta} &= \sqrt{L^2 - \frac{p_\phi^2}{\sin^2\theta}}, \end{aligned} \tag{2.0.18}$$

where L^2 is the squared angular momentum,

$$L^2 = r^4\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}, \tag{2.0.19}$$

which is an integral of motion as well as its z-component $L_z = p_\phi$ and of course H. This results will lead then to the complete solution of the equations of motion by quadrature.

Let us summarize by introducing some notation used in [121]. Given a Hamiltonian system with f degrees of freedom and integrable in the Liouvilles's sense, it will possess a set of f independent constants of motion H_j commuting with respect to the Poisson structure

$$\{H_j, H_k\} = 0, \qquad j,k = 1,\ldots,f.$$

A system of canonical variables $(\boldsymbol{p}, \boldsymbol{q})$ will be called separated if there exist a set of f relations of the type

$$\Phi_j(q_j, p_j, h_1, \ldots, h_f) = 0, \qquad j = 1,\ldots,f \tag{2.0.20}$$

binding together each pair (p_j, q_j) and the constant values taken by the integrals of motion $H_j = h_j$.

The quantum side

In the classical theory, the SoV can be produced by a more or less complicated canonical transformation, as we briefly discussed above. Once we step into the quantum regime, the search for the appropriate transformation is substituted by the definition of a unitary operator. The papers of Gutzwiller [63, 64] on 3,4-particle Toda lattice and of Komarov [75, 76] on Goryachev-Chaplygin top provided the first example of how such a unitary operator could be guessed by using known CSoV results and its scheme.

Consider a quantum mechanical system which possess a CSCO (complete set of commuting observables) formed by f operators H_j for $j = 1,\ldots,f$. Consider now a set of canonical coordinates organizable in f pairs (x_j, p_j) for $j = 1,\ldots,N$, such that

$$[x_j, x_k] = [p_j, p_k] = 0, \qquad [p_j, x_j] = -i\delta_j k, \qquad \forall j,k = 1,\ldots,f. \tag{2.0.21}$$

Now, suppose that the common spectrum of $\{x_j\}_{j=1}^f$ is simple, in a way that the whole Hilbert space is isomorphic to a space of functions on spec$\{x_j\}_{j=1}^f$. The momenta p_j are then realized as the usual differential operators $p_j = \partial/\partial x_j$. Then suppose, there exist a set of f polynomials of the form

$$\Phi_j(p_j, x_j, H_1, \ldots, H_f) = 0, \qquad j = 1,\ldots,f, \tag{2.0.22}$$

where the "enlisting" order $p_j x_j H_1 \ldots H_f$ coincide with the actual ordering. Now consider a common eigenfunction of the H_js $\Psi(x_1, \ldots, x_f)$, such that

$$H_j\Psi = h_j\Psi, \tag{2.0.23}$$

then from (2.0.22) we get

$$\Phi_j(-\frac{\partial}{\partial x_j}, x_j, h_1, \dots, h_f)\Psi(x_1, \dots, x_f) = 0, \qquad j = 1, \dots, f, \tag{2.0.24}$$

which permits the separation of variables

$$\Psi(x_1, \dots, x_f) = \prod_{j=1}^{f} \Psi(x_j). \tag{2.0.25}$$

The original multidimensional spectral problem has been reduced to a set of f one-dimensional multi-parameter spectral problems

$$\Phi_j(-\frac{\partial}{\partial x_j}, x_j, h_1, \dots, h_f)\Psi_j(x_j) = 0, \qquad j = 1, \dots, f. \tag{2.0.26}$$

This construction at this stage remains quite a qualitative one, since it establishes only some local separation of variables and for the actual global ones one should request further conditions [122].

This approach has been applied to quantum systems which are considered integrable under the light of QISM machinery as well. For example the results on the Toda chain and the GC top by Gutzwiller and Komarov have been re-established in the QISM formalislm by Sklyanin in [115] and [114]. In the next section we will se how this program can be used for the quantum integrable models generated by the Yangian $\mathcal{Y}[sl(2)]$.

2.1 The exact construction of SoV for $\mathcal{Y}[sl(2)]$ models

In this section, we will summarize, following [117], the *step-by-step* construction that has to be implemented in order to build the SoV representation for models with the symmetry defined by the Yangian $\mathcal{Y}[sl(2)]$. Simply speaking, such models can be characterized by a QISM construction with the rational R-matrix (solution of the Yang-Baxter equation)

$$R_{12}^{\text{XXX}}(\lambda) = \lambda \mathbb{1} + \eta \mathbb{P}_{12} = \begin{pmatrix} \lambda+\eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda+\eta \end{pmatrix} \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2), \tag{2.1.1}$$

where $\mathbb{P}_{12}$ is the permutation operator defined in chapter **1**. Consider the monodromy matrix as a polynomial in λ

$$M(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = \sum_{n=1}^{N} \lambda^n M_n = \sum_{n=1}^{N} \lambda^n \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}, \tag{2.1.2}$$

which will define a representation for the QISM algebra $\mathcal{M}$ in a finite-dimensional space W. The commutation relations among the components of $M(\lambda)$ are encoded, as usual, in the Yang-Baxter relation that it satisfies

$$R_{12}^{\text{XXX}}(\lambda-\mu)M_1(\lambda)M_2(\lambda) = M_2(\lambda)M_1(\lambda)R_{12}^{\text{XXX}}(\lambda-\mu). \tag{2.1.3}$$

The initial key point in order to develop the quantum SoV analysis is the commutativity of the coefficients of the operator $B(\lambda)$,

$$[B_n, B_m] = 0, \qquad \forall (n, m) \in \{1, \dots, N\}, \tag{2.1.4}$$

which derive from

$$[B(\lambda), B(\mu)] = 0, \tag{2.1.5}$$

as it follows easily from the quadratic relation (2.1.3).

Proposition 2.1.1. *The operator roots* $\{x_n\}_{n=1}^N$ *of* $B(\lambda)$ *provide the separated variables.*

The explanation and the proof of proposition 2.1.1 will be given in the next sections.

2.1.1 The *operator roots*

Following Sklyanin's argument, in order to avoid particular or degenerate cases, which would request an *ad hoc* analysis, we need to introduce and impose some conditions.

Condition 2.1.1. *The coefficient* M_N *in* (2.1.2) *is a number matrix, the quantum determinant q-det* $(M(\lambda))$ *is a numerical function.*

This condition is quite general and holds true for all the irreducible representations of $\mathcal{M}$. Next one will ensure nondegeneracy, in the sense that $B(\lambda)$ and q-det $(M(\lambda))$ are polynomials of maximal degree.

Condition 2.1.2. *Given the polynomial definition of the monodromy matrix* (2.1.2)*, we require that*

$$B_N \neq 0, \qquad det(M_N) \neq 0$$

In order to define more precisely what the operator roots are, it is convenient to introduce the following notation to express better the polynomiality of $B(\lambda))$

$$\hat{b}_n = (-1)^N \frac{B_{N-n}}{B_N}, \qquad n = 1, \dots, N, \tag{2.1.6}$$

which, once inserted in expression (2.1.2), implies the representation

$$B(\lambda) = B_N \left(\lambda^N - \hat{b}_1 \lambda^{N-1} + \hat{b}_2^{N-2} + \dots + \hat{b}_1 \lambda\right). \tag{2.1.7}$$

From the commutation relations (2.1.5), it follows that the whole set of polynomials $\{\hat{b}\}_{n=1}^N$ commute among each other

$$[\hat{b}_n, \hat{b}_m] = 0, \qquad \forall m, n = 1, \dots, N. \tag{2.1.8}$$

The first direct consequence of these commutation relations resides in the fact that they can be diagonilised simultaneously, remember that they are operator polynomials, and they will share a common spectrum

$$\mathbf{B} = \text{spec}\{\hat{b}_n\}_{n=1}^N \tag{2.1.9}$$

Always following Sklyanin, it is better at this point, to keep these notes as simple as possible, to introduce a more restrictive condition

Condition 2.1.3. *The operators* $\{\hat{b}_n\}_{n=1}^N$ *have a complete set of common eigenfunctions and to every point* $\mathbf{b} = (b_1, \dots, b_N) \in \mathbf{B} \subset \mathbb{C}^N$ *there corresponds only one eigenfunction.*

This condition seems to be quite restrictive, but it is satisfied by different models, being the nondegeneracy of the spectrum of the Bethe roots a built-in property of the method. In general different separated variables and relative shifts will label different orthogonal states. In particular it will be shown how it is satisfied for the models treated in chapters **3** and **4**. This set up permits us to see how the polynomials $\{\hat{b}_n\}_{n=1}^N$ and in particular eigenvalues $\{b_n\}_{n=1}^N$ form the structure of the representation space W of our problem. In particular W is isomorphic, up to a multiplicative non-zero constant, to the space of function $\text{Fun}(\mathbf{B})$ on the set $\mathbf{B} \subset \mathbb{C}^N$. An easy way to realize the representation is given by considering the operators $\hat{b}_n$ as multiplicative operators

$$\hat{b}_j f(\mathbf{b}) = b_j f(\mathbf{b}), \tag{2.1.10}$$

where $\mathbf{b}$ has been defined in condition (2.1.3) and $f \in \text{Fun}(\mathbf{B})$ is one of the common eigenfunction. The identity (2.1.10) allows to "mix" the notations for the operators $\hat{b}_j$s and eigenvalues b_js because of the simple multiplicative nature of their action. Because of this isomorphism, in the following, we will use just the eigenvalue symbols which will play the role of operators or numerical values depending on the situation.

At the moment, we have expressed the $B(\lambda)$ operator as a polynomial in λ of the form (2.1.7), the operator roots are somehow "trapped" inside the $\{b_n\}_{n=1}^N$. Then, we should find a way to make them explicit and realize a different representation for $B(\lambda)$, which has to be a product on the roots. To do so, we can define the following map

$$\begin{aligned} \Theta :& \mathbb{C}^N \to \mathbb{C}^N \\ & \mathbf{x} \;\; \to \mathbf{b}, \end{aligned} \tag{2.1.11}$$

realized by the equality

$$b_n(x) = s_n(x) \tag{2.1.12}$$

where the $s_n(x)$ is the elementary symmetric polynomial (ESP) of degree $n \in \{1, \dots, N\}$ in N variables $\{x_n\}_{n=1}^N$

$$\begin{aligned} s_0(\mathbf{x}) &= 1 \\ s_1(\mathbf{x}) &= x_1 + x_2 + \dots + x_N = \sum_{1 \le j \le N} x_n, \\ s_2(\mathbf{x}) &= x_1 x_2 + \dots = \sum_{1 \le j < k \le N} x_j x_k \\ &\vdots \\ s_k(\mathbf{x}) &= \sum_{1 \le j_1 < j_2 < \dots < j_k \le N} x_{j_1} x_{j_2} \dots x_{j_k} \\ &\vdots \\ s_N(\mathbf{x}) &= x_1 x_2 \dots x_N. \end{aligned} \tag{2.1.13}$$

In words, a symmetric polynomial (SP) P in N variables $\{x_n\}_{n=1}^N$ is a polynomial which remains unchanged for permutations σ of the indices $j = 1, \dots, N$: $P(x_{\sigma(1)}, \dots, x_{\sigma(N)}) = P(x_1, \dots, x_N)$. The ESP's are the basing building blocks for SP's, i.e. any SP can be expressed by sums and multiplication by constants of ESP's as it follows from the *fundamental theorem of symmetric polynomials* [82].

Remark 2.1.1. Note that this passage depicted by Sklyanin is purely formal as it is understandable by equality (2.1.12). Indeed, consider a polynomial of degree N

$$P(x) = a_N x^N + a_{N-1} x^{N-1} + \dots + a_1 x + a_0,$$

which, after the fundamental theorem of algebra, is known to possess N, not necessarily distinct, complex roots $\{x_n\}_{n=1}^N$. Then one can introduce the *Vieta's formulae* with the purpose to link the coefficients in the polynomial expression written above and its roots. They read

$$\begin{cases} x_1 + x_2 + \cdots + x_n = -\dfrac{a_{N-1}}{a_N}, \\ (x_1x_2 + x_1x_3 + \cdots + x_1x_N) + (x_2x_3 + \cdots + x_2x_N) + \cdots + x_{N-1}x_N = \dfrac{a_{N-2}}{a_N}, \\ \vdots \\ x_1x_2x_3 \ldots x_N = (-1)^N \dfrac{a_0}{a_N}, \end{cases} \tag{2.1.14}$$

or in general

$$\sum_{1\le j_1<j_2<\cdots<j_k\le N} x_{j_1}x_{j_2}\ldots x_{j_k} = (-1)^k \frac{a_{N-k}}{a_N}, \quad \forall k = 1,\ldots,N. \tag{2.1.15}$$

Now, we can notice that the lhs of (2.1.14) and (2.1.15) coincide with the elementary symmetric polynomials in (2.1.13), while at the rhs we find exactly the definitions of the b_ns we gave in (2.1.6). So the $\{b_n\}_{n=1}^N$ are the elementary symmetric polynomials $\{s_n\}_{n=1}^N$ for a certain fixed normal ordering. Normal ordering that has to be considered since the operator nature of the $\{\hat{b}_n\}_{n=1}^N$.

Let us now introduce the last condition in order to characterize the operator roots

Condition 2.1.4. *Pre-image* $\mathbf{X} = \Theta^{-1}(\mathbf{B})$ *contains no multiple points that is each* $\mathbf{b} \in \mathbf{B}$ *has exactly* N *pre-images.*

This last condition induces isomorphism between W =Fun$(\mathbf{B})$ and the space of symmetric functions SymFun$\mathbf{X}$. Now, given that spec$\{x_n\}_{n=1}^N = \mathbf{X}$ we arrive at the wanted diagonal representation

$$B(\lambda) = B_N \prod_{n=1}^N (\lambda - x_n). \tag{2.1.16}$$

Let us mention that, techicacally speaking, the above definition holds exclusively in the representation space W =SymFun$(\mathbf{X})$, but it could be assumed, as well, as a working definition in the extended *non-physical* space

$$\widetilde{W} = \text{Fun}(\mathbf{X}) \supset W = \text{SymFun}(\mathbf{X}). \tag{2.1.17}$$

The reason to introduce this bigger space resides in the definition of the image of the operators which lie on the diagonal of the monodromy matrix. To explain this statement better, we need to study the behaviour of the operators $A(\lambda)$ and $D(\lambda)$ evaluated in the operator roots $\{x_n\}_{n=1}^N$, and analyze their action on the elements of W.

2.1.2 The conjugated *momenta*

The analysis and construction of the SoV representation for the diagonal terms of the monodromy matrix is a natural step, since, as usual, these operators are the one appearing in the transfer matrix

$$\mathcal{T}(\lambda) = \text{tr}_0\{M(\lambda)\} = A(\lambda) + D(\lambda). \tag{2.1.18}$$

Consider now the following notations

$$\begin{aligned} X_n^- &= \sum_{j=1}^N x_n^j A_j \equiv [A(\lambda)]_{\lambda=x_n}, \\ X_n^+ &= \sum_{j=1}^N x_n^j D_j \equiv [D(\lambda)]_{\lambda=x_n}, \end{aligned} \tag{2.1.19}$$

which were defined in [117], as *conjugated momenta* to the separated variables x_ns. The reason why the notation $X^{\pm}$ was used will be clear later, as, in this representation, these two operators behave like ladder operators. As mentioned above and as it will be clear in a bit, the action of these momenta is defined from W to $\widetilde{W}$. Before giving the first main theorem of the theory, let us introduce some more notation which will be useful in the following

$$T_n^{\pm} : (x_1, \dots, x_n, \dots, x_N) \to (x_1, \dots, x_n + \eta, \dots, x_N). \tag{2.1.20}$$

Theorem 2.1.1. *The separated roots $\{x_n\}_{n=1}^N$ and the conjugated momenta $\{X_n\}_{n=1}^N$ satisfy the following commutation relations*

$$X_m^{\pm} x_n = (x_n \pm \eta \delta_{mn}) X_m^{\pm}, \qquad \forall m, n = 1, \dots, N. \tag{2.1.21}$$

Proof. Consider the commutation relations between the operators $A(\lambda)$ and $B(\mu)$ that can be extracted from the YB relation (2.1.3), which read

$$A(\mu)B(\lambda) = \frac{\lambda - \mu - \eta}{\lambda - \mu} B(\lambda)A(\mu) + \frac{\eta}{\mu - \lambda} B(\mu)A(\lambda). \tag{2.1.22}$$

Now, by substituting $\mu = x_n$ we get

$$(x_n - \lambda) X_n^- B(\lambda) = (x_n - \lambda + \eta) B(\lambda) X_n^-, \tag{2.1.23}$$

where we used the fact that $B(x_n) = 0$ and the definition (2.1.19). By substituting the explicit form for $B(\lambda)$ (2.1.16), simplifying the common terms of the lhs and rhs and by expanding both sides in powers of λ we get

$$X_n^- s(\mathbf{x}) = s(T_n^- \mathbf{x}) X_n^-, \tag{2.1.24}$$

for any symmetric polynomial $s(\mathbf{x})$. Note that we are free to interchange the term symmetric polynomial with symmetric function since the number of "variables", i.e. the number of elements of $\mathbf{X}$, is finite [125, 92]. From identity (2.1.24) we can understand what mentioned before, that the conjugated momenta are operators acting in $W \to \widetilde{W}$, in fact we see that the image of the symmetric function $s(\mathbf{x})$ on the lhs is another function $s(T_n^- \mathbf{x})$ which is not anymore symmetric because of the shift in η on the nth root. Same arguments can be applied for the X_n^+ operator by exploiting the other commutation relation

$$D(\mu)B(\lambda) = \frac{\mu - \lambda - \eta}{\mu - \lambda} B(\lambda)A(\mu) + \frac{\eta}{\lambda - \mu} B(\mu)A(\lambda). \tag{2.1.25}$$

□

Of course the next passage it would be to give the commutation relations for the conjugated momenta, but before we have to better implement their representation in a way to extend their action from W to $\widetilde{W}$. Consider the following constant function

$$\omega(\mathbf{x}) = 1, \qquad \forall \mathbf{x} \in \mathbf{X}, \tag{2.1.26}$$

which is of course symmetric and then belongs to W. Now consider the following image functions of ω for the $X_n^{\pm}$

$$\Delta_n^{\pm}(\mathbf{x}) = [X_n^{\pm} \omega](\mathbf{x}), \qquad \forall \mathbf{x} \in \mathbf{X}, \tag{2.1.27}$$

which, as we will see in a bit, determine completely the action of $X_n^{\pm}$ on any element s of W. Before proving the last statement we need to characterize the polynomials $s(\mathbf{x})$ in relation to the constant function ω, consider the operator $\hat{s} = s(\hat{x}_1, \dots, \hat{x}_N)$, then it follows that

$$s(\mathbf{x}) = [\hat{s}\omega](\mathbf{x}). \tag{2.1.28}$$

This last realization permits us to establish the following chain of identities

$$\left[X_n^{\pm}s\right](\mathbf{x}) = \left[X_n^{\pm}\hat{s}\omega\right](\mathbf{x}) \underset{(2.1.24)}{=} s\left(T_n^{\pm}\mathbf{x}\right)\left[X_n^{\pm}\omega\right](\mathbf{x}) = s\left(T_n^{\pm}\mathbf{x}\right)\Delta_n^{\pm}(\mathbf{x}). \tag{2.1.29}$$

This last representation can be then considered valid as well in the extended space $\widetilde{W}$, so that the action of the $X_n^{\pm}$s is defined on any function $s(\mathbf{x} \in \text{Fun}(\mathbf{X})$. Either the commutation relations (2.1.24) can be taken as valid in the extended space. As a final step, before introducing the second main theorem, we can state that the conjugated momenta $X_n^{\pm}$s can be expressed, in this representation, as

$$X_n^{\pm} = \Delta_n^{\pm}T_n^{\pm}. \tag{2.1.30}$$

Theorem 2.1.2. *The conjugated momenta* $\{X_n^+, X_n^-\}_{n=1}^N$ *satisfy the following commutation relations*

$$[X_m^{\pm}, X_n^{\pm}] = 0, \qquad \forall m, n = 1, \dots, N, \tag{2.1.31}$$

$$[X_m^{\pm}, X_n^{\mp}] = 0, \qquad \forall m, n = 1, \dots, N \qquad m \neq n \tag{2.1.32}$$

$$X_n^{\pm}X_n^{\mp} = \Delta(x_n \pm \eta/2), \tag{2.1.33}$$

where Δ *is the quantum determinant of the model, which reads*

$$\begin{aligned}\Delta(\lambda) = q\text{-}det\,(M(\lambda)) &= A(\lambda+\eta/2)D(\lambda-\eta/2) - B(\lambda+\eta/2)C(\lambda-\eta/2) \\ &= D(\lambda-\eta/2)A(\lambda-\eta/2) - C(\lambda-\eta/2)B(\lambda+\eta/2)\end{aligned} \tag{2.1.34}$$

Proof. The first commutation relations follow from the fact that the operators $A(\lambda)$, $D(\lambda)$ and $\{x_n\}_{n=1}^N$ commute among themselves

$$[A(\lambda), A(\mu)] = 0, \qquad [D(\lambda), D(\mu)] = 0, \qquad [x_n, x_m] = 0,\ \forall m, n.$$

The first two are a consequence of the YB relation and holding for any value of λ and so in the separated variables as well. The third one, instead, follows directly by the definition of the operators roots as multiplicative operators. From these relations it follows, for example for the X^+

$$[D(\lambda)D(\mu)]_{\substack{\lambda=x_n\\ \mu=x_m}} = \sum_{k,t} x_n^k x_m^t D_k D_m = \sum_{k,t} x_m^t x_n^k D_k D_m = \sum_t x_m^t X_n^+ D_m = X_n^+ X_m^+,$$

where we have exploited the commutation relations (2.1.21) for $m \neq n$. This expression will be equivalent to the commuted one that can be built by $[D(\mu)D(\lambda)]_{\substack{\lambda=x_n\\ \mu=x_m}}$.

N.B. One might get confused by the fact that we used actively some commutation relations for the x_ns while they were supposed to be numerical values only. The fact is that, if not applied to any eigenfunction, one should consider the isomorphism generated by $x_n = [\hat{x}_n\omega](\mathbf{x})$. So when we claimed to have used commutation relations (2.1.21) in the last passage, we were implicitly doing the more formal pre-operation $x_m X_n^+ = [\hat{x}_m X_n^+ \omega](\mathbf{x})$ and then actively use the commutation relations.

Relation (2.1.32) follows similarly once we consider the commutation relation

$$[D(\mu), A(\lambda)] = \frac{\eta}{\lambda-\mu}\left(B(\mu)C(\lambda) - B(\lambda)C(\mu)\right) \tag{2.1.35}$$

and the fact that the Bs goes to zero when evaluated in the separated variables.

Relations (2.1.33) can be proven by considering the quantum determinant evaluated with two different shifts in η. Consider, for first

$$\Delta(\lambda - \eta/2) = A(\lambda)D(\lambda-\eta) - B(\lambda)C(\lambda-\eta).$$

Now, by substituting $\lambda = x_n$ it leaves us just the first term on the rhs and we get

$$\Delta(x_n - \eta/2) = [A(\lambda)D(\lambda-\eta)]_{\lambda=x_n} = \sum_{k,t} x_n^k (x_n-\eta)^t A_k D_t = \sum_{k,t}(x_n-\eta)^t x_n^k A_k D_t$$
$$= \sum_t (x_n-\eta)^t X_n^- D_t \underset{(2.1.21)}{=} \sum_t X_n^- x_n^t D_t = X_n^- X_n^+,$$

where the same remark used above in the proof for identity (2.1.31) holds. A similar chain of identities can be applied to the expression for $X_n^+ X_n^-$, by starting with

$$\Delta(\lambda) = D(\lambda-\eta/2)A(\lambda-\eta/2) - C(\lambda-\eta/2)B(\lambda+\eta/2).$$

□

2.1.3 SoV representation

Let us summarize, for first, the commutation relations established in the previous parts

$$\begin{array}{cc} [x_n, x_m] = 0 & \forall m, n \\ X_m^{\pm} x_n = (x_n \pm \eta\delta_{mn}) X_m^{\pm} & \forall m, n \\ [X_m^{\pm}, X_n^{\pm}] = 0, & \forall m, n \\ [X_m^{\pm}, X_n^{\mp}] = 0, & \forall m, n \quad m \neq n \\ X_n^{\pm}, X_n^{\mp} = \Delta(x_n \pm \eta/2), & \forall m, n \end{array} \tag{2.1.36}$$

which define the SoV algebra χ_Δ generated by $\{x_n, X_n^{\pm}\}_{n=1}^N$ and labelled by the function $\Delta(\lambda)$.

Next step consists then in defining a proper SoV representation for the algebra χ_Δ. As we are interested in finite-dimensional representations which are as well non-degenerate, meaning that the spectrum $\mathbf{X}$ doesn't contain multiple points, the task of building a representation for χ_Δ coincide with the search of appropriate functions with proprieties defined by relations (2.1.36). These conditions are displayed in next Proposition.

Proposition 2.1.2. *In order to build a proper finite- dimensional non-degenerate SoV representation, the functions $\Delta_n^{\pm}$ for $n = 1, \dots, N$ and $\Delta(\lambda)$ should satisfy the following relations*

$$\Delta_m^{\pm}(\mathbf{x})\Delta_n^{\pm}(T_m^{\pm}\mathbf{x}) = \Delta_n^{\pm}(\mathbf{x})\Delta_m^{\pm}(T_n^{\pm}\mathbf{x}), \quad \forall n, m, \ \forall \mathbf{x} \in \mathbf{X} \tag{2.1.37}$$
$$\Delta_m^{\pm}(\mathbf{x})\Delta_n^{\mp}(T_m^{\pm}\mathbf{x}) = \Delta_n^{\mp}(\mathbf{x})\Delta_m^{\pm}(T_n^{\mp}\mathbf{x}), \quad \forall n, m, \ m \neq n \ \forall \mathbf{x} \in \mathbf{X} \tag{2.1.38}$$
$$\Delta_m^{\pm}(\mathbf{x})\Delta_n^{\mp}(T_m^{\pm}\mathbf{x}) = \Delta(x_n \pm \eta/2) \quad \forall \mathbf{x} \in \mathbf{X}. \tag{2.1.39}$$

Moreover, one should request

$$\Delta_n^{\pm}(\mathbf{x}) = 0, \qquad \forall \mathbf{x} \in \mathbf{X}_n^{\pm}, \tag{2.1.40}$$

where the $\mathbf{X}_n^{\pm}$s are the border set *of points*

$$\mathbf{X}_n^{\pm} = \left\{ \mathbf{x} \in \mathbf{X} | T_n^{\pm}\mathbf{x} \in \mathbb{C}^N \backslash \mathbf{X} \right\}. \tag{2.1.41}$$

The function that parametrizes the quantum determinant has to satisfy the following

$$\Delta(\mathbf{x}) = 0, \qquad \forall \mathbf{x} \in \bigcup_{n=1}^{N} \left(\left(\mathbf{X}_n^+ + \frac{\eta}{2} \right) \cup \left(\mathbf{X}_n^- - \frac{\eta}{2} \right) \right). \tag{2.1.42}$$

Proof. Relations (2.1.37)-(2.1.39) follow directly from the commutation relations (2.1.36), once we use the representation (2.1.30). These relations are not anyway sufficient in order to define finite-dimensional irreducible representations, one should ask further conditions on the functions defined. Indeed, when the operators $T_n^{\pm}$s move a point $\mathbf{x}$ outside the set $\mathbf{X}$, we should require the functions $\Delta_n^{\pm}$ to go to zero. Which is exactly what it is stated in (2.1.40). Consequently, identity (2.1.42) follows. □

Example: an irreducible finite-dimensional SoV representation

To conclude this chapter we would like to display an example developped by Sklyanin showing how it is explicitly possible to build a proper SoV representation for the algebra χ_Δ in terms of functions satisfying the whole set of conditions defined in proposition 2.1.2.

Consider the set of values $\{\zeta_n^\pm\}_{n=1}^N$, subject to the rule

$$(\zeta_n^+ - \zeta_n^-) = 2l_n\eta, \qquad 2l_n \in \mathbb{N}, \tag{2.1.43}$$

then we can build the following string set

$$\Lambda_n = \{\zeta_n^-, \zeta_n^- + \eta, \dots, \zeta_n^+ - \eta, \zeta_n^+\}, \qquad |\Lambda_n| = 2l_n + 1, \tag{2.1.44}$$

and such that

$$\forall\, m \neq n \Rightarrow \Lambda_m \cap \Lambda_n = \emptyset. \tag{2.1.45}$$

The set of the separated variables then can be built as

$$\mathbf{X} = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_N, \qquad |\mathbf{X}| = \prod_{n=1}^N (2l_n + 1). \tag{2.1.46}$$

Then, we just lack the definition of the functions $\Delta_n^\pm(\lambda)$s and $\Delta(\lambda)$

$$\Delta_n^\pm(\mathbf{x}) = \Delta_\pm(x_n), \tag{2.1.47}$$

$$\Delta_\pm(\lambda) = g_\pm \prod_{n=1}^N (\lambda - \zeta_n^\pm), \tag{2.1.48}$$

$$\Delta(\lambda) = \Delta_-(\lambda + \eta/2)\Delta_+(\lambda - \eta/2), \tag{2.1.49}$$

where $g_\pm$ are non-zero parameters which depends on the explicit transfer matrix under study.

Theorem 2.1.3. *The functions $\Delta_n^\pm$ define an irreducible representation of the algebra χ_Δ in the space Fun$\mathbf{X}$.*

Proof. The functions (2.1.47)-(2.1.49) constitute a representation of χ_Δ since they satisfy the conditions listed in proposition 2.1.2, given the definition of $\mathbf{X}$ in (2.1.46). The irriducibility can be proven ad absurdum, by supposing the existence of an invariant subspace $V \subset$Fun$\mathbf{X}$. By acting with the commutative subalgebra generated by the x_ns on the functions $f(\mathbf{x}) \in V$ we would conclude that these functions will go to zero on some subset $\mathbf{Y} \subset \mathbf{X}$. On the other side, by acting with the $X_n^\pm$s, we would realize that V can't be invariant under this action since, for condition (2.1.45), the only possible zeroes of the functions $\Delta_n^\pm$ are in the sets $X_n^\pm$. □

This theorem does not specify if this IRREP is unique. It is known that is possible to generate a family of equivalent IRREPs by multiplication for a function with no zeroes on $\mathbf{X}$ but still it doesn't tell if that family is the complete one or not. On the other hand these are quite technical points that don't prevent us to actually bring the SoV method to accomplishment and finally characterize the spectrum of a given transfer matrix.

2.2 The twisted periodic Heisenberg model

Always following the review paper by Sklyanin [117], we would like here to introduce some application of the results established in the previous section. We will explicitly build the representation for the periodic twisted XXX magnet and then characterize its spectrum by SoV means.

First of all, let us define what the twisted periodic model is. It consists of the usual closed Heisenberg isotropic chain where particular quasi-periodicity conditions are considered, i.e. relations that are satisfied by the operators sitting on the junction of the chain (site $N+1$ and site 1). For example the *antiperiodic* XXX chain will be defined by the following Hamiltonian

$$\begin{gathered} H_{\text{XXX}}^{\text{anti}} = \sum_{j=1}^{N} \left[\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^y \sigma_{j+1}^y \right] \in \text{End}(\mathcal{H}), \\ \sigma_{N+1}^a = (-1)^{1-\delta_{ax}}, \quad a = x, y, z \end{gathered} \tag{2.2.1}$$

where $\mathcal{H}$ is some Hilbert space depending on which spin representation one chooses and δ_{ij} is the Kronecker's delta function. In our case we won't specify any Hamiltonian since we would like to keep the discussion as general as possible under the point of view of the representation space and boundary conditions. Although, we will see how some conditions on the boundary has to be imposed in order to satisfy 2.1.2.

The QISM algebra $\mathcal{M}$ related to this model, is generated by the following monodromy matrix

$$M(\lambda) = KL_N(\lambda - \xi_N) \dots L_1(\lambda - \xi_1) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \in \text{End}(\mathcal{V} \otimes \mathcal{H}) \tag{2.2.2}$$

where $\mathcal{V} = \mathbb{C}^2$ is the auxiliary space, $\{\xi_n\}_{n=1}^N$ are the inhomogeneities and the (2×2) numerical matrix K encodes the boundary conditions of the quantum system. The L-operators appearing in (2.2.2) are the quantum Lax operators which read

$$L(\lambda) = \lambda + \eta \sum_{a=x,y,z} S_a \sigma^a = \begin{pmatrix} \lambda + \eta S_z & \eta S_- \\ \eta S_+ & \lambda - \eta S_z \end{pmatrix}, \qquad S_\pm = S_x \pm i S_y, \tag{2.2.3}$$

where the operators S_a for $a = x, y, z$ generate the finite dimensional irreducible representations of the Lie algebra *sl(2)*. The dimension of the algebra can be related to the value taken by the Casimir operator $|S^2| = l(l+1)$ and therefore dim$= 2l+1$. Notice that when $l = 1/2$ we get the fundamental representation of the chain, i.e. spin-$1/2$, with which we dealt with in chapter **1**. The corresponding quantum determinant reads

$$\Delta(\lambda) = \det(K) \prod_{n=1}^{N} \text{q-det}\,(L(\lambda - \xi_n)) = \det(K) \prod_{n=1}^{N} (\lambda - \xi_n - l_n\eta - \eta/2)(\lambda - \xi_n + l_n\eta + \eta/2), \tag{2.2.4}$$

being q-det $(K) = \det(K)$.

With this set-up we have automatically generated a family of finite-dimensional (dim$= \prod_n (2l_n + 1)$) representations for the QISM algebra $\mathcal{M}$

$$M\left(\lambda | K \in \mathbb{C}^{2,2}, N \in \mathbb{Z}_+, \{l_n \in \mathbb{Z}/2\}_{n=1}^N, \{\xi_n \in \mathbb{C}\}_{n=1}^N\right), \tag{2.2.5}$$

parametrized by the matrix K the number of "sites" N, the spins l_n and the inhomogeneities ξ_n. Let us remark that, technically speaking, these irreducible representations correspond to the IRREPs of the the Yangian $\mathcal{Y}[sl(2)]$ just in the case $K = \mathbb{1}$. Paradoxically, this case can't be dealt with the SoV method, meaning that our title of section §2.1 has to be understood as merely labeling the R-matrix that appears in the QISM formalism and the Yang-Baxter relation which define the algebra's relations.

Now we can reproduce the theorem **4.1**$_{[117]}$.

Theorem 2.2.1. *Let the monodromy matrix* $M(\lambda)$ *defined in* (2.2.2) *and* (2.2.5) *be the representation of the QISM algebra* $\mathcal{M}$ *and the following conditions satisfied:*

(I) $det(K) \neq 0$, $K_{12} \neq 0$;

(II) The sets $\{\Lambda_n\}_{n=1}^N$, *defined in* (2.1.44), *are non-intersecting, i.e. condition* (2.1.45), *for*

$$\zeta_n^{\pm} = \xi_n \pm l_n \eta. \tag{2.2.6}$$

The spectrum of the operators $\{x_n\}_{n=1}^N$ *denoted by* $\tilde{\mathbf{X}}$ *is defined as*

$$\tilde{\mathbf{X}} = \bigcup_{\sigma \in S_N} \sigma \mathbf{X}, \tag{2.2.7}$$

being $\mathbf{X}$ *defined by eq.*(2.1.46).

The corresponding representation of the algebra χ_Δ *is the direct sum of* $N!$ *IRREPs with spec*$\{x_n\}_{n=1}^N = \sigma\mathbf{X}$, *defined by theorem 2.1.3, being* $g_\pm$ *arbitrary non-zero parameters s.t.*

$$det(K) = g_+ g_- \tag{2.2.8}$$

The proof of this theorem can be found in [117] and it won't be reproduced here, since it doesn't need further comments and it is not of main interest for the following. Let us just remark that conditions (I) and (II) are imposed in order to ensure that conditions (2.1.1)-(2.1.3) are fulfilled. Moreover, notice that the sets $\sigma\mathbf{X}$ do not intersect, which is an important point for our construction.

Let us move to finally consider the eigenproblem related to the monodromy matrix $\mathcal{T}(\lambda) = A(\lambda) + D(\lambda)$. The spectral analysis consists in the study of the spectral problem given by

$$\mathcal{T}(\lambda)\varphi = \tau(\lambda)\varphi. \tag{2.2.9}$$

Now, by inserting $\lambda = x_n$, and exploiting the consequences of the SoV representation we get the following set of equations

$$\tau(x_n)\varphi(\mathbf{x}) = \Delta_n^-(\mathbf{x})\varphi(T^-\mathbf{x}) + \Delta_n^+(\mathbf{x})\varphi(T^-\mathbf{x}), \qquad \forall n = 1, \dots, N \tag{2.2.10}$$

which admits separation of variables

$$\varphi(\mathbf{x}) = \prod_{n=1}^N Q_n(x_n), \tag{2.2.11}$$

and then resulting in a set of N independent *Baxter-like* equations

$$\begin{gathered} \tau(x_n)Q(x_n) = \Delta_-(x_n)Q_n(x_n - \eta) + \Delta_+(x_n)Q_n(x_n + \eta), \\ \forall x_n \in \Lambda_n, \qquad n \in \{1, \dots, N\}. \end{gathered} \tag{2.2.12}$$

This is the main result which is at the moment obtainable through SoV in order to characterize the spectrum of the transfer matrix for the problem under study. We referred to equations (2.2.12) as Baxter-like since they coincide, in form, with Baxter equations on a finite set of points, i.e. the $\{x_n\}$. Furthermore, the Baxter equations, involving the polynomials $Q(\lambda)$ with zeroes on the Bethe roots, need the the Bethe ansatz equations to be solved and an analytic solution can be found just in the thermodynamic limit $N \to \infty$. Our analysis seems to suit better the discrete finite problem, not involving any BAE and being eventually subject to numerical solution. But, the main advantage of the method consists in building, by construction, a complete spectrum. This last statement follows from the fact that we built a one-to-one correspondence between the eigenfunctions φ in the multidimensional problem and those of the one-dimensional one. This argument can't be in general applied for systems solved by ABA, where the completeness has to be studied case by case.

CHAPTER 3

NON-DIAGONAL OPEN SPIN-1/2 XXZ CHAIN BY *SEPARATION OF VARIABLES* METHOD

In this chapter we will give the full treatment of the eigenproblem associated to the XXZ spin-1/2 chain with the most generic integrable boundary conditions. As mentioned in chapter **1**, the solution can't be obtained by a standard use of the QISM machinery, *i.e.* the algebraic Bethe ansatz. With non-diagonal terms in the Hamiltonian, it's not possible to define a proper reference state, which is a key point for ABA applicability. Then, by following the steps of [101], it is possible to exploit the separation of variables method by Sklyanin, introduced in the previous chapter, in order to retrieve the full complete spectrum and eigenvectors associated to the transfer matrix of the inhomogeneous chain. In order to do so, it has been necessary to add some gauge transformations to the game, and exploit the gauge freedom coming from them, in the fashion of [23]. In fact the SoV method is well implementable on the open chain when one of the two boundary matrices is triangular [101], so, in order to circumvent this kind of constraint on the boundary parameters, the gauge freedom artificially introduced can be successfully used for the task. The results presented here can be found in a publication of S.F., N.Kitanine and G.Niccoli [45].

3.1 Open XXZ spin chains and reflection algebra.

The quantum system that we want to describe and analyze in this chapter is defined by the following quantum Hamiltonian

$$\begin{aligned} H^{\text{G.B.}}_{\text{XXZ}} = & \sum_{i=1}^{N-1} \left[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cosh(\eta) \left(\sigma_i^z \sigma_{i+1}^z - 1 \right) \right] \\ & + \frac{\sinh(\eta)}{\sinh(\xi_-)} \Big(\sigma_1^z \cosh \xi_- + 2\kappa_- (\sigma_1^x \cosh \tau_- + i \sigma_1^y \sinh \tau_-) \Big) \\ & + \frac{\sinh(\eta)}{\sinh(\xi_+)} \Big(\sigma_N^z \cosh \xi_+ + 2\kappa_+ (\sigma_N^x \cosh \tau_+ + i \sigma_N^y \sinh \tau_+) \Big) \in (\mathcal{H}). \end{aligned} \tag{3.1.1}$$

The model defined by (3.1.1) lives in the Hilbert space $\mathcal{H} = \mathbb{C}^{2^{\otimes N}}$, which is the tensor product of N spin-1/2 representation spaces $\mathcal{H}_{1/2} = \mathbb{C}^2$. The sigma operators σ_i^a, with $i \in \{1, \dots, N\}$ and $a \in \{x, y, z\}$, are the usual Pauli matrices which act non-trivially in the ith space of the tensor product

$$\sigma_i^a = \underset{1}{\mathbb{1}} \otimes \cdots \otimes \underset{i}{\sigma^a} \otimes \cdots \otimes \underset{N}{\mathbb{1}}.$$

The parameters $\{\zeta_\pm, \kappa_\pm, \tau_\pm\}$ encode the interaction with the boundaries.

Remark 3.1.1. The difference between $H_{\text{XXZ}}^{\text{G.B.}}$ (3.1.1) and $H_{\text{XXZ}}^{\text{D.B.}}$ (1.4.1) resides in the presence of non diagonal boundary terms. To be more precise the Hamiltonian in (3.1.1) contains the most general as possible integrable boundary magnetic field. This gives birth to a far richer behavior of the model, being not anymore isolated and stationary, but *communicating with the exterior world*. Indeed, the usual XXZ bulk coexist with spin currents interactions, flowing along the chain, which are introduced and removed, from the left and from the right, by the full set of boundary terms appearing in (3.1.1). This point of view becomes explicit once we map our model on some classical exclusion process model, see [32] for a review, where these terms coincide exactly with the *injection/ejection* rate coefficients.

Definition of the algebra

Let's briefly recall what was already introduced in **1**, with the due generalizations and changes. As we understood, the main ingredient of QISM is the R-matrix, which for the current analysis will still be the trigonometric solution to Yang-Baxter equation

$$R_{12}(\lambda)R_{23}(\lambda-\mu)R_{13}(\mu) = R_{13}(\mu)R_{23}(\lambda-\mu)R_{12}(\lambda), \tag{3.1.2}$$

where

$$R_{12}(\lambda) = \begin{pmatrix} \sinh(\lambda+\eta) & 0 & 0 & 0 \\ 0 & \sinh(\lambda) & \sinh(\eta) & 0 \\ 0 & \sinh(\eta) & \sinh(\lambda) & 0 \\ 0 & 0 & 0 & \sinh(\lambda+\eta) \end{pmatrix} \in (End(\mathcal{V}_1 \otimes \mathcal{V}_2))$$

being the *auxiliary spaces* $\mathcal{V}_i = \mathbb{C}^2$ for $i = 1, 2, 3$.

Dealing with open chains, a very important piece in the definition of the algebra comes from the boundary matrices that can be defined via the most generic scalar solution to reflection equation

$$R_{12}(\lambda-\mu)K_1(\lambda)R_{21}(\lambda+\mu)K_2(\mu) = K_2(\mu)R_{21}(\lambda+\mu)K_1(\lambda)R_{12}(\lambda-\mu)$$

which reads [34, 33, 58]

$$K(\lambda;\zeta,\kappa,\tau) = \frac{1}{\sinh\zeta}\begin{pmatrix} \sinh(\lambda+\zeta) & \kappa e^{\tau}\sinh(2\lambda) \\ \kappa e^{-\tau}\sinh(2\lambda) & \sinh(\zeta-\lambda) \end{pmatrix}. \tag{3.1.3}$$

Let the two families of solutions be as in (1.4.4)

$$K_-(\lambda) = K(\lambda-\eta/2;\zeta_-,\kappa_-,\tau_-) \quad K_+(\lambda) = K(\lambda+\eta/2;\zeta_+,\kappa_+,\tau_+), \tag{3.1.4}$$

which contain the full set of boundary parameters appearing in (3.1.1). The bulk monodromy matrix

$$M_0(\lambda) = R_{0N}(\lambda-\xi_N-\eta/2)\dots R_{01}(\lambda-\xi_1-\eta/2) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{\mathcal{V}_0}$$

and its dual

$$\hat{M}_0(\lambda) = (-1)^N \sigma_0^y [M_0(-\lambda)]^{t_0} \sigma_0^y$$

contribute to the definition of the two isometric double-row monodromy matrices

$$\mathcal{U}_-(\lambda) = M(\lambda)K_-(\lambda)\hat{M}(\lambda) = \begin{pmatrix} \mathcal{A}_-(\lambda) & \mathcal{B}_-(\lambda) \\ \mathcal{C}_-(\lambda) & \mathcal{D}_-(\lambda) \end{pmatrix}$$
$$\mathcal{U}_+^{t_0}(\lambda) = M^{t_0}(\lambda)K_+^{t_0}(\lambda)\hat{M}^{t_0}(\lambda) = \begin{pmatrix} \mathcal{A}_+(\lambda) & \mathcal{C}_+(\lambda) \\ \mathcal{B}_+(\lambda) & \mathcal{D}_+(\lambda) \end{pmatrix}$$

both solutions of the reflection equations, see (1.3.2) and (1.3.3). The transfer matrix

$$\begin{aligned}\mathcal{T}(\lambda) &= \mathrm{tr}_0 \left\{K_+(\lambda)M(\lambda)K_-(\lambda)\hat{M}(\lambda)\right\} \\ &= \mathrm{tr}_0 \left\{K_+(\lambda)\mathcal{U}_-(\lambda)\right\} = \mathrm{tr}_0 \left\{\mathcal{U}_+(\lambda)K_-(\lambda)\right\}\end{aligned}$$

generates the whole family of conserved charges and, in particular, by means of a trace identity, provides the link with the Hamiltonian (3.1.1)

Proposition 3.1.1. *The following identity holds*

$$H_{XXZ}^{G.B.} = \frac{2(\sinh\eta)^{1-2N}}{tr\left\{K_-(\eta/2)\right\} tr\left\{K_+(\eta/2)\right\}} \frac{d}{d\lambda}\mathcal{T}(\lambda)|_{\substack{\lambda=\eta/2 \\ \xi_1,\ldots,\xi_N=0}} + const.$$

Proof. It follows directly from the *proof* of proposition 1.4.1, since the key property $K_-(\eta/2) = \mathbb{1}$ holds in the general non-diagonal case either. □

Some basic properties

Here we will replicate some of the most important properties of the XXZ reflection algebra, keeping in consideration a slight change in notation due to the non-diagonal boundary terms in (3.1.4). The commutation relations (1.4.7) among the generators of the algebra $\mathcal{A}(\lambda)$, $\mathcal{B}(\lambda)$, $\mathcal{C}(\lambda)$ and $\mathcal{D}(\lambda)$ are not affected by the modified set-up, implying that the following expression for the quantum determinant still holds

$$\begin{aligned}\frac{\text{q-det}\,(\mathcal{U}_-(\lambda))}{\sinh(2\lambda-2\eta)} =& \mathcal{A}_-(\epsilon\lambda+\eta/2)\mathcal{A}_-(\eta/2-\epsilon\lambda) + \mathcal{B}_-(\epsilon\lambda+\eta/2)\mathcal{C}_-(\eta/2-\epsilon\lambda) \\ =& \mathcal{D}_-(\epsilon\lambda+\eta/2)\mathcal{D}_-(\eta/2-\epsilon\lambda) + \mathcal{C}_-(\epsilon\lambda+\eta/2)\mathcal{B}_-(\eta/2-\epsilon\lambda)\end{aligned} \tag{3.1.5}$$

where $\epsilon = \pm 1$ and proposition 1.4.3 has been used. The central element here defined takes the explicit form

$$\begin{aligned}\text{q-det}\,(\mathcal{U}_-(\lambda)) =& \text{q-det}\,(K_-(\lambda))\text{q-det}\,(M_0(\lambda))\text{q-det}\,(M_0(-\lambda)) \\ =& \sinh(2\lambda-\eta)\mathsf{A}_-(\lambda+\eta/2)\mathsf{A}_-(-\lambda+\eta/2),\end{aligned} \tag{3.1.6}$$

where q-det $(M_0(\lambda)) = a(\lambda+\eta/2)d(\lambda-\eta/2)$ and

$$\text{q-det}\,(K_\pm(\lambda)) = \mp\sinh(2\lambda \pm 2\eta/2)g_\pm(\lambda+\eta/2)g_\mp(-\lambda+\eta/2). \tag{3.1.7}$$

The following functions have been used

$$\mathsf{A}_-(\lambda) = g_-(\lambda)a(\lambda)d(-\lambda),\ d(\lambda) = a(\lambda-\eta),\ a(\lambda) = \prod_{j=1}^{N}\sinh(\lambda-\xi_j+\eta/2), \tag{3.1.8}$$

and

$$g_\pm(\lambda) = \frac{\sinh(\lambda+\alpha_\pm-\eta/2)\cosh(\lambda+\beta_\pm-\eta/2)}{\sinh\alpha_\pm\cosh\beta_\pm} \tag{3.1.9}$$

where the $\alpha_\pm$ and $\beta_\pm$ are related to $\zeta_\pm$ and $\kappa_\pm$ through

$$\sinh\alpha_\pm\cosh\beta_\pm = \frac{\sinh\zeta_\pm}{2\kappa_\pm}, \quad \cosh\alpha_\pm\sinh\beta_\pm = \frac{\cosh\zeta_\pm}{2\kappa_\pm}. \tag{3.1.10}$$

Expression (3.1.7) can be proven again by direct computation as done for (1.4.17), for example by using the equality

$$K_\pm(\lambda\mp\eta/2)K_\pm(-\lambda\mp\eta/2) = \frac{\text{q-det}\,(K_\pm(\lambda))}{\sinh(\mp2\lambda\pm2\eta)} = g_\pm(\lambda+\eta/2)g_\pm(-\lambda+\eta/2).$$

Properties (1.4.12), (1.4.18) and proposition 1.4.3 still hold in the non-diagonal case as well. In next two propositions some parity and Hermitian conjugations properties will be given; they were firstly proven in [70] in the diagonal case and they were useful in [101] to build the separation of variable characterization of the open boundary XXZ spin chain with a triangualr K-matrix.

Proposition 3.1.2. *The transfer matrix $\mathcal{T}(\lambda)$ of the generic open boundaries XXZ spin-1/2 chain is even*

$$\mathcal{T}(-\lambda) = \mathcal{T}(\lambda). \tag{3.1.11}$$

Proof. Let us, first of all, introduce the following notation

$$\begin{aligned} K_\pm(\lambda) = \frac{1}{\sinh\zeta_\pm}&\begin{pmatrix}\sinh(\lambda+\zeta_\pm\pm\eta/2) & \kappa_\pm e^{\tau_\pm}\sinh(2\lambda\pm\eta)\\ \kappa_\pm e^{-\tau_\pm}\sinh(2\lambda\pm\eta) & \sinh(\zeta_\pm-\lambda\mp\eta/2)\end{pmatrix}\\ &= \begin{pmatrix} a_\pm(\lambda) & b_\pm(\lambda)\\ c_\pm(\lambda) & d_\pm(\lambda)\end{pmatrix} \end{aligned} \tag{3.1.12}$$

Then the transfer matrix can be rewritten as

$$\mathcal{T}(\lambda) = \mathcal{T}_\backslash^{(\pm)}(\lambda) + b_\mp(\lambda)\mathcal{C}_\pm + c_\mp(\lambda)\mathcal{B}_\pm(\lambda), \tag{3.1.13}$$

with the diagonal part being

$$\begin{aligned} \mathcal{T}_\backslash^{(\pm)}(\lambda) =& a_\mp(\lambda)\mathcal{A}_\pm(\lambda) + d_\mp(\lambda)\mathcal{D}_\pm(\lambda)\\ =&\hat{a}_\mp(\lambda)\mathcal{A}_\pm(\lambda) + \hat{a}_\mp(-\lambda)\mathcal{A}_\pm(-\lambda)\\ =&\hat{d}_\mp(\lambda)\mathcal{D}_\pm(\lambda) + \hat{d}_\mp(-\lambda)\mathcal{D}_\pm(-\lambda) \end{aligned} \tag{3.1.14}$$

where the second and third lines can be simply proven by some algebra manipulations, keeping in mind proposition 1.4.3 and given

$$\begin{aligned} \hat{a}_\pm(\lambda) =& \frac{\sinh(2\lambda\pm2\eta)\sinh(\lambda+\zeta_\pm\mp\eta/2)}{\sinh2\lambda\sinh\zeta_\pm},\\ \hat{d}_\pm(\lambda) =& \frac{\sinh(2\lambda\pm2\eta)\sinh(\zeta_\pm-\lambda\pm\eta/2)}{\sinh2\lambda\sinh\zeta_\pm}. \end{aligned} \tag{3.1.15}$$

It's then clear that $\mathcal{T}_\backslash^{(\pm)}$ is even in λ. To end the proof one has just to consider the following identities

$$b_\mp(-\lambda)\mathcal{C}_\pm(-\lambda) = b_\mp(\lambda)\mathcal{C}_\pm(\lambda), \quad c_\mp(-\lambda)\mathcal{B}_\pm(-\lambda) = c_\mp(\lambda)\mathcal{B}_\pm(\lambda).$$

□

Proposition 3.1.3. *Under Hermitian conjugation the monodromy matrix $\mathcal{U}_{\pm}(\lambda)$ satisfy the following properties:*

1. Massless regime - i.e. *for $\eta \in i\mathbb{R}$, it holds*
$$\mathcal{U}_{\pm}(\lambda)^{\dagger} = [\mathcal{U}_{\pm}(-\lambda^{*})]^{t_0}. \tag{3.1.16}$$
for $\{\xi_1, \ldots, \xi_n, i\zeta_{\pm}, i\kappa_{\pm}, i\tau_{\pm}\} \in \mathbb{R}^{N+3}$.
2. Massive regime - i.e. *for $\eta \in \mathbb{R}$, it holds*
$$\mathcal{U}_{\pm}(\lambda)^{\dagger} = [\mathcal{U}_{\pm}(\lambda^{*})]^{t_0}. \tag{3.1.17}$$
for $\{i\xi_1, \ldots, i\xi_n, \zeta_{\pm}, \kappa_{\pm}, \tau_{\pm}\} \in \mathbb{R}^{N+3}$.

Furthermore, for the same constraints on parameters and both in the massless or massive regime, it holds:
$$\mathcal{T}(\lambda)^{\dagger} = \mathcal{T}(\lambda^{*}). \tag{3.1.18}$$

Proof. Consider the following equalities

(I) for $\{i\eta, \xi_1, \ldots, \xi_n, i\zeta_{\pm}, i\kappa_{\pm}, i\tau_{\pm}\} \in \mathbb{R}^{N+4}$ it holds:
$$[R_{0n}(\lambda - \xi_n - \eta/2)]^{\dagger} = -[R_{0n}(-\lambda^{*} + \xi_n - \eta/2)]^{t_0},$$
$$K_{\pm}(\lambda)^{\dagger} = [K_{\pm}(-\lambda^{*})]^{t_0};$$

(II) for $\{\eta, i\xi_1, \ldots, i\xi_n, \zeta_{\pm}, \kappa_{\pm}, \tau_{\pm}\} \in \mathbb{R}^{N+4}$ it holds:
$$[R_{0n}(\lambda - \xi_n - \eta/2)]^{\dagger} = [R_{0n}(\lambda^{*} + \xi_n - \eta/2)]^{t_0},$$
$$K_{\pm}(\lambda)^{\dagger} = [K_{\pm}(\lambda^{*})]^{t_0}.$$

Then it follows
$$\begin{aligned} M_0(\lambda)^{\dagger} &= [R_{01}(\lambda - \xi_1 - \eta/2)]^{\dagger} \ldots [R_{0N}(\lambda - \xi_N - \eta/2)]^{\dagger} \\ &= \left(-\frac{\eta}{\eta^{*}}\right)^{N} \left[\hat{M}_0(\left(\frac{\eta}{\eta^{*}}\right)\lambda^{*})\right]^{t_0}, \end{aligned} \tag{3.1.19}$$
and
$$\mathcal{U}_{\pm}(\lambda)^{\dagger} = \left.\left\{ \left[\hat{M}(\left(\frac{\eta}{\eta^{*}}\right)\lambda^{*})\right]^{t_0} \left[K_{\pm}(\left(\frac{\eta}{\eta^{*}}\right)\lambda^{*})\right]^{t_0} \left[M(\left(\frac{\eta}{\eta^{*}}\right)\lambda^{*})\right]^{t_0} \right\}\right|_{\text{q-reverse-order}}. \tag{3.1.20}$$
The prescription *q-reverse-order* means that, once the matrix products in $\mathcal{V}_0$ are ultimated in the rhs of (3.1.20), one should let the operator entries of $\left[\hat{M}(\left(\frac{\eta}{\eta^{*}}\right)\lambda^{*})\right]^{t_0}$ go through those of $\left[M(\left(\frac{\eta}{\eta^{*}}\right)\lambda^{*})\right]^{t_0}$ to the right. With this order prescription, the following equality will clearly hold
$$\mathcal{U}_{\pm}(\lambda)^{\dagger} = \left[\mathcal{U}_{\pm}(\left(\frac{\eta}{\eta^{*}}\right)\lambda^{*})\right]^{t_0}, \tag{3.1.21}$$
then
$$\mathcal{T}(\lambda)^{\dagger} = \text{tr}_0 \left\{ \left[K_{\mp}(\left(\frac{\eta}{\eta^{*}}\right)\lambda^{*})\right]^{t_0} \left[\mathcal{U}_{\pm}(\left(\frac{\eta}{\eta^{*}}\right)\lambda^{*})\right]^{t_0} \right\} = \mathcal{T}(\left(\frac{\eta}{\eta^{*}}\right)\lambda^{*}) = \mathcal{T}(\lambda^{*})$$
where, for the last equality, equation (3.1.11) has been used. □

3.2 Gauge transformations

As we mentioned before, in order to solve the eigenproblem associated to the transfer matrix $\mathcal{T}(\lambda)$ via the SoV method, we have to introduce some gauge transformations. This is required in order to by-pass the constraint on the boundary parameters that would be necessary otherwise; *i.e* the requirement that one of the K matrices has to be triangular. This condition comes from the fact that the reflection transfer matrix under study is in general a sum over the four generators of the reflection algebra, see (3.1.13). In the SoV representation, only the action of the operators $\mathcal{B}(\mathcal{C})$, $\mathcal{A}$ and $\mathcal{D}$ on the eigentstates of $\mathcal{B}(\mathcal{C})$ is trivial, while the action of $\mathcal{C}(\mathcal{B})$ is not simply defined. Now, being the eigenstates of the transfer matrix built out of the SoV $\mathcal{B}$ or $\mathcal{C}$ eigenstates, one has to eliminate the non-trivial contribution in the transfer matrix in order of being able to study its spectrum. This elimination is just the triangularity constraint one has to require. This condition will be fixed at the end as well but just on a gauge parameters level, leaving the original, physical boundary parameters, untouched.

The gauge transformations that will be employed in the following sections will be merely acting at a representation level, the auxiliary space $\mathcal{V}_0 \simeq \mathbb{C}^2$, while the Hilbert space will be left unchanged. The main idea is that the transfer matrix, and then its spectrum, should be invariant under the action of such transformations. This can be written, in a simplified form, as

$$\mathcal{T}_{\text{gauge}}(\lambda) = \text{tr}_0\{SM(\lambda)S^{-1}SK_-(\lambda)S^{-1}\hat{M}(\lambda)S^{-1}SK_+(\lambda)S^{-1}\} = \mathcal{T}(\lambda)$$

where S is some toy gauge matrix. We will see that the structure will be a little more involved since lots of subtleties emerge on representation level; but the philosophy is essentially the same. As stated above, the transformations that will be used in our construction act purely in the auxiliary space. Despite their look, they must not be confused with the so-called *face-vertex* transformations [14, 134, 51], which constitute the connection between some open spin chain model and *solid on solid* (SoS) ones. In fact, the face-vertex transformations act in the Hilbert space as well, resulting in a more profound change of the algebraic structure of the theory. For example we will never arrive to deal with the *dynamical* YB equation [48] and, consequently, neither the algebra generated by it.

3.2.1 Definitions

The form of the gauge transformations we are going to introduce are equivalent to the one used in [23] and they are organized in two matrices

$$\bar{G}(\lambda|\beta) = (X(\lambda|\beta), Y(\lambda|\beta)) \tag{3.2.1a}$$

$$\tilde{G}(\lambda|\beta) = (X(\lambda|\beta+1), Y(\lambda|\beta-1)) \tag{3.2.1b}$$

where the column vectors X and Y are defined as

$$X(\lambda|\beta) = \begin{pmatrix} e^{-[\lambda+(\alpha+\beta)\eta]} \\ 1 \end{pmatrix}, \quad Y(\lambda|\beta) = \begin{pmatrix} e^{-[\lambda+(\alpha-\beta)\eta]} \\ 1 \end{pmatrix}, \tag{3.2.2}$$

$\forall(\alpha, \beta) \in \mathbb{C}^2$. It's clear that the gauge parameters, in the definitions above, coincide with the sum and difference of α and β, but in the following we will act just on β leaving α constant if not stated otherwise. Then the notation will include just the β dependence. It will be useful to give the explicit expressions for the inverses of (3.2.1) as well, they read

$$\bar{G}^{-1}(\lambda|\beta) = \begin{pmatrix} \bar{Y}(\lambda|\beta) \\ \bar{X}(\lambda|\beta) \end{pmatrix}, \quad \tilde{G}^{-1}(\lambda|\beta) = \begin{pmatrix} \tilde{Y}(\lambda|\beta-1) \\ \tilde{X}(\lambda|\beta+1) \end{pmatrix} \tag{3.2.3}$$

where

$$\bar{X}(\lambda|\beta) = \frac{e^{(\lambda+\alpha\eta)}}{2\sinh\beta\eta}\left(1, -e^{-[\lambda+(\alpha+\beta)\eta]}\right),$$
$$\bar{Y}(\lambda|\beta) = \frac{e^{(\lambda+\alpha\eta)}}{2\sinh\beta\eta}\left(-1, e^{-[\lambda+(\alpha-\beta)\eta]}\right), \tag{3.2.4}$$

$$\tilde{X}(\lambda|\beta) = e^{\eta}\frac{\sinh\beta\eta}{\sinh(\beta-1)\eta}\bar{X}(\lambda|\beta),\ \tilde{Y}(\lambda|\beta) = e^{\eta}\frac{\sinh\beta\eta}{\sinh(\beta+1)\eta}\bar{Y}(\lambda|\beta). \tag{3.2.5}$$

Moreover, it's simple to compute the following properties

$$\bar{Y}(\lambda|\beta)X(\lambda|\beta) = 1, \qquad \bar{Y}(\lambda|\beta)Y(\lambda|\beta) = 0, \tag{3.2.6a}$$
$$\bar{X}(\lambda|\beta)X(\lambda|\beta) = 0, \qquad \bar{X}(\lambda|\beta)Y(\lambda|\beta) = 1, \tag{3.2.6b}$$
$$X(\lambda|\beta)\bar{Y}(\lambda|\beta) + Y(\lambda|\beta)\bar{X}(\lambda|\beta) = \mathbb{1}, \tag{3.2.6c}$$

and

$$\tilde{Y}(\lambda|\beta-1)X(\lambda|\beta+1) = 1, \qquad \tilde{Y}(\lambda|\beta-1)Y(\lambda|\beta-1) = 0, \tag{3.2.7a}$$
$$\tilde{X}(\lambda|\beta+1)X(\lambda|\beta+1) = 0, \qquad \tilde{X}(\lambda|\beta+1)Y(\lambda|\beta-1) = 1, \tag{3.2.7b}$$
$$X(\lambda|\beta+1)\tilde{Y}(\lambda|\beta-1) + Y(\lambda|\beta-1)\tilde{X}(\lambda|\beta+1) = \mathbb{1}, \tag{3.2.7c}$$

where $\mathbb{1}$ is the identity matrix as usual.

3.2.2 Gauge transformed boundary operators

Let us now consider the explicit construction of the gauge transformed bulk and boundary operators. The starting point is the transformation rule for the basic building blocks of the algebra, the L operators $R_{0j}(\lambda - \xi_j - \eta/2)$, which has been reproduced here

$$R_{0j}(\lambda - \xi_j - \eta/2|\beta) = \tilde{G}^{-1}(\lambda - \eta/2|\beta + N - j)R_{0j}(\lambda - \xi_j - \eta/2) \times G(\lambda - \eta/2|\beta + N - j + 1) \tag{3.2.8}$$

and consequently the bulk monodromy matrix becomes

$$M(\lambda|\beta) = \tilde{G}^{-1}(\lambda - \eta/2|\beta)M(\lambda)\tilde{G}(\lambda - \eta/2|\beta + N) = \begin{pmatrix} A(\lambda|\beta) & B(\lambda|\beta) \\ C(\lambda|\beta) & D(\lambda|\beta) \end{pmatrix}. \tag{3.2.9}$$

The gauged bulk operators appearing in (3.2.9) can be easily expressed in terms of the ungauged ones

$$A(\lambda|\beta) = \tilde{Y}(\lambda - \eta/2|\beta - 1)M(\lambda)X(\lambda - \eta/2|\beta + N + 1), \tag{3.2.10a}$$
$$B(\lambda|\beta) = \tilde{Y}(\lambda - \eta/2|\beta - 1)M(\lambda)Y(\lambda - \eta/2|\beta + N - 1), \tag{3.2.10b}$$
$$C(\lambda|\beta) = \tilde{X}(\lambda - \eta/2|\beta + 1)M(\lambda)X(\lambda - \eta/2|\beta + N + 1), \tag{3.2.10c}$$
$$D(\lambda|\beta) = \tilde{X}(\lambda - \eta/2|\beta + 1)M(\lambda)Y(\lambda - \eta/2|\beta + N - 1). \tag{3.2.10d}$$

In a similar way we can apply the second gauge transformation to the right-to-left bulk monodromy matrix

$$\hat{M}(\lambda|\beta) = \bar{G}^{-1}(\eta/2 - \lambda|\beta + N)M(\lambda)\bar{G}(\eta/2 - \lambda|\beta) = \begin{pmatrix} \bar{A}(\lambda|\beta) & \bar{B}(\lambda|\beta) \\ \bar{C}(\lambda|\beta) & \bar{D}(\lambda|\beta) \end{pmatrix}. \tag{3.2.11}$$

Similarly to (3.2.10) we can express:

$$\bar{A}(\lambda|\beta) = \bar{Y}(\eta/2-\lambda|\beta+N)\hat{M}(\lambda)X(\eta/2-\lambda|\beta), \tag{3.2.12a}$$
$$\bar{B}(\lambda|\beta) = \bar{Y}(\eta/2-\lambda|\beta+N)\hat{M}(\lambda)Y(\eta/2-\lambda|\beta), \tag{3.2.12b}$$
$$\bar{C}(\lambda|\beta) = \bar{X}(\eta/2-\lambda|\beta+N)\hat{M}(\lambda)X(\eta/2-\lambda|\beta), \tag{3.2.12c}$$
$$\bar{D}(\lambda|\beta) = \bar{X}(\eta/2-\lambda|\beta+N)\hat{M}(\lambda)Y(\eta/2-\lambda|\beta). \tag{3.2.12d}$$

The gauged transformed monodromy matrix is defined as follows

$$\mathsf{U}_-(\lambda|\beta) = \tilde{G}^{-1}(\lambda-\eta/2|\beta)\mathcal{U}_-(\lambda)\tilde{G}(\eta/2-\lambda|\beta) = \begin{pmatrix} \hat{\mathcal{A}}(\lambda|\beta+2) & \hat{\mathcal{B}}(\lambda|\beta) \\ \hat{\mathcal{C}}(\lambda|\beta+2) & \hat{\mathcal{D}}(\lambda|\beta) \end{pmatrix}. \tag{3.2.13}$$

with the consequent definitions

$$\hat{\mathcal{A}}(\lambda|\beta) = \tilde{Y}(\lambda-\eta/2|\beta-3)\mathcal{U}_-(\lambda)X(\eta/2-\lambda|\beta-1), \tag{3.2.14a}$$
$$\hat{\mathcal{B}}(\lambda|\beta) = \tilde{Y}(\lambda-\eta/2|\beta-1)\mathcal{U}_-(\lambda)Y(\eta/2-\lambda|\beta-1), \tag{3.2.14b}$$
$$\hat{\mathcal{C}}(\lambda|\beta) = \tilde{X}(\lambda-\eta/2|\beta-1)\mathcal{U}_-(\lambda)X(\eta/2-\lambda|\beta-1), \tag{3.2.14c}$$
$$\hat{\mathcal{D}}(\lambda|\beta) = \tilde{X}(\lambda-\eta/2|\beta+1)\mathcal{U}_-(\lambda)Y(\eta/2-\lambda|\beta-1). \tag{3.2.14d}$$

This is indeed the correct definition, as it turned out to be clear during calculations and being in line with the construction in [23], but it generates a non-trivial gauged boundary-bulk decomposition since

$$\begin{pmatrix} \hat{\mathcal{A}}_-(\lambda|\beta+2) \\ \hat{\mathcal{C}}_-(\lambda|\beta+2) \end{pmatrix} = M(\lambda|\beta)\bar{K}_-(\lambda|\beta)\begin{pmatrix} \bar{A}_-(\lambda|\beta+1) \\ \bar{C}_-(\lambda|\beta+1) \end{pmatrix} \tag{3.2.15}$$

$$\begin{pmatrix} \hat{\mathcal{B}}_-(\lambda|\beta) \\ \hat{\mathcal{D}}_-(\lambda|\beta) \end{pmatrix} = M(\lambda|\beta)K_-(\lambda|\beta)\begin{pmatrix} \bar{A}_-(\lambda|\beta-1) \\ \bar{C}_-(\lambda|\beta-1) \end{pmatrix}, \tag{3.2.16}$$

where we have used

$$K_-(\lambda|\beta) = \tilde{G}^{-1}(\lambda-\eta/2|\beta+N)K_-(\lambda)\bar{G}(\eta/2-\lambda|\beta+N-1) \tag{3.2.17}$$
$$\bar{K}(\lambda|\beta) = \tilde{G}^{-1}(\lambda-\eta/2|\beta+N)K_-(\lambda)\bar{G}(\eta/2-\lambda|\beta+N+1). \tag{3.2.18}$$

Remark 3.2.1. In order to understand how (3.2.15) and (3.2.16) follow directly from the definition (3.2.13), consider the explicit decomposition, for example, of $\hat{\mathcal{D}}_-(\lambda|\beta)$:

$$\begin{aligned}
\hat{\mathcal{D}}_-(\lambda|\beta) =& \tilde{X}(\lambda-\eta/2|\beta+1)M(\lambda)K_-(-\lambda)\hat{M}(\lambda)Y(\eta/2-\lambda|\beta-1) \\
=& \tilde{X}(\lambda-\eta/2|\beta+1)M(\lambda)\,[\,Y(\lambda-\eta/2|\beta+N-1)\tilde{X}(\lambda-\eta/2|\beta+N+1) \\
& \qquad + X(\lambda-\eta/2|\beta+N-1)\tilde{Y}(\lambda-\eta/2|\beta+N+1)\,] \\
& \times K_-(\lambda)\,[\,Y(\eta/2-\lambda|\beta+N-1)\bar{X}(\eta/2-\lambda|\beta+N-1) \\
& \qquad + X(\eta/2-\lambda|\beta+N-1)\bar{Y}(\eta/2-\lambda|\beta+N-1)\,] \\
& \times \hat{M}(\lambda)Y(\eta/2-\lambda|\beta-1) \\
=& D(\lambda|\beta)\tilde{X}(\lambda-\eta/2|\beta+N+1)K_-(\lambda)Y(\eta/2-\lambda|\beta-1)\bar{D}(\lambda|\beta-1) \\
&+ D(\lambda|\beta)\tilde{X}(\lambda-\eta/2|\beta+N+1)K_-(\lambda)X(\eta/2-\lambda|\beta-1)\bar{B}(\lambda|\beta-1) \\
&+ C(\lambda|\beta)\,\tilde{Y}(\lambda-\eta/2|\beta+N+1)K_-(\lambda)Y(\eta/2-\lambda|\beta-1)\bar{D}(\lambda|\beta-1) \\
&+ C(\lambda|\beta)\,\tilde{Y}(\lambda-\eta/2|\beta+N+1)K_-(\lambda)X(\eta/2-\lambda|\beta-1)\bar{B}(\lambda|\beta-1),
\end{aligned}$$

where we have inserted a couple of orthogonality identities, one of type (3.2.6c) and one of type (3.2.7c), in a way to be consistent with the objects already defined in (3.2.10), (3.2.12) and (3.2.14);

from the expression found it's then possible to read out the definition of $K_-(\lambda|\beta)$ given in (3.2.17). Following the same reasoning and studying the explicit expression of $\mathcal{A}(\lambda|\beta+2)$ for example, $\bar{K}(\lambda|\beta)$ in (3.2.18) can be checked as well.

For convenience the operator in (3.2.13) should be renormalized as follows

$$\mathcal{U}_-(\lambda|\beta) = e^{-\lambda+\eta/2}\mathsf{U}_-(\lambda|\beta) = \begin{pmatrix} \mathcal{A}(\lambda|\beta+2) & \mathcal{B}(\lambda|\beta) \\ \mathcal{C}(\lambda|\beta+2) & \mathcal{D}(\lambda|\beta) \end{pmatrix}. \tag{3.2.19}$$

3.2.3 Properties of the gauged operators

We want to give here the equivalent properties satisfied by the gauged operators, as it was done in §1.2.2, §1.2.3 and §3.1 for the reflection algebra generators.

Lemma 3.2.1. *The following incomplete set of commutation relations holds for the gauged operators defined in* (3.2.13)

(I)

$$\mathcal{B}_-(\lambda_2|\beta)\mathcal{B}_-(\lambda_1|\beta-2) = \mathcal{B}_-(\lambda_1|\beta)\mathcal{B}_-(\lambda_2|\beta-2), \tag{3.2.20}$$

(II)

$$\begin{aligned}\mathcal{A}_-(\lambda_2|\beta+2)\mathcal{B}_-(\lambda_1|\beta) = {} & \frac{\sinh(\lambda_1-\lambda_2+\eta)\sinh(\lambda_2+\lambda_1-\eta)}{\sinh(\lambda_1-\lambda_2)\sinh(\lambda_1+\lambda_2)}\mathcal{B}_-(\lambda_1|\beta)\mathcal{A}_-(\lambda_2|\beta) \\ & + \frac{\sinh(\lambda_1+\lambda_2-\eta)\sinh(\lambda_1-\lambda_2+(\beta-1)\eta)\sinh\eta}{\sinh(\lambda_2-\lambda_1)\sinh(\lambda_1+\lambda_2)\sinh(\beta-1)\eta} \\ & \times\mathcal{B}_-(\lambda_2|\beta)\mathcal{A}_-(\lambda_1|\beta) \\ & + \frac{\sinh\eta\sinh(\lambda_1+\lambda_2-\beta\eta)}{\sinh(\lambda_1+\lambda_2)\sinh(\beta-1)\eta}\mathcal{B}_-(\lambda_2|\beta)\mathcal{D}_-(\lambda_1|\beta),\end{aligned} \tag{3.2.21}$$

(III)

$$\begin{aligned}\mathcal{B}_-(\lambda_1|\beta)\mathcal{D}_-(\lambda_2|\beta) = {} & \frac{\sinh(\lambda_1-\lambda_2+\eta)\sinh(\lambda_2+\lambda_1-\eta)}{\sinh(\lambda_1-\lambda_2)\sinh(\lambda_1+\lambda_2)}\mathcal{D}_-(\lambda_2|\beta+2)\mathcal{B}_-(\lambda_1|\beta) \\ & + \frac{\sinh(\lambda_1+\lambda_2-\eta)\sinh(\lambda_2-\lambda_1+(\beta+1)\eta)}{\sinh(\lambda_1-\lambda_2)\sinh(\lambda_1+\lambda_2)\sinh(\beta+1)\eta} \\ & \times\mathcal{D}_-(\lambda_1|\beta+2)\mathcal{B}_-(\lambda_2|\beta) \\ & + \frac{\sinh\eta\sinh(\lambda_1+\lambda_2+\beta\eta)}{\sinh(\lambda_1+\lambda_2)\sinh(\beta+1)\eta}\mathcal{A}_-(\lambda_1|\beta+2)\mathcal{B}_-(\lambda_2|\beta),\end{aligned} \tag{3.2.22}$$

(IV)

$$\begin{aligned}& \mathcal{A}_-(\lambda_1|\beta+2)\mathcal{A}_-(\lambda_2|\beta+2) \\ & \qquad - \frac{\sinh\eta\sinh(\lambda_1+\lambda_2-\beta\eta)}{\sinh(\lambda_1+\lambda_2)\sinh(\beta-1)\eta}\mathcal{B}_-(\lambda_1|\beta)\mathcal{C}_-(\lambda_2|\beta+2) = \\ & \mathcal{A}_-(\lambda_2|\beta+2)\mathcal{A}_-(\lambda_1|\beta+2) \\ & \qquad - \frac{\sinh\eta\sinh(\lambda_1+\lambda_2-\beta\eta)}{\sinh(\lambda_1+\lambda_2)\sinh(\beta-1)\eta}\mathcal{B}_-(\lambda_2|\beta)\mathcal{C}_-(\lambda_1|\beta+2).\end{aligned} \tag{3.2.23}$$

These expressions coincides with the commutation relations of the dynamical reflection algebra [51].

Proof. The commutation relations can be obtained as well by multiplying the reflection equation from the left and from the right by certain gauges. The appropriate choices can be found in table 3.1; the subscripts, in the column/row gauge terms, label the spaces where they act. As an illustrative example

	Left	**Right**
(I)	$\tilde{Y}_1(\lambda_1-\eta/2\|\beta-2)\tilde{Y}_2(\lambda_2-\eta/2\|\beta-1)$	$Y_1(-\lambda_1+\eta/2\|\beta-2)Y_2(-\lambda_2+\eta/2\|\beta-3)$
(II)	$\tilde{Y}_1(\lambda_1-\eta/2\|\beta-2)\tilde{Y}_2(\lambda_2-\eta/2\|\beta-1)$	$Y_1(-\lambda_1+\eta/2\|\beta-2)X_2(-\lambda_2+\eta/2\|\beta-1)$
(III)	$\tilde{Y}_1(\lambda_1-\eta/2\|\beta)\tilde{X}_2(\lambda_2-\eta/2\|\beta+3)$	$Y_1(-\lambda_1+\eta/2\|\beta)Y_2(-\lambda_2+\eta/2\|\beta-1)$
(IV)	$\tilde{Y}(\lambda_1-\eta/2\|\beta-2)\tilde{Y}_2(\lambda_2-\eta/2\|\beta-1)$	$X_1(-\lambda_1+\eta/2\|\beta)X_2(-\lambda_2+\eta/2\|\beta+1)$

Table 3.1: Left and Right multiplication of the reflection equation in order to get the commutation relations of 3.2.1.

we will explicitly show how to establish relation (3.2.21). In order to do so we will have to use the trigonometric version of the *face-vertex correspondence relations*, see, for example, [11, 12, 43, 23, 51]. For completeness the entire set of relevant relations are reproduced in Appendix A.1. Consider, first of all, the shift

$$(\lambda_1,\lambda_2)=(\mu_1+\eta/2,\mu_2+\eta/2)\in\mathbb{C}^2,\quad \text{s.t.}\quad \mathcal{U}_-(\lambda)\to\mathcal{U}_-(\mu+\eta/2)\overset{\text{notation}}{\equiv}\mathcal{U}_-(\mu), \tag{3.2.24}$$

introduced for an easier understanding of calculations. It follows that the double row monodromy matrix satisfies the reflection equation

$$R_{12}(\mu_1-\mu_2)\mathcal{U}_-(\mu_1)R_{21}(\mu_1+\mu_2)\mathcal{U}_-(\mu_2)=\\ \mathcal{U}_-(\mu_2)R_{21}(\mu_1+\mu_2)\mathcal{U}_-(\mu_1)R_{12}(\mu_1-\mu_2).$$

Now, multiply the above equation for $\tilde{Y}_1(\mu_1|\beta-2)\tilde{Y}_2(\mu_2|\beta-1)$ from the left and for $Y_1(-\mu_1|\beta-2)X_2(-\mu_2|\beta-1)$ from the right. The subscripts label the space where the gauges act. What we get on the two sides of the reflection equation can be schematized as it follows

Left hand side

$$\begin{aligned}&\tilde{Y}_1(\mu_1|\beta-2)\tilde{Y}_2(\mu_2|\beta-1)\times\\ &\qquad[R_{12}(\mu_1-\mu_2)\mathcal{U}_-(\mu_1)R_{21}\,(\mu_1+\mu_2)\mathcal{U}_-(\mu_2)]\\ &\qquad\qquad\times Y_1(-\mu_1|\beta-2)X_2(-\mu_2|\beta-1)\\ &=N_1\times[\tilde{Y}_1(\mu_1|\beta-1)\mathcal{U}_-(\mu_1)\\ &\qquad\times\tilde{Y}(\mu_2|\beta-2)\;R_{21}(\mu_1+\mu_2)Y_1(-\mu_1|\beta-2)\\ &\qquad\qquad\times\mathcal{U}_-(\mu_2)X_2(-\mu_2|\beta-1)]\end{aligned}$$

where $N_1=\sinh(\mu_1-\mu_2+\eta)$ comes from the relation (A.1.1f)

$$\tilde{Y}_1(\mu_1|\beta-1)\tilde{Y}_2(\mu_2|\beta)R_{12}(\mu_1-\mu_2)=\sinh(\mu_1-\mu_2+\eta)\tilde{Y}_2(\mu_2|\beta-1)\tilde{Y}_1(\mu_1|\beta)$$

for $\beta\to\beta-1$. Now by using relation (A.1.1j)

$$\tilde{Y}_1(\mu_1|\beta-1)R_{12}(\mu_1-\mu_2)Y_2(\mu_2|\beta-1)=\frac{\sinh(\mu_1-\mu_2)\sinh(\beta-1)\eta}{\sinh\beta\eta}Y_1(\mu_1|\beta)\tilde{Y}_2(-\mu_2|\beta-2)$$

with the prescriptions $1 \leftrightarrow 2$, $\mu_1 \to -\mu_1$, $\beta \to \beta - 1$ and $N_2 = \frac{\sinh(\mu_1+\mu_2)\sinh(\beta-2)\eta}{\sinh(\beta-1)\eta}$ we get

$$N_1 {}_\times N_2 {}_\times [\tilde{Y}_1(\mu_1|\beta-1)\mathcal{U}_-(\mu_1)Y_1(-\mu_1|\beta-1) {}_\times \tilde{Y}_2(\mu_2|\beta-3)\mathcal{U}_-(\mu_2)X_2(-\mu_2|\beta-1)].$$

By comparing what we got with the definition of $\mathcal{U}_-(\lambda|\beta)$ (3.2.19) and by reintroducing the proper shift in $\eta/2$ of the spectral parameter we arrive at:

$$e^{\eta-\lambda_1-\lambda_2} {}_\times (l.h.s.) \equiv \frac{\sinh(\mu_1-\mu_2+\eta)\sinh(\mu_1+\mu_2-\eta)}{\sinh(\beta+1)\eta}\mathcal{B}_-(\lambda_1|\beta)\mathcal{A}_-(\lambda_2|\beta).$$

Right hand side

The *r.h.s.* can be transformed in the very same way by using relations (A.1.1b), (A.1.1i) and (A.1.1j), resulting in

$$\begin{aligned} e^{\eta-\lambda_1-\lambda_2} {}_\times (r.h.s.) \equiv & \\ & \frac{\sinh(\lambda_1-\lambda_2)\sinh(\lambda_1+\lambda_2)\sinh(\beta-2)\eta}{\sinh(\beta-1)\eta}\mathcal{A}_-(\lambda_2|\beta+2)\mathcal{B}_-(\lambda_1|\beta) \\ & -\frac{\sinh(\lambda_1-\lambda_2)\sinh(\lambda_1+\lambda_2-\beta\eta)\sinh(\beta-2)\eta\sinh\eta}{\sinh^2(\beta-1)\eta}\mathcal{B}_-(\lambda_2|\beta)\mathcal{D}_-(\lambda_1|\beta) \\ & \frac{\sinh(\lambda_1-\lambda_2+(\beta-1)\eta)\sinh(\lambda_1+\lambda_2-\eta)\sinh(\beta-2)\eta\sinh\eta}{\sinh^2(\beta-1)\eta}\mathcal{B}_-(\lambda_2|\beta)\mathcal{A}_-(\lambda_1|\beta). \end{aligned}$$

Finally by equating the two sides we arrive at the wanted result.

□

Consider now the gauged equivalent of the *algebraic adjunct* of the double row monodromy matrix introduced in proposition 1.4.2

$$\begin{aligned} e^{(-\lambda-\eta/2)}\tilde{\mathcal{U}}_-(\lambda|\beta) &= \tilde{G}^{-1}(-\lambda-\eta/2|\beta)\tilde{\mathcal{U}}_-(\lambda)\tilde{G}(\lambda+\eta/2|\beta) \\ &= \begin{pmatrix} \tilde{Y}(-\lambda-\eta/2|\beta-1) \\ \tilde{X}(-\lambda-\eta/2|\beta+1) \end{pmatrix}\tilde{\mathcal{U}}_-(\lambda) \\ &\quad {}_\times (X(\lambda+\eta/2|\beta+1), Y(-\lambda+\eta/2|\beta-1)) \end{aligned} \tag{3.2.25}$$

where we used formula (1.4.14)

$$\begin{aligned} \tilde{\mathcal{U}}_-(\lambda) = & \\ & \begin{pmatrix} -\sinh\eta\mathcal{A}_-(\lambda)+\sinh(2\lambda)\mathcal{D}_-(\lambda) & -\sinh(2\lambda+\eta)\mathcal{B}_-(\lambda) \\ -\sinh(2\lambda+\eta)\mathcal{C}_-(\lambda) & -\sinh\eta\mathcal{D}_-(\lambda)+\sinh(2\lambda)\mathcal{A}_-(\lambda) \end{pmatrix} \\ & \overset{(1.4.19)}{=} \sinh(2\lambda-\eta)\mathcal{U}_-(-\lambda). \end{aligned} \tag{3.2.26}$$

Let us now organize in a set of propositions the results concerning the quantum determinant and some additional properties of the gauged algebra, equivalently to what was done in proposition 1.4.2.

Proposition 3.2.1. *The inverse transformed double-row monodromy matrix can be written in terms of the quantum determinant of the reflection algebra*

$$\mathcal{U}_-^{-1}(\lambda+\eta/2|\beta) = \frac{\tilde{\mathcal{U}}(\lambda-\eta/2|\beta)}{q\text{-}det\,(\mathcal{U}_-(\lambda)} = \frac{\sinh(2\lambda-2\eta)}{q\text{-}det\,(\mathcal{U}_-(\lambda)}\mathcal{U}_-(-\lambda+\eta/2|\beta), \tag{3.2.27}$$

where the following representation holds for the quantum determinant, for both $\epsilon = \pm$*:*

$$\begin{aligned}\frac{\text{q-det}\,(\mathcal{U}_-(\lambda))}{\sinh(2\lambda-2\eta)} =&\mathcal{A}_-(\epsilon\lambda+\eta/2|\beta+2)\mathcal{A}_-(\eta/2-\epsilon\lambda|\beta+2)+\mathcal{B}_-(\epsilon\lambda+\eta/2|\beta)\mathcal{C}_-(\eta/2-\epsilon\lambda|\beta+2)\\ =&\mathcal{D}_-(\epsilon\lambda+\eta/2|\beta)\mathcal{D}_-(\eta/2-\epsilon\lambda|\beta)+\mathcal{C}_-(\epsilon\lambda+\eta/2|\beta+2)\mathcal{B}_-(\eta/2-\epsilon\lambda|\beta).\end{aligned} \tag{3.2.28}$$

Proof. By using the definition of the gauge transformed boundary monodromy matrix introduced above, it's simple to show that

$$\mathcal{U}_-(\lambda+\eta/2|\beta) = e^{-\lambda}\tilde{G}^{-1}(\lambda|\beta)\mathcal{U}_-(\lambda+\eta/2)\tilde{G}(-\lambda|\beta),$$

and then, remembering equations (1.4.12) and (1.4.15), the following expression holds:

$$\begin{aligned}\mathcal{U}_-(\lambda+\eta/2|\beta)\mathcal{U}_-(-\lambda+\eta/2|\beta) =&\tilde{G}^{-1}(\lambda|\beta)\mathcal{U}_-(\lambda+\eta/2)\mathcal{U}_-(-\lambda+\eta/2)\tilde{G}(\lambda|\beta)\\ =&\frac{\text{q-det}\,(\mathcal{U}_-(\lambda)}{\sinh(2\lambda-2\eta)},\end{aligned} \tag{3.2.29}$$

and similarly

$$\mathcal{U}_-(-\lambda+\eta/2|\beta)\mathcal{U}_-(\lambda+\eta/2|\beta) = \frac{\text{q-det}\,(\mathcal{U}_-(\lambda)}{\sinh(2\lambda-2\eta)}. \tag{3.2.30}$$

By using the existent relation (3.2.26), we complete the proof of (3.2.27).

Finally by exploiting the explicit product (3.2.29) or (3.2.30) we arrive at the expressions in (3.2.28). □

Proposition 1.4.3 can be also generalized to the gauged algebra.

Proposition 3.2.2. *The gauged operators* $\mathcal{A}_-(\lambda|\beta)$*,* $\mathcal{B}_-(\lambda|\beta)$*,* $\mathcal{C}_-(\lambda|\beta)$ *and* $\mathcal{D}_-(\lambda|\beta)$ *satisfy the following parity relations:*

$$\mathcal{A}_-(\lambda|\beta) = \frac{\sinh(2\lambda-\eta)\sinh(\beta-1)\eta}{\sinh 2\lambda\sinh(\beta-2)\eta}\mathcal{D}_-(-\lambda|\beta) - \frac{\sinh\eta\sinh(2\lambda-(\beta-1)\eta)}{\sinh 2\lambda\sinh(\beta-2)\eta}\mathcal{D}_-(\lambda|\beta), \tag{3.2.31a}$$

$$\mathcal{D}_-(\lambda|\beta) = \frac{\sinh(2\lambda-\eta)\sinh(\beta-1)\eta}{\sinh 2\lambda\sinh\beta\eta}\mathcal{A}_-(-\lambda|\beta) + \frac{\sinh\eta\sinh(2\lambda+(\beta-1)\eta)}{\sinh 2\lambda\sinh\beta\eta}\mathcal{A}_-(\lambda|\beta), \tag{3.2.31b}$$

$$\mathcal{B}_-(-\lambda|\beta) = -\frac{\sinh(2\lambda+\eta)}{\sinh(2\lambda-\eta)}\mathcal{B}_-(\lambda|\beta), \tag{3.2.31c}$$

$$\mathcal{C}_-(-\lambda|\beta) = -\frac{\sinh(2\lambda+\eta)}{\sinh(2\lambda-\eta)}\mathcal{C}_-(\lambda|\beta). \tag{3.2.31d}$$

Proof. We can establish formulas (3.2.31) by explicit calculations, exploiting the equality $\tilde{\mathcal{U}}_-(\lambda|\beta) = \sinh(2\lambda-\eta)\mathcal{U}_-(-\lambda|\beta)$ and the following identities

$$\left(\tilde{\mathcal{U}}_-(\lambda|\beta)\right)_{12} = -\sinh(2\lambda+\eta)\mathcal{B}(\lambda|\beta), \tag{3.2.32}$$

$$\left(\tilde{\mathcal{U}}_-(\lambda|\beta)\right)_{21} = -\sinh(2\lambda+\eta)\mathcal{C}(\lambda|\beta), \tag{3.2.33}$$

$$\begin{aligned}\left(\tilde{\mathcal{U}}_-(\lambda|\beta)\right)_{22} = \Bigg(&\frac{\sinh 2\lambda\sinh(\beta-2)\eta}{\sinh(\beta-1)\eta}\mathcal{A}(\lambda|\beta)\\ &+\frac{\sinh\eta\sinh(2\lambda-(\beta-1)\eta)}{\sinh(\beta-1)\eta}\mathcal{D}(\lambda|\beta)\Bigg).,\end{aligned} \tag{3.2.34}$$

which can be established by expanding both the elements of $\tilde{\mathcal{U}}_-(\lambda|\beta)$ and $\mathcal{U}_-(\lambda|\beta)$ in terms of the ungauged elements of $\mathcal{U}_-(\lambda)$. □

The gauged boundary-monodromy matrix possesses a β-parity symmetry as well.

Proposition 3.2.3. *The following identity holds*

$$\mathcal{U}_-(\lambda|-\beta+2) = \sigma_0^x \mathcal{U}_-(\lambda|\beta)\sigma_0^x, \quad \sigma_0^x \in End(\mathcal{V}_0) \tag{3.2.35}$$

or in terms of the matrix elements

$$\mathcal{B}_-(\lambda|\beta) = \mathcal{C}_-(\lambda|-\beta+2), \quad \mathcal{A}_-(\lambda|\beta) = \mathcal{D}_-(\lambda|-\beta+2). \tag{3.2.36}$$

Proof. The proof is a trivial consequence of the following simple identity

$$Y(\lambda|\beta) = X(\lambda|-\beta). \tag{3.2.37}$$

□

3.2.4 Boundary transfer matrix and gauged operators

It is possible to write down the explicit expression of the boundary transfer matrix $\mathcal{T}(\lambda)$ in terms of the gauged generators of the reflection algebra. In order to do so, one has to define a transformation rule for the $K_+(\lambda)$ matrix as well. It turns out, given the definition of $\mathcal{U}_-(\lambda|\beta)$ in (3.2.13), that there are two possible relevant ways to do so. We will call these two transformation rules *left* and *right* as it will be clear in the following parts. Once again these definitions are not trivial as one could expect, but necessary to build a consistent representation.

Consider the vectors

$$\hat{X}(\lambda|\beta+2) = e^{\lambda+\eta}\frac{\sinh(\beta-1)\eta}{\sinh\beta\eta}X(\lambda|\beta+2), \qquad \underline{X}(\lambda|\beta) = e^{-\lambda}\bar{X}(\lambda|\beta), \tag{3.2.38}$$

$$\hat{Y}(\lambda|\beta-2) = e^{\lambda+\eta}\frac{\sinh(\beta+1)\eta}{\sinh\beta\eta}Y(\lambda|\beta-2), \qquad \underline{Y}(\lambda|\beta) = e^{-\lambda}\bar{Y}(\lambda|\beta). \tag{3.2.39}$$

Then we can define the following 2×2 matrices

$$K_+^{(L)}(\lambda|\beta) = \begin{pmatrix} \tilde{Y}(\eta/2-\lambda|\beta-2)K_+(\lambda)\hat{X}(\lambda-\eta/2|\beta+2) & \tilde{Y}(\eta/2-\lambda|\beta)K_+(\lambda)\hat{Y}(\lambda-\eta/2|\beta-2) \\ \tilde{X}(\eta/2-\lambda|\beta)K_+(\lambda)\hat{X}(\lambda-\eta/2|\beta+2) & \tilde{X}(\eta/2-\lambda|\beta+2)K_+(\lambda)\hat{Y}(\lambda-\eta/2|\beta-2) \end{pmatrix}, \tag{3.2.40}$$

$$K_+^{(R)}(\lambda|\beta) = \begin{pmatrix} \bar{Y}(\eta/2-\lambda|\beta)K_+(\lambda)X(\lambda-\eta/2|\beta) & \bar{Y}(\eta/2-\lambda|\beta)K_+(\lambda)Y(\lambda-\eta/2|\beta-2) \\ \bar{X}(\eta/2-\lambda|\beta)K_+(\lambda)X(\lambda-\eta/2|\beta+2) & \bar{X}(\eta/2-\lambda|\beta+2)K_+(\lambda)Y(\lambda-\eta/2|\beta) \end{pmatrix}. \tag{3.2.41}$$

In appendix A.2 it's possible to find the explicit expressions for the entries of these two matrices.

Lemma 3.2.2. *The boundary transfer matrix admits the following two representations in terms of the gauged generators:*

$$\begin{aligned}\mathcal{T}(\lambda) =& [K_+^{(L)}(\lambda|\beta-1)]_{11}\mathcal{A}_-(\lambda|\beta) + [K_+^{(L)}(\lambda|\beta-1)]_{22}\mathcal{D}_-(\lambda|\beta) \\ &+ [K_+^{(L)}(\lambda|\beta-1)]_{21}\mathcal{B}_-(\lambda|\beta-2) + [K_+^{(L)}(\lambda|\beta-1)]_{12}\mathcal{C}_-(\lambda|\beta+2),\end{aligned} \tag{3.2.42}$$

and

$$\begin{aligned}\mathcal{T}(\lambda) =& [K_+^{(R)}(\lambda|\beta-1)]_{11}\mathcal{A}_-(\lambda|\beta) + [K_+^{(R)}(\lambda|\beta-1)]_{22}\mathcal{D}_-(\lambda|\beta) \\ &+ [K_+^{(R)}(\lambda|\beta-1)]_{21}\mathcal{B}_-(\lambda|\beta) + [K_+^{(R)}(\lambda|\beta-1)]_{12}\mathcal{C}_-(\lambda|\beta).\end{aligned} \tag{3.2.43}$$

Proof. To prove expression (3.2.42) we introduce a new gauge matrix

$$\hat{G}(\lambda|\beta) = \left(\hat{X}(\lambda|\beta+2), \hat{Y}(\lambda|\beta-2)\right), \tag{3.2.44}$$

and

$$\hat{G}^{-1}(\lambda|\beta) = \begin{pmatrix} \tilde{Y}(\lambda|\beta-2) \\ \tilde{X}(\lambda|\beta+2) \end{pmatrix}, \tag{3.2.45}$$

which satisfy the following orthogonality condition

$$\hat{G}(\lambda|\beta)\hat{G}^{-1}(\lambda|\beta) = e^{\lambda} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.2.46}$$

Then we can rewrite the rhs of (3.2.42) as follows

$$\begin{aligned} &\mathcal{A}_-(\lambda|\beta)[K_+^{(L)}(\lambda|\beta-1)]_{11} + \mathcal{D}_-(\lambda|\beta)[K_+^{(L)}(\lambda|\beta-1)]_{22} \\ &\qquad + \mathcal{B}_-(\lambda|\beta-2)[K_+^{(L)}(\lambda|\beta-1)]_{21} + \mathcal{C}_-(\lambda|\beta+2)[K_+^{(L)}(\lambda|\beta-1)]_{12} \\ &= e^{-\lambda+\eta/2}\big(\tilde{Y}(\lambda-\eta/2|\beta-3)\mathcal{U}_-(\lambda)K_+(\lambda)\hat{X}(\lambda-\eta/2|\beta+1) \\ &\qquad + \tilde{X}(\lambda-\eta/2|\beta+1)\mathcal{U}_-(\lambda)K_+(\lambda)\hat{Y}(\lambda-\eta/2|\beta-3)\big) \\ &= e^{-\lambda+\eta/2}\mathrm{tr}_0\{\hat{G}^{-1}(\lambda-\eta/2|\beta-1)\mathcal{U}_-(\lambda)K_+(\lambda)\hat{G}(\lambda-\eta/2|\beta-1)\} \\ &\qquad = \mathrm{tr}_0\{\mathcal{U}_-(\lambda)K_+(\lambda)\} = \mathcal{T}(\lambda), \end{aligned}$$

where relation (3.2.46) has been used.

Similarly for expression (3.2.41), it holds

$$\begin{aligned} &[K_+^{(R)}(\lambda|\beta-1)]_{11}\mathcal{A}_-(\lambda|\beta) + [K_+^{(R)}(\lambda|\beta-1)]_{22}\mathcal{D}_-(\lambda|\beta) \\ &\qquad + [K_+^{(R)}(\lambda|\beta-1)]_{21}\mathcal{B}_-(\lambda|\beta) + [K_+^{(R)}(\lambda|\beta-1)]_{12}\mathcal{C}_-(\lambda|\beta) \\ &= \bar{Y}(\eta/2-\lambda|\beta-1)K_+(\lambda)\mathcal{U}_-(\lambda)X(\eta/2-\lambda|\beta-1) \\ &\qquad + \bar{X}(\eta/2-\lambda|\beta-1)K_+(\lambda)\mathcal{U}_-(\lambda)Y(\eta/2-\lambda|\beta-1) \\ &= \mathrm{tr}_0\{\bar{G}^{-1}(\eta/2-\lambda|\beta-1)K_+(\lambda)\mathcal{U}_-(\lambda)\bar{G}(\eta/2-\lambda|\beta-1)\} \\ &\qquad = \mathrm{tr}_0\{K_+(\lambda)\mathcal{U}_-(\lambda)\} = \mathcal{T}(\lambda), \end{aligned}$$

where relation (3.2.6c) has been used. □

By exploiting the properties of the gauged generators it is possible to produce some other useful representations.

Proposition 3.2.4. *The most general transfer matrix can be written in the following form*

$$\begin{aligned} \mathcal{T}(\lambda) =&\mathsf{a}_+(\lambda|\beta-1)\mathcal{A}_-(\lambda|\beta) + \mathsf{a}_+(-\lambda|\beta-1)\mathcal{A}_-(-\lambda|\beta) \\ &+ [K_+^{(L)}(\lambda|\beta-1)]_{21}\mathcal{B}_-(\lambda|\beta-2) + [K_+^{(L)}(\lambda|\beta-1)]_{12}\mathcal{C}_-(\lambda|\beta+2), \end{aligned} \tag{3.2.47}$$

and

$$\begin{aligned} \mathcal{T}(\lambda) =&\mathsf{d}_+(\lambda|\beta-1)\mathcal{D}_-(\lambda|\beta) + \mathsf{d}_+(-\lambda|\beta-1)\mathcal{D}_-(-\lambda|\beta) \\ &+ [K_+^{(R)}(\lambda|\beta-1)]_{21}\mathcal{B}_-(\lambda|\beta) + [K_+^{(R)}(\lambda|\beta-1)]_{12}\mathcal{C}_-(\lambda|\beta), \end{aligned} \tag{3.2.48}$$

where we have defined

$$\mathsf{a}_+(\lambda|\beta) = \frac{e^{\lambda-\eta/2}\sinh(2\lambda+\eta)}{\sinh 2\lambda \sinh(\beta+1)\eta \sinh\zeta_+}\,[\sinh\zeta_+\cosh(\lambda-\eta/2)\sinh(\lambda+\eta/2+\beta\eta) - (\cosh\zeta_+\sinh(\lambda-\eta/2)\cosh(\lambda+\eta/2+\beta\eta) + \kappa_+\sinh(2\lambda-\eta)\sinh(\tau_+ +\alpha\eta+2\eta)]\,, \quad (3.2.49)$$

$$\mathsf{d}_+(\lambda|\beta) = \frac{e^{\lambda-\eta/2}\sinh(2\lambda+\eta)}{\sinh 2\lambda \sinh(\beta-1)\eta \sinh\zeta_+}\,[\sinh\zeta_+\cosh(\lambda-\eta/2)\sinh(-\lambda-\eta/2+\beta\eta) - (\cosh\zeta_+\sinh(\lambda-\eta/2)\cosh(-\lambda-\eta/2+\beta\eta) + \kappa_+\sinh(2\lambda-\eta)\sinh(\tau_+ +\alpha\eta)]\,. \quad (3.2.50)$$

The proof of this proposition is purely computational, it suffices to introduce (3.2.31b) and (3.2.31a) into, respectively, (3.2.42) and (3.2.43) and use the explicit expressions for the matrices $K_+^{(L)}(\lambda|\beta)$ and $K_+^{(R)}(\lambda|\beta)$ given in appendix A.2.

3.3 SoV representation of the gauge transformed reflection algebra

In this section we construct explicitly the SoV representation of the gauged reflection algebra. In general it is associated to the construction of the eigenstates of the operators $\mathcal{B}$ (or $\mathcal{C}$). However the gauge transformations and the particular structure of the reference states leads to a slightly different result. Instead of the eigenstates we construct right and left *pseudo-eigenstates* for these operators. More precisely, for any generic value of β we will construct a basis in the Hilbert space $\mathcal{H}$

$$\langle\, \beta, \mathbf{h}\,|\,, \quad \mathbf{h}\equiv(h_1,\ldots,h_N), \quad h_j\in\{0,1\},$$

formed by states that we will call left pseudo-eigenstates of $\mathcal{B}_-(\lambda|\beta)$ if they satisfy the identities

$$\langle\, \beta, \mathbf{h}\,|\,\mathcal{B}_-(\lambda|\beta) = \mathsf{B}_{\mathbf{h}}(\lambda|\beta)\,\langle\, \beta-2, \mathbf{h}\,|\,, \quad (3.3.1)$$

where, for all the possible $\mathbf{h}$, the $\mathsf{B}_{\mathbf{h}}(\lambda|\beta)$ are the pseudo-eigenvalues of $\mathcal{B}_-(\lambda|\beta)$. Similarly we can define the basis of right pseudo-eigenstates.

The limits of applicability of the method are summarized in the following theorem.

Theorem 3.3.1. *Let the inhomogeneities* $\{\xi_1,\ldots,\xi_N\}\in\mathbb{C}^N$ *be s.t.*

$$\xi_a \neq \xi_b + r\eta \quad \forall a\neq b\in\{1,\ldots,N\} \quad \text{and } r\in\{-1,0,1\}, \quad (3.3.2)$$

then:
I_a) for any $\alpha,\beta\in\mathbb{C}$ such that for any integer k

$$(\alpha-\beta)\eta \neq (N-1)\eta - \tau_- - (-1)^k(\alpha_- + \beta_-) + i\pi k, \quad (3.3.3)$$

the one parameter family of the gauge transformed generators of the reflection algebra $\mathcal{B}_-(\lambda|\beta)$ is left pseudo-diagonalizable and its pseudo-spectrum is simple.

II_a) for any fixed $\alpha,\beta\in\mathbb{C}$ such that for any integer k

$$(\alpha-\beta)\eta \neq -(N+1)\eta - \tau_- - (-1)^k(\alpha_- + \beta_-) + i\pi k, \quad (3.3.4)$$

the one parameter family of the gauge transformed generators of the reflection algebra $\mathcal{B}_-(\lambda|\beta)$ is right pseudo-diagonalizable and its pseudo-spectrum is simple.

I_b) for any fixed $\alpha, \beta \in \mathbb{C}$ such that for any integer k

$$(\alpha+\beta)\eta \neq (N+1)\eta - \tau_- - (-1)^k(\alpha_- + \beta_-) + i\pi k, \tag{3.3.5}$$

the one parameter family of the gauge transformed generators of the reflection algebra $\mathcal{C}_-(\lambda|\beta)$ is left pseudo-diagonalizable and its pseudo-spectrum is simple.

II_b) for any $\alpha, \beta \in \mathbb{C}$ such that for any integer k

$$(\alpha+\beta)\eta \neq -(N-1)\eta - \tau_- - (-1)^k(\alpha_- + \beta_-) + i\pi k, \tag{3.3.6}$$

the one parameter family of the gauge transformed generators of the reflection algebra $\mathcal{C}_-(\lambda|\beta)$ is right pseudo-diagonalizable and its pseudo-spectrum is simple.

In all these cases we can construct a SoV representation of the gauge transformed reflection algebra.

The proof and some necessary clarifications of the statements contained in this theorem are given by the explicit constructions of the SoV representation in the next sections. In fact, we build explicitly the representations only for the cases I$_a$) and II$_a$) since for cases I$_b$) and II$_b$) the construction can be induced from the others due to the symmetries.

3.3.1 Reference states

The existence of a reference state is fundamental for the ABA to work, as it was pointed out in **1**. The usual vectors used as reference states are the pseudo-vacuums "all-spin-up" and "all-spin-down" states,

$$|\,0\,\rangle = \bigotimes_{n=1}^{N} |\uparrow\rangle_n = \bigotimes_{n=1}^{N} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_n, \quad |\,\bar{0}\,\rangle = \bigotimes_{n=1}^{N} |\downarrow\rangle_n = \bigotimes_{n=1}^{N} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_n,$$

and their dual left counterpart

$$\langle\,0\,| = \bigotimes_{n=1}^{N} \langle\uparrow|_n = \bigotimes_{n=1}^{N} (1,0)_n, \quad \langle\,\bar{0}\,| = \bigotimes_{n=1}^{N} \langle\downarrow|_n = \bigotimes_{n=1}^{N} (0,1)_n.$$

These reference states are not anymore eigenstates of the transfer matrix with the most generic boundary conditions, and this is the reason why ABA can't be applied successfully in this case. Moreover the insertion of gauge transformations makes the reference states for the bulk operators useless. We need to introduce then some new reference state for the gauge deformed bulk operators in order to build properly the SoV representation.

Definition 3.3.1.

$$\begin{aligned}\langle\,\beta\,| = \bigotimes_{n=1}^{N} (g(\xi_n|\beta+N-n)\langle\downarrow|_n - \langle\uparrow|_n) &= \bigotimes_{n=1}^{N} (-1, g(\xi_n|\beta+N-n))_n \\ &= N_\beta \langle\,0\,| \bigotimes_{n=1}^{N} \bar{G}_n^{-1}(\xi_n|\beta+N-n)\end{aligned} \tag{3.3.7}$$

where $g(\xi_n|\beta) = e^{-[\xi_n+(\alpha-\beta)\eta]}$ and $\bar{G}_n^{-1}(\xi_n)$ is the gauge transformation acting on the local quantum space $\mathcal{H}_n$ and N_β is the normalization factor

$$N_\beta = 2^N e^{-\alpha N\eta - \sum_j^N \xi_j} \prod_{n=1}^{N} \sinh(\beta+N-n)\eta. \tag{3.3.8}$$

Proposition 3.3.1. *The state* $\langle\, \beta \mid$ *is a simultaneous* $B(\lambda|\beta)$ *and* $\bar{B}(\lambda|\beta)$ *left reference state:*

$$\langle\, \beta \mid B(\lambda|\beta) = \langle\, \beta \mid \bar{B}(\lambda|\beta) = 0, \tag{3.3.9}$$

$$\langle\, \beta \mid A(\lambda|\beta) = \frac{\sinh(N+\beta)\eta}{\sinh\beta\eta} \prod_{n=1}^{N} \sinh(\lambda - \xi_n + \eta/2) \,\langle\, \beta - 1 \mid, \tag{3.3.10}$$

$$\langle\, \beta \mid D(\lambda|\beta) = \prod_{n=1}^{N} \sinh(\lambda - \xi_n - \eta/2) \,\langle\, \beta + 1 \mid, \tag{3.3.11}$$

$$\langle\, \beta \mid \bar{A}(\lambda|\beta) = \frac{\sinh\beta\eta}{\sinh(N+\beta)\eta} \prod_{n=1}^{N} \sinh(\lambda + \xi_n + \eta/2) \,\langle\, \beta + 1 \mid, \tag{3.3.12}$$

$$\langle\, \beta \mid \bar{D}(\lambda|\beta) = \prod_{n=1}^{N} \sinh(\lambda + \xi_n - \eta/2) \,\langle\, \beta - 1 \mid. \tag{3.3.13}$$

Proof. The proposition can be checked easily for local R-matrix by direct computation. Let's introduce the local vector

$$\langle\, \beta \mid_n = \left(-1, e^{-[\xi_n + (\alpha - (\beta + N - n)\eta]}\right)_n \tag{3.3.14}$$

then it follows

$$\begin{aligned}
&\langle\, \beta \mid_n \tilde{G}^{-1}(\lambda - \eta/2|\beta + N - n) R_{0n}(\lambda - \xi_n - \eta/2) \tilde{G}(\lambda - \eta/2|\beta + N - n + 1) \\
&= \begin{pmatrix} \frac{\sinh(\beta+N-n+1)\eta \sinh(\lambda-\xi_n+\eta/2)}{\sinh(\beta+N-n)\eta} \,\langle\, \beta - 1 \mid_n & 0 \\ \star & \sinh(\lambda - \xi_n + \eta/2) \,\langle\, \beta + 1 \mid_n \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
&- \langle\, \beta \mid_n \bar{G}^{-1}(\eta/2 - \lambda|\beta + N - n + 1) \sigma_0^y R_{0n}^{t_0}(-\lambda - \xi_n - \eta/2) \sigma_0^y \bar{G}(\eta/2 - \lambda|\beta + N - n) \\
&= \langle\, \beta \mid_n \bar{G}^{-1}(\eta/2 - \lambda|\beta + N - n + 1) R_{0n}(\lambda + \xi_n - \eta/2) \bar{G}(\eta/2 - \lambda|\beta + N - n) \\
&= \begin{pmatrix} \frac{\sinh(\beta+N-n)\eta \sinh(\lambda+\xi_n+\eta/2)}{\sinh(\beta+N-n+1)\eta} \,\langle\, \beta + 1 \mid_n & 0 \\ \star & \sinh(\lambda + \xi_n + \eta/2) \,\langle\, \beta - 1 \mid_n \end{pmatrix}.
\end{aligned}$$

The proof easily follows. □

The same construction can be implemented on the right as well.

Definition 3.3.2.

$$\begin{aligned}
\mid \beta \,\rangle = \bigotimes_{n=1}^{N} (f(\xi_n|\beta + N - n) \mid \uparrow\, \rangle_n - \mid \downarrow\, \rangle_n) &= \bigotimes_{n=1}^{N} \begin{pmatrix} f(\xi_n|\beta + N - n) \\ 1 \end{pmatrix} \\
&= \bigotimes_{n=1}^{N} \bar{G}_n(\xi_n|\beta + N - n) \mid 0 \,\rangle
\end{aligned} \tag{3.3.15}$$

where $f(\xi_n|\beta) = e^{-[\xi_n + (\alpha+\beta)\eta]}$.

We can then generate the results equivalent to proposition 3.3.1 for the right reference state too.

Proposition 3.3.2. *The state* $|\ \beta+1\ \rangle$ *is a simultaneous* $C(\lambda|\beta)$ *and* $\bar{C}(\lambda|\beta)$ *left reference state:*

$$C(\lambda|\beta)\,|\ \beta+1\ \rangle=\bar{C}(\lambda|\beta)\,|\ \beta+1\ \rangle=0, \tag{3.3.16}$$

$$A(\lambda|\beta)\,|\ \beta+1\ \rangle=\prod_{n=1}^{N}\sinh(\lambda-\xi_n+\eta/2)\,|\ \beta+2\ \rangle, \tag{3.3.17}$$

$$D(\lambda|\beta)\,|\ \beta+1\ \rangle=\frac{\sinh(N+\beta)\eta}{\sinh\beta\eta}\prod_{n=1}^{N}\sinh(\lambda-\xi_n-\eta/2)\,|\ \beta\ \rangle, \tag{3.3.18}$$

$$\bar{A}(\lambda|\beta)\,|\ \beta+1\ \rangle=\prod_{n=1}^{N}\sinh(\lambda+\xi_n+\eta/2)\,|\ \beta\ \rangle, \tag{3.3.19}$$

$$\bar{D}(\lambda|\beta)\,|\ \beta+1\ \rangle=\frac{\sinh\beta\eta}{\sinh(N+\beta)\eta}\prod_{n=1}^{N}\sinh(\lambda+\xi_n-\eta/2)\,|\ \beta+2\ \rangle. \tag{3.3.20}$$

Proof. The proposition can be checked easily for local R-matrix by direct computation. Let's introduce the local vector

$$|\ \beta\ \rangle_n=\begin{pmatrix} e^{-[\xi_n+(\alpha+(\beta+N-n))\eta]} \\ 1\end{pmatrix}_n \tag{3.3.21}$$

then it follows

$$\begin{aligned}&\tilde{G}^{-1}(\lambda-\eta/2|\beta+N-n)R_{0n}(\lambda-\xi_n-\eta/2)\tilde{G}(\lambda-\eta/2|\beta+N-n+1)\,|\ \beta+1\ \rangle_n\\ &=\begin{pmatrix}\sinh(\lambda-\xi_n+\eta/2)\,|\ \beta+2\ \rangle_n & \star\\ 0 & \frac{\sinh(\beta+N-n+1)\eta\,\sinh(\lambda-\xi_n-\eta/2)}{\sinh(\beta+N-n)\eta}\,|\ \beta\ \rangle_n\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}&-\bar{G}^{-1}(\eta/2-\lambda|\beta+N-n+1)\sigma_0^y R_{0n}^{t_0}(-\lambda-\xi_n-\eta/2)\sigma_0^y\bar{G}(\eta/2-\lambda|\beta+N-n)\,|\ \beta+1\ \rangle_n\\ &=\bar{G}^{-1}(\eta/2-\lambda|\beta+N-n+1)R_{0n}(\lambda+\xi_n-\eta/2)\bar{G}(\eta/2-\lambda|\beta+N-n)\,|\ \beta+1\ \rangle_n\\ &=\begin{pmatrix}\sinh(\lambda+\xi_n+\eta/2)\,|\ \beta\ \rangle_n & \star\\ 0 & \frac{\sinh(\beta+N-n)\eta\,\sinh(\lambda+\xi_n-\eta/2)}{\sinh(\beta+N-n+1)\eta}\,|\ \beta+2\ \rangle_n\end{pmatrix}.\end{aligned}$$

□

3.3.2 $\mathcal{B}_-(\lambda|\beta)$-SoV representations of the gauge transformed reflection algebra

In the following two subsections we will give the explicit construction of the left and right $\mathcal{B}_-(\lambda|\beta)$ SoV representations for the states that, as we will see, generate the whole Hilbert space.

Left $\mathcal{B}_-(\lambda|\beta)$-SoV representation of the gauge transformed reflection algebra

In this subsection we construct the left $\mathcal{B}_-(\lambda|\beta)$-pseudo-eigenbasis.

Theorem 3.3.2. *Left* $\mathcal{B}_-(\lambda|\beta)$ *SoV-basis The following states:*

$$\langle\ \beta,h_1,\dots,h_N\ |=\langle\ \beta\ |\prod_{n=1}^{N}\left(\frac{\mathcal{A}_-(\eta/2-\xi_n|\beta+2)}{\mathsf{A}_-(\eta/2-\xi_n)}\right)^{h_n}, \tag{3.3.22}$$

where $\langle\, \beta \,|$ *is the state defined in* (3.3.7) *and the function* $\mathsf{A}_-(\lambda)$ *is given by* (3.1.8). *If* (3.3.2) *and* (3.3.3) *are satisfied, these states define a basis of* $\mathcal{H}$ *formed out of pseudo-eigenstates of* $\mathcal{B}_-(\lambda|\beta)$*:*

$$\langle\, \beta, \mathbf{h} \,|\, \mathcal{B}_-(\lambda|\beta) = \mathsf{B}_{\mathbf{h}}(\lambda|\beta) \,\langle\, \beta - 2, \mathbf{h} \,|\,, \tag{3.3.23}$$

where $\langle\, \beta, \mathbf{h} \,| = \langle\, \beta, h_1, \ldots, h_N \,|$, $\mathbf{h} = (h_1, \ldots, h_N)$, $h_j \in \{0, 1\}$ *and*

$$\begin{aligned}\mathsf{B}_{\mathbf{h}}(\lambda|\beta) = &(-1)^N e^{(\beta+N)\eta} a_{\mathbf{h}}(\lambda) a_{\mathbf{h}}(-\lambda) \\ &\times \frac{\sinh(2\lambda-\eta)\left(2\kappa_- \sinh\left[(N+\beta-\alpha-1)\eta - \tau_-\right] - e^{\zeta_-}\right)}{2\sinh\zeta_- \sinh(\beta\eta)},\end{aligned} \tag{3.3.24}$$

with

$$a_{\mathbf{h}}(\lambda) = \prod_{n=1}^{N} \sinh(\lambda - \xi_n - (h_n - \frac{1}{2})\eta). \tag{3.3.25}$$

Proof. It is worth writing explicitly the (boundary-bulk) decomposition of the gauge transformed reflection algebra generator (3.2.16)

$$\begin{aligned}e^{\lambda-\eta/2}\mathcal{B}_-(\lambda|\beta) = &K_-(\lambda|\beta)_{12} A(\lambda|\beta)\bar{D}(\lambda|\beta-1) + K_-(\lambda|\beta)_{11} A(\lambda|\beta)\bar{B}(\lambda|\beta-1) \\ &+ K_-(\lambda|\beta)_{21} B(\lambda|\beta)\bar{B}(\lambda|\beta-1) + K_-(\lambda|\beta)_{22} B(\lambda|\beta)\bar{D}(\lambda|\beta-1).\end{aligned} \tag{3.3.26}$$

Then, the formulae (3.3.9) - (3.3.13) imply that $\langle\, \beta \,|$ is a $\mathcal{B}_-(\lambda|\beta)$-pseudo-eigenstate with non-zero eigenvalue

$$\langle\, \beta \,|\, \mathcal{B}_-(\lambda|\beta) = \mathsf{B}_{\mathbf{0}}(\lambda|\beta) \,\langle\, \beta - 2 \,|\,, \tag{3.3.27}$$

where

$$\mathsf{B}_{\mathbf{0}}(\lambda|\beta) = (-1)^N e^{-\lambda+\eta/2} \frac{\sinh(\beta+N)\eta}{\sinh(\beta\eta)} K_-(\lambda|\beta)_{12} a_{\mathbf{0}}(\lambda) a_{\mathbf{0}}(-\lambda), \tag{3.3.28}$$

with $a_{\mathbf{0}}(\lambda)$ given by (3.3.25) for all $h_j = 0$ and

$$e^{-\lambda+\eta/2} K_-(\lambda|\beta)_{12} = \frac{e^{(\beta+N)\eta} \sinh(2\lambda-\eta)(2\kappa_- \sinh\left[(N+\beta-\alpha-1)\eta - \tau_-\right] - e^{\zeta_-})}{2\sinh(N+\beta)\eta \sinh\zeta_-}. \tag{3.3.29}$$

In order to prove the result (3.3.24) one has to actively use the commutation relation (3.2.21). Let's compute explicitly the eigenvalue of $\mathcal{B}_-(\lambda|\beta)$ for this particular state:

$$\langle\, \beta, \mathbf{h}_1 \,| = \langle\, \beta \,| \left(\frac{\mathcal{A}_-(\eta/2 - \xi_1|\beta+2)}{\mathsf{A}_-(\eta/2 - \xi_1)} \right). \tag{3.3.30}$$

where $\mathbf{h}_1 = \left\{h_1 = 1, \{h_j = 0\}_{j=2,\ldots,N}\right\}$. From the commutation relations between $\mathcal{A}_-(\eta/2 - \xi_1|\beta + 2)$ and $\mathcal{B}_-(\lambda|\beta)$ it results

$$\begin{aligned}\mathcal{A}_-(\eta/2-\xi_1|\beta+2)\mathcal{B}_-(\lambda|\beta) = &\frac{\sinh(\lambda+\xi_1+\eta/2)\sinh(\lambda-\xi_1-\eta/2)}{\sinh(\lambda+\xi_1-\eta/2)\sinh(\lambda-\xi_1+\eta/2)} \\ &\qquad\times \mathcal{B}_-(\lambda|\beta)\mathcal{A}_-(\eta/2-\xi_1|\beta) \\ &-\frac{\sinh(\lambda-\xi_1-\eta/2)\sinh(\lambda+\xi_1-\eta/2+(\beta-1)\eta)\sinh\eta}{\sinh(\lambda+\xi_1-\eta/2)\sinh(\lambda-\xi_1+\eta/2)\sinh(\beta-1)\eta} \\ &\qquad\times \mathcal{B}_-(\eta/2-\xi_1|\beta)\mathcal{A}_-(\lambda|\beta) \\ &+\frac{\sinh\eta\sinh(\lambda-\xi_1+\eta/2-\beta\eta)}{\sinh(\lambda-\xi_1+\eta/2)\sinh(\beta-1)\eta}\mathcal{B}_-(\eta/2-\xi_1|\beta)\mathcal{D}_-(\lambda|\beta).\end{aligned}$$

Now by using the boundary-bulk decomposition (3.3.26) and the formulae (3.3.9) - (3.3.13) it's easy to see how the second and third terms in the rhs, in the commutation relation desplayed above, go to zero. The remaining term generates the eigenvalue B_0 times a factor which corrects the "contributions" in h_1 resulting in the wanted $a_{\mathbf{h}_1}$. By applying this reasoning to a generic state of type (3.3.22) we arrive at (3.3.24). □

This theorem permits us to prove that the condition (3.3.3) is essential to for the SoV applicability in the $\mathcal{B}_-(\lambda|\beta)$ left representation. Indeed, by using the re-parametrization (3.1.9) and by inserting it in (3.3.24), it's possible to see that the condition (3.3.3) inhibits the eigenvalue $\mathsf{B}_{\mathbf{h}}(\lambda)$ from being identically zero. In other words, in these particular points, the operator $\mathcal{B}_-(\lambda|\beta)$ would be nihilpotent and so it would be impossible to diagonalize and successively build the SoV representation associated to it. An identical argument can be applied to the other SoV represenations as well.

Remark 3.3.1. It is important to point out that the states $\langle\ \beta,\mathbf{h}\ |$ are well defined non-zero states and they are invariant to permutations of the order of the operators $\mathcal{A}_-(\eta/2-\xi_b|\beta+2)$ as it follows from the commutation relations (3.2.23).

Remark 3.3.2. Under the condition (3.3.2), the identities (3.3.23) also imply that the set of states $\langle\ \beta,\mathbf{h}\ |$ forms a set of 2^N independent states, i.e. a $\mathcal{B}_-(\lambda|\beta)$-pseudo-eigenbasis of $\mathcal{H}$.

Remark 3.3.3. From the expression (3.3.24) for $\mathsf{B}_{\mathbf{h}}(\lambda)$ it's possible to read out the operator roots of $\mathcal{B}_-(\lambda|\beta)$, or better their eigenvalues. This is a central point in the SoV formulation since they constitute the so called *separated variables* of the model. We can find a common definition for all of them by introducing the following

$$\zeta_n^{(h_n)} = \varphi_n\left[\xi_n + (h_n - \frac{1}{2})\eta\right] \quad \forall n \in \{1,\dots,2N\}, \tag{3.3.31}$$

where $h_{n+N} \equiv h_n \in \{0,1\}$, and

$$\varphi_a = 1 \quad \text{for} \quad a \le N \quad \text{and} \quad \varphi_a = -1 \quad \text{for} \quad a > N. \tag{3.3.32}$$

At last, in order to understand the normalization factor in (3.3.22) it's useful to consider the next theorem.

Theorem 3.3.3. *The action of the reflection algebra generators $\mathcal{A}_-(\lambda|\beta+2)$ on the generic state $\langle\ \beta,\mathbf{h}\ |$, is given by the following expression*

$$\begin{aligned}
\langle\ \beta,\mathbf{h}\ |\,\mathcal{A}_-(\lambda|\beta+2) = & \sum_{a=1}^{2N}\frac{\sinh(2\lambda-\eta)\sinh(\lambda+\zeta_a^{(h_a)})}{\sinh(2\zeta_a^{(h_a)}-\eta)\sinh 2\zeta_a^{(h_a)}} \\
& \times \prod_{\substack{b=1\\ b\neq a\ modN}}^{N}\frac{\cosh 2\lambda-\cosh 2\zeta_b^{(h_b)}}{\cosh 2\zeta_a^{(h_a)}-\cosh 2\zeta_b^{(h_b)}}\mathsf{A}_-(\zeta_a^{(h_a)})\,\langle\ \beta,\mathbf{h}\ |\,T_a^{-\varphi_a} \\
& + \textit{q-det}\,(M(0))\cosh(\lambda-\eta/2)\prod_{b=1}^{N}\frac{\cosh 2\lambda-\cosh 2\zeta_b^{(h_b)}}{\cosh\eta-\cosh 2\zeta_b^{(h_b)}}\,\langle\ \beta,\mathbf{h}\ | \\
& + (-1)^{N+1}\coth\zeta_-\textit{q-det}\,(M(i\pi/2))\sinh(\lambda-\eta/2) \\
& \qquad \times\prod_{b=1}^{N}\frac{\cosh 2\lambda-\cosh 2\zeta_b^{(h_b)}}{\cosh\eta+\cosh 2\zeta_b^{(h_b)}}\,\langle\ \beta,\mathbf{h}\ |\,,
\end{aligned} \tag{3.3.33}$$

where we used the definitions contained in remark 3.3.3 and

$$\langle\, \beta, h_1, \ldots, h_a, \ldots, h_N \mid T_a^{\pm} = \langle\, \beta, h_1, \ldots, h_a \pm 1, \ldots, h_N \mid . \tag{3.3.34}$$

Proof. First of all consider the fact that the operator $\mathcal{A}_-(\lambda|\beta)$ is a trigonometric polynomial of the form

$$\mathcal{A}_-(\lambda|\beta) = \sum_{a=0}^{2N+1} e^{2a-2N-1} \mathcal{A}_{-,a} \quad , \tag{3.3.35}$$

then it is sufficient to characterize its action on a generic state $\langle\, \beta, \mathbf{h} \mid$ in $2N+2$ points in order to build the correct interpolation formula (3.3.33). The action of $\mathcal{A}_-(\zeta_b^{(h_b)}|\beta+2)$ for $b \in \{1, \ldots, 2N\}$ follows by the definition of the states $\langle\, \beta, \mathbf{h} \mid$, the reflection algebra commutation relations (3.2.23), the quantum determinant relations (3.2.28) and the identities:

$$\langle\, \beta \mid \mathcal{A}_-(\xi_n - \eta/2|\beta+2) = 0, \quad \langle\, \beta \mid \mathcal{A}_-(\eta/2 - \xi_n|\beta+2) \neq 0 \tag{3.3.36}$$

which are a consequence of the boundary-bulk decomposition introduced in section §3.2.2

$$\begin{aligned} e^{\lambda-\eta/2} \mathcal{A}_-(\lambda|\beta+2) = {} & \bar{K}_-(\lambda|\beta)_{11} A(\lambda|\beta) \bar{A}(\lambda|\beta+1) + \bar{K}_-(\lambda|\beta)_{12} A(\lambda|\beta) \bar{C}(\lambda|\beta+1) \\ & + \bar{K}_-(\lambda|\beta)_{21} B(\lambda|\beta) \bar{A}(\lambda|\beta+1) + \bar{K}_-(\lambda|\beta)_{22} B(\lambda|\beta) \bar{C}(\lambda|\beta+1). \end{aligned} \tag{3.3.37}$$

Moreover, by using the identities:

$$\mathcal{U}_-(\eta/2) = \text{q-det}\ (M(0))\ \mathbb{1}_0, \quad \mathcal{U}_-(\eta/2 + i\pi/2) = i \coth \zeta_- \text{q-det}\ (M(i\pi/2))\ \sigma_0^z, \tag{3.3.38}$$

and

$$\tilde{Y}(0|\beta-1) X(0|\beta+1) = 1, \quad \tilde{Y}(i\pi/2|\beta-1) \sigma_0^z X(-i\pi/2|\beta+1) = -1 \tag{3.3.39}$$

we get the formula (see [139])

$$\begin{aligned} \langle\, \beta, \mathbf{h} \mid \mathcal{A}_-(\lambda|\beta+2) = {} & \sum_{a=1}^{2N} \frac{\sinh(2\lambda - \eta)}{\sinh(2\zeta_a^{(h_a)} - \eta)} \prod_{\substack{b=1 \\ b \neq a}}^{2N} \frac{\sinh(\lambda - \zeta_b^{(h_b)})}{\sinh(\zeta_a^{(h_a)} - \zeta_b^{(h_b)})} \mathsf{A}_-(\zeta_a^{(h_a)}) \langle\, \beta, \mathbf{h} \mid T_a^{-\varphi_a} \\ & + \text{q-det}\ (M(0)) \cosh(\lambda - \eta/2) \prod_{b=1}^{2N} \frac{\sinh(\lambda - \zeta_b^{(h_b)})}{\sinh(\eta/2 - \zeta_b^{(h_b)})} bra\beta, \mathbf{h} \\ & \coth \zeta_- \text{q-det}\ (M(i\pi/2)) \sinh(\lambda - \eta/2) \prod_{b=1}^{2N} \frac{\sinh(\lambda - \zeta_b^{(h_b)})}{\sinh(\eta/2 + i\pi/2 - \zeta_b^{(h_b)})} \langle\, \beta, \mathbf{h} \mid . \end{aligned}$$

Then, it is a simple exercise to rewrite this in the form (3.3.33). □

Right $\mathcal{B}_-(\lambda|\beta)$-SoV representation of the gauge transformed reflection algebra

Theorem 3.3.4. *Right $\mathcal{B}_-(\lambda|\beta)$ SoV-basis We define the states:*

$$|\, \beta, h_1, \ldots, h_N \,\rangle = \prod_{n=1}^{N} \left(\frac{\mathcal{D}_-(\xi_n + \eta/2|\beta)}{f_n(\beta) \mathsf{A}_-(\eta/2 - \xi_n)} \right)^{(1-h_n)} |\, -\beta + 2 \,\rangle , \tag{3.3.40}$$

where

$$f_n(\beta) = \frac{\sinh(2\xi_n + \eta) \sinh \beta\eta}{\sinh(2\xi_n - \eta) \sinh(2\xi_n + \beta\eta)}, \tag{3.3.41}$$

and $h_n \in \{0,1\}$, $n \in \{1,\dots,N\}$. *If* (3.3.2) *and* (3.3.4) *are satisfied, then this set of states defines a basis of* $\mathcal{H}$ *and they are* $\mathcal{B}_-(\lambda|\beta)$ *right pseudo-eigenstates*

$$\mathcal{B}_-(\lambda|\beta) \mid \beta, \mathbf{h} \,\rangle = \mid \beta+2, \mathbf{h} \,\rangle \, \bar{\mathsf{B}}_{\mathbf{h}}(\lambda|\beta), \tag{3.3.42}$$

where

$$\begin{aligned} \bar{\mathsf{B}}_{\mathbf{h}}(\lambda|\beta) = (-1)^N e^{(\beta-N)\eta} \prod_{n=1}^{N} \left(\frac{f_n(\beta+2)}{f_n(\beta)}\right)^{1-h_n} a_{\mathbf{h}}(\lambda) a_{\mathbf{h}}(-\lambda) \\ \times \frac{\sinh(2\lambda-\eta)\left(2\kappa_- \sinh\left[(\beta-(1+N+\alpha))\eta - \tau_-\right] - e^{\zeta_-}\right)}{2\sinh\zeta_- \sinh\beta\eta}. \end{aligned} \tag{3.3.43}$$

Proof. The proof is similar to the one for the left SoV basis. First we prove that $\langle\, -\beta+2 \mid$ is a right $\mathcal{B}_-(\lambda|\beta)$ pseudo-eigenstate. From the Proposition 3.3.2 and the boundary-bulk decomposition (3.2.15):

$$\begin{aligned} e^{\lambda-\eta/2}\mathcal{C}_-(\lambda|\beta) = \bar{K}_-(\lambda|\beta-2)_{21} D(\lambda|\beta-2)\bar{A}(\lambda|\beta-1) + \bar{K}_-(\lambda|\beta-2)_{22} D(\lambda|\beta-2)\bar{C}(\lambda|\beta-1) \\ + \bar{K}_-(\lambda|\beta-2)_{12} C(\lambda|\beta-2)\bar{C}(\lambda|\beta-1) + \bar{K}_-(\lambda|\beta-2)_{11} C(\lambda|\beta-2)\bar{A}(\lambda|\beta-1). \end{aligned} \tag{3.3.44}$$

It follows that the state $\mid \beta \,\rangle$ is a right $\mathcal{C}_-(\lambda|\beta)$-pseudo-eigenstate; i.e. it holds:

$$\mathcal{C}_-(\lambda|\beta) \mid \beta \,\rangle = \mid \beta-2 \,\rangle \, \mathsf{C}_{\mathbf{0}}(\lambda|\beta) \tag{3.3.45}$$

where:

$$\mathsf{C}_{\mathbf{0}}(\lambda|\beta) = (-1)^N e^{-\lambda+\eta/2} \bar{K}_-(\lambda|\beta-2)_{21} \frac{\sinh(N+\beta-2)\eta}{\sinh(\beta-2)\eta} a_{\mathbf{1}}(\lambda) a_{\mathbf{1}}(-\lambda), \tag{3.3.46}$$

and $a_{\mathbf{1}}(\lambda)$ is given by (3.3.25) for all $h_j = 1$ and

$$e^{-\lambda+\eta/2}\bar{K}_-(\lambda|\beta-2)_{21} = \frac{e^{-(\beta+N-2)\eta}\sinh(2\lambda-\eta)\left(2\kappa_- \sinh\left[(N+\beta+\alpha-1)\eta + \tau_-\right] + e^{\zeta_-}\right)}{2\sinh\zeta_- \sinh(N+\beta-2)\eta}. \tag{3.3.47}$$

From the identity (3.2.36), it follows that the formula (3.3.45) is equivalent to the following one:

$$\mathcal{B}_-(\lambda|\beta) \mid -\beta+2 \,\rangle = \mid -\beta \,\rangle, \quad \mathsf{C}_{\mathbf{0}}(\lambda|-\beta+2). \tag{3.3.48}$$

By using the identities (3.3.48) and the commutation relations (3.2.22) and the formulae:

$$\mathcal{D}_-(-\xi_n - \eta/2|\beta)|-\beta+2\rangle = 0, \quad \mathcal{D}_-(\xi_n + \eta/2|\beta)|-\beta+2\rangle \neq 0, \tag{3.3.49}$$

the states (3.3.40) are proved to be non-zero $\mathcal{B}_-(\lambda|\beta)$-pseudo-eigenstates with pseudo -eigenvalues $\bar{\mathsf{B}}_{\mathbf{h}}(\lambda|\beta)$ which form a basis of $\mathcal{H}$. □

To define the action of the operators $\mathcal{D}_-(\lambda|\beta)$ on the generic state $\mid \beta, \mathbf{h} \,\rangle$ we will need to introduce a set of values

$$\mathsf{D}_-(\zeta_a^{(h_a)}) = \left[f_a(\beta))\right]^{\varphi_a} \mathsf{A}_-(-\zeta_a^{(1-h_a)}), \qquad a = 1,\dots,2N. \tag{3.3.50}$$

It is important to underline that this set of values cannot be seen as values of some analytic function D_-, however to construct the SoV representation we will need only these points.

Theorem 3.3.5. *The action of the reflection algebra generators $\mathcal{D}_{-}(\lambda|\beta)$ on the generic state $|\,\beta,\mathbf{h}\,\rangle$, can be written as follows*

$$\begin{aligned}\mathcal{D}_{-}(\lambda|\beta)\,|\,\beta,\mathbf{h}\,\rangle &= \sum_{a=1}^{2N} T_a^{-\varphi_a}\,|\,\beta,\mathbf{h}\,\rangle \frac{\sinh(2\lambda-\eta)\sinh(\lambda+\zeta_a^{(h_a)})}{\sinh(2\zeta_a^{(h_a)}-\eta)\sinh 2\zeta_a^{(h_a)}} \\ &\times \prod_{\substack{b=1\\ b\neq a\ modN}}^{N} \frac{\cosh 2\lambda-\cosh 2\zeta_b^{(h_b)}}{\cosh 2\zeta_a^{(h_a)}-\cosh 2\zeta_b^{(h_b)}}\mathsf{D}_{-}(\zeta_a^{(h_a)}) \\ &+|\,\beta,\mathbf{h}\,\rangle\, \textit{q-det}\ (M(0))\cosh(\lambda-\eta/2)\prod_{b=1}^{N}\frac{\cosh 2\lambda-\cosh 2\zeta_b^{(h_b)}}{\cosh\eta-\cosh 2\zeta_b^{(h_b)}} \\ &+(-1)^N\,|\,\beta,\mathbf{h}\,\rangle \coth\zeta_{-}\textit{q-det}\ (M(i\pi/2))\sinh(\lambda-\eta/2)\prod_{b=1}^{N}\frac{\cosh 2\lambda-\cosh 2\zeta_b^{(h_b)}}{\cosh\eta+\cosh 2\zeta_b^{(h_b)}},\end{aligned} \tag{3.3.51}$$

where we used the definitions contained in remark 3.3.3 and

$$T_a^{\pm}\,|\,\beta,h_1,\ldots,h_a,\ldots,h_N\,\rangle = |\,\beta,h_1,\ldots,h_a\pm 1,\ldots,h_N\,\rangle\,. \tag{3.3.52}$$

Proof. The form of the action of $\mathcal{D}_{-}(\zeta_a^{(h_a)}|\beta)$ on $|\,\beta,\mathbf{h}\,\rangle$ is just a consequence of the definition of the states and the quantum determinant. Finally, the formula (3.3.51) is just a rewriting of the following interpolation formula for the action on $|\,\beta,\mathbf{h}\,\rangle$:

$$\begin{aligned}\mathcal{D}_{-}(\lambda|\beta)\,|\,\beta,\mathbf{h}\,\rangle &= \sum_{a=1}^{2N} T_a^{-\varphi_a}|\beta,\mathbf{h}\rangle\frac{\sinh(2\lambda-\eta)}{\sinh(2\zeta_a^{(h_a)}-\eta)}\prod_{\substack{b=1\\ b\neq a}}^{2N}\frac{\sinh(\lambda-\zeta_b^{(h_b)})}{\sinh(\zeta_a^{(h_a)}-\zeta_b^{(h_b)})}f_a^{\varphi_a}(\beta)\mathsf{A}_{-}(-\zeta_a^{(1-h_a)}) \\ &+|\,\beta,\mathbf{h}\,\rangle\,\text{q-det}\ (M(0))\cosh(\lambda-\eta/2)\prod_{b=1}^{2N}\frac{\sinh(\lambda-\zeta_b^{(h_b)})}{\sinh(\eta/2-\zeta_b^{(h_b)})} \\ &-|\,\beta,\mathbf{h}\,\rangle\coth\zeta_{-}\text{q-det}\ (M(i\pi/2))\sinh(\lambda-\eta/2)\prod_{b=1}^{2N}\frac{\sinh(\lambda-\zeta_b^{(h_b)})}{\sinh(\eta/2+i\pi/2-\zeta_b^{(h_b)})}.\end{aligned}$$

□

Remark 3.3.4. After having proven theorems 3.3.2 and 3.3.4, it results more clear the reason why certain existence conditions where to be imposed in theorem 3.3.1, in order to build the SoV representations. In fact, keeping in mind definition (3.1.10), conditions (3.3.3) and (3.3.4) coincide to ask, respectively, that the left and right $\mathcal{B}_{-}(\lambda|\beta)$ SoV representations not to be nihilpotent. In particular they are the conditions for which the left and right eigenvalues of the $\mathcal{B}_{-}$ operator on the SoV states are not identically zero. If we turned the inequality sign into an equality we would get, respectively, $K_{-}(\lambda|\beta)_{12}=0$ and $\tilde{K}_{-}(\lambda|\beta)_{21}=0$ as it is clear from definitions (3.3.29) and (3.3.47).

3.3.3 Change of basis properties

In order to study the properties of the SoV basis we introduce first the standard spin basis for the 2-dimensional linear space $\mathcal{H}_n$, the quantum space in the site n of the chain,

$$\sigma_n^z\,|\,k,n\,\rangle = (2k-1)\,|\,k,n\,\rangle\,,\quad k\in\{0,1\}. \tag{3.3.53}$$

Similarly, we introduce the dual σ_n^z-eigenvectors $\langle\,k,n\,|$,

$$\langle\,k,n\,|\,\sigma_n^z = (2k-1)\,\langle\,k,n\,|\,,\quad k\in\{0,1\}. \tag{3.3.54}$$

The tensor products of the local basis vectors constitute an orthogonal basis in $\mathcal{H}$

$$| \mathbf{k} \rangle = \otimes_{n=1}^{N} | k_n, n \rangle , \qquad \langle \mathbf{k} | = \otimes_{n=1}^{N} \langle k_n, n | \quad \text{where} \quad \mathbf{k} = \{k_1, \ldots, k_N\}, \tag{3.3.55}$$

and

$$\langle \mathbf{k}' | \mathbf{k} \rangle = \prod_{n=1}^{N} \delta_{k_n, k'_n} \quad \forall k_n, k'_n \in \{0, 1\}. \tag{3.3.56}$$

We define the following $2^N \times 2^N$ matrices $U^{(L,\beta)}$ and $U^{(R,\beta)}$

$$\begin{cases} \langle \beta, \mathbf{h} | = \langle \mathbf{h} | U^{(L,\beta)} = \sum_{i=1}^{2^N} U^{(L,\beta)}_{\varkappa(\mathbf{h}),i} \langle \varkappa^{-1}(i) | , \\ | \beta, \mathbf{h} \rangle = U^{(R,\beta)} | \mathbf{h} \rangle = \sum_{i=1}^{2^N} U^{(R,\beta)}_{i,\varkappa(\mathbf{h})} | \varkappa^{-1}(i) \rangle , \end{cases} \tag{3.3.57}$$

which is useful to implement the change of basis to the SoV-basis starting from the original spin basis:

$$\langle \mathbf{h} | = \otimes_{n=1}^{N} \langle h_n, n | \text{ and } | \mathbf{h} \rangle = \otimes_{n=1}^{N} | h_n, n \rangle , \tag{3.3.58}$$

where $\varkappa$ is the following isomorphism between the sets $\{0,1\}^N$ and $\{1, \ldots, 2^N\}$:

$$\varkappa : \mathbf{h} \in \{0,1\}^N \to \varkappa(\mathbf{h}) = 1 + \sum_{a=1}^{N} 2^{(a-1)} h_a \in \{1, \ldots, 2^N\}. \tag{3.3.59}$$

Note that the matrices $U^{(L,\beta)}$ and $U^{(R,\beta)}$ are invertible matrices for the pseudo-diagonalizability of $\mathcal{B}_-(\lambda|\beta)$

$$U^{(L,\beta)} \mathcal{B}_-(\lambda|\beta) = \Delta^L_{\mathcal{B}_-}(\lambda|\beta) U^{(L,\beta-2)}, \; \mathcal{B}_-(\lambda|\beta) U^{(R,\beta)} = U^{(R,\beta+2)} \Delta^R_{\mathcal{B}_-}(\lambda|\beta). \tag{3.3.60}$$

Here $\Delta^{L/R}_{\mathcal{B}_-}(\lambda|\beta)$ are the $2^N \times 2^N$ diagonal matrices with elements

$$\left(\Delta^L_{\mathcal{B}_-}(\lambda|\beta) \right)_{i,j} = \delta_{i,j} \mathsf{B}_{\varkappa^{-1}(i)}(\lambda|\beta), \quad \left(\Delta^R_{\mathcal{B}_-}(\lambda|\beta) \right)_{i,j} = \delta_{i,j} \bar{\mathsf{B}}_{\varkappa^{-1}(i)}(\lambda|\beta), \tag{3.3.61}$$

$\forall i, j \in \{1, \ldots, 2^N\}$.

The main result of this section is the following proposition:

Proposition 3.3.3. *The* $2^N \times 2^N$ *matrix*

$$M \equiv U^{(L,\beta-2)} U^{(R,\beta)} \tag{3.3.62}$$

is diagonal and it is characterized by

$$M_{\varkappa(\mathbf{h})\varkappa(\mathbf{k})} = \langle \beta - 2, \mathbf{h} | \beta, \mathbf{k} \rangle = \delta_{\varkappa(\mathbf{h})\varkappa(\mathbf{k})} Z(\beta - 2) \prod_{1 \le b < a \le N} \frac{1}{\eta_a^{(h_a)} - \eta_b^{(h_b)}}, \tag{3.3.63}$$

with the normalization constant

$$Z(\beta) = \prod_{1 \le b < a \le N} (\eta_a^{(1)} - \eta_b^{(1)}) \langle \beta | \left(\prod_{n=1}^{N} \mathcal{A}_-(\eta/2 - \xi_n | \beta + 2) / \mathsf{A}_-(\eta/2 - \xi_n) \right) | -\beta \rangle , \tag{3.3.64}$$

and

$$\eta_a^{(h_a)} \equiv \cosh 2 \left[(\xi_a + (h_a - \frac{1}{2}) \eta \right] . \tag{3.3.65}$$

Proof. First we prove that the matrix M is diagonal. In order to do it we compute the matrix element $\langle\ \beta,\mathbf{h}\mid \mathcal{B}_{-}(\lambda|\beta)\mid \beta,\mathbf{k}\ \rangle$ which lead to the following identity

$$\mathsf{B}_{\mathbf{h}}(\lambda|\beta)\,\langle\beta-2,\mathbf{h}|\beta,\mathbf{k}\rangle=\bar{\mathsf{B}}_{\mathbf{k}}(\lambda|\beta)\,\langle\beta,\mathbf{h}|\beta+2,\mathbf{k}\rangle\,, \tag{3.3.66}$$

which implies

$$\langle\beta-2,\mathbf{h}|\beta,\mathbf{k}\rangle\propto\delta_{\varkappa(\mathbf{h})\varkappa(\mathbf{k})}, \tag{3.3.67}$$

since from the condition $\mathbf{h}\neq\mathbf{k}$ it follows that $\exists n\in\{1,\ldots,N\}$ such that $h_n\neq k_n$ and then

$$\mathsf{B}_{\mathbf{h}}(\zeta_n^{(h_n)}|\beta)\neq 0,\quad \bar{\mathsf{B}}_{\mathbf{k}}(\zeta_n^{(k_n)}|\beta)=0. \tag{3.3.68}$$

To compute the diagonal elements $M_{\varkappa(\mathbf{h})\varkappa(\mathbf{h})}$, we compute the matrix elements

$$\theta_a(\beta)=\langle\ \beta-2,h_1,\ldots,h_a=1,\ldots,h_N\mid\mathcal{D}_{-}(\xi_a+\eta/2|\beta)\mid\beta,h_1,\ldots,h_a=0,\ldots,h_N\ \rangle\,,$$

where $a\in\{1,\ldots,N\}$. Using the right action of the operator $\mathcal{D}_{-}(\xi_a+\eta/2|\beta)$ and the condition (3.3.67), we get

$$\begin{aligned}\theta_a(\beta)=f_a^{-1}(\beta)\mathsf{A}_{-}(\eta/2+\xi_a)\frac{\sinh\eta}{\sinh(2\xi_a-\eta)}\prod_{\substack{b=1\\b\neq a}}^{N}\frac{\cosh 2\zeta_a^{(1)}-\cosh 2\zeta_b^{(h_b)}}{\cosh 2\zeta_a^{(0)}-\cosh 2\zeta_b^{(h_b)}}\\ \times\langle\beta-2,h_1,\ldots,h_a=1,\ldots,h_N|\beta,h_1,\ldots,h_a=1,\ldots,h_N\rangle\,,\end{aligned} \tag{3.3.69}$$

while using the decomposition (3.2.31b) and the fact that

$$\langle\ \beta-2,h_1,\ldots,h_a=1,\ldots,h_N\mid\mathcal{A}_{-}(-(\xi_a+\eta/2)|\beta)=0 \tag{3.3.70}$$

it holds

$$\begin{aligned}&\langle\ \beta-2,h_1,\ldots,h_a=1,\ldots,h_N\mid\mathcal{D}_{-}(\xi_a+\eta/2|\beta)\\ &=\frac{\sinh\eta\sinh(2\xi_a+\beta\eta)}{\sinh(2\xi_a+\eta)\sinh(\beta)\eta}\langle\ \beta-2,h_1,\ldots,h_a=1,\ldots,h_N\mid\mathcal{A}_{-}(\xi_a+\eta/2|\beta) &(3.3.71)\\ &=\frac{\sinh\eta\sinh(2\xi_a+\beta\eta)}{\sinh(2\xi_a+\eta)\sinh(\beta)\eta}\mathsf{A}_{-}(\eta/2+\xi_a)\langle\ \beta-2,h_1,\ldots,h_a=0,\ldots,h_N\mid, &(3.3.72)\end{aligned}$$

and then we get:

$$\begin{aligned}\theta_a(\beta)=\frac{\sinh\eta\sinh(2\xi_a+(\beta)\eta)}{\sinh(2\xi_a+\eta)\sinh(\beta)\eta}\mathsf{A}_{-}(\eta/2+\xi_a)\\ \times\langle\beta-2,h_1,\ldots,h_a=0,\ldots,h_N|\beta,h_1,\ldots,h_a=0,\ldots,h_N\rangle\,.\end{aligned} \tag{3.3.73}$$

Now, by taking the ratio of (3.3.69) and (3.3.73) we arrive at

$$\frac{\langle\beta-2,h_1,\ldots,h_a=1,\ldots,h_N|\beta,h_1,\ldots,h_a=1,\ldots,h_N\rangle}{\langle\beta-2,h_1,\ldots,h_a=0,\ldots,h_N|\beta,h_1,\ldots,h_a=0,\ldots,h_N\rangle}=\prod_{\substack{b=1\\b\neq a}}^{N}\frac{\cosh 2\zeta_a^{(0)}-\cosh 2\zeta_b^{(h_b)}}{\cosh 2\zeta_a^{(1)}-\cosh 2\zeta_b^{(h_b)}}, \tag{3.3.74}$$

from which one can prove:

$$\frac{\langle\beta-2,h_1,\ldots,h_N|\beta,h_1,\ldots,h_N\rangle}{\langle\beta-2,1,\ldots,1|\beta,1,\ldots,1\rangle}=\prod_{1\leq b<a\leq N}\frac{\eta_a^{(1)}-\eta_b^{(1)}}{\eta_a^{(h_a)}-\eta_b^{(h_b)}}. \tag{3.3.75}$$

This proves the proposition as it is easy to see that

$$\langle\beta-2,1,\ldots,1|\beta,1,\ldots,1\rangle = Z(\beta-2)\prod_{1\le b<a\le N}\frac{1}{\eta_a^{(1)}-\eta_b^{(1)}}, \tag{3.3.76}$$

by our definition of the normalization $Z(\beta)$. □

3.3.4 SoV-decomposition of the identity

The identity operator $\mathbb{1}$ admits the following representation in terms of left and right SoV-basis:

$$\mathbb{1} = \sum_{i=1}^{2^N}\mu\left|\beta,\varkappa^{-1}(i)\right\rangle\left\langle\beta-2,\varkappa^{-1}(i)\right|, \tag{3.3.77}$$

where the $\mu = \left(\langle\beta-2,\varkappa^{-1}(i)\,|\beta,\varkappa^{-1}(i)\rangle\right)^{-1}$ is the Sklyanin's measure [115, 123, 22] analogous for our 6-vertex reflection algebra representations. Now using the result of the previous section we can write it explicitly

$$\mathbb{I} = \frac{1}{Z(\beta-2)}\sum_{h_1,\ldots,h_N=0}^{1}\prod_{1\le b<a\le N}(\eta_a^{(h_a)}-\eta_b^{(h_b)})\mid\beta,h_1,\ldots,h_N\,\rangle\,\langle\,\beta-2,h_1,\ldots,h_N\mid. \tag{3.3.78}$$

3.4 SoV representations for $\mathcal{T}(\lambda)$-spectral problem

In [115, 117] Sklyanin has introduced a method to construct quantum separation of variable (SoV) representations for the spectral problem of the transfer matrices associated to the representations of the Yang-Baxter algebra. For the most general representations of the reflection algebra with non-diagonal boundary matrices the quantum SoV representations are constructed here following the same approach developed in [101] but we use the gauge transformation to eliminate one of the non-diagonal entries of K_+. It means that we fix either $\alpha-\beta$ or $\alpha+\beta$. It is important to underline that the second gauge parameter remains free and can be used either to eliminate the second non-diagonal entry of K_+ or the corresponding entry of K_-. However we do not need to fix this second parameter to construct the eigenvectors of the transfer matrix.

The precise gauge fixing conditions that have to be imposed can be found in the following couple of theorems.

Theorem 3.4.1. *Under the most general boundary conditions, and if the gauge parameters $\alpha,\beta\in\mathbb{C}$ satisfy the following condition for an integer k*

$$(\alpha-\beta+2)\eta = -\tau_+ + (-1)^k(\alpha_+-\beta_+) + i\pi k, \tag{3.4.1}$$

then $K_+^{(L)}(\lambda|\beta-1)_{12} = K_+^{(R)}(\lambda|\beta-1)_{12} = 0$ and

I_a) the left representation for which the one parameter family $\mathcal{B}_-(\lambda|\beta-2)$ is pseudo-diagonal defines a left SoV representation for the spectral problem of the transfer matrix $\mathcal{T}(\lambda)$.

II_a) the right representation for which the one parameter family $\mathcal{B}_-(\lambda|\beta)$ is pseudo-diagonal defines a right SoV representation for the spectral problem of the transfer matrix $\mathcal{T}(\lambda)$.

Similarly we can formulate the same theorem for the $\mathcal{C}_-(\lambda|\beta)$ SoV representations:

Theorem 3.4.2. *Under the most general boundary conditions, if the gauge parameters $\alpha, \beta \in \mathbb{C}$ satisfy the following condition for an integer k*

$$(\alpha+\beta)\eta = -\tau_+ + (-1)^k(\alpha_+ - \beta_+) + i\pi k, \tag{3.4.2}$$

then $K_+^{(L)}(\lambda|\beta-1)_{21} = K_+^{(R)}(\lambda|\beta-1)_{21} = 0$ and

I_b) the left representation for which the one parameter family $\mathcal{C}_-(\lambda|\beta+2)$ is pseudo-diagonal defines a left SoV representation for the spectral problem of the transfer matrix $\mathcal{T}(\lambda)$.

II_b) the right representation for which the one parameter family $\mathcal{C}_-(\lambda|\beta)$ is pseudo-diagonal defines a right SoV representation for the spectral problem of the transfer matrix $\mathcal{T}(\lambda)$.

The proof of the Theorem 3.4.1 and the explicit constructions of the SoV solutions of the spectral problem for the transfer matrix $\mathcal{T}(\lambda)$ will be given in the following subsection. Theorem 3.4.2 can be proven in a similar way.

3.4.1 Transfer matrix spectrum in $\mathcal{B}_-(\lambda|\beta)$-SoV-representations

Theorem 3.4.3. *Let $\Sigma_{\mathcal{T}}$ be the set of the eigenvalue functions of the transfer matrix $\mathcal{T}(\lambda)$, then any $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ is an even function of λ of the form*

$$\begin{aligned}
\tau(\lambda) = &\sum_{a=1}^{N} \frac{\cosh^2 2\lambda - \cosh^2\eta}{\cosh^2 2\zeta_a^{(0)} - \cosh^2\eta} \prod_{\substack{b=1\\ b\neq a}}^{N} \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(0)}}{\cosh 2\zeta_a^{(0)} - \cosh 2\zeta_b^{(0)}} \tau(\zeta_a^{(0)}), \\
&+ (\cosh 2\lambda + \cosh\eta) \prod_{b=1}^{N} \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(0)}}{\cosh\eta - \cosh 2\zeta_b^{(0)}} \textit{q-det}\,(M(0)) \\
&+ (-1)^N (\cosh 2\lambda - \cosh\eta) \prod_{b=1}^{N} \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(0)}}{\cosh\eta + \cosh 2\zeta_b^{(0)}} \coth\zeta_- \coth\zeta_+ \textit{q-det}\,(M(i\pi/2)) \\
&+ 2^{(1-N)} \frac{\kappa_+\kappa_- \cosh(\tau_+ - \tau_-)}{\sinh\xi_+ \sinh\xi_-} (\cosh^2 2\lambda - \cosh^2\eta) \prod_{b=1}^{N} (\cosh 2\lambda - \cosh 2\zeta_b^{(0)}).
\end{aligned} \tag{3.4.3}$$

If the condition (3.3.2) *is satisfied, then $\mathcal{T}(\lambda)$ has simple spectrum and $\Sigma_{\mathcal{T}}$ is given by the solutions of the discrete system of equations*

$$\tau(\pm\zeta_a^{(0)})\tau(\pm\zeta_a^{(1)}) = \mathbf{A}(\zeta_a^{(1)})\mathbf{A}(-\zeta_a^{(0)}), \quad \forall a \in \{1, \dots, N\}, \tag{3.4.4}$$

in the class of functions of the form (3.4.3)*, where the coefficient $\mathbf{A}(\lambda)$ is defined by*

$$\mathbf{A}(\lambda) \equiv \mathsf{a}_+(\lambda|\beta-1)\mathsf{A}_-(\lambda), \tag{3.4.5}$$

and satisfies the quantum determinant condition

$$\frac{\textit{q-det}\,(K_+(\lambda))\textit{q-det}\,(\mathcal{U}_-(\lambda))}{\sinh(2\lambda+\eta)\sinh(\eta-2\lambda)} = \mathbf{A}(\lambda+\eta/2)\mathbf{A}(-\lambda+\eta/2). \tag{3.4.6}$$

I) Under the condition (3.3.4)*, the vector*

$$|\,\tau\,\rangle = \sum_{h_1,\dots,h_N=0}^{1} \prod_{a=1}^{N} Q_\tau(\zeta_a^{(h_a)}) \prod_{1\le b<a\le N} (\eta_a^{(h_a)} - \eta_b^{(h_b)})\,|\,\beta, h_1, \dots, h_N\,\rangle\,, \tag{3.4.7}$$

defines, uniquely up to an overall normalization, the right $\mathcal{T}$-eigenstate corresponding to $\tau(\lambda) \in \Sigma_{\mathcal{T}}$. The coefficients in (3.4.7) *are characterized by*

$$Q_\tau(\zeta_a^{(1)})/Q_\tau(\zeta_a^{(0)}) = \tau(\zeta_a^{(0)})\mathbf{A}(-\zeta_a^{(0)}). \tag{3.4.8}$$

II) Under the condition (3.3.3)*, the covector*

$$\langle\ \tau\ | = \sum_{h_1,\dots,h_N=0}^{1} \prod_{a=1}^{N} \bar{Q}_\tau(\zeta_a^{(h_a)}) \prod_{1\le b<a\le N} (\eta_a^{(h_a)} - \eta_b^{(h_b)}) \langle\ \beta-2, h_1, \dots, h_N\ |, \tag{3.4.9}$$

defines, uniquely up to an overall normalization, the left $\mathcal{T}$-eigenstate corresponding to $\tau(\lambda) \in \Sigma_{\mathcal{T}}$. The coefficients in (3.4.9) *are characterized by*

$$\bar{Q}_{\tau_-}(\zeta_a^{(1)})/\bar{Q}_{\tau_-}(\zeta_a^{(0)}) = \tau(\zeta_a^{(0)})/\mathbf{D}(\zeta_a^{(1)}), \tag{3.4.10}$$

where

$$\mathbf{D}(\zeta_a^{(h_a)}) \equiv \mathsf{d}_+(\zeta_a^{(h_a)}|\beta-1)\mathsf{D}_-(\zeta_a^{(h_a)}). \tag{3.4.11}$$

Proof. The transfer matrix $\mathcal{T}(\lambda)$ is an even function of λ so the same is true for the $\tau(\lambda) \in \Sigma_{\mathcal{T}}$. Moreover, it is simple to observe that the transfer matrix admits the following asymptotic

$$\lim_{\lambda\to\pm\infty} e^{\mp 2\lambda(N+2)}\mathcal{T}(\lambda) = 2^{-(2N+1)}\frac{\kappa_+\kappa_- \cosh(\tau_+ - \tau_-)}{\sinh\zeta_+ \sinh\zeta_-}, \tag{3.4.12}$$

while, using formulas (3.3.38), after some simple computations, it's possible to show that

$$\mathcal{T}(\pm\eta/2) = 2\cosh\eta\,\text{q-det}\ (M(0)), \tag{3.4.13}$$

$$\mathcal{T}(\pm(\eta/2 - i\pi/2)) = -2\cosh\eta\coth\zeta_-\coth\zeta_+\text{q-det}\ (M(i\pi/2)). \tag{3.4.14}$$

These identities together with the known functional form of $\mathcal{T}(\lambda)$ with respect to λ imply that $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ satisfy the characterization (3.4.3). In the $\mathcal{B}_-$-SoV representations the spectral problem for $\mathcal{T}(\lambda)$ is reduced to a discrete system of 2^N Baxter-like equations

$$\tau(\zeta_n^{(h_n)})\Psi_\tau(\mathbf{h}) = \mathbf{A}(\zeta_n^{(h_n)})\Psi_\tau(\mathsf{T}_n^-(\mathbf{h})) + \mathbf{A}(-\zeta_n^{(h_n)})\Psi_\tau(\mathsf{T}_n^+(\mathbf{h})), \tag{3.4.15}$$

for any $n \in \{1,\dots,N\}$ and $\mathbf{h} \in \{0,1\}^N$, in the coefficients (*wave-functions*) $\Psi_\tau(\mathbf{h})$ of the $\mathcal{T}$-eigenstate $|\ \tau\ \rangle$ associated to $\tau(\lambda) \in \Sigma_{\mathcal{T}}$. Here, we have used the notations

$$\mathsf{T}_n^\pm(\mathbf{h}) = (h_1, \dots, h_n \pm 1, \dots, h_N). \tag{3.4.16}$$

This system trivially follows when we recall the identities

$$\mathbf{A}_-(\zeta_n^{(0)}) = \mathbf{A}_-(-\zeta_n^{(1)}) = 0, \tag{3.4.17}$$

and we compute the matrix elements

$$\langle\ \beta-2, h_1, \dots, h_n, \dots, h_N\ |\ \mathcal{T}(\pm\zeta_n^{(h_n)})\ |\ \tau\ \rangle. \tag{3.4.18}$$

Indeed, from the decomposition (3.2.47), we have

$$\begin{aligned}
\tau(\pm\zeta_n^{(0)})\Psi_\tau(h_1,\dots,h_n=0,\dots,h_N) = \\
&= \langle\ \beta-2, h_1,\dots,h_n=0,\dots,h_N\ |\ \mathcal{T}(-\zeta_n^{(0)})\ |\ \tau\ \rangle \\
&= \mathsf{a}_+(-\zeta_n^{(0)}|\beta-1)\langle\ \beta-2, h_1,\dots,h_n=0,\dots,h_N\ |\ \mathcal{A}_-(-\zeta_n^{(0)})\ |\ \tau\ \rangle \\
&= \mathbf{A}(-\zeta_n^{(0)})\Psi_\tau(h_1,\dots,h_n=1,\dots,h_N),
\end{aligned} \tag{3.4.19}$$

and

$$\begin{aligned}
\tau(\pm\zeta_n^{(1)})\Psi_\tau(h_1,\ldots,h_n=1,\ldots,h_N) &= \\
&= \langle\ \beta-2,h_1,\ldots,h_n=1,\ldots,h_N\ |\ \mathcal{T}(\zeta_n^{(1)})\ |\ \tau\ \rangle \\
&= \mathsf{a}_+(\zeta_n^{(1)}|\beta-1)\ \langle\ \beta-2,h_1,\ldots,h_n=1,\ldots,h_N\ |\ \mathcal{A}_-(\zeta_n^{(1)})\ |\ \tau\ \rangle \\
&= \mathbf{A}(\zeta_n^{(1)})\Psi_\tau(h_1,\ldots,h_n=0,\ldots,h_N).
\end{aligned} \tag{3.4.20}$$

Clearly the previous system of equations (3.4.15) is equivalent to the following system of homogeneous equations

$$\begin{pmatrix} \tau(\pm\zeta_n^{(0)}) & -\mathbf{A}(-\zeta_n^{(0)}) \\ -\mathbf{A}(\zeta_n^{(1)}) & \tau(\pm\zeta_n^{(1)}) \end{pmatrix} = \begin{pmatrix} \Psi_{\tau-}(h_1,\ldots,h_n=0,\ldots,h_1) \\ \Psi_{\tau-}(h_1,\ldots,h_n=1,\ldots,h_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.4.21}$$

for any $n\in\{1,\ldots,N\}$ with $h_{m\neq n}\in\{0,1\}$. The condition $\tau(\lambda)\in\Sigma_{\mathcal{T}_-}$ implies that the determinants of the 2×2 matrices in (3.4.21) must be zero for any $n\in\{1,\ldots,N\}$, which is equivalent to (3.4.4). Moreover, the rank of the matrices in (3.4.21) is 1 as

$$\mathbf{A}(-\zeta_n^{(0)})\neq 0 \quad \text{and} \quad \mathbf{A}(\zeta_n^{(1)})\neq 0, \tag{3.4.22}$$

and then (up to an overall normalization) the solution is unique:

$$\frac{\Psi_\tau(h_1,\ldots,h_n=1,\ldots,h_N)}{\Psi_\tau(h_1,\ldots,h_n=0,\ldots,h_N)} = \frac{\tau(\zeta_a^{(0)})}{\mathbf{A}(-\zeta_a^{(0)})}, \tag{3.4.23}$$

for any $n\in\{1,\ldots,N\}$ with $h_{m\neq n}\in\{0,1\}$. So fixed $\tau(\lambda)\in\Sigma_{\mathcal{T}}$ there exists (up to normalization) one and only one corresponding $\mathcal{T}$-eigenstate $|\ \tau\ \rangle$ with coefficients of the factorized form given in (3.4.7)-(3.4.8); i.e. the $\mathcal{T}$-spectrum is simple.

Vice versa, if $\tau(\lambda)$ is in the set of functions (3.4.3) and satisfies (3.4.4), then the state $|\ \tau\ \rangle$ defined by eqs. (3.4.7) to (3.4.8) satisfies

$$\begin{aligned}
\langle\ \beta-2,h_1,\ldots,h_n,\ldots,h_N\ |\ \mathcal{T}(\zeta_n^{(h_n)})\ |\ \tau\ \rangle &= \begin{cases} \mathbf{A}(-\zeta_n^{(0)})\Psi_\tau(h_1,\ldots,h_n=1,\ldots,h_N) & \text{for } h_n=0 \\ \mathbf{A}(\zeta_n^{(1)})\Psi_\tau(h_1,\ldots,h_n=0,\ldots,h_N) & \text{for } h_n=1 \end{cases} \\
&= \begin{cases} \mathbf{A}(-\zeta_n^{(0)})\dfrac{\tau(\zeta_a^{(0)})}{\mathbf{A}(-\zeta_a^{(0)})}\Psi_\tau(h_1,\ldots,h_n=0,\ldots,h_N) & \text{for } h_n=0 \\ \mathbf{A}(\zeta_n^{(1)})\dfrac{\tau(\zeta_a^{(0)})}{\mathbf{A}(\zeta_a^{(1)})}\Psi_\tau(h_1,\ldots,h_n=1,\ldots,h_N) & \text{for } h_n=1 \end{cases} \\
&= \tau(\zeta_n^{(h_n)})\Psi_\tau(h_1,\ldots,h_n,\ldots,h_N)\ \ \forall n\in\{1,\ldots,N\}.
\end{aligned}$$

By considering next the functional form respect to λ of the transfer matrix

$$\mathcal{T}(\lambda) = \sum_{b=1}^{N+3}\mathcal{T}_b(\cosh 2\lambda)^{b-1}, \tag{3.4.24}$$

we arrive at the identity

$$\langle\ \beta-2,h_1,\ldots,h_N\ |\ \mathcal{T}(\lambda)\ |\ \tau\ \rangle = \tau(\lambda)\Psi_\tau(h_1,\ldots,h_n,\ldots,h_N)\ \forall\lambda\in\mathbb{C}, \tag{3.4.25}$$

for any $\mathcal{B}_-(\lambda|\beta-2)$ pseudo-eigenstate $\langle\ \beta-2,h_1,\ldots,h_N\ |$, i.e. $\tau(\lambda)\in\Sigma_{\mathcal{T}}$ and $|\ \tau\ \rangle$ is the corresponding eigenstate of the transfer matrix $\mathcal{T}$. The proof for the left $\mathcal{T}$-eigenstates is equivalent to what we just shown.

Finally, it is important to point out that the quantum determinant condition (3.4.6) is a simple consequence of the identity

$$\det_q K_+(\lambda) = -\frac{\sinh(2\lambda+\eta)\sinh(2\lambda-\eta)}{\sinh(2\lambda-2\eta)}\mathsf{a}_+(\lambda+\eta/2|\beta-1)\mathsf{a}_+(-\lambda+\eta/2|\beta-1) \quad (3.4.26)$$

which can be proven by direct computations when the condition (3.4.1) is satisfied. □

This theorem implies that each eigenvalue and eigenstate of the transfer matrix can be characterized in terms of a set of parameters $\{x_1,\ldots,x_N\}$ satisfying a system of quadratic equations. This system replaces the Bethe equations in this case. This new set of equations has been defined in the following corollary.

Corollary 3.4.1. *The set $\Sigma_{\mathcal{T}}$ of the eigenvalue functions of the transfer matrix $\mathcal{T}(\lambda)$ admits the following characterization*

$$\Sigma_{\mathcal{T}} = \left\{\tau(\lambda) : \tau(\lambda) = f(\lambda) + \sum_{a=1}^{N} g_a(\lambda)x_a,\ \forall\{x_1,\ldots,x_N\}\in\Sigma_T\right\}, \quad (3.4.27)$$

where we have used

$$g_a(\lambda) = \frac{\cosh^2 2\lambda - \cosh^2\eta}{\cosh^2 2\zeta_a^{(0)} - \cosh^2\eta}\prod_{\substack{b=1\\ b\neq a}}^{N}\frac{\cosh 2\lambda - \cosh 2\zeta_b^{(0)}}{\cosh 2\zeta_a^{(0)} - \cosh 2\zeta_b^{(0)}} \quad \text{for } a\in\{1,\ldots,N\}, \quad (3.4.28)$$

$$\begin{aligned} f(\lambda) =& \frac{(\cosh 2\lambda + \cosh\eta)}{2\cosh\eta}\prod_{b=1}^{N}\frac{\cosh 2\lambda - \cosh 2\zeta_b^{(0)}}{\cosh\eta - \cosh 2\zeta_b^{(0)}}\tau(\eta/2) \\ &- (-1)^N\frac{(\cosh 2\lambda - \cosh\eta)}{2\cosh\eta}\prod_{b=1}^{N}\frac{\cosh 2\lambda - \cosh 2\zeta_b^{(0)}}{\cosh\eta + \cosh 2\zeta_b^{(0)}}\tau(\eta/2 + i\pi/2) \\ &+ 2^{(1-N)}\frac{\kappa_+\kappa_-\cosh(\tau_+ - \tau_-)}{\sinh\xi_+\sinh\xi_-}(\cosh^2 2\lambda - \cosh^2\eta)\prod_{b=1}^{N}(\cosh 2\lambda - \cosh 2\zeta_b^{(0)}), \end{aligned} \quad (3.4.29)$$

and Σ_T is the set of the solutions to the following inhomogeneous system of N quadratic equations

$$x_n\sum_{a=1}^{N} g_a(\zeta_n^{(1)})x_a + x_n f(\zeta_n^{(1)}) = q_n, \quad q_n = \frac{\det_q K_+(\xi_n)\det_q\mathcal{U}_-(\xi_n)}{\sinh(\eta+2\xi_n)\sinh(\eta-2\xi_n)},\ \forall n\in\{1,\ldots,N\}, \quad (3.4.30)$$

in N parameters $\{x_1,\ldots,x_N\}$.

3.4.2 SoV applicability and Nepomechie's constraint

Combining together conditions for the existence of SoV basis (3.3.3)-(3.3.6) and the choice of the gauge parameters necessary to construct the eigenstates of the transfer matrix (3.4.1), (3.4.2) we obtain the limits of applicability of the SoV method. This particular situation happens to coincide with the domain of applicability of the algebraic Bethe ansatz, studied by Nepomechie *et al.* in [98, 99]. More precisely the following theorem holds;

Theorem 3.4.4. *The SoV constructions corresponding to the cases I_a and I_b fails to exist if and only if the following condition on the parameters of the boundary matrices are satisfied*

$$(N+1)\,\eta = \tau_- - \tau_+ + (-1)^k(\alpha_- + \beta_-) - (-1)^m(\alpha_+ - \beta_+) + i\pi(k+m), \tag{3.4.31}$$

where $(k,m) \in \mathbb{Z}^2$.

Similarly, the SoV constructions corresponding to the cases II_a and II_b fails to exist if and only if the following condition on the parameters of the boundary matrices is satisfied

$$(1-N)\,\eta = \tau_- - \tau_+ + (-1)^k(\alpha_- + \beta_-) - (-1)^m(\alpha_+ - \beta_+) + i\pi(\hat{k}+\hat{m}). \tag{3.4.32}$$

where $(\hat{k},\hat{m}) \in \mathbb{Z}^2$.

Then, our SoV scheme to construct the spectrum (eigenvalues and eigenstates) of the transfer matrix $\mathcal{T}(\lambda)$ cannot be used if and only if the conditions (3.4.31) *and* (3.4.32) *are simultaneously satisfied.*

Remark 3.4.1. In our notations for the boundary parameters the Nepomechie's constraints read

$$\mathsf{k}\eta = \tau_- - \tau_+ + \epsilon_-(\alpha_- + \beta_-) + \epsilon_+(\alpha_+ - \beta_+), \text{ mod } 2\pi i \text{ and } \mathsf{k} = N-1+2r \text{ with } r \in \mathbb{Z} \tag{3.4.33}$$

so that we recover the relations (3.4.31) and (3.4.32) respectively for $r = 1$ and $r = 1-N$. The previous theorem says that the SoV construction works also when the boundary parameters satisfy one Nepomechie's condition: if $r \neq 1$ and $r \neq 1-N$ we can use both the left and right SoV construction, if $r = 1$ we can use the right SoV construction and if $r = 1-N$ we can use the left SoV construction. The only problem in our SoV schema appears if the two Nepomechie's conditions for $r = 1$ and $r = 1-N$ are simultaneously satisfied. As it's clear, this least scenario can occur just for very special values of η. Finally, the special case when only one of these two conditions is satisfied maybe of particular interest as in this situation there are two simultaneous descriptions and it is possible to compare the construction of eigenvalues and eigenstates by the separation of variables and by the algebraic Bethe ansatz.

3.5 Scalar Products

One of the main reasons of interest in the SoV method is that it seems to provide a possibility to go beyond the spectral analysis constructing dynamic observables of the physical system. The following theorem represents the first step in the solution of this problem.

Theorem 3.5.1. *Let $\langle\, \omega \,|$ and $|\, \rho \,\rangle$ be an arbitrary covector and vector of separate forms:*

$$\langle\, \omega \,| = \sum_{h_1,\dots,h_{\mathsf{N}}=0}^{1} \prod_{a=1}^{N} \omega_a(\zeta_a^{(h_a)}) \prod_{1\leq b<a\leq N} (\eta_a^{(h_a)} - \eta_b^{(h_b)}) \langle \beta-2, h_1, \dots, h_N|, \tag{3.5.1}$$

$$|\, \rho \,\rangle = \sum_{h_1,\dots,h_{\mathsf{N}}=0}^{1} \prod_{a=1}^{N} \rho_a(\zeta_a^{(h_a)}) \prod_{1\leq b<a\leq N} (\eta_a^{(h_a)} - \eta_b^{(h_b)}) |\beta, h_1, \dots, h_N\rangle, \tag{3.5.2}$$

in the $\mathcal{B}$-pseudo-eigenbasis, then the action of $\langle\, \omega \,|$ on $|\, \rho \,\rangle$ reads:

$$\langle \omega|\rho\rangle = Z(\beta-2) \det_N ||\mathcal{M}_{a,b}^{(\omega,\rho)}|| \quad \textit{with} \quad \mathcal{M}_{a,b}^{(\omega,\rho)} = \sum_{h=0}^{1} \omega_a(\zeta_a^{(h)}) \rho_a(\zeta_a^{(h)}) (\eta_a^{(h)})^{(b-1)}. \tag{3.5.3}$$

The above formula still holds if the left and right states are transfer matrix eigenstates.

Proof. The formula (3.3.63) and the SoV-decomposition of the states $\langle\, \omega \mid$ and $\mid \rho\, \rangle$ implies that

$$\langle \omega | \rho \rangle = Z(\beta - 2) \sum_{h_1,\dots,h_N=0}^{1} V(\eta_1^{(h_1)}, \dots, \eta_N^{(h_N)}) \prod_{a=1}^{N} \omega_a(\zeta_a^{(h_a)}) \rho_a(\zeta_a^{(h_a)}), \tag{3.5.4}$$

where

$$V(x_1, \dots, x_N) \equiv \prod_{1 \le b < a \le N} (x_a - x_b) \tag{3.5.5}$$

is the Vandermonde determinant and due to the multilinearity of the determinant (3.5.3) follows. □

The normalization coefficient $Z(\beta - 2)$ is an artifact of the gauge transformation, for any interesting quantity (form-factors, correlation functions) represented as a ratio of two scalar products this constant will disappear.

CHAPTER 4

NON-DIAGONAL OPEN SPIN-1/2 XYZ CHAIN BY *SEPARATION OF VARIABLES* METHOD

This chapter will deal with the construction of the SoV representations of an other open model: the XYZ spin chain. This will lead to the complete solution of the eigenproblem associated to the boundary transfer matrix of the inhomogeneous chain. As it will be clear in the following, the whole machinery will reflect complete analogy with the one built for the open XXZ chain treated in the previous chapter **3**. Remarkably, this analogy is almost *one-to-one* and this twin construction will differ mainly for the passage to an algebra based on elliptic functions rather than trigonometric. Another exciting aspect is that the implementation of the SoV method is methodologically simple and makes the open XYZ simpler to solve than the periodic one even if the closed chain, at a first sight, might look as an easier problem. Since the objects to introduce will be, on an algebra level, the same we needed in chapter **3**, the notation will be kept essentially unchanged. The reader should not mix elements of the two chapters, if not stated otherwise. In Appendix B.1 we fix the notation for the Jacobi theta functions that will be largely used in the following sections and collect some relevant properties. The results presented here can be found in a publication of S.F. and G.Niccoli [46].

4.1 Open XYZ spin chains and reflection algebra.

The quantum system that we want to describe and analyze in this chapter was defined originally in [67], and it's defined by the Hamiltonian

$$\begin{aligned} H^{\text{G.B.}}_{\text{XYZ}} = &\sum_{i=1}^{N-1}\left[\mathbf{J}_x\,\sigma_i^x\sigma_{i+1}^x+\mathbf{J}_y\,\sigma_i^y\sigma_{i+1}^y+\mathbf{J}_z\,\sigma_i^z\sigma_{i+1}^z\right]\\ &+\frac{\operatorname{sn}(\tilde{\eta})}{\operatorname{sn}(\tilde{\xi}_-)}\left(\sigma_1^z\operatorname{cn}\tilde{\xi}_-\operatorname{dn}\tilde{\xi}_-+2\kappa_-(\sigma_1^x\cosh\tau_-+i\sigma_1^y\theta_1\tau_-)\right)\\ &+\frac{\operatorname{sn}(\tilde{\eta})}{\operatorname{sn}(\tilde{\xi}_+)}\left(\sigma_1^z\operatorname{cn}\tilde{\xi}_+\operatorname{dn}\tilde{\xi}_++2\kappa_+(\sigma_N^x\cosh\tau_++i\sigma_N^y\theta_1\tau_+)\right)\in\operatorname{End}(\mathcal{H}). \end{aligned} \tag{4.1.1}$$

where

$$\mathbf{J}_x = 1 + k\,\mathrm{sn}^2\,\tilde{\eta}, \quad \mathbf{J}_y = 1 - k\,\mathrm{sn}^2\,\tilde{\eta}, \quad \mathbf{J}_z = \mathrm{cn}\,\tilde{\eta}\,\mathrm{dn}\,\tilde{\eta}. \tag{4.1.2}$$

As seen for the XXZ model, the quantum system defined by (4.1.1) lives in the Hilbert space $\mathcal{H} = \mathbb{C}^{2^{\otimes N}}$, which is the tensor product of N spin-1/2 representation spaces $\mathcal{H}_{1/2} = \mathbb{C}^2$.

In the definition (4.1.1) the tilde notation $\tilde{x} = 2\mathbf{K}_k x$ has been used

$$\mathrm{sn}\,\tilde{\lambda} \equiv \frac{1}{\sqrt{k}}\frac{\theta_1(\lambda|2\omega)2}{\theta_4(\lambda|2\omega)}, \quad \mathrm{cn}\,\tilde{\lambda} \equiv \sqrt{\frac{k'}{k}}\frac{\theta_2(\lambda|2\omega)}{\theta_4(\lambda|2\omega)}, \quad \mathrm{dn}\,\tilde{\lambda} \equiv \sqrt{k'}\frac{\theta_3(\lambda|2\omega)}{\theta_4(\lambda|2\omega)}, \tag{4.1.3}$$

$$k \equiv \frac{\theta_2^2(0|2\omega)}{\theta_3^2(0|2\omega)}, \quad k' \equiv \frac{\theta_4^2(0|2\omega)}{\theta_3^2(0|2\omega)}, \quad k^2 + k'^2 = 1, \quad \mathbf{K}_k = \frac{1}{2}\theta_3^2(0|2\omega). \tag{4.1.4}$$

4.1.1 Definition of the elliptic reflection algebra

The first object that has to be introduced in order to build the suitable QISM representation, as we saw for the XXZ model, is the R-matrix. In this case we are interested to the elliptic solution of the Yang-Baxter equation. Since the close connection between the quantum system under study and the statistical mechanical model known as 8-*vertex model* the R-matrix will be labelled with the superscript $8V$. It reads

$$R_{12}^{(8\mathsf{V})}(\lambda) = \begin{pmatrix} \mathsf{a}(\lambda) & 0 & 0 & \mathsf{d}(\lambda) \\ 0 & \mathsf{b}(\lambda) & \mathsf{c}(\lambda) & 0 \\ 0 & \mathsf{c}(\lambda) & \mathsf{b}(\lambda) & 0 \\ \mathsf{d}(\lambda) & 0 & 0 & \mathsf{a}(\lambda) \end{pmatrix} \in \mathrm{End}(\mathcal{V}_1 \otimes \mathcal{V}_2) \tag{4.1.5}$$

with

$$\mathsf{a}(\lambda) = \frac{2\theta_4(\eta|2\omega)\theta_4(\lambda|2\omega)\theta_1(\lambda+\eta|2\omega)}{\theta_2(0|\omega)\theta_4(0|2\omega)}, \quad \mathsf{b}(\lambda) = \frac{2\theta_4(\eta|2\omega)\theta_1(\lambda|2\omega)\theta_4(\lambda+\eta|2\omega)}{\theta_2(0|\omega)\theta_4(0|2\omega)},$$
$$\mathsf{c}(\lambda) = \frac{2\theta_1(\eta|2\omega)\theta_4(\lambda|2\omega)\theta_4(\lambda+\eta|2\omega)}{\theta_2(0|\omega)\theta_4(0|2\omega)}, \quad \mathsf{d}(\lambda) = \frac{2\theta_1(\eta|2\omega)\theta_1(\lambda|2\omega)\theta_1(\lambda+\eta|2\omega)}{\theta_2(0|\omega)\theta_4(0|2\omega)}, \tag{4.1.6}$$

where again $\mathcal{V}_i = \mathbb{C}^2$. The R-matrix reproduced above appears often in literature in its elliptic trigonometric form and it might be useful to give its form using the functions (4.1.3):

$$\mathsf{a}(\lambda) = f(\lambda)\,\mathrm{sn}(\tilde{\lambda}+\tilde{\eta}), \quad \mathsf{b}(\lambda) = f(\lambda)\,\mathrm{sn}\,\tilde{\lambda},$$
$$\mathsf{c}(\lambda) = f(\lambda)\,\mathrm{sn}\,\tilde{\eta}, \quad \mathsf{d}(\lambda) = kf(\lambda)\,\mathrm{sn}(\tilde{\lambda}+\tilde{\eta})\,\mathrm{sn}\,\tilde{\lambda}\,\mathrm{sn}\,\tilde{\eta}, \tag{4.1.7}$$

where

$$f(\lambda) = \frac{2\sqrt{k}\theta_4(\eta|2\omega)\theta_4(\lambda|2\omega)\theta_4(\lambda+\eta|2\omega)}{\theta_2(0|\omega)\theta_4(0|2\omega)}. \tag{4.1.8}$$

The 8-vertex R-matrix has very similar symmetry properties to the 6-vertex one, see (1.2.5); they can be easily generated by direct computation and they take the following form

Permutation op. point $\quad R_{12}^{(8\mathsf{V})}(0) = \theta_1(\eta|\omega)\cdot\mathbb{P}_{12}$ (4.1.9a)

Antisymetrizer op. point $\quad R_{12}^{(8\mathsf{V})}(-\eta) = (-\theta_1(\eta|\omega))\cdot P_{12}^{-}$ (4.1.9b)

Unitarity $\quad R_{12}^{(8\mathsf{V})}(\lambda)R_{12}^{(8\mathsf{V})}(-\lambda) = -\theta_1(\lambda+\eta|\omega)\theta_1(\lambda-\eta|\omega)\mathbb{1}$ (4.1.9c)

Crossing Unitarity $\quad \sigma_1^y R_{12}^{(8\mathsf{V})}(\lambda)\sigma_1^y = -(R_{12}^{(8\mathsf{V})})^{t_2}(-\lambda-\eta)$ (4.1.9d)

PT-Symmetry $\quad (R_{12}^{(8\mathsf{V})})^{t_1 t_2}(\lambda) = \mathbb{P}_{12}R_{12}^{(8\mathsf{V})}\mathbb{P}_{12} = R_{12}^{(8\mathsf{V})}(\lambda)$ (4.1.9e)

$\mathbb{Z}_2$-Symmetry $\quad \sigma_1^j\sigma_2^j R_{12}^{(8\mathsf{V})}(\lambda)\sigma_1^j\sigma_2^j = R_1^{(8\mathsf{V})}2(\lambda)$, with $j = x, y, z$ (4.1.9f)

We see that the mapping from eqs. (1.2.5) and (4.1.9) is obtained by letting $\sinh(x) \to \theta_1(x|\omega)$.

We can now introduce the appropriate boundary K-matrix, which is the generic scalar elliptic solution to the following reflection equation

$$R_{12}^{(8V)}(\lambda-\mu)K_1(\lambda)R_{21}^{(8V)}(\lambda+\mu)K_2(\mu) = K_2(\mu)R_{21}^{(8V)}(\lambda+\mu)K_1(\lambda)R_{12}^{(8V)}(\lambda-\mu), \quad (4.1.10)$$

which reads

$$\begin{aligned} K(\lambda;\alpha_1,\alpha_2,\alpha_3) = F(\lambda)\Bigg\{ &\mathbb{1} + c_x(\alpha_1,\alpha_2,\alpha_3)\frac{\theta_1(\lambda)}{\theta_4(\lambda)}\sigma^x + c_y(\alpha_1,\alpha_2,\alpha_3)\frac{\theta_1(\lambda)}{\theta_3(\lambda)}\sigma^y \\ &+ c_z(\alpha_1,\alpha_2,\alpha_3)\frac{\theta_1(\lambda)}{\theta_2(\lambda)}\sigma^z \Bigg\}, \end{aligned} \quad (4.1.11)$$

with $\theta_i(\lambda) = \theta_i(\lambda|\omega)$ and

$$F(\lambda) = \frac{\theta_1(2\lambda)}{2\theta_1(\lambda)}; \quad (4.1.12)$$

$$c_x(\alpha_1,\alpha_2,\alpha_3) = \prod_{l=1}^{3}\frac{\theta_4(\alpha_l)}{\theta_1(\alpha_l)}; \quad (4.1.13)$$

$$c_y(\alpha_1,\alpha_2,\alpha_3) = -i\prod_{l=1}^{3}\frac{\theta_3(\alpha_l)}{\theta_1(\alpha_l)}; \quad (4.1.14)$$

$$c_z(\alpha_1,\alpha_2,\alpha_3) = -\prod_{l=1}^{3}\frac{\theta_2(\alpha_l)}{\theta_1(\alpha_l)}. \quad (4.1.15)$$

The expression in (4.1.11) was established by *Hou et al.* in [66], while the equivalent K-matrix used in [46] was introduced by *Inami* and *Konno* in [67]. For completeness, in Appendix B.2 we give the explicit transformation one has to take into account in order to pass from one to another expression. Once again, following [116], two classes of solutions of the reflection equations can be constructed

$$K_\pm(\lambda) = K(\lambda \pm \eta/2;\alpha_1^\pm,\alpha_2^\pm,\alpha_3^\pm) = \begin{pmatrix} a_\pm(\lambda) & b_\pm(\lambda) \\ c_\pm(\lambda) & d_\pm(\lambda) \end{pmatrix} \quad (4.1.16)$$

where the functions appearing in the matrix form are defined in (4.1.11). **N.B.** The link between the sets of parameters $\{\zeta_\pm,\kappa_\pm,\tau_\pm\}$ appearing in (4.1.1) and $\{\alpha_1^\pm\alpha_2^\pm,\alpha_3^\pm\}$ is given in Appendix B.2 as well.

The *bulk* monodromy matrix $M_0(\lambda) \in \mathrm{End}(\mathcal{V}_0 \otimes \mathcal{H})$ and its adjoint $\hat{M}_0(\lambda) \in \mathrm{End}(\mathcal{V}_0 \otimes \mathcal{H})$

$$M_0(\lambda) = R_{0N}^{(8V)}(\lambda-\xi_N-\eta/2)\ldots R_{01}^{(8V)}(\lambda-\xi_1-\eta/2) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \quad (4.1.17)$$

$$\hat{M}_0(\lambda) = R_{01}^{(8V)}(\lambda+\xi_1-\eta/2)\ldots R_{0N}^{(8V)}(\lambda+\xi_N-\eta/2) = (-1)^N\sigma_0^y M_0(\lambda)\sigma_0^y, \quad (4.1.18)$$

satisfy the 8-vertex Yang-Baxter relation

$$R_{12}^{(8V)}(\lambda-\mu)M_1(\lambda)M_2(\mu) = M_2(\mu)M_1(\lambda)R_{12}^{(8V)}(\lambda-\mu). \quad (4.1.19)$$

The boundary monodromy matrices $\mathcal{U}_\pm(\lambda) \in \mathrm{End}(\mathcal{V}_0 \otimes \mathcal{H})$ are defined as usual

$$\mathcal{U}_-(\lambda) = M(\lambda)K_-(\lambda)\hat{M}(\lambda) = \begin{pmatrix} \mathcal{A}_-(\lambda) & \mathcal{B}_-(\lambda) \\ \mathcal{C}_-(\lambda) & \mathcal{D}_-(\lambda) \end{pmatrix}, \quad (4.1.20)$$

$$\mathcal{U}_+^{t_0}(\lambda) = M^{t_0}(\lambda)K_+^{t_0}(\lambda)\hat{M}^{t_0}(\lambda) = \begin{pmatrix} \mathcal{A}_+(\lambda) & \mathcal{C}_+(\lambda) \\ \mathcal{B}_+(\lambda) & \mathcal{D}_+(\lambda) \end{pmatrix}. \quad (4.1.21)$$

The matrices $\mathcal{U}_-(\lambda)$ and $\mathcal{V}_+(\lambda) \equiv \mathcal{U}_+^{t_0}(\lambda)$ are solutions of the 8-vertex reflection equation

$$R_{12}^{(8V)}(\lambda-\mu)\mathcal{U}_-^1(\lambda)R_{21}^{(8V)}(\lambda+\mu-\eta)\mathcal{U}_-^2(\mu) = \mathcal{U}_-^2(\mu)R_{21}^{(8V)}(\lambda+\mu-\eta)\mathcal{U}_-^1(\lambda)R_{12}^{(8V)}(\lambda-\mu). \tag{4.1.22}$$

Finally the transfer matrix can be built as for the XXZ model

$$\mathcal{T}(\lambda) = \mathrm{tr}_0\{K_+(\lambda)M(\lambda)K_-(\lambda)\hat{M}(\lambda)\} = \mathrm{tr}_0\{K_+(\lambda)\mathcal{U}_-(\lambda)\} = \mathrm{tr}_0\{\mathcal{U}_+(\lambda)K_-(\lambda)\} \in \mathrm{End}(\mathcal{H}). \tag{4.1.23}$$

In the homogeneous limit $\{\xi_n\}_{n=1,\dots,N} \to 0$ the spectrum related to the transfer matrix (4.1.23) coincides with the one associated to the hamiltonian (4.1.1) as it was established in the periodic case, through a *trace identity*, in [43].

Remark 4.1.1. To be precise in order to reconstruct the open version of the trace identity established in [43], and reproducing the Hamiltonian (4.1.1) one should use a different normalization for the R-matrix, *i.e.* $\frac{1}{f(\lambda)}R_{12}^{(8V)}(\lambda)$, and the K-matrix parametrization introduced in Appendix B.2.

In our notation it's possible as well to reproduce a trace identity which would result in a different parametrization of the XYZ-Hamiltonian. It's indeed straightforward to show, by the same exact steps of Prop. 3.1.1, that the following result holds true:

$$\tilde{H}^{\text{G.B.}}_{\text{XYZ}} \propto \frac{d}{d\lambda}\tilde{\mathcal{T}}(\lambda)|_{\substack{\lambda=\eta/2\\ \xi_1,\dots,\xi_N=0}} + \text{const.}$$

where $\tilde{\mathcal{T}}$ is the transfer matrix built with the renormalized R-matrix $\tilde{R}(\lambda) = \frac{1}{g(\lambda)}R(\lambda)$, for $g(\lambda) = \theta_4(\lambda+\eta/2|2\omega)\theta_4(\lambda-\eta/2|2\omega)$, and

$$\tilde{H}^{\text{G.B.}}_{\text{XYZ}} = \sum_{i=1}^{N-1}\left[\tilde{\mathbf{J}}_x\,\sigma_i^x\sigma_{i+1}^x + \tilde{\mathbf{J}}_y\,\sigma_i^y\sigma_{i+1}^y + \tilde{\mathbf{J}}_z\,\sigma_i^z\sigma_{i+1}^z\right] + \left(h_1^-\sigma_1^x + h_2^-\sigma_1^y + h_3^-\sigma_1^z + h_1^+\sigma_N^x + h_2^+\sigma_N^y + h_3^+\sigma_N^z\right),$$

where

$$\tilde{\mathbf{J}}_x = \frac{\theta_4(\eta)}{\theta_4(\eta|2\omega)}\frac{\theta_2(0)\theta_3(0)}{2\theta_4(0|2\omega)},\quad \tilde{\mathbf{J}}_y = \frac{\theta_3(\eta)}{\theta_4(\eta|2\omega)}\frac{\theta_2(0)\theta_4(0)}{2\theta_4(0|2\omega)},\quad \tilde{\mathbf{J}}_z = \frac{\theta_2(\eta)}{\theta_4(\eta|2\omega)}\frac{\theta_4(0|2\omega)}{2},$$

$$h_1^\gamma = \left(\varphi(\eta)\frac{\theta_1(2\eta)}{\theta_4(\eta)}\right)^{\delta_{(+,\gamma)}}(\theta_2(0)\theta_3(0))^{\delta_{(-,\gamma)}}\,c_x,\quad h_2^\gamma = \left(\varphi(\eta)\frac{\theta_1(2\eta)}{\theta_3(\eta)}\right)^{\delta_{(+,\gamma)}}(\theta_2(0)\theta_4(0))^{\delta_{(-,\gamma)}}\,c_y,$$

$$h_3^\gamma = \left(\varphi(\eta)\frac{\theta_1(2\eta)}{\theta_2(\eta)}\right)^{\delta_{(+,\gamma)}}(\theta_3(0)\theta_4(0))^{\delta_{(-,\gamma)}}\,c_z,\qquad \text{for}\quad \gamma=\pm,$$

$$\text{where}\quad \varphi(\eta) = \frac{\theta_2(\eta)}{\theta_4(\eta|2\omega)}\theta_4(0|2\omega)\quad\text{and}\quad \delta_{(\gamma,\gamma')} = \begin{cases}0, & \text{for } \gamma\neq\gamma'\\ 1, & \text{for } \gamma=\gamma'\end{cases},$$

and c_x, c_y and c_z were defined in (4.1.13)-(4.1.15).

4.1.2 Basic properties and quantum determinants

Let us briefly introduce some properties of the R-matrix and the boundary matrices given by their quasi-periodic nature. In order to obtain the following is sufficient to make the explicit calculations taking into account the quasi-periodicity properties of the theta functions, see (B.1.5)-(B.1.6h) in App. B.1. The R-matrix has the following quasi-periodicity properties

$$R_{0a}^{(8V)}(\lambda+\pi) = -\sigma_0^z R_{0a}^{(8V)}(\lambda)\sigma_0^z, \tag{4.1.24}$$

$$R_{0a}^{(8V)}(\lambda+\omega\pi) = -e^{-i(2\lambda+\eta+i\pi\omega)}\sigma_0^x R_{0a}^{(8V)}(\lambda)\sigma_0^x, \tag{4.1.25}$$

$$R_{0a}^{(8V)}(\lambda+\omega\pi+\pi) = e^{-i(2\lambda+\eta+i\pi\omega)}\sigma_0^y R_{0a}^{(8V)}(\lambda)\sigma_0^y, \tag{4.1.26}$$

and consequently the *bulk* monodromy matrix behaves like

$$M_0(\lambda+\pi) = (-1)^N \sigma_0^z M_0(\lambda)\sigma_0^z, \tag{4.1.27}$$

$$M_0(\lambda+\omega\pi) = (-1)^N e^{-2iN(\lambda+\frac{\omega\pi}{2})} e^{2i\sum_{n=1}^N \xi_n} \sigma_0^x M_0(\lambda)\sigma_0^x, \tag{4.1.28}$$

$$M_0(\lambda+\omega\pi+\pi) = e^{-2iN(\lambda+\frac{\omega\pi}{2})} e^{2i\sum_{n=1}^N \xi_n} \sigma_0^y M_0(\lambda)\sigma_0^y. \tag{4.1.29}$$

In the same way we can give the following quasi-periodic properties of the K-matrix (4.1.11)

$$K(\lambda+\pi) = -\sigma_0^z K(\lambda)\sigma_0^z, \tag{4.1.30}$$

$$K(\lambda+\omega\pi) = -e^{-2i(3\lambda+\frac{3}{2}\omega\pi)}\sigma_0^x K(\lambda)\sigma_0^x, \tag{4.1.31}$$

$$K(\lambda+\omega\pi+\pi) = -e^{-2i(3\lambda+\frac{3}{2}\omega\pi)}\sigma_0^y K(\lambda)\sigma_0^y. \tag{4.1.32}$$

Let us now introduce some functions that will be useful in order to define the quantum determinants for this algebra. Consider

$$\widehat{\mathsf{A}}_-(\lambda) = g_-(\lambda)a(\lambda)d(-\lambda), \quad d(\lambda) = a(\lambda-\eta), \quad a(\lambda) = \prod_{j=1}^N \theta_1(\lambda-\xi_n+\eta/2), \tag{4.1.33}$$

where

$$g_\pm(\lambda) = \prod_{l=1}^3 \frac{\theta_1(\alpha_l^\pm+\lambda-\eta/2)}{\theta_1(\alpha_l^\pm)}, \tag{4.1.34}$$

then it is possible to establish the following

Proposition 4.1.1. *The reflection algebra generators are related by the following parity relations*

$$\mathcal{A}_-(\lambda) = \frac{c(2\lambda)\mathcal{D}_-(\lambda)+p(\lambda)\mathcal{D}_-(-\lambda)}{b(2\lambda)}, \qquad \mathcal{D}_-(\lambda) = \frac{c(2\lambda)\mathcal{A}_-(\lambda)+p(\lambda)\mathcal{A}_-(-\lambda)}{b(2\lambda)}, \tag{4.1.35}$$

$$\mathcal{B}_-(\lambda) = \frac{a(2\lambda)\mathcal{C}_-(\lambda)+p(\lambda)\mathcal{C}_-(-\lambda)}{d(2\lambda)}, \qquad \mathcal{C}_-(\lambda) = \frac{a(2\lambda)\mathcal{B}_-(\lambda)+p(\lambda)\mathcal{B}_-(-\lambda)}{d(2\lambda)}, \tag{4.1.36}$$

where

$$\begin{aligned} p(\lambda) &= \frac{-\mathsf{c}(2\lambda)a_-(\lambda)+\mathsf{b}(2\lambda)d_-(\lambda)}{a_-(-\lambda)} = \frac{-\mathsf{c}(2\lambda)d_-(\lambda)+\mathsf{b}(2\lambda)a_-(\lambda)}{d_-(-\lambda)} \\ &= \frac{-\mathsf{a}(2\lambda)b_-(\lambda)+\mathsf{d}(2\lambda)c_-(\lambda)}{b_-(-\lambda)} = \frac{-\mathsf{a}(2\lambda)c_-(\lambda)+\mathsf{d}(2\lambda)b_-(\lambda)}{c_-(-\lambda)} = \theta_1(2\lambda-\eta). \end{aligned} \tag{4.1.37}$$

The following inversion relation holds

$$\mathcal{U}_{-}^{-1}(\lambda+\eta/2)=\frac{p(\lambda-\eta/2)}{q\text{-}det\,(\mathcal{U}_{-}(\lambda))}\mathcal{U}_{-}(\eta/2-\lambda), \tag{4.1.38}$$

where in the reflection algebra generated by the elements of $\mathcal{U}_{-}(\lambda)$ *the quantum determinant takes the form:*

$$\frac{q\text{-}det\,(\mathcal{U}_{-}(\lambda))}{p(\lambda-\eta/2)}=\mathcal{A}_{-}(\epsilon\lambda+\eta/2)\mathcal{A}_{-}(\eta/2-\epsilon\lambda)+\mathcal{B}_{-}(\epsilon\lambda+\eta/2)\mathcal{C}_{-}(\eta/2-\epsilon\lambda) \tag{4.1.39}$$

$$=\mathcal{D}_{-}(\epsilon\lambda+\eta/2)\mathcal{D}_{-}(\eta/2-\epsilon\lambda)+\mathcal{C}_{-}(\epsilon\lambda+\eta/2)\mathcal{B}_{-}(\eta/2-\epsilon\lambda), \tag{4.1.40}$$

for $\epsilon=\pm 1$. *The quantum determinant is a central element of the algebra*

$$[q\text{-}det\,(\mathcal{U}_{-}(\lambda)),\mathcal{U}_{-}(\mu)]=0, \tag{4.1.41}$$

moreover, it admits the following explicit expression

$$q\text{-}det\,(\mathcal{U}_{-}(\lambda))=p(\lambda-\eta/2)\widehat{\mathsf{A}}_{-}(\lambda+\eta/2)\widehat{\mathsf{A}}_{-}(-\lambda+\eta/2). \tag{4.1.42}$$

Proof. This proposition is analog to what was proven in chapters **1** and **3** for the XXZ model, and can be proven by following once again the steps drawn by Sklyanin in [116] but in the 8-vertex case. Here we will give just the essential computational ingredients that differ from what already seen. The following relation holds

$$K_{\pm}^{-1}(\lambda\mp\eta/2)=\frac{p(\mp\lambda-\eta/2)}{\text{q-det}\,(K_{\pm}(\lambda))}K_{\pm}(-\lambda\mp\eta/2), \tag{4.1.43}$$

since

$$K_{\pm}(\lambda\mp\eta/2)K_{\pm}(-\lambda\mp\eta/2)=\prod_{l=1}^{3}\frac{\theta_1(\alpha_l^{\pm}+\lambda)\theta_1(\alpha_l^{\pm}-\lambda)}{\theta_1(\alpha_l^{\pm})\theta_1(\alpha_l^{\pm})}=g_{\pm}(\lambda+\eta/2)g_{\pm}(-\lambda+\eta/2),$$

and

$$\text{q-det}\,(K_{\pm}(\lambda))=p(\mp\lambda\pm\eta/2)g_{\pm}(\lambda+\eta/2)g_{\pm}(-\lambda+\eta/2). \tag{4.1.44}$$

N.B. In order to prove the statement (4.1.44) one can use different approaches, all defined in the Sklyanin paper [116]. For example it's simple to use the fact that

$$\begin{aligned}\tilde{K}_{\pm}(\lambda)&=\text{tr}_2\{P_{12}^{-}K_{\pm}^{2}(\lambda)R_{12}^{(8V)}(\mp 2\lambda)\}=\\&\begin{pmatrix}-\mathsf{c}(\mp 2\lambda)a_{\pm}(\lambda)+\mathsf{b}(\mp 2\lambda)d_{\pm}(\lambda) & -\mathsf{a}(\mp 2\lambda)b_{\pm}(\lambda)+\mathsf{d}(\mp 2\lambda)c_{\pm}(\lambda)\\ -\mathsf{a}(\mp 2\lambda)c_{\pm}(\lambda)+\mathsf{d}(\mp 2\lambda)b_{\pm}(\lambda) & -\mathsf{c}(\mp 2\lambda)d_{\pm}(\lambda)+\mathsf{b}(\mp 2\lambda)a_{\pm}(\lambda)\end{pmatrix}\\&=p(\mp\lambda)\begin{pmatrix}a_{\pm}(-\lambda) & b_{\pm}(-\lambda)\\ c_{\pm}(-\lambda) & d_{\pm}(-\lambda)\end{pmatrix}=p(\mp\lambda)K_{\pm}(-\lambda),\end{aligned} \tag{4.1.45}$$

from which (4.1.37) follows and by using one of the standard definition of the quantum determinant for object which are solution of a reflection equation

$$\text{q-det}\,(K_{\pm}(\lambda))=\tilde{K}_{\pm}(\lambda\pm\eta/2)K_{\pm}(\lambda\mp\eta/2)=K_{\pm}(\lambda\mp\eta/2)\tilde{K}_{\pm}(\lambda\pm\eta/2).$$

Identity (4.1.38) is obtained by considering that

$$\begin{aligned}
&\mathcal{U}_-(\eta/2+\lambda)\mathcal{U}_-(\eta/2-\lambda) \\
&\underset{(4.1.48)}{=} \text{q-det}\,(M_0(-\lambda))M_0(\lambda+\eta/2)K_-(\lambda+\eta/2)K_-(\eta/2-\lambda)\hat{M}_0(\eta/2-\lambda) \\
&\underset{(4.1.43)}{=} \text{q-det}\,(M_0(-\lambda))\frac{\text{q-det}\,(K_-(\lambda))}{p(\lambda-\eta/2)}M_0(\lambda+\eta/2)\hat{M}_0(\eta/2-\lambda) \underset{(4.1.48)}{=} \frac{\text{q-det}\,(\mathcal{U}_-(\lambda))}{p(\lambda-\eta/2)},
\end{aligned} \tag{4.1.46}$$

where

$$\text{q-det}\,(\mathcal{U}_-(\lambda)) \equiv \text{q-det}\,(K_-(\lambda))\text{q-det}\,(M_0(\lambda))\text{q-det}\,(M_0)(-\lambda), \tag{4.1.47}$$

and we have used that

$$\begin{aligned}
\hat{M}(\pm\lambda+\eta/2) &= (-1)^N \begin{pmatrix} D(-\eta/2\mp\lambda) & -B(-\eta/2\mp\lambda) \\ -C(-\eta/2\mp\lambda) & A(-\eta/2\mp\lambda) \end{pmatrix} \\
&= (-1)^N \text{q-det}\,(M_0(\mp\lambda))M^{-1}(\mp\lambda+\eta/2),
\end{aligned} \tag{4.1.48}$$

where

$$\text{q-det}\,(M_0(\lambda)) = A(\lambda+\eta/2)D(\lambda-\eta/2) - B(\lambda+\eta/2)C(\lambda-\eta/2) \tag{4.1.49}$$

$$= a(\lambda+\eta/2)d(\lambda-\eta/2), \tag{4.1.50}$$

is the bulk quantum determinant, proven to be central for the 6-vertex case in chapter **1**. By putting together expressions (4.1.50) and (4.1.44), it's straightforward to arrive to result (4.1.42). Expressions (4.1.35) and (4.1.36) follow by considering the "algebraic adjoint" $\tilde{\mathcal{U}}_-(\lambda)$ of the boundary monodromy matrix $\mathcal{U}_-(\lambda)$

$$\begin{aligned}
\tilde{\mathcal{U}}_-(\lambda) = \text{tr}_2\{P_{12}^{-}\overset{2}{\mathcal{U}}_-(\lambda)R_{12}^{(8V)}(2\lambda)\} &= \begin{pmatrix} \tilde{\mathcal{D}}_-(\lambda) & -\tilde{\mathcal{B}}_-(\lambda) \\ -\tilde{\mathcal{C}}_-(\lambda) & \tilde{\mathcal{A}}_-(\lambda) \end{pmatrix} \\
&\begin{pmatrix} -\mathsf{c}(2\lambda)\mathcal{A}_-(\lambda)+\mathsf{b}(2\lambda)\mathcal{D}_-(\lambda) & -\mathsf{a}(2\lambda)\mathcal{B}_-(\lambda)+\mathsf{d}(2\lambda)\mathcal{C}_-(\lambda) \\ -\mathsf{a}(2\lambda)\mathcal{C}_-(\lambda)+\mathsf{d}(2\lambda)\mathcal{B}_-(\lambda) & -\mathsf{c}(2\lambda)\mathcal{D}_-(\lambda)+\mathsf{b}(2\lambda)\mathcal{A}_-(\lambda) \end{pmatrix}
\end{aligned} \tag{4.1.51}$$

and the fact that

$$\tilde{\mathcal{U}}_-(\lambda-\eta/2)\mathcal{U}_-(\lambda+\eta/2) = \text{q-det}\,\mathcal{U}_-(\lambda), \tag{4.1.52}$$

and so from identity (4.1.38) it follows

$$\tilde{\mathcal{U}}_-(\lambda) = p(\lambda)\mathcal{U}_-(-\lambda), \tag{4.1.53}$$

which concludes our proof. □

Similar statements hold for the reflection algebra generated by $\mathcal{U}_+(\lambda)$, as they are simply consequences of the previous proposition being $\mathcal{U}_+^{t_0}(-\lambda)$ solution of the same reflection equation of $\mathcal{U}_-(\lambda)$.

Lemma 4.1.1. *The most general boundary transfer matrix $\mathcal{T}(\lambda)$ satisfies the following properties*

1. *Parity;*

$$\mathcal{T}(\lambda) = \mathcal{T}(-\lambda). \tag{4.1.54}$$

2. *Periodicity;*

$$\mathcal{T}(\lambda+\pi) = \mathcal{T}(\lambda). \tag{4.1.55}$$

3. *Quasi-periodicity;*

$$\mathcal{T}(\lambda+\omega\pi) = \left(e^{-i\omega\pi}e^{-2i\lambda}\right)^{2N+6}\mathcal{T}(\lambda) \tag{4.1.56}$$

Proof. Identities (4.1.55) and (4.1.56) follow directly from properties (4.1.24)-(4.1.32). The parity relation (4.1.54) can be proven by by direct computation

$$\begin{aligned}\mathcal{T}(-\lambda) &= \text{tr}_0\{K_+(-\lambda)\mathcal{U}_-(-\lambda)\} = \frac{\text{tr}_0\{K_+(-\lambda)\tilde{\mathcal{U}}_-(\lambda)\}}{p(\lambda)} \\ &= p^{-1}(\lambda)\Big(\mathcal{A}_-(\lambda)a_+(\lambda)\frac{d_+(-\lambda)b(2\lambda)-a_+(-\lambda)\,c(2\lambda)}{a_+(\lambda)} \\ &+\mathcal{D}_-(\lambda)d_+(\lambda)\frac{a_+(-\lambda)b(2\lambda)-d_+(-\lambda)\,c(2\lambda)}{d_+(\lambda)}+\mathcal{B}_-(\lambda)c_+(\lambda)\frac{b_+(-\lambda)d(2\lambda)-c_+(-\lambda)\,d(2\lambda)}{c_+(\lambda)} \\ &+\mathcal{C}_-(\lambda)b_+\,(\lambda)\frac{c_+(-\lambda)d(2\lambda)-b_+(-\lambda)\,d(2\lambda)}{b_+(\lambda)}\Big) \\ &= \mathcal{A}_-(\lambda)a_+(\lambda)+\mathcal{D}_-(\lambda)d_+(\lambda)+\mathcal{B}_-(\lambda)c_+(\lambda)+\mathcal{C}_-(\lambda)b_+(\lambda)=\mathcal{T}(\lambda)\end{aligned}$$

once we observe that

$$\begin{aligned}p(\lambda) &= \frac{-\mathsf{c}(2\lambda)a_+(-\lambda)+\mathsf{b}(2\lambda)d_+(-\lambda)}{a_+(\lambda)} = \frac{-\mathsf{c}(2\lambda)d_+(-\lambda)+\mathsf{b}(2\lambda)a_+(-\lambda)}{d_+(\lambda)} \\ &= \frac{-\mathsf{a}(2\lambda)b_+(-\lambda)+\mathsf{d}(2\lambda)c_+(-\lambda)}{b_+(\lambda)} = \frac{-\mathsf{a}(2\lambda)c_+(-\lambda)+\mathsf{d}(2\lambda)b_+(-\lambda)}{c_+(\lambda)} = \theta_1(2\lambda-\eta).\end{aligned} \tag{4.1.57}$$

as a direct consequence of the identities (4.1.45) and completely equivalent to (4.1.37). Alternatively one can notice the existence of the isomorphism

$$\begin{aligned}a_+(-\lambda|\alpha_1^+,\alpha_2^+,\alpha_3^+) &= d_-(\lambda|\widetilde{\alpha}_1^-,\widetilde{\alpha}_2^-,\widetilde{\alpha}_3^-), \quad c_+(-\lambda|\alpha_1^+,\alpha_2^+,\alpha_3^+) = -c_-(\lambda|\widetilde{\alpha}_1^-,\widetilde{\alpha}_2^-,\widetilde{\alpha}_3^-), \\ d_+(-\lambda|\alpha_1^+,\alpha_2^+,\alpha_3^+) &= a_-(\lambda|\widetilde{\alpha}_1^-,\widetilde{\alpha}_2^-,\widetilde{\alpha}_3^-), \quad b_+(-\lambda|\alpha_1^+,\alpha_2^+,\alpha_3^+) = -b_-(\lambda|\widetilde{\alpha}_1^-,\widetilde{\alpha}_2^-,\widetilde{\alpha}_3^-),\end{aligned} \tag{4.1.58}$$

once we identify $\widetilde{\alpha}_i^- \equiv \alpha_i^+$, for $i \in \{1,2,3\}$. □

4.2 Gauge transformations

As it was done in Section §3.2 we will introduce here the definition of some gauge transformations needed in order to solve the eigenproblem associated to the transfer matrix (4.1.23) by keeping the most generic boundary conditions. Furthermore, the gauged algebra will be defined as well and some important properties studied in details.

4.2.1 Notations

Let us introduce the following 2×2 matrices

$$\bar{G}(\lambda|\beta) \equiv (X(\lambda|\beta), Y(\lambda|\beta)), \qquad \tilde{G}(\lambda|\beta) \equiv (X(\lambda|\beta+1), Y(\lambda|\beta-1)), \tag{4.2.1}$$

$$\bar{G}^{-1}(\lambda|\beta) \equiv \begin{pmatrix}\bar{Y}(\lambda|\beta)\\ \bar{X}(\lambda|\beta)\end{pmatrix}, \qquad \tilde{G}^{-1}(\lambda|\beta) \equiv \begin{pmatrix}\tilde{Y}(\lambda|\beta-1)\\ \tilde{X}(\lambda|\beta+1)\end{pmatrix}, \tag{4.2.2}$$

where

$$X(\lambda|\beta) \equiv \begin{pmatrix}\theta_2(\lambda+(\alpha+\beta)\eta|2\omega)\\ \theta_3(\lambda+(\alpha+\beta)\eta|2\omega)\end{pmatrix}, \qquad Y(\lambda|\beta) \equiv \begin{pmatrix}\theta_2(\lambda+(\alpha-\beta)\eta|2\omega)\\ \theta_3(\lambda+(\alpha-\beta)\eta|2\omega)\end{pmatrix}, \tag{4.2.3}$$

and

$$\bar{X}(\lambda|\beta) \equiv \frac{\left(\theta_3(\lambda+(\alpha+\beta)\eta|2\omega) \quad -\theta_2(\lambda+(\alpha+\beta)\eta|2\omega)\right)}{\theta_1(\lambda+\alpha\eta)\theta_1(\beta\eta)}, \tag{4.2.4}$$

$$\tilde{X}(\lambda|\beta) = \frac{\theta_1(\lambda+\alpha\eta)\theta_1(\beta\eta)}{\theta_1(\lambda+(\alpha+1)\eta)\theta_1((\beta-1)\eta)}\bar{X}(\lambda|\beta), \tag{4.2.5}$$

$$\bar{Y}(\lambda|\beta) \equiv \frac{\left(-\theta_3(\lambda+(\alpha-\beta)\eta|2\omega) \quad \theta_2(\lambda+(\alpha-\beta)\eta|2\omega)\right)}{\theta_1(\lambda+\alpha\eta)\theta_1(\beta\eta)}, \tag{4.2.6}$$

$$\tilde{Y}(\lambda|\beta) = \frac{\theta_1(\lambda+\alpha\eta)\theta_1(\beta\eta)}{\theta_1(\lambda+(\alpha+1)\eta)\theta_1((1+\beta)\eta)}\bar{Y}(\lambda|\beta). \tag{4.2.7}$$

Here, α and β are arbitrary complex number and we omit the index α as it won't play an explicit role. These covectors/vectors satisfy the very same properties (3.2.6) and (3.2.7)

$$\bar{Y}(\lambda|\beta)X(\lambda|\beta) = 1, \qquad \bar{Y}(\lambda|\beta)Y(\lambda|\beta) = 0, \tag{4.2.8a}$$

$$\bar{X}(\lambda|\beta)X(\lambda|\beta) = 0, \qquad \bar{X}(\lambda|\beta)Y(\lambda|\beta) = 1, \tag{4.2.8b}$$

$$X(\lambda|\beta)\bar{Y}(\lambda|\beta) + Y(\lambda|\beta)\bar{X}(\lambda|\beta) = \mathbb{1}, \tag{4.2.8c}$$

and

$$\tilde{Y}(\lambda|\beta-1)X(\lambda|\beta+1) = 1, \qquad \tilde{Y}(\lambda|\beta-1)Y(\lambda|\beta-1) = 0, \tag{4.2.9a}$$

$$\tilde{X}(\lambda|\beta+1)X(\lambda|\beta+1) = 0, \qquad \tilde{X}(\lambda|\beta+1)Y(\lambda|\beta-1) = 1, \tag{4.2.9b}$$

$$X(\lambda|\beta+1)\tilde{Y}(\lambda|\beta-1) + Y(\lambda|\beta-1)\tilde{X}(\lambda|\beta+1) = \mathbb{1}, \tag{4.2.9c}$$

where $\mathbb{1}$ is the identity matrix as usual.

4.2.2 Baxter's gauge transformation

In [13, 14, 15] Baxter defined some gauge transformations in order to tackle the spectral problem related to the transfer matrix of the 8-vertex yang-Baxter algebra representations. the use of gauge transformations allows, in particular, to define pseudo-reference states and then open the way to the use of the ABA method as derived in [43]. The Baxter's gauge transformations were used also in [47] to analyze the spectral problem associated to the 8-vertex reflection algebra in the ABA framework and in [23] in the 6-vertex case. The Baxter's gauge transformations have the following matrix form

$$R^{(8V)}_{0a}(\lambda_{12})S_0(\lambda_1|\alpha,\beta)S_a(\lambda_2|\alpha,\beta+\sigma^z_0) = S_a(\lambda_2|\alpha,\beta)S_0(\lambda_1|\alpha,\beta+\sigma^z_a)R^{(6VD)}_{0a}(\lambda_{12}|\beta), \tag{4.2.10}$$

where

$$S_0(\lambda|\alpha,\beta) \equiv \left(Y(\lambda|\beta) \quad X_\beta(\lambda|\beta)\right). \tag{4.2.11}$$

In (4.2.10) $R^{(6VD)}_{12}(\lambda_{12}|\beta)$ is the elliptic solution of the following dynamical 6-vertex Yang-Baxter equation [49]

$$R^{(6VD)}_{12}(\lambda_{12}|\beta+\sigma^z_a)R^{(6VD)}_{1a}(\lambda_1|\beta)R^{(6VD)}_{2a}(\lambda_2|\beta+\sigma^z_1)$$
$$= R^{(6VD)}_{2a}(\lambda_2|\beta)R^{(6VD)}_{1a}(\lambda_1|\beta+\sigma^z_2)R^{(6VD)}_{12}(\lambda_{12}|\beta), \tag{4.2.12}$$

and it has the form

$$R^{(6VD)}_{12}(\lambda|\beta) = \begin{pmatrix} \mathsf{a}(\lambda) & 0 & 0 & 0 \\ 0 & \mathsf{b}(\lambda|\beta) & \mathsf{c}(\lambda|\beta) & 0 \\ 0 & \mathsf{c}(\lambda|-\beta) & \mathsf{b}(\lambda|-\beta) & 0 \\ 0 & 0 & 0 & \mathsf{a}(\lambda) \end{pmatrix} \tag{4.2.13}$$

where $\mathsf{a}(\lambda)$, $\mathsf{b}(\lambda|\beta)$ and $\mathsf{c}(\lambda|\beta)$ are defined by

$$\mathsf{a}(\lambda)=\theta_1(\lambda+\eta),\quad \mathsf{b}(\lambda|\beta)=\frac{\theta_1(\lambda)\theta_1((\beta+1)\eta)}{\theta_1(\beta\eta)},\quad \mathsf{c}(\lambda|\beta)=\frac{\theta_1(\eta)\theta_1(\beta\eta+\lambda)}{\theta_1(\beta\eta)}. \tag{4.2.14}$$

Historically, Baxter has used first a vectorial representation for these transformations, which explicitly reads:

$$R_{12}(\lambda_{12})X_1(\lambda_1|\beta)X_2(\lambda_2|\beta-1)=\mathsf{a}(\lambda_{12})X_2(\lambda_2|\beta)X_1(\lambda_1|\beta-1), \tag{4.2.15a}$$

$$\begin{aligned}R_{12}(\lambda_{12})X_1(\lambda_1|\beta)Y_2(\lambda_2|\beta-1)&=\mathsf{b}(\lambda_{12}|-\beta)Y_2(\lambda_2|\beta)X_1(\lambda_1|\beta+1)\\&+\mathsf{c}(\lambda_{12}|\beta)X_2(\lambda_2|\beta)Y_1(\lambda_1|\beta-1),\end{aligned} \tag{4.2.15b}$$

$$\begin{aligned}R_{12}(\lambda_{12})Y_1(\lambda_1|\beta)X_2(\lambda_2|\beta+1)&=\mathsf{b}(\lambda_{12}|\beta)X_2(\lambda_2|\beta)Y_1(\lambda_1|\beta-1)\\&+\mathsf{c}(\lambda_{12}|-\beta)Y_2(\lambda_2|\beta)X_1(\lambda_1|\beta+1),\end{aligned} \tag{4.2.15c}$$

$$R_{12}(\lambda_{12})Y_1(\lambda_1|\beta)Y_2(\lambda_2|\beta+1)=\mathsf{a}(\lambda_{12})Y_2(\lambda_2|\beta)Y_1(\lambda_1|\beta+1), \tag{4.2.15d}$$

this clarifies the original use of the terminology intertwining vectors for these gauge transformations.

Remark 4.2.1. One should notice that the relations (4.2.15), with all the others that can be generated by considering $\tilde{X}$ and $\tilde{Y}$, coincide exactly to the relations displayed in appendix A.1 provided that we take into account the map $\sinh(\cdot)\to\theta_1(\cdot|\omega)$. This remarkable fact will become even more evident when commutation relations of the gauged operators will be considered.

Remark 4.2.2. As already stated at the beginning of section §3.2 for the XXZ model, also in this case the gauge transformations defined by Baxter and the one used here are functionally the same. But one should pay attention to the fact that, in the SoV construction, we never pass to study a dynamical algebra, since the gauges will be applied exclusively in the auxiliary space.

4.2.3 Gauge transformed boundary operators

In a completely equivalent way to section §3.2.2, we can define all the gauged transformed element of the algebra. Consider, for first, the following gauge transformed bulk monodromy matrices

$$M(\lambda|\beta)=\tilde{G}^{-1}(\lambda-\eta/2|\beta)M(\lambda)\tilde{G}(\lambda-\eta/2|\beta+N)=\begin{pmatrix}A(\lambda|\beta) & B(\lambda|\beta)\\ C(\lambda|\beta) & D(\lambda|\beta)\end{pmatrix}. \tag{4.2.16}$$

and

$$\hat{M}(\lambda|\beta)=\bar{G}^{-1}(\eta/2-\lambda|\beta+N)M(\lambda)\bar{G}(\eta/2-\lambda|\beta)=\begin{pmatrix}\bar{A}(\lambda|\beta) & \bar{B}(\lambda|\beta)\\ \bar{C}(\lambda|\beta) & \bar{D}(\lambda|\beta)\end{pmatrix}. \tag{4.2.17}$$

The boundary monodromy matrix in the gauged theory becomes

$$\mathsf{U}_-(\lambda|\beta)=\tilde{G}^{-1}(\lambda-\eta/2|\beta)\mathcal{U}_-(\lambda)\tilde{G}(\eta/2-\lambda|\beta)=\begin{pmatrix}\hat{\mathcal{A}}(\lambda|\beta+2) & \hat{\mathcal{B}}(\lambda|\beta)\\ \hat{\mathcal{C}}(\lambda|\beta+2) & \hat{\mathcal{D}}(\lambda|\beta)\end{pmatrix}. \tag{4.2.18}$$

Expression (4.2.18) defines, once again, a non-trivial gauged boundary-bulk decomposition, since

$$\begin{pmatrix}\hat{\mathcal{A}}_-(\lambda|\beta+2)\\ \hat{\mathcal{C}}_-(\lambda|\beta+2)\end{pmatrix}=M(\lambda|\beta)\bar{K}_-(\lambda|\beta)\begin{pmatrix}\bar{A}_-(\lambda|\beta+1)\\ \bar{C}_-(\lambda|\beta+1)\end{pmatrix} \tag{4.2.19}$$

$$\begin{pmatrix}\hat{\mathcal{B}}_-(\lambda|\beta)\\ \hat{\mathcal{D}}_-(\lambda|\beta)\end{pmatrix}=M(\lambda|\beta)K_-(\lambda|\beta)\begin{pmatrix}\bar{A}_-(\lambda|\beta-1)\\ \bar{C}_-(\lambda|\beta-1)\end{pmatrix}, \tag{4.2.20}$$

where we have used

$$K_{-}(\lambda|\beta) = \tilde{G}^{-1}(\lambda - \eta/2|\beta + N)K_{-}(\lambda)\bar{G}(\eta/2 - \lambda|\beta + N - 1) \tag{4.2.21}$$

$$\bar{K}(\lambda|\beta) = \tilde{G}^{-1}(\lambda - \eta/2|\beta + N)K_{-}(\lambda)\bar{G}(\eta/2 - \lambda|\beta + N + 1). \tag{4.2.22}$$

At last, it's favorable to re-normalize the gauged boundary monodromy matrix as it follows

$$\mathcal{U}_{-}(\lambda|\beta) = r(\lambda)\mathsf{U}_{-}(\lambda|\beta) = \begin{pmatrix} \mathcal{A}(\lambda|\beta+2) & \mathcal{B}(\lambda|\beta) \\ \mathcal{C}(\lambda|\beta+2) & \mathcal{D}(\lambda|\beta) \end{pmatrix}, \tag{4.2.23}$$

where

$$r(\lambda) = \theta_1(\lambda + (\alpha + 1/2)\eta). \tag{4.2.24}$$

4.2.4 Properties of the gauged transformed operators

In the following lemma we present the minimal set of commutation relations, which are necessary to build the SoV representations.

Lemma 4.2.1. *The following incomplete set of commutation relations holds for the gauged operators defined in* (4.2.23)

(I)

$$\mathcal{B}_{-}(\lambda_2|\beta)\mathcal{B}_{-}(\lambda_1|\beta-2) = \mathcal{B}_{-}(\lambda_1|\beta)\mathcal{B}_{-}(\lambda_2|\beta-2), \tag{4.2.25}$$

(II)

$$\begin{aligned}\mathcal{A}_{-}(\lambda_2|\beta+2)\mathcal{B}_{-}(\lambda_1|\beta) = &\frac{\theta_1(\lambda_1-\lambda_2+\eta)\theta_1(\lambda_2+\lambda_1-\eta)}{\theta_1(\lambda_1-\lambda_2)\theta_1(\lambda_1+\lambda_2)}\mathcal{B}_{-}(\lambda_1|\beta)\mathcal{A}_{-}(\lambda_2|\beta)\\ &+\frac{\theta_1(\lambda_1+\lambda_2-\eta)\theta_1(\lambda_1-\lambda_2+(\beta-1)\eta)\theta_1(\eta)}{\theta_1(\lambda_2-\lambda_1)\theta_1(\lambda_1+\lambda_2)\theta_1((\beta-1)\eta)}\\ &\times\mathcal{B}_{-}(\lambda_2|\beta)\mathcal{A}_{-}(\lambda_1|\beta)\\ &+\frac{\theta_1(\eta)\theta_1(\lambda_1+\lambda_2-\beta\eta)}{\theta_1(\lambda_1+\lambda_2)\theta_1((\beta-1)\eta)}\mathcal{B}_{-}(\lambda_2|\beta)\mathcal{D}_{-}(\lambda_1|\beta),\end{aligned} \tag{4.2.26}$$

(III)

$$\begin{aligned}\mathcal{B}_{-}(\lambda_1|\lambda_1)\mathcal{D}_{-}(\lambda_2|\beta) = &\frac{\theta_1(\lambda_1-\lambda_2+\eta)\theta_1(\lambda_2+\lambda_1-\eta)}{\theta_1(\lambda_1-\lambda_2)\theta_1(\lambda_1+\lambda_2)}\mathcal{D}_{-}(\lambda_2|\beta+2)\mathcal{B}_{-}(\lambda_1|\beta)\\ &+\frac{\theta_1(\lambda_1+\lambda_2-\eta)\theta_1(\lambda_2-\lambda_1+(\beta+1)\eta)}{\theta_1(\lambda_1-\lambda_2)\theta_1(\lambda_1+\lambda_2)\theta_1((\beta+1)\eta)}\\ &\times\mathcal{D}_{-}(\lambda_1|\beta+2)\mathcal{B}_{-}(\lambda_2|\beta)\\ &+\frac{\theta_1(\eta)\theta_1(\lambda_1+\lambda_2+\beta\eta)}{\theta_1(\lambda_1+\lambda_2)\theta_1((\beta+1)\eta)}\mathcal{A}_{-}(\lambda_1|\beta+2)\mathcal{B}_{-}(\lambda_2|\beta),\end{aligned} \tag{4.2.27}$$

(IV)

$$\begin{aligned}&\mathcal{A}_{-}(\lambda_1|\beta+2)\mathcal{A}_{-}(\lambda_2|\beta+2)\\ &\quad-\frac{\theta_1(\eta)\theta_1(\lambda_1+\lambda_2-\beta\eta)}{\theta_1(\lambda_1+\lambda_2)\theta_1((\beta-1)\eta)}\mathcal{B}_{-}(\lambda_1|\beta)\mathcal{C}_{-}(\lambda_2|\beta+2) =\\ &\mathcal{A}_{-}(\lambda_2|\beta+2)\mathcal{A}_{-}(\lambda_1|\beta+2)\\ &\quad-\frac{\theta_1(\eta)\theta_1(\lambda_1+\lambda_2-\beta\eta)}{\theta_1(\lambda_1+\lambda_2)\theta_1((\beta-1)\eta)}\mathcal{B}_{-}(\lambda_2|\beta)\mathcal{C}_{-}(\lambda_1|\beta+2).\end{aligned} \tag{4.2.28}$$

Proof. In order to prove these expressions we can follow exactly the arguments and calculations developed in the *Proof* of Lemma 3.2.1, taking into account the map, introduced above, $\sinh(\cdot) \to \theta_1(\cdot)$. The first two commutation relations were first presented in paper [47]. □

Note that these commutation relations for the gauge transformed generators of the 8-vertex reflection algebra exactly coincides with those of the gauge transformed 6-vertex ones once we transform the function $\theta_1()$ in $\sinh()$. This observation and the remark that the first coefficients both in (4.2.26) and in (4.2.27) do not depend from the gauge parameters and coincide (under the same elliptic to trigonometric transformation) with those appearing in commutation relations of the original 6-vertex reflection algebra are at the basis of the strong similarity in all the SoV representations of reflection algebra generators.

Next Propositions will give all the remaining details and results about the properties of the gauged operators and the quantum determinants.

Proposition 4.2.1. *The inverse transformed double-row monodromy matrix can be written in terms of the quantum determinant of the* 8*-vertex reflection algebra*

$$\mathsf{U}_-^{-1}(\lambda+\eta/2|\beta) = \frac{\tilde{\mathsf{U}}_-(\lambda-\eta/2|\beta)}{q\text{-}det\,(\mathcal{U}_-(\lambda))} = \frac{p(\lambda-\eta/2)}{q\text{-}det\,(\mathcal{U}_-(\lambda))}\mathsf{U}_-(\eta/2-\lambda|\beta), \tag{4.2.29}$$

where

$$\tilde{\mathsf{U}}_-(\lambda|\beta) \equiv \tilde{G}^{-1}(-\lambda-\eta/2|\beta)\tilde{\mathcal{U}}_-(\lambda)\tilde{G}(\eta/2+\lambda|\beta) \tag{4.2.30}$$

and the quantum determinant admits the representation, for both $\epsilon = \pm 1$

$$\begin{aligned}\frac{q\text{-}det\,(\mathcal{U}_-(\lambda))r(\lambda+\eta/2)r(-\lambda+\eta/2)}{p(\lambda-\eta/2)} \\ &= \mathcal{A}_-(\epsilon\lambda+\eta/2|\beta+2)\mathcal{A}_-(\eta/2-\epsilon\lambda|\beta+2) + \mathcal{B}_-(\epsilon\lambda+\eta/2|\beta)\mathcal{C}_-(\eta/2-\epsilon\lambda|\beta+2) \\ &= \mathcal{D}_-(\epsilon\lambda+\eta/2|\beta)\mathcal{D}_-(\eta/2-\epsilon\lambda|\beta) + \mathcal{C}_-(\epsilon\lambda+\eta/2|\beta+2)\mathcal{B}_-(\eta/2-\epsilon\lambda|\beta).\end{aligned} \tag{4.2.31}$$

Proof. Let us start by proving expression (4.2.29). By definition it holds

$$\begin{aligned}\tilde{\mathsf{U}}_-(\lambda-\eta/2|\beta) &\equiv \tilde{G}^{-1}(-\lambda|\beta)\tilde{\mathcal{U}}_-(\lambda-\eta/2)\tilde{G}(\lambda|\beta), \\ \mathsf{U}_-(\lambda+\eta/2|\beta) &\equiv \tilde{G}^{-1}(\lambda|\beta)\mathcal{U}_-(\lambda+\eta/2)\tilde{G}(-\lambda|\beta),\end{aligned}$$

and then

$$\begin{aligned}\mathsf{U}_-(\lambda+\eta/2|\beta)\tilde{\mathsf{U}}_-(\lambda-\eta/2|\beta) &= \tilde{G}^{-1}(\lambda|\beta)\mathcal{U}_-(\lambda+\eta/2)\tilde{\mathcal{U}}_-(\lambda-\eta/2)\tilde{G}(\lambda|\beta) \\ &= \tilde{G}^{-1}(\lambda|\beta)\text{q-det}\,(\mathcal{U}_-(\lambda))\tilde{G}(\lambda|\beta) \\ &= \text{q-det}\,(\mathcal{U}_-(\lambda)),\end{aligned} \tag{4.2.32}$$

and similarly

$$\begin{aligned}\tilde{\mathsf{U}}_-(\lambda-\eta/2|\beta)\mathsf{U}_-(\lambda+\eta/2|\beta) &= \tilde{G}^{-1}(-\lambda|\beta)\tilde{\mathcal{U}}_-(\lambda-\eta/2)\mathcal{U}_-(\lambda+\eta/2)\tilde{G}(-\lambda|\beta) \\ &= \tilde{G}^{-1}(-\lambda|\beta)\text{q-det}\,(\mathcal{U}_-(\lambda))\tilde{G}(-\lambda|\beta) \\ &= \text{q-det}\,(\mathcal{U}_-(\lambda)).\end{aligned} \tag{4.2.33}$$

From these identities the expressions (4.2.31) for the quantum determinant in terms of gauge transformed operators directly follow. □

Proposition 4.2.2. *$\mathcal{A}_-(\lambda|\beta)$ and $\mathcal{D}_-(\lambda|\beta)$ satisfy the following interrelated parity relations*

$$\mathcal{A}_-(\lambda|\beta) = -\frac{\theta_1(\eta)\theta_1\left(2\lambda-(\beta-1)\eta\right)}{\theta_1\left(2\lambda\right)\theta_1\left((\beta-2)\eta\right)}\mathcal{D}_-(\lambda|\beta) + \frac{\theta_1\left(2\lambda-\eta\right)\theta_1\left((\beta-1)\eta\right)}{\theta_1\left(2\lambda\right)\theta_1\left((\beta-2)\eta\right)}\mathcal{D}_-(-\lambda|\beta), \tag{4.2.34a}$$

$$\mathcal{D}_-(\lambda|\beta) = \frac{\theta_1(\eta)\theta_1\left(2\lambda+(\beta-1)\eta\right)}{\theta_1\left(2\lambda\right)\theta_1\left(\beta\eta\right)}\mathcal{A}_-(\lambda|\beta) + \frac{\theta_1\left(2\lambda-\eta\right)\theta_1\left((\beta-1)\eta\right)}{\theta_1\left(2\lambda\right)\theta_1\left(\beta\eta\right)}\mathcal{A}_-(-\lambda|\beta), \tag{4.2.34b}$$

$$\mathcal{B}_-(-\lambda|\beta) = -\frac{\theta_1(2\lambda+\eta)}{\theta_1\left(2\lambda-\eta\right)}\mathcal{B}_-(\lambda|\beta), \tag{4.2.34c}$$

$$\mathcal{C}_-(-\lambda|\beta) = -\frac{\theta_1(2\lambda+\eta)}{\theta_1\left(2\lambda-\eta\right)}\mathcal{C}_-(\lambda|\beta). \tag{4.2.34d}$$

Proof. Let us define

$$f_\alpha(\lambda) \equiv \frac{r(\lambda)}{r(-\lambda)} = \frac{\theta_1\left((\alpha+1/2)\eta+\lambda\right)}{\theta_1\left((\alpha+1/2)\eta-\lambda\right)}, \tag{4.2.35}$$

and consider the identities

$$\left(\tilde{\mathsf{U}}_-(\lambda|\beta)\right)_{12} = -f_\alpha(\lambda)\theta_1\left(2\lambda+\eta\right)\hat{\mathcal{B}}_-(\lambda|\beta), \quad \left(\tilde{\mathsf{U}}_-(\lambda|\beta)\right)_{21} = -f_\alpha(\lambda)\theta_1\left(2\lambda+\eta\right)\hat{\mathcal{C}}_-(\lambda|\beta), \tag{4.2.36}$$

$$\left(\tilde{\mathsf{U}}_-(\lambda|\beta)\right)_{22} = f_\alpha(\lambda)\left(\frac{\theta_1\left(2\lambda\right)\theta_1\left((\beta-2)\eta\right)}{\theta_1\left((\beta-1)\eta\right)}\hat{\mathcal{A}}_-(\lambda|\beta) + \frac{\theta_1\left(\eta\right)\theta_1\left(2\lambda-(\beta-1)\eta\right)}{\theta_1\left((\beta-1)\eta\right)}\hat{\mathcal{D}}_-(\lambda|\beta)\right), \tag{4.2.37}$$

that can be shown by direct computation expanding both the elements of $\tilde{\mathsf{U}}_-(\lambda|\beta)$ and $\mathcal{U}_-(\lambda|\beta)$ in terms of the ungauged elements of $\mathcal{U}_-(\lambda)$. Then the formulae (4.2.34) are simply derived by using the above identities and that

$$\tilde{\mathsf{U}}_-(\lambda|\beta) = p(\lambda)\begin{pmatrix}\tilde{Y}(-\lambda-\eta/2|\beta-1)\\ \tilde{X}(-\lambda-\eta/2|\beta+1)\end{pmatrix}\mathcal{U}_-(-\lambda)\begin{pmatrix}X(\eta/2+\lambda|\beta+1) & Y(\eta/2+\lambda|\beta-1)\end{pmatrix} \tag{4.2.38}$$

$$= p(\lambda)\mathsf{U}_-(-\lambda|\beta). \tag{4.2.39}$$

□

N.B. The presence of the term $f_\alpha(\lambda)$ in expressions (4.2.36) and (4.2.37) justifies the renormalization of the gauged operators introduced in (4.2.23).

In conclusion, we can present a Lemma about the *β-parity relation* of the gauged operators.

Lemma 4.2.2. *The gauge transformed generators satisfy the following symmetry*

$$\mathcal{U}_-(\lambda|-\beta+2) = \sigma^x\mathcal{U}_-(\lambda|\beta)\sigma^x \tag{4.2.40}$$

which in terms of matrix elements reads

$$\mathcal{B}_-(\lambda|\beta) = \mathcal{C}_-(\lambda|-\beta+2), \qquad \mathcal{A}_-(\lambda|\beta) = \mathcal{D}_-(\lambda|-\beta+2). \tag{4.2.41}$$

Proof. The relation is a trivial consequence of the following simple identities

$$\tilde{Y}(\lambda|\beta) = \tilde{X}(\lambda|-\beta), \qquad Y(\lambda|\beta) = X(\lambda|-\beta); \tag{4.2.42}$$

e.g. we have that

$$\begin{aligned}\hat{\mathcal{B}}_-(\lambda|\beta) &= \tilde{Y}(\lambda-\eta/2|\beta-1)\mathcal{U}_-(\lambda)Y_(\eta/2-\lambda|\beta-1)\\ &= \tilde{X}(\lambda-\eta/2|(-\beta+2)-1)\mathcal{U}_-(\lambda)X(\eta/2-\lambda|(-\beta+2)-1)\\ &= \hat{\mathcal{C}}_-(\lambda|-\beta+2).\end{aligned} \tag{4.2.43}$$

□

4.2.5 Boundary transfer matrix and gauged operators

Let us introduce the vectors

$$\hat{Y}(\lambda|\beta-2) = \frac{\theta_1((\beta+1)\eta)Y(\lambda|\beta-2)}{\theta_1(\beta\eta)\theta_1(\lambda+(\alpha+2)\eta)}, \qquad \underline{Y}(\lambda|\beta) = \frac{\bar{Y}(\lambda|\beta)}{\theta_1(-\lambda+(\alpha+1)\eta)}, \tag{4.2.44}$$

$$\hat{X}(\lambda|\beta+2) = \frac{\theta_1((\beta-1)\eta)X(\lambda|\beta+2)}{\theta_1(\beta\eta)\theta_1(\lambda+(\alpha+2)\eta)}, \qquad \underline{X}(\lambda|\beta) = \frac{\bar{X}(\lambda|\beta)}{\theta_1(-\lambda+(\alpha+1)\eta)}, \tag{4.2.45}$$

and the following two gauge transformations on the boundary matrix K_+

$$K_+^{(L)}(\lambda|\beta) = \begin{pmatrix} \tilde{Y}(\eta/2-\lambda|\beta-2)K_+(\lambda)\hat{X}(\lambda-\eta/2|\beta+2) & \tilde{Y}(\eta/2-\lambda|\beta)K_+(\lambda)\hat{Y}(\lambda-\eta/2|\beta-2) \\ \tilde{X}(\eta/2-\lambda|\beta)K_+(\lambda)\hat{X}(\lambda-\eta/2|\beta+2) & \tilde{X}(\eta/2-\lambda|\beta+2)K_+(\lambda)\hat{Y}(\lambda-\eta/2|\beta-2) \end{pmatrix}, \tag{4.2.46}$$

$$K_+^{(R)}(\lambda|\beta) = \begin{pmatrix} \underline{Y}(\eta/2-\lambda|\beta)K_+(\lambda)X(\lambda-\eta/2|\beta) & \underline{Y}(\eta/2-\lambda|\beta)K_+(\lambda)Y(\lambda-\eta/2|\beta-2) \\ \underline{X}(\eta/2-\lambda|\beta)K_+(\lambda)X(\lambda-\eta/2|\beta+2) & \underline{X}(\eta/2-\lambda|\beta)K_+(\lambda)Y(\lambda-\eta/2|\beta) \end{pmatrix}. \tag{4.2.47}$$

then the following proposition holds

Proposition 4.2.3. *In terms of the gauge transformed reflection algebra generators, the boundary transfer matrix $\mathcal{T}(\lambda)$ admit the decompositions*

$$\begin{aligned}\mathcal{T}(\lambda) =&[K_+^{(L)}(\lambda|\beta-1)]_{11}\mathcal{A}_-(\lambda|\beta) + [K_+^{(L)}(\lambda|\beta-1)]_{22}\mathcal{D}_-(\lambda|\beta)\\ &+ [K_+^{(L)}(\lambda|\beta-1)]_{21}\mathcal{B}_-(\lambda|\beta-2) + [K_+^{(L)}(\lambda|\beta-1)]_{12}\mathcal{C}_-(\lambda|\beta+2),\end{aligned} \tag{4.2.48}$$

and

$$\begin{aligned}\mathcal{T}(\lambda) =&[K_+^{(R)}(\lambda|\beta-1)]_{11}\mathcal{A}_-(\lambda|\beta) + [K_+^{(R)}(\lambda|\beta-1)]_{22}\mathcal{D}_-(\lambda|\beta)\\ &+ [K_+^{(R)}(\lambda|\beta-1)]_{21}\mathcal{B}_-(\lambda|\beta) + [K_+^{(R)}(\lambda|\beta-1)]_{12}\mathcal{C}_-(\lambda|\beta).\end{aligned} \tag{4.2.49}$$

Proof. To prove the two decompositions of the transfer matrix we first remark that the following identities hold

$$\begin{pmatrix}\hat{X}(\lambda|\beta+2) & \hat{Y}(\lambda|\beta-2)\end{pmatrix}\begin{pmatrix}\tilde{Y}(\lambda|\beta-2)\\ \tilde{X}(\lambda|\beta+2)\end{pmatrix} = \frac{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}{\theta_1(\lambda+(\alpha+1)\eta)}, \tag{4.2.50}$$

The formulae (4.2.8) and (4.2.50) imply the following chain of identities

$$\begin{aligned}
&\mathcal{A}_-(\lambda|\beta)[K_+^{(L)}(\lambda|\beta-1)]_{11}+\mathcal{B}_-(\lambda|\beta-2)[K_+^{(L)}(\lambda|\beta-1)]_{21}\\
&\qquad\qquad\qquad+\mathcal{D}_-(\lambda|\beta)[K_+^{(L)}(\lambda|\beta-1)]_{22}+\mathcal{C}_-(\lambda|\beta+2)[K_+^{(L)}(\lambda|\beta-1)]_{12}\\
&=\frac{\tilde{Y}(\lambda-\eta/2|\beta-3)\mathcal{U}_-(\lambda)K_+(\lambda)\hat{X}(\lambda-\eta/2|\beta+1)}{(\theta_1(\lambda+(\alpha+1/2)\eta))^{-1}}\\
&\qquad\qquad+\frac{\tilde{X}(\lambda-\eta/2|\beta+1)\mathcal{U}_-(\lambda)K_+(\lambda)\hat{Y}(\lambda-\eta/2|\beta-3)}{(\theta_1(\lambda+(\alpha+1/2)\eta))^{-1}}\\
&=\frac{\mathrm{tr}_0\{\begin{pmatrix}\tilde{Y}(\lambda-\eta/2|\beta-3)\\ \tilde{X}(\lambda-\eta/2|\beta+1)\end{pmatrix}\mathcal{U}_-(\lambda)K_+(\lambda)\begin{pmatrix}\hat{X}(\lambda-\eta/2|\beta+1) & \hat{Y}(\lambda-\eta/2|\beta-3)\end{pmatrix}\}}{(\theta_1(\lambda+(\alpha+1/2)\eta))^{-1}}\\
&=\frac{\mathrm{tr}_0\{\begin{pmatrix}\hat{X}(\lambda-\eta/2|\beta+1) & \hat{Y}(\lambda-\eta/2|\beta-3)\end{pmatrix}\begin{pmatrix}\tilde{Y}(\lambda-\eta/2|\beta-3)\\ \tilde{X}(\lambda-\eta/2|\beta+1)\end{pmatrix}\mathcal{U}_-(\lambda)K_+(\lambda)\}}{(\theta_1(\lambda+(\alpha+1/2)\eta))^{-1}}\\
&=\mathrm{tr}_0\{\mathcal{U}_-(\lambda)K_+(\lambda)\}=\mathcal{T}(\lambda).
\end{aligned} \tag{4.2.51}$$

Similarly, the formulae (4.2.8) imply

$$\begin{aligned}
&[K_+^{(R)}(\lambda|\beta-1)]_{11}\mathcal{A}_-(\lambda|\beta)+[K_+^{(R)}(\lambda|\beta-1)]_{22}\mathcal{D}_-(\lambda|\beta)\\
&\qquad\qquad\qquad+[K_+^{(R)}(\lambda|\beta-1)]_{21}\mathcal{B}_-(\lambda|\beta)+[K_+^{(R)}(\lambda|\beta-1)]_{12}\mathcal{C}_-(\lambda|\beta)\\
&=\bar{Y}(\eta/2-\lambda|\beta-1)K_+(\lambda)\mathcal{U}_-(\lambda)X(\eta/2-\lambda|\beta-1)\\
&\qquad\qquad\qquad+\bar{X}(\eta/2-\lambda|\beta-1)K_+(\lambda)\mathcal{U}_-(\lambda)Y(\eta/2-\lambda|\beta-1)\\
&=\mathrm{tr}_0\{\begin{pmatrix}\bar{Y}(\eta/2-\lambda|\beta-1)\\ \bar{X}(\eta/2-\lambda|\beta-1)\end{pmatrix}K_+(\lambda)\mathcal{U}_-(\lambda)\begin{pmatrix}X(\eta/2-\lambda|\beta-1) & Y(\eta/2-\lambda|\beta-1)\end{pmatrix}\}\\
&=\mathrm{tr}_0\{\begin{pmatrix}X(\lambda-\eta/2|\beta-1) & Y(\lambda-\eta/2|\beta-1)\end{pmatrix}\begin{pmatrix}\bar{Y}(\lambda-\eta/2|\beta-1)\\ \bar{X}(\lambda-\eta/2|\beta-1)\end{pmatrix}\mathcal{U}_-(\lambda)K_+(\lambda)\}\\
&=\mathrm{tr}_0\{\mathcal{U}_-(\lambda)K_+(\lambda)\}=\mathcal{T}(\lambda).
\end{aligned} \tag{4.2.52}$$

□

Proposition 4.2.4. *The following two explicitly even in λ representations of the transfer matrix hold*

$$\begin{aligned}
\mathcal{T}(\lambda)=&\ \mathsf{a}_+(\lambda|\beta-1)\mathcal{A}_-(\lambda|\beta)+\mathsf{a}_+(-\lambda|\beta-1)\mathcal{A}_-(-\lambda|\beta)\\
&+[K_+^{(L)}(\lambda|\beta-1)]_{12}\mathcal{C}_-(\lambda|\beta+2)+[K_+^{(L)}(\lambda|\beta-1)]_{21}\mathcal{B}_-(\lambda|\beta-2),
\end{aligned} \tag{4.2.53}$$

$$\begin{aligned}
\mathcal{T}(\lambda)=&\ \mathsf{d}_+(\lambda|\beta-1)\mathcal{D}_-(\lambda|\beta)+\mathsf{d}_+(-\lambda|\beta-1)\mathcal{D}_-(-\lambda|\beta)\\
&+[K_+^{(R)}(\lambda|\beta-1)]_{12}\mathcal{C}_-(\lambda|\beta)+[K_+^{(R)}(\lambda|\beta-1)]_{21}\mathcal{B}_-(\lambda|\beta),
\end{aligned} \tag{4.2.54}$$

where we have defined

$$\mathsf{a}_+(\lambda|\beta)=\frac{\theta_1(2\lambda+\eta)\,\theta_1(\beta\eta)}{\theta_1(2\lambda)\,\theta_1((\beta+1)\eta)}[K_+^{(L)}(-\lambda|\beta)]_{22}, \tag{4.2.55}$$

$$\mathsf{d}_+(\lambda|\beta)=\frac{\theta_1(2\lambda+\eta)\,\theta_1(\beta\eta)}{\theta_1(2\lambda)\,\theta_1((\beta-1)\eta)}[K_+^{(R)}(-\lambda|\beta)]_{11}. \tag{4.2.56}$$

Proof. The decompositions of the transfer matrix given in the previous proposition can be rewritten in the following way

$$\begin{aligned}\mathcal{T}(\lambda) =& \left([K_+^{(L)}(\lambda|\beta-1)]_{11} + \frac{\theta_1(\eta)\theta_1(2\lambda+(\beta-1)\eta)}{\theta_1(2\lambda)\theta_1((\beta)\eta)}[K_+^{(L)}(\lambda|\beta-1)]_{22}\right)\mathcal{A}_-(\lambda|\beta)\\ &+\left(\frac{\theta_1(2\lambda-\eta)\theta_1((\beta-1)\eta)}{\theta_1(2\lambda)\theta_1((\beta)\eta)}[K_+^{(L)}(\lambda|\beta-1)]_{22}\right)\mathcal{A}_-(-\lambda|\beta)\\ &+[K_+^{(L)}(\lambda|\beta-1)]_{21}\mathcal{B}_-(\lambda|\beta-2)+[K_+^{(L)}(\lambda|\beta-1)]_{12}\mathcal{C}_-(\lambda|\beta+2),\end{aligned} \tag{4.2.57}$$

$$\begin{aligned}\mathcal{T}(\lambda) =& \left([K_+^{(R)}(\lambda|\beta-1)]_{22} - \frac{\theta_1(\eta)\theta_1(2\lambda-(\beta-1)\eta)}{\theta_1(2\lambda)\theta_1((\beta-2)\eta)}[K_+^{(R)}(\lambda|\beta-1)]_{11}\right)\mathcal{D}_-(\lambda|\beta)\\ &+\left(\frac{\theta_1(2\lambda-\eta)\theta_1((\beta-1)\eta)}{\theta_1(2\lambda)\theta_1((\beta-2)\eta)}[K_+^{(R)}(\lambda|\beta-1)]_{11}\right)\mathcal{D}_-(-\lambda|\beta)\\ &+[K_+^{(R)}](\lambda|\beta-1)_{21}\mathcal{B}_-(\lambda|\beta)+[K_+^{(R)}(\lambda|\beta-1)]_{12}\mathcal{C}_-(\lambda|\beta),\end{aligned} \tag{4.2.58}$$

once we use the properties (4.2.34). Finally, consider the identities

$$\begin{aligned}&[K_+^{(L)}(\lambda|\beta-1)]_{11} + \frac{\theta_1(\eta)\theta_1(2\lambda+(\beta-1)\eta)}{\theta_1(2\lambda)\theta_1((\beta)\eta)}[K_+^{(L)}(\lambda|\beta-1)]_{22}\\ &\qquad\qquad = \frac{\theta_1(2\lambda+\eta)\theta_1((\beta-1)\eta)}{\theta_1(2\lambda)\theta_1(\beta\eta)}[K_+^{(L)}(-\lambda|\beta-1)]_{22},\end{aligned} \tag{4.2.59}$$

$$\begin{aligned}&[K_+^{(R)}(\lambda|\beta-1)]_{22} - \frac{\theta_1(\eta)\theta_1(2\lambda-(\beta-1)\eta)}{\theta_1(2\lambda)\theta_1((\beta-2)\eta)}[K_+^{(R)}(\lambda|\beta-1)]_{11}\\ &\qquad\qquad = \frac{\theta_1(2\lambda+\eta)\theta_1((\beta-1)\eta)}{\theta_1(2\lambda)\theta_1((\beta-2)\eta)}[K_+^{(R)}(-\lambda|\beta-1)]_{11},\end{aligned} \tag{4.2.60}$$

which can be proven by exploiting the connection existing between $\tilde{K}_+(\lambda)$ and $K_+(-\lambda)$, see (4.1.45), and its consequences among the gauged operators, as it was done in proposition 4.2.2 for the gauged elements of the monodromy matrix. For example, it's possible to prove the following expression holds

$$\begin{aligned}[\tilde{K}_+^{(L)}(\lambda|\beta)]_{22} &= \tilde{X}(\lambda+\eta/2|\beta+2)\tilde{K}_+(\lambda)\hat{Y}(-\lambda-\eta/2|\beta-2)\\ &= -\left(\frac{\theta_1(2\lambda)\theta_1((\beta+1)\eta)}{\theta_1(\beta\eta)}[K_+^{(L)}(\lambda|\beta)]_{11} + \frac{\theta_1(2\lambda+\beta\eta)\theta_1(\eta)}{\theta_1(\beta\eta)}[K_+^{(L)}(\lambda|\beta-1)]_{22}\right)\\ &= p(-\lambda)[K_+^{(L)}(-\lambda|\beta-1)]_{22},\end{aligned} \tag{4.2.61}$$

by expanding both the elements of $[\tilde{K}_+^{(L)}(\lambda\beta)]_{22}$, $[K_+^{(L)}(\lambda|\beta)]_{11}$ and $[K_+^{(L)}(\lambda|\beta)]_{22}$ in terms of the ungauged elements of $K_+(\lambda)$ and using (4.1.45) to connect with $[K_+^{(L)}(-\lambda|\beta-1)]_{22}$. The same arguments can be applied to $K_+^{(R)}(\lambda|\beta)$.

The remaining two terms in $\mathcal{B}_-$ and $\mathcal{C}_-$ in (4.2.55)-(4.2.56) are even as well a it can be shown by considering the explicit expressions for the gauged K_+-matrices components and the parity relations (4.2.34c)-(4.2.34d). For example, consider the term $[K_+^{(L)}(\lambda|\beta-1)]_{12}\mathcal{C}(\lambda|\beta+2)$ and send $\lambda\to-\lambda$, we get

$$\begin{aligned}
&[K_+^{(L)}(-\lambda|\beta-1)]_{12}\mathcal{C}(-\lambda|\beta+2)\\
&\qquad=\left(-\frac{\theta_1(2\lambda-\eta)}{\theta_1(2\lambda+\eta)}\right)[K_+^{(L)}(\lambda|\beta-1)]_{12}\left(-\frac{\theta_1(2\lambda+\eta)}{\theta_1(2\lambda-\eta)}\right)\mathcal{C}(\lambda|\beta+2)\\
&\qquad=[K_+^{(L)}(\lambda|\beta-1)]_{12}\mathcal{C}(\lambda|\beta+2),
\end{aligned}$$

where we have used the explicit expression (4.2.69) (see the proof of next lemma) and property (4.2.34d). The same can be established for all the other terms considering the following identities

$$[K_+^{(L)}(\lambda|\beta)]_{21}=[K_+^{(L)}(\lambda|-\beta)]_{12}; \tag{4.2.62}$$

$$\begin{aligned}
[K_+^{(R)}(\lambda|\beta)]_{12}&=\underline{Y}(\eta/2-\lambda|\beta)K_+(\lambda)Y(\lambda-\eta/2|\beta-2)\\
&=\frac{1}{\theta_1(\lambda+(\alpha+1/2)\eta)}\bar{Y}(\eta/2-\lambda|\beta)K_+(\lambda)Y(\lambda-\eta/2|\beta-2)\\
&=\frac{\theta_1(-\lambda+(\alpha+3/2)\eta)\theta_1((\beta+1)\eta)}{\theta_1(-\lambda+(\alpha+1/2)\eta)\theta_1(\lambda+(\alpha+1/2)\eta)\theta_1(\beta\eta)}\\
&\qquad\times\tilde{Y}(\eta/2-\lambda|\beta)K_+(\lambda)Y(\lambda-\eta/2|\beta-2) \qquad (4.2.63)\\
&=\frac{\theta_1(-\lambda+(\alpha+3/2)\eta)\theta_1(\lambda+(\alpha+3/2)\eta)}{\theta_1(-\lambda+(\alpha+1/2)\eta)\theta_1(\lambda+(\alpha+1/2)\eta)}\\
&\qquad\times\tilde{Y}(\eta/2-\lambda|\beta)K_+(\lambda)\hat{Y}(\lambda-\eta/2|\beta-2)\\
&=\frac{\theta_1(-\lambda+(\alpha+3/2)\eta)\theta_1(\lambda+(\alpha+3/2)\eta)}{\theta_1(-\lambda+(\alpha+1/2)\eta)\theta_1(\lambda+(\alpha+1/2)\eta)}[K_+^{(L)}(\lambda|\beta)]_{12};
\end{aligned}$$

$$[K_+^{(R)}(\lambda|\beta)]_{21}=[K_+^{(R)}(\lambda|-\beta)]_{12}. \tag{4.2.64}$$

□

The functions $\mathsf{a}_+(\lambda|\beta)$ and $\mathsf{d}_+(\lambda|\beta)$ will be crucial in the SoV description of the transfer matrix spectrum and so will be the following properties

Lemma 4.2.3. *Using the freedom in the choice of the gauge parameters to fix*

$$(\alpha-\beta+2)\eta=\sum_{l=1}^{3}\left(\epsilon_l^+\alpha_l^+\right)+(2p+1)\pi+k\omega\pi,\qquad \prod_{l=1}^{3}\epsilon_l^+=1,\qquad \forall(p,k)\in\mathbb{Z},\ \epsilon_l^+\in\{-1,1\}, \tag{4.2.65}$$

which comes from the condition

$$[K_+^{(L)}(\lambda|\beta-1)]_{12}=[K_+^{(R)}(\lambda|\beta-1)]_{12}=0, \tag{4.2.66}$$

keeping completely arbitrary the six boundary parameters, the following quantum determinant conditions are satisfied

$$\frac{q\text{-}det\,(K_+(\lambda))p(\lambda-\eta/2)}{\theta_1(\eta-2\lambda)\theta_1(2\lambda+\eta)r(\lambda+\eta/2)r(-\lambda+\eta/2)}=\mathsf{a}_+(\lambda+\eta/2|\beta-1)\mathsf{a}_+(-\lambda+\eta/2|\beta-1) \tag{4.2.67}$$

$$=\mathsf{d}_+(\lambda+\eta/2|\beta-1)\mathsf{d}_+(-\lambda+\eta/2|\beta-1), \tag{4.2.68}$$

where we remember expression (4.1.44)

$$\det_q K_+(\lambda) = p(-\lambda - \eta/2)g_+(\lambda + \eta/2)g_+(-\lambda + \eta/2).$$

Proof. In order to understand how conditions (4.2.66) coincide with equation (4.2.65), one has to consider and study the zeroes of $[K_+^{(L)}(\lambda|\beta-1)]_{12}$ and $[K_+^{(R)}(\lambda|\beta-1)]_{12}$. As we understand from equation (4.2.63) the two elements are proportional, then the same gauge fixing will send both the representations $[K_+^{(L)}(\lambda|\beta-1)]_{12}$ and $[K_+^{(R)}(\lambda|\beta-1)]_{12}$ to zero in the same time. Now consider

$$\begin{aligned}
[K_+^{(L)}(\lambda|\beta-1)]_{12} &= \tilde{Y}(\eta/2-\lambda|\beta-1)K_+(\lambda)\hat{Y}(\lambda-\eta/2|\beta-3)\\
&= \frac{1}{\theta_1(\lambda+(\alpha+3/2)\eta)\theta_1(-\lambda+(\alpha+3/2)\eta)\theta_1((\beta-1)\eta)}\\
&\times\{-\theta_2(\lambda-\eta/2+(\alpha-\beta-1)\eta|2\omega)\theta_3(\eta/2-\lambda+(\alpha-\beta-3)\eta|2\omega)a_+(\lambda)\\
&\quad+\theta_2(\lambda-\eta/2+(\alpha-\beta-1)\eta|2\omega)\theta_2(\eta/2-\lambda+(\alpha-\beta-3)\eta|2\omega)c_+(\lambda)\\
&\quad-\theta_3(\lambda-\eta/2+(\alpha-\beta-1)\eta|2\omega)\theta_3(\eta/2-\lambda+(\alpha-\beta-3)\eta|2\omega)b_+(\lambda)\\
&\quad+\theta_3(\lambda-\eta/2+(\alpha-\beta-1)\eta|2\omega)\theta_2(\eta/2-\lambda+(\alpha-\beta-3)\eta|2\omega)d_+(\lambda)\},
\end{aligned}$$

which, after some manipulations with the theta functions and the explicit form of $a_+(\lambda)$, $b_+(\lambda)$, $c_+(\lambda)$ and $d_+(\lambda)$, takes the form

$$\begin{aligned}
[K_+^{(L)}(\lambda|\beta-1)]_{12} &= \frac{\theta_1(2\lambda+\eta)}{2\theta_1(\lambda+(\alpha+3/2)\eta)\theta_1(-\lambda+(\alpha+3/2)\eta)\theta_1((\beta-1)\eta)\prod_{l=1}^3\alpha_l^+}\\
&\times\{\theta_1((\alpha-\beta+2)\eta)\prod_{l=1}^3\theta_1(\alpha_l^+)+\theta_2((\alpha-\beta+2)\eta)\prod_{l=1}^3\theta_2(\alpha_l^+)\\
&\quad+\theta_3((\alpha-\beta+2)\eta)\prod_{l=1}^3\theta_3(\alpha_l^+)-\theta_4((\alpha-\beta+2)\eta)\prod_{l=1}^3\theta_4(\alpha_l^+)\},
\end{aligned} \tag{4.2.69}$$

and by using properties (B.1.5a) - (B.1.6h) we get

$$\begin{aligned}
[K_+^{(L)}(\lambda|\beta-1)]_{12} &= \frac{\theta_1(2\lambda+\eta)}{2\theta_1(\lambda+(\alpha+3/2)\eta)\theta_1(-\lambda+(\alpha+3/2)\eta)\theta_1((\beta-1)\eta)\prod_{l=1}^3\alpha_l^+}\\
&\times\{-\theta_1((\alpha-\beta+2)\eta-\pi)\prod_{l=1}^3\theta_1(\alpha_l^+)-\theta_4((\alpha-\beta+2)\eta-\pi)\prod_{l=1}^3\theta_1(\alpha_l^+)\\
&\quad+\theta_1((\alpha-\beta+2)\eta+\frac{\pi}{2})\prod_{l=1}^3\theta_1(\alpha_l^++\frac{\pi}{2})+\theta_4((\alpha-\beta+2)\eta+\frac{\pi}{2})\prod_{l=1}^3\theta_4(\alpha_l^++\frac{\pi}{2})\}.
\end{aligned} \tag{4.2.70}$$

It is then easy to see when the above expression goes to zero, keeping in mind the equality (B.1.10a)

$$\begin{aligned}
&\theta_1(\lambda_1)\theta_1(\lambda_2)\theta_1(\lambda_3)\theta_1(\lambda_1+\lambda_2+\lambda_3)+\theta_4(\lambda_1)\theta_4(\lambda_2)\theta_4(\lambda_3)\theta_4(\lambda_1+\lambda_2+\lambda_3)\\
&\qquad=\theta_4(0)\theta_4(\lambda_1+\lambda_2)\theta_4(\lambda_1+\lambda_3)\theta_4(\lambda_2+\lambda_3).
\end{aligned}$$

For example, the first non trivial solution to $[K_+^{(L)}(\lambda|\beta-1)]_{12}=0$ is obtained if we set, in the two line L1 and L2 of eq. (4.2.70)

$$\begin{array}{lllll}
\text{L1:} & \lambda_1=\alpha_1, & \lambda_2=\alpha_2, & \lambda_3=\alpha_3, & \lambda_1+\lambda_2+\lambda_3=(\alpha-\beta+2)\eta-\pi,\\
\text{L2:} & \lambda_1=\alpha_1+\frac{\pi}{2}, & \lambda_2=\alpha_2+\frac{\pi}{2}, & \lambda_3=\alpha_3+\frac{\pi}{2}, & \lambda_1+\lambda_2+\lambda_3+\frac{3\pi}{2}=(\alpha-\beta+2)\eta+\frac{1}{2}\pi,
\end{array}$$

which both result in

$$(\alpha-\beta+2)\eta=\sum_{l}^{3}\alpha_l^{+}+\pi,$$

which coincides with (4.2.65) for $p=0$ and $\epsilon_l^{+}=1\ \forall l\in\{1,2,3\}$. In general it should be stressed that $[K_{+}^{(L)}(\lambda|\beta-1)]_{12}$ has functional dependence in the spectral parameter λ factorized with respect to the other parameters and it is an elliptic polynomials in the boundary and the gauge parameters. Then for any fixed value of the α_1^{+}, α_2^{+} and α_3^{+} one can always fix β such that the condition (4.2.66) is satisfied for any value of λ.

Let us now prove only the identity (4.2.67) as the other one follows similarly. From the definitions of these functions it holds

$$\begin{aligned}
&\mathsf{a}_{+}(\lambda+\eta/2|\beta-1)\mathsf{a}_{+}(-\lambda+\eta/2|\beta-1)\\
&=\frac{\tilde{X}(\eta+\lambda|\beta+1)K_{+}(-\lambda-\eta/2)Y(-\lambda+\eta|\beta-1)}{\theta_1(2\lambda+\eta)r(-\lambda+\eta/2)(p(\lambda-\eta/2))^{-1}}\\
&\qquad\times\frac{\tilde{X}(-\lambda+\eta|\beta+1)K_{+}(\lambda-\eta/2)Y(\lambda+\eta|\beta-1)}{\theta_1(-2\lambda+\eta)r(\lambda+\eta/2)(p(-\lambda-\eta/2))^{-1}}\\
&=\frac{\tilde{X}(\eta+\lambda|\beta+1)K_{+}(-\lambda-\eta/2)K_{+}(\lambda-\eta/2)Y(\lambda+\eta|\beta-1)}{r(\lambda+\eta/2)r(-\lambda+\eta/2)\theta_1(\eta-2\lambda)\theta_1(2\lambda+\eta)\left(p(-\lambda-\eta/2)p(\lambda-\eta/2)\right)^{-1}}\\
&=\frac{\text{q-det}\,(K_{+}(\lambda))p(\lambda-\eta/2)\tilde{X}(\eta+\lambda|\beta+1)Y(\lambda+\eta|\beta-1)}{\theta_1(\eta-2\lambda)\theta_1(2\lambda+\eta)r(\lambda+\eta/2)r(-\lambda+\eta/2)}\\
&=\frac{\text{q-det}\,(K_{+}(\lambda))p(\lambda-\eta/2)}{\theta_1(\eta-2\lambda)\theta_1(2\lambda+\eta)r(\lambda+\eta/2)r(-\lambda+\eta/2)}.
\end{aligned}\tag{4.2.71}$$

The first line is obtained by noticing that

$$Y(\lambda-\eta|\beta-2)=Y(\lambda+\eta|\beta),\tag{4.2.72}$$

which simply follows the definition (4.2.3). The second line is obtained by using the identity (4.2.8c) once we add to the first line the following term

$$\frac{\tilde{X}(\eta+\lambda|\beta+1)K_{+}(-\lambda-\eta/2)X(-\lambda+\eta|\beta+1)\tilde{Y}(-\lambda+\eta|\beta-1)K_{+}(\lambda-\eta/2)Y(\lambda+\eta|\beta-1)}{r(\lambda+\eta/2)r(-\lambda+\eta/2)\theta_1(\eta-2\lambda)\theta_1(2\lambda+\eta)\left(p(-\lambda-\eta/2)p(\lambda-\eta/2)\right)^{-1}},\tag{4.2.73}$$

which is zero for the condition (4.2.66) being

$$\begin{aligned}
&\tilde{Y}(-\lambda+\eta|\beta-1)K_{+}(\lambda-\eta/2)Y(\lambda+\eta|\beta-1)\\
&\qquad=\tilde{Y}(-\lambda+\eta|\beta-1)K_{+}(\lambda-\eta/2)Y(\lambda-\eta|\beta-3)\\
&\qquad\qquad=P(\lambda)[K_{+}^{(L)}(\lambda-\eta/2|\beta-1)]_{12}=0,
\end{aligned}\tag{4.2.74}$$

with $P(\lambda)=\frac{\theta_1((\beta-1)\eta)\theta_1(\lambda+(\alpha+1)\eta)\theta_4(2\lambda-2\eta|2\omega)}{\theta_1((\beta)\eta)}$, since $Y(\lambda-\eta|\beta-3)=P(\lambda)\hat{Y}(\lambda-\eta|\beta-3)$.

□

4.3 SoV representation of the gauge transformed elliptic reflection algebra

Given the definitions (4.2.21) and (4.2.22) of the gauged transformed *left* boundary matrices $K_-(\lambda|\beta)$ and $\bar{K}_-(\lambda|\beta)$, we can produce the following theorem.

Theorem 4.3.1. *Let the inhomogeneities* $\{\xi_1, \dots, \xi_N\} \in \mathbb{C}^N$ *satisfy the following conditions:*

$$\xi_a \neq \xi_b + r\eta \quad \forall a \neq b \in \{1, \dots, N\} \quad \text{and } r \in \{-1, 0, 1\}, \tag{4.3.1}$$

then I_a*) for any* $\alpha, \beta \in \mathbb{C}$ *such that*

$$(\alpha - \beta)\eta \neq (N-1)\eta - \sum_{l=1}^{3}(\epsilon_l^- \alpha_l^-) + (2p+1)\pi + k\tau, \qquad \prod_{l=1}^{3} \epsilon_l^+ = 1, \\ \forall (p,k) \in \mathbb{Z},\ \epsilon_l^+ \in \{-1, 1\}, \tag{4.3.2}$$

corresponding to

$$[K_-(\lambda|\beta)]_{12} \neq 0, \tag{4.3.3}$$

the one parameter family of the gauge transformed generators of the reflection algebra $\mathcal{B}_-(\lambda|\beta)$ *is left pseudo-diagonalizable and its pseudo-spectrum is simple.*

II_a*) for any* $\alpha, \beta \in \mathbb{C}$ *such that*

$$(\alpha - \beta)\eta \neq -(N+1)\eta - \sum_{l=1}^{3}(\epsilon_l^- \alpha_l^-) + (2p+1)\pi + k\tau, \qquad \prod_{l=1}^{3} \epsilon_l^+ = 1, \\ \forall (p,k) \in \mathbb{Z},\ \epsilon_l^+ \in \{-1, 1\}, \tag{4.3.4}$$

corresponding to

$$[\bar{K}_-(\lambda| - \beta)]_{12} \neq 0, \tag{4.3.5}$$

the one parameter family of the gauge transformed generators of the reflection algebra $\mathcal{B}_-(\lambda|\beta)$ *is right pseudo-diagonalizable and its pseudo-spectrum is simple.*

I_b*) for any* $\alpha, \beta \in \mathbb{C}$ *such that*

$$(\alpha + \beta)\eta \neq (N+1)\eta - \sum_{l=1}^{3}(\epsilon_l^- \alpha_l^-) + (2p+1)\pi + k\tau, \qquad \prod_{l=1}^{3} \epsilon_l^+ = 1, \\ \forall (p,k) \in \mathbb{Z},\ \epsilon_l^+ \in \{-1, 1\}, \tag{4.3.6}$$

corresponding to

$$[K_-(\lambda| - \beta + 2)]_{12} \neq 0, \tag{4.3.7}$$

the one parameter family of the gauge transformed generators of the reflection algebra $\mathcal{C}_-(\lambda|\beta)$ *is right pseudo-diagonalizable and its pseudo-spectrum is simple.*

II_b*) for any* $\alpha, \beta \in \mathbb{C}$ *such that*

$$(\alpha + \beta)\eta \neq -(N-1)\eta - \sum_{l=1}^{3}(\epsilon_l^- \alpha_l^-) + (2p+1)\pi + k\tau, \qquad \prod_{l=1}^{3} \epsilon_l^+ = 1, \\ \forall (p,k) \in \mathbb{Z},\ \epsilon_l^+ \in \{-1, 1\}, \tag{4.3.8}$$

corresponding to

$$[\bar{K}_-(\lambda|\beta - 2)]_{12} \neq 0, \tag{4.3.9}$$

the one parameter family of the gauge transformed generators of the reflection algebra $\mathcal{C}_-(\lambda|\beta)$ is right pseudo-diagonalizable and its pseudo-spectrum is simple.

In all these cases we can construct a SoV representation of the gauge transformed reflection algebra.

The proof and some necessary clarifications of the statements contained in this theorem are given by the explicit constructions of the SoV representation in the next subsections. In fact, we build explicitly the representations only for the cases I_a) and II_a) since for cases I_b) and II_b) the construction can be induced from the others due to the symmetries.

4.3.1 Reference states

As it was done for the XXZ model in section §3.3.1, we define here the reference pseudo-states, which define the first step in the construction of the SoV representation.

Definition 4.3.1.

$$\langle\, \beta \,| = N_\beta \bigotimes_{n=1}^{N} \bar{Y}_n(\xi_n|\beta + N - n) \tag{4.3.10}$$

where $\bar{Y}_n(\xi_n|\beta)$ is the gauge transformation acting on the local quantum space $\mathcal{H}_n$ and N_β is the normalization factor

$$N_\beta = \prod_{n=1}^{N} \theta_1((\beta + N - n)\eta). \tag{4.3.11}$$

Proposition 4.3.1. *The state $\langle\, \beta \,|$ is a simultaneous $B(\lambda|\beta)$ and $\bar{B}(\lambda|\beta)$ left reference state:*

$$\langle\, \beta \,| B(\lambda|\beta) = \langle\, \beta \,| \bar{B}(\lambda|\beta) = 0, \tag{4.3.12}$$

$$\langle\, \beta \,| A(\lambda|\beta) = \frac{\theta_1((N+\beta)\eta)}{\theta_1(\beta\eta)} \prod_{n=1}^{N} \theta_1(\lambda - \xi_n + \eta/2) \langle\, \beta - 1 \,|, \tag{4.3.13}$$

$$\langle\, \beta \,| D(\lambda|\beta) = \prod_{n=1}^{N} \theta_1(\lambda - \xi_n - \eta/2) \langle\, \beta + 1 \,|, \tag{4.3.14}$$

$$\langle\, \beta \,| \bar{A}(\lambda|\beta) = \frac{\theta_1(\beta\eta)}{\theta_1((N+\beta)\eta)} \prod_{n=1}^{N} \theta_1(\lambda + \xi_n + \eta/2) \langle\, \beta + 1 \,|, \tag{4.3.15}$$

$$\langle\, \beta \,| \bar{D}(\lambda|\beta) = \prod_{n=1}^{N} \theta_1(\lambda + \xi_n - \eta/2) \langle\, \beta - 1 \,|. \tag{4.3.16}$$

Proof. The proposition can be checked easily for local R-matrix by direct computation. Let's introduce the local vector

$$\langle\, \beta \,|_n = \theta_1((\beta + N - n)\eta) \bar{Y}_n(\xi_n|\beta) \tag{4.3.17}$$

then it follows

$$\begin{aligned} &\langle\, \beta \,|_n \tilde{G}^{-1}(\lambda - \eta/2|\beta + N - n) R_{0n}^{(8V)}(\lambda - \xi_n - \eta/2) \tilde{G}(\lambda - \eta/2|\beta + N - n + 1) \\ &= \begin{pmatrix} \frac{\theta_1((\beta+N-n+1)\eta)\theta_1(\lambda-\xi_n+\eta/2)}{\theta_1((\beta+N-n)\eta)} \langle\, \beta - 1 \,|_n & \underline{0}_n \\ \star & \sinh(\lambda - \xi_n + \eta/2) \langle\, \beta + 1 \,|_n \end{pmatrix}. \end{aligned}$$

This identity can be established by direct computation or, in an easier fashion, by exploiting the elliptic face-vertex relations and the definitions of the gauged bulk operators. Let us start by considering the explicit calculations for the left action of $A_n(\lambda|\beta)$. Once set $\gamma = \beta + N - n$, by definition we have that

$$\langle\, \beta\, |_n\, A_n(\lambda\beta) \equiv \theta_1(\gamma\eta)\left(\bar{Y}_n(\xi_n|\gamma)\tilde{Y}_0(\lambda-\eta/2|\gamma-1)R_{0n}(\lambda-\eta/2-\xi_n)X(\lambda-\eta/2|\beta+2)\right). \tag{4.3.18}$$

By exploiting the connection between the $\bar{Y}$'s and the $\tilde{Y}$'s we get

$$\begin{aligned}\langle\, \beta\, |_n\, A_n(\lambda\beta) = \frac{\theta_1(\xi_n+(\alpha+1)\eta)}{\theta_1(\xi_n+\alpha\eta)}\theta_1((\gamma+1)\eta)\\ \tilde{Y}_n(\xi_n|\gamma)\tilde{Y}_0(\lambda-\eta/2|\gamma-1)R_{0n}(\lambda-\eta/2-\xi_n)X(\lambda-\eta/2|\gamma+2).\end{aligned}$$

Now by using the elliptic version of (A.1.1f), we remember obtained by letting $\sinh(\cdot) \to \theta_1(\cdot)$, we arrive at

$$\langle\, \beta\, |_n\, A_n(\lambda\beta) = \frac{\theta_1(\xi_n+(\alpha+1)\eta)}{\theta_1(\xi_n+\alpha\eta)}\theta_1((\gamma+1)\eta)\theta_1(\lambda-\xi_n-\eta/2)\tilde{Y}_n(\xi_n|\gamma-1)$$

where, after the orthogonality condition (4.2.9a), the result $\tilde{Y}(\lambda-\eta/2|\gamma)X(\lambda-\eta/2\gamma+2) = 1$ has been used. Finally by reverting back all notation, in order to be coherent with the definition of the state (4.3.10)-(4.3.11) we arrive at the result displayed above. Similarly for the action of $D_n(\lambda|\beta)$ one should exploit the elliptic face-vertex relation equivalent to (A.1.1e) after noticing that

$$\begin{aligned}&\tilde{Y}(\xi_n|\gamma)\tilde{X}(\lambda-\eta/2|\gamma+1)R_{0n}(\lambda-\xi_n-\eta/2)\\ &\qquad\equiv \tilde{Y}(\xi_n-\eta|\gamma-1)\tilde{X}(\lambda-\eta/2-\eta|\gamma+2)R_{0n}(\lambda-\xi_n-\eta/2),\end{aligned}$$

and, for the orthogonality condition (4.2.9a), that $\tilde{Y}_0(\lambda-\eta/2\gamma)Y_0(\lambda-\eta/2|\gamma) = 0$. The same can be calculated for the gauged elements of the dual bulk monodromy matrix as well, resulting in

$$\begin{aligned}&-\ \langle\, \beta\, |_n\, \bar{G}^{-1}(\eta/2-\lambda|\beta+N-n+1)\sigma_0^y[R_{0n}^{(8V)}(-\lambda-\xi_n-\eta/2)]^{t_0}\sigma_0^y\bar{G}(\eta/2-\lambda|\beta+N-n)\\ &=\ \langle\, \beta\, |_n\, \bar{G}^{-1}(\eta/2-\lambda|\beta+N-n+1)R_{0n}^{(8V)}(\lambda+\xi_n-\eta/2)\bar{G}(\eta/2-\lambda|\beta+N-n)\\ &=\begin{pmatrix}\frac{\theta_1((\beta+N-n)\eta)\theta_1(\lambda+\xi_n+\eta/2)}{\theta_1((\beta+N-n+1)\eta)}\langle\, \beta+1\, |_n & \underline{0}_n\\ \star & \theta_1(\lambda+\xi_n+\eta/2)\langle\, \beta-1\, |_n\end{pmatrix}.\end{aligned}$$

□

In order to have a complete SoV construction, a right reference has to be defined as well.

Definition 4.3.2.

$$|\, \beta\, \rangle = \bigotimes_{n=1}^{N} X_n(\xi_n|\beta+N-n). \tag{4.3.19}$$

We can then generate the results equivalent to proposition 4.3.1 for the right reference state too.

Proposition 4.3.2. *The state* $|\,\beta+1\,\rangle$ *is a simultaneous* $C(\lambda|\beta)$ *and* $\bar{C}(\lambda|\beta)$ *left reference state:*

$$C(\lambda|\beta)\,|\,\beta+1\,\rangle=\bar{C}(\lambda|\beta)\,|\,\beta+1\,\rangle=0, \tag{4.3.20}$$

$$A(\lambda|\beta)\,|\,\beta+1\,\rangle=\prod_{n=1}^{N}\sinh(\lambda-\xi_n+\eta/2)\,|\,\beta+2\,\rangle\,, \tag{4.3.21}$$

$$D(\lambda|\beta)\,|\,\beta+1\,\rangle=\frac{\sinh(N+\beta)\eta}{\sinh\beta\eta}\prod_{n=1}^{N}\sinh(\lambda-\xi_n-\eta/2)\,|\,\beta\,\rangle\,, \tag{4.3.22}$$

$$\bar{A}(\lambda|\beta)\,|\,\beta+1\,\rangle=\prod_{n=1}^{N}\sinh(\lambda+\xi_n-\eta/2)\,|\,\beta\,\rangle\,, \tag{4.3.23}$$

$$\bar{D}(\lambda|\beta)\,|\,\beta+1\,\rangle=\frac{\sinh\beta\eta}{\sinh(N+\beta)\eta}\prod_{n=1}^{N}\sinh(\lambda+\xi_n-\eta/2)\,|\,\beta+2\,\rangle\,. \tag{4.3.24}$$

Proof. The proposition can be checked as it was done for the left representation. Consider the local vector

$$|\,\beta+1\,\rangle_n=X_n(\xi_n|\beta+N+-n+1)$$

then it follows

$$\begin{aligned}&\tilde{G}^{-1}(\lambda-\eta/2|\beta+N-n)R_{0n}^{(8V)}(\lambda-\xi_n-\eta/2)\tilde{G}(\lambda-\eta/2|\beta+N-n+1)\,|\,\beta+1\,\rangle_n\\&\quad=\begin{pmatrix}\theta_1(\lambda-\xi_n+\eta/2)\,\langle\,\beta+2\,|_n & \star\\ \underline{0}_n & \frac{\theta_1((\beta+N-n+1)\eta)\theta_1(\lambda-\xi_n-\eta/2)}{\theta_1((\beta+N-n)\eta)}\,|\,\beta\,\rangle_n\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}&-\bar{G}^{-1}(\eta/2-\lambda|\beta+N-n+1)\sigma_0^y[R_{0n}^{(8V)}(-\lambda-\xi_n-\eta/2)]^{t_0}\sigma_0^y\\&\qquad\qquad\times\;\bar{G}(\eta/2-\lambda|\beta+N-n)\,|\,\beta+1\,\rangle_n\\&=\bar{G}^{-1}(\eta/2-\lambda|\beta+N-n+1)R_{0n}^{(8V)}(\lambda+\xi_n-\eta/2)\bar{G}(\eta/2-\lambda|\beta+N-n)\,|\,\beta+1\,\rangle_n\\&=\begin{pmatrix}\theta_1(\lambda+\xi_n+\eta/2)\,|\,\beta+2\,\rangle_n & \star\\ \underline{0}_n & \frac{\theta_1((\beta+N-n)\eta)\theta_1(\lambda+\xi_n-\eta/2)}{\theta_1((\beta+N-n+1)\eta)}\,|\,\beta+2\,\rangle_n\end{pmatrix}.\end{aligned}$$

□

4.3.2 $\mathcal{B}_-(\lambda|\beta)$-SoV representations of the gauge transformed reflection algebra

The left $\mathcal{B}_-(|\beta)$-pseudo-eigenbasis is here constructed and the representation of the gauge transformed boundary operator $\mathcal{A}_-(\lambda|\beta)$ in this basis is determined. In the following we will need the following notations

$$\zeta_{-1}\equiv\eta/2,\quad\zeta_{-2}\equiv(\eta-\pi)/2,\quad\zeta_{-3}\equiv(\eta-\pi\omega)/2,\quad\zeta_{-4}\equiv(\eta-\pi\omega-\pi)/2 \tag{4.3.25}$$

and

$$\zeta_n^{(h_n)}=\varphi_n\left[\xi_n+(h_n-\frac{1}{2})\eta\right]\quad\forall n\in\{1,\ldots,2N\}, \tag{4.3.26}$$

where $h_{n+N}\equiv h_n\in\{0,1\}$, and

$$\varphi_a=1\quad\text{for}\quad a\leq N\quad\text{and}\quad\varphi_a=-1\quad\text{for}\quad a>N. \tag{4.3.27}$$

In a completely analog way to XXZ SoV analysis, the objects in (4.3.26) will coincide with the eigenvalues of the operator roots of $\mathcal{B}_-(\lambda|\beta)$, *i.e.* the *separated variables*.

It will be also useful ti define the concept of order-M elliptic polynomial.

Definition 4.3.3. Given a function $f(\lambda) : \mathbb{C} \to \mathbb{C}$, it will be said to be equivalent to or to behave like an order-M elliptic polynomial of periods π and $\tau\pi$ if it satisfies the following quasi-periodicity properties

$$f(\lambda+\pi) = \pm f(\lambda), \quad f(\lambda+\tau\pi) = \left(\pm e^{-2i(\lambda)}/q\right)^M e^{2i\alpha_f} f(\lambda), \tag{4.3.28}$$

where $q = e^{i\tau\pi}$ and the term α_f depends on the explicit expression of the function $f(\lambda)$. The same polynomial $f(\lambda)$ can be considered with periods π and $2\pi\tau$ as well, resulting in the following properties

$$f(\lambda+\pi) = \pm f(\lambda), \quad f(\lambda+2\tau\pi) = \left(e^{-2i(\lambda)}/\hat{q}^2\right)^{2M} e^{2i\hat{\alpha}_f} f(\lambda), \tag{4.3.29}$$

where $\hat{q} = e^{2i\tau\pi}$.

Left $\mathcal{B}_-(\lambda|\beta)$-SoV representation of the gauge transformed reflection algebra

In this subsection we construct the left $\mathcal{B}_-(\lambda|\beta)$-pseudo-eigenbasis.

Theorem 4.3.2. *Left $\mathcal{B}_-(\lambda|\beta)$ SoV-basis The following states:*

$$\langle\, \beta, h_1, \dots, h_N \,| = \langle\, \beta \,| \prod_{n=1}^{N} \left(\frac{\mathcal{A}_-(\eta/2 - \xi_n|\beta+2)}{\mathsf{A}_-(\eta/2-\xi_n)} \right)^{h_n}, \tag{4.3.30}$$

where $\langle\, \beta \,|$ *is the state defined in* (4.3.10) *and*

$$\mathsf{A}_-(\lambda) = r(\lambda)\widehat{\mathsf{A}}_-(\lambda) \tag{4.3.31}$$

with the function $\widehat{\mathsf{A}}_-$ being given by (4.1.33). *Let us assume that* (4.3.1) *and* (4.3.2) *are satisfied, then the states* (4.3.30) *define a basis formed out of pseudo-eigenstates of $\mathcal{B}_-(\lambda|\beta)$*

$$\langle\, \beta, \mathbf{h} \,| \mathcal{B}_-(\lambda|\beta) = \mathsf{B}_{\beta,\mathbf{h}}(\lambda) \,\langle\, \beta-2, \mathbf{h} \,|, \tag{4.3.32}$$

where $\bar{\beta}\mathbf{h} \equiv \langle\, \beta, h_1, ..., h_N \,|$ for $\mathbf{h} \equiv (h_1, ..., h_N)$ and

$$\mathsf{B}_{\beta,\mathbf{h}}(\lambda) \equiv (-1)^N \frac{\theta_1((N+\beta)\eta)}{\theta_1(\beta\eta)} \theta_1\left(\lambda + (\alpha+1/2)\eta\right) [K_-(\lambda|\beta)]_{12} a_{\mathbf{h}}(\lambda) a_{\mathbf{h}}(-\lambda), \tag{4.3.33}$$

with

$$a_{\mathbf{h}}(\lambda) \equiv \prod_{n=1}^{N} \theta_1(\lambda - \xi_n - (h_n - \frac{1}{2})\eta). \tag{4.3.34}$$

Moreover, $\mathcal{B}_-(\lambda|\beta)$ is an order $4N+8$ elliptic polynomial of periods π and $2\pi\omega$

$$\mathcal{B}_-(\lambda+\pi|\beta) = \mathcal{B}_-(\lambda|\beta), \quad \mathcal{B}_-(\lambda+2\pi\omega|\beta) = \left(e^{-2i\lambda}/q^2\right)^{4N+8} e^{2i\alpha_{\mathcal{B}_-(\beta)}} \mathcal{B}_-(\lambda|\beta), \tag{4.3.35}$$

where $q \equiv e^{i\pi\omega}$ and $\alpha_{\mathcal{B}_-(\beta)} = 4\eta$. $\mathcal{A}_-(\lambda|\beta)$ is an order $4N+8$ elliptic polynomial of periods π and $2\pi\omega$

$$\mathcal{A}_-(\lambda+2\pi\omega|\beta) = \left(e^{-2i\lambda}/q^2\right)^{4N+8} e^{2i\alpha_{\mathcal{A}_-(\beta)}} \mathcal{A}_-(\lambda|\beta), \tag{4.3.36}$$

$$\mathcal{A}_-(\lambda+\pi|\beta) = \mathcal{A}_-(\lambda|\beta), \text{ where } \alpha_{\mathcal{A}_-(\beta)} \equiv 2(\beta+2)\eta. \tag{4.3.37}$$

Defined the operator $\mathcal{A}_-^{(0)}(\lambda|\beta+2)$ *by the following action on the generic state* $\langle\,\beta,\mathbf{h}\,|$

$$\langle\,\beta,\mathbf{h}\,|\,\mathcal{A}_-^{(0)}(\lambda|\beta+2)\equiv\sum_{a=1}^{4}\frac{\theta_1(2(\beta+4)\eta-\lambda-\sum_{b=1,b\neq a}^{4}\zeta_{-b})}{\theta_1(2(\beta+4)\eta-\sum_{b=1}^{4}\zeta_{-b})}\frac{a_{\mathbf{h}}(\lambda)a_{\mathbf{h}}(-\lambda)}{a_{\mathbf{h}}(\zeta_{-a})a_{\mathbf{h}}(-\zeta_{-a})}\times\prod_{b=1,b\neq a}^{4}\frac{\theta_1(\lambda-\zeta_{-b})}{\theta_1(\zeta_{-a}-\zeta_{-b})}\langle\,\beta,\mathbf{h}\,|\,\mathcal{A}_-(\zeta_{-a}|\beta+2),\tag{4.3.38}$$

then the operator

$$\widetilde{\mathcal{A}}_-(\lambda|\beta+2)\equiv\mathcal{A}_-(\lambda|\beta+2)-\mathcal{A}_-^{(0)}(\lambda|\beta+2),\tag{4.3.39}$$

has the following action on the generic state $\langle\,\beta,\boldsymbol{h}\,|$

$$\begin{aligned}\langle\,\beta,\mathbf{h}\,|\widetilde{\mathcal{A}}_-(\lambda|\beta+2)&\\ =\sum_{a=1}^{2N}&\frac{\theta_4(2\lambda-\eta|2\omega)\theta_1(2\lambda-\eta|2\omega)\theta_1(2(\beta+4)\eta+\zeta_a^{(h_a)}-\lambda-\sum_{b=1}^{4}\zeta_{-b})}{\theta_4(2\zeta_a^{(h_a)}-\eta|2\omega)\theta_1(2\zeta_a^{(h_a)}-\eta|2\omega)\theta_1(2(\beta+4)\eta-\sum_{b=1}^{4}\zeta_{-b})}\\ &\times\frac{\theta_1(\lambda+\zeta_a^{(h_a)})\theta_2^{2(N-1)}(\lambda)}{\theta_1(2\zeta_a^{(h_a)})\theta_2^{2(N-1)}(\zeta_a^{(h_a)})}\prod_{\substack{b=1\\ b\neq a\,modN}}^{N}\frac{\frac{\theta_4^2(\lambda)}{\theta_2^2(\lambda)}-\frac{\theta_4^2(\zeta_b^{(h_b)})}{\theta_2^2(\zeta_b^{(h_b)})}}{\frac{\theta_4^2(\zeta_a^{(h_a)})}{\theta_2^2(\zeta_a^{(h_a)})}-\frac{\theta_4^2(\zeta_b^{(h_b)})}{\theta_2^2(\zeta_b^{(h_b)})}}\mathsf{A}_-(\zeta_a^{(h_a)})\langle\,\beta,\mathbf{h}\,|\,T_a^{-\varphi_a}\end{aligned}\tag{4.3.40}$$

where

$$\langle\,\beta,h_1,...,h_a,...,h_N\,|\,T_a^{\pm}=\langle\,\beta,h_1,...,h_a\pm1,...,h_N\,|\,.\tag{4.3.41}$$

Proof. Let us start in pointing out that the states $\langle\,\beta,\mathbf{h}\,|$ are well defined states; i.e. their definition does not depend on the order of operator $\mathcal{A}_-(-\zeta_b^{(0)}|\beta+2)$ as one can verify directly from the commutation relations (4.2.28). The following boundary-bulk decomposition

$$\begin{aligned}\frac{\mathcal{B}_-(\lambda|\beta)}{\theta_1\left(\lambda+(\alpha+1/2)\eta\right)}=&K_-(\lambda|\beta)_{22}B(\lambda|\beta)\bar{D}(\lambda|\beta-1)+K_-(\lambda|\beta)_{11}A(\lambda|\beta)\bar{B}(\lambda|\beta-1)\\&+K_-(\lambda|\beta)_{21}B(\lambda|\beta)\bar{B}(\lambda|\beta-1)+K_-(\lambda|\beta)_{12}A(\lambda|\beta)\bar{D}(\lambda|\beta-1),\end{aligned}\tag{4.3.42}$$

of the gauge transformed reflection algebra generator $\mathcal{B}_-(\lambda|\beta)$ in terms of the gauge transformed bulk generators and the formulae (4.3.12)-(4.3.16) imply that $\langle\,be\,|$ is a $\mathcal{B}_-(\lambda|\beta)$-pseudo-eigenstate

$$\langle\,\beta\,|\,\mathcal{B}_-(\lambda)\equiv\mathsf{B}_{\beta,\mathbf{0}}(\lambda)\,\langle\,\beta-2\,|\,,\tag{4.3.43}$$

with non-zero pseudo-eigenvalue

$$\mathsf{B}_{\beta,\mathbf{0}}(\lambda)=(-1)^N\frac{\theta_1((N+\beta)\eta)}{\theta_1(\beta\eta)}[K_-(\lambda|\beta)]_{12}\theta_1\left(\lambda+(\alpha+1/2)\eta\right)a_{\mathbf{0}}(\lambda)a_{\mathbf{0}}(-\lambda),\tag{4.3.44}$$

where

$$[K_-(\lambda|\beta)]_{12} = \frac{\theta_1(2\lambda-\eta)}{2\theta_1(\lambda+(\alpha+1/2)\eta)\theta_1((\beta+N)\eta)\prod_{l=1}^{3}\alpha_l^+} \times \{\theta_3((\alpha-\beta-N+1)\eta)\prod_{l=1}^{3}\theta_3(\alpha_l^-)+\theta_2((\alpha-\beta-N+1)\eta)\prod_{l=1}^{3}\theta_2(\alpha_l^-) - \theta_1((\alpha-\beta-N+1)\eta)\prod_{l=1}^{3}\theta_1(\alpha_l^-)-\theta_4((\alpha-\beta-N+1)\eta)\prod_{l=1}^{3}\theta_4(\alpha_l^-)\}, \tag{4.3.45}$$

from which condition (4.3.2) follows. We can use now step by step the procedure used for the XXZ model in section §3.3.2 to prove the validity of (4.3.33) starting from the gauged transformed reflection algebra commutation relations. Under the condition (4.3.1) the operators zeros of $\mathcal{B}_-(\lambda|\beta)$ (the separate quantum variables) have disjoint spectrum. Indeed, under these conditions we can prove that the set of non-zero pseudo-eigenvalues of $\mathcal{B}_-(\lambda|\beta)$ defined by the $\mathsf{B}_{\beta,\mathbf{h}}(\lambda)$, when $\mathbf{h}=(h_1,...,h_N)$ takes values in $\{0,1\}^{\otimes N}$, defines a set of $2N$ different elliptic polynomials. Then we can prove that the set of states $\langle\ \beta,\mathbf{h}\ |$ forms a set of $2N$ independent states, i.e. a $\mathcal{B}_-(\lambda|\beta)$-pseudo-eigenbasis of the left representation space. Let us stress the fact that this last property is essential to apply the SoV method as we need to use this SoV basis to represent all the transfer matrix eigenstates. Moreover, the definition of the states $\langle\ \beta,\mathbf{h}\ |$ and the commutation relation (4.2.26) allow to define the action of $\mathcal{A}_-(\zeta_b^{(h_b)}|\beta+2)$ for $b\in\{1,...,2N\}$ once we use the quantum determinant relations and the following conditions

$$\langle\ \beta\ |\ \mathcal{A}_-(\xi_n-\eta/2|\beta+2)=0, \quad \langle\ \beta\ |\ \mathcal{A}_-(\eta/2-\xi_n|\beta+2)\neq 0 \tag{4.3.46}$$

which trivially follows from the boundary-bulk decomposition

$$\frac{\mathcal{A}_-(\lambda|\beta+2)}{\theta\,(\lambda+(\alpha+1/2)\eta)} = \bar{K}_-(\lambda|\beta)_{11}A(\lambda|\beta)\bar{A}(\lambda|\beta+1)+\bar{K}_-(\lambda|\beta)_{22}B(\lambda|\beta)\bar{C}(\lambda|\beta+1) + \bar{K}_-(\lambda|\beta)_{21}B(\lambda|\beta)\bar{A}(\lambda|\beta+1)+\bar{K}_-(\lambda|\beta)_{12}A(\lambda|\beta)\bar{C}(\lambda|\beta+1). \tag{4.3.47}$$

The fact that the operator $\mathcal{B}_-(\lambda|\beta)$ is an order $4N+8$ elliptic polynomials of periods π and $2\pi\omega$ which satisfies (4.3.35) can be simply derived from the functional form of its pseudo-eigenvalues and the quasi-periodicity properties of the K-matrices (4.1.30)-(4.1.32). One should just use the identities (B.1.5) , which we display here

$$\theta_a(x+\pi|2\omega)=(-1)^{\delta_{a,1}+\delta_{a,2}}\,\theta_a(x|2\omega), \quad \theta_a(x+2\pi\omega|2\omega)=(-1)^{\delta_{a,1}+\delta_{a,4}}\,e^{-2i(x+\pi\omega)}\theta_a(x|2\omega), \tag{4.3.48}$$

from which also follows

$$\theta_1(x+\pi)=-\theta_1(x),\ \ \theta_1(x+2\pi\omega)=e^{-4i(x+\pi\omega)}\theta_1(x). \tag{4.3.49}$$

One should pay attention to the fact that the eigenvalues formulae (4.3.33) and (4.3.44) might be considered as polynomials of periods π and $\pi\omega$ as well. In particular the semi-period $\pi\omega$ seems to be more natural than its double since all functions appearing in the above expressions contain exclusively theta functions with period $\pi\omega$. The reasons for choosing the doubled quasi-period, and consequently the doubled order of the polynomial, resides mainly on the gentler form of the coefficient $\alpha_{\mathcal{B}_-}$ appearing in (4.3.35), which would contain unwanted $\pi\omega$ terms. Anyway this choice won't affect our analysis and the interpolation formula should be understood as a working form expression for $\mathcal{A}_-(\lambda|\beta)$. Indeed, we will see how the transfer matrix is effectively an elliptic polynomial of half quasi-period $\pi\omega$. The fact that the operator $\mathcal{A}_-(\lambda|\beta)$ is an order $4N+8$ elliptic polynomial of periods π and $2\pi\omega$

which satisfies (4.3.36)-(4.3.37) can be simply derived from (4.3.35) by using the commutation relations (4.2.26). Indeed, shifting the variable λ_2 in $\lambda_2 + 2\pi\omega$ and using the transformation properties (4.3.35) and (4.3.49), we get

$$
\begin{aligned}
&f_{\mathcal{A}_-(\beta+2)}(\lambda_2)\mathcal{A}_-(\lambda_2|\beta+2)\mathcal{B}_-(\lambda_1|\beta) = \\
&\quad \frac{\theta(\lambda_1-\lambda_2+\eta)\theta(\lambda_2+\lambda_1-\eta)}{\theta(\lambda_1-\lambda_2)\theta(\lambda_1+\lambda_2)} e^{8i\eta} f_{\mathcal{A}_-(\beta)}(\lambda_2)\mathcal{B}_-(\lambda_1|\beta)\mathcal{A}_-(\lambda_2|\beta) \\
&\quad + \frac{\theta(\lambda_1+\lambda_2-\eta)\theta(\lambda_1-\lambda_2+(\beta-1)\eta)\theta(\eta)}{\theta(\lambda_2-\lambda_1)\theta(\lambda_1+\lambda_2)\theta((\beta-1)\eta)e^{-4i\beta\eta}} f_{\mathcal{B}_-(\beta)}(\lambda_2)\mathcal{B}_-(\lambda_2|\beta)\mathcal{A}_-(\lambda_1|\beta) \\
&\quad + \frac{\theta(\eta)\theta(\lambda_1+\lambda_2-\beta\eta)}{\theta(\lambda_1+\lambda_2)\theta((\beta-1)\eta)} e^{4i\beta\eta} f_{\mathcal{B}_-(\beta)}(\lambda_2)\mathcal{B}_-(\lambda_2|\beta)\mathcal{D}_-(\lambda_1|\beta).
\end{aligned}
$$

where $f_{\mathcal{A}_-(\beta)}(\lambda)$ is defined by

$$\mathcal{A}_-(\lambda+2\pi\omega|\beta) = f_{\mathcal{A}_-(\beta)}(\lambda)\mathcal{A}_-(\lambda|\beta), \tag{4.3.50}$$

which implies

$$f_{\mathcal{A}_-(\beta)}(\lambda) \equiv \left(e^{-2i\lambda}/q^2\right)^{4N+8} e^{2i\alpha_{\mathcal{A}_-(\beta)}} \text{ where } \alpha_{\mathcal{A}_-(\beta)} \equiv 2(\beta+4)\eta. \tag{4.3.51}$$

Now, by the definition (4.3.38) it is simple to argue that the operator $\mathcal{A}_-^{(0)}(\lambda|\beta)$ is also an order $4N+8$ elliptic polynomial of periods π and $2\pi\omega$ which satisfies (4.3.36) and (4.3.37). Then the same is true for $\widetilde{\mathcal{A}}_-(\lambda|\beta)$. These properties together with the identities

$$\widetilde{\mathcal{A}}_-(\,\zeta_{-a}|\beta) \equiv 0 \text{for any } a \in \{1,...,8\}, \tag{4.3.52}$$

imply the expression (4.3.40), where the following interpolation formula has to be taken into account [5]

$$\mathcal{P}(\lambda) = \sum_{a=1}^{\mathsf{M}} \frac{\theta_1(\alpha_{\mathcal{P}}+x_a-\lambda-\sum_{n=1}^{\mathsf{M}} x_n)}{\theta_1(\alpha_{\mathcal{P}}-\sum_{n=1}^{\mathsf{M}} x_n)} \prod_{b\neq a} \frac{\theta_1(\lambda-x_b)}{\theta_1(x_a-x_b)} \mathcal{P}(x_a), \tag{4.3.53}$$

which holds true for any order M elliptic polynomial such that

$$\mathcal{P}(\lambda+\pi) = (-1)^{\mathsf{M}}\mathcal{P}(\lambda), \;\; \mathcal{P}(\lambda+2\pi\omega) = \left(e^{-2i\lambda}/q^2\right)^{2\mathsf{M}} e^{2i\alpha_{\mathcal{P}}}\mathcal{P}(\lambda). \tag{4.3.54}$$

□

Right $\mathcal{B}_-(\lambda|\beta)$-SoV representation of the gauge transformed reflection algebra

The right $\mathcal{B}_-(|\beta)$-pseudo-eigenbasis is here constructed and the representation of the gauge transformed boundary operator $\mathcal{D}_-(\lambda|\beta)$ in this basis is determined. Let us use the following notation

$$\overline{|\beta\rangle} \equiv |-\beta+2\rangle, \tag{4.3.55}$$

where $|\ \beta\ \rangle$ is the right reference state defined in (4.3.19).

Theorem 4.3.3. *Right $\mathcal{B}_-(\lambda|\beta)$ SoV-basis The following states:*

$$|\ \beta, h_1, ..., h_N\ \rangle \equiv \prod_{n=1}^{N} \left(\frac{\mathcal{D}_-(\xi_n+\eta/2|\beta)}{k_n^{(\beta)}\mathsf{A}_-(\eta/2-\xi_n)}\right)^{(1-h_n)} \overline{|\beta\rangle}, \tag{4.3.56}$$

where

$$k_a^{(\beta)} \equiv \frac{\theta_1\,(2\xi_a+\eta)\,\theta_1\,(\beta\eta)\,\theta_1(2(4-\beta)\eta - \sum_{b=1}^{4}\zeta_{-b} - 2\xi_a)\theta_1(\eta)\theta_2^{2(N-1)}(\zeta_a^{(1)})}{\theta_1(\eta)\theta_1\,(2\xi_a+\beta\eta)\,\theta_1(2(4-\beta) - \sum_{b=1}^{4}\zeta_{-b})\theta_1(2\zeta_a^{(0)})\theta_2^{2(N-1)}(\zeta_a^{(0)})}, \tag{4.3.57}$$

$h_n \in \{0,1\}$, $n \in \{1,...,N\}$. *If conditions* (4.3.1) *and* (4.3.4) *then the states* $|\,\beta,\mathbf{h}\,\rangle$ *define a basis formed out of* $\mathcal{B}_-(\lambda|\beta)$*-pseudo-eigenstates*

$$\mathcal{B}_-(\lambda|\beta)\,|\,\beta,\mathbf{h}\,\rangle = |\,\beta+2,\mathbf{h}\,\rangle\,\mathsf{B}_{\beta,\mathbf{h}}(\lambda), \tag{4.3.58}$$

where

$$\mathsf{B}_{\beta,\mathbf{h}}(\lambda) \equiv (-1)^N\,[\bar{K}_-(\lambda|-\beta)]_{21}\frac{\theta\,(\lambda+(\alpha+1/2)\eta)\,\theta_1(\eta(\beta-N))}{\theta_1\,(\beta\eta)\left(\prod_{n=1}^{N} k_n^{(\beta)}/k_n^{(\beta+2)}\right)}a_{\mathbf{h}}(\lambda)a_{\mathbf{h}}(-\lambda). \tag{4.3.59}$$

The operator $\mathcal{D}_-(\lambda|\beta)$ *is an order* $4N+8$ *elliptic polynomials of periods* π *and* $2\pi\omega$

$$\mathcal{D}_-(\lambda+2\pi\omega|\beta) = \left(-e^{-2i\lambda}/q^2\right)^{4N+8} e^{2i\alpha_{\mathcal{D}_-(\beta)}}\mathcal{D}_-(\lambda|\beta), \tag{4.3.60}$$

$$\mathcal{D}_-(\lambda+\pi|\beta) = \mathcal{D}_-(\lambda|\beta), \;\; \textit{where}\; \alpha_{\mathcal{D}_-(\beta)} \equiv 2(4-\beta)\eta. \tag{4.3.61}$$

Defined the operator$\mathcal{D}_-^{(0)}(\lambda|\beta)$ *by the following action on the generic state* $|\,\beta,\mathbf{h}\,\rangle$

$$\begin{aligned}\mathcal{D}_-^{(0)}(\lambda|\beta+2)\,|\,\beta,\mathbf{h}\,\rangle \equiv \sum_{a=1}^{4}&\frac{\theta_1(2(4-\beta)\eta-\lambda-\sum_{b=1,b\neq a}^{4}\zeta_{-b})}{\theta_1(2(4-\beta)\eta-\sum_{b=1}^{4}\zeta_{-b})}\frac{a_{\mathbf{h}}(\lambda)a_{\mathbf{h}}(-\lambda)}{a_{\mathbf{h}}(\zeta_{-a})a_{\mathbf{h}}(-\zeta_{-a})}\\ &\times\prod_{b=1,b\neq a}^{4}\frac{\theta_1(\lambda-\zeta_{-b})}{\theta_1(\zeta_{-a}-\zeta_{-b})}\mathcal{D}_-(\zeta_{-a}|\beta+2)\,|\,\beta,\mathbf{h}\,\rangle,\end{aligned} \tag{4.3.62}$$

then the operator

$$\widetilde{\mathcal{D}}_-(\lambda|\beta) \equiv \mathcal{D}_-(\lambda|\beta) - \mathcal{D}_-^{(0)}(\lambda|\beta), \tag{4.3.63}$$

has the following action on the generic state $|\,\beta,\mathbf{h}\,\rangle$

$$\begin{aligned}&\widetilde{\mathcal{D}}_-(\lambda|\beta+2)\,|\,\beta,\mathbf{h}\,\rangle\\ &\quad=\sum_{a=1}^{2N}\frac{\theta_4(2\lambda-\eta|2\omega)\theta_1(2\lambda-\eta|2\omega)\theta_1(2(4-\beta)\eta+\zeta_a^{(h_a)}-\lambda-\sum_{b=1}^{8}\zeta_{-b})}{\theta_4(2\zeta_a^{(h_a)}-\eta|2\omega)\theta_1(2\zeta_a^{(h_a)}-\eta|2\omega)\theta_1(2(2-\beta)\eta-\sum_{b=1}^{4}\zeta_{-b})}\\ &\quad\times\frac{\theta_1(\lambda+\zeta_a^{(h_a)})\theta_2^{2(N-1)}(\lambda)}{\theta_1(2\zeta_a^{(h_a)})\theta_2^{2(N-1)}(\zeta_a^{(h_a)})}\prod_{\substack{b=1\\ b\neq a\ modN}}^{N}\frac{\frac{\theta_4^2(\lambda)}{\theta_2^2(\lambda)}-\frac{\theta_4^2(\zeta_b^{(h_b)})}{\theta_2^2(\zeta_b^{(h_b)})}}{\frac{\theta_4^2(\zeta_a^{(h_a)})}{\theta_2^2(\zeta_a^{(h_a)})}-\frac{\theta_4^2(\zeta_b^{(h_b)})}{\theta_2^2(\zeta_b^{(h_b)})}}\mathsf{D}_-(\zeta_a^{(h_a)})T_a^{-\varphi_a}\,|\,\beta,\mathbf{h}\,\rangle\end{aligned} \tag{4.3.64}$$

where

$$\mathsf{D}_-(\zeta_a^{(h_a)}) = (k_a^{(\beta)})^{\varphi_a}\mathsf{A}_-(\zeta_a^{(h_a)}-2\varphi_a\xi_a), \qquad T_a^{\pm}\,|\,\beta,h_1,...,h_a,...,h_N\,\rangle = |\,\beta,h_1,...,h_a\pm 1,...,h_N\,\rangle\,. \tag{4.3.65}$$

Proof. The proof follows as in the previous theorem. Let us first prove that $\overline{|\,\beta\,\rangle}$ is a right $\mathcal{B}_-(\lambda|\beta)$-pseudo-eigenstate. From the Proposition 4.3.2 and the following boundary-bulk decomposition

$$\begin{aligned}&\frac{\mathcal{C}_-(\lambda|\beta)}{\theta\left(\lambda+(\alpha+1/2)\eta\right)}\\&\quad=\bar{K}_-(\lambda|\beta-2)_{11}C(\lambda|\beta-2)\bar{A}(\lambda|\beta-1)+\bar{K}_-(\lambda|\beta-2)_{22}D(\lambda|\beta-2)\bar{C}(\lambda|\beta-1)\\&\quad+\bar{K}_-(\lambda|\beta-2)_{12}C(\lambda|\beta-2)\bar{C}(\lambda|\beta-1)+\bar{K}_-(\lambda|\beta-2)_{21}D(\lambda|\beta-2)\bar{A}(\lambda|\beta-1),\end{aligned} \tag{4.3.66}$$

it follows that the state $|\,\beta\,\rangle$ is a right $\mathcal{C}_-(\lambda|\beta)$-pseudo-eigenstate; i.e. it holds

$$\mathcal{C}_-(\lambda|\beta)|\beta\rangle=|\beta-2\rangle\mathsf{C}_\beta(\lambda) \tag{4.3.67}$$

where

$$\mathsf{C}_\beta(\lambda)=(-1)^N\left[\bar{K}_-(\lambda|\beta-2)\right]_{21}\theta_1\left(\lambda+(\alpha+1/2)\eta\right)\frac{\theta_1(\eta(N+\beta-2))}{\theta_1\left(\eta(\beta-2)\right)}a_{\mathbf{1}}(\lambda)a_{\mathbf{1}}(-\lambda). \tag{4.3.68}$$

The boundary matrix element appearing in (4.3.68) satisfies the identity

$$[\bar{K}_-(\lambda|\beta)]_{21}=[K_-(\lambda|-\beta+2N)]_{12} \tag{4.3.69}$$

from which the condition (4.3.4) can be understood, once the shift $\beta\to-\beta+2$ has been taken into account.

From the identities (4.2.41), it follows that the formula (4.3.67) is equivalent to the following one

$$\mathcal{B}_-(\lambda|\beta)\overline{|\,\beta\,\rangle}=\overline{|\,\beta+2\,\rangle}\mathsf{C}_{-\beta+2}(\lambda). \tag{4.3.70}$$

Then by using the identities (4.3.70), the commutation relations (4.2.27) and the formulae

$$\mathcal{D}_-(-\xi_n-\eta/2|\beta)\overline{|\,\beta\,\rangle}=0,\qquad\mathcal{D}_-(\xi_n+\eta/2|\beta)\overline{|\,\beta\,\rangle}\neq0, \tag{4.3.71}$$

the states (4.3.56) are proven to be non-zero $\mathcal{B}_-(\lambda|\beta)$-pseudo-eigenstates with pseudo-eigenvalues $\mathsf{B}_{\beta,\mathbf{h}}(\lambda)$ which then forms a basis of $\mathcal{H}$. The fact that the operator $\mathcal{D}_-(\lambda|\beta)$ is an order $4N+8$ elliptic polynomials of periods π and $2\pi\omega$ which satisfies (4.3.60) and (4.3.61) can be simply derived from (4.3.35) by using the commutation relations (4.2.27). Indeed, shifting the variable λ_2 in $\lambda_2+2\pi\omega$ and using the transformation properties (4.3.35) and (4.3.49), we get

$$\begin{aligned}&f_{\mathcal{D}_-(\beta)}(\lambda_2)\mathcal{B}_-(\lambda_1|\beta)\mathcal{D}_-(\lambda_2|\beta)\\&\quad=\frac{\theta_1(\lambda_1-\lambda_2+\eta)\theta_1(\lambda_2+\lambda_1-\eta)}{\theta_1(\lambda_1-\lambda_2)\theta_1(\lambda_1+\lambda_2)}e^{8i\eta}f_{\mathcal{D}_-(\beta+2)}(\lambda_2)\mathcal{D}_-(\lambda_2|\beta+2)\mathcal{B}_-(\lambda_1|\beta)\\&\quad-\frac{\theta_1(\lambda_2-\lambda_1+(1+\beta)\eta)\theta_1(\lambda_2+\lambda_1-\eta)}{\theta_1(\lambda_1-\lambda_2)\theta_1(\lambda_2+\lambda_1)\theta_1((1+\beta)\eta)}e^{-4i\beta\eta}f_{\mathcal{B}_-(\beta)}(\lambda_2)\mathcal{D}_-(\lambda_1|\beta+2)\mathcal{B}_-(\lambda_2|\beta)\\&\quad-\frac{\theta_1(\eta)\theta_1(\lambda_2+\lambda_1+\beta\eta)}{\theta_1(\lambda_2+\lambda_1)\theta_1((1+\beta)\eta)}e^{-4i\beta\eta}f_{\mathcal{B}_-(\beta)}(\lambda_2)\mathcal{A}_-(\lambda_1|\beta+2)\mathcal{B}_-(\lambda_2|\beta),\end{aligned}$$

where we have defined

$$\mathcal{D}_-(\lambda+2\pi\omega|\beta)=f_{\mathcal{D}_-(\beta)}(\lambda)\mathcal{D}_-(\lambda|\beta). \tag{4.3.72}$$

Then the following result follows

$$f_{\mathcal{D}_-(\beta)}(\lambda)\equiv\left(-e^{-2i\lambda}/q^2\right)^{4N+8}e^{2i\alpha_{\mathcal{D}_-(\beta)}}\text{where }\ \alpha_{\mathcal{D}_-(\beta)}\equiv2(4-\beta)\eta. \tag{4.3.73}$$

By the definition (4.3.62) it is simple to argue that the operators $\mathcal{D}_{-}^{(0)}(\lambda|\beta)$ is also an order $4N+8$ elliptic polynomials of periods π and $2\pi\omega$ which satisfies (4.3.60) and (4.3.61); then the same is true for $\widetilde{\mathcal{D}}_{-}(\lambda|\beta)$. These properties together with the identities

$$\widetilde{\mathcal{D}}_{-}(\zeta_{-a}|\beta) \equiv 0 \forall a \in \{1,...,8\}, \tag{4.3.74}$$

imply the interpolation formula (4.3.64). □

4.3.3 Change of basis properties

In this section we deal again with the change of basis analysis (spin basis to SoV basis) discussed in section §3.3.3. The notation concerning the identifications of the two basis is essentially the same and it won't be repeated. We can then proceed to produce a proposition equivalent, for the XYZ model, to proposition 3.3.3.

Proposition 4.3.3. *The* $2^N \times 2^N$ *matrix*

$$M \equiv U^{(L,\beta-2)} U^{(R,\beta)} \tag{4.3.75}$$

is diagonal and it is characterized by

$$M_{\varkappa(\mathbf{h})\varkappa(\mathbf{k})} = \langle \beta-2, \mathbf{h}|\beta, \mathbf{k}\rangle = \delta_{\varkappa(\mathbf{h})\varkappa(\mathbf{k})} Z(\beta-2) \prod_{1\leq b<a\leq N} \frac{1}{\eta_a^{(h_a)} - \eta_b^{(h_b)}}, \tag{4.3.76}$$

with the normalization constant

$$Z(\beta) = \prod_{1\leq b<a\leq N} (\eta_a^{(1)} - \eta_b^{(1)}) \langle \beta | \left(\prod_{n=1}^{N} \mathcal{A}_{-}(\eta/2-\xi_n|\beta+2)/\mathsf{A}_{-}(\eta/2-\xi_n) \right) | -\beta \rangle, \tag{4.3.77}$$

and

$$\eta_a^{(h_a)} \equiv \frac{\theta_4^2((\xi_a + (h_a - \frac{1}{2})\eta)}{\theta_2^2((\xi_a + (h_a - \frac{1}{2})\eta)}. \tag{4.3.78}$$

Proof. The occurence of $\delta_{\varkappa(\mathbf{h})\varkappa(\mathbf{k})}$ in (4.3.76) fresults from the identities among matrix elements

$$\bar{\mathsf{B}}_{\beta,\mathbf{k}}(\lambda|\beta) \langle \beta, \mathbf{h}|\beta+2, \mathbf{k}\rangle = \langle \beta, \mathbf{h}|\mathcal{B}_{-}(\lambda|\beta)|\beta, \mathbf{k}\rangle = \mathsf{B}_{\beta,\mathbf{h}}(\lambda|\beta) \langle \beta-2, \mathbf{h}|\beta, \mathbf{k}\rangle, \tag{4.3.79}$$

indeed the condition $\mathbf{h}\neq\mathbf{k}$ implies $\exists n \in \{1,...,N\}$ such that $h_n \neq k_n$ and then it follows

$$\bar{\mathsf{B}}_{\beta,\mathbf{k}}(\zeta_n^{(k_n)}|\beta) = 0, \qquad \mathsf{B}_{\beta,\mathbf{h}}(\zeta_n^{(h_n)}|\beta) \neq 0, \tag{4.3.80}$$

and so

$$\langle \beta-2, \mathbf{h}|\beta, \mathbf{k}\rangle \propto \delta_{\varkappa(\mathbf{h})\varkappa(\mathbf{k})}. \tag{4.3.81}$$

The diagonal elements $M_{\varkappa(\mathbf{h})\varkappa(\mathbf{h})}$ are obtained by computing

$$\theta_a^{(\beta)} \equiv \langle \beta-2, h_1, ..., h_a = 1, ..., h_N | \widetilde{\mathcal{D}}_{-}(\xi_a + \eta/2|\beta) | \beta, h_1, ..., h_a = 0, ..., h_N \rangle \tag{4.3.82}$$

for any $a \in \{1,...,N\}$. Being

$$\langle \beta-2, h_1, ..., h_a = 1, ..., h_N | \tilde{\mathcal{D}}_{-}(\xi_a+\eta/2|\beta) = \langle \beta-2, h_1, ..., h_a = 1, ..., h_N | \mathcal{D}_{-}(\xi_a+\eta/2|\beta), \tag{4.3.83}$$

then using the decomposition (4.2.34b) and the fact that

$$\langle\ \beta-2,h_1,...,h_a=1,...,h_N\ |\ \mathcal{A}_-(-(\xi_a+\eta/2)|\beta)=0 \tag{4.3.84}$$

it holds

$$\langle\ \beta-2,h_1,...,h_a=1,...,h_N\ |\ \tilde{\mathcal{D}}_-(\xi_a+\eta/2|\beta) \tag{4.3.85}$$

$$=\frac{\theta_1(\eta)\theta_1\left(2\xi_a+\beta\eta\right)}{\theta_1\left(2\xi_a+\eta\right)\theta_1\left(\beta\eta\right)}\langle\ \beta-2,h_1,...,h_a=1,...,h_N\ |\ \mathcal{A}_-(\xi_a+\eta/2|\beta) \tag{4.3.86}$$

$$=\frac{\theta_1(\eta)\theta_1\left(2\xi_a+\beta\eta\right)}{\theta_1\left(2\xi_a+\eta\right)\theta_1\left(\beta\eta\right)}\mathsf{A}_-(\eta/2+\xi_a)\langle\beta-2,h_1,...,h_a=0,...,h_N|, \tag{4.3.87}$$

and then we get

$$\theta_a^{(\beta)}=\frac{\theta_1(\eta)\theta\left(2\xi_a+\beta\eta\right)}{\theta_1\left(2\xi_a+\eta\right)\theta_1\left(\beta\eta\right)}\mathsf{A}_-(\eta/2+\xi_a)\left\langle\beta-2,h_1,...,h_a=0,...,h_N|\beta,h_1,...,h_a=0,...,h_N\right\rangle. \tag{4.3.88}$$

On the other hand the right action of the operator $\widetilde{\mathcal{D}}_-(\xi_a+\eta/2|\beta)$ and the condition (4.3.81) imply *

$$\theta_a^{(\beta)}=\left(k_a^{(\beta)}\right)^{-1}\mathsf{A}_-(\eta/2+\xi_a)\frac{\theta_1(2(4-\beta)-\sum_{b=1}^4\zeta_{-b}-2\xi_a)\theta_1(\eta)\theta_2^{2(N-1)}(\zeta_a^{(1)})}{\theta_1(2(4-\beta)-\sum_{b=1}^4\zeta_{-b})\theta_1(2\zeta_a^{(0)})\theta_2^{2(N-1)}(\zeta_a^{(0)})}$$
$$\times\prod_{\substack{b=1\\b\neq a}}^{N}\frac{\frac{\theta_4^2(\lambda)}{\theta_2^2(\lambda)}-\frac{\theta_4^2(\zeta_b^{(h_b)})}{\theta_2^2(\zeta_b^{(h_b)})}}{\frac{\theta_4^2(\zeta_a^{(h_a)})}{\theta_2^2(\zeta_a^{(h_a)})}-\frac{\theta_4^2(\zeta_b^{(h_b)})}{\theta_2^2(\zeta_b^{(h_b)})}}\left\langle\beta-2,h_1,...,h_a=1,...,h_N|\beta,h_1,...,h_a=1,...,h_N\right\rangle \tag{4.3.89}$$

so that it holds

$$\frac{\langle\beta-2,h_1,...,h_a=0,...,h_N|\beta,h_1,...,h_a=0,...,h_N\rangle}{\langle\beta-2,h_1,...,h_a=1,...,h_N|\beta,h_1,...,h_a=1,...,h_N\rangle}=\prod_{\substack{b=1\\b\neq a}}^{N}\frac{\frac{\theta_4^2(\lambda)}{\theta_2^2(\lambda)}-\frac{\theta_4^2(\zeta_b^{(h_b)})}{\theta_2^2(\zeta_b^{(h_b)})}}{\frac{\theta_4^2(\zeta_a^{(h_a)})}{\theta_2^2(\zeta_a^{(h_a)})}-\frac{\theta_4^2(\zeta_b^{(h_b)})}{\theta_2^2(\zeta_b^{(h_b)})}}. \tag{4.3.90}$$

From (4.3.90) one can prove

$$\frac{\langle\beta-2,h_1,...,h_N|\beta,h_1,...,h_N\rangle}{\langle\beta-2,1,...,1|\beta,1,...,1\rangle}=\prod_{1\leq b<a\leq N}\frac{\eta_a^{(1)}-\eta_b^{(1)}}{\eta_a^{(h_a)}-\eta_b^{(h_b)}}. \tag{4.3.91}$$

This last identity implies (4.3.76) being

$$\langle\beta-2,1,...,1|\beta,1,...,1\rangle=Z(\beta-2)\prod_{1\leq b<a\leq N}\frac{1}{\eta_a^{(1)}-\eta_b^{(1)}}, \tag{4.3.92}$$

by our definition of the normalization $Z(\beta)$. □

---

*. From expression (4.3.89) it follows the definition (4.3.57) for $k_n^{(\beta)}$ itself.

4.3.4 SoV-decomposition of the identity

The identity operator $\mathbb{1}$ admits the following representation in terms of left and right SoV-basis:

$$\mathbb{1}=\sum_{i=1}^{2^N}\mu\left|\beta,\varkappa^{-1}(i)\right\rangle\left\langle\beta-2,\varkappa^{-1}(i)\right|, \tag{4.3.93}$$

where the $\mu=\left(\langle\beta-2,\varkappa^{-1}(i)\,|\beta,\varkappa^{-1}(i)\rangle\right)^{-1}$ is the Sklyanin's measure analogous in our 8-vertex reflection algebra representations. Now using the result of the previous section we can write it explicitly

$$\mathbb{I}=\frac{1}{Z(\beta-2)}\sum_{h_1,\dots,h_N=0}^{1}\prod_{1\leq b<a\leq N}(\eta_a^{(h_a)}-\eta_b^{(h_b)})\mid\beta,h_1,\dots,h_N\rangle\langle\beta-2,h_1,\dots,h_N\mid. \tag{4.3.94}$$

4.4 SoV representations for $\mathcal{T}(\lambda)$-spectral problem

In this section, we show how the SoV approach allows to write eigenvalues and eigenstates for the transfer matrix associated to the most general representation of the 8-vertex reflection algebra once the gauge transformations are used. The SoV characterization here presented is the natural generalization to the 8-vertex reflection algebra case of those derived for the 6-vertex case in section §3.4.

Theorem 4.4.1. *Under the most general boundary conditions, and if the gauge parameters $\alpha,\beta\in\mathbb{C}$ satisfy the following condition for $(k,p)\in\mathbb{Z}^2$ and $\epsilon_l^+\in\{-1,1\}$*

$$(\alpha-\beta+2)\eta=\sum_{l=1}^{3}\epsilon_l^+\alpha_l^++(2k+1)\pi+p\omega\pi,\quad\prod_{l=1}^{3}\epsilon_l^+=1 \tag{4.4.1}$$

then $K_+^{(L)}(\lambda|\beta-1)_{12}=K_+^{(R)}(\lambda|\beta-1)_{12}=0$ and

I_a) the left representation for which the one parameter family $\mathcal{B}_-(\lambda|\beta-2)$ is pseudo-diagonal defines a left SoV representation for the spectral problem of the transfer matrix $\mathcal{T}(\lambda)$.

II_a) the right representation for which the one parameter family $\mathcal{B}_-(\lambda|\beta)$ is pseudo-diagonal defines a right SoV representation for the spectral problem of the transfer matrix $\mathcal{T}(\lambda)$.

Under the most general boundary conditions, if the gauge parameters $\alpha,\beta\in\mathbb{C}$ satisfy the following condition for $(k,p)\in\mathbb{Z}^2$ and $\epsilon_l^+\in\{-1,1\}$

$$(\alpha+\beta)\eta=\sum_{l=1}^{3}\epsilon_l^+\alpha_l^++(2k+1)\pi+p\omega\pi,\quad\prod_{l=1}^{3}\epsilon_l^+=1 \tag{4.4.2}$$

then $K_+^{(L)}(\lambda|\beta-1)_{21}=K_+^{(R)}(\lambda|\beta-1)_{21}=0$ and

I_b) the left representation for which the one parameter family $\mathcal{C}_-(\lambda|\beta+2)$ is pseudo-diagonal defines a left SoV representation for the spectral problem of the transfer matrix $\mathcal{T}(\lambda)$.

II_b) the right representation for which the one parameter family $\mathcal{C}_-(\lambda|\beta)$ is pseudo-diagonal defines a right SoV representation for the spectral problem of the transfer matrix $\mathcal{T}(\lambda)$.

Here, we will present these SoV constructions in this way proving the theorem only in the cases I_a) and II_a) as for the cases I_c) and II_c) these can be inferred mainly by using the β-symmetries defined in Lemma 4.2.2.

Lemma 4.4.1. *Let us denote with $\Sigma_{\mathcal{T}}$ the set of the eigenvalue functions of the transfer matrix $\mathcal{T}(\lambda)$, then any $\mathtt{t}(\lambda) \in \Sigma_{\mathcal{T}}$ is even in λ and it satisfies the following quasi-periodicity properties in λ w.r.t. the periods π and $\pi\omega$*

$$\mathtt{t}(\lambda+\pi) = \mathtt{t}(\lambda),\ \mathtt{t}(\lambda+\pi\omega) = \left(e^{-2i\lambda}/q\right)^{2N+6}\mathtt{t}(\lambda). \tag{4.4.3}$$

Moreover, the following identities hold

$$\mathtt{t}(\pm\frac{\eta}{2}) = (-1)^N\frac{\theta_1(2\eta)}{\theta_1(\eta)}det_q M_0(0), \tag{4.4.4a}$$

$$\mathtt{t}(\pm(\frac{\eta}{2}-\frac{\pi}{2})) = \frac{\theta_1(2\eta)}{\theta_1(\eta)}\prod_{\gamma=\pm}\prod_{l=1}^{3}\frac{\theta_1(\alpha_l^\gamma-\frac{\pi}{2})}{\theta_1(\alpha_l^\gamma)}det_q M_0(\frac{\pi}{2}), \tag{4.4.4b}$$

$$\begin{aligned}\mathtt{t}(\pm(\frac{\eta}{2}-\frac{\pi\omega}{2})) = {}&\frac{\theta_1(2\eta)}{\theta_1(\eta)}\prod_{\gamma=\pm}\prod_{l=1}^{3}\frac{\theta_1(\alpha_l^\gamma-\frac{\omega\pi}{2})}{\theta_1(\alpha_l^\gamma)}\\ &\times det_q M_0(\frac{\omega\pi}{2})\times e^{-2i\sum_{j=1}^N\xi_j}e^{i(N+3)\eta}e^{-i\sum_{\gamma=\pm}\sum_{l=1}^3\alpha_l^\gamma},\end{aligned} \tag{4.4.4c}$$

$$\begin{aligned}\mathtt{t}(\pm(\frac{\eta}{2}-\frac{\pi\omega}{2}-\frac{\pi}{2})) = {}&\frac{\theta_1(2\eta)}{\theta_1(\eta)}\prod_{\gamma=\pm}\prod_{l=1}^{3}\frac{\theta_1(\alpha_l^\gamma-\frac{\omega\pi}{2}-\frac{\pi}{2})}{\theta_1(\alpha_l^\gamma)}\\ &\times det_q M_0(\frac{\omega\pi}{2}+\frac{\pi}{2})\times e^{-2i\sum_{j=1}^N\xi_j}e^{i(N+3)\eta}e^{-i\sum_{\gamma=\pm}\sum_{l=1}^3\alpha_l^\gamma},\end{aligned} \tag{4.4.4d}$$

where we have re-expressed all the boundary functions $c_x^\pm$, $c_y^\pm$ and $c_z^\pm$ in terms of $\theta_1(x|\omega)$.

Proof. The function $\mathtt{t}(\lambda)$ is an elliptic polynomial of order $2N+6$ even in λ since the transfer matrix $\mathcal{T}(\lambda)$ has the same property as it was stated in Lemma (4.1.1). The identities (4.4.4) can be proven by direct computation and showing how the transfer matrix become trivial, *i.e.* a function times the identity operator, in these points. One has to study the single elements of the transfer matrix evaluated in one of the point of interest and exploit the R-matrix properties (4.1.9) and the bulk quantum determinant relation (4.1.49). Let us reproduce the main passages of the calculations. Consider, first of all, the following identities

$$\begin{aligned}K_-(\eta/2) = \mathbb{1},\quad K_-((\eta-\pi)/2) = c_z^-\times\sigma^z,\\ K_-((\eta-\omega\pi)/2) = c_x^-\times\sigma^x,\quad K_-((\eta-\pi-\omega\pi)/2) = c_y^-\times\sigma^y.\end{aligned} \tag{4.4.5}$$

Then we can plug them in the transfer matrix definition to get

$$\begin{aligned}&\mathcal{T}(\pm\eta/2) = \mathrm{tr}_0\left\{K_+(\eta/2)M(\eta/2)\hat{M}(\eta/2)\right\}\times\mathbb{1},\\ &\mathcal{T}(\pm(\eta/2-\pi/2)) = c_z^-\mathrm{tr}_0\left\{K_+(\eta/2-\pi/2)M(\eta/2-\pi/2)\sigma^z\hat{M}(\eta/2-\pi/2)\right\},\\ &\mathcal{T}(\pm(\eta/2-\omega\pi/2)) = c_x^-\mathrm{tr}_0\{K_+(\eta/2-\omega\pi/2)\\ &\qquad\qquad\times M(\eta/2-\omega\pi/2)\sigma^x\hat{M}(\eta/2-\omega\pi/2)\},\\ &\mathcal{T}(\pm(\eta/2-\pi/2-\omega\pi/2)) = c_z^-\mathrm{tr}_0\{K_+(\eta/2-\pi/2-\omega\pi/2)\\ &\qquad\qquad\times M(\eta/2-\pi/2-\omega\pi/2)\sigma^y\hat{M}(\eta/2-\pi/2-\omega\pi/2)\}.\end{aligned}$$

Then by exploiting the properties (4.1.27)-(4.1.32) and the bulk quantum determinant relation (4.1.48), the result follows. □

4.4.1 Transfer matrix spectrum in $\mathcal{B}_-(\lambda|\beta)$-SoV-representations

After Lemma 4.4.1 we learnt that the transfer matrix eigenvalue is a $2N+6$-order elliptic polynomial of periods π and $\omega\pi$ and it takes values in the points $\pm\zeta_{-a}$ for $a = 1,2,3$ and 4 which are independent from the particular choice of $\mathsf{t}(\lambda) \in \Sigma_{\mathcal{T}}$. We can define then the following function

$$j(\lambda) = \sum_{a=1}^{4} l_{-a}(\lambda) t(\zeta_{-a}), \tag{4.4.6}$$

where

$$l_{-a}(\lambda) = \prod_{\substack{c=1\\c\neq a}}^{4} \frac{\theta_1(\lambda-\zeta_{-c})\theta_1(\lambda+\zeta_{-c})}{\theta_1(\zeta_{-a}-\zeta_{-c})\theta_1(\zeta_{-a}+\zeta_{-c})} \prod_{b=1}^{N} \frac{\theta_1(\lambda-\zeta_b^{(0)})\theta_1(\lambda+\zeta_b^{(0)})}{\theta_1(\zeta_{-a}-\zeta_b^{(0)})\theta_1(\zeta_{-a}+\zeta_b^{(0)})}. \tag{4.4.7}$$

One can observe that the elliptic polynomial $j(\lambda)$ is independent from the particular choice of $\mathsf{t}(\lambda) \in \Sigma_{\mathcal{T}}$. We can now prove the following complete characterization of the transfer matrix spectrum.

Theorem 4.4.2. *$\mathcal{T}(\lambda)$ has simple spectrum if* (4.3.1) *is satisfied and $\Sigma_{\mathcal{T}}$ admits the following characterization*

$$\Sigma_{\mathcal{T}} \equiv \left\{ \mathsf{t}(\lambda) : \mathsf{t}(\lambda) = j(\lambda) + \sum_{a=1}^{N} l_a(\lambda) x_a, \quad \forall \{x_1, ..., x_N\} \in \Sigma_T \right\}, \tag{4.4.8}$$

with

$$l_a(\lambda) = \prod_{c=1}^{4} \frac{\theta_1(\lambda-\zeta_{-c})\theta_1(\lambda+\zeta_{-c})}{\theta_1(\zeta_a^{(0)}-\zeta_{-c})\theta_1(\zeta_a^{(0)}+\zeta_{-c})} \prod_{\substack{b=1\\b\neq a}}^{N} \frac{\theta_1(\lambda-\zeta_b^{(0)})\theta_1(\lambda+\zeta_b^{(0)})}{\theta_1(\zeta_a^{(0)}-\zeta_b^{(0)})\theta_1(\zeta_a^{(0)}+\zeta_b^{(0)})}, \tag{4.4.9}$$

and Σ_T is the set of the solutions to the following inhomogeneous system of N quadratic equations

$$x_n \sum_{a=1}^{N} l_a(\zeta_n^{(1)}) x_a + x_n j(\zeta_n^{(1)}) = q_n, \;\; q_n \equiv \mathbf{A}(\zeta_n^{(1)})\mathbf{A}(-\zeta_n^{(0)}), \quad \forall n \in \{1, ..., N\}, \tag{4.4.10}$$

in the N unknown $\{x_1, ..., x_N\}$, where $\mathbf{A}(\lambda)$ is defined by

$$\mathbf{A}(\lambda) \equiv \mathsf{a}_+(\lambda|\beta-1)\mathsf{A}_-(\lambda), \tag{4.4.11}$$

and it satisfies the quantum determinant condition

$$\frac{\text{q-det}\,(K_+(\lambda))\text{q-det}\,(\mathcal{U}_-(\lambda))}{\theta_1(\eta+2\lambda)\theta_1(\eta-2\lambda)} = \mathbf{A}(\eta/2-\lambda)\mathbf{A}(\lambda+\eta/2). \tag{4.4.12}$$

R) If (4.3.4) *is verified, the vector*

$$|\,\mathsf{t}\,\rangle = \sum_{h_1,...,h_N=0}^{1} \prod_{a=1}^{N} \mathsf{Q}_\mathsf{t}(\zeta_a^{(h_a)}) \prod_{1\leq b<a\leq N} (\eta_a^{(h_a)} - \eta_b^{(h_b)}) \,|\, \beta+2, h_1, ..., h_N \,\rangle, \tag{4.4.13}$$

with coefficients

$$\mathsf{Q}_\mathsf{t}(\zeta_a^{(1)})/\mathsf{Q}_\mathsf{t}(\zeta_a^{(0)}) = \mathsf{t}(\zeta_a^{(0)})/\mathbf{A}(-\zeta_a^{(0)}), \tag{4.4.14}$$

is the right $\mathcal{T}$-eigenstate corresponding to $\mathsf{t}(\lambda) \in \Sigma_{\mathcal{T}}$ uniquely defined up to an overall normalization.

L) If (4.3.2) *is verified, the covector*

$$\bar{\mathsf{t}} = \sum_{h_1,...,h_N=0}^{1} \prod_{a=1}^{N} \bar{\mathsf{Q}}_{\mathsf{t}}(\zeta_a^{(h_a)}) \prod_{1\leq b<a\leq N} (\eta_a^{(h_a)} - \eta_b^{(h_b)}) \langle\, \beta, h_1, ..., h_N \mid, \tag{4.4.15}$$

with coefficients

$$\bar{\mathsf{Q}}_{\mathsf{t}}(\zeta_a^{(1)})/\bar{\mathsf{Q}}_{\mathsf{t}}(\zeta_a^{(0)}) = \mathsf{t}(\zeta_a^{(0)})/\left(\mathbf{D}(\zeta_a^{(1)})\right) \tag{4.4.16}$$

where

$$\mathbf{D}(\zeta_a^{(1)}) = \mathsf{d}_+(\zeta_a^{(1)}|\beta-1)\mathsf{D}_-(\zeta_a^{(1)} \tag{4.4.17}$$

is the left $\mathcal{T}$-eigenstate corresponding to $\mathsf{t}(\lambda) \in \Sigma_{\mathcal{T}}$ uniquely defined up to an overall normalization.

Proof. The separate variables characterization of the spectral problem for $\mathcal{T}(\lambda)$ is reduced to the discrete system of 2^N Baxter-like equations

$$\mathsf{t}(\zeta_n^{(h_n)})\Psi_{\mathsf{t}}(\mathbf{h}) = \mathbf{A}(\zeta_n^{(h_n)})\Psi_{\mathsf{t}}(\mathsf{T}_n^-(\mathbf{h})) + \mathbf{A}(-\zeta_n^{(h_n)})\Psi_{\mathsf{t}}(\mathsf{T}_n^+(\mathbf{h})), \tag{4.4.18}$$

for any $n \in \{1,...,N\}$ and $\mathbf{h} \in \{0,1\}^N$. Here, the (*wave-functions*) $\Psi_{\mathsf{t}}(\mathbf{h})$ are the coefficient of the $\mathcal{T}$-eigenstate $\mid \mathsf{t}\,\rangle$ corresponding to the $\mathsf{t}(\lambda) \in \Sigma_{\mathcal{T}}$ in the right $\mathcal{B}_-$-SoV representation and the following notations are introduced

$$\mathsf{T}_n^{\pm}(\mathbf{h}) \equiv (h_1, \ldots, h_n \pm 1, \ldots, h_N). \tag{4.4.19}$$

This system of separate equations is derived from the identities:

$$\mathbf{A}_-(\zeta_n^{(0)}) = \mathbf{A}_-(-\zeta_n^{(1)}) = 0, \tag{4.4.20}$$

once we compute the matrix elements

$$\langle\, \beta, h_1, ..., h_n, ..., h_N \mid \mathcal{T}(\pm\zeta_n^{(h_n)}) \mid \mathsf{t}\,\rangle. \tag{4.4.21}$$

Indeed the decomposition (4.2.53) implies

$$\begin{aligned}
\mathsf{t}(\pm\zeta_n^{(0)})&\Psi_{\mathsf{t}}(h_1, ..., h_n = 0, ..., h_N) \\
&= \langle\, \beta, h_1, ..., h_n = 0, ..., h_N \mid \mathcal{T}(-\zeta_n^{(0)}) \mid \mathsf{t}\,\rangle \\
&= \mathsf{a}_+(-\zeta_n^{(0)}) \langle\, \beta, h_1, ..., h_n = 0, ..., h_N \mid \mathcal{A}_-(-\zeta_n^{(0)}) \mid \mathsf{t}\,\rangle \\
&= \mathbf{A}(-\zeta_n^{(0)})\Psi_{\mathsf{t}}(h_1, ..., h_n = 1, ..., h_N) \\
&= \mathbf{A}(-\zeta_n^{(0)})\Psi_{\mathsf{t}}(h_1, ..., h_n = 1, ..., h_N) + \mathbf{A}(\zeta_n^{(0)})\Psi_{\mathsf{t}}(h_1, ..., h_n = -1, ..., h_N),
\end{aligned} \tag{4.4.22}$$

and

$$\begin{aligned}
\mathsf{t}(\pm\zeta_n^{(1)})&\Psi_{\mathsf{t}}(h_1, ..., h_n = 1, ..., h_N) \\
&= \langle\, \beta, h_1, ..., h_n = 1, ..., h_N \mid \mathcal{T}(\zeta_n^{(1)}) \mid \mathsf{t}\,\rangle \\
&= \mathsf{a}_+(\zeta_n^{(1)}) \langle\, \beta, h_1, ..., h_n = 1, ..., h_N \mid \mathcal{A}_-(\zeta_n^{(1)}) \mid \mathsf{t}\,\rangle \\
&= \mathbf{A}(\zeta_n^{(1)})\Psi_{\mathsf{t}}(h_1, ..., h_n = 0, ..., h_N) \\
&= \mathbf{A}(\zeta_n^{(1)})\Psi_{\mathsf{t}}(h_1, ..., h_n = 0, ..., h_N) + \mathbf{A}(-\zeta_n^{(1)})\Psi_{\mathsf{t}}(h_1, ..., h_n = 2, ..., h_N).
\end{aligned}$$

The system (4.4.18) is clearly equivalent to the system of homogeneous equations

$$\begin{pmatrix} \mathsf{t}(\pm\zeta_n^{(0)}) & -\mathbf{A}(-\zeta_n^{(0)}) \\ -\mathbf{A}(\zeta_n^{(1)}) & \mathsf{t}(\pm\zeta_n^{(1)}) \end{pmatrix} \begin{pmatrix} \Psi_{\mathsf{t}}(h_1,...,h_n=0,...,h_1) \\ \Psi_{\mathsf{t}}(h_1,...,h_n=1,...,h_1) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.4.23}$$

for any $n \in \{1,...,N\}$ with $h_{r\neq n} \in \{0,1\}$. Then the determinants of the 2×2 matrices in (4.4.23) must be zero for any $n \in \{1,...,N\}$ if $\mathsf{t}(\lambda) \in \Sigma_{\mathcal{T}}$, *i.e.* it holds

$$\mathsf{t}(\pm\zeta_a^{(0)})\mathsf{t}(\pm\zeta_a^{(1)}) = \mathbf{A}(\zeta_a^{(1)})\mathbf{A}(-\zeta_a^{(0)}), \quad \forall a \in \{1,...,N\}. \tag{4.4.24}$$

Being

$$\mathbf{A}(-\zeta_n^{(0)}) \neq 0 \text{ and } \mathrm{A}(\zeta_n^{(1)}) \neq 0, \tag{4.4.25}$$

then the matrices in (4.4.23) have all rank 1 and up to an overall normalization the solution is unique

$$\frac{\Psi_{\mathsf{t}}(h_1,...,h_n=1,...,h_N)}{\Psi_{\mathsf{t}}(h_1,...,h_n=0,...,h_N)} = \frac{\mathsf{t}(\zeta_a^{(0)})}{\mathbf{A}(-\zeta_a^{(0)})}, \tag{4.4.26}$$

for any $n \in \{1,...,N\}$ with $h_{r\neq n} \in \{0,1\}$. So for any fixed $\mathsf{t}(\lambda) \in \Sigma_{\mathcal{T}}$ the associate eigenspace is one dimensional ($\mathcal{T}(\lambda)$ has simple spectrum) and $|\,\mathsf{t}\,\rangle$ defined by (4.4.13) and (4.4.14) is the only corresponding eigenstate up to normalization.

Formula (4.4.8) is the correct interpolation formula for $\mathsf{t}(\lambda)$ since, as we discussed in Lemma (4.4.1), it is an order $2N+6$ elliptic polynomial even in λ with periods π and $\omega\pi$, being by definition $x_a := \mathsf{t}(\zeta_a^{(0)})$ for all $a = 1,\ldots,N$. Instead, conditions (4.4.10) coincide with (4.4.24) once we use, in the *lhs*, the expression (4.4.8) to express $\mathsf{t}(\pm\zeta_a^{(0)})$ and $\mathsf{t}(\pm\zeta_a^{(1)})$.

Let prove now the reverse inclusion of set of functions, i.e. let us prove that if $\mathsf{t}(\lambda)$ is in the set of functions characterized by (4.4.8) and (4.4.10) then it is an element of $\Sigma_{\mathcal{T}}$. Indeed, taking the state $|\mathsf{t}\rangle$ defined by (4.4.13) and (4.4.14) the following identities are satisfied

$$\langle\, \beta, h_1,...,h_N \,|\, \mathcal{T}(\pm\zeta_n^{(h_n)}) \,|\, \mathsf{t}\,\rangle = \mathsf{t}(\pm\zeta_n^{(h_n)})\,\langle \beta, h_1,...,h_N|\mathsf{t}\rangle\,, \quad \forall n \in \{1,...,N\}, \tag{4.4.27}$$

and

$$\langle\, \beta, h_1,...,h_N \,|\, \mathcal{T}(\pm\zeta_{-a}) \,|\, \mathsf{t}\,\rangle = \mathsf{t}(\pm\zeta_{-a})\,\langle \beta, h_1,...,h_N|\mathsf{t}\rangle\,, \quad \forall a \in \{1,...,4\},$$

which imply

$$\langle\, \beta, h_1,...,h_N \,|\, \mathcal{T}(\lambda) \,|\, \mathsf{t}\,\rangle = \mathsf{t}(\lambda)\,\langle \beta, h_1,...,h_N|\mathsf{t}\rangle \quad \forall \lambda \in \mathbb{C}, \tag{4.4.28}$$

for any $\mathcal{B}_-(\lambda|\beta)$-pseudo-eigenstate $\langle\, \beta, h_1,...,h_N\,|$, i.e. $\mathsf{t}(\lambda) \in \Sigma_{\mathcal{T}}$ and $|\,\mathsf{t}\,\rangle$ is the corresponding $\mathcal{T}$-eigenstate. Finally, let us point out that the quantum determinant condition (4.4.12) follows from the definition (4.4.11) and the quantum determinant conditions (4.1.42) and (4.2.67), where this last identity holds when (4.2.65)-(4.2.66) is satisfied as proven in Lemma 4.2.3. Concerning the left $\mathcal{T}$-eigenstates the proof is done as above. Here one has to compute the matrix elements

$$\langle\, \mathsf{t} \,|\, \mathcal{T}(\zeta_n^{(h_n)}) \,|\, \beta+2, h_1,...,h_N\,\rangle\,, \tag{4.4.29}$$

which by using the right $\mathcal{B}(|\beta)$-representation read

$$\mathsf{t}(\zeta_n^{(h_n)})\bar{\Psi}_{\mathsf{t}}(\mathbf{h}) = \mathbf{D}(\zeta_n^{(h_n)})\bar{\Psi}_{\mathsf{t}}(\mathsf{T}_n^-(\mathbf{h})) + \mathbf{D}(-\zeta_n^{(h_n)})\bar{\Psi}_{\mathsf{t}}(\mathsf{T}_n^+(\mathbf{h})), \quad \forall n \in \{1,...,N\} \tag{4.4.30}$$

where

$$\bar{\Psi}_{\mathsf{t}}(\mathbf{h}) \equiv \langle \mathsf{t}|\beta+2, h_1,...,h_N\rangle\,, \mathbf{D}(\pm\zeta_a^{(h_a)}) \equiv \mathsf{d}_+(\pm\zeta_a^{(h_a)})\mathsf{D}_-(\pm\zeta_a^{(h_a)}). \tag{4.4.31}$$

□

4.4.2 SoV applicability constraint

Combining together conditions for the existence of SoV basis (4.3.2)-(4.3.8) and the choice of the gauge parameters necessary to construct the eigenstates of the transfer matrix (4.4.1), (4.4.2) we obtain the limits of applicability of the SoV method. Unfortunately, this particular situation should coincide with the domain of applicability of the algebraic Bethe ansatz, studied by Fan, Hou, Shi and Yang in [47]. At the moment, more analysis is requested and we are not in the position to build a parallel to Nepomechie's conditions of section §3.4.2. More precisely the following theorem holds

Theorem 4.4.3. *The SoV constructions corresponding to the cases I_a and I_b fails to exist if and only if the following condition on the parameters of the boundary matrices are satisfied*

$$(N+1)\,\eta = \sum_{\gamma=\pm}\sum_{l=1}^{3}\epsilon_l^{\gamma}\alpha_l^{\gamma} + 2(k+m)\pi + (p+n)\omega\pi, \quad \prod_{l=1}^{3}\epsilon_l^{-} = 1, \prod_{l=1}^{3}\epsilon_l^{+} = 1, \tag{4.4.32}$$

where $(k,m,p,n)\in\mathbb{Z}^4$ and $\epsilon_l^{\gamma}\in\{-1,1\}$.

Similarly, the SoV constructions corresponding to the cases II_a and II_b fails to exist if and only if the following condition on the parameters of the boundary matrices is satisfied

$$(1-N)\,\eta = \sum_{\gamma=\pm}\sum_{l=1}^{3}\epsilon_l^{\gamma}\alpha_l^{\gamma} + 2(\hat{k}+\hat{m})\pi + (\hat{p}+\hat{n})\omega\pi, \quad \prod_{l=1}^{3}\epsilon_l^{-} = 1, \prod_{l=1}^{3}\epsilon_l^{+} = 1, \tag{4.4.33}$$

where $(\hat{k},\hat{m},\hat{p},\hat{n})\in\mathbb{Z}^4$ and $\epsilon_l^{\gamma}\in\{-1,1\}$..

Then, our SoV scheme to construct the spectrum (eigenvalues and eigenstates) of the transfer matrix $\mathcal{T}(\lambda)$ cannot be used if and only if the conditions (4.4.32) *and* (4.4.33) *are simultaneously satisfied.*

4.5 Scalar Products

Also in this case, it's possible to produce explicitly the determinant formulae for the SoV states. It's remarkable to notice that the scalar products we are going to introduce are formally equivalent to the one introduced in section §3.5 for the trigonometric algebra.

Theorem 4.5.1. *Let $\langle\,\omega\,|$ and $|\,\rho\,\rangle$ be an arbitrary covector and vector of separate forms:*

$$\langle\,\omega\,| = \sum_{h_1,\dots,h_N=0}^{1}\prod_{a=1}^{N}\omega_a(\zeta_a^{(h_a)})\prod_{1\le b<a\le N}(\eta_a^{(h_a)}-\eta_b^{(h_b)})\langle\beta-2,h_1,\dots,h_N|, \tag{4.5.1}$$

$$|\,\rho\,\rangle = \sum_{h_1,\dots,h_N=0}^{1}\prod_{a=1}^{N}\rho_a(\zeta_a^{(h_a)})\prod_{1\le b<a\le N}(\eta_a^{(h_a)}-\eta_b^{(h_b)})|\beta,h_1,\dots,h_N\rangle, \tag{4.5.2}$$

in the $\mathcal{B}$-pseudo-eigenbasis, then the action of $\langle\,\omega\,|$ on $|\,\rho\,\rangle$ reads:

$$\langle\omega|\rho\rangle = Z(\beta-2)\det_N||\mathcal{M}_{a,b}^{(\omega,\rho)}|| \quad \textit{with} \quad \mathcal{M}_{a,b}^{(\omega,\rho)} = \sum_{h=0}^{1}\omega_a(\zeta_a^{(h)})\rho_a(\zeta_a^{(h)})(\eta_a^{(h)})^{(b-1)}. \tag{4.5.3}$$

The above formula still holds if the left and right states are transfer matrix eigenstates.

Proof. The formula (4.3.76) and the SoV-decomposition of the states $\langle\,\omega\,|$ and $|\,\rho\,\rangle$ implies that

$$\langle\omega|\rho\rangle = Z(\beta-2)\sum_{h_1,\dots,h_N=0}^{1}V(\eta_1^{(h_1)},\dots,\eta_N^{(h_N)})\prod_{a=1}^{N}\omega_a(\zeta_a^{(h_a)})\rho_a(\zeta_a^{(h_a)}), \tag{4.5.4}$$

where

$$V(x_1, ..., x_N) \equiv \prod_{1 \leq b < a \leq N} (x_a - x_b) \tag{4.5.5}$$

is the Vandermonde determinant and due to the multilinearity of the determinant (3.5.3) follows. □

CONCLUSIONS & OUTLOOK

The main results of the thesis at hand concern the study of the 6-vertex and 8-vertex representations of the reflection algebra. We have considered the quantum models associated to the most general integrable boundary conditions of the spin-$1/2$ quantum chains and we have developed the SoV construction for them. This method permitted us to retrieve the complete characterization of their spectrum (transfer matrix eigenvalues and eigenstates) in terms of a set of solutions to an inhomogeneous system of N quadratic equations in N unknowns, where N is the number of sites in the chain.

Let us here summarize, step-by-step, the key points and results of the application of the SoV technique, obtained in chapter **3** and **4**, in the light of the general scheme of the method, formally developed for the Yangian $\mathcal{Y}[sl(2)]$ by Sklyanin in [117] and reproduced in chapter **2**. We will reproduce and reference the results concerning the *left* representation, except at the end where, in order to explicitly point out the separation of variables for the wave function, we will evidently need the notion of *right* state as well.

Step 1: the commuting generating operators

As we learned in chapter **2**, the first and essential key-point of the SoV method is the identification of a proper generator of the Yang-Baxter algebra to diagonalize. This operator must commute with itself, for different values of the spectral parameter, in order to ensure the global diagonalizability

$$[B(\lambda), B(\mu)] = 0, \qquad \forall (\lambda, \mu) \in \mathbb{C}^2.$$

In our case, the reference algebra was the reflection algebra, in its 6-vertex or 8-vertex representations, with the injection of a gauge transformation. The good generator had to be taken from the gauged double-row-monodromy matrix $\mathcal{U}_-(\lambda|\beta)$. It was necessary to soften the constraint to find a self-commuting operator in the place of a pseudo-commuting one, resulting in a pseudo-diagonalizability condition. Indeed it was not difficult to show that the operator $\mathcal{B}_-(\lambda|\beta)$ is such that

$$\mathcal{B}_-(\lambda|\beta)\mathcal{B}_-(\mu|\beta-2) = \mathcal{B}_-(\mu|\beta)\mathcal{B}_-(\lambda|\beta-2), \qquad \forall (\lambda, \mu) \in \mathbb{C}^2.$$

XXZ: See eq. (3.2.20).
XYZ: See eq. (4.2.25).

Step 2: the separated variables

The next fundamental step is to understand how to read out or pinpoint the so-called operator roots of B operators. The definition of these operator zeroes can be understood once the polynomial nature of the monodromy matrix and then of its entries has been established. in other words we see the definition of those as the operators that permits the representation

$$B(\lambda) \propto \prod_{n=1}^{N} f(\hat{x}_n - \lambda),$$

where the nature of the functions f changes according the nature of the algebra one is dealing with. in the case of the $\mathcal{Y}[sl(2)]$ representation, we showed how it is just a rational function $f(\lambda) = \lambda$. Although this is not the case for our analysis of the open XXZ and XYZ chains. Indeed, the trigonometric 6-vertex algebra will result in $f_{6V}(\lambda) = \sinh(\lambda)$, while the 8-vertex elliptic algebra in $f_{8V}(\lambda) = \theta_1(\lambda)$. The operator roots where easily established once the complete set of pseudo -eigenstates of the operator $\mathcal{B}_-(\lambda|\beta)$ was built. Indeed we proved that

$$\langle\ \beta, h_1, \ldots, h_N \mid \mathcal{B}_-(\lambda|\beta) = \mathsf{B}_{\mathbf{h}}(\lambda|\beta) \langle\ \beta - 2, h_1, \ldots, h_N \mid$$

where

$$\mathsf{B}_{\mathbf{h}}(\lambda|\beta) \propto \begin{cases} \prod_{n=1}^{N} \sinh(\lambda - \zeta_n^{(h_n)}) \sinh(\lambda + \zeta_n^{(h_n)}), & \text{for } \mathbf{XXZ}; \\ \prod_{n=1}^{N} \theta_1(\lambda - \zeta_n^{(h_n)}) \theta_1(\lambda + \zeta_n^{(h_n)}), & \text{for } \mathbf{XYZ}, \end{cases}$$

where, for both models, we defined

$$\zeta_n^{(h_n)} = \pm(\xi_n - (h_n - \frac{1}{2})\eta).$$

From these expressions it is easy to read out directly the eigenvalues of operator roots. In reference to representation (2.1.44) we could define the following string set

$$\Lambda_n = \{-\xi_n - \eta/2, -\xi_n + \eta/2, \xi_n + \eta/2, \xi_n - \eta/2\},$$

even if it does not follow the rule given in chapter **2**.

Step 3: The conjugated momenta

The definition of the ladder operators that constitutes the conjugated momenta to the separated variables came as a further step quite naturally, since we rather built directly the representation of them. in particular we generated some interpolation formula exploiting their trigonometric or elliptic polynomial nature. In particular, out of these formulas, one can extract the behaviour of the *off-diagonal* terms of the monodromy matrix. Strictly speaking, mainly due to the reflection algebra properties, we had to deal with just the $\mathcal{A}_-$ operator (we used the $\mathcal{D}_-$ for the right representation) working as a ladder operator in both direction according to which root it was evaluated in. We found that

$$\langle\ \beta; h_1, \ldots, h_N \mid \mathcal{A}_-(\pm\zeta_n^{(h_n)})|\beta + 2) = A_-(\pm\zeta_n^{(h_n)}) \langle\ \beta; h_1, \ldots, h_N \mid T_n^{\pm}$$

and in particular

$$A_-(\zeta_n^{(h_n=0)}) = 0, \qquad A_-(-\zeta_n^{(h_n=1)}) = 0.$$

These functions satisfies as well the reflection algebra version of the quantum determinant condition (2.1.39)

$$\mathbf{A}(-\zeta_n^{(h_n=0)})\mathbf{A}(\zeta_n^{(h_n=1)}) = \begin{cases} \dfrac{\text{q-det }(K_+(\xi_n))\text{q-det }(\mathcal{U}_-(\xi_n))}{\sinh(\eta+2\xi_n)\sinh(\eta-2\xi_n)}, & \text{for } \mathbf{XXZ}; \\ \dfrac{\text{q-det }(K_+(\xi_n))\text{q-det }(\mathcal{U}_-(\xi_n))}{\theta_1(\eta+2\xi_n)\theta_1(\eta-2\xi_n)}, & \text{for } \mathbf{XYZ}; \end{cases}$$

where, remarkably, after having fixed the gauge, the product of the functions $\mathbf{A}(\lambda) = A_-(\lambda)\mathsf{a}_+(\lambda|\beta - 1)$, does not depend on the gauge parameters anymore.
XXZ: See eq. (3.4.6).
XYZ: See eq. (4.4.12).

Step 4: the separated equations

the last point in program presented by Sklyanin is given by the actual characterization of the spectrum of the model under study in the SoV representation. This characterization turned out to be quite general for every kind of models treatable with the method and look like a set of N Baxter-like equations

$$\tau(x_n)\varphi(\mathbf{x}) = \Delta_n^-(\mathbf{x})\varphi(T^-\mathbf{x}) + \Delta_n^+(\mathbf{x})\varphi(T^-\mathbf{x}), \qquad \forall n = 1,\dots,N$$

which admit separation of variables, meanining that once defined

$$\varphi(\mathbf{x}) = \prod_{n=1}^{N} Q_n(x_n),$$

we get

$$\tau(x_n)Q(x_n) = \Delta_-(x_n)Q_n(x_n-\eta) + \Delta_+(x_n)Q_n(x_n+\eta),$$
$$\forall x_n \in \Lambda_n, \qquad n \in \{1,\dots,N\}.$$

In our case we get completely analogous results, which read

$$\tau(\zeta_n^{(h_n)})\Psi_\tau(\mathbf{h}) = \mathbf{A}(\zeta_n^{(h_n)})\Psi_\tau(\mathsf{T}_n^-(\mathbf{h})) + \mathbf{A}(-\zeta_n^{(h_n)})\Psi_\tau(\mathsf{T}_n^+(\mathbf{h})), \qquad n \in \{1,\dots,N\}.$$

for both models. The wavefunction Ψ_τ is, by construction is equal to

$$\Psi_\tau(\mathbf{h}) = \prod_{n=1}^{N} Q_t(\zeta_n^{(h_n)}).$$

XXZ: See eq. (3.4.7) and (3.4.15).
XYZ: See eq. (4.4.13) and (4.4.18).

A new type of characterization was pinpointed in our work. This new characterization is equivalent to the one just presented but it present some interesting points. It reads

$$\Sigma_{\mathcal{T}} = \left\{ \tau(\lambda) : \tau(\lambda) = f(\lambda) + \sum_{a=1}^{N} g_a(\lambda)x_a,\ \forall\{x_1,\dots,x_N\} \in \Sigma_T \right\},$$

with the opportune definitions for the functions into play. See corollary 3.4.1 for the XXZ model and theorem 4.4.2 for the XYZ case. The unknowns $\{x_a\}_{a=1}^N$ have to satisfy a set of interlaced quadratic equations

$$x_n \sum_{a=1}^{N} g_a(\zeta_n^{(1)})x_a + x_n f(\zeta_n^{(1)}) = q_n, \quad q_n = \mathbf{A}(\zeta_n^{(1)})\mathbf{A}(-\zeta_n^{(0)}),\ \forall n \in \{1,\dots,N\},$$

which somehow substitute the role usually played by the Bethe ansatz equations.

Step 5: the scalar products

As a final result we arrived to establish a compact, simple determinant formula for the scalar products in the SoV basis. The general structure of the two formulas for the models studied are the same; this fact has not to surprise the reader since it is usual in the SoV picture to have identical structures in models quite different from each others and , in particular, the scalar products. See, for example, the works on Sine-Gordon [104, 100], XXX model [103], antiperiodic XXZ [102] and open XXZ [101]. What we got can be summarized as

$$\langle\omega|\rho\rangle = Z(\beta-2)\det_N ||\mathcal{M}^{(\omega,\rho)}_{a,b}|| \quad \text{with} \quad \mathcal{M}^{(\omega,\rho)}_{a,b} = \sum_{h=0}^{1} \omega_a(\zeta_a^{(h)})\rho_a(\zeta_a^{(h)})(\eta_a^{(h)})^{(b-1)},$$

where

$$\eta_a^{(h)} = \begin{cases} \cosh 2\left[(\xi_a + (h_a - \frac{1}{2})\right], & \text{for } \mathbf{XXZ}; \\ \dfrac{\theta_4^2((\xi_a + (h_a - \frac{1}{2}))}{\theta_2^2((\xi_a + (h_a - \frac{1}{2}))}, & \text{for } \mathbf{XYZ}. \end{cases}$$

The Q-operator

We should remark now that the previous characterization, for the most generic boundary conditions and values of η was not proven to be equivalent to any Bethe equations system. In other terms, we did not build a viable Q-operator that would eventually interpolate our Baxter-like system of equations. This point is fundamental if one wants to link the SoV solution the more popular BA analysis. In a recent paper [74], Kitanine, Maillet and Niccoli, showed how such a description is possible for the non-diagonal generic boundaries XXX and XXZ models. They explicitly built a polynomial Q-operator satisfying an inhomogeneous Baxter equation and leading then to the formulation of some Bethe ansatz equations. in the case of the XXZ model, they defined the following Q-operator

$$Q(\lambda) = 2^N \prod_{n=1}^{N} (\cosh 2\lambda - \cosh 2\lambda_n),$$

where the $\{\lambda_n\}_{n=1}^N$ constitute the set of Bethe roots. The inhomogeneous Baxter equation it satisfies is given by

$$\tau(\lambda)Q(\lambda) = \mathbf{A}(\lambda)Q(\lambda-\eta) + \mathbf{A}(-\lambda)Q(\lambda+\eta) + F(\lambda).$$

Remarkably, these equations, reduce to the usual homogeneous BAxter equations when the SoV construction seizes to exist or to be valid, i.e. when conditions (3.4.31) or (3.4.32) are satisfied. As we discussed in section §3.4.2, when these conditions are true the system fails, but, on the other side, the method developed by Nepomechie in [98, 99] is well defined and somehow, then, almost complementary to SoV. In [25] an equivalent formulation in terms of inhomogeneous Baxter equation was made, but the Q-operator could be written just on conjecture level rather than a rigorous construction. In the same paper the same work was done for the non-diagonal open XYZ model we studied in our last chapter. Although also in this case just a conjectured construction of the Q-operator was possible. It should be, then, an interesting job trying to extend the results of [74] to the XYZ as well.

Let us point out that the nature of the inhomogeneous term of these Baxter's equations resides in the fact that a polynomial solution was implemented. It remains an unresolved task trying to soften this polynomiality condition in order to retrieve homogeneous Baxter's equations.

Inverse problem and form factors

The results summarized so far define a good set-up in order to develop a method of calculation of matrix elements of local operators acting on the transfer matrix eigenstates. The high similarity, that we have the chance to appreciate and remark during the dissertation, in the SoV representations of the 6-vertex and 8-vertex algebras, as well as the SoV pseudo-measure entering the decomposition of the identity, permits to tackle in parallel this problem. The first step then would consist in the *inverse problem*, i.e. in the reconstruction of local operators in terms of Sklyanin's separated variables. This is a crucial and fundamental step as it allows to identify the local quantum operators in terms of the global generators of the SoV representations. It would be nice to reproduce in our case some reconstruction formula like it was done in [72, 93] for the closed chain. Once this problem is solved, then it will be simple to compute algebraically the action of some local operators on the transfer matrix eigenstates and then write them as a linear combination of SoV states. From that expression one can hope to write the form factors by actively exploiting our results on the scalar products, i.e. the action of left separate states on right ones. Let us point out that the inverse problem is not so far to be solved, at least all the needed ingredients have already been established. Indeed, by using the decomposition of the gauged operators in terms of the ungauged ones and exploiting the results obtained in [101], one should have everything to tackle the problem. This last statement means that *a priori* we already know how to describe the matrix elements of a class of local operators for the most general reflection algebra representations for both 6-vertex and 8-vertex type; calculations and details should just be developed from our work and [101].

Only when this program is completed, the problem of calculation of form factors of local operators on transfer matrix eigenstates will come next. A satisfactory answer to this extension of the theory will probably reside in the establishment of some determinant form. This last statement has, of course, to be understood in perspective of the existing results in bibliography, e.g. see [72, 70, 71].

Appendix

APPENDIX A

THE OPEN XXZ CHAIN: COMPLEMENTS

A.1 The face vertex correspondence relations

We reproduce here the list of the *face-vertex correspondence relations* in their trigonometric form used in section §3.2.3, in the fashion of [23].

$$R_{12}(\mu_1-\mu_2)X_1(\mu_1|\beta+2)X_2(\mu_2|\beta+1) = \sinh(\mu_1-\mu_2+\eta)X_2(\mu_2|\beta+2)X_1(\mu_1|\beta+1), \tag{A.1.1a}$$

$$\begin{aligned} &R_{12}(\mu_1-\mu_2)X_1(\mu_1|\beta)Y_2(\mu_2|\beta-1) \\ &\quad= \frac{\sinh(\mu_1-\mu_2)\sinh(\beta-1)\eta}{\sinh\beta\eta}Y_2(\mu_2|\beta)X_1(\mu_1|\beta+1) \\ &\quad+ \frac{\sinh(\mu_1-\mu_2+\beta\eta)\sinh\eta}{\sinh\beta\eta}X_2(\mu_2|\beta)Y_1(\mu_1|\beta-1), \end{aligned} \tag{A.1.1b}$$

$$R_{12}(\mu_1-\mu_2)Y_1(\mu_1|\beta-2)Y_2(\mu_2|\beta-1) = \sinh(\mu_1-\mu_2+\eta)Y_2(\mu_2|\beta-2)Y_1(\mu_1|\beta-1), \tag{A.1.1c}$$

$$\tilde{X}_1(\mu_1|\beta+1)\tilde{X}_2(\mu_2|\beta)R_{12}(\mu_1-\mu_2) = \sinh(\mu_1-\mu_2+\eta)\tilde{X}_2(\mu_2|\beta+1)\tilde{X}_1(\mu_1|\beta), \tag{A.1.1d}$$

$$\begin{aligned} &\tilde{X}_1(\mu_1|\beta+1)\tilde{Y}_2(\mu_2|\beta-2)R_{12}(\mu_1-\mu_2) \\ &\quad= \frac{\sinh(\mu_1-\mu_2)\sinh(\beta+1)\eta}{\sinh\beta\eta}\tilde{Y}_2(\mu_2|\beta-1)\tilde{X}_1(\mu_1|\beta+2) \\ &\quad+ \frac{\sinh(\mu_1-\mu_2+\beta\eta)\sinh\eta}{\sinh\beta\eta}\tilde{X}_2(\mu_2|\beta+1)\tilde{Y}_1(\mu_1|\beta-2), \end{aligned} \tag{A.1.1e}$$

$$\tilde{Y}_1(\mu_1|\beta-1)\tilde{Y}_2(\mu_2|\beta)R_{12}(\mu_1-\mu_2) = \sinh(\mu_1-\mu_2+\eta)\tilde{Y}_2(\mu_2|\beta-1)\tilde{Y}_1(\mu_1|\beta), \tag{A.1.1f}$$

$$\tilde{X}(\mu_1|\beta+1)R_{12}(\mu_1-\mu_2)X(\mu_2|\beta+1) = \frac{\sinh(\mu_1-\mu_2)\sinh(\beta+1)\eta}{\sinh\beta\eta}X_2(\mu_2|\beta)\tilde{X}_1(\mu_1|\beta+2), \quad \text{(A.1.1g)}$$

$$\begin{aligned}\tilde{X}_1(\mu_1|\beta+1)R_{12}(\mu_1-\mu_2)Y_2\mu_2|\beta-1) &= \sinh(\mu_1-\mu_2+\eta)Y_2(\mu_2|\beta-2)\tilde{X}_1(\mu_1|\beta)\\ &+\frac{\sinh(\mu_1-\mu_2+\beta\eta)\sinh\eta}{\sinh\beta\eta}X_2(\mu_2|\beta)\tilde{Y}_1(\mu_1|\beta-2),\end{aligned} \quad \text{(A.1.1h)}$$

$$\begin{aligned}\tilde{Y}_1(\mu_1|\beta-1)R_{12}(\mu_1-\mu_2)X_2\mu_2|\beta+1) &= \sinh(\mu_1-\mu_2+\eta)X_2(\mu_2|\beta+2)\tilde{Y}_1(\mu_1|\beta)\\ &+\frac{\sinh(\mu_2-\mu_1+\beta\eta)\sinh\eta}{\sinh\beta\eta}Y_2(\mu_2|\beta)\tilde{X}_1(\mu_1|\beta+2),\end{aligned} \quad \text{(A.1.1i)}$$

$$\tilde{Y}(\mu_1|\beta-1)R_{12}(\mu_1-\mu_2)Y(\mu_2|\beta-1) = \frac{\sinh(\mu_1-\mu_2)\sinh(\beta-1)\eta}{\sinh\beta\eta}Y_2(\mu_2|\beta)\tilde{Y}_1(\mu_1|\beta-2). \quad \text{(A.1.1j)}$$

A.2 Gauged transformed K_+ matrices

Here we give the explicit expressions for the entries of the matrices $K_+^{(L)}(\lambda|\beta)$ (3.2.40) and $K_+^{(R)}(\lambda|\beta)$ (3.2.41).

$$\begin{aligned}[K_+^{(L)}(\lambda|\beta)]_{11} = {}& \frac{e^{\lambda-\eta/2}}{\sinh\beta\eta\sinh\zeta_+}\,[\sinh\zeta_+\cosh(\lambda+\eta/2)\sinh(\lambda-\eta/2\beta\eta)\\ &-(\cosh\zeta_+\sinh(\lambda+\eta/2)\cosh(\lambda-\eta/2+\beta\eta)+\kappa_+\sinh(2\lambda+\eta)\sinh(\tau_++(\alpha+2)\eta)]\,,\end{aligned} \quad \text{(A.2.1a)}$$

$$[K_+^{(L)}(\lambda|\beta)]_{12} = \frac{e^{(\lambda-\eta/2+(\beta+1)\eta)}\sinh(2\lambda+\eta)[\kappa_+\sinh((\beta-1-\alpha)\eta-\tau_+)-e^{\zeta_+}/2]}{\sinh\beta\eta\sinh\zeta_+}, \quad \text{(A.2.1b)}$$

$$[K_+^{(L)}(\lambda|\beta)]_{21} = \frac{e^{(\lambda-\eta/2-(\beta+1)\eta)}\sinh(2\lambda+\eta)[\kappa_+\sinh((\beta+1+\alpha)\eta+\tau_+)+e^{\zeta_+}/2]}{\sinh\beta\eta\sinh\zeta_+}, \quad \text{(A.2.1c)}$$

$$\begin{aligned}[K_+^{(L)}(\lambda|\beta)]_{22} = {}& \frac{e^{\lambda-\eta/2}}{\sinh\beta\eta\sinh\zeta_+}\,[\sinh\zeta_+\cosh(\lambda+\eta/2)\sinh(-\lambda+\eta/2\beta\eta)\\ &-(\cosh\zeta_+\sinh(\lambda+\eta/2)\cosh(-\lambda+\eta/2+\beta\eta)+\kappa_+\sinh(2\lambda+\eta)\sinh(\tau_++(\alpha+2)\eta)]\,,\end{aligned} \quad \text{(A.2.1d)}$$

and

$$[K_+^{(R)}(\lambda|\beta)]_{11} = e^{\lambda-\eta/2}\frac{e^{\zeta_+}\sinh(\beta-1)\eta-e^{-\zeta_+}\sinh(2\lambda+\beta\eta)-2\kappa_+\sinh(2\lambda+\beta)\sinh(\tau_++\alpha\eta)}{2\sin\beta\eta\sinh\zeta_+}, \quad \text{(A.2.2a)}$$

$$[K_+^{(R)}(\lambda|\beta)]_{12} = [K_+^{(L)}(\lambda|\beta)]_{12}, \quad \text{(A.2.2b)}$$

$$[K_+^{(R)}(\lambda|\beta)]_{21} = [K_+^{(L)}(\lambda|\beta)]_{21}, \quad \text{(A.2.2c)}$$

$$[K_+^{(R)}(\lambda|\beta)]_{22} = e^{\lambda-\eta/2}\frac{e^{\zeta_+}\sinh(\beta+1)\eta-e^{-\zeta_+}\sinh(2\lambda-\beta\eta)+2\kappa_+\sinh(2\lambda+\beta)\sinh(\tau_++\alpha\eta)}{2\sin\beta\eta\sinh\zeta_+}. \quad \text{(A.2.2d)}$$

APPENDIX B

THE OPEN XYZ CHAIN: COMPLEMENTS

B.1 The elliptic Jacobi theta functions

The definitions and notation for the elliptic functions used in the thesis at hand are taken integrally from [60]. Here we will reproduce parts of identities, properties and formula that can be found in section (8.18-8.19)$_{[60]}$.

The Jacobi theta functions are defined as the sums (for $|q| < 1$) of the following series:

$$\theta_1(\lambda|\,\omega) = \frac{1}{i}\sum_{n=-\infty}^{+\infty}(-1)^n e^{i\omega\pi(n+\frac{1}{2})^2}e^{i(2n+1)\lambda}, \quad \text{(B.1.1)}$$

$$\theta_2(\lambda|\,\omega) = \sum_{n=-\infty}^{+\infty} e^{i\omega\pi(n+\frac{1}{2})^2}e^{i(2n+1)\lambda}, \quad \text{(B.1.2)}$$

$$\theta_3(\lambda|\,\omega) = \sum_{n=-\infty}^{+\infty} e^{i\omega\pi n^2}e^{2in\lambda}, \quad \text{(B.1.3)}$$

$$\theta_4(\lambda|\,\omega) = \sum_{n=-\infty}^{+\infty}(-1)^n e^{i\omega\pi n^2}e^{2in\lambda}, \quad \text{(B.1.4)}$$

where ω is the *parameter* of the *gnome* $q = e^{i\omega\pi}$.

Quasi-periodicity

Given $q = e^{i\omega\pi}$ ($Im\,\omega > 0$) and $\theta_i(\lambda|\,\omega) = \theta_i(\lambda)$, theta functions are *quasi-periodic* functions of ω and λ.

$$\theta_1(\lambda+\pi) = -\theta_1(\lambda), \tag{B.1.5a}$$

$$\theta_1(\lambda+\omega\pi) = -\frac{1}{q}e^{-2i\lambda}\theta_1(\lambda) \tag{B.1.5b}$$

$$\theta_2(\lambda+\pi) = -\theta_2(\lambda), \tag{B.1.5c}$$

$$\theta_2(\lambda+\omega\pi) = \frac{1}{q}e^{-2i\lambda}\theta_2(\lambda), \tag{B.1.5d}$$

$$\theta_3(\lambda+\pi) = \theta_3(\lambda), \tag{B.1.5e}$$

$$\theta_3(\lambda+\omega\pi) = \frac{1}{q}e^{-2i\lambda}\theta_3(\lambda), \tag{B.1.5f}$$

$$\theta_4(\lambda+\pi) = \theta_4(\lambda), \tag{B.1.5g}$$

$$\theta_4(\lambda+\omega\pi) = -\frac{1}{q}e^{-2i\lambda}\theta_4(\lambda), \tag{B.1.5h}$$

$$\theta_1(\lambda+\frac{1}{2}\pi) = \theta_2(\lambda), \tag{B.1.6a}$$

$$\theta_1(\lambda+\frac{1}{2}\omega\pi) = iq^{-1/4}e^{-i\lambda}\theta_4(\lambda), \tag{B.1.6b}$$

$$\theta_2(\lambda+\frac{1}{2}\pi) = -\theta_1(\lambda), \tag{B.1.6c}$$

$$\theta_2(\lambda+\frac{1}{2}\omega\pi) = q^{-1/4}e^{-i\lambda}\theta_3(\lambda), \tag{B.1.6d}$$

$$\theta_3(\lambda+\frac{1}{2}\pi) = \theta_4(\lambda), \tag{B.1.6e}$$

$$\theta_3(\lambda+\frac{1}{2}\omega\pi) = q^{-1/4}e^{-i\lambda}\theta_2(\lambda), \tag{B.1.6f}$$

$$\theta_4(\lambda+\frac{1}{2}\pi) = \theta_3(\lambda), \tag{B.1.6g}$$

$$\theta_4(\lambda+\frac{1}{2}\omega\pi) = iq^{-1/4}e^{-i\lambda}\theta_1(\lambda). \tag{B.1.6h}$$

Parity relations

$$\theta_1(-\lambda) = -\theta_1(\lambda), \tag{B.1.7a}$$

$$\theta_2(-\lambda) = \theta_2(\lambda), \tag{B.1.7b}$$

$$\theta_3(-\lambda) = \theta_3(\lambda), \tag{B.1.7c}$$

$$\theta_4(-\lambda) = \theta_4(\lambda). \tag{B.1.7d}$$

Zeroes

$$\theta_1(\lambda) = 0 \quad \text{for} \quad 2m\frac{\pi}{2} + 2n\frac{\pi\omega}{2}, \tag{B.1.8a}$$

$$\theta_2(\lambda) = 0 \quad \text{for} \quad (2m-1)\frac{\pi}{2} + 2n\frac{\pi\omega}{2}, \tag{B.1.8b}$$

$$\theta_3(\lambda) = 0 \quad \text{for} \quad (2m-1)\frac{\pi}{2} + (2n-1)\frac{\pi\omega}{2}, \tag{B.1.8c}$$

$$\theta_4(\lambda) = 0 \quad \text{for} \quad 2m\frac{\pi}{2} + (2n-1)\frac{\pi\omega}{2}. \tag{B.1.8d}$$

with $(m,n) \in \mathbb{Z}^2$.

Identities involving products of theta functions

$$\theta_1(\lambda|\,\omega)\theta_1(\mu|\,\omega) = \theta_3(\lambda+\mu|2\omega)\theta_2(\lambda-\mu|2\omega) - \theta_2(\lambda+\mu|2\omega)\theta_3(\lambda-\mu|2\omega), \tag{B.1.9a}$$

$$\theta_1(\lambda|\,\omega)\theta_2(\mu|\,\omega) = \theta_1(\lambda+\mu|2\omega)\theta_4(\lambda-\mu|2\omega) + \theta_4(\lambda+\mu|2\omega)\theta_1(\lambda-\mu|2\omega), \tag{B.1.9b}$$

$$\theta_2(\lambda|\,\omega)\theta_2(\mu|\,\omega) = \theta_2(\lambda+\mu|2\omega)\theta_3(\lambda-\mu|2\omega) + \theta_3(\lambda+\mu|2\omega)\theta_2(\lambda-\mu|2\omega), \tag{B.1.9c}$$

$$\theta_3(\lambda|\,\omega)\theta_3(\mu|\,\omega) = \theta_3(\lambda+\mu|2\omega)\theta_3(\lambda-\mu|2\omega) + \theta_2(\lambda+\mu|2\omega)\theta_2(\lambda-\mu|2\omega), \tag{B.1.9d}$$

$$\theta_3(\lambda|\,\omega)\theta_4(\mu|\,\omega) = \theta_4(\lambda+\mu|2\omega)\theta_4(\lambda-\mu|2\omega) - \theta_1(\lambda+\mu|2\omega)\theta_1(\lambda-\mu|2\omega), \tag{B.1.9e}$$

$$\theta_4(\lambda|\,\omega)\theta_4(\mu|\,\omega) = \theta_3(\lambda+\mu|2\omega)\theta_3(\lambda-\mu|2\omega) - \theta_2(\lambda+\mu|2\omega)\theta_2(\lambda-\mu|2\omega), \tag{B.1.9f}$$

$$\theta_1(\lambda+\mu)\theta_1(\lambda-\mu)\theta_4^2(0) = \theta_3^2(\lambda)\theta_2^2(\mu) - \theta_2^2(\lambda)\theta_3^2(\mu) = \theta_1^2(\lambda)\theta_4^2(\mu) - \theta_4^2(\lambda)\theta_1^2(\mu), \tag{B.1.9g}$$

$$\theta_2(\lambda+\mu)\theta_2(\lambda-\mu)\theta_4^2(0) = \theta_4^2(\lambda)\theta_2^2(\mu) - \theta_1^2(\lambda)\theta_3^2(\mu) = \theta_1^2(\lambda)\theta_4^2(\mu) - \theta_3^2(\lambda)\theta_1^2(\mu), \tag{B.1.9h}$$

$$\theta_3(\lambda+\mu)\theta_3(\lambda-\mu)\theta_4^2(0) = \theta_4^2(\lambda)\theta_3^2(\mu) - \theta_1^2(\lambda)\theta_2^2(\mu) = \theta_3^2(\lambda)\theta_4^2(\mu) - \theta_2^2(\lambda)\theta_1^2(\mu), \tag{B.1.9i}$$

$$\theta_4(\lambda+\mu)\theta_4(\lambda-\mu)\theta_4^2(0) = \theta_4^2(\lambda)\theta_4^2(\mu) - \theta_1^2(\lambda)\theta_1^2(\mu) = \theta_3^2(\lambda)\theta_3^2(\mu) - \theta_2^2(\lambda)\theta_2^2(\mu), \tag{B.1.9j}$$

$$\theta_1(\lambda+\mu)\theta_1(\lambda-\mu)\theta_3^2(0) = \theta_1^2(\lambda)\theta_3^2(\mu) - \theta_3^2(\lambda)\theta_1^2(\mu) = \theta_4^2(\lambda)\theta_2^2(\mu) - \theta_2^2(\lambda)\theta_4^2(\mu), \tag{B.1.9k}$$

$$\theta_2(\lambda+\mu)\theta_2(\lambda-\mu)\theta_3^2(0) = \theta_2^2(\lambda)\theta_3^2(\mu) - \theta_4^2(\lambda)\theta_1^2(\mu) = \theta_3^2(\lambda)\theta_2^2(\mu) - \theta_1^2(\lambda)\theta_4^2(\mu), \tag{B.1.9l}$$

$$\theta_3(\lambda+\mu)\theta_3(\lambda-\mu)\theta_3^2(0) = \theta_1^2(\lambda)\theta_1^2(\mu) + \theta_3^2(\lambda)\theta_3^2(\mu) = \theta_2^2(\lambda)\theta_2^2(\mu) + \theta_4^2(\lambda)\theta_4^2(\mu), \tag{B.1.9m}$$

$$\theta_4(\lambda+\mu)\theta_4(\lambda-\mu)\theta_3^2(0) = \theta_1^2(\lambda)\theta_2^2(\mu) + \theta_3^2(\lambda)\theta_4^2(\mu) = \theta_2^2(\lambda)\theta_1^2(\mu) + \theta_4^2(\lambda)\theta_3^2(\mu), \tag{B.1.9n}$$

$$\theta_1(\lambda+\mu)\theta_1(\lambda-\mu)\theta_2^2(0) = \theta_1^2(\lambda)\theta_2^2(\mu) - \theta_2^2(\lambda)\theta_1^2(\mu) = \theta_4^2(\lambda)\theta_3^2(\mu) - \theta_3^2(\lambda)\theta_4^2(\mu), \tag{B.1.9o}$$

$$\theta_2(\lambda+\mu)\theta_2(\lambda-\mu)\theta_2^2(0) = \theta_2^2(\lambda)\theta_2^2(\mu) - \theta_1^2(\lambda)\theta_1^2(\mu) = \theta_3^2(\lambda)\theta_3^2(\mu) - \theta_4^2(\lambda)\theta_4^2(\mu), \tag{B.1.9p}$$

$$\theta_3(\lambda+\mu)\theta_3(\lambda-\mu)\theta_2^2(0) = \theta_3^2(\lambda)\theta_2^2(\mu) + \theta_4^2(\lambda)\theta_1^2(\mu) = \theta_2^2(\lambda)\theta_3^2(\mu) + \theta_1^2(\lambda)\theta_4^2(\mu), \tag{B.1.9q}$$

$$\theta_4(\lambda+\mu)\theta_4(\lambda-\mu)\theta_2^2(0) = \theta_4^2(\lambda)\theta_2^2(\mu) + \theta_3^2(\lambda)\theta_1^2(\mu) = \theta_1^2(\lambda)\theta_3^2(\mu) + \theta_2^2(\lambda)\theta_4^2(\mu), \tag{B.1.9r}$$

and the following couple of *Faddeev-Takhtadzan* relations [43].

$$\theta_1(\lambda)\theta_1(\mu)\theta_1(\alpha)\theta_1(\lambda+\mu+\alpha) + \theta_4(\lambda)\theta_4(\mu)\theta_4(\alpha)\theta_4(\lambda+\mu+\alpha) = \theta_4(0)\theta_4(\lambda+\mu)\theta_4(\lambda+\alpha)\theta_4(\mu+\alpha), \tag{B.1.10a}$$

$$\theta_1(\lambda)\theta_1(\mu)\theta_4(\alpha)\theta_4(\lambda+\mu+\alpha) + \theta_4(\lambda)\theta_4(\mu)\theta_1(\alpha)\theta_1(\lambda+\mu+\alpha) = \theta_4(0)\theta_4(\lambda+\mu)\theta_1(\lambda+\alpha)\theta_1(\mu+\alpha), \tag{B.1.10b}$$

B.2 Bridge between notations for the boundary matrices

In this section of the appendix we will reproduce the notational bridge that links the K-matrix introduced in chapter **4**, solution found in [66], and $\hat{K}$ which was used in [46] and introduced at first in [67]

$$\begin{aligned}\hat{K}(\lambda|\zeta,\kappa,\tau) &= \begin{pmatrix} \frac{\theta_4(\zeta|2\omega)\theta_4(-\lambda+\zeta|2\omega)\theta_1(\lambda+\zeta|2\omega)}{\theta_1(\zeta|2\omega)} & \frac{\kappa e^{\tau}\theta_1(2\lambda|2\omega)(\theta_4^2(\lambda|2\omega)-e^{-2\tau}\theta_1^2(\lambda|2\omega))}{\theta_1(\zeta|2\omega)\theta_4^{-3}(\zeta|2\omega)\theta_4^2(0|2\omega)\theta_4(2\lambda|2\omega)} \\ \frac{\kappa e^{-\tau}\theta_1(2\lambda|2\omega)(\theta_4^2(\lambda|2\omega)-e^{2\tau}\theta_1^2(\lambda|2\omega))}{\theta_1(\zeta|2\omega)\theta_4^{-3}(\zeta|2\omega)\theta_4^2(0|2\omega)\theta_4(2\lambda|2\omega)} & \frac{\theta_4(\zeta|2\omega)\theta_4(-\lambda+\zeta|2\omega)\theta_1(\lambda+\zeta|2\omega)}{\theta_1(\zeta|2\omega)} \end{pmatrix} \\ &= \begin{pmatrix} \hat{k}_1 & \hat{k}_2 \\ \hat{k}_3 & \hat{k}_4 \end{pmatrix}\end{aligned} \tag{B.2.1}$$

The two representations are completely equivalent and it's possible to pass from one to another explicitly. Basically, we want to rewrite (B.2.1) in the form:

$$\begin{aligned}\hat{K} &= \begin{pmatrix} K_1+K_2 & K_3-iK_4 \\ K_3+iK_4 & K_1-K_2 \end{pmatrix} = K_1\left(\mathbb{1} + \frac{K_2}{K_1}\sigma_z + \frac{K_3}{K_1}\sigma_x + \frac{K_4}{K_1}\sigma_y\right) \\ &= \hat{F}(\lambda)\left\{\mathbb{1} + \hat{c}_x\frac{\theta_1(\lambda)}{\theta_4(\lambda)}\sigma^x + \hat{c}_y\frac{\theta_1(\lambda)}{\theta_3(\lambda)}\sigma^y + \hat{c}_z\frac{\theta_1(\lambda)}{\theta_2(\lambda)}\sigma^z\right\}.\end{aligned} \tag{B.2.2}$$

By explicit calculations it's possible to arrive to the following

$$\begin{cases} K_1 = \dfrac{k_1+k_4}{2} = \theta_2(\lambda)\theta_1(\zeta)\dfrac{\theta_3(\zeta/2)\theta_4(\zeta/2)}{\theta_1(\zeta/2)\theta_2(\zeta/2)}; \\ \dfrac{K_2}{K_1} = \dfrac{k_1-k_4}{2} = \dfrac{\theta_2(\zeta)}{\theta_1(\zeta)}\dfrac{\theta_1(\lambda)}{\theta_2(\lambda)}; \\ \dfrac{K_3}{K_1} = \dfrac{k_2+k_3}{2} = k\cosh(\tau)\dfrac{\theta_3^2(\zeta/2)\theta_4^2(\zeta/2)}{\theta_1(\zeta)}\dfrac{\theta_4(0)}{\theta_4^2(0)\theta_3^2(0)}\dfrac{\theta_1(\lambda)}{\theta_4(\lambda)}; \\ \dfrac{K_4}{K_1} = \dfrac{k_3-k_2}{2i} = -k\sinh(\tau)\dfrac{\theta_3^2(\zeta/2)\theta_4^2(\zeta/2)}{\theta_1(\zeta)}\dfrac{\theta_3(0)}{\theta_4^2(0)\theta_3^2(0)}\dfrac{\theta_3(\lambda)}{\theta_4(\lambda)}; \end{cases} \tag{B.2.3}$$

and then

$$\begin{cases} \hat{F}(\lambda) = \theta_2(\lambda)\theta_1(\zeta)\dfrac{\theta_3(\zeta/2)\theta_4(\zeta/2)}{\theta_1(\zeta/2)\theta_2(\zeta/2)}; \\ \hat{c}_x = k\cosh(\tau)\dfrac{\theta_3^2(\zeta/2)\theta_4^2(\zeta/2)}{\theta_1(\zeta)}\dfrac{\theta_4(0)}{\theta_4^2(0)\theta_3^2(0)}; \\ \hat{c}_y = -k\sinh(\tau)\dfrac{\theta_3^2(\zeta/2)\theta_4^2(\zeta/2)}{\theta_1(\zeta)}\dfrac{\theta_3(0)}{\theta_4^2(0)\theta_3^2(0)}; \\ \hat{c}_z = \dfrac{\theta_2(\zeta)}{\theta(\zeta)}. \end{cases} \tag{B.2.4}$$

Notice that we basically turned all Jacobi theta functions with quasi-period 2ω appearing in (B.2.1) into theta functions with quasi-period ω, by using the formulas and theta functions properties in appendix B.1; in particular (B.1.9b) and (B.1.9e) were used. The final step in order to get the exact link between (B.2.1) and (4.1.11) consists in the renormalization

$$\begin{cases} \hat{F}(\lambda) \longrightarrow F(\lambda); \\ \hat{c}_x(\zeta,\kappa,\tau) \longrightarrow c_x(\alpha_1,\alpha_2,\alpha_3); \\ \hat{c}_y(\zeta,\kappa,\tau) \longrightarrow c_y(\alpha_1,\alpha_2,\alpha_3); \\ \hat{c}_z(\zeta,\kappa,\tau) \longrightarrow c_z(\alpha_1,\alpha_2,\alpha_3); \end{cases} \tag{B.2.5}$$

Remark B.2.1. This choice of normalization, in particular the choice in the first line of (B.2.5), changes the analytic property of the K-matrix. We should point out that the points $\pm(\eta/2-\omega/2)$ and $\pm(\eta/2-\omega/2-\pi/2)$ which are simple poles in (B.2.1) are now regular points in (4.1.11). In this way, we won't have to deal with limits or residues anymore once we try to interpolate our transfer matrix eigenvalue function on some set of points which usually contain the 4 points listed above, *i.e.* the $\pm\xi_{-3}$ and $\pm\xi_{-4}$ in [46].

Remark B.2.2. The formula (4.1.11) coincides exactly with the one used in [25] once we exploit the relations between the various Jacobi theta functions.

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